A solution to the Collins–Tosatti conjecture

Joint work with D. V. Vu

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A toy example

The toy example

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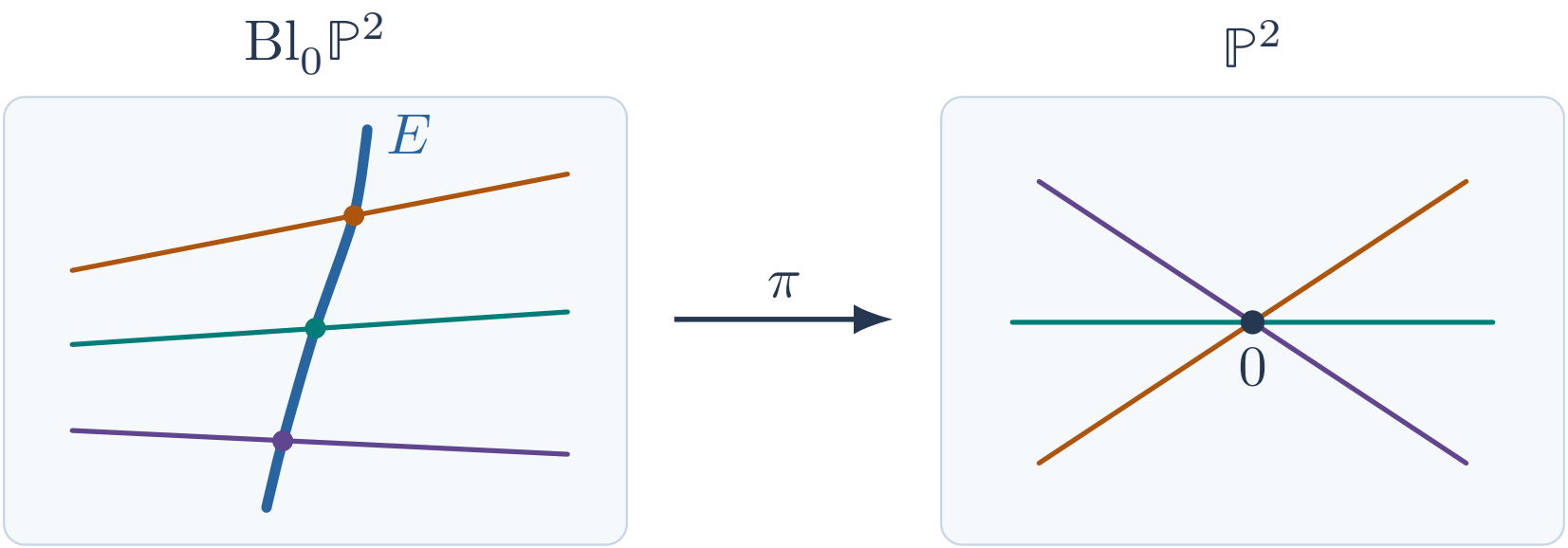
The toy example

Consider the blow-up $\pi\colon \mathrm{Bl}_0\mathbb{P}^2 \to \mathbb{P}^2$ and the pullback of the hyperplane class $H = \pi^*\mathcal{O}(1)$.

The exceptional curve E is canonically associated with H:

1. (Volume aspect) The only curve with zero volume in the class H is E, with $H \cdot E = 0$.
2. (Singularity aspect) All Kähler currents in H are singular along E.

The blow-up and its exceptional curve



$$\pi^{-1}(0) = E \simeq \mathbb{P}^1$$

Main theorem in one sentence

Theorem (Main theorem in one sentence)

On a compact Kähler manifold, the same phenomenon holds for any class:

The locus defined by the Singularity = The locus defined by the Volume

Two descriptions of the obstruction

Consider a pseudoeffective class α on a compact Kähler manifold X.
Pseudoeffective: α admits a positive current.

Non-Kähler locus

$E_{\mathrm{nK}}(\alpha)$: Where every Kähler current in α is singular.

Null locus

$\mathrm{Null}(\alpha)$: The union of irreducible subvarieties with zero restricted volume.

Restricted volume

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For a nef class

$$\mathrm{vol}_{X|V}(\alpha) = \int_V \alpha^{\dim V}.$$

In general, the definition is more involved:

$$\mathrm{vol}_{X|V}(\alpha) = \sup_T \mathrm{Mass}(T|_V),$$

where T runs over all Kähler currents with analytic singularities in α.

The Collins–Tosatti conjecture

Theorem (Vu–Xia, 2026)

$$E_{\mathrm{nK}}(\alpha) = \mathrm{Null}(\alpha).$$

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Known cases

α is nef: Collins–Tosatti.

The algebraic case: Ein–Lazarsfeld–Mustaţă–Nakamaye–Popa.

The algebraic case, α nef: Nakamaye.

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The non-trivial inclusion

Given an irreducible component V of $E_{\mathrm{nK}}(\alpha)$, we wish to show that $\mathrm{vol}_{X|V}(\alpha) = 0$.

Equivalently, suppose that $\mathrm{vol}_{X|V}(\alpha) > 0$, we wish to construct a Kähler current in α that is smooth at a point of V.

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From local information near V, we need to conclude global information on X.

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Solution

We need an extension theorem for Kähler currents.

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The nef case

Assume that α is nef, $V \subseteq X$, $\mathrm{vol}_{X|V}(\alpha) > 0$.

Goal

Construct a Kähler current in α that is smooth at a point of V.

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Proof sketch

1. Construct a Kähler current on V using Demailly–Păun.
2. Extend the Kähler current to a global one using a nef extension theorem.

Demailly–Păun

By a simple reduction, we may assume that α is nef and big, and V is smooth. By assumption,

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Theorem (Demailly–Păun)

The class $\alpha|_V$ contains a Kähler current, and hence a Kähler current with analytic singularities by Demailly's approximation.

Analytic singularities: Locally, the potential is of the form $\varphi = c \log \sum |f_i|^2 + O(1)$ for some holomorphic functions f_i.

The nef extension theorem

Let α be nef and big, and V a smooth component of $E_{\mathrm{nK}}(\alpha)$.

Theorem (Collins–Tosatti, Darvas–Reboulet–Witt Nyström–X.–Zhang)

A Kähler current T with analytic singularities in $\alpha|_V$ can be extended to a global Kähler current in α whose restriction is smooth near a general point of V under mild assumptions.

Together with Demailly–Păun, we conclude a contradiction.

How the two potentials glue

By a simple birational modification, it suffices to extend Kähler currents T on V with divisorial poles.

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1. **Local.** By explicit local coordinates, we may extend T to a small neighborhood U of V in X with potential φ_U.
2. **Global.** Choose a Kähler current in α with mild singularities with potential φ_G.

Gluing

Using nefness, we can control the singularities of φ_G very precisely, so that

φ_G wins

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Near the boundary of U, $\varphi_U - C \leq \varphi_G$.

As a consequence, we can glue φ_U and $\varphi_G + C$ to get the desired extension.

Gluing the local and global potentials

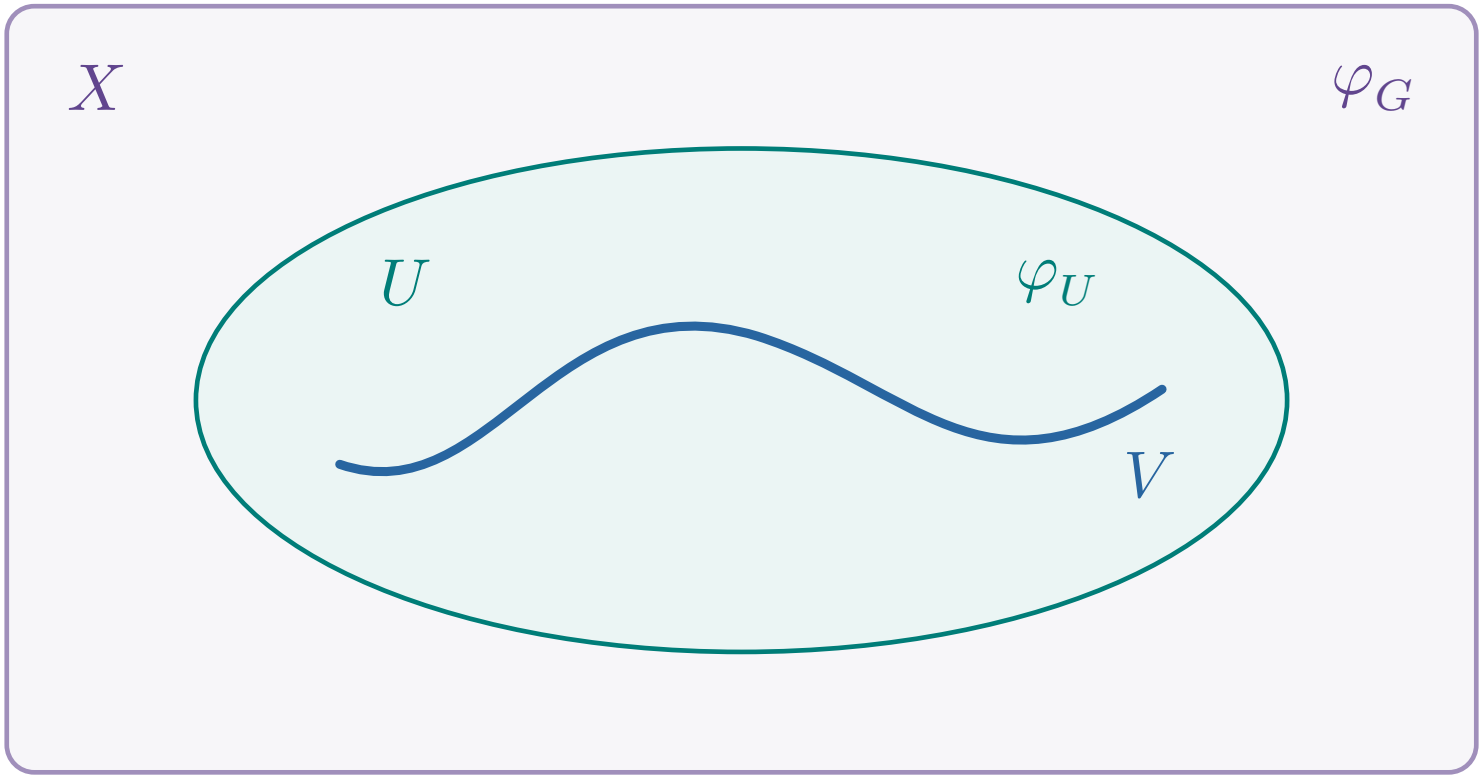


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Why the nef extension argument breaks down

Assume from now on that α is big but not nef. Let V be a smooth component of $E_{\mathrm{nK}}(\alpha)$ with $\mathrm{vol}_{X|V}(\alpha) > 0$.

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Problem

Even the least singular current $T_{\min} \in \alpha$ may have poles along V.

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Gluing

The potential φ_G does not win φ_U in general. Gluing is impossible without a new idea.

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AI has generated generic counterexamples to the extension problem.

A refined version of Demailly–Păun

We first establish a refined version of Demailly–Păun, which allows us to construct a Kähler current on V with prescribed singularities.

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Theorem (Very rough version)

Given a current T on V satisfying some mild assumptions, we can construct a more singular Kähler current in the same class with a controllable lower bound of positivity.

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Theorem (Very rough version)

Given a current T on V satisfying some mild assumptions, we can construct a more singular Kähler current in the same class with a controllable lower bound of positivity.

Key inputs:

- a weak version of the transcendental Morse inequality: Xiao and Popovici;
- the theory of singularities: Darvas and me.

The idea of the general big proof

The global potential is constructed via Demailly approximation of $T_{\min}$.

Theorem

There is a version of Demailly approximation which allows a qualitative control of the loss of positivity.

More precisely, the loss of positivity is of order $1/M$ at the M-th level of approximation.

Idea of the proof

The proof follows the same strategy as the nef case, but with a more delicate gluing argument.

Step 1

Choose a Demailly approximation Φ_q of the current with minimal singularities in α. The loss of positivity is of order $1/q$.

Idea of the proof

The proof follows the same strategy as the nef case, but with a more delicate gluing argument.

Step 1

Choose a Demailly approximation Φ_q of the current with minimal singularities in α. The loss of positivity is of order $1/q$.

Step 2

Suppose that $\mathrm{vol}_{X|V}(\alpha) > 0$. Using the theory of trace operators, and the refined version of Demailly–Păun, we can construct a Kähler current V with singularities prescribed using $\Phi_q|_V$.

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Step 3

The local potential and the global one do not glue yet. We pass to a birational model and make subtle modifications to so that the global potential wins the local one. The gluing is then possible.

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The proof is then complete.

Thank you!

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