Monotonicity of Monge–Ampère products

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Table of Contents

Motivation

Fix

- a compact Kähler manifold X of dimension n;
- a $(1,1)$-cohomology class α on X.

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Here the Monge–Ampère volume $\operatorname{vol} T$ is defined in your favorite way (non-pluripolar/Cao/Darvas–X.).

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Example

$X = \mathbb{P}^1$, $T_c = \omega_{\mathrm{FS}} + \mathrm{dd^c}\varphi_c$, and $\varphi_c \approx c \log |z|^2$. Then

$$\operatorname{vol} T_c = 1 - c.$$

Motivation

Intuition

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Main question

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Let's come back to the example with $c < d$:

Example

$X = \mathbb{P}^1$, $T_c = \omega_{\mathrm{FS}} + \mathrm{dd}^{\mathrm{c}}\varphi_c$, and $\varphi_c \approx c \log |z|^2$. Then

$$\operatorname{vol} T_c - \operatorname{vol} T_d \text{ (volume difference)} = d - c \text{ (Lelong number difference)}.$$

Main theorem

A similar result is true in general.

Theorem (X. 2025)

$$\textit{volume difference} = \sum_{D/X} \textit{Lelong number difference} \cdot \textit{restricted volume}.$$

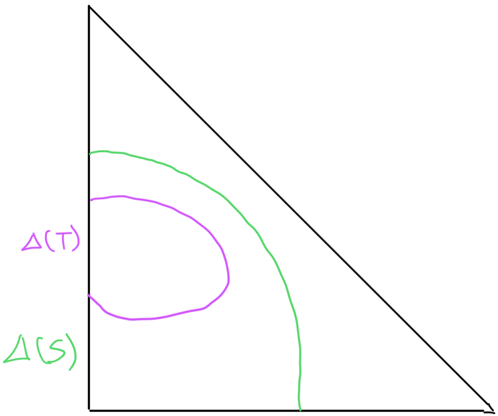
Here D runs over all prime divisors over X.

A prime divisor D over X is a prime divisor on a birational modification $Y \to X$ of X.

Table of Contents

The toric situation

Suppose that everything is toric. Say we consider $(\mathbb{P}^2, \mathcal{O}(1))$.
From the toric dictionary, the currents T and S correspond to convex bodies:



The toric situation

Theorem

$\operatorname{vol}\Delta(T)$ *is (proportional to)* $\operatorname{vol} T$.

Dictionary

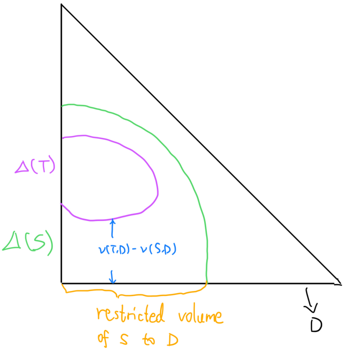
Monotonicity theorem $\iff$ $\Delta(T) \subseteq \Delta(S)$;
Quantitative monotonicity theorem $\iff$ compute the volume of $\Delta(S) - \Delta(T)$.

How do the components show up

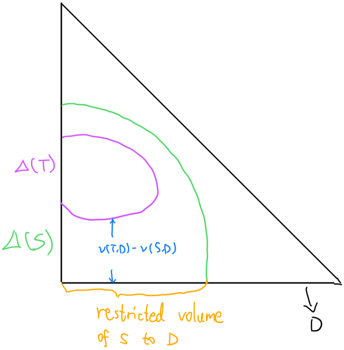
Theorem (Goal)

$$\textit{volume difference} = \sum_{D/X} \textit{Lelong number difference} \cdot \textit{restricted volume}.$$

Recall that facets of the triangle correspond to divisors.



How do the components show up



Minkowski's integral formula tells us that the volume difference can be written as an integral over the unit sphere using these components.

Guess

Theorem (First guess)

If T is more singular than S, then

$$\operatorname{vol}(T_1, \dots, T_{n-1}, S) - \operatorname{vol}(T_1, \dots, T_{n-1}, T) = \sum_{D \subseteq X} (\nu(T, D) - \nu(S, D)) \cdot \operatorname{vol}_{X|D}(T_1, \dots, T_{n-1}).$$

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The restricted volume $\mathrm{vol}_{X|D}(T_1,\dots,T_{n-1})$ is difficult to define, but keep in mind:

- It vanishes if $\nu(T_i,D)>0$ for some i;
- If all T_i's have analytic singularities, it is

$$\int_D T_1|_D\wedge\cdots\wedge T_{n-1}|_D.$$

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But

This is wrong!

Two reasons:

1. The volume is a birational invariant, $\sum_{D \subseteq X}$ is not;
2. The volume is invariant if we remove the divisorial parts from the T_i's, but the restricted volume is not.

Solutions

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Solution

Take restricted volume only after removing the divisorial parts of the T_i's along D.

In fancier terms, this means, we consider restricted volumes of the b-divisors $\mathbb{D}(T_i)$.

Main theorem

Theorem (Main theorem)

If T is more singular than S, then

$$\mathrm{vol}(T_1,\ldots,T_{n-1},S)-\mathrm{vol}(T_1,\ldots,T_{n-1},T)\geq \sum_{D/X}(\nu(T,D)-\nu(S,D))\cdot\mathrm{vol}_{X|D}\left(\mathbb{D}(T_1),\ldots,\mathbb{D}(T_{n-1})\right).$$

Equality holds if either X is projective or if the transcendental Morse inequality holds.

The volume here is in the sense of J. Cao (equivalently in the sense of Darvas–X.): Take Demailly approximations of the currents, and the volume is defined as the limit.

Main theorem

The 1D case is already non-trivial.

Corollary

Suppose that X is a Riemann surface, then

$$\operatorname{vol} S - \operatorname{vol} T = \sum_{x \in X} (\nu(T, x) - \nu(S, x)).$$

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Corollary

Suppose that X is a Riemann surface, then

$$\operatorname{vol} S - \operatorname{vol} T = \sum_{x \in X} \left(\nu(T,x) - \nu(S,x)\right).$$

This generalizes

Example

$X = \mathbb{P}^1$, $T_c = \omega_{\mathrm{FS}} + \mathrm{dd}^{\mathrm{c}}\varphi_c$, and $\varphi_c \approx c \log |z|^2$. Then

$\operatorname{vol} T_c - \operatorname{vol} T_d$ (volume difference) $= d - c$ (Lelong number difference).

Fancy interpretation

If X is projective, X^{an} is the Berkovich analytification of X, then we can define a measure

$$\mu_{T_1,\ldots,T_{n-1}} := \sum_{D/X} \mathrm{vol}_{X|D}\left(\mathbb{D}(T_1),\ldots,\mathbb{D}(T_{n-1})\right)\delta_D.$$

Reformulation as suggested by C. Favre

$$\mathrm{vol}(T_1,\ldots,T_{n-1},S) - \mathrm{vol}(T_1,\ldots,T_{n-1},T) = \int_{X^{\mathrm{an}}} \left(S^{\mathrm{an}} - T^{\mathrm{an}}\right) \mathrm{d}\mu_{T_1,\ldots,T_{n-1}}.$$

It is natural to investigate the dynamical behavior of this measure.

Table of Contents

One application

Our main theorem gives a large number of new inequalities. But let me mention a most exciting one:

Theorem

Suppose that T is more singular than S, denote the cohomology class $\{T\} = \{S\}$ by α, D is a prime divisor over X, then

$$\frac{\operatorname{vol} S - \operatorname{vol} T}{\operatorname{vol} S} \geq (\nu(T, D) - \nu(S, D))^n \cdot \frac{1}{2^{n-1}\left(\nu_{\max}(\alpha, D) - \nu_{\min}(\alpha, D)\right)}.$$

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This greatly generalizes previous results of Vu and Su. The second factor on the RHS is explicit and "universal".

Interesting fact

The proof relies on the volume formula of transcendental Okounkov bodies.

Idea of the proof (If you know what Okounkov bodies are)

The main theorem tells

$$\operatorname{vol} S - \operatorname{vol} T \geq (\nu(T, D) - \nu(S, D)) \cdot \text{Restricted volume}.$$

It suffices to estimate Restricted volume from below.

Idea of the proof (If you know what Okounkov bodies are)

The main theorem tells

$$\operatorname{vol} S - \operatorname{vol} T \geq (\nu(T, D) - \nu(S, D)) \cdot \text{Restricted volume}.$$

It suffices to estimate Restricted volume from below.
But Restricted volume is the volume of a slice in the Okounkov body. We get the desired lower bound using

Elementary fact

vol (slices of a convex body P) $\geq$ a function of $\operatorname{vol} P$.

The volume of the Okounkov body is computed in Darvas–Reboulet–Witt Nyström–X.–Zhang (2024).

References

1. Transcendental b-divisors I. (2025);
2. Transcendental b-divisors II. (2025);

Both are available on my homepage. The first was the subject of my KASS talk of May 2025.


Tack!