Zero-mass conjecture — AI-assisted counterexample

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The physical picture: potential and charge

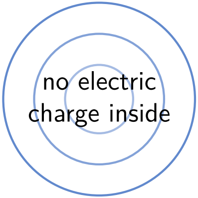
Gauss's law

$$\underset{\text{electric potential}}{\Delta u} = \underset{\text{charge density}}{\rho}$$

The message

The Laplacian of the potential records where the charge is located.

No charge means a harmonic potential



a harmonic potential

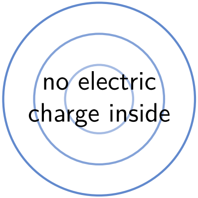
No charge in the disk

If no electric charge is present, then

$$\Delta u = 0.$$

Such a potential is called harmonic. Its value at the center equals its average on every surrounding circle.

No charge means a harmonic potential



a harmonic potential

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So what is a singularity?

It is where the potential behaves as if a point charge were hidden.

Three subharmonic functions in $\mathbb{C}$

A function is subharmonic when its value at a point is no larger than every surrounding circular average.
For a smooth function, this is equivalent to

$$\Delta u \geq 0.$$

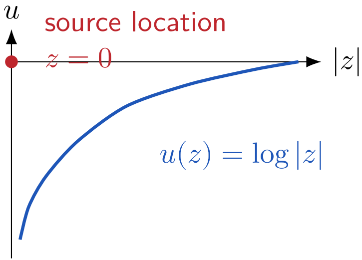
Examples

$u(z) = \operatorname{Re} z$	$\Delta u = 0$	harmonic: no charge
$u(z) = \|z\|^2$	$\Delta u = 4$	smooth positive charge
$u(z) = \log \|z\|$	$\Delta u = 0$ away from 0	all charge at 0

Subharmonic functions may have poles

The last example is allowed to take the value $-\infty$ at $z = 0$.

In one variable, a logarithmic pole is a point charge



Its Riesz measure

$$\mu_u := \frac{1}{2\pi}\Delta u, \qquad \mu_{\log|z|} = \delta_0.$$

- $\log|z|$ measures logarithmic depth;
- δ_0 records one unit of mass at 0.

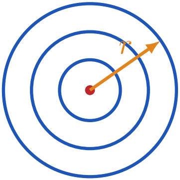
Two descriptions of one phenomenon

The logarithmic singularity and the point charge describe the same event.

A circle can read the hidden charge

For a subharmonic u near 0, allowing an isolated pole there, listen to its circular average:

$$m(r) := \frac{1}{2\pi}\int_0^{2\pi} u(re^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$



the circles shrink toward the pole

Listen as the circles shrink

Compare how fast the average falls with $\log r$. Its limiting slope records exactly how much point charge sits at the center.

The Lelong number

We call this limiting slope the Lelong number:

$$\nu(u,0) = \lim_{r\downarrow 0}\frac{m(r)}{\log r} = \mu_u(\{0\}).$$

Holomorphic zeros are integer point charges

If f is holomorphic near 0, then

$$f(z) = z^m g(z), \qquad g(0) \neq 0,$$

and hence

$$\log|f(z)| = m \log|z| + \underbrace{\log|g(z)|}_{\text{harmonic near } 0}.$$

Slope detector

$$\nu(\log|f|, 0) = m$$

Mass detector

$$\mu_{\log|f|}(\{0\}) = m$$

Three names for the same integer

Order of the zero = logarithmic slope = point mass.

The complete one-variable picture

Riesz decomposition

The Riesz measure of every subharmonic function splits into

a point mass $\nu(u,0)\delta_0$ at 0 $\quad + \quad$ a remainder with no atom at 0.

Therefore

$$\boxed{\mu_u(\{0\}) = \nu(u,0).}$$

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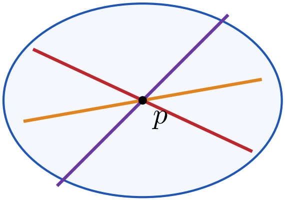
Slope and point mass are the same quantity

$$\nu(u, 0) = 0 \quad \Longleftrightarrow \quad \mu_u(\{0\}) = 0.$$

In $\mathbb{C}$, there is no way to separate the two detectors.

From $\mathbb{C}$ to $\mathbb{C}^2$: keep every complex line



test every complex line through p

The idea

A function $u(z_1, z_2)$ is plurisubharmonic (psh) when, along every complex line, it becomes a one-variable subharmonic function.

The bridge

Every complex direction still carries the one-variable theory we have just learned.

The first detector survives: logarithmic slope

For a psh function near $0 \in \mathbb{C}^2$,

$$\boxed{\nu(u,0) = \liminf_{z\to 0} \frac{u(z)}{\log\|z\|}.}$$

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Two poles that both tend to $-\infty$

singularity	Lelong number
$u = c\log\|z\| + O(1)$	c
$u = -\sqrt{-\log\|z\|}$ near 0	0

Zero does not mean harmless

$\nu(u,0) = 0$ does not mean that u is bounded.

The second detector changes: mass becomes nonlinear

In $\mathbb{C}$

$$\mu_u = \frac{1}{2\pi}\Delta u$$

is a linear detector.

In $\mathbb{C}^2$

$$\mathrm{MA}(u) = (\mathrm{d}\mathrm{d}^{\mathrm{c}} u)^2$$

is a nonlinear detector coupling two complex directions.

The number we care about

The mass sitting exactly at the pole is

$$\tau(u,0) := \mathrm{MA}(u)(\{0\}).$$

Now the conjecture can be stated

For isolated singularities with a well-defined Monge–Ampère measure, comparison gives

$$\boxed{\nu(u,0)^2 \leq \tau(u,0).}$$

$$\begin{array}{c} \nu(u,0) > 0 \\ \text{at least logarithmic strength} \end{array} \xrightarrow{\text{known}} \begin{array}{c} \tau(u,0) > 0 \\ \text{atomic mass} \end{array}$$

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Guedj–Rashkovskii zero-mass conjecture

$$\nu(u,0) = 0 \quad \overset{?}{\Longrightarrow} \quad \tau(u,0) = 0.$$

The question accumulated a long history

2001	2010	2016	2019–26
Rashkovskii considers the zero-mass question.	Guedj studies the zero-mass implication.	It appears in a major problem list.	Positive special cases.

Why it became celebrated

Every successful theorem strengthened the feeling that zero logarithmic slope should leave no atom behind.

Known results lived in orderly settings

Three recurring kinds of structure

- models built from explicit **formulas or equations**;
- models with strong **symmetry**;
- models that were sufficiently **regular and controlled**.

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- models that were sufficiently **regular and controlled**.

The possibility that remained hidden

Perhaps those assumptions were not technical conveniences. Perhaps they prevented a real mechanism of failure.

The answer: maximally negative

Theorem (Li–Xia, 2026)

There exists a psh function u on the bidisk with exactly one pole and

$$u^{-1}(-\infty) = \{0\}, \qquad \nu(u,0) = 0, \qquad \mathrm{MA}(u) = \delta_0.$$

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Not a near miss

The pole has zero Lelong number, yet the entire Monge–Ampère measure is the Dirac mass at that pole.

July 10: I gave AI a problem I expected to be false

Why this problem?

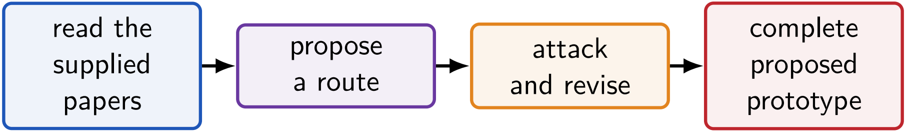
I had long suspected that the zero-mass conjecture was false: no known principle explained why such a weak reading should control the mass.

new model GPT-5.6-sol → local Rethlas clone (PKU) → research task "Find a counterexample"

The setup

Both the idea generator and the checker used GPT-5.6-sol at xhigh reasoning, with recent papers supplied as references.

Nine hours of autonomous search



Not a single answer to a single prompt

One role proposed ideas; another tried to break them. The cycle repeated autonomously.

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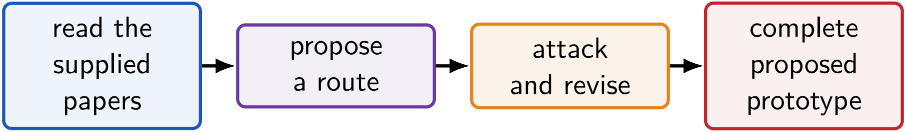

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By that evening

After about **nine hours**, the system returned a complete proposed counterexample.

The decisive move: step just outside the toric case

What was known

Long Li and collaborators had proved the conjecture for **toric** singularities—a highly symmetric class.

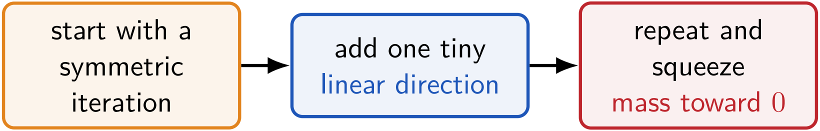
The AI's move

Keep the toric skeleton, then add a tiny perturbation that breaks the symmetry.

The change of viewpoint

Do not abandon the positive theorem. Move just beyond its boundary.

The construction in one picture



Two effects, controlled separately

The surviving direction controls the Lelong number; repeated iteration concentrates the Monge–Ampère mass at the origin.

A formidable prototype, then a simple theorem

The AI prototype

More than **twenty interacting parameters**, tuned with great care. The mechanism worked, but the proof was hard to see through.

The human compression

Over the next few days, I rebuilt the idea as a much simpler example. Long Li and I completed the proof and the paper.

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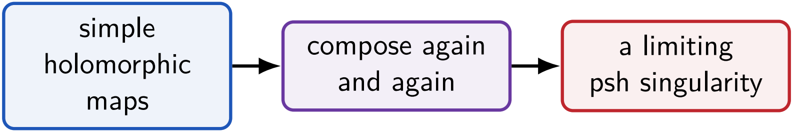
What surprised me most

AI did not merely survive a long calculation. It found the natural route—the route that, afterward, looks as if it should have been tried all along.

Our counterexample is built by complex dynamics

The construction

Rather than write down the final singularity at once, we let repeated compositions of holomorphic maps build it.



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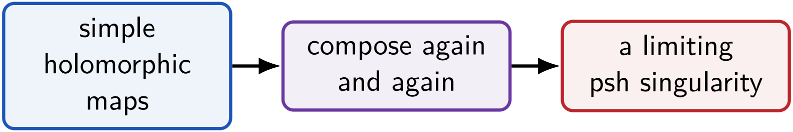


This made an older history suddenly relevant

Complex dynamics had already been used to create surprising geometry in $\mathbb{C}^2$.

Complex dynamics had created unexpected domains before

1919	1933	1988	1997
Fatou gives the first proper domain biholomorphic to $\mathbb{C}^2$.	Bieberbach gives another early example.	Rosay–Rudin develop a rich theory and new examples.	Stensønes engineers a smooth boundary by iteration.

The Fatou–Bieberbach surprise

A proper domain in $\mathbb{C}^2$ can nevertheless be biholomorphic to all of $\mathbb{C}^2$—and iteration is the engine.

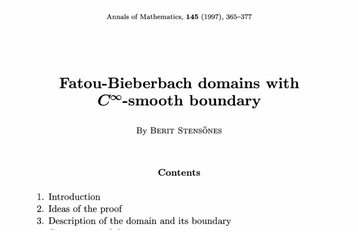
Stensønes's iteration, without the machinery

$$F_k = H_k \circ H_{k-1} \circ \cdots \circ H_2.$$

add a very high power of w to z → then add a very high power of z to w → repeat

New steps are almost invisible on regions already protected, but become enormous outside. The largest region where the compositions converge becomes the domain.

Only afterward, Mattias Jonsson recognized the echo



Annals of Mathematics, **145** (1997), 365–377

Fatou-Bieberbach domains with C^∞-smooth boundary

By Berit Stensönes

Contents

1. Introduction
2. Ideas of the proof
3. Description of the domain and its boundary

The observation

Stensønes's construction and ours share a broad dynamical grammar:

- the map changes at every stage;
- successive scales are very far apart;
- the limit creates a new geometric object.

A retrospective connection

This is an older structural precursor recognized after our counterexample was found.

The same dynamical grammar does a different job

Stensønes, 1997

- every step is reversible;
- iteration carves out a domain;
- goal: a smooth boundary.

Li–Xia, 2026

- every step folds space many-to-one;
- the folding accumulates;
- goal: hidden mass at a pole.

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Precursor, not provenance

A structural precursor only: there is no documented causal link.

Who did what

Discovery and proof

Rethlas proposed an unexpected iteration prototype.
Long Li and Mingchen Xia extracted the mechanism, simplified it, and proved the theorem.

Reading and context

Per Åhag streamlined the final Lelong argument.
Mattias Jonsson recognized the older structural analogue.

My takeaway

AI exposed a mechanism; mathematics made it understandable, accountable, and historical.

What to remember

In one variable

$$\mu_u(\{0\}) = \nu(u,0).$$

Logarithmic slope and point mass are the same detector.

In two variables

$$\nu(u,0) = 0 \not\Longrightarrow \tau(u,0) = 0.$$

Nonlinear coupling can separate them.

The conclusion

The zero-mass conjecture is false, and the counterexample reveals exactly where one-variable intuition breaks.

Thank you!