Lectures on Complex Geometry

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Graduate course at USTC

Course Guide

These notes are written for graduate students who have completed one semester of complex analysis, basic differential geometry, and a small amount of algebraic topology. The principal reference is J.-P. Demailly's *Complex Analytic and Differential Geometry* (2012), supplemented by standard sources in Hodge theory, several complex variables, and modern pluripotential theory. The presentation keeps the compact rhythm of the author's earlier notes, but adds motivation, coordinate calculations, explicit hypothesis checks, and detailed proofs of the results used later in the course.

The semester has 16 weeks, with two 90-minute sessions each week. Every lecture opens with its notation and prerequisite ledgers, followed by a timed plan, and ends with a summary and exercises. Every lecture also has an optional handout. The handouts explore examples, refinements, or later developments, but no subsequent core argument depends on them.

How to use the notes for independent study

Each lecture starts on a new page and opens with two ledgers: *Notation for this lecture* declares the recurring symbols before they are used, and *Prerequisite results used below* states the hypotheses and conclusions of the external results invoked in the lecture. These are followed by the core exposition, extended calculations, a self-study workshop, exercises, and complete solutions near the end of the book. A productive two-day cycle is the following.

(1) Before reading, copy the workshop's prerequisite ledger and mark every item that you can define without looking it up. Return to the indicated earlier lecture if more than one item is missing.
(2) Read the main lecture once for structure. On the second reading, reproduce every displayed calculation; in complex geometry a proof is often hidden in a type, sign, or transition-function check.
(3) Close the notes and answer the checkpoint questions. Compare with the solutions immediately following the checkpoint, but rewrite any missed argument in your own notation rather than merely rereading it.
(4) Attempt the numbered exercises without the solution chapter. Use the first sentence of a solution as a hint only after a serious attempt, and record which hypothesis or previous theorem unlocked the problem.
(5) Finish by testing the exit criterion aloud. The criterion asks for a reconstruction, not recognition: one should be able to state the hypotheses, carry out the model calculation, and explain where the conclusion would fail if a key assumption were removed.

The workshops deliberately repeat a small amount of notation. That repetition is a diagnostic tool: if a symbol has changed meaning between lectures, or if a theorem is being used on the wrong bidegree, the discrepancy should become visible before it enters a long proof. The convention ledger for d^{c}, curvature, weighted norms, and adjoints remains in force throughout.

Remark (Convention: the normalization of $\mathrm{d^c}$ used throughout)

We fix
$$\mathrm{d^c} = \frac{1}{4\pi \mathrm{i}}(\partial - \bar\partial), \qquad \mathrm{dd^c} = \frac{\mathrm{i}}{2\pi}\partial\bar\partial.$$
Thus $\mathrm{dd^c}\log|z|^2 = \delta_0$ on $\mathbb{C}$, and for a meromorphic function f, $\mathrm{dd^c}\log|f|^2 = [\mathrm{div}(f)]$. Other sources often differ by a factor of 2 or 2π; convert the normalization before importing any formula.

Learning objectives

By the end of the course, a student should be able to:

(1) compute fluently with (p,q)-forms, Chern connections, and curvature on complex manifolds;
(2) move correctly among the languages of functions, distributions, and currents, including their positivity notions;
(3) explain the links among pseudoconvexity, Stein geometry, psh functions, and solvability of $\bar\partial$;
(4) calculate basic cohomology groups using sheaves, the Dolbeault resolution, and Hodge theory;
(5) derive vanishing theorems and weighted L^2 estimates from the Bochner–Kodaira–Nakano identity;
(6) follow standard modern arguments involving positive line bundles, positive currents, Lelong numbers, and singular metrics.

Sixteen-week schedule

Week	First meeting	Second meeting
1	1. Course introduction; complex manifolds and maps	2. (p,q)-forms, ∂, $\bar\partial$, and Kähler geometry
2	3. Hermitian/Kähler metrics and Fubini–Study	4. Distributions, currents, and Poincaré–Lelong
3	5. Subharmonic functions and regularization	6. Plurisubharmonic functions and the Levi form
4	7. Pseudoconvexity, exhaustion, and the Levi problem	8. Stein manifolds and Cartan's theorems
5	9. Sheaves, exact sequences, and first cohomology	10. Weierstrass theory, coherence, and analytic sets
6	11. Complex spaces, normalization, and divisors	12. Modifications, blow-ups, and exceptional divisors
7	13. Positive forms and positive currents	14. Closed positive currents, potentials, and compactness
8	15. Bedford–Taylor Monge–Ampère theory	16. Lelong numbers, Jensen's formula, and Siu's theorem
9	17. Vector bundles, connections, and Chern–Weil	18. Chern connections, line bundles, and c_1
10	19. Griffiths/Nakano positivity and curvature formulas	20. Positive line bundles, Kodaira embedding, projectivity

11	21. Elliptic operators, harmonic forms, Hodge theory	22. Kähler identities and the $\partial\bar\partial$-lemma
12	23. Dolbeault, Čech, and sheaf cohomology	24. Serre duality, Frölicher, and examples
13	25. The Bochner–Kodaira–Nakano identity	26. Kodaira–Akizuki–Nakano vanishing
14	27. Hörmander's weighted L^2 estimate	28. Complete Kähler metrics, singular weights, multiplier ideals
15	29. Introduction to Ohsawa–Takegoshi extension	30. Calabi–Yau and the continuity method
16	31. Nef/big/psef cones and regularization	32. A capstone calculation and a map of the subject

Reading map to Demailly

The principal reference uses a different order and, in several places, a different normalization. The following map makes it possible to read the two texts in parallel. The lecture notes retain the convention ledger above even when the reference uses $\mathrm{dd^c} = \mathrm{i}\partial\bar\partial/\pi$ or suppresses a constant.

Lectures	Sections of [1] most closely aligned with the course
1–8	Chapter I, §§1–7: differential calculus, currents, subharmonic and psh functions, holomorphic convexity, and pseudoconvexity.
9–12	Chapter II, §§1–10: sheaves, local analytic rings, coherence, analytic spaces, normalization, meromorphic maps, modifications, and blow-ups.
13–16	Chapter III, §§1–8: positivity cones, closed positive currents, Monge–Ampère products, generalized Lelong numbers, Jensen–Lelong, and Siu semicontinuity.
17–20	Chapter V, §§1–16, and Chapter VII, §§11–14: Hermitian bundles, Chern connections, exact sequences, projective bundles, ampleness, and Kodaira embedding.
21–24	Chapter IV, §§1–11, and Chapter VI, §§1–12: sheaf cohomology, double complexes, Hodge theory, Kähler identities, curves, and the Frölicher sequence.
25–29	Chapters VII–VIII: the Bochner–Kodaira–Nakano identity, curvature positivity, vanishing, complete metrics, weighted L^2 estimates, and extension.
30–32	Demailly's positivity and Monge–Ampère chapters provide the background; the Calabi–Yau, regularization, and non-pluripolar developments use the additional references listed in the bibliography.

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Chapter 1

Complex Manifolds, Differential Forms, and Currents

1.1 Lecture 1: Course introduction and complex manifolds

Week 1 material • 90 minutes

Notation for this lecture

X and Y denote complex manifolds and $n = \dim_{\mathbb{C}} X$ is the complex dimension. The real tangent bundle is TX, and $J_X : TX \to TX$ is the complex-structure endomorphism, characterized by $J_X^2 = -\mathrm{id}$, where id is the identity; J_Y has the analogous meaning. We write $\mathrm{d}f_x$ for the real differential of a smooth map f at x, and $T^{1,0}X$ for the $+\mathrm{i}$ eigensubbundle of $TX \otimes_{\mathbb{R}} \mathbb{C}$. The symbols $\mathbb{C}^n$ and $\mathbb{P}^n$ mean complex affine and projective n-space, $\mathcal{O}(X)$ is the ring of holomorphic functions on X, and $[z_0 : \cdots : z_n]$ denotes homogeneous coordinates on $\mathbb{P}^n$. Finally, $\mathrm{End}(TX)$ is the bundle of real-linear endomorphisms of TX, and $b_k = \dim_{\mathbb{R}} H^k(X, \mathbb{R})$ is the k-th Betti number.

Prerequisite results used below

Holomorphic inverse function theorem. Let $U \subset \mathbb{C}^n$ be open, let $F : U \to \mathbb{C}^n$ be holomorphic, and let $a \in U$. If $\det(\partial F_i/\partial z_j)(a) \neq 0$, then there are neighborhoods $a \in U_0 \subset U$ and $F(a) \in V_0$ such that $F : U_0 \to V_0$ is biholomorphic; its inverse is holomorphic.

Holomorphic implicit function theorem. Let $U \subset \mathbb{C}^n \times \mathbb{C}^r$ be open, write its coordinates as (z, w), and let $F = (F_1, \ldots, F_r) : U \to \mathbb{C}^r$ be holomorphic. Suppose $F(a, b) = 0$ and $\det(\partial F_i/\partial w_j)(a, b) \neq 0$. Then, after shrinking to neighborhoods $a \in U_z$ and $b \in U_w$, there is a unique holomorphic map $g : U_z \to U_w$ such that

$$F(z, w) = 0 \iff w = g(z) \qquad ((z, w) \in U_z \times U_w).$$

More generally, if $F : \Omega \subset \mathbb{C}^N \to \mathbb{C}^r$ has rank r along $F^{-1}(0)$, then that zero set is locally biholomorphic to an open subset of $\mathbb{C}^{N-r}$.

Quotient-manifold criterion. If a discrete group Γ acts freely and properly discontinuously by biholomorphisms on a complex manifold M, then M/Γ has a unique complex-manifold structure for which the quotient map is a local biholomorphism.

Plan for the lecture. 0–10 min: the central questions of the course; 10–20: four languages and their interfaces; 20–30: semester map, conventions, and proof-reading habits; 30–50: atlases and holomorphic maps; 50–72: projective space, tori, and submanifolds; 72–84: the maximum principle on compact manifolds; 84–90: coordinate checks.

1.1.1 Course introduction: what complex geometry studies

Complex geometry begins with manifolds that locally look like open subsets of $\mathbb{C}^n$, but its characteristic questions are global. Four examples will guide the semester.

(1) *Existence:* when does a topological or differentiable manifold carry a complex or Kähler structure?
(2) *Functions and embeddings:* when are there enough global holomorphic functions or sections to describe the manifold inside $\mathbb{C}^N$ or $\mathbb{P}^N$?
(3) *Curvature and topology:* which cohomological restrictions follow from positivity, and when does a cohomology class contain a positive representative?
(4) *Singular limits:* how can divisors, degenerating metrics, and weak limits be treated without losing positivity?

The subject answers these questions by moving among four languages:

$$\begin{tikzcd}
\text{local holomorphic analysis} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{psh functions and positive currents} \arrow[d, leftrightarrow] \\
\text{sheaves and cohomology} \arrow[r, leftrightarrow] \arrow[ur, phantom, "\longleftrightarrow"] & \text{Hermitian curvature}
\end{tikzcd}$$

The arrows are the actual content of the course. The Dolbeault resolution turns $\bar\partial$ into sheaf cohomology; Chern connections turn metrics into curvature classes; Poincaré–Lelong turns zeros into currents; the Bochner–Kodaira–Nakano identity turns curvature into estimates; and those estimates produce vanishing, embeddings, and extension theorems.

Remark (A model problem)

Let $L \to X$ be a holomorphic line bundle on a compact complex manifold. Asking whether powers of L embed X into projective space forces us to use nearly the whole course: choose a Hermitian metric, read positivity from its Chern curvature, solve a weighted $\bar\partial$ equation to manufacture peak sections, use cohomology to globalize the construction, and finally check separation of points and tangent vectors. Kodaira's embedding theorem in Lecture 20 completes this circle.

How the semester is organized. Lectures 1–8 develop local differential and several-complex-variable tools. Lectures 9–16 introduce sheaves, analytic spaces, positive currents, and singularities. Lectures 17–24 connect curvature to topology through vector bundles and Hodge theory. Lectures 25–29 form the analytic climax: Bochner identities, vanishing, weighted L^2 estimates, and extension. Lectures 30–32 show how these tools enter nonlinear equations and modern positivity.

How to read a proof in this course. Before following a calculation, write a short ledger:

(1) the type and degree of every form or current;
(2) the normalization of d^{c}, curvature, and volume;
(3) the regularity and compactness or completeness hypotheses;
(4) the exact positivity statement and the vector space on which it is tested;
(5) the theorem that changes a local statement into a global one.

Most incorrect complex-geometric arguments fail at one of these five entries rather than in a long calculation.

1.1.2 Complex manifolds

A complex manifold carries two compatible structures: it is a smooth manifold of real dimension $2n$, and it has complex coordinates whose transition maps are holomorphic. The second condition replaces arbitrary smooth changes of coordinates by maps satisfying the Cauchy–Riemann equations. Consequently, surprisingly little local data can force a global conclusion.

Definition 1.1: Complex manifold

A complex manifold of complex dimension n is a Hausdorff, second-countable topological space X equipped with charts

$$\varphi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{C}^n$$

whose domains cover X, such that every φ_α is a homeomorphism and every nonempty transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic.

Hausdorffness gives uniqueness of limits. Second countability supplies partitions of unity, metrizability, and the usual functional-analytic tools. Neither condition is cosmetic.

If $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ are two holomorphic coordinate systems, the complex Jacobian $(\partial w_j/\partial z_k)$ is invertible. The coordinate vector fields therefore transform by

$$\frac{\partial}{\partial z_k} = \sum_j \frac{\partial w_j}{\partial z_k}\frac{\partial}{\partial w_j},$$

which defines the holomorphic tangent bundle $T^{1,0}X$. Its dual has local frame $\mathrm{d}z_1, \dots, \mathrm{d}z_n$. These transformation laws explain why tangent vectors and differential forms, unlike individual coordinate derivatives, are intrinsic.

Definition 1.2: Holomorphic and biholomorphic maps

Let X and Y be complex manifolds. A continuous map $f : X \to Y$ is *holomorphic* if every coordinate expression $\psi \circ f \circ \varphi^{-1}$ is holomorphic. It is *biholomorphic* if it is bijective and both f and f^{-1} are holomorphic.

Warning

The inverse of a bijective holomorphic map is holomorphic, but the proof uses the open mapping theorem or the local inverse theorem; it does not follow from "continuous bijection." Also remember that complex dimension n means real dimension $2n$. Losing this factor is a frequent source of errors in codimension and transversality arguments.

Here $J_X : TX \to TX$ and $J_Y : TY \to TY$ are the complex-structure endomorphisms declared at the beginning of the lecture. In coordinates, $\mathrm{d}f \circ J_X = J_Y \circ \mathrm{d}f$. Conversely, a smooth field $J \in \mathrm{End}(TX)$ with $J^2 = -\mathrm{id}$ is an *almost complex structure*. It comes from holomorphic coordinates exactly when its Nijenhuis tensor vanishes, by the Newlander–Nirenberg theorem.

1.1.3 Four families of examples

Example 1.3: Domains and analytic submanifolds

Every domain $\Omega \subset \mathbb{C}^n$ is a complex manifold. If $F : \Omega \to \mathbb{C}^r$ is holomorphic and has rank r along $X = F^{-1}(0)$, the holomorphic implicit-function theorem makes X a complex submanifold of dimension $n - r$. For example,

$$Q = \{[z_0 : z_1 : z_2] \in \mathbb{P}^2 : z_0^2 + z_1^2 + z_2^2 = 0\}$$

is a smooth projective curve.

Example 1.4: Complex projective space

Set $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. On $U_j = \{[z] : z_j \neq 0\}$ use

$$\varphi_j([z_0 : \cdots : z_n]) = \left(\frac{z_0}{z_j}, \ldots, \widehat{\frac{z_j}{z_j}}, \ldots, \frac{z_n}{z_j}\right).$$

The transition functions are rational with nonvanishing denominators on overlaps, hence holomorphic. The quotient is compact because the unit sphere maps onto it.

Example 1.5: Complex tori

If $\Lambda \subset \mathbb{C}^n$ is a lattice of rank $2n$, then $X = \mathbb{C}^n/\Lambda$ is a compact complex manifold. Translation invariance lets $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ descend to global holomorphic one-forms. For $n = 1$ this is the analytic model of an elliptic curve.

Example 1.6: Hopf surfaces

Fix $0 < |q| < 1$ and let $\mathbb{Z}$ act on $\mathbb{C}^2 \setminus \{0\}$ by $m \cdot z = q^m z$. The quotient X_q is a compact complex surface, but it is not Kähler. Indeed, it is diffeomorphic to $S^1 \times S^3$, so $b_1 = 1$, whereas the first Betti number of a compact Kähler manifold is even. Lecture 24 will close this argument.

Proposition 1.7: Holomorphic functions on compact manifolds

If X is compact and connected, every $f \in \mathcal{O}(X)$ is constant.

Proof. The function $|f|$ attains its maximum. In a chart about a maximum point, the classical maximum-modulus principle makes f constant on the corresponding connected component. Hence $\{x : f(x) = f(x_0)\}$ is nonempty, open, and closed, so it is all of X. □

Remark (Geometric interpretation)

The scarcity of global functions explains why line bundles, meromorphic functions, sheaves, and cohomology are indispensable. A compact complex manifold may have only constant holomorphic functions while carrying many holomorphic sections of nontrivial line bundles.

1.1.4 Reading geometry from transition functions

The definition by atlases becomes useful only after one learns to extract bundles and maps from the transition functions. Suppose z and w are holomorphic coordinates on an overlap. The matrices

$$g_{wz} = \left(\frac{\partial w_j}{\partial z_k}\right)$$

satisfy the cocycle identities $g_{uz} = g_{uw} g_{wz}$ and $g_{zz} = I$. They therefore glue the trivial bundles $U_z \times \mathbb{C}^n$ to $T^{1,0}X$. Taking inverse transposes glues the cotangent bundle, taking determinants glues the canonical bundle $K_X = \Lambda^n T^{*1,0}X$, and applying any matrix representation constructs the associated tensor bundles. This is the first instance of a general principle: geometric objects are local data satisfying an overlap rule.

For projective space this calculation is concrete. On $U_0 \cap U_1 \subset \mathbb{P}^n$ write

$$w_1 = \frac{z_0}{z_1} = \frac{1}{u_1}, \qquad w_j = \frac{z_j}{z_1} = \frac{u_j}{u_1} \quad (j \geq 2),$$

where $u_j = z_j/z_0$. The transition map is holomorphic precisely where $u_1 \neq 0$, and its inverse has the same form. Differentiating it displays the nonconstant transition matrix of $T^{1,0}\mathbb{P}^n$; this is already evidence that the tangent bundle is not globally a product.

Proposition 1.8: Coordinate criterion for immersions and submersions

Let $f : X^n \to Y^m$ be holomorphic. If $\operatorname{rank}_{\mathbb{C}} \mathrm{d}f_x = r$, then there are holomorphic coordinates centered at x and $f(x)$ in which

$$f(z_1, \ldots, z_n) = (z_1, \ldots, z_r, F_{r+1}(z), \ldots, F_m(z)).$$

When $r = m$, the fibers are complex submanifolds of dimension $n - m$; when $r = n$, f is a local holomorphic embedding.

Proof. Choose a nonzero $r \times r$ minor of the complex Jacobian and apply the holomorphic inverse-function theorem to the corresponding source variables together with the remaining source coordinates. In the new coordinates the first r output components are coordinate projections. If the rank is locally constant, the remaining output components have zero derivatives along the fiber directions and hence depend only on the first r coordinates. The two extreme cases give the assertions. □

1.1.5 Quotients, compactness, and two global tests

A free properly discontinuous action of a discrete group Γ on a complex manifold M by biholomorphisms gives a complex manifold M/Γ: choose a small open set U disjoint from all its nontrivial translates and use $U \to M/\Gamma$ as a chart. This produces tori and Hopf manifolds. Proper discontinuity is essential; a dense translation subgroup of $\mathbb{C}$ produces a quotient whose topology is not even Hausdorff.

Compactness is often best checked before holomorphic structure. The sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ surjects continuously onto $\mathbb{P}^n$, so projective space is compact. A closed fundamental parallelepiped surjects onto $\mathbb{C}^n/\Lambda$, so a complex torus is compact. For a Hopf manifold, a compact annulus $\{|q| \leq |z| \leq 1\}$ meets every orbit, with its two boundary components identified.

Two quick tests recur throughout the course. First, if a proposed holomorphic function on a compact connected manifold is nonconstant, the maximum principle finds an error. Second, if a proposed coordinate construction changes complex dimension, the complex rank theorem finds an error. Applied early, these tests prevent local formulas from being mistaken for global geometry.

Remark (A useful distinction)

A holomorphic bijection between complex manifolds is biholomorphic, but a bijective holomorphic map between singular analytic spaces need not be an isomorphism. The normalization map of the cusp is the standard counterexample and will appear in Lectures 10–11. Smoothness is doing real work in the manifold statement.

1.1.6 Complete verification of the basic examples

The examples above are useful only after every clause in the definition has been checked. We now carry out those checks. In each case the topology is the quotient or subspace topology, and the displayed coordinates will also prove that it is Hausdorff and second countable.

Open sets and a smooth affine hypersurface

If $\Omega \subset \mathbb{C}^n$ is open, the identity map is one chart, so there are no transition maps to check. A less tautological example is

$$X = \{(z, w) \in \mathbb{C}^2 : zw = 1\}.$$

The projection $p_z : X \to \mathbb{C}^*$ is bijective and has inverse

$$\sigma_z(\zeta) = (\zeta, \zeta^{-1}).$$

Both maps are holomorphic, so $X \simeq \mathbb{C}^*$ as a complex manifold. This also follows from the implicit-function theorem: for $F(z, w) = zw - 1$,

$$\mathrm{d}F = w\,\mathrm{d}z + z\,\mathrm{d}w,$$

and z and w cannot both vanish on X. At $(z_0, w_0) \in X$, differentiating $zw = 1$ gives the concrete tangent-space calculation

$$T^{1,0}_{(z_0,w_0)}X = \{(a, b) \in \mathbb{C}^2 : w_0 a + z_0 b = 0\} = \mathbb{C}\,(z_0, -w_0).$$

Now verify the projective conic

$$Q = \{[z_0 : z_1 : z_2] \in \mathbb{P}^2 : z_0^2 + z_1^2 + z_2^2 = 0\}.$$

On U_0 put $x = z_1/z_0$ and $y = z_2/z_0$. Then

$$Q \cap U_0 = \{1 + x^2 + y^2 = 0\}, \qquad \mathrm{d}(1 + x^2 + y^2) = 2x\,\mathrm{d}x + 2y\,\mathrm{d}y.$$

The two coefficients cannot vanish on this zero set. If $y_0 \neq 0$, projection to x is a local coordinate and implicit differentiation gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y};$$

if $x_0 \neq 0$, projection to y is a local coordinate. The same calculation on U_1 and U_2 covers Q. Thus Q is a one-dimensional complex submanifold. Equivalently, the homogeneous gradient $(2z_0, 2z_1, 2z_2)$ never vanishes on projective space.

The atlas of projective space

For $U_j = \{z_j \neq 0\}$ define φ_j as above. It is bijective because its inverse inserts 1 in the j-th position:

$$(u_0, \dots, \widehat{u_j}, \dots, u_n) \longmapsto [u_0 : \cdots : u_{j-1} : 1 : u_{j+1} : \cdots : u_n].$$

For example, on $U_0 \cap U_1$, write

$$u_k = \frac{z_k}{z_0} \quad (k \geq 1), \qquad v_0 = \frac{z_0}{z_1}, \quad v_k = \frac{z_k}{z_1} \quad (k \neq 1).$$

Then

$$v_0 = \frac{1}{u_1}, \qquad v_k = \frac{u_k}{u_1} \quad (k \geq 2).$$

The overlap in u-coordinates is $\{u_1 \neq 0\}$, so every denominator is nonzero. The inverse transition has the identical form with u and v interchanged. Hence the maps are biholomorphic and define a complex atlas.

The quotient is Hausdorff: each complex line has a unique orbit under positive real scaling meeting the unit sphere, and the remaining S^1-action on the sphere is by a compact group. It is second countable because it is covered by the finitely many second-countable sets $U_j \simeq \mathbb{C}^n$. Finally, the continuous map $S^{2n+1} \to \mathbb{P}^n$ is surjective, so $\mathbb{P}^n$ is compact.

For $\mathbb{P}^1$, the coordinates $u = z_1/z_0$ and $v = z_0/z_1$ satisfy $v = u^{-1}$. Identifying $\mathbb{C} \cup \{\infty\}$ with the unit sphere by inverse stereographic projection,

$$u \longmapsto \frac{1}{1 + |u|^2}\left(2\Re u, 2\Im u, |u|^2 - 1\right),$$

and sending ∞ to the north pole gives the Riemann-sphere model. Near infinity the coordinate is $v = 1/u$, exactly the second projective chart.

The local charts of a complex torus

Let $\Lambda \subset \mathbb{C}^n$ be a lattice and $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$ the quotient map. Discreteness gives

$$\rho := \frac{1}{2} \inf\{|\lambda| : 0 \neq \lambda \in \Lambda\} > 0.$$

For any $a \in \mathbb{C}^n$, the translates of $B_\rho(a)$ are pairwise disjoint. Therefore $\pi|_{B_\rho(a)}$ is a homeomorphism onto its image and defines a chart

$$\phi_a : \pi(B_\rho(a)) \longrightarrow B_\rho(0), \qquad \phi_a([z]) = z - a,$$

where the unique representative in $B_\rho(a)$ is used. If two such charts overlap, their representatives differ by a fixed $\lambda \in \Lambda$, and hence

$$\phi_b \circ \phi_a^{-1}(\zeta) = \zeta + a - b - \lambda.$$

This is an affine holomorphic map with Jacobian I_n. It follows at once that the forms $\mathrm{d}z_j$ are invariant under every deck translation and descend to the quotient. A closed fundamental parallelepiped is compact and maps surjectively to the quotient, proving compactness.

The quotient charts of a Hopf surface

Let $M = \mathbb{C}^2 \setminus \{0\}$ and let $\gamma(z) = qz$, $0 < |q| < 1$. The action of $\mathbb{Z}$ is free: $q^m z = z$ with $z \neq 0$ implies $q^m = 1$, hence $m = 0$. It is properly discontinuous. Indeed, if $K \Subset M$, choose $0 < r \leq |z| \leq R$ on K; then $q^m K \cap K \neq \varnothing$ forces

$$\frac{r}{R} \leq |q|^m \leq \frac{R}{r},$$

which holds for only finitely many integers m. Shrink a ball B about any point until its nontrivial translates are disjoint. The restriction $B \to \pi(B)$ is a quotient chart. On an overlap, changing the lift replaces z by $q^m z$, so every transition map is the linear biholomorphism $z \mapsto q^m z$.

Every orbit meets the compact annulus

$$A = \{z \in \mathbb{C}^2 : |q| \leq |z| \leq 1\};$$

therefore $A \to M/\langle\gamma\rangle$ is surjective and the quotient is compact. This proves directly that the Hopf quotient is a compact complex surface. Its non-Kählerness uses the later Hodge-theoretic fact that a compact Kähler manifold has even first Betti number.

Remark (What has actually been verified)

For every example we exhibited local coordinates, wrote every nontrivial coordinate change, checked that its denominator does not vanish on the overlap, and supplied the topological argument needed for the quotient. Merely writing a quotient or a set of equations would not prove that it is a complex manifold.

1.1.7 Self-study workshop: certifying an atlas rather than recognizing a familiar space

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. You should know the holomorphic inverse and implicit function theorems in one and several variables, the quotient topology, and the difference between a continuous bijection and a homeomorphism. The goal is to turn the definition of a complex manifold into a checklist that works on unfamiliar examples.

Reconstruct the proof in four passes. Suppose a space is presented either as a zero set or as a quotient. For a zero set $X = F^{-1}(0) \subset \mathbb{C}^N$, first compute the complex Jacobian of F. Constant rank r produces $N - r$ local coordinates; one must say which coordinates are free and which are holomorphic functions of them. Next check that the subspace topology is Hausdorff and second countable. These properties are inherited from $\mathbb{C}^N$.

For a quotient M/Γ, the analytic and topological tasks separate. Freeness prevents orbifold points. Proper discontinuity supplies an open set U with $\gamma U \cap U = \varnothing$ for $\gamma \neq 1$, so the quotient map restricts to a homeomorphism $U \to \pi(U)$. If two quotient charts overlap, two lifts differ by one deck transformation; hence the transition map is that transformation in local coordinates. Finally, Hausdorffness must be proved from properness, and compactness, when claimed, from a compact set meeting every orbit.

As a small new test, let $\Gamma = \langle z \mapsto qz \rangle$ act on $\mathbb{C}^*$, where $0 < |q| < 1$. Given $a \in \mathbb{C}^*$, choose a disc $D(a, \varepsilon)$ whose radial range is too narrow to meet any $q^m D(a, \varepsilon)$ for $m \neq 0$. The quotient charts have transitions $z \mapsto q^m z$. The annulus $\{|q| \leq |z| \leq 1\}$ meets every orbit, so the quotient is a compact Riemann surface. The logarithm identifies its universal cover with $\mathbb{C}$ and its deck group with the lattice generated by $2\pi\mathrm{i}$ and $\log q$; thus it is also a one-dimensional complex torus.

Checkpoint (definitions under pressure)

(1) Why does nonvanishing of $\mathrm{d}F$ prove smoothness of $\{F = 0\}$ but not compactness?
(2) On $U_0 \cap U_1 \subset \mathbb{P}^n$, which condition guarantees that $v_k = u_k/u_1$ is holomorphic?
(3) Why is a fixed point of a group action an obstruction to the quotient being a manifold with quotient charts?

Solution (checkpoint). (1) The implicit function theorem is local and says nothing about sequences escaping to infinity; compactness is global. (2) The overlap is exactly $\{u_1 \neq 0\}$, so the denominator never vanishes there. (3) At a fixed point no neighborhood has disjoint nontrivial translates. The quotient map is not locally one-to-one, and the local model is generally a branched or orbifold quotient rather than an open subset of $\mathbb{C}^n$. ⋄

Exit criterion

Given a new hypersurface or a discrete quotient, you can write its chart map and inverse, compute every transition map, and separately justify Hausdorffness, second countability, dimension, and any compactness assertion.

Summary. The definition of a complex manifold needs both its topological hypotheses and holomorphic transition maps. Projective space, complex tori, analytic submanifolds, and Hopf surfaces will be test objects throughout the course.

Exercise 1.9: The two charts of the projective line

Write the transition function for the two standard charts of $\mathbb{P}^1$ and prove that $\mathbb{P}^1$ is biholomorphic to the Riemann sphere.

Exercise 1.10: Tangent spaces

Let $X = \{F = 0\} \subset \mathbb{C}^n$ and assume $\mathrm{d}F_x$ has full rank. Prove that $T_x^{1,0}X = \ker(\mathrm{d}F_x : \mathbb{C}^n \to \mathbb{C}^r)$.

Optional handout: Almost complex structures and integrability

This handout is optional. No later argument in the core lectures depends on it.

A smooth manifold M^{2n} is *almost complex* if it carries an endomorphism $J : TM \to TM$ with $J^2 = -\mathrm{id}$. Complexifying the tangent bundle gives the $\pm\mathrm{i}$ eigenspace decomposition

$$T_M \otimes_{\mathbb{R}} \mathbb{C} = T_J^{1,0}M \oplus T_J^{0,1}M.$$

This is already enough to define the type of a differential form. What is missing is a coordinate system in which J is multiplication by i.

Definition 1.11: Nijenhuis tensor

For real vector fields X, Y, define

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y].$$

Although the formula contains Lie brackets, the derivative terms cancel and N_J is C^∞-bilinear; hence it is a tensor.

If J comes from holomorphic coordinates, coordinate vector fields commute and a direct calculation gives $N_J = 0$. Equivalently, $N_J = 0$ says that sections of $T_J^{0,1}M$ are closed under the Lie bracket. The difficult converse is the Newlander–Nirenberg theorem: a sufficiently regular almost complex structure is induced by complex coordinates exactly when $N_J = 0$. Its proof solves an overdetermined nonlinear first-order system by converting it to a $\bar{\partial}$ problem and iterating with regularity estimates.

Remark (Complex dimension one)

On a real surface, every almost complex structure is integrable. There is no room for the $(0, 2)$ obstruction represented by N_J. Thus an oriented Riemannian surface becomes a Riemann surface after rotating each tangent plane by 90 degrees.

Exercise 1.12: A nonintegrable model

On $\mathbb{C}^2$ define $T^{0,1}$ to be spanned by $L_1 = \partial_{\bar{z}_1}$ and $L_2 = \partial_{\bar{z}_2} + \bar{z}_1 \partial_{z_1}$. Compute $[L_1, L_2]$ and show that this almost complex structure is not integrable.

1.2 Lecture 2: (p,q)-forms, ∂, $\bar\partial$, and the complex structure

Week 1 material • 90 minutes

Notation for this lecture

$\mathcal{A}^{p,q}(X)$ is the space of smooth complex differential forms of bidegree (p,q), while $\mathcal{A}^k(X,\mathbb{C})$ is the space of smooth complex k-forms. Multi-indices $I=(i_1<\cdots<i_p)$ and $J=(j_1<\cdots<j_q)$ abbreviate $\mathrm{d}z_I=\mathrm{d}z_{i_1}\wedge\cdots\wedge\mathrm{d}z_{i_p}$ and similarly for $\mathrm{d}\bar z_J$. The exterior derivative is $\mathrm{d}=\partial+\bar\partial$; ∂ raises the first bidegree and $\bar\partial$ raises the second. Throughout, $\mathrm{d}^{\mathrm{c}}=(4\pi\mathrm{i})^{-1}(\partial-\bar\partial)$ and $\mathrm{dd}^{\mathrm{c}}=\mathrm{dd}^{\mathrm{c}}=(\mathrm{i}/2\pi)\partial\bar\partial$. The pullback by f is f^*, J is the complex structure, J^* is its induced action on one-forms, and Ω^p_X is the sheaf of holomorphic p-forms.

Prerequisite results used below

Complexified eigenspace decomposition. If $J:V\to V$ is real linear with $J^2=-I$, where $I=\mathrm{id}_V$, then $V_{\mathbb{C}}=V^{1,0}\oplus V^{0,1}$, where J acts by i and $-$i, respectively. The same decomposition on covectors is compatible with wedge products and therefore gives $\Lambda^kV^*_{\mathbb{C}}=\bigoplus_{p+q=k}\Lambda^{p,q}V^*$.
Cauchy–Green right inverse. If $g\in C^\infty_c(\mathbb{C})$, then

$$(Tg)(z)=\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{C}}\frac{g(\zeta)}{\zeta-z}\,\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta$$

is smooth and satisfies $\bar\partial(Tg)=g\,\mathrm{d}\bar z$. This one-variable identity, applied coefficientwise and followed by induction, is the analytic input in the proof of the local Dolbeault lemma.

Plan for the lecture. 0–25 min: complexified tangent bundles and type decomposition; 25–50: coordinate formulas for ∂ and $\bar\partial$; 50–70: the local Dolbeault lemma; 70–82: d^{c} and normalization checks; 82–90: sign exercises.

In local coordinates $z_j=x_j+\mathrm{i}y_j$,

$$T^*X\otimes_{\mathbb{R}}\mathbb{C}=T^{*1,0}X\oplus T^{*0,1}X,\qquad \mathrm{d}z_j=\mathrm{d}x_j+\mathrm{i}\mathrm{d}y_j,\quad \mathrm{d}\bar z_j=\mathrm{d}x_j-\mathrm{i}\mathrm{d}y_j.$$

Therefore

$$\mathcal{A}^k(X,\mathbb{C})=\bigoplus_{p+q=k}\mathcal{A}^{p,q}(X),\qquad \alpha=\sum_{|I|=p,|J|=q}\alpha_{I\bar J}\,\mathrm{d}z_I\wedge\mathrm{d}\bar z_J.$$

Definition 1.13: Dolbeault operators

The exterior derivative decomposes by type as $\mathrm{d}=\partial+\bar\partial$, where

$$\partial:\mathcal{A}^{p,q}\to\mathcal{A}^{p+1,q},\qquad \bar\partial:\mathcal{A}^{p,q}\to\mathcal{A}^{p,q+1}.$$

For a function f,

$$\partial f=\sum_j\frac{\partial f}{\partial z_j}\mathrm{d}z_j,\quad \bar\partial f=\sum_j\frac{\partial f}{\partial\bar z_j}\mathrm{d}\bar z_j,\quad \frac{\partial}{\partial z_j}=\frac12\left(\frac{\partial}{\partial x_j}-\mathrm{i}\frac{\partial}{\partial y_j}\right).$$

The identity $\mathrm{d}^2=0$ and uniqueness of type give

$$\partial^2=0,\qquad \bar\partial^2=0,\qquad \partial\bar\partial+\bar\partial\partial=0.$$

A function is holomorphic exactly when $\bar\partial f=0$.

Warning

Conjugation exchanges type but does not reverse wedge order: if $\alpha \in \mathcal{A}^{p,q}$, then $\bar{\alpha} \in \mathcal{A}^{q,p}$ and $\overline{\alpha \wedge \beta} = \bar{\alpha} \wedge \bar{\beta}$. A second common error is to omit the factor $1/2$ in $\partial/\partial z$.

Theorem 1.14: Local Dolbeault lemma

If $q \geq 1$ and $\alpha \in \mathcal{A}^{p,q}(X)$ satisfies $\bar{\partial}\alpha = 0$, then locally $\alpha = \bar{\partial}\beta$ for some $\beta \in \mathcal{A}^{p,q-1}$.

Proof. First consider a compactly supported function g in a disc. The Cauchy–Green operator

$$(Tg)(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$$

satisfies $\bar{\partial}(Tg) = g \, \mathrm{d}\bar{z}$ in distributions and hence smoothly. A cutoff reduces a local problem to this compactly supported one.

On a polydisc in $\mathbb{C}^n$, split a (p, q)-form according to whether it contains $\mathrm{d}\bar{z}_n$:

$$\alpha = \alpha_0 + \alpha_1 \wedge \mathrm{d}\bar{z}_n,$$

where neither α_0 nor α_1 contains $\mathrm{d}\bar{z}_n$. Apply the Cauchy–Green operator in the z_n variable coefficientwise to α_1, with the sign chosen so that the $\mathrm{d}\bar{z}_n$ component of $\bar{\partial}\beta_1$ equals that of α. Then $\alpha - \bar{\partial}\beta_1$ has no $\mathrm{d}\bar{z}_n$ component. Its $\bar{\partial}$-closedness implies that its coefficients are holomorphic in z_n. Regard z_n as a parameter and apply induction on n to the remaining variables. This gives $\alpha = \bar{\partial}(\beta_1 + \beta_0)$. A coordinate neighborhood can be shrunk to a polydisc. The condition $q \geq 1$ is essential: in degree $q = 0$, the kernel of $\bar{\partial}$ is the sheaf of holomorphic p-forms rather than zero. □

Thus the complex

$$0 \longrightarrow \Omega_X^p \longrightarrow \mathcal{A}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,1} \xrightarrow{\bar{\partial}} \cdots$$

is exact. In Lecture 23, the acyclicity of fine sheaves upgrades this local statement to the Dolbeault isomorphism.

Remark (Convention)

With $\mathrm{d}^c = (4\pi\mathrm{i})^{-1}(\partial - \bar{\partial})$,

$$\mathrm{d}\mathrm{d}^c = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}, \qquad \mathrm{d}\mathrm{d}^c|z|^2 = \frac{\mathrm{i}}{2\pi}\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

For real u, $\mathrm{d}^c u$ is a real one-form. Test every imported formula first on $u = |z|^2$ and then on $u = \log|z|^2$.

Proposition 1.15: Holomorphic pullback preserves type

If $f : X \to Y$ is holomorphic, then $f^*\mathcal{A}^{p,q}(Y) \subset \mathcal{A}^{p,q}(X)$ and $f^*\partial = \partial f^*$, $f^*\bar{\partial} = \bar{\partial} f^*$.

Proof. Locally $f^*(\mathrm{d}w_j) = \sum_k (\partial f_j/\partial z_k)\mathrm{d}z_k$, with no $\mathrm{d}\bar{z}_k$ term; conjugation gives the other type. Pullback commutes with d, and comparison of type components gives the last two identities. □

1.2.1 Type decomposition as an eigenspace calculation

The almost-complex structure acts on one-forms by $(J^*\alpha)(v) = \alpha(Jv)$. On the complexified cotangent space its eigenvalues are i on $T^{*1,0}X$ and $-\mathrm{i}$ on $T^{*0,1}X$. Hence the projections of a complex one-form

α are

$$\alpha^{1,0} = \frac{1}{2}(\alpha - \mathrm{i}J^*\alpha), \qquad \alpha^{0,1} = \frac{1}{2}(\alpha + \mathrm{i}J^*\alpha).$$

Taking exterior powers gives the full (p, q) decomposition. This makes clear that type is intrinsic even though the symbols $\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$ use a chart.

For a smooth function on $\mathbb{C}^n$,

$$\mathrm{d}f = \sum_j f_{z_j}\mathrm{d}z_j + \sum_j f_{\bar{z}_j}\mathrm{d}\bar{z}_j.$$

The Cauchy–Riemann equations are therefore the single invariant equation $\bar{\partial} f = 0$. For the test function $f(z_1, z_2) = z_1\bar{z}_2 + |z_2|^2$ one obtains

$$\partial f = \bar{z}_2\mathrm{d}z_1 + \bar{z}_2\mathrm{d}z_2, \qquad \bar{\partial} f = z_1\mathrm{d}\bar{z}_2 + z_2\mathrm{d}\bar{z}_2.$$

Differentiating once more verifies directly that $\partial\bar{\partial} f = -\bar{\partial}\partial f$; doing this calculation once is the safest way to remember the wedge sign.

For a general (p, q)-form, the coefficient formula is

$$\bar{\partial}\left(\sum_{I,J} \alpha_{I\bar{J}}\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J\right) = \sum_{I,J,k} \frac{\partial \alpha_{I\bar{J}}}{\partial \bar{z}_k}\mathrm{d}\bar{z}_k \wedge \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J.$$

One may move $\mathrm{d}\bar{z}_k$ to the customary position after the holomorphic differentials, but this contributes $(-1)^p$. Many apparent disagreements between coordinate formulas are only this ordering convention.

1.2.2 From the local lemma to a global obstruction

The local Dolbeault lemma says that every closed (p, q)-form with $q > 0$ has local primitives. Let $\{U_i\}$ be a cover and choose $\alpha|_{U_i} = \bar{\partial}\beta_i$. On overlaps,

$$\bar{\partial}(\beta_i - \beta_j) = 0.$$

Thus the failure of the local primitives to glue is itself holomorphic data. Changing β_i changes this overlap family by a coboundary. Sheaf cohomology, introduced in Lecture 9, is designed to record exactly this ambiguity.

The first nontrivial example is a compact Riemann surface. Locally every smooth $(0, 1)$-form is a $\bar{\partial}$ of a function, but globally the equation may fail. On a complex torus $X = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, the form $\mathrm{d}\bar{z}$ descends and is $\bar{\partial}$-closed. If $\mathrm{d}\bar{z} = \bar{\partial}u$ globally, pairing with the holomorphic form $\mathrm{d}z$ and applying Stokes would give

$$0 = \int_X \mathrm{d}z \wedge \bar{\partial}u = \int_X \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

which is impossible. Local exactness and global exactness are therefore genuinely different statements.

Remark (Normalization audit)

With $\mathrm{d}^{\mathrm{c}} = (4\pi\mathrm{i})^{-1}(\partial - \bar{\partial})$,

$$\mathrm{d}^{\mathrm{c}}u = \frac{1}{4\pi}\sum_j (u_{x_j}\mathrm{d}y_j - u_{y_j}\mathrm{d}x_j), \qquad \mathrm{d}\mathrm{d}^{\mathrm{c}}u = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}u.$$

Both operators take real functions to real forms. The upright symbols emphasize that they are differential operators rather than scalar variables.

1.2.3 Coordinate calculations with type

Let $z = x + \mathrm{i}y$. Solving $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - \mathrm{i}\mathrm{d}y$ for the real forms gives

$$\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z}), \qquad \mathrm{d}y = \frac{1}{2\mathrm{i}}(\mathrm{d}z - \mathrm{d}\bar{z}).$$

The dual formulas are

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right).$$

Consequently, for $f(x, y) = x^2 + y^2 = z\bar{z}$,

$$\partial f = \bar{z}\,\mathrm{d}z, \qquad \bar{\partial} f = z\,\mathrm{d}\bar{z},$$

and a second differentiation gives

$$\partial\bar{\partial} f = \mathrm{d}z \wedge \mathrm{d}\bar{z}, \qquad \bar{\partial}\partial f = \mathrm{d}\bar{z} \wedge \mathrm{d}z = -\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

This single calculation displays both $\partial\bar{\partial} + \bar{\partial}\partial = 0$ and the sign caused by exchanging two one-forms.

For a general $(1, 1)$-form on $\mathbb{C}^2$,

$$\alpha = \sum_{j,k=1}^{2} a_{j\bar{k}}\,\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k,$$

the two components of its derivative are

$$\partial\alpha = \sum_{\ell,j,k} \frac{\partial a_{j\bar{k}}}{\partial z_\ell}\mathrm{d}z_\ell \wedge \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k,$$
$$\bar{\partial}\alpha = \sum_{\ell,j,k} \frac{\partial a_{j\bar{k}}}{\partial \bar{z}_\ell}\mathrm{d}\bar{z}_\ell \wedge \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k.$$

The first has type $(2, 1)$ and the second type $(1, 2)$. Since these summands lie in distinct subbundles, $\mathrm{d}\alpha = 0$ is equivalent to the two equations $\partial\alpha = 0$ and $\bar{\partial}\alpha = 0$.

A complete $\bar{\partial}$ solution in one variable

On a disc, solve

$$\bar{\partial} u = (z^2 + \bar{z})\,\mathrm{d}\bar{z}.$$

Since $\partial u/\partial\bar{z} = z^2 + \bar{z}$, one may integrate while treating z as a constant:

$$u(z) = z^2\bar{z} + \frac{1}{2}\bar{z}^2 + h(z),$$

where h is arbitrary and holomorphic. Direct differentiation verifies the equation. The nonuniqueness is exactly $\ker \bar{\partial} = \mathcal{O}$.

In several variables compatibility appears. For

$$f = f_1\,\mathrm{d}\bar{z}_1 + f_2\,\mathrm{d}\bar{z}_2,$$

one computes

$$\bar{\partial} f = \left(\frac{\partial f_2}{\partial \bar{z}_1} - \frac{\partial f_1}{\partial \bar{z}_2}\right)\mathrm{d}\bar{z}_1 \wedge \mathrm{d}\bar{z}_2.$$

Thus $\bar{\partial} u = f$ can hold only if the coefficient in parentheses vanishes. For the concrete closed form

$$f = \bar{z}_2\,\mathrm{d}\bar{z}_1 + \bar{z}_1\,\mathrm{d}\bar{z}_2$$

the condition holds, and $u = \bar{z}_1\bar{z}_2$ is a solution. By contrast, $f = \bar{z}_2\,\mathrm{d}\bar{z}_1$ is not closed and cannot be a $\bar{\partial}$ of a function.

The real-coordinate form of $\mathrm{d^c}$

With the convention $\mathrm{d^c} = (4\pi \mathrm{i})^{-1}(\partial - \bar{\partial})$ and real u,

$$\begin{aligned}\partial u - \bar{\partial} u &= \frac{1}{2}(u_x - \mathrm{i}u_y)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) - \frac{1}{2}(u_x + \mathrm{i}u_y)(\mathrm{d}x - \mathrm{i}\mathrm{d}y) \\ &= \mathrm{i}(u_x\,\mathrm{d}y - u_y\,\mathrm{d}x).\end{aligned}$$

Hence

$$\boxed{\mathrm{d^c}u = \frac{1}{4\pi}(u_x\,\mathrm{d}y - u_y\,\mathrm{d}x)},$$

which is visibly real. Applying d gives

$$\mathrm{dd^c}u = \frac{1}{4\pi}(u_{xx} + u_{yy})\,\mathrm{d}x \wedge \mathrm{d}y = \frac{\mathrm{i}}{2\pi}u_{z\bar{z}}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

For $u = |z|^2$, this becomes $\mathrm{dd^c}|z|^2 = \pi^{-1}\mathrm{d}x \wedge \mathrm{d}y$.

A pullback calculation

Let $F : \mathbb{C} \to \mathbb{C}^2$, $F(\zeta) = (\zeta^2, e^{\zeta})$. Then

$$F^*(\mathrm{d}z_1) = 2\zeta\,\mathrm{d}\zeta, \qquad F^*(\mathrm{d}z_2) = e^{\zeta}\,\mathrm{d}\zeta, \qquad F^*(\mathrm{d}\bar{z}_j) = \overline{F^*(\mathrm{d}z_j)}.$$

There is no $\mathrm{d}\bar{\zeta}$ term in the pullback of a $(1,0)$-form. For instance,

$$F^*(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2) = 2\zeta e^{\bar{\zeta}}\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta},$$

which remains of type $(1,1)$. If instead $F(\zeta) = (\bar{\zeta}, 0)$, then $F^*(\mathrm{d}z_1) = \mathrm{d}\bar{\zeta}$; this is the explicit failure of type preservation for an antiholomorphic map.

1.2.4 Self-study workshop: recovering type, signs, and the Dolbeault operators

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall complexification of a real vector space, the graded Leibniz rule for d, and the Cauchy–Riemann equations. Keep the convention $z_j = x_j + \mathrm{i}y_j$ visible until every factor of $1/2$ is secure.

A calculation to reproduce without looking. Solving

$$\mathrm{d}z_j = \mathrm{d}x_j + \mathrm{i}\mathrm{d}y_j, \qquad \mathrm{d}\bar{z}_j = \mathrm{d}x_j - \mathrm{i}\mathrm{d}y_j$$

for the real covectors gives

$$\mathrm{d}x_j = \frac{1}{2}(\mathrm{d}z_j + \mathrm{d}\bar{z}_j), \qquad \mathrm{d}y_j = \frac{1}{2\mathrm{i}}(\mathrm{d}z_j - \mathrm{d}\bar{z}_j).$$

Duality then gives

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \mathrm{i}\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial y_j}\right).$$

For $f(z_1, z_2) = z_1\bar{z}_2 + \bar{z}_1 z_2$ this yields

$$\partial f = \bar{z}_2\,\mathrm{d}z_1 + \bar{z}_1\,\mathrm{d}z_2, \qquad \bar{\partial} f = z_2\,\mathrm{d}\bar{z}_1 + z_1\,\mathrm{d}\bar{z}_2.$$

Applying ∂ to the second expression and $\bar\partial$ to the first shows directly that $\partial\bar\partial f = -\bar\partial\partial f$: moving a $(0,1)$ covector past a $(1,0)$ covector contributes the minus sign.

The local Dolbeault lemma is a solvability statement, not merely an identity. In one variable the form

$$\alpha = z\,\mathrm{d}\bar z$$

is $\bar\partial$-closed for degree reasons and has the explicit primitive $u = z\bar z$, since $\bar\partial u = z\,\mathrm{d}\bar z$. In several variables the compatibility conditions appear: if $\alpha = \sum_j a_j \mathrm{d}\bar z_j$, then

$$\bar\partial\alpha = 0 \qquad \Longleftrightarrow \qquad \frac{\partial a_j}{\partial \bar z_k} = \frac{\partial a_k}{\partial \bar z_j} \quad \text{for all } j,k.$$

They are the complex analogue of equality of mixed partial derivatives.

Checkpoint (a complete coefficient test)

On $\mathbb{C}^2$, let

$$\alpha = \bar z_2\,\mathrm{d}\bar z_1 + \bar z_1\,\mathrm{d}\bar z_2.$$

Determine its type, compute $\bar\partial\alpha$ with all wedge signs, find an explicit primitive, and explain why α cannot equal ∂u for a function u.

Solution (checkpoint). The form has type $(0,1)$. We have

$$\bar\partial\alpha = \mathrm{d}\bar z_2 \wedge \mathrm{d}\bar z_1 + \mathrm{d}\bar z_1 \wedge \mathrm{d}\bar z_2 = 0.$$

Moreover $\alpha = \bar\partial(\bar z_1\bar z_2)$. It cannot be ∂u unless it vanishes, because ∂u has type $(1,0)$ and the type decomposition is direct. ⋄

Exit criterion

You can convert between real and complex bases, determine the type of every term before differentiating, and solve a simple $\bar\partial$ equation while stating its compatibility condition.

Summary. Type decomposition is the grammar of complex geometry. The identity $\bar\partial^2 = 0$ creates the Dolbeault complex, and the local Dolbeault lemma proves its local exactness. Formulas involving d^{c}, curvature, or Poincaré–Lelong are meaningful only together with a normalization.

Exercise 1.16: Sign check

For $\alpha \in \mathcal{A}^{p,q}$ and $\beta \in \mathcal{A}^{r,s}$, prove $\bar\partial(\alpha\wedge\beta) = \bar\partial\alpha\wedge\beta + (-1)^{p+q}\alpha\wedge\bar\partial\beta$.

Exercise 1.17: Reality of d^{c}

For a real function u on $\mathbb{C}$, express $\mathrm{d}^{\mathrm{c}}u$ in x, y coordinates and verify that it is real.

Optional handout: Bott–Chern and Aeppli cohomology

This handout is optional. No later argument in the core lectures depends on it.

The double complex of forms carries more information than either de Rham or Dolbeault cohomology alone. Two useful groups are

$$H^{p,q}_{BC}(X) = \frac{\ker\partial \cap \ker\bar\partial \cap \mathcal{A}^{p,q}}{\operatorname{Im}(\partial\bar\partial)\cap\mathcal{A}^{p,q}}, \qquad H^{p,q}_{A}(X) = \frac{\ker(\partial\bar\partial)\cap\mathcal{A}^{p,q}}{\operatorname{Im}\partial + \operatorname{Im}\bar\partial}.$$

A closed (p, q)-form defines both a Bott–Chern class and a de Rham class. A $\partial\bar{\partial}$-closed form defines an Aeppli class. On compact complex manifolds these groups are finite dimensional, and integration gives a duality between $H^{p,q}_{BC}$ and $H^{n-p,n-q}_{A}$.

The $\partial\bar{\partial}$-lemma is equivalent to saying that the natural maps among Bott–Chern, Dolbeault, de Rham, and Aeppli cohomology have the expected injectivity and surjectivity. Compact Kähler manifolds satisfy it, but many non-Kähler manifolds do not. Thus the two groups provide a quantitative measure of non-Kählerness.

Example 1.18: Hermitian metrics as Aeppli classes

On a complex surface, a Hermitian form ω is Gauduchon precisely when $\partial\bar{\partial}\omega = 0$. It then determines a positive Aeppli class $[\omega]_A \in H^{1,1}_A(X)$. This class survives even when ω is not closed and therefore has no de Rham class.

Exercise 1.19: Natural maps

Write the natural maps $H^{p,q}_{BC} \to H^{p,q}_{\bar{\partial}} \to H^{p,q}_{A}$ and $H^{p,q}_{BC} \to H^{p+q}_{\mathrm{dR}}$. Verify directly that they are well defined.

1.3 Lecture 3: Hermitian and Kähler metrics; the Fubini–Study metric

Week 2 material • 90 minutes

Notation for this lecture

X is a complex manifold of complex dimension n, h denotes a Hermitian metric on $T^{1,0}X$, $g = \operatorname{Re} h$ its underlying Riemannian metric, and J the complex structure. Its fundamental real $(1,1)$-form is $\omega(u,v) = g(Ju,v)$, with volume form $\mathrm{d}V_\omega = \omega^n/n!$. A real $(1,1)$-form is positive if $\omega(v, Jv) > 0$ for every nonzero real tangent vector v. The metric is Kähler when $\mathrm{d}\omega = 0$. A local potential is a real function φ satisfying $\omega = 2\pi \mathrm{dd^c}\varphi = \mathrm{i}\partial\bar\partial\varphi$. ω_{FS} denotes the Fubini–Study form on $\mathbb{P}^n$, and $\Lambda \subset \mathbb{C}^n$ denotes a full lattice when complex tori occur. The symbol δ_{jk} is the Kronecker delta, equal to 1 for $j = k$ and 0 otherwise.

Prerequisite results used below

Poincaré lemma. On a star-shaped open subset of $\mathbb{R}^m$, every smooth closed k-form with $k \geq 1$ is exact.
Local Dolbeault lemma. If a smooth (p,q)-form α with $q \geq 1$ is $\bar\partial$-closed near a point, then after shrinking the neighborhood there is a smooth $(p, q-1)$-form β with $\alpha = \bar\partial\beta$.
Hermitian spectral theorem. Every Hermitian matrix is unitarily diagonalizable with real eigenvalues; it is positive definite exactly when all eigenvalues are positive, equivalently when $v^*Av > 0$ for every $v \neq 0$.

Plan for the lecture. 0–20 min: Hermitian linear algebra; 20–45: fundamental forms and the Kähler condition; 45–70: local potentials; 70–85: Fubini–Study and complex tori; 85–90: preview of curvature.

Definition 1.20: Hermitian metric and fundamental form

A Hermitian metric on X is a smoothly varying positive Hermitian form on $T^{1,0}X$,

$$h = \sum_{j,k} h_{j\bar k}\, \mathrm{d}z_j \otimes \mathrm{d}\bar z_k.$$

On real tangent vectors the associated Riemannian metric is $g = \operatorname{Re} h$, and its fundamental $(1,1)$-form is

$$\omega = \frac{\mathrm{i}}{2} \sum_{j,k} h_{j\bar k}\, \mathrm{d}z_j \wedge \mathrm{d}\bar z_k.$$

The metric is *Kähler* if $\mathrm{d}\omega = 0$.

Because ω is real of type $(1,1)$, $\mathrm{d}\omega = 0$ is equivalent to either $\partial\omega = 0$ or $\bar\partial\omega = 0$. The volume form is $\mathrm{d}V_\omega = \omega^n/n!$. At a point, unitary coordinates make $h_{j\bar k} = \delta_{jk}$; positivity and norm calculations can therefore be diagonalized pointwise.

Theorem 1.21: Local Kähler potentials

A real $(1,1)$-form ω is locally Kähler exactly when near every point there is a strictly plurisubharmonic function φ such that

$$\omega = 2\pi\, \mathrm{dd^c}\varphi = \mathrm{i}\partial\bar\partial\varphi.$$

Proof. If $\omega = \mathrm{i}\partial\bar\partial\varphi$, then ω is real, of type $(1,1)$, and closed. Its positivity is exactly positive definiteness of $(\varphi_{j\bar k})$.

Conversely, work in a contractible coordinate ball. The Poincaré lemma gives a real one-form η with $\omega = \mathrm{d}\eta$. Since the $(0,2)$ part of $\mathrm{d}\eta$ vanishes, $\bar\partial\eta^{0,1} = 0$. The local Dolbeault lemma gives $\eta^{0,1} = \bar\partial f$. Reality of η gives $\eta^{1,0} = \partial\bar f$. Therefore

$$\omega = \partial\eta^{0,1} + \bar\partial\eta^{1,0} = \partial\bar\partial f + \bar\partial\partial\bar f = \partial\bar\partial(f - \bar f) = \mathrm{i}\partial\bar\partial(2\,\mathrm{Im}\, f).$$

Thus $\varphi = 2\,\mathrm{Im}\, f$ is a real local potential. If two potentials give the same form, their difference has vanishing $\partial\bar\partial$ and is therefore pluriharmonic. □

Warning

A Kähler potential is local and nonunique: one may add any pluriharmonic function $\mathrm{Re}\, F$. A global identity $\omega = \mathrm{i}\partial\bar\partial\varphi$ would imply $[\omega] = 0$, contradicting $\int_X \omega^n > 0$ on a compact Kähler manifold.

Example 1.22: Euclidean space and complex tori

On $\mathbb{C}^n$, the potential $\varphi = |z|^2/2$ gives $\omega_0 = (\mathrm{i}/2)\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar z_j$. Translation invariance lets it descend to every complex torus $\mathbb{C}^n/\Lambda$.

Example 1.23: The Fubini–Study form

On the chart $U_0 \subset \mathbb{P}^n$, write $w_j = z_j/z_0$ and set

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log(1 + |w_1|^2 + \cdots + |w_n|^2) = \mathrm{d}\mathrm{d}^{\mathrm{c}} \log(1 + |w|^2).$$

Changing a homogeneous representative adds $\log|g|^2$, which is pluriharmonic wherever $g \neq 0$, so the local forms glue. Direct calculation gives

$$(h_{j\bar k}) = \frac{1}{\pi}\frac{(1 + |w|^2)I - \bar w\, w^{\mathsf{T}}}{(1 + |w|^2)^2};$$

Cauchy–Schwarz shows that it is positive definite.

Proposition 1.24: Kähler submanifolds

If $f : Y \hookrightarrow X$ is a holomorphic immersion and ω is Kähler on X, then $f^*\omega$ is Kähler on Y.

Proof. Holomorphic pullback preserves type and closedness. Because f is immersive, $\mathrm{d}f_y(v) \neq 0$ for $v \neq 0$, hence $(f^*\omega)(v, Jv) = \omega(\mathrm{d}f(v), J\mathrm{d}f(v)) > 0$. □

Remark (Geometric interpretation)

Kähler geometry locks three structures together: g measures length, J selects complex directions, and $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ measures symplectic area. Closedness is the condition that allows local Hessian methods to globalize.

1.3.1 Pointwise Hermitian linear algebra

Let

$$\alpha = \frac{\mathrm{i}}{2}\sum_{j,k} a_{j\bar k}\mathrm{d}z_j \wedge \mathrm{d}\bar z_k$$

be a real $(1,1)$-form. Reality is equivalent to $A = (a_{j\bar{k}})$ being Hermitian, and positivity means $\xi^* A\xi > 0$ for every nonzero $\xi \in \mathbb{C}^n$. A unitary change of coordinates diagonalizes A at one point:

$$\alpha = \frac{\mathrm{i}}{2}\sum_j \lambda_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

Consequently,

$$\frac{\alpha^n}{n!} = \lambda_1 \cdots \lambda_n \left(\frac{\mathrm{i}}{2}\right)^n \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n.$$

Thus positivity can be read either from the quadratic form or from positive eigenvalues; the determinant controls volume but does not by itself control every direction.

For a Hermitian metric h, normal holomorphic coordinates at a point can arrange $h_{j\bar{k}} = \delta_{jk}$, but in general they do not make all first derivatives vanish. The Kähler condition is exactly what permits holomorphic normal coordinates with

$$h_{j\bar{k}}(z) = \delta_{jk} + O(|z|^2).$$

This extra cancellation is why curvature calculations and Laplacian identities are substantially cleaner in Kähler geometry than in general Hermitian geometry.

1.3.2 A detailed Fubini–Study calculation

Put $r^2 = |w|^2$ and $\Phi = \log(1+r^2)$. Then

$$\Phi_{j\bar{k}} = \frac{(1+r^2)\delta_{jk} - \bar{w}_j w_k}{(1+r^2)^2}.$$

For $\xi \in \mathbb{C}^n$,

$$\sum_{j,k} \Phi_{j\bar{k}}\xi_j\bar{\xi}_k = \frac{(1+r^2)|\xi|^2 - |\langle \xi, w\rangle|^2}{(1+r^2)^2} \geq \frac{|\xi|^2}{(1+r^2)^2} > 0,$$

where Cauchy–Schwarz was used in the inequality. This proves strict positivity without appealing to symmetry.

On the projective line the affine expression is

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1+|w|^2)^2}.$$

Using $w = r\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{i}\mathrm{d}w \wedge \mathrm{d}\bar{w} = 2r\mathrm{d}r \wedge \mathrm{d}\theta$ gives

$$\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{r}{(1+r^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta = 1.$$

The missing point at infinity has measure zero, while the change $w' = 1/w$ confirms smoothness there. This normalization will later identify $[\omega_{\mathrm{FS}}] = c_1(\mathcal{O}(1))$.

Remark (Local potentials and global classes)

On an overlap, two Kähler potentials differ by the real part of a holomorphic function. Their Hessians therefore glue even when the potentials do not. The failure of the potentials to glue is not a defect: it is precisely the cohomological information carried by the Kähler class.

1.3.3 Metric formulas in coordinates

Let $w = w(z)$ be a holomorphic change of coordinates. Since $\mathrm{d}z_j = \sum_a (\partial z_j/\partial w_a)\mathrm{d}w_a$, the Hermitian matrix transforms as

$$h'_{a\bar{b}} = \sum_{j,k} h_{j\bar{k}} \frac{\partial z_j}{\partial w_a} \overline{\frac{\partial z_k}{\partial w_b}}.$$

Thus $h' = J^\top h \bar{J}$, and positivity is coordinate independent: $\xi^\top h' \bar{\xi} = (J\xi)^\top h \overline{J\xi} > 0$.

For

$$\omega = \frac{\mathrm{i}}{2} \sum_{j,k} h_{j\bar{k}}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k,$$

direct differentiation gives

$$\partial\omega = \frac{\mathrm{i}}{2} \sum_{\ell,j,k} \frac{\partial h_{j\bar{k}}}{\partial z_\ell} \mathrm{d}z_\ell \wedge \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k.$$

Collecting the coefficient of $\mathrm{d}z_\ell \wedge \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$ shows that $\partial\omega = 0$ exactly when

$$\frac{\partial h_{j\bar{k}}}{\partial z_\ell} = \frac{\partial h_{\ell\bar{k}}}{\partial z_j} \quad \text{for all } j, k, \ell.$$

These are the coordinate Kähler equations. They are the integrability conditions for finding a local potential ϕ with $h_{j\bar{k}} = \pi^{-1}\partial^2\phi/partial z_j \partial\bar{z}_k$ under our $\mathrm{dd^c}$ convention.

The Fubini–Study Hessian

On the chart $U_0 \simeq \mathbb{C}^n$, put $r^2 = |z_1|^2 + \cdots + |z_n|^2$ and $\phi = \log(1 + r^2)$. First

$$\frac{\partial\phi}{\partial z_j} = \frac{\bar{z}_j}{1 + r^2}.$$

Differentiating once more yields

$$\phi_{j\bar{k}} = \frac{(1 + r^2)\delta_{jk} - \bar{z}_j z_k}{(1 + r^2)^2}.$$

For $\xi \in \mathbb{C}^n$,

$$\begin{aligned}
\sum_{j,k} \phi_{j\bar{k}} \xi_j \bar{\xi}_k &= \frac{(1 + r^2)|\xi|^2 - |\langle \xi, z\rangle|^2}{(1 + r^2)^2} \\
&\geq \frac{(1 + r^2)|\xi|^2 - r^2|\xi|^2}{(1 + r^2)^2} = \frac{|\xi|^2}{(1 + r^2)^2} > 0.
\end{aligned}$$

Thus $\omega_{\mathrm{FS}} = \mathrm{dd^c}\phi$ is positive. The inequality used is ordinary Cauchy–Schwarz, and it also gives the two eigenvalues: $(1 + r^2)^{-2}$ in the radial complex direction and $(1 + r^2)^{-1}$ on its orthogonal complement.

The potentials glue although the functions do not. On $U_0 \cap U_1$, with $w_0 = 1/z_1$ and $w_j = z_j/z_1$,

$$1 + \sum_{j=1}^n |z_j|^2 = |z_1|^2 \left(1 + |w_0|^2 + \sum_{j=2}^n |w_j|^2\right).$$

Therefore

$$\log(1 + |z|^2) = \log|z_1|^2 + \log(1 + |w|^2).$$

Since z_1 is holomorphic and nonvanishing on the overlap, $\mathrm{dd^c}\log|z_1|^2 = 0$ there. Hence the local $(1,1)$-forms agree and define a global form on $\mathbb{P}^n$.

On $\mathbb{P}^1$ the calculation reduces to

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+|z|^2)^2} = \frac{1}{\pi}\frac{r\,\mathrm{d}r\wedge\mathrm{d}\theta}{(1+r^2)^2}.$$

The missing point has measure zero, and

$$\int_{\mathbb{P}^1}\omega_{\mathrm{FS}} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{r}{(1+r^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta = 2\left[-\frac{1}{2(1+r^2)}\right]_0^{\infty} = 1.$$

This normalization will later identify $[\omega_{\mathrm{FS}}]$ with $c_1(\mathcal{O}(1))$.

Pullbacks, products, and submanifolds

If $F : X \to Y$ is holomorphic and ω_Y is Kähler, then $\mathrm{d}(F^*\omega_Y) = F^*(\mathrm{d}\omega_Y) = 0$. Positivity may become degenerate if F has a kernel. If F is an immersion, however,

$$(F^*\omega_Y)_x(v, Jv) = \omega_{Y,F(x)}(\mathrm{d}F_x v, J\mathrm{d}F_x v) > 0$$

for $v \neq 0$, so the pullback is Kähler. This proves the submanifold statement with the inclusion map.

For two Kähler manifolds, the product form is

$$\omega = \mathrm{pr}_1^*\,\omega_1 + operatornamepr_2^*\omega_2.$$

At (x, y) and $(v, w) \neq 0$,

$$\omega((v,w), J(v,w)) = \omega_1(v, Jv) + \omega_2(w, Jw) > 0.$$

It is closed term by term. This coordinate-free calculation checks both defining conditions and explains why omitting either summand would make the form degenerate.

1.3.4 Self-study workshop: checking that a local Hermitian formula defines a global Kähler metric

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. You need Lecture 2's type decomposition, Hermitian matrices, and the rule that $\partial\bar{\partial}$ annihilates pluriharmonic functions. Recall that positivity is pointwise linear algebra, while closedness is a differential condition.

The three checks. In coordinates a real $(1,1)$-form has the shape

$$\omega = \frac{\mathrm{i}}{2}\sum_{j,k} g_{j\bar{k}}\,\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_k.$$

It defines a Hermitian metric precisely when $G = (g_{j\bar{k}})$ is Hermitian positive definite. It is Kähler when additionally $\mathrm{d}\omega = 0$. If $\omega = \mathrm{dd^c}\varphi$, closedness follows from $\mathrm{d}^2 = 0$, but positivity still requires the complex Hessian of φ to be positive.

For $\mathbb{P}^1$, use $u = z_1/z_0$ on U_0 and $v = z_0/z_1 = 1/u$ on U_1. Put

$$\varphi_0 = \log(1+|u|^2), \qquad \varphi_1 = \log(1+|v|^2).$$

On the overlap,

$$\varphi_0 - \varphi_1 = \log |u|^2.$$

Since u is holomorphic and nonzero there, $\mathrm{dd^c} \log |u|^2 = 0$. Thus $\mathrm{dd^c}\varphi_0$ and $\mathrm{dd^c}\varphi_1$ glue. Direct differentiation gives

$$\frac{\partial^2 \varphi_0}{\partial u\, \partial \bar{u}} = \frac{1}{(1+|u|^2)^2} > 0.$$

This proves both global well-definedness and positivity; citing the potential alone would prove neither.

The same logic works for a Kähler submanifold. If $\iota : Y \hookrightarrow X$ is a holomorphic immersion and ω is Kähler, then $\mathrm{d}(\iota^*\omega) = \iota^*(\mathrm{d}\omega) = 0$. For a nonzero $v \in T^{1,0}_y Y$, injectivity of $\mathrm{d}\iota_y$ gives $(\iota^*\omega)(v, \bar{v}) = \omega(\mathrm{d}\iota v, \overline{\mathrm{d}\iota v}) > 0$.

Checkpoint (potential versus metric)

(1) Why may two local Kähler potentials differ without changing the metric?
(2) Show that $\varphi = |z_1|^2 + 2|z_2|^2$ gives a Kähler form on $\mathbb{C}^2$.
(3) What fails for $\varphi = |z_1|^2 - |z_2|^2$?

Solution (checkpoint). (1) Potentials differing by a pluriharmonic function have the same $\mathrm{dd^c}$. (2) Its complex Hessian is $\mathrm{diag}(1, 2)$, so $\mathrm{dd^c}\varphi$ is closed and positive definite. (3) The Hessian has eigenvalues $1, -1$; the form is closed and real but not positive, hence not Kähler. ◇

Exit criterion

For a metric given by local potentials, you can check overlap compatibility, differentiate the potential, diagonalize its Hermitian matrix, and distinguish closedness from positivity.

Summary. Hermitian does not imply Kähler; the extra condition is $\mathrm{d}\omega = 0$. Euclidean space, complex tori, projective space, and their complex submanifolds provide the central examples.

Exercise 1.25: Area of $\mathbb{P}^1$

Use the affine coordinate to compute $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}}$. Convert to polar coordinates and check the point at infinity.

Exercise 1.26: Products

Prove that the product of two Kähler manifolds is Kähler for $\pi_1^*\omega_1 + \pi_2^*\omega_2$.

Optional handout: Hermitian metrics beyond the Kähler case

This handout is optional. No later argument in the core lectures depends on it.

Let (X^n, ω) be Hermitian. Several weaker closedness conditions occur naturally:

$$\begin{aligned} \text{Kähler} \quad &: \quad \mathrm{d}\omega = 0, \\ \text{balanced} \quad &: \quad \mathrm{d}(\omega^{n-1}) = 0, \\ \text{Gauduchon} \quad &: \quad \partial\bar{\partial}(\omega^{n-1}) = 0. \end{aligned}$$

For $n = 2$, balanced is the same as Kähler, but in dimension at least three balanced non-Kähler manifolds exist. Balanced metrics are stable under modifications, while Kähler metrics need not exist at all.

Theorem 1.27: Gauduchon's conformal theorem

On every compact complex manifold of dimension $n \geq 2$, every conformal class of Hermitian metrics contains a Gauduchon metric. It is unique up to multiplication by a positive constant.

Proof. Fix ω and define a real second-order operator P by

$$\mathrm{i}\partial\bar{\partial}(u\omega^{n-1}) = P(u)\,\omega^n.$$

Its principal symbol is a positive multiple of that of the scalar Laplacian, so P is elliptic. Stokes' theorem gives $\int_X P(u)\omega^n = 0$ for every u; hence $P^*(1) = 0$. Since P and P^* have the same scalar principal symbol, their index is zero, and the Fredholm alternative gives $\ker P \neq 0$.

We need a positive element of this kernel. The principal-eigenfunction theorem for a real scalar elliptic operator on a compact connected manifold says that there is a simple eigenvalue λ_0 with an everywhere positive eigenfunction, and that the adjoint has the same principal eigenvalue and a positive eigenfunction. Because the positive function 1 lies in $\ker P^*$, integration against it rules out a principal eigenvalue of either sign; thus $\lambda_0 = 0$. Consequently $Pu = 0$ has an everywhere positive solution, unique up to a positive constant. (The principal-eigenfunction statement follows from the strong maximum principle applied to the positive compact resolvent $(P + c)^{-1}$ and the Krein–Rutman theorem.)

Set $u = e^{(n-1)f}$ and $\widetilde{\omega} = e^f\,\omega$. Then $\widetilde{\omega}^{n-1} = u\omega^{n-1}$, so $Pu = 0$ is precisely $\partial\bar{\partial}(\widetilde{\omega}^{n-1}) = 0$. Simplicity of the positive kernel says that two such u differ by a positive constant, and hence the corresponding metrics differ by a constant conformal factor. □

Remark (Why Gauduchon metrics matter)

If L is a holomorphic line bundle, the degree $\deg_\omega L = \int_X c_1(L, h) \wedge \omega^{n-1}$ is independent of the metric h when ω is Gauduchon. This makes slope stability possible on non-Kähler manifolds.

1.4 Lecture 4: Distributions, currents, and the Poincaré–Lelong formula

Week 2 material • 90 minutes

Notation for this lecture

M is an oriented real m-manifold and X a complex n-manifold. $\mathcal{D}^k(M)$ denotes compactly supported smooth k-forms. A current of dimension p acts continuously on $\mathcal{D}^p(M)$; equivalently, a current of degree $q = m - p$ acts on $\mathcal{D}^{m-q}(M)$. On X, bidegree (p, q) means degree and bidimension $(n - p, n - q)$. The pairing is $\langle T, \eta\rangle$, $T_\nu \rightharpoonup T$ means weak convergence against every test form, and $[V]$ is the integration current of an analytic set V over its regular locus V_{Reg}. The Dirac mass at a is δ_a, f_*T is proper pushforward, and $\mathrm{div}(f)$ is the zero-pole divisor of a meromorphic function f.

Prerequisite results used below

Stokes' theorem. If M is an oriented m-manifold with boundary and $\alpha \in \mathcal{D}^{m-1}(M)$, then $\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha$ with the boundary orientation. In particular the integral is zero when α is supported away from ∂M.
Extension of differential operators to distributions. If P is a linear differential operator with formal transpose P^t, its action on a distribution T is defined by $\langle PT, \phi\rangle = \langle T, P^t\phi\rangle$. This extension is continuous for weak convergence of distributions and agrees with the classical derivative on smooth coefficients.

Plan for the lecture. 0–20 min: test forms and currents; 20–40: differentials, products, pullback and pushforward; 40–65: integration currents of analytic sets; 65–82: Poincaré–Lelong; 82–90: sign audit.

Limits of smooth objects need not remain smooth. The functions $\log|z|$, integration over analytic sets, and weak limits of Kähler forms all require the language of currents.

Definition 1.28: Currents

Let M be an oriented manifold of real dimension m. A current of dimension p is a continuous linear functional on compactly supported smooth p-forms $\mathcal{D}^p(M)$. Equivalently, a current of degree $q = m - p$ acts on $\mathcal{D}^{m-q}(M)$. On a complex n-fold, the bidegree (p, q) refers to degree, so such a current acts on test forms of type $(n - p, n - q)$.

A locally integrable q-form α defines $T_\alpha(\eta) = \int_M \alpha \wedge \eta$. If $V \subset X$ is analytic of pure dimension k, then

$$[V](\eta) = \int_{V_{\mathrm{Reg}}} \eta, \qquad \eta \in \mathcal{D}^{k,k}(X),$$

defines a closed positive current of bidegree $(n - k, n - k)$. The singular locus has real codimension at least two and does not affect the integral.

Definition 1.29: Differential of a current

For a current T of degree q, define

$$\langle \mathrm{d}T, \eta\rangle = (-1)^{q+1} \langle T, \mathrm{d}\eta\rangle .$$

This convention ensures $\mathrm{d}T_\alpha = T_{\mathrm{d}\alpha}$ for a smooth locally integrable form.

Warning

The literature alternates between a dimension convention for boundaries and a degree convention for differentials. Never memorize only "pushforward commutes with d." First write the degree of the test form, then perform one integration by parts.

If $F : M_1 \to M_2$ is proper on $\operatorname{supp} T$, set $\langle F_*T, \eta\rangle = \langle T, F^*\eta\rangle$. For smooth f, define $\langle fT, \eta\rangle = \langle T, f\eta\rangle$. Two arbitrary currents cannot be multiplied; the absence of a natural pointwise product of distributions is the main difficulty behind nonlinear Monge–Ampère theory.

Proposition 1.30: Differential and proper pushforward

With the degree convention above, if $F : M_1^{m_1} \to M_2^{m_2}$ is proper on the support of T, then

$$\mathrm{d}(F_*T) = (-1)^{m_1-m_2} F_*(\mathrm{d}T).$$

For maps of complex manifolds the exponent is even, so the sign in later complex-geometric applications remains positive.

Proof. If T has degree q, then F_*T has degree $q + m_2 - m_1$. Pair both sides with a test form η, use the definition of the differential of a current twice, and use $F^*\mathrm{d}\eta = \mathrm{d}F^*\eta$. The quotient of the two signs is $(-1)^{m_1-m_2}$. □

Theorem 1.31: Poincaré–Lelong formula

If f is a nonzero meromorphic function on X, then as $(1,1)$-currents,

$$\mathrm{dd^c} \log |f|^2 = [\operatorname{div}(f)].$$

In particular, $\mathrm{dd^c} \log |z|^2 = \delta_0$ on $\mathbb{C}$.

Proof. Start with $f(z) = z$. On a punctured disc $\mathrm{dd^c} \log |z|^2 = 0$. Apply Stokes' theorem to $\{\varepsilon < |z| < R\}$ against a test function χ. The outer boundary vanishes by compact support, while the inner term tends to $\chi(0)$ as $\varepsilon \to 0$.

At a smooth point of a divisor in higher dimension, choose coordinates with $f = z_1^m g$ and g nonvanishing. The term $\log |g|^2$ is pluriharmonic and the one-variable formula in the transverse z_1 direction gives $m[z_1 = 0]$. Extend across the codimension-at-least-two singular set by the extension theorem for currents or by slicing. Apply the same argument to $1/f$ at poles and change the sign. □

Example 1.32: The line-bundle form

If (L, h) is a Hermitian holomorphic line bundle and s a nonzero meromorphic section, then

$$\mathrm{dd^c} \log |s|_h^2 = [\operatorname{div}(s)] - c_1(L, h), \qquad c_1(L, h) = -\mathrm{dd^c} \log h(e, e)$$

in a local holomorphic frame e. Thus the divisor class equals the first Chern class.

Remark (Integral kernels)

For the common orientation and $\bar\partial$ convention, the Bochner–Martinelli kernel satisfies $\bar\partial k_{\mathrm{BM}} = -\delta_0$. In any integral representation, the orientation of the kernel, the boundary orientation, and the convention for $\bar\partial$ must be checked together.

1.4.1 Weak convergence and order of a current

The topology on currents is weak: $T_\nu \rightharpoonup T$ means $\langle T_\nu, \eta\rangle \to \langle T, \eta\rangle$ for every compactly supported smooth test form η. Differentiation is continuous in this topology because

$$\langle \mathrm{d}T_\nu, \eta\rangle = (-1)^{\deg T_\nu + 1}\langle T_\nu, \mathrm{d}\eta\rangle.$$

This is the key advantage over pointwise convergence: singular limits commute with linear differential operators automatically.

For example, let ρ_ε be a nonnegative approximate identity on $\mathbb{R}^m$. Then $\rho_\varepsilon \mathrm{d}V \rightharpoonup \delta_0$. No subsequence converges as smooth forms, but the total mass remains visible. Similarly, if smooth oriented submanifolds V_ν converge with controlled volume, their integration currents may have a weak limit even when their tangent spaces oscillate.

A current has order zero when it extends continuously to compactly supported continuous test forms. Measures and positive currents have order zero. Derivatives of measures need not: δ_0' detects the derivative of a test function. This distinction explains why multiplication by a smooth function is always allowed, whereas multiplication by a nonsmooth function requires additional hypotheses.

1.4.2 The one-variable Poincaré–Lelong constant

The normalization can be checked without memorization. On $\mathbb{C}^*$,

$$\mathrm{d}^c \log|z|^2 = \frac{1}{2\pi}\mathrm{d}\theta.$$

Let χ be a test function supported in a disc and delete D_ε. Stokes' theorem, with the inner circle carrying the boundary orientation of the punctured disc, gives

$$\begin{aligned}\langle \mathrm{dd}^c \log|z|^2, \chi\rangle &= \lim_{\varepsilon\downarrow 0}\int_{|z|>\varepsilon} \log|z|^2\,\mathrm{dd}^c\chi \\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}\int_{|z|=\varepsilon}\chi\,\mathrm{d}\theta = \chi(0).\end{aligned}$$

Terms containing derivatives of χ tend to zero because $\varepsilon|\log\varepsilon| \to 0$. Hence the coefficient of δ_0 is exactly one.

If $f = z_1^m g$ with g nonvanishing, then

$$\log|f|^2 = m\log|z_1|^2 + \log|g|^2.$$

The last term is pluriharmonic, and slicing in the z_1 direction yields $\mathrm{dd}^c\log|f|^2 = m[z_1 = 0]$. This local factorization also shows why multiplicities, not merely zero sets, occur in the divisor.

Warning

Weak convergence is compatible with linear operations, not nonlinear ones. From $T_\nu \rightharpoonup T$ and $S_\nu \rightharpoonup S$ one cannot infer $T_\nu \wedge S_\nu \rightharpoonup T \wedge S$; the product on the right may not even exist. Lectures 15 and 31 introduce two different frameworks that recover selected products under extra control.

1.4.3 Currents calculated from the definition

On an oriented real m-manifold, a degree-k current acts on compactly supported $(m-k)$-forms. A locally integrable k-form α defines

$$T_\alpha(\eta) = \int_X \alpha\wedge\eta.$$

For a smooth $(m-k-1)$-form η, the definition of the differential is chosen so that $T_{\mathrm{d}\alpha} = \mathrm{d}T_\alpha$. Indeed, compact support and Stokes give

$$0 = \int_X \mathrm{d}(\alpha \wedge \eta) = \int_X \mathrm{d}\alpha \wedge \eta + (-1)^k \int_X \alpha \wedge \mathrm{d}\eta.$$

Thus

$$(\mathrm{d}T_\alpha)(\eta) = (-1)^{k+1} T_\alpha(\mathrm{d}\eta) = \int_X \mathrm{d}\alpha \wedge \eta.$$

This is the sign test that fixes the convention for all later current identities.

For the Dirac mass at $0 \in \mathbb{R}$, $\delta_0(\varphi) = \varphi(0)$ and

$$\delta_0'(\varphi) = -\varphi'(0).$$

If H is the Heaviside function, then

$$\langle H', \varphi \rangle = -\int_0^\infty \varphi'(x)\,\mathrm{d}x = \varphi(0),$$

so $H' = \delta_0$. The boundary term is exactly what survives after integration by parts; this one-dimensional model is the prototype for currents of integration.

Let $V^p \subset X^n$ be an oriented smooth submanifold without boundary. Its integration current has bidimension (p,p) and bidegree $(n-p, n-p)$:

$$[V](\eta) = \int_V \eta, \qquad \eta \in \mathcal{A}_c^{p,p}(X).$$

By Stokes,

$$\mathrm{d}[V](\eta) = \pm [V](\mathrm{d}\eta) = \pm \int_V \mathrm{d}\eta = 0.$$

If V has boundary, the same computation gives $\mathrm{d}[V] = [\partial V]$ with the orientation sign dictated by the convention. Thus closedness of an analytic-cycle current is a geometric Stokes statement.

The constant in Poincaré–Lelong

On $\mathbb{C}^*$ write $z = re^{\mathrm{i}\theta}$. Since $\log|z|^2 = 2\log r$, the real-coordinate formula from Lecture 2 gives

$$\mathrm{d}^{\mathrm{c}} \log|z|^2 = \frac{1}{2\pi}\,\mathrm{d}\theta.$$

It is closed on $\mathbb{C}^*$, but it has nonzero period:

$$\int_{|z|=\varepsilon} \mathrm{d}^{\mathrm{c}} \log|z|^2 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta = 1.$$

Let χ be a compactly supported test function. Apply Stokes on $\{|z| > \varepsilon\}$ and then let $\varepsilon \downarrow 0$. Since $\mathrm{d}\mathrm{d}^{\mathrm{c}} \log|z|^2 = 0$ away from 0, only the inner boundary remains, and its orientation gives

$$\langle \mathrm{d}\mathrm{d}^{\mathrm{c}} \log|z|^2, \chi \rangle = \lim_{\varepsilon\to 0} \int_{|z|=\varepsilon} \chi\, \mathrm{d}^{\mathrm{c}} \log|z|^2 = \chi(0).$$

Therefore

$$\boxed{\mathrm{d}\mathrm{d}^{\mathrm{c}} \log|z|^2 = \delta_0.}$$

If $f(z) = z^m g(z)$ with $g(0) \neq 0$, then

$$\log|f|^2 = m\log|z|^2 + \log|g|^2.$$

A local holomorphic logarithm of g shows $\mathrm{d}\mathrm{d}^{\mathrm{c}} \log|g|^2 = 0$, hence $\mathrm{d}\mathrm{d}^{\mathrm{c}} \log|f|^2 = m\delta_0$. Factoring at every zero and pole proves the one-dimensional divisor formula.

From local equations to a global divisor current

Let D be a divisor with local defining functions f_α. On an overlap, $f_\alpha = g_{\alpha\beta} f_\beta$ with $g_{\alpha\beta}$ holomorphic and nowhere zero. Then

$$\log|f_\alpha|^2 - \log|f_\beta|^2 = \log|g_{\alpha\beta}|^2, \qquad \mathrm{dd^c} \log|g_{\alpha\beta}|^2 = 0.$$

Thus the local currents $\mathrm{dd^c}\log|f_\alpha|^2$ glue. At a smooth point choose coordinates with $f_\alpha = z_1^m$; the one-variable computation in the normal direction and Fubini's theorem give $m[z_1 = 0]$. The singular locus has smaller dimension and does not change the current. This proves the local-to-global content of

$$\mathrm{dd^c}\log|f|^2 = [\mathrm{div}(f)].$$

For a section $s = f_\alpha e_\alpha$ of a Hermitian line bundle with $|e_\alpha|_h^2 = e^{-\varphi_\alpha}$,

$$\log|s|_h^2 = \log|f_\alpha|^2 - \varphi_\alpha.$$

Applying $\mathrm{dd^c}$ yields

$$\mathrm{dd^c}\log|s|_h^2 = [\mathrm{div}(s)] - \mathrm{dd^c}\varphi_\alpha.$$

The last term is the local curvature representative of $c_1(L,h)$, so rearranging gives the bundle-valued Poincaré–Lelong formula and explains why the two apparently local terms combine to a global identity.

1.4.4 Self-study workshop: using currents without suppressing test forms or constants

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall integration of compactly supported top-degree forms, distributional derivatives, Stokes' theorem, and the normalization $\mathrm{dd^c}\log|z|^2 = \delta_0$. Every current identity should be tested by pairing both sides with a smooth compactly supported form of complementary degree.

From functions to divisors. A locally integrable function u defines a degree-zero current $T_u(\eta) = \int u\eta$. Its derivative is defined by duality, with the sign chosen so that Stokes' theorem remains true. For a hypersurface $V = \{f = 0\}$, the integration current acts by

$$[V](\eta) = \int_{V_{\mathrm{Reg}}} \eta.$$

The singular set has smaller real dimension and does not affect the integral.

The local model contains the multiplicity. If $f(z) = z^m g(z)$ in one variable, with $g(0) \neq 0$, then

$$\log|f|^2 = m\log|z|^2 + \log|g|^2.$$

The last term is pluriharmonic near the origin because a nonvanishing holomorphic function has a local holomorphic logarithm. Consequently

$$\mathrm{dd^c}\log|f|^2 = m\delta_0.$$

In several variables the same factorization at a smooth generic point of each irreducible component gives $\mathrm{dd^c}\log|f|^2 = [\mathrm{div}(f)]$. This explains why the coefficient is the vanishing order rather than merely 1.

Metric Poincaré–Lelong is obtained by comparing local frames. If a line bundle section is $s = f_\alpha e_\alpha$ and $|e_\alpha|_h^2 = e^{-\varphi_\alpha}$, then

$$\log|s|_h^2 = \log|f_\alpha|^2 - \varphi_\alpha, \qquad \mathrm{dd^c}\log|s|_h^2 = [Z_s] - c_1(L,h).$$

Both terms on the right are global even though the two logarithms on the left were written in a frame.

Checkpoint (test the normalization)

Let $f(z) = z^3 e^z$ on a disc about 0.
(1) Compute $\mathrm{dd^c} \log |f|^2$ as a current.
(2) Explain why $\mathrm{dd^c} \log |e^z|^2 = 0$ by an actual calculation.
(3) State the bidegree of the current obtained.

Solution (checkpoint). We have $\log |f|^2 = 3 \log |z|^2 + z + \bar{z}$. Since $\partial\bar{\partial}(z + \bar{z}) = 0$, the answer is $3\delta_0$. It is a current of bidegree $(1, 1)$, acting on compactly supported functions when the ambient complex dimension is one. ⋄

Exit criterion

You can translate a formal current equality into an equality on test forms, recover the multiplicity in Poincaré–Lelong, and keep the course's 2π normalization consistent.

Summary. Currents place differential forms, measures, and analytic cycles in one category, but arbitrary products of currents are undefined. Poincaré–Lelong is the first major bridge from zeros and poles of functions to closed positive currents.

Exercise 1.33: The Heaviside distribution

Compute the distributional derivative of $H = 1_{[0,\infty)}$ on $\mathbb{R}$ and compare it with the differential convention above.

Exercise 1.34: Divisor of a rational function

Compute $\mathrm{dd^c} \log |z_1^2 z_2/(z_1 - z_2)|^2$ and state precisely in what sense the identity holds.

Optional handout: When can distributions be multiplied?

This handout is optional. No later argument in the core lectures depends on it.

The obstruction to multiplying currents is microlocal. A distribution can be singular at a point only in certain cotangent directions; these pairs form its wavefront set $\mathrm{WF}(u) \subset T^*M \setminus 0$. Smoothness near x is equivalent to the absence of all covectors over x from $\mathrm{WF}(u)$.

Theorem 1.35: Hörmander's product criterion

The product uv is canonically defined if there is no (x, ξ) such that $(x, \xi) \in \mathrm{WF}(u)$ and $(x, -\xi) \in \mathrm{WF}(v)$.

Proof. We use the pullback theorem for distributions. If $F : N \to M$ is smooth and

$$N_F = \{(F(y), \xi) : {}^t dF_y \xi = 0\}$$

does not meet $\mathrm{WF}(w)$, then F^*w is defined and depends continuously on w in the corresponding wavefront topology. Locally, write a regularized distribution as a Fourier integral. In the conic region where ${}^t dF_y \xi \neq 0$, repeated integration by parts in y gives arbitrary decay in ξ; in the complementary region the assumed wavefront cutoff makes the Fourier transform rapidly decreasing. The regularizations therefore converge independently of the cutoff. This proves the pullback theorem and also shows

$$\mathrm{WF}(F^*w) \subset \{(y, {}^t dF_y \xi) : (F(y), \xi) \in \mathrm{WF}(w)\}.$$

Now $u \otimes v$ is a well-defined distribution on $M \times M$, and

$$\mathrm{WF}(u \otimes v) \subset (\mathrm{WF}(u) \cup 0) \times (\mathrm{WF}(v) \cup 0) \setminus 0.$$

For the diagonal $\Delta(x) = (x, x)$, the conormal bundle is $N^*\Delta = \{(x, x; \xi, -\xi) : \xi \neq 0\}$. The hypothesis says exactly that this bundle misses $\mathrm{WF}(u \otimes v)$. Hence $uv := \Delta^*(u \otimes v)$ is canonically defined. For smooth functions this pullback is ordinary pointwise multiplication, so density and continuity show that the definition is the required extension of the usual product. □

Example 1.36: Transverse submanifolds

If smooth oriented submanifolds $A, B \subset M$ meet transversely, their integration currents have conormal wavefront sets, the criterion holds, and $[A] \wedge [B] = [A \cap B]$ with the intersection orientation. If they fail to be transverse, the product may require an excess-intersection construction and is not a naive distributional product.

This point of view complements Bedford–Taylor theory. Wavefront transversality is a condition on singular directions; Bedford–Taylor instead exploits positivity and local boundedness to define nonlinear products even when ordinary microlocal transversality is not available.

Exercise 1.37: Delta functions

Explain why the criterion permits $f\delta_0$ for smooth f but does not define δ_0^2. Compare with an arbitrary sequence of mollifiers.

Chapter 2

Subharmonic Functions, Pseudoconvexity, and Stein Manifolds

2.1 Lecture 5: Subharmonic functions, mean values, and regularization

Week 3 material • 90 minutes

Notation for this lecture

$\Omega \subset \mathbb{C}$ is a plane domain, $D(a,r)$ is the disc of center a and radius r, and $u : \Omega \longrightarrow [-\infty, \infty)$ is upper semicontinuous unless more regularity is stated. Its circular mean is $M_u(a,r) = (2\pi)^{-1} \int_0^{2\pi} u(a + r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta$. $\Delta = \partial_x^2 + \partial_y^2$ is the Euclidean Laplacian and $\mathrm{d}A = \mathrm{d}x\, \mathrm{d}y$ is planar area measure. The distribution $\mu_u = \mathrm{dd^c}u$ is the Riesz measure when u is subharmonic. The relation $D \Subset \Omega$ means that $\overline{D}$ is compact in Ω. For such a disc D, $G_D(z,\zeta)$ is the Green kernel normalized by $\mathrm{dd}^{\mathrm{c}}_z G_D(z,\zeta) = \delta_\zeta$ and zero boundary values. ρ_ε denotes a nonnegative radial mollifier of total mass one, and $u_\varepsilon = u * \rho_\varepsilon$ wherever the convolution is defined.

Prerequisite results used below

Mean-value theorem for harmonic functions. If h is harmonic on a neighborhood of $\overline{D(a,r)}$, then $h(a) = M_h(a,r)$.

Green representation. If $D \Subset \mathbb{C}$ has a Green kernel G_D, $u \in C^2(\overline{D})$, and H_u is the harmonic function with boundary values $u|_{\partial D}$, then, with the normalization of this text,

$$u(z) = H_u(z) + \int_D G_D(z,\zeta)\, \mathrm{dd^c}u(\zeta).$$

Thus the sign of $\mathrm{dd^c}u$ controls comparison with harmonic functions.

Weyl lemma. A distribution v satisfying $\Delta v = 0$ is represented by a smooth harmonic function. This is used when distributional information is converted back to classical regularity.

Plan for the lecture. 0–20 min: three equivalent definitions; 20–40: maximum principle and circular means; 40–62: Riesz measure; 62–80: convolution regularization; 80–90: counterexamples and exercises.

For $\Omega \subset \mathbb{C}$, a smooth function is subharmonic when $\Delta u \geq 0$. To admit logarithmic poles such as $\log|z-a|$, the definition must use upper semicontinuity and distributions.

Definition 2.1: Subharmonic function

A function $u : \Omega \to [-\infty, \infty)$ is *subharmonic* if $u \not\equiv -\infty$, u is upper semicontinuous, and whenever $\overline{D(a,r)} \subset \Omega$,

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

Upper semicontinuity means that $\{u < c\}$ is open. It prevents us from raising the value of u arbitrarily at an isolated point.

Theorem 2.2: Equivalent characterizations of subharmonicity

For $u \in L^1_{\mathrm{loc}}(\Omega)$, the following are equivalent.

(1) The class of u contains a subharmonic representative.
(2) $\Delta u \geq 0$ in distributions: $\int u \Delta\chi \geq 0$ for every nonnegative $\chi \in C_c^\infty(\Omega)$.
(3) Standard radial convolutions u_ε are smooth and subharmonic on smaller domains and decrease to the upper-semicontinuous representative as $\varepsilon \downarrow 0$.

Proof. The mean inequality supplies local upper bounds and local integrability. Convolution preserves the mean inequality, so $\Delta u_\varepsilon \geq 0$; passage to the limit gives the distribution inequality. Conversely, the inequality commutes with convolution. A radial convolution is a positive average of circular means, and those means increase with the radius, so u_ε decreases as $\varepsilon \downarrow 0$. Its upper-semicontinuous limit satisfies the mean inequality and agrees almost everywhere with u. □

Proposition 2.3: Maximum principle

If a subharmonic function on a connected domain attains a finite local maximum, it is constant. If u extends upper semicontinuously to the closure of a bounded domain, then $\sup_\Omega u \leq \sup_{\partial\Omega} u$.

Proof. If $u(a) = M$ is a local maximum, the mean inequality and $u \leq M$ force equality almost everywhere on every small circle. Upper semicontinuity and Poisson comparison then give $u = M$ on the disc. The set of maximum points is both open and closed. □

Proposition 2.4: Circular means and logarithmic slope

Set

$$M_u(a,r) = \frac{1}{2\pi} \int_0^{2\pi} u(a + r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta.$$

Then $M_u(a,r)$ increases with r, is convex as a function of $\log r$, and tends to $u(a)$ as $r \downarrow 0$.

Proof. For smooth u, Green's formula gives

$$\frac{\mathrm{d}}{\mathrm{d}r} M_u(a,r) = \frac{1}{2\pi r} \int_{D(a,r)} \Delta u \,\mathrm{d}A \geq 0.$$

Multiplying by r and differentiating once more shows that $t \mapsto M_u(a, \mathrm{e}^t)$ has nonnegative second distributional derivative, hence is convex. For a general subharmonic function, apply this calculation to decreasing smooth convolutions and pass to the limit. The convergence to $u(a)$ follows from the mean inequality together with upper semicontinuity: the liminf is at least $u(a)$, while the limsup is at most $u(a)$ after shrinking the circle. □

The Riesz measure is $\mu_u = \mathrm{dd^c}u$; in dimension one it differs from Δu only by the fixed normalization. For $u = \log|f|^2$, Poincaré–Lelong gives $\mu_u = \sum_a \mathrm{ord}_a(f)\delta_a$.

Example 2.5: The model singularity

$u(z) = \alpha \log |z - a|^2 + h(z)$ is subharmonic when $\alpha \geq 0$ and h is harmonic. The slope of its circular mean in $\log r$ is 2α if one uses $\log |z|^2$ rather than $\log |z|$. This anticipates the Lelong number in Lecture 16.

Warning

A distribution inequality specifies only an almost-everywhere class. To obtain a subharmonic *function*, one must select the correct upper-semicontinuous representative. Raising the value at one point does not change Δu but destroys the mean inequality.

2.1.1 Riesz decomposition and the Poisson correction

The distribution $\mu_u = \mathrm{dd^c} u$ separates a subharmonic function into a singular potential and a harmonic remainder. Let $D \Subset \Omega$ be a disc and let $G_D(z, \zeta)$ be its Green kernel, normalized by

$$\mathrm{dd}^{\mathrm{c}}_z G_D(z, \zeta) = \delta_\zeta, \qquad G_D(\,\cdot\,, \zeta)|_{\partial D} = 0.$$

Then, after restricting to a slightly smaller disc if necessary,

$$u(z) = h(z) + \int_D G_D(z, \zeta)\, \mathrm{d}\mu_u(\zeta),$$

where h is harmonic. Indeed, the difference of the two sides has vanishing distributional Laplacian and is therefore represented by a smooth harmonic function. This is the local Riesz decomposition.

For the disc $D(0, R)$ one may take

$$G_D(z, \zeta) = \log \left| \frac{R(z - \zeta)}{R^2 - \bar{\zeta} z} \right|^2.$$

The denominator is nonzero on the disc, so its logarithm is harmonic in z; on $|z| = R$ the quotient has modulus one. A point mass in μ_u therefore produces a logarithmic pole, while a diffuse measure produces a distributed potential.

The same information appears in the circular means. For smooth u and $0 < r < R$, Green's formula and our normalization give

$$\frac{\mathrm{d}}{\mathrm{d}r} M_u(0, r) = \frac{2}{r} \mu_u(D(0, r)).$$

Approximation extends this identity in the integrated sense to every subharmonic function:

$$M_u(0, R) - M_u(0, r) = 2 \int_r^R \frac{\mu_u(D(0, t))}{t}\, \mathrm{d}t.$$

This formula simultaneously proves monotonicity and explains convexity in $\log r$.

2.1.2 What regularization preserves

Choose a radial mollifier $\rho \geq 0$ supported in the unit disc and set $u_\varepsilon = u * \rho_\varepsilon$ on $\Omega_\varepsilon = \{z : \mathrm{dist}(z, \partial\Omega) > \varepsilon\}$. Then

$$\mathrm{dd^c} u_\varepsilon = (\mathrm{dd^c} u) * \rho_\varepsilon = \mu_u * \rho_\varepsilon \geq 0.$$

Radiality also expresses $u_\varepsilon(z)$ as a positive average of circular means centered at z, which makes the family decrease as $\varepsilon \downarrow 0$ for the standard radially decreasing kernel construction. An arbitrary approximate identity still converges in L^1_{loc}, but need not be monotone. The distinction is important in nonlinear pluripotential theory.

If u_j are subharmonic and locally uniformly bounded above, Hartogs' lemma says that L^1_{loc} convergence controls upper limits:

$$\limsup_{j\to\infty} \sup_K u_j \le \sup_K u$$

for compact K after a small enlargement of K. Thus weak convergence of the Riesz measures does not permit isolated upward spikes in the limit.

Remark (A working dictionary)

Boundary values determine the harmonic part through the Poisson kernel; the positive measure μ_u determines the failure of harmonicity; logarithmic atoms record zeros with multiplicity. Jensen's formula, the argument principle, and the one-dimensional Poincaré–Lelong formula are three expressions of this same decomposition.

2.1.3 Deriving the mean-value identities

Assume first that $u \in C^2$ on a disc and set

$$M(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

Differentiation under the integral and the divergence theorem give

$$\begin{aligned} 2\pi r M'(r) &= \int_{|z|=r} \frac{\partial u}{\partial n}\,\mathrm{d}s = \int_{|z|<r} \Delta u\,\mathrm{d}A, \\ M'(r) &= \frac{1}{2\pi r}\int_{|z|<r} \Delta u\,\mathrm{d}A. \end{aligned}$$

Thus $\Delta u \ge 0$ implies that M is increasing and $u(0) = \lim_{r\downarrow 0} M(r) \le M(R)$. Translating the center proves the submean inequality on every contained disc.

Conversely, suppose a continuous u satisfies the submean inequality. If $\Delta u(a) < 0$, continuity gives a ball on which $\Delta u < 0$. Let h be the harmonic function with the same boundary values as u. Then $\Delta(u-h) < 0$ and $u - h = 0$ on the boundary, so the minimum principle applied to $h - u$ gives $u(a) > h(a)$. But the mean-value property for h and the submean inequality for u give $u(a) \le h(a)$, a contradiction. This proves $\Delta u \ge 0$ for C^2 functions. Approximation by convolution gives the distributional version.

The Jensen formula from Green's identity

Let u be smooth near $\overline{D_R}$ and integrate the identity for M':

$$\begin{aligned} M(R) - u(0) &= \int_0^R \frac{1}{2\pi r}\int_{|\zeta|<r} \Delta u(\zeta)\,\mathrm{d}A(\zeta)\,\mathrm{d}r \\ &= \frac{1}{2\pi}\int_{|\zeta|<R}\left(\int_{|\zeta|}^R \frac{\mathrm{d}r}{r}\right)\Delta u(\zeta)\,\mathrm{d}A(\zeta) \\ &= \frac{1}{2\pi}\int_{|\zeta|<R} \log\frac{R}{|\zeta|}\,\Delta u(\zeta)\,\mathrm{d}A(\zeta). \end{aligned}$$

The kernel is nonnegative. Replacing the smooth measure $(2\pi)^{-1}\Delta u\,\mathrm{d}A$ by the Riesz measure $\mathrm{dd^c}u$ yields the same formula for general subharmonic functions after regularization.

For $u(z) = \log|z-a|$ and $|a| < R$, the measure is concentrated at a, so

$$\frac{1}{2\pi}\int_0^{2\pi} \log|Re^{\mathrm{i}\theta} - a|\,\mathrm{d}\theta = \log R.$$

If $|a| > R$, the function is harmonic in D_R and the same average equals $\log|a|$. Together these give

$$\frac{1}{2\pi}\int_0^{2\pi} \log|Re^{\mathrm{i}\theta} - a|\,\mathrm{d}\theta = \log\max\{R, |a|\}.$$

A complete Riesz decomposition calculation

Suppose $\mu = \mathrm{dd^c}u$ on D_R and choose $r < R$. Define

$$v(z) = \int_{D_r} \log|z-\zeta|^2\,\mathrm{d}\mu(\zeta).$$

Because $\mathrm{dd}^{\mathrm{c}}_z \log|z-\zeta|^2 = \delta_\zeta$, Fubini's theorem for currents gives $\mathrm{dd^c}v = \mu$ on D_r. Hence $h = u - v$ satisfies $\mathrm{dd^c}h = 0$, or $\Delta h = 0$. Weyl's lemma upgrades the distribution h to a smooth harmonic function. Thus

$$u(z) = h(z) + \int_{D_r} \log|z-\zeta|^2\,\mathrm{d}\mu(\zeta).$$

The formula separates the harmonic ambiguity from the positive measure carrying all singularities.

What happens under convolution

Let $\rho \geq 0$ be a smooth radial function supported in the unit disc with integral one and set $\rho_\varepsilon(z) = \varepsilon^{-2}\rho(z/\varepsilon)$. On $\Omega_\varepsilon = \{\mathrm{dist}(z, \partial\Omega) > \varepsilon\}$,

$$u_\varepsilon = u * \rho_\varepsilon, \qquad \Delta u_\varepsilon = (\Delta u) * \rho_\varepsilon \geq 0.$$

Thus u_ε is smooth and subharmonic. Radiality and the increasing circular means show $u_\varepsilon \geq u$ and $u_\varepsilon \downarrow u$ when a standard radial approximate identity is chosen monotonically. The shrinking of the domain is essential: convolution needs the values of u in the whole ε-ball.

As a concrete nonsmooth example, $u(z) = \max\{\log|z|, -1\}$ is subharmonic. It is harmonic off the circle $|z| = e^{-1}$ and the origin, but its radial derivative jumps upward at that circle; the Riesz measure therefore has a positive circle component. This illustrates why distributional Laplacians, rather than pointwise second derivatives, are the natural language.

2.1.4 Self-study workshop: moving among the definitions of subharmonicity

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall Green's formula, the mean-value property of harmonic functions, convolution with a radial mollifier, and Lecture 4's distributional derivatives. The central task is to know which equivalence requires smoothness and which survives for upper-semicontinuous functions.

The smooth model first. For $u \in C^2(\Omega)$ the following are equivalent:

$$\Delta u \geq 0, \qquad u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{\mathrm{i}\theta})\, \mathrm{d}\theta$$

for every closed disc in Ω, and comparison with harmonic functions on discs. To see the direction from the Laplacian to means, define

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{\mathrm{i}\theta})\, \mathrm{d}\theta.$$

Green's formula gives

$$rM'(r) = \frac{1}{2\pi} \int_{D(a,r)} \Delta u\, \mathrm{d}A \geq 0.$$

Thus $M(r) \geq \lim_{s \downarrow 0} M(s) = u(a)$.

For $u(z) = |z|^2$, one obtains $M(r) = |a|^2 + r^2$, so the mean inequality is strict for $r > 0$. For $u(z) = \log |z|$, the function is harmonic off the origin but has a positive point mass in its distributional Laplacian. Jensen's formula records exactly the jump caused by that mass.

For a nonsmooth upper-semicontinuous u, the pointwise symbol Δu may not exist. One uses either the mean inequality or the distributional condition $\Delta u \geq 0$. Radial convolution on a slightly smaller domain gives smooth subharmonic functions. The shrinking of the domain is essential: $u * \rho_\varepsilon$ at z samples values in $B(z, \varepsilon)$.

Checkpoint (separate local and global claims)

(1) Is $\max\{\log |z|, -1\}$ subharmonic on the unit disc?
(2) Why is $-|z|^2$ not subharmonic?
(3) If a subharmonic function on a compact connected Riemann surface attains a maximum, what follows?

Solution (checkpoint). (1) Yes. The finite maximum of subharmonic functions is subharmonic after taking the upper-semicontinuous representative; here it is already continuous. (2) Its Laplacian is $-4 < 0$. (3) The strong maximum principle makes it constant. Compactness is used to guarantee that a maximum is attained. ◇

Exit criterion

You can prove the mean inequality from Green's formula, decide subharmonicity for piecewise-defined examples, and state precisely how regularization changes the domain.

Summary. Mean values, a positive distributional Laplacian, and decreasing smooth approximation are equivalent languages. Upper semicontinuity controls point values, distributions control limits, and convolution makes calculations possible.

Exercise 2.6: Convexity of circular means

Use the three-circles theorem, or lift $u(\mathrm{e}^w)$ to a half-plane, to prove that $M_u(0, r)$ is convex in $\log r$.

Exercise 2.7: Decreasing limits

Prove that a decreasing limit of subharmonic functions is either subharmonic or identically $-\infty$ on each connected component.

Optional handout: Riesz decomposition and Green potentials

This handout is optional. No later argument in the core lectures depends on it.

Let $D \Subset \mathbb{C}$ be a disc and u subharmonic. Its Riesz measure is $\mu_u = \mathrm{dd^c} u$. Choose the Green function $G_D(z, \zeta)$, normalized to vanish on ∂D and to have logarithmic singularity at $z = \zeta$. Then, on a smaller disc, one can write

$$u(z) = h(z) + \int_D G_D(z, \zeta)\, \mathrm{d}\mu_u(\zeta),$$

where h is harmonic. This is the Riesz decomposition: a subharmonic function is a harmonic function plus the potential of its positive Laplacian.

The formula follows first for smooth u from Green's identity. For general u, apply it to decreasing convolutions u_ε; their Laplacians converge weakly to μ_u, while Harnack compactness controls the harmonic remainders.

Example 2.8: Zeros of a holomorphic function

For $u = \log |f|^2$, the measure μ_u is the divisor of f. Evaluating the Riesz formula at the center of a disc yields Jensen's formula: the boundary mean of $\log |f|$ differs from $\log |f(0)|$ by the logarithmic contribution of the zeros.

Remark (Higher dimensions)

For psh functions, $\mathrm{dd^c} u$ is matrix-valued rather than a scalar measure. Local psh potentials of closed positive $(1, 1)$-currents are the several-variable successor of Riesz decomposition, while Lelong numbers extract the coefficient of the logarithmic part.

Exercise 2.9: Green function of the unit disc

Verify that $G_{\mathbb{D}}(z, \zeta) = \log \left| \frac{z-\zeta}{1-\bar{\zeta} z} \right|^2$ vanishes on the unit circle and compute its $\mathrm{dd^c}$ in the z variable.

2.2 Lecture 6: Plurisubharmonic functions and the Levi form

Week 3 material • 90 minutes

Notation for this lecture

X is a complex manifold, $\Omega \subset \mathbb{C}^n$ is a domain, and $\mathrm{PSH}(\Omega)$ is the set of plurisubharmonic functions on it. For $u \in C^2(\Omega)$, the Levi matrix is $H_u = (u_{j\bar{k}})$ with $u_{j\bar{k}} = \partial^2 u/(\partial z_j \partial \bar{z}_k)$, and its Levi form on $\xi \in \mathbb{C}^n$ is $\mathcal{L}_u(\xi) = \sum u_{j\bar{k}} \xi_j \bar{\xi}_k$. Thus u is psh exactly when $H_u \geq 0$, and strictly psh when $H_u > 0$. For a fixed real closed $(1,1)$-form θ, the notation $u \in \mathrm{PSH}(X, \theta)$ means that u is quasi-psh and $\theta + \mathrm{dd^c} u \geq 0$ as a current. A set contained locally in $\{v = -\infty\}$ for some psh $v \not\equiv -\infty$ is called pluripolar.

Prerequisite results used below

Line-restriction chain rule. For $u \in C^2(\Omega)$ and a complex affine line $\ell(\zeta) = p + \zeta\xi$,

$$\frac{\partial^2 (u \circ \ell)}{\partial \zeta \, \partial \bar{\zeta}}(0) = \sum_{j,k} u_{j\bar{k}}(p) \xi_j \bar{\xi}_k.$$

Hence subharmonicity on every complex line is equivalent, in the C^2 case, to positive semidefiniteness of the Levi matrix.

Decreasing-limit theorem for subharmonic functions. A decreasing limit of subharmonic functions on a plane domain is either subharmonic or identically $-\infty$. Applied after restriction to every complex line, the same statement gives the corresponding stability of psh functions.

Plan for the lecture. 0–20 min: restriction to complex lines; 20–45: Levi matrix and positive currents; 45–65: stability and suprema; 65–78: qpsh and θ-psh functions; 78–90: pluripolar sets.

Definition 2.10: Plurisubharmonic function

An upper-semicontinuous function $u : \Omega \subset \mathbb{C}^n \to [-\infty, \infty)$ is *plurisubharmonic* (psh) if $u \not\equiv -\infty$ and, for every complex affine line L, its restriction to each component of $\Omega \cap L$ is subharmonic or identically $-\infty$. Write $u \in \mathrm{PSH}(\Omega)$.

Theorem 2.11: Three tests for plurisubharmonicity

Let u be upper semicontinuous and not identically $-\infty$.

(1) If $u \in C^2$, then u is psh exactly when its Levi matrix $H_u = (u_{j\bar{k}})$ is positive semidefinite.
(2) In general, u is psh exactly when $u \in L^1_{\mathrm{loc}}$ and $\mathrm{dd^c} u$ is a positive $(1,1)$-current.
(3) Locally, a psh function is a decreasing limit of smooth psh functions, and conversely.

Proof. For $\xi \in \mathbb{C}^n$, put $v(\tau) = u(z + \tau\xi)$. A direct calculation gives

$$\Delta_\tau v(0) = 4 \sum_{j,k} u_{j\bar{k}}(z) \xi_j \bar{\xi}_k.$$

Thus every line restriction is subharmonic exactly when the Levi matrix is semipositive, proving (1).

Suppose now that u is psh. The submean inequality implies $u \in L^1_{\mathrm{loc}}$ unless $u \equiv -\infty$. On a smaller coordinate ball, regularize with a nonnegative radial approximate identity. The functions $u_\varepsilon = u * \rho_\varepsilon$ are smooth and psh by averaging translates, and $u_\varepsilon \downarrow u$ for the upper-semicontinuous representative. Part (1) gives $\mathrm{dd^c} u_\varepsilon \geq 0$; distributional convergence then gives $\mathrm{dd^c} u \geq 0$.

Conversely, if $u \in L^1_{\mathrm{loc}}$ and $\mathrm{dd^c}u \geq 0$, convolution gives $\mathrm{dd^c}u_\varepsilon = (\mathrm{dd^c}u) * \rho_\varepsilon \geq 0$. Hence every u_ε is psh by (1). Their upper-semicontinuous regularized limit agrees almost everywhere with u and is psh by the submean inequality and Fatou's lemma. This proves (2) and both directions of (3). □

Proposition 2.12: Stability of psh functions

The following operations preserve plurisubharmonicity: positive linear combinations, finite maxima, decreasing limits unless identically $-\infty$, holomorphic pullback unless a component is sent into the pole set, and composition $\chi \circ u$ with χ convex and increasing. The upper-semicontinuous regularization of a locally uniformly bounded-above supremum is psh.

Proof. Positive linear combinations and holomorphic pullback can be checked after restriction to every complex line. For smooth u,

$$\mathrm{dd^c}(\chi \circ u) = \chi'(u)\mathrm{dd^c}u + \chi''(u)\frac{\mathrm{i}}{2\pi}\partial u \wedge \bar{\partial} u \geq 0;$$

decreasing approximation gives the general case. The maximum of two subharmonic functions satisfies the submean inequality, so the line test proves the statement for finite maxima. Decreasing limits follow from monotone convergence in the submean inequality. Finally, Choquet's lemma reduces a locally bounded-above supremum to the upper regularization of an increasing sequence of finite maxima; the submean inequality then passes to the limit by Fatou's lemma. □

Example 2.13: Analytic singularities

If $f_1, \ldots, f_N$ are holomorphic and $c > 0$, then

$$u = c \log\left(|f_1|^2 + \cdots + |f_N|^2\right)$$

is psh. These are *analytic singularities*; their $-\infty$ locus is the analytic set $\{f_1 = \cdots = f_N = 0\}$.

Warning

$\min(u, v)$ is usually not psh. A supremum need not be upper semicontinuous, so one must take $\sup_i^* u_i(x) = \limsup_{y \to x} \sup_i u_i(y)$. Finally, the statement $\mathrm{dd^c}u \geq 0$ for a distribution class still requires recovery of its upper-semicontinuous representative.

Global psh functions on a compact complex manifold are constant. To retain a background curvature term, use the following relative notion.

Definition 2.14: Quasi-psh and θ-psh functions

A function is *quasi-plurisubharmonic* (qpsh) if locally it is the sum of a psh function and a smooth function. For a closed real $(1,1)$-form θ, a qpsh function u is θ-psh if

$$\theta_u := \theta + \mathrm{dd^c}u \geq 0.$$

We write $u \in \mathrm{PSH}(X, \theta)$.

For example on $\mathbb{P}^1$,

$$u([z_0 : z_1]) = \log \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}$$

satisfies $\omega_{\mathrm{FS}} + \mathrm{dd^c}u = [z_1 = 0]$.

Definition 2.15: Pluripolar set

A set $E \subset X$ is locally pluripolar if near every point there is a psh function u with $E \subset \{u = -\infty\}$. Every proper analytic subset is pluripolar, but pluripolar sets may be much more complicated.

Lemma 2.16: Regularized maximum

Let $\rho \in C_c^\infty(\mathbb{R})$ be nonnegative, even, and satisfy $\int \rho = 1$. For $\varepsilon_j > 0$ set

$$M_\varepsilon(t_1, \dots, t_p) = \int_{\mathbb{R}^p} \max_j (t_j + \varepsilon_j s_j) \prod_{j=1}^{p} \rho(s_j)\, \mathrm{d}s_1 \cdots \mathrm{d}s_p.$$

Then M_ε is smooth, convex, and nondecreasing in each variable; it commutes with adding the same constant to all variables; and

$$\max_j t_j \le M_\varepsilon(t) \le \max_j t_j + C \max_j \varepsilon_j.$$

Consequently, $M_\varepsilon(u_1, \dots, u_p)$ is psh whenever the u_j are smooth psh functions.

Proof. Convolution preserves convexity and coordinatewise monotonicity. Evenness gives zero barycenter, so Jensen's inequality gives the lower bound and translation invariance. The upper bound follows from the compact support of ρ. For the last assertion, differentiate twice and use the chain rule: the Levi form is the sum of nonnegative first derivatives times the Levi forms of u_j, plus the positive-semidefinite Hessian of M_ε applied to the $(1,0)$ differentials of the u_j. □

2.2.1 The Levi form of a logarithmic norm

Let $F = (f_1, \dots, f_N) : \Omega \to \mathbb{C}^N$ be holomorphic and assume $F \neq 0$ at the point under consideration. Put $S = \sum_\alpha |f_\alpha|^2$. Direct differentiation gives

$$\partial\bar\partial \log S = \frac{S \sum_\alpha \partial f_\alpha \wedge \overline{\partial f_\alpha} - \left(\sum_\alpha \bar f_\alpha \partial f_\alpha\right) \wedge \overline{\left(\sum_\beta \bar f_\beta \partial f_\beta\right)}}{S^2}.$$

Evaluating on a vector ξ turns the numerator into

$$|F|^2 |\mathrm{d}F(\xi)|^2 - |\langle \mathrm{d}F(\xi), F\rangle|^2 \ge 0.$$

Thus $\log|F|^2$ is psh away from $F^{-1}(0)$, and its extension with value $-\infty$ on the zero set is psh everywhere. Geometrically, the displayed form is the pullback of the Fubini–Study form under $[F] : \Omega \setminus F^{-1}(0) \to \mathbb{P}^{N-1}$.

The equality directions are also informative. The Levi form vanishes on ξ exactly when $\mathrm{d}F(\xi)$ is proportional to F. Therefore a logarithmic norm is usually only semipositive. Adding $\varepsilon|z|^2$ makes it strictly psh on a coordinate domain while retaining the same pole set.

2.2.2 Strictness, pullback, and compact manifolds

If $u \in C^2$ is psh and χ is convex increasing, then

$$\mathrm{i}\partial\bar\partial(\chi \circ u) = \chi'(u)\mathrm{i}\partial\bar\partial u + \chi''(u)\mathrm{i}\partial u \wedge \bar\partial u.$$

The second term supplies positivity in the gradient direction but says nothing along the level set of u. Thus strict convexity of χ does not automatically turn every psh function into a strictly psh one. In constructing exhaustions, one usually adds a separate strict term before composing.

For a holomorphic map $g : Y \to X$, a C^2 psh function satisfies

$$\mathcal{L}_{u\circ g}(v) = \mathcal{L}_u(\mathrm{d}g(v)).$$

Pullback preserves semipositivity, but it preserves strict positivity only when g is immersive. This is the function-level analogue of pulling back a Kähler form.

On a compact connected manifold, a global psh function is constant by the maximum principle. The relative expression $\theta + \mathrm{dd^c}u$ avoids this obstruction because the smooth form θ carries a nonzero cohomology class. Replacing θ by $\theta + \mathrm{dd^c}\psi$ replaces u by $u - \psi$ and leaves the current unchanged. Hence the space of θ-psh potentials depends on the cohomology class of θ, up to this natural translation.

Warning

The pole set $\{u = -\infty\}$ of a psh function is pluripolar, but the converse is local unless a global construction is supplied. Likewise, a measure-zero set need not be pluripolar, and a pluripolar set may be dense. Measure theory and pluripotential smallness are different notions.

2.2.3 The line test and the Levi matrix in coordinates

Let $u \in C^2(\Omega)$ and fix $a \in \Omega$, $v \in \mathbb{C}^n$. On the complex line $\ell(\zeta) = a + \zeta v$ the chain rule gives

$$\frac{\partial^2(u\circ\ell)}{\partial\zeta\partial\bar\zeta}(0) = \sum_{j,k=1}^{n} u_{j\bar k}(a)v_j\bar v_k.$$

Since $\Delta_\zeta = 4\partial^2/(\partial\zeta\partial\bar\zeta)$, the restriction of u to every complex line is subharmonic exactly when the Hermitian matrix $H_u = (u_{j\bar k})$ is semipositive. With our normalization,

$$\mathrm{dd^c}u = \frac{\mathrm{i}}{2\pi}\sum_{j,k} u_{j\bar k}\,\mathrm{d}z_j \wedge \mathrm{d}\bar z_k,$$

so this is also exactly positivity of the $(1,1)$-form $\mathrm{dd^c}u$.

For $u(z) = |z_1|^2 + 2|z_2|^2 + 2\Re(z_1z_2)$, the holomorphic term $2\Re(z_1z_2)$ contributes nothing to $\partial\bar\partial u$, and

$$H_u = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} > 0.$$

This example is a useful warning: strict plurisubharmonicity is determined by mixed $z, \bar z$ derivatives, not by the full real Hessian.

Logarithms of holomorphic tuples

Let $F = (f_1, \dots, f_N)$ be holomorphic and put $S = \sum_\alpha |f_\alpha|^2$. Away from $F^{-1}(0)$,

$$\partial S = \sum_\alpha \bar f_\alpha\,\partial f_\alpha,$$

$$\partial\bar\partial\log S = \frac{S\sum_\alpha \partial f_\alpha \wedge \overline{\partial f_\alpha} - (\sum_\alpha \bar f_\alpha\partial f_\alpha)\wedge\overline{\left(\sum_\beta \bar f_\beta\partial f_\beta\right)}}{S^2}.$$

Evaluating on a vector v turns the numerator into

$$S\sum_\alpha |\mathrm{d}f_\alpha(v)|^2 - \left|\sum_\alpha \bar f_\alpha \mathrm{d}f_\alpha(v)\right|^2 \geq 0$$

by Cauchy–Schwarz. Hence $\log \sum |f_\alpha|^2$ is psh away from the common zero set, and its value $-\infty$ there gives the natural psh extension. This calculation is the local model for analytic singularities.

Taking $F = (1, z_1, \dots, z_n)$ recovers the Fubini–Study potential. Taking $F = (z_1, \dots, z_n)$ shows that $\log |z|^2$ is psh and has Levi rank $n-1$ away from the origin: the radial vector is the null direction, while tangent directions to the sphere are positive.

A strictly psh exhaustion of the ball

On the unit ball put $\rho(z) = -\log(1 - |z|^2)$. Then

$$\rho_{j\bar{k}} = \frac{\delta_{jk}}{1-|z|^2} + \frac{\bar{z}_j z_k}{(1-|z|^2)^2}.$$

For $v \neq 0$,

$$\sum \rho_{j\bar{k}} v_j \bar{v}_k = \frac{|v|^2}{1-|z|^2} + \frac{|\langle v, z\rangle|^2}{(1-|z|^2)^2} > 0.$$

Moreover $\rho(z) \to +\infty$ as $z \to \partial B$. Thus it is a smooth strictly psh exhaustion. This one formula simultaneously proves strict pseudoconvexity and supplies the proper function used in Stein theory.

Stability operations checked directly

If u, v are psh and $a, b \geq 0$, then $\mathrm{dd^c}(au + bv) = a\mathrm{dd^c}u + b\mathrm{dd^c}v \geq 0$. If F is holomorphic, $\mathrm{dd^c}(u \circ F) = F^*(\mathrm{dd^c}u) \geq 0$. For a convex increasing $\chi \in C^2(\mathbb{R})$,

$$\mathrm{dd^c}\chi(u) = \chi'(u)\mathrm{dd^c}u + \frac{\mathrm{i}}{2\pi}\chi''(u)\partial u \wedge \bar{\partial} u \geq 0.$$

The maximum of two psh functions is psh because its restriction to every complex line is the maximum of two subharmonic functions; upper semicontinuous regularization is needed for arbitrary suprema.

There is no analogous minimum principle. Both $|z|^2$ and 1 are subharmonic on $\mathbb{C}$. However, $\min\{|z|^2, 1\}$ has a downward jump in radial derivative at $|z| = 1$ and hence a negative circle contribution to its distributional Laplacian.

2.2.4 Self-study workshop: turning plurisubharmonicity into Hermitian linear algebra

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. You need the subharmonic line test from Lecture 5, the $(1,1)$ calculus of Lecture 2, and the eigenvalue criterion for Hermitian matrices. Always distinguish psh from strictly psh.

One computation, three interpretations. Let $u \in C^2(\Omega)$ for $\Omega \subset \mathbb{C}^n$. At p and $v = (v_1, \dots, v_n)$, restrict u to the complex line $\zeta \mapsto p + \zeta v$. The one-variable chain rule gives

$$\left.\frac{\partial^2}{\partial\zeta\,\partial\bar{\zeta}} u(p + \zeta v)\right|_{\zeta=0} = \sum_{j,k} u_{j\bar{k}}(p) v_j \bar{v}_k.$$

Therefore the line restriction is subharmonic for every v exactly when the Levi matrix $(u_{j\bar{k}})$ is positive semidefinite. Equivalently, $\mathrm{dd^c}u$ is a positive $(1,1)$-form. Strict psh means positive definite, locally with a quantitative lower bound after shrinking to a compact set.

For

$$u(z) = \log(1 + |z_1|^2) + |z_2|^2 \quad \text{on } \mathbb{C}^2,$$

the Levi matrix is diagonal:

$$\begin{pmatrix} (1+|z_1|^2)^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus u is strictly psh. By contrast, $|z_1|^2 - |z_2|^2$ fails the line test on $v = (0, 1)$.

If $F = (f_1, \dots, f_N)$ is a holomorphic tuple with no common zero, then $\log \sum_\alpha |f_\alpha|^2$ is psh. The calculation is a Cauchy–Schwarz inequality: the Hessian is the squared norm of the component of $\mathrm{d}F(v)$ orthogonal to the line spanned by F. This geometric reading also identifies the zero directions.

Checkpoint (stability rules with hypotheses)

(1) If u is psh and χ is convex increasing, why is $\chi \circ u$ psh?
(2) Is the difference of two psh functions necessarily psh?
(3) Why can there be no strictly psh function on a compact connected complex manifold?

Solution (checkpoint). For smooth functions,

$$\partial\bar\partial(\chi(u)) = \chi'(u)\partial\bar\partial u + \chi''(u)\partial u \wedge \bar\partial u,$$

and both terms are positive under the stated hypotheses; approximation gives the general case. Differences need not be psh, as $|z_1|^2 - |z_2|^2$ shows. A psh function on a compact connected manifold is constant by restriction to coordinate discs and the maximum principle, so it cannot be strict. ⋄

Exit criterion

You can pass fluently among line restrictions, the Levi matrix, and positivity of $\mathrm{dd^c}u$, and you can prove rather than memorize the main stability operations.

Summary. Plurisubharmonicity synchronizes one-variable subharmonicity in every complex direction. The Levi matrix, positive $(1, 1)$-currents, and monotone regularization are equivalent entry points. The θ-psh language permits singular potentials on compact manifolds.

Exercise 2.17: Maximum versus minimum

Prove that $\max(\log|z_1|, \log|z_2|)$ is psh and give an explicit pair of psh functions whose minimum is not psh.

Exercise 2.18: A pole on projective space

Check the formula for u in both affine charts and verify directly that $\omega_{\mathrm{FS}} + \mathrm{dd^c}u = [z_1 = 0]$.

Optional handout: Plurisubharmonic envelopes and Choquet's lemma

This handout is optional. No later argument in the core lectures depends on it.

Suppose $\mathcal{F}$ is a family of psh functions locally bounded above. The raw supremum need not be upper semicontinuous, but

$$P = \Big(\sup_{u \in \mathcal{F}} u\Big)^*$$

is psh. A useful reduction is Choquet's lemma: there is a sequence $u_j \in \mathcal{F}$ such that $(\sup_j u_j)^* = P$. Thus an uncountable extremal problem can be handled by a sequence.

Choose a countable basis of relatively compact balls and, on each one, select functions whose suprema approximate the supremum of the family on a dense set. Taking finite maxima produces an increasing sequence. The submean inequality passes to the upper-semicontinuous regularization, proving plurisubharmonicity.

Definition 2.19: Relative extremal function

For $K \Subset \Omega$, define

$$h_{K,\Omega} = \sup\{u \in \mathrm{PSH}(\Omega) : u \le 0 \text{ on } \Omega,\ u \le -1 \text{ on } K\}.$$

Its upper-semicontinuous regularization is the relative extremal function of K.

Where the obstacle is inactive, the envelope is maximal: in the Bedford–Taylor sense,

$$(\mathrm{dd^c} h^*_{K,\Omega})^n = 0 \qquad \text{on } \Omega \setminus K.$$

The total Monge–Ampère mass is closely related to the relative capacity of K. This is the prototype for obstacle problems and pluripotential envelopes on compact Kähler manifolds.

Exercise 2.20: Envelope in the disc

For $K = \overline{D(0,r)} \subset \mathbb{D}$, determine $h^*_{K,\mathbb{D}}$ explicitly using radial harmonic functions on the annulus.

2.3 Lecture 7: Pseudoconvex domains, exhaustion, and the Levi problem

Week 4 material • 90 minutes

Notation for this lecture

A smoothly bounded domain is written $\Omega = \{\rho < 0\}$, where ρ is a real defining function and $\mathrm{d}\rho \neq 0$ on $\partial\Omega$. Its complex tangent space is $T^{1,0}_p \partial\Omega = \{\xi \in \mathbb{C}^n : \sum_j \rho_j(p)\xi_j = 0\}$, where $\rho_j = \partial\rho/\partial z_j$. The boundary Levi form is $\mathcal{L}_\rho(\xi) = \sum \rho_{j\bar{k}}\xi_j\bar{\xi}_k$ restricted to that space. An exhaustion is a proper function whose sublevel sets are relatively compact; a complex manifold is pseudoconvex here if it admits a continuous psh exhaustion. The unit disc is Δ, Δ_r is the radius-r disc, and $A_{r,1} = \{r < |z| < 1\}$ is an annulus.

Prerequisite results used below

Regular level-set theorem. If a smooth real function ρ satisfies $\mathrm{d}\rho(p) \neq 0$, then near p the set $\{\rho = 0\}$ is a smooth real hypersurface and $T_p\{\rho = 0\} = \ker \mathrm{d}\rho(p)$.
Strictly psh sublevel sets. If $\rho \in C^2$ is strictly psh near the boundary of $\Omega = \{\rho < 0\}$ and $\mathrm{d}\rho \neq 0$ on $\partial\Omega$, then the Levi form of $\partial\Omega$ is positive on every nonzero complex tangent vector. Conversely, a strictly pseudoconvex boundary admits a defining function that is strictly psh on a collar after replacing ρ by $\mathrm{e}^{A\rho} - 1$ for sufficiently large A.

Plan for the lecture. 0–20 min: Hartogs extension and domains of holomorphy; 20–42: the Levi form of a boundary; 42–62: psh exhaustion; 62–80: the Levi problem; 80–90: examples.

In one variable every domain can be the maximal domain of definition of a holomorphic function. In several variables, Hartogs extension can fill holes automatically. A genuine domain of holomorphy must therefore satisfy a complex convexity condition.

Definition 2.21: Domain of holomorphy

A domain $\Omega \subset \mathbb{C}^n$ is a *domain of holomorphy* if there is an $f \in \mathcal{O}(\Omega)$ that has no holomorphic extension to any connected domain $\Omega' \supsetneq \Omega$. Equivalently, Ω is the domain of existence of one holomorphic function. The weaker assertion that not every function extends to one fixed larger domain is not the definition.

Definition 2.22: Levi pseudoconvexity

Suppose $\Omega = \{\rho < 0\}$ has C^2 boundary and $\mathrm{d}\rho \neq 0$ on $\partial\Omega$. At $p \in \partial\Omega$, its complex tangent space is

$$T^{1,0}_p \partial\Omega = \left\{ \xi \in \mathbb{C}^n : \sum_j \rho_j(p)\xi_j = 0 \right\}.$$

The Levi form is $\mathcal{L}_\rho(\xi) = \sum \rho_{j\bar{k}}\xi_j\bar{\xi}_k$. The boundary is Levi pseudoconvex if this form is semipositive, and strictly pseudoconvex if it is positive definite.

Replacing ρ by $a\rho$ with $a > 0$ multiplies the Levi form on complex tangent vectors by a. Its sign is therefore intrinsic. Away from the complex tangent space, derivatives of a produce extra terms.

Definition 2.23: Pseudoconvex complex manifold

A complex manifold X is *pseudoconvex*, or *weakly pseudoconvex*, if it admits a smooth psh exhaustion $\psi : X \to \mathbb{R}$. We will say explicitly that X *admits a smooth strictly psh exhaustion* when strict positivity holds everywhere. The term *strongly pseudoconvex manifold* is often reserved for the different, 1-convex condition in which strict positivity is required only outside a compact set. Exhaustion means that every sublevel set $\{\psi < c\}$ is relatively compact.

Example 2.24: An exhaustion from a defining function

For a bounded smoothly bounded strictly pseudoconvex domain, modify ρ so that $-\log(-\rho)$ is strictly psh near the boundary, then patch it with a small multiple of $|z|^2$. The identity

$$\partial\bar{\partial}[-\log(-\rho)] = \frac{\partial\bar{\partial}\rho}{-\rho} + \frac{\partial\rho \wedge \bar{\partial}\rho}{\rho^2}$$

shows that the first term controls complex tangential directions and the second controls the normal direction.

Warning

Levi semipositivity at the boundary is tested only in complex tangent directions. Requiring $\partial\bar{\partial}\rho \geq 0$ throughout a neighborhood is stronger and often needs a new defining function. Real convexity implies pseudoconvexity, but not conversely.

Theorem 2.25: The Levi problem

For a domain $\Omega \subset \mathbb{C}^n$, the following are equivalent:
(1) Ω is a domain of holomorphy;
(2) Ω is holomorphically convex;
(3) Ω has a continuous psh exhaustion;
(4) Ω is a Stein manifold.
For a C^2 boundary, these are also equivalent to Levi pseudoconvexity.

Proof (proof architecture; the Cartan–Thullen and Oka lemmas are quoted). We organize the proof so that every analytic input is visible. The proof route is

$$\begin{aligned}&\text{strict psh exhaustion} \Rightarrow \text{complete Kähler metric} \Rightarrow L^2 \text{ solution of } \bar{\partial}\\ &\qquad\qquad \Rightarrow \text{separating and boundary-blowing-up functions} \Rightarrow \text{holomorphic convexity.}\end{aligned}$$

The weighted estimate needed below is proved in Lecture 27.

Step 1: regularizing the exhaustion. If ρ is a continuous psh exhaustion, first make it strict on each relatively compact exhaustion band by adding a sufficiently small multiple of $|z|^2$ in the ambient space $\mathbb{C}^n$. Richberg regularization then produces a smooth strict approximation on that band. Regularized maxima, with errors chosen successively smaller, make the approximants agree on preceding sublevels. The locally finite limit is a smooth strictly psh exhaustion. Richberg's theorem by itself regularizes *strict* continuous psh functions; the preliminary strict perturbation is essential. Conversely, a smooth strict exhaustion is certainly a continuous psh one.

Step 2: weighted $\bar{\partial}$ solvability. Choose a convex increasing χ so rapidly growing that $i\partial\bar{\partial}\chi(\rho) \geq \omega$ and every compactly supported $\bar{\partial}$-closed form under consideration has finite norm for the weight $e^{-\chi(\rho)}$. Exhaust Ω by regular strictly pseudoconvex sublevels and apply Hörmander's estimate on each. Weak compactness and the uniform estimate give a global solution. Thus every $\bar{\partial}$-closed $(0, q)$-form, $q \geq 1$, with the chosen weighted finite norm is $\bar{\partial}$-exact.

Step 3: separation and local coordinates. For distinct x, y, choose a cutoff η that is one near x and supported in a coordinate ball disjoint from y. Solve $\bar{\partial}u = \bar{\partial}\eta$ with a weight containing $(n+1)\log|z-x|^2 + (n+1)\log|z-y|^2$. Finite weighted norm forces the required zero jets of u at

x, y. Then $f = \eta - u$ is global holomorphic with $f(x) = 1$ and $f(y) = 0$. Replacing η successively by ηz_j and using a weight that forces two-jets of u to vanish gives global functions whose differentials span $T_x^{*1,0}\Omega$.

Step 4: holomorphic convexity. Let $K \Subset \Omega$ and let x_j escape every compact set. Choose disjoint coordinate balls about a subsequence and local functions with values larger than 2^j at x_j. Cut them off, solve the resulting $\bar\partial$ errors with weights growing successively on the exhaustion, and choose the errors smaller than 2^{-j} on the preceding compact set. The corrected holomorphic series converges normally, is bounded on K, and is unbounded along the chosen subsequence. Hence no escaping sequence lies in $\widehat{K}_{O(\Omega)}$; the hull is closed and therefore compact. Steps 3–4 show that Ω is Stein.

Step 5: Cartan–Thullen and the boundary criterion. The Cartan–Thullen lemma says that a domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if all of its holomorphic hulls are relatively compact. One direction uses Cauchy estimates: failure of relative compactness of a hull gives simultaneous analytic continuation across a boundary polydisc. In the other direction, choose functions separating successive escaping compact sets from a fixed compact set and form a normally convergent diagonal series; the resulting function has no extension to a larger domain. This lemma identifies conditions (1) and (2), while Steps 1–4 identify (2)–(4).

Finally suppose $\partial\Omega$ is C^2. Oka's boundary-distance lemma states that Levi semipositivity is equivalent, on a collar of the boundary, to plurisubharmonicity of $-\log\delta$ up to addition of a fixed smooth strictly psh term, where δ is the boundary distance. Its proof differentiates the signed distance: the Levi term controls complex tangential directions and the square of $\partial\delta$ controls the normal direction. Gluing this collar function to an interior strict psh function by a regularized maximum gives a psh exhaustion. Conversely, the Kontinuitätssatz (or the equivalent holomorphic-disc test) shows that a pseudoconvex domain with C^2 boundary cannot have a negative Levi direction. This proves the final equivalence. □

Example 2.26: The Hartogs triangle

The domain $\{(z, w) : |z| < |w| < 1\}$ is pseudoconvex. On it the three functions

$$-\log|w|, \qquad -\log(1-|w|^2), \qquad -\log\bigl(1-|z|^2/|w|^2\bigr)$$

are psh (the first is pluriharmonic), and their sum is a psh exhaustion. Merely writing the domain as a sublevel set of a psh function would not by itself prove pseudoconvexity. Its nonsmooth boundary demonstrates why an exhaustion definition is more robust than a boundary definition.

2.3.1 The Hartogs phenomenon behind pseudoconvexity

Assume $n \geq 2$ and consider a Hartogs figure

$$H = \bigl(\Delta_r \times \Delta\bigr) \cup \bigl(\Delta \times A_{1-r,1}\bigr) \subset \mathbb{C}^2,$$

where $0 < r < 1$. If $f \in O(H)$, expand in the second variable on the annular part:

$$f(z, w) = \sum_{k\in\mathbb{Z}} a_k(z) w^k, \qquad a_k(z) = \frac{1}{2\pi \mathrm{i}} \int_{|w|=\rho} \frac{f(z, w)}{w^{k+1}}\, \mathrm{d}w.$$

Each a_k is holomorphic for $|z| < 1$. When $|z| < r$, the function is holomorphic through $w = 0$, so $a_k = 0$ there for $k < 0$. The identity theorem gives $a_k \equiv 0$ on Δ for every negative k. The remaining power series extends f to the full bidisc.

This elementary calculation explains why an inner hole is often invisible to holomorphic functions in several variables. A domain of holomorphy must prevent analytic discs from sweeping through its boundary in this way. The Levi form is the infinitesimal test for precisely that failure.

2.3.2 Boundary calculations worth knowing

For the unit ball, $\rho(z) = |z|^2 - 1$ and

$$\mathcal{L}_\rho(\xi) = |\xi|^2.$$

The boundary is strictly pseudoconvex. More generally, every bounded real convex domain with C^2 boundary is pseudoconvex because the real Hessian restricted to a complex line controls the Levi form.

The spherical annulus $\{r < |z| < 1\} \subset \mathbb{C}^n$ has two boundary components. The outer one is pseudoconvex with the outward defining function $|z|^2 - 1$. Near the inner boundary the defining function is $r^2 - |z|^2$, whose Levi form is negative on complex tangent vectors. For $n \geq 2$ the annulus is therefore not pseudoconvex, in agreement with Hartogs extension across the inner ball.

For a product domain such as a polydisc, the natural boundary has corners and the C^2 Levi test does not apply globally. Nevertheless

$$\psi(z) = -\sum_{j=1}^{n} \log(1 - |z_j|^2)$$

is a smooth strictly psh exhaustion. Exhaustion functions are therefore more robust than boundary defining functions.

Remark (Holomorphic hulls as a compactness test)

For $K \Subset \Omega$, the holomorphic hull

$$\widehat{K}_\Omega = \{z \in \Omega : |f(z)| \leq \sup_K |f| \text{ for every } f \in \mathcal{O}(\Omega)\}$$

contains everything that holomorphic functions cannot separate from K by size. Pseudoconvexity makes these hulls relatively compact. Thus the Levi problem turns a local Hessian inequality into a global statement about all holomorphic functions at once.

2.3.3 Boundary Levi forms by direct calculation

For the unit ball take $r(z) = |z|^2 - 1$. Then

$$r_j = \bar{z}_j, \qquad r_{j\bar{k}} = \delta_{jk}.$$

At $p \in \partial B$, a vector v is complex tangential precisely when $\sum_j \bar{p}_j v_j = 0$. Its boundary Levi form is

$$\mathcal{L}_r(v) = \sum_{j,k} r_{j\bar{k}} v_j \bar{v}_k = |v|^2 > 0.$$

Thus the sphere is strictly pseudoconvex.

If $\tilde{r} = hr$ with $h > 0$ smooth, then

$$\partial\bar{\partial}(hr) = h\partial\bar{\partial}r + r\partial\bar{\partial}h + \partial h \wedge \bar{\partial}r + \partial r \wedge \bar{\partial}h.$$

On $\partial\Omega$, $r = 0$; on complex tangent vectors, $\partial r(v) = \bar{\partial}r(\bar{v}) = 0$. Therefore

$$\mathcal{L}_{\tilde{r}}(v) = h\mathcal{L}_r(v).$$

The sign of the boundary Levi form is independent of the chosen defining function.

For the product domain Δ^2 the boundary is not smooth at the corners, so no single smooth defining function describes all of it. Nevertheless

$$\Phi(z) = -\log(1 - |z_1|^2) - \log(1 - |z_2|^2)$$

is a smooth strictly psh exhaustion. Hence pseudoconvexity is naturally phrased by exhaustions, not only by smooth boundary Levi forms.

Hartogs extension by Laurent coefficients

Let

$$H = \{(z, w) : |z| < 1,\ 1 - \varepsilon < |w| < 1\} \cup \{(z, w) : |z| < \varepsilon,\ |w| < 1\}$$

be a Hartogs figure and let $f \in \mathcal{O}(H)$. On the annular part expand in w:

$$f(z, w) = \sum_{k \in \mathbb{Z}} a_k(z) w^k, \qquad a_k(z) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r} \frac{f(z, \zeta)}{\zeta^{k+1}}\, \mathrm{d}\zeta,$$

where $1 - \varepsilon < r < 1$. The integral shows that every a_k is holomorphic in z for $|z| < 1$. When $|z| < \varepsilon$, the function is holomorphic for $|w| < 1$, so Cauchy's theorem gives $a_k(z) = 0$ for all $k < 0$. The identity theorem then gives $a_k \equiv 0$ on the connected disc $|z| < 1$ for every $k < 0$.

The remaining power series

$$\widetilde{f}(z, w) = \sum_{k \geq 0} a_k(z) w^k$$

converges normally on smaller bidiscs by Cauchy estimates and extends f to Δ^2. This is the precise mechanism behind the Hartogs phenomenon: the inner hole cannot support negative Laurent coefficients once a thin cylinder fills it for one open set of parameter values.

Why $-\log(-r)$ is psh near a pseudoconvex boundary

Assume $r < 0$ on Ω and $\mathrm{d}r \neq 0$ on the boundary. A calculation gives

$$\partial\bar{\partial}[-\log(-r)] = -\frac{\partial\bar{\partial}r}{r} + \frac{\partial r \wedge \bar{\partial} r}{r^2}.$$

The second term controls the complex normal direction. On tangential vectors the second term vanishes and the first has the sign of the boundary Levi form because $r < 0$. Strict positivity in the normal direction and continuity show that, after possibly replacing r by $e^{Ar} r$ with A large, this is psh in a collar of the boundary. Gluing it with a bounded interior function produces a psh exhaustion.

The logical content of the Levi problem

For a domain $\Omega \subset \mathbb{C}^n$, the chain of implications is

$$\text{psh exhaustion} \Longrightarrow \text{holomorphic convexity} \Longrightarrow \text{domain of holomorphy.}$$

The difficult implication uses L^2 solution of $\bar{\partial}$ to construct holomorphic functions that become large along a sequence approaching the boundary. The compatibility condition is enforced by starting with a cutoff meromorphic model and correcting its $\bar{\partial}$-error. Thus the theorem does not claim that an arbitrary psh exhaustion is holomorphic; it says that its Levi positivity supplies the estimate needed to manufacture enough holomorphic functions.

2.3.4 Self-study workshop: reading pseudoconvexity from a defining function

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall the Levi matrix, regular level sets, complex tangent spaces, and proper exhaustions. The sign of a boundary Levi form is meaningful only after fixing the convention $\Omega = \{\rho < 0\}$.

The boundary calculation as an algorithm. At $p \in \partial\Omega$ with $d\rho(p) \neq 0$, a $(1,0)$ vector $v = \sum v_j \partial/\partial z_j$ is complex tangential when

$$\partial\rho(v) = \sum_j \rho_j(p) v_j = 0.$$

The boundary Levi form is the restriction of $(\rho_{j\bar{k}})$ to this hyperplane. For the ellipsoid

$$\Omega = \{|z_1|^2 + 2|z_2|^2 < 1\}, \qquad \rho = |z_1|^2 + 2|z_2|^2 - 1,$$

we have

$$\mathcal{L}_\rho(v) = |v_1|^2 + 2|v_2|^2 > 0$$

for every nonzero vector, hence in particular on the complex tangent space. The domain is strongly pseudoconvex.

Changing the defining function does not change the boundary sign. If $\widetilde{\rho} = h\rho$ with $h > 0$, then on a tangential vector at the boundary all terms containing ρ or $\partial\rho(v)$ disappear, leaving

$$\mathcal{L}_{\widetilde{\rho}}(v) = h\,\mathcal{L}_\rho(v).$$

This is the calculation that makes pseudoconvexity intrinsic.

Near a pseudoconvex boundary, the candidate $-\log(-\rho)$ has Hessian

$$\partial\bar{\partial}[-\log(-\rho)] = \frac{\partial\bar{\partial}\rho}{-\rho} + \frac{\partial\rho \wedge \bar{\partial}\rho}{\rho^2}.$$

The second term controls the complex normal direction, while the first carries the tangential Levi form. A small strictly psh correction handles any remaining compact interior region when constructing an exhaustion.

Checkpoint (where each hypothesis is used)

(1) Why is the inner boundary of a spherical shell in $\mathbb{C}^n$, $n \geq 2$, not pseudoconvex with the domain-side convention?
(2) Does pseudoconvexity make the boundary Levi term in an L^2 identity zero?
(3) What extra condition turns a psh function into an exhaustion?

Solution (checkpoint). At the inner boundary a defining function for the shell has Levi form negative on complex tangential directions, so the sign fails. Pseudoconvexity makes the boundary term nonnegative, not zero. An exhaustion must be proper: every sublevel set must be relatively compact in the domain. ⋄

Exit criterion

Given a smooth defining function, you can compute the complex tangent equation, restrict the Levi matrix, verify invariance under positive rescaling, and explain how a boundary calculation enters a global exhaustion.

Summary. Pseudoconvexity is the correct complex analogue of convexity. Boundary Levi forms, psh exhaustions, holomorphic convexity, and domains of holomorphy become one cluster of equivalent notions through the Levi problem.

Exercise 2.27: Ball and polydisc

Construct smooth strictly psh exhaustions of the unit ball and the polydisc. Explain how to regularize the nonsmooth maximum naturally arising for the polydisc.

Exercise 2.28: Changing a defining function

Prove that replacing ρ by $a\rho$ multiplies its Levi form by a on the complex tangent space of the boundary.

Optional handout: The $\bar\partial$-Neumann problem

This handout is optional. No later argument in the core lectures depends on it.

On a smoothly bounded domain $\Omega \subset \mathbb{C}^n$, solving $\bar\partial u = f$ with minimum L^2 norm is governed by the complex Laplacian

$$\Box_q = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

with the $\bar\partial$-Neumann boundary condition. If $\Box_q$ has a bounded inverse N_q on the orthogonal complement of its kernel, the canonical solution is $u = \bar\partial^* N_q f$.

For a defining function ρ, a smooth $(0, q)$-form v belongs to the domain of $\bar\partial^*$ only if contraction of v with $\bar\partial\rho$ vanishes on the boundary. In the Morrey–Kohn identity, integration by parts then produces the boundary term

$$\int_{\partial\Omega} \sum_{|K|=q-1} \sum_{j,k} \rho_{j\bar k} v_{jK} \overline{v_{kK}} \frac{\mathrm{d}S}{|\nabla\rho|}.$$

Equivalently one may normalize the defining function along the boundary and absorb this positive density into surface measure. Pseudoconvexity makes the term nonnegative. This is the boundary counterpart of curvature positivity in the Bochner–Kodaira–Nakano identity.

Strict pseudoconvexity gives subelliptic estimates: roughly, one controls a positive fraction of a Sobolev derivative of v by $\|\bar\partial v\| + \|\bar\partial^* v\|$. Compactness of the relevant Sobolev embedding then implies that N_q is compact. Weak pseudoconvexity can lose subellipticity and leads to subtle questions about boundary type.

Remark (Two solution mechanisms)

Hörmander's weighted estimate obtains coercivity from an interior psh weight. The $\bar\partial$-Neumann method can obtain it from the boundary Levi form. Many arguments mix the two sources of positivity.

2.4 Lecture 8: Stein manifolds and Cartan's theorems

Week 4 material • 90 minutes

Notation for this lecture

X is a complex manifold, $\mathcal{O}_X$ its structure sheaf, and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. The stalk at x is $\mathcal{F}_x$ and $\Gamma(X,\mathcal{F}) = \mathcal{F}(X)$ is the space of global sections. A coherent sheaf is locally finitely generated with finitely generated relation sheaves. For a closed complex subspace $Y \subset X$, $\mathcal{I}_Y$ is its ideal sheaf; $\mathfrak{m}_x = \mathcal{I}_{\{x\},x}$ is the maximal ideal of germs vanishing at x. The holomorphic hull of a compact set K is $\widehat{K}_{\mathcal{O}} = \{x : |f(x)| \le \sup_K |f| \text{ for all } f \in \mathcal{O}(X)\}$. $H^q(X,\mathcal{F})$ denotes sheaf cohomology. *Cartan A* means generation by global sections and *Cartan B* means $H^q(X,\mathcal{F}) = 0$ for $q \ge 1$ on a Stein manifold.

Prerequisite results used below

Long exact cohomology sequence. Every short exact sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ induces functorially

$$0 \to H^0(X,\mathcal{F}') \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}'') \xrightarrow{\partial_{\mathrm{conn}}} H^1(X,\mathcal{F}') \to \cdots .$$

Exactness says, in particular, that a section of $\mathcal{F}''$ lifts globally precisely when its connecting class vanishes.
Identity principle. If two holomorphic functions on a connected complex manifold agree on a nonempty open set, then they agree everywhere. Equivalently, a holomorphic function vanishing on a nonempty open set is identically zero.

Plan for the lecture. 0–18 min: definition of Stein; 18–38: equivalent characterizations; 38–63: Cartan A and B; 63–78: embedding and interpolation; 78–90: audit of the week's proof chain.

Compact complex manifolds have too few holomorphic functions. Stein manifolds lie at the other extreme: their holomorphic functions are rich enough to make them resemble affine algebraic varieties.

Definition 2.29: Stein manifold

A complex manifold X is *Stein* if:
(1) for every compact K, the holomorphic hull $\widehat{K}_{\mathcal{O}(X)} = \{x : |f(x)| \le \sup_K |f| \text{ for all } f \in \mathcal{O}(X)\}$ is compact;
(2) global holomorphic functions separate points;
(3) at each x, the differentials of global holomorphic functions span $T_x^{*1,0}X$.

Theorem 2.30: Analytic and geometric characterizations of Stein manifolds

For a connected complex manifold X, the following are equivalent:
(1) X is Stein;
(2) X has a smooth strictly psh exhaustion;
(3) X is biholomorphic to a closed complex submanifold of some $\mathbb{C}^N$;
(4) every coherent analytic sheaf satisfies Cartan's Theorems A and B.

Proof (proof architecture; the proper embedding theorem is the quoted input). The implication from a smooth strictly psh exhaustion to the defining Stein properties was proved in the Levi problem: weighted $\bar\partial$ solutions separate points and tangent vectors, and the boundary-growing construction gives holomorphic convexity.

Assume next that X is Stein. The Bishop–Narasimhan–Remmert proper embedding theorem constructs finitely many global functions whose joint map $X \to \mathbb{C}^N$ is a proper injective immersion. Its proof repeats the separation argument on an exhaustion, adds boundary-growing functions for properness, and then uses a finite-dimensional generic-projection theorem with dimension estimates for the double-point and tangent loci. Properness makes the image closed, and a proper injective immersion is a biholomorphism onto that closed complex submanifold.

A closed complex submanifold $Y \subset \mathbb{C}^N$ has the smooth strictly psh exhaustion $|z|^2|_Y$: strict positivity restricts to $T^{1,0}Y$, and closedness makes the restriction proper. This proves the equivalence of the first three conditions.

Cartan A and B for a Stein manifold are proved immediately below. Conversely, assume they hold for every coherent sheaf. Theorem A applied to the ideal sheaves of two points and to $\mathfrak{m}_x/\mathfrak{m}_x^2$ gives separation of points and spanning of cotangent spaces. Theorem B solves the additive Cousin problems used in the exhaustion construction above, hence gives holomorphic convexity. Thus the defining Stein conditions hold. All four conditions are equivalent. □

Theorem 2.31: Cartan's Theorems A and B

Let X be Stein and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.

Theorem A. For every $x \in X$, the stalk $\mathcal{F}_x$ is generated by germs of global sections in $H^0(X, \mathcal{F})$.

Theorem B. $H^q(X, \mathcal{F}) = 0$ for every $q \geq 1$.

Proof (proof architecture via the coherent-sheaf Runge lemma). Choose a strongly pseudoconvex exhaustion $K_1 \Subset K_2 \Subset \cdots$ of X. We first record the analytic lemma used at every stage: if K is holomorphically convex, $U \supset K$, and $\mathcal{F}$ is coherent, then sections of $\mathcal{F}$ on U can be approximated uniformly on K by global sections. Indeed, coherence gives a finite presentation by free sheaves near K. Lift the section to a vector of holomorphic functions on finitely many coordinate balls, patch with a cutoff, and solve its $\bar\partial$ error with a weight that is very large outside U. The weighted estimate makes the correction arbitrarily small on K. Relations between two lifts lie in the coherent kernel of the presentation, so the same argument there makes the construction independent of the lift. This is the coherent-sheaf Runge lemma.

To prove Theorem B, take a locally finite Stein-coordinate cover and a Čech q-cocycle c, $q \geq 1$. On a neighborhood of K_1, finite polydisc intersections are acyclic by the Dolbeault lemma, so solve $c = \delta b_1$ there. Suppose b_j has been constructed near K_j. Solve once near K_{j+1} and compare the two solutions; their difference is a cocycle of one lower degree. The Runge lemma changes the new solution by a coboundary so that it differs from b_j by less than 2^{-j} on K_{j-1}. Descending induction on the Čech degree performs this correction at every stage. The corrected cochains form a normally convergent sequence on compact sets; its limit b satisfies $\delta b = c$. Hence all positive Čech cohomology vanishes. Using a Leray refinement gives $H^q(X, \mathcal{F}) = 0$.

For Theorem A fix $x \in X$. Apply Theorem B to

$$0 \longrightarrow \mathfrak{m}_x\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \longrightarrow 0.$$

The right-hand term is a finite-dimensional skyscraper sheaf and the kernel is coherent. Since $H^1(X, \mathfrak{m}_x\mathcal{F}) = 0$, global sections of $\mathcal{F}$ surject onto the fiber $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$. Lift a basis; Nakayama's lemma says that their germs generate $\mathcal{F}_x$. This proves Theorem A. □

Theorem A globalizes local equations; Theorem B removes gluing obstructions. Together they express the analytic analogue of affineness.

Example 2.32: Mittag–Leffler and Cousin problems

If meromorphic functions f_i on a Stein cover satisfy $f_i - f_j \in \mathcal{O}(U_i \cap U_j)$, their differences form a Čech one-cocycle in $\mathcal{O}_X$. Theorem B gives $f_i - f_j = g_i - g_j$, so $f_i - g_i$ glue to a global meromorphic function. The multiplicative Cousin problem has an additional topological obstruction in $H^2(X, \mathbb{Z})$.

Proposition 2.33: Interpolation on a discrete set

Let $A = \{a_j\}$ be a discrete subset of a Stein manifold. Any prescribed finite-order jets at the points of A are realized by one $f \in \mathcal{O}(X)$.

Proof. Let $\mathcal{I}$ be the ideal sheaf of the prescribed vanishing orders. In the long exact sequence of $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$, Theorem B gives $H^1(X, \mathcal{I}) = 0$. Hence $H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X/\mathcal{I})$ is surjective. □

Remark (Geometric interpretation)

A Stein manifold is a noncompact complex manifold on which holomorphic functions can serve as coordinates. A strictly psh exhaustion provides convexity; L^2 estimates turn convexity into $\bar\partial$-solvability; Cartan B turns solvability into global gluing.

2.4.1 How Cartan B removes a gluing obstruction

Let $Y \subset X$ be a closed complex submanifold of a Stein manifold and let $\mathcal{I}_Y$ be its ideal sheaf. The exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

gives

$$\mathcal{O}(X) \longrightarrow \mathcal{O}(Y) \longrightarrow H^1(X, \mathcal{I}_Y).$$

Theorem B makes the last group zero, so every holomorphic function on Y extends to X. The statement is qualitative: no norm estimate is supplied. Lecture 29 adds an L^2 estimate and thereby obtains information invisible to Theorem B.

The same pattern works for interpolation. If Z is a finite nonreduced zero-dimensional subspace, the quotient $\mathcal{O}_X/\mathcal{I}_Z$ stores prescribed finite jets. Surjectivity of

$$\mathcal{O}(X) \longrightarrow H^0(X, \mathcal{O}_X/\mathcal{I}_Z)$$

produces a global holomorphic function with those jets. Choosing Z to be two reduced points separates points; choosing a double point produces prescribed differentials.

2.4.2 From global generators to an embedding

Cartan A says that a coherent sheaf is generated by global sections at every point. Apply it first to the ideal of a point and then to its square. The quotients

$$\mathcal{I}_x/\mathcal{I}_x^2 \simeq T_x^{*1,0}X, \qquad \mathcal{O}_X/\mathcal{I}_x \simeq \mathbb{C}_x$$

show that global functions can separate tangent directions and values. On a compact piece K, finitely many such functions work simultaneously. A suitable finite map

$$F = (f_1, \ldots, f_N) : X \longrightarrow \mathbb{C}^N$$

is then immersive and injective on K.

Holomorphic convexity supplies additional global functions that become large outside successive compact sets. Adjoining them makes F proper. A proper injective holomorphic immersion is a closed holomorphic embedding. This is the mechanism behind the embedding characterization of Stein manifolds; both local generation and global compactness are needed.

The basic examples fit the definition cleanly. Every domain in $\mathbb{C}^n$ that is pseudoconvex is Stein; closed complex submanifolds of $\mathbb{C}^N$ are Stein; products and unbranched coverings of Stein manifolds are Stein. In contrast, a compact connected positive-dimensional complex manifold is not Stein: its holomorphic functions are constant and therefore cannot separate points.

Remark (A practical coherent-sheaf workflow)

To globalize local data on a Stein manifold: encode the data as a quotient of a coherent sheaf, identify the kernel, write the short exact sequence, and use Theorem B to kill the connecting obstruction. Theorem A is used when generators are needed; Theorem B is used when compatibility or extension is needed.

2.4.3 Checking the Stein axioms in model cases

The space $\mathbb{C}^n$ is holomorphically separable because a coordinate z_j separates any two distinct points. It is holomorphically convex: if $K \Subset \mathbb{C}^n$ and $x \in \widehat{K}_{\mathcal{O}}$, then applying the coordinate functions gives $|x_j| \le \sup_K |z_j|$ for every j, so the hull is closed and bounded, hence compact.

For $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the two global holomorphic functions z and z^{-1} give the bounds

$$\inf_K |z| \le |x| \le \sup_K |z| \qquad (x \in \widehat{K}_{\mathcal{O}}).$$

Thus the hull stays in a compact annulus. The function $|z|^2 + |z|^{-2}$ is a proper strictly subharmonic exhaustion, giving the analytic criterion as well.

If $Y \subset X$ is a closed complex submanifold of a Stein manifold, Cartan A applied to its ideal sheaf provides local defining functions from global ones. Holomorphic separability restricts from X to Y, and holomorphic convexity follows because $\widehat{K}_{\mathcal{O}(Y)} \subset \widehat{K}_{\mathcal{O}(X)} \cap Y$ after extending the finitely many functions needed near the hull. This proves that Y is Stein; the extension step is a use of Cartan B, not a purely topological fact.

The first Cousin problem as a cocycle calculation

Let $\{U_i\}$ be a Stein cover and suppose meromorphic functions m_i have holomorphic differences $g_{ij} = m_i - m_j$ on overlaps. Then

$$g_{ij} + g_{jk} + g_{ki} = 0$$

on triple overlaps, so (g_{ij}) is a Čech 1-cocycle with values in $\mathcal{O}$. Cartan B gives $H^1(X, \mathcal{O}) = 0$, hence there are $h_i \in \mathcal{O}(U_i)$ with

$$g_{ij} = h_i - h_j.$$

The functions $m_i - h_i$ agree on overlaps and glue to a global meromorphic function m. Its local principal parts are the prescribed ones. Every sign in this argument can be checked directly:

$$(m_i - h_i) - (m_j - h_j) = g_{ij} - (h_i - h_j) = 0.$$

For multiplicative data $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$, solving $g_{ij} = h_i/h_j$ is obstructed by $H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$ rather than $H^1(X, \mathcal{O})$. Taking logarithms may introduce integral multiples of $2\pi\mathrm{i}$; the exponential sequence records exactly this additional obstruction.

Interpolation on a discrete set

Let $D = \{a_1, a_2, \ldots\} \subset X$ be closed and discrete, and let $\mathcal{I}_D$ be its ideal sheaf. The sequence

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

gives

$$H^0(X, \mathcal{O}_X) \longrightarrow H^0(D, \mathcal{O}_D) \longrightarrow H^1(X, \mathcal{I}_D).$$

The ideal sheaf is coherent, so Cartan B kills the last group. Since a holomorphic function on a discrete reduced space is just an arbitrary collection of values, every sequence (c_j) is the restriction of some $f \in \mathcal{O}(X)$. The same argument with $\mathcal{I}_D^{m+1}$ prescribes finite jets of order m.

How Theorem A begins an embedding proof

Fix $x \in X^n$ and let $\mathfrak{m}_x$ be the maximal ideal sheaf. The quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is the holomorphic cotangent space at x. Theorem A gives global functions $f_1, \ldots, f_N$ whose germs generate $\mathfrak{m}_x$. Their differentials span $T_x^{*1,0}X$, so some n of them have linearly independent differentials. The map

$$F = (f_{i_1}, \ldots, f_{i_n}) : X \to \mathbb{C}^n$$

is therefore a local embedding at x by the holomorphic inverse-function theorem. Holomorphic separability supplies global functions distinguishing pairs of points. On a relatively compact set, compactness reduces these local and pairwise choices to finitely many functions, yielding a proper holomorphic embedding after an exhaustion argument. The algebraic input is generation of $\mathfrak{m}_x$ and $\mathfrak{m}_x^2$; the analytic input is the rank calculation.

2.4.4 Self-study workshop: using Cartan's theorems as problem-solving tools

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall holomorphic convexity, coherent ideal sheaves, short exact sequences, and the meaning of a stalk. It is not enough to remember the slogans for Theorems A and B; one must identify the coherent sheaf to which each theorem is applied.

Three standard translations. To separate a point x from a closed analytic set A, use the coherent ideal sheaf $\mathcal{I}_A$. Theorem A says that germs of global sections generate $(\mathcal{I}_A)_x$. If $x \notin A$, this stalk is $\mathcal{O}_{X,x}$, so a suitable global $f \in \mathcal{I}_A(X)$ satisfies $f(x) \neq 0$ while $f|_A = 0$.

To prescribe a finite jet at x, use

$$0 \longrightarrow \mathfrak{m}_x^{k+1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathfrak{m}_x^{k+1} \longrightarrow 0.$$

The quotient is a coherent skyscraper sheaf. Theorem B kills $H^1(X, \mathfrak{m}_x^{k+1})$, so the restriction map on global sections is surjective. Thus an abstract vanishing theorem becomes a concrete interpolation statement.

For a first Cousin problem, local meromorphic functions f_i with holomorphic differences $f_i - f_j$ define a Čech 1-cocycle in $\mathcal{O}_X$. If X is Stein, $H^1(X, \mathcal{O}_X) = 0$, hence $f_i - f_j = g_i - g_j$ for holomorphic g_i. The functions $f_i - g_i$ glue to one global meromorphic function. Every sign in the correction can be checked on an overlap.

Checkpoint (choose the sheaf)

(1) Which sheaf controls a holomorphic function with a prescribed value and first derivative at x?
(2) Which cohomology group obstructs gluing additive Cousin data?
(3) Why is a compact connected positive-dimensional complex manifold not Stein?

Solution (checkpoint). Use $\mathcal{O}_X/\mathfrak{m}_x^2$ and the exact sequence with kernel $\mathfrak{m}_x^2$. Additive Cousin data define a class in $H^1(X, \mathcal{O}_X)$. A compact connected complex manifold has only constant holomorphic functions, so it cannot separate two points and therefore fails a Stein axiom. ◇

Exit criterion

For a separation, interpolation, or gluing problem, you can write the relevant short exact sequence, name the coherent kernel, identify the obstruction group, and state exactly where Theorem A or B removes the obstruction.

Summary. Stein geometry has four faces: holomorphic convexity, strict psh exhaustion, closed embedding into $\mathbb{C}^N$, and vanishing of higher coherent cohomology. We use these interfaces now and prove the analytic engine in Lectures 27–28.

Exercise 2.34: Open Riemann surfaces

Consult the Behnke–Stein theorem and explain why every open Riemann surface is Stein. Why is the Levi condition automatic in complex dimension one?

Exercise 2.35: Closed complex submanifolds

Assume X is Stein and $Y \subset X$ is a closed complex submanifold. Restrict a strictly psh exhaustion to prove that Y is Stein.

Optional handout: Runge approximation and the Oka viewpoint

This handout is optional. No later argument in the core lectures depends on it.

Let X be Stein. A compact set $K \subset X$ is $\mathcal{O}(X)$-convex if $K = \widehat{K}_{\mathcal{O}(X)}$. The Oka–Weil theorem says that every function holomorphic on a neighborhood of such a K can be approximated uniformly on K by functions in $\mathcal{O}(X)$. For $X = \mathbb{C}$, this is Runge's theorem without prescribed poles.

The proof has the same local-to-global shape as the rest of the course. Cover K by small coordinate sets, approximate locally, and represent the mismatch by a $\bar\partial$-closed form supported near the boundary of the cover. Solve $\bar\partial$ with a weight that is small on K and large away from it. The correction is then globally holomorphic and uniformly small on K.

Oka theory reverses the emphasis. Stein manifolds are flexible as *sources*; Oka manifolds are flexible as *targets*. A complex manifold Y is Oka if continuous maps from Stein spaces to Y can, under suitable hypotheses, be deformed to holomorphic maps with approximation and interpolation. Complex Lie groups and their homogeneous spaces are basic examples.

Example 2.36: The exponential map

A holomorphic map from a Stein manifold to $\mathbb{C}^*$ is a nowhere-zero holomorphic function. The exponential sequence separates the analytic problem of finding a logarithm from its integral topological obstruction. This is a simple model for the Oka principle: once the topological obstruction vanishes, analytic flexibility solves the holomorphic problem.

Exercise 2.37: Polynomial convexity of the ball

Show directly that the closed unit ball in $\mathbb{C}^n$ is polynomially convex by separating an exterior point with a complex linear polynomial.

Chapter 3

Sheaves, Coherence, Analytic Spaces, and Blow-ups

3.1 Lecture 9: Sheaves, exact sequences, and a first look at cohomology

Week 5 material • 90 minutes

Notation for this lecture

$\mathcal{F}, \mathcal{G}, \mathcal{H}$ denote sheaves on a space X, $\mathcal{F}(U)$ their sections on an open set U, and $\mathcal{F}_x$ the stalk at x. For an open cover $\mathfrak{U} = \{U_i\}$, a Čech q-cochain is a family $c = (c_{i_0 \dots i_q})$ of sections on $U_{i_0} \cap \cdots \cap U_{i_q}$; δ is the alternating Čech coboundary. The resulting cohomology is $\check{H}^q(\mathfrak{U}, \mathcal{F})$, while $H^q(X, \mathcal{F})$ denotes derived-functor sheaf cohomology. The functor of global sections is $\Gamma(X, -)$, and the connecting homomorphism associated with a short exact sequence is denoted ∂_{conn} so that it is not confused with the Dolbeault operator ∂.

Prerequisite results used below

Stalkwise exactness criterion. A sequence of sheaves $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact if and only if the induced sequence of stalks $\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$ is exact for every $x \in X$. Surjectivity therefore means local liftability, not necessarily the existence of one global lift.

Long exact sequence of derived functors. For a left-exact functor F with right derived functors $R^q F$, every short exact sequence $0 \to A' \to A \to A'' \to 0$ gives natural connecting maps and an exact sequence $0 \to F(A') \to F(A) \to F(A'') \to R^1 F(A') \to \cdots$. Taking $F = \Gamma(X, -)$ defines sheaf cohomology.

Plan for the lecture. 0–18 min: why sheaves; 18–40: stalks and sheafification; 40–62: exact sequences; 62–78: locally free and coherent sheaves; 78–90: the first Čech obstruction.

"Locally available but globally impossible to glue" is a recurring situation in complex geometry. A sheaf packages the local objects on every open set, their restriction maps, and the rules for unique gluing.

Definition 3.1: Sheaf

A sheaf $\mathcal{F}$ on X assigns a set, group, ring, or module $\mathcal{F}(U)$ to every open set and a restriction map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ whenever $V \subset U$. Restrictions are functorial and satisfy:

(1) sections agreeing on every member of an open cover are equal;

(2) compatible sections $s_i \in \mathcal{F}(U_i)$ glue uniquely to $s \in \mathcal{F}(\bigcup U_i)$.

Examples include the sheaves C_X of continuous functions, $\mathcal{A}_X$ of smooth functions, $\mathcal{O}_X$ of holomorphic functions, Ω_X^p of holomorphic p-forms, an ideal sheaf $\mathcal{I}_Z$, and a skyscraper sheaf supported at a point.

Definition 3.2: Stalk and germ

The stalk at x is

$$\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}(U).$$

It consists of sections on neighborhoods of x, identified when they agree after shrinking. The germ of s at x is written s_x.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ induces $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$. A sequence of sheaves is exact if and only if every stalk sequence is exact. This fact turns local geometry into algebra.

Warning

The pointwise image presheaf of a sheaf morphism need not satisfy the gluing axiom; $\operatorname{im}\varphi$ means its sheafification. Having local preimages on small pieces does not imply one preimage on the whole open set. Cohomology measures precisely this failure.

Definition 3.3: Locally free and coherent sheaves

An $\mathcal{O}_X$-module $\mathcal{E}$ is locally free of rank r if it is locally isomorphic to $\mathcal{O}_X^{\oplus r}$; these are the sheaves of holomorphic sections of vector bundles. A sheaf $\mathcal{F}$ is *coherent* if it is locally finitely generated and the relations among every finite set of local sections are locally finitely generated. Equivalently, on a complex manifold it has local finite presentations

$$\mathcal{O}_U^{\oplus p} \xrightarrow{A} \mathcal{O}_U^{\oplus q} \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

Coherence is stronger than finite generation: it controls relations among equations by finitely many holomorphic relations.

Proposition 3.4: The ideal sequence

For a closed analytic subspace $i : Z \hookrightarrow X$ with ideal sheaf $\mathcal{I}_Z$,

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

is exact. If X is Stein, then $\mathcal{O}(X) \to \mathcal{O}(Z)$ is surjective.

Proof. At a stalk, $\mathcal{O}_{X,x}/\mathcal{I}_{Z,x} = \mathcal{O}_{Z,x}$, proving exactness. On a Stein space the obstruction group $H^1(X, \mathcal{I}_Z)$ in the associated long exact sequence vanishes by Cartan B. □

3.1.1 Čech cocycles as gluing obstructions

For a cover $\mathfrak{U} = \{U_i\}$ and an abelian sheaf $\mathcal{F}$, set

$$C^q(\mathfrak{U}, \mathcal{F}) = \prod_{i_0<\cdots<i_q} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q}),$$

with coboundary

$$(\delta c)_{i_0\cdots i_{q+1}} = \sum_{k=0}^{q+1} (-1)^k c_{i_0\cdots\widehat{i_k}\cdots i_{q+1}}.$$

Then $\delta^2 = 0$. Differences $s_i - s_j$ of local sections are one-coboundaries; whether prescribed difference data have this form is measured by $\check{H}^1$.

Indeed, in $(\delta^2 c)_{i_0 \ldots i_{q+2}}$ every restriction obtained by omitting i_a and i_b occurs twice. The two orders of omission have opposite signs, so the terms cancel in pairs. This elementary cancellation is the source of the sign convention in every later double complex.

3.1.2 Why stalks are the correct test for exactness

A presheaf remembers restriction maps; a sheaf adds the assertion that compatible local sections glue uniquely. Passing to germs removes choices of neighborhood. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf morphism, its kernel is already a sheaf, but the pointwise presheaf image need not be: a section may be locally in the image without having one global preimage. The sheaf image is obtained by sheafifying this presheaf.

A sequence

$$\mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}''$$

is exact if and only if

$$\mathcal{F}'_x \xrightarrow{\alpha_x} \mathcal{F}_x \xrightarrow{\beta_x} \mathcal{F}''_x$$

is exact for every x. One direction is immediate. Conversely, if $s \in \mathcal{F}(U)$ has $\beta(s) = 0$, stalkwise exactness supplies near each $x \in U$ a section t_x with $\alpha(t_x) = s$. The t_x need not glue, but the conclusion needed for exactness of sheaves is only that s belongs *locally* to the image. This distinction is precisely why surjectivity of sheaves does not imply surjectivity on global sections.

3.1.3 The connecting class written on overlaps

Consider a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \longrightarrow 0$$

and a global section $s \in H^0(X, \mathcal{F}'')$. Choose an open cover $\{U_i\}$ and local lifts $t_i \in \mathcal{F}(U_i)$. On overlaps,

$$t_i - t_j \in \mathcal{F}'(U_i \cap U_j), \qquad (t_i - t_j) + (t_j - t_k) + (t_k - t_i) = 0.$$

Thus $(t_i - t_j)$ is a Čech one-cocycle. Changing the local lifts changes it by a coboundary, so its class

$$\delta(s) \in H^1(X, \mathcal{F}')$$

is intrinsic. It vanishes exactly when corrections from $\mathcal{F}'(U_i)$ make the t_i agree, equivalently when s has a global lift. This is the concrete meaning of the connecting morphism in the long exact cohomology sequence.

For the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp(2\pi i\,\cdot)} \mathcal{O}_X^* \longrightarrow 1,$$

the same construction begins with logarithms of transition functions. Their integral differences on triple overlaps produce the first Chern class. Thus the line-bundle cocycle, its logarithms, and an integral two-class are three levels of the same gluing problem.

Remark (Coherence is uniform finite generation)

Finite generation of every stalk separately is weaker than coherence. Coherence says that generators and all relations among them can be chosen on one neighborhood. This uniformity is what permits kernels, quotients, and cohomology of analytic data to remain finite-dimensional or finitely presented.

3.1.4 Sheaf axioms tested on concrete data

For an open set $U \subset X$, let $C^0(U)$ be the continuous functions on U. If functions $f_i \in C^0(U_i)$ agree on all overlaps, define $f(x) = f_i(x)$ for $x \in U_i$. Agreement makes this independent of i, continuity is local, and uniqueness is pointwise. Thus C^0 is a sheaf.

By contrast, the assignment

$$U \longmapsto \{\text{constant functions on } U\}$$

is only a presheaf if "constant" means one constant on all of U. If $U = U_1 \sqcup U_2$ and one chooses different constants on the two components, the local sections agree on every overlap but do not glue to a globally constant function. Its sheafification is the sheaf of locally constant functions.

The stalk of $\mathcal{O}_{\mathbb{C}}$ at 0 is $\mathbb{C}\{z\}$. Every nonzero germ has a unique form

$$f(z) = z^m u(z), \qquad u(0) \neq 0.$$

The units are exactly the germs with nonzero value at 0, and the unique maximal ideal is $\mathfrak{m}_0 = (z)$. In $\mathbb{C}^n$ the same evaluation map gives

$$0 \longrightarrow \mathfrak{m}_0 \longrightarrow \mathcal{O}_{\mathbb{C}^n,0} \xrightarrow{f \mapsto f(0)} \mathbb{C} \longrightarrow 0, \qquad \mathfrak{m}_0 = (z_1, \ldots, z_n).$$

Taylor expansion identifies $\mathfrak{m}_0/\mathfrak{m}_0^2$ with the cotangent space by

$$[f] \longmapsto \sum_j \frac{\partial f}{\partial z_j}(0)\, \mathrm{d}z_j.$$

Why exactness is checked on stalks

Consider sheaf morphisms $\mathcal{F} \xrightarrow{a} \mathcal{G} \xrightarrow{b} \mathcal{H}$. If the stalk sequence is exact and $g \in \mathcal{G}(U)$ satisfies $b(g) = 0$, then for each $x \in U$ the germ g_x equals $a_x(f_x)$ for some germ f_x. Choose a representative f_x on a neighborhood U_x. The sections $g|_{U_x}$ and $a(f_x)$ have the same germ at x, so after shrinking U_x they agree. This proves local exactness of sheaves. It does not usually produce one global f: the differences $f_x - f_y$ on overlaps form a gluing obstruction.

For a closed complex subspace $Y \subset X$ with ideal sheaf $\mathcal{I}_Y$, the sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0$$

is exact on stalks. In the model $Y = \{z_1 = \cdots = z_r = 0\} \subset \mathbb{C}^n$, restriction is the substitution $z_1 = \cdots = z_r = 0$, and its kernel is precisely $(z_1, \ldots, z_r)$. Surjectivity follows because a germ in the remaining variables is already a holomorphic germ in all variables, independent of the first r.

The connecting class with all choices visible

Given $0 \to \mathcal{F} \to \mathcal{G} \xrightarrow{q} \mathcal{H} \to 0$ and $s \in H^0(X, \mathcal{H})$, choose a cover $\{U_i\}$ and lifts $t_i \in \mathcal{G}(U_i)$ with $q(t_i) = s|_{U_i}$. On overlaps,

$$q(t_i - t_j) = 0,$$

so exactness gives $f_{ij} \in \mathcal{F}(U_i \cap U_j)$ with image $t_i - t_j$. On triple overlaps,

$$f_{ij} + f_{jk} + f_{ki} = 0,$$

because the corresponding sum of the t's telescopes. Hence (f_{ij}) is a Čech 1-cocycle.

Changing the lift to $t_i + a_i$, $a_i \in \mathcal{F}(U_i)$, changes f_{ij} to $f_{ij} + a_i - a_j$, a coboundary. Thus the class $\delta(s) = [(f_{ij})] \in H^1(X, \mathcal{F})$ is well defined. It vanishes exactly when $f_{ij} = a_j - a_i$; then the corrected lifts $t_i + a_i$ agree and glue to a global lift. This proves both directions of the obstruction statement.

The exponential sequence locally

The map $\exp(2\pi \mathrm{i}\,\cdot) : \mathcal{O}_X \to \mathcal{O}_X^*$ is locally surjective: a nonvanishing holomorphic function on a small ball has a holomorphic logarithm because $g^{-1}\mathrm{d}g$ is closed and the ball is simply connected. Its kernel consists of locally constant integer-valued functions. Therefore

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp(2\pi \mathrm{i}\,\cdot)} \mathcal{O}_X^* \longrightarrow 1$$

is exact as a sequence of sheaves. The associated connecting map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ sends a line bundle to its first Chern class. The local logarithms explain concretely why an integral cocycle appears.

3.1.5 Self-study workshop: turning local gluing failures into cohomology classes

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall the sheaf axioms, germs and stalks, kernels and cokernels, and an open cover with ordered indices. Keep separate three levels: sections on an open set, germs at a point, and Čech cochains on intersections.

Why exactness is local but splitting is not. A sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \longrightarrow 0$$

is exact precisely when the sequence of stalks is exact at every point. Surjectivity of β therefore means that a germ of a section of $\mathcal{F}''$ has a lift near each point; it does not promise one lift over the whole open set.

Let $s \in H^0(X, \mathcal{F}'')$. Choose an open cover and local lifts $t_i \in \mathcal{F}(U_i)$. On $U_i \cap U_j$,

$$\beta(t_i - t_j) = s - s = 0,$$

so exactness gives $t_i - t_j = \alpha(a_{ij})$ for a unique $a_{ij} \in \mathcal{F}'(U_i \cap U_j)$. On triple overlaps,

$$a_{ij} + a_{jk} + a_{ki} = 0.$$

Thus (a_{ij}) is a Čech 1-cocycle. Replacing t_i by $t_i + \alpha(b_i)$ changes a_{ij} by the coboundary $b_i - b_j$, so the cohomology class is independent of all choices. It vanishes exactly when the local lifts can be corrected to glue. This is the connecting map $\delta : H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}')$.

Checkpoint (track the choices)

(1) Where is exactness at $\mathcal{F}$ used in the construction?
(2) If $a_{ij} = b_i - b_j$, which corrected lifts glue?
(3) Why does stalkwise surjectivity not imply surjectivity on global sections?

Solution (checkpoint). Exactness identifies $t_i - t_j \in \ker\beta$ with a section from $\operatorname{im}\alpha$. If $a_{ij} = b_i - b_j$, then $t_i - \alpha(b_i)$ have zero differences and glue. Global surjectivity can fail because the local lifts may have a nonzero connecting class in $H^1(X, \mathcal{F}')$. ◇

Exit criterion

Starting from a short exact sequence and a global quotient section, you can construct the connecting cocycle, prove independence of choices, and interpret its vanishing as a gluing statement.

Summary. Sheaves encode local objects and gluing, stalks turn exactness into pointwise algebra, and coherence supplies finite presentations. Cohomology is not decoration: it is the obstruction space for turning local solutions into global ones.

Exercise 3.5: The exponential sequence

Prove exactness of $0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{f\mapsto\exp(2\pi i f)} \mathcal{O}_X^* \to 1$. The key point is the local existence of a logarithm.

Exercise 3.6: Nakayama at a stalk

If $\mathcal{F}$ is coherent, explain why a basis of $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ lifts to finitely many generators near x.

Optional handout: Line bundles, divisors, and the exponential sequence

This handout is optional. No later argument in the core lectures depends on it.

The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{f\mapsto\exp(2\pi i f)} \mathcal{O}_X^* \longrightarrow 1$$

gives the exact segment

$$H^1(X, \mathcal{O}_X) \longrightarrow \operatorname{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

Here $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ classifies holomorphic line bundles. The map c_1 takes transition functions g_{ij}, chooses local logarithms $g_{ij} = \exp(2\pi i f_{ij})$, and records the integral two-cocycle $f_{ij} + f_{jk} + f_{ki}$.

The connected component

$$\operatorname{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$$

parametrizes topologically trivial holomorphic line bundles, with the usual image interpretation if the integral map has a kernel. On a compact Kähler manifold it is a complex torus. The quotient $\operatorname{NS}(X) = \operatorname{Pic}(X)/\operatorname{Pic}^0(X)$ is the Néron–Severi group and embeds into $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$.

Example 3.7: The projective line

Since $H^1(\mathbb{P}^1, \mathcal{O}) = H^2(\mathbb{P}^1, \mathcal{O}) = 0$ and $H^2(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}$, the first Chern class gives $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$. The generator is $\mathcal{O}(1)$.

Exercise 3.8: Divisor line bundle

Given a Cartier divisor represented by meromorphic functions f_i, construct the line bundle with transition functions f_i/f_j and its canonical meromorphic section. Check that principal divisors give the trivial bundle.

3.2 Lecture 10: Weierstrass theory, coherence, and analytic sets

Week 5 material • 90 minutes

Notation for this lecture

$\mathcal{O}_{\mathbb{C}^n,0}$ is the local ring of holomorphic germs at the origin and $\mathbb{C}\{z_1, \dots, z_n\}$ is the ring of convergent power series there. A unit is a germ nonzero at the base point. We distinguish $z' = (z_1, \dots, z_{n-1})$ from the chosen Weierstrass variable z_n. A monic polynomial $P(z', z_n) = z_n^d + a_1(z')z_n^{d-1} + \cdots + a_d(z')$ with $a_j(0) = 0$ is a Weierstrass polynomial of degree d. For an ideal I, $(f_1, \dots, f_r)$ means the ideal it generates. If I is an ideal in a ring A, then A/I is the quotient ring or module. $\operatorname{ord}_0 f$ denotes the vanishing order of a nonzero germ f at the origin.

Prerequisite results used below

Holomorphic implicit function theorem. Let $F : U \subset \mathbb{C}^{n+r} \to \mathbb{C}^r$ be holomorphic, $F(a) = 0$, and suppose some $r \times r$ minor of its complex Jacobian is invertible at a. Then $F^{-1}(0)$ is, near a, the graph of a holomorphic map in the remaining n variables.
Identity theorem for germs. If a holomorphic function on a connected domain vanishes on a nonempty open subset, it vanishes identically. Hence the local ring of germs on an irreducible smooth germ has no zero divisors, and equality of convergent power series can be checked on any sufficiently small representative.

Plan for the lecture. 0–22 min: local rings of convergent power series; 22–48: Weierstrass preparation and division; 48–68: Oka coherence; 68–82: dimension and regular points of analytic sets; 82–90: computations.

The ring of germs at $0 \in \mathbb{C}^n$ is $\mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\}$, with maximal ideal $\mathfrak{m} = (z_1, \dots, z_n)$. It is a Noetherian local domain, but this is not a formal consequence of the polynomial case; convergence is controlled by Weierstrass theory.

Definition 3.9: Weierstrass polynomial

A Weierstrass polynomial of degree d in z_n is

$$P(z', z_n) = z_n^d + a_1(z')z_n^{d-1} + \cdots + a_d(z'), \qquad a_j(0) = 0.$$

Theorem 3.10: Weierstrass preparation and division

Suppose $f \in \mathbb{C}\{z', z_n\}$ is regular of order d in z_n, meaning $f(0, z_n) = z_n^d g(z_n)$ with $g(0) \neq 0$. Then uniquely $f = uP$, where u is a unit and P is a degree-d Weierstrass polynomial. For every h, there are unique q and $r \in \mathbb{C}\{z'\}[z_n]$ with $\deg_{z_n} r < d$ such that $h = qf + r$.

Proof. Choose radii so that $f(z', z_n)$ has exactly d zeros in the z_n-disc for every small z', counted with multiplicity. The argument principle expresses the power sums of these zeros as Cauchy integrals depending holomorphically on z'. Newton identities then produce the holomorphic coefficients of the monic polynomial P having those zeros. The quotient $u = f/P$ is holomorphic and nonvanishing by removable singularities.

For division, first divide boundary Laurent series by P and use Cauchy projection to separate nonnegative and negative powers. Uniform estimates on a smaller polydisc show that the resulting q and the d coefficient functions of r are convergent. If two divisions existed, their difference would say that a polynomial of degree $< d$ is divisible by the monic degree-d polynomial P, hence is zero. These estimates, not formal polynomial division alone, are what preserve convergence. More explicitly, on

$|z_n| = R$ write

$$\frac{h(z', z_n)}{P(z', z_n)} = \sum_{k \in \mathbb{Z}} c_k(z') z_n^k.$$

The nonnegative Laurent part is the quotient q. Multiplying the negative part by P leaves only the powers $1, z_n, \ldots, z_n^{d-1}$; their coefficients form r. Cauchy's estimates on a slightly smaller polydisc give normal convergence and holomorphic dependence on z'. Replacing P by $f = uP$ divides the quotient by the unit u. This proves existence and uniqueness of both preparation and division. □

Corollary 3.11: Noetherianity of the local ring

$\mathbb{C}\{z_1, \ldots, z_n\}$ is Noetherian and is a unique factorization domain.

Proof. We argue by induction on n. The ring $\mathbb{C}\{z_1\}$ is a discrete valuation ring: every nonzero germ is a unit times a power of z_1. Assume $R' = \mathbb{C}\{z_1, \ldots, z_{n-1}\}$ is Noetherian and let $I \subset R'\{z_n\}$ be a nonzero ideal. After a generic linear change of variables, choose $f \in I$ regular of order d in z_n and replace it by its Weierstrass polynomial. Divide every $g \in I$ by f:

$$g = qf + r_0 + r_1 z_n + \cdots + r_{d-1} z_n^{d-1}.$$

The coefficient vectors $(r_0, \ldots, r_{d-1})$ form an R'-submodule $M \subset (R')^d$. Since R' is Noetherian, M is generated by finitely many remainder vectors, represented by $g_1, \ldots, g_N \in I$. Subtracting an $R'\{z_n\}$-linear combination of the g_j from any $g \in I$ leaves a multiple of f. Thus $I = (f, g_1, \ldots, g_N)$, proving Noetherianity.

For unique factorization, again use induction. Write a nonzero nonunit as uP by Weierstrass preparation, where u is a unit and $P \in R'[z_n]$ is monic. By the induction hypothesis R' is a UFD, so Gauss's lemma factors P uniquely into monic irreducibles in $R'[z_n]$. Any factorization in $R'\{z_n\}$ can itself be prepared; uniqueness in Weierstrass preparation reduces it to a polynomial factorization of P. Hence the analytic local ring has existence and uniqueness of factorization. □

Theorem 3.12: Oka's coherence theorem

The structure sheaf $\mathcal{O}_X$ of a complex manifold is coherent. Consequently, kernels, images, and cokernels of finite holomorphic matrices, as well as ideal sheaves of analytic sets, are coherent.

Proof. Coherence is local, so work on a polydisc and induct on the number n of variables. For $n = 0$ the assertion is linear algebra. Suppose a finite tuple $f = (f_1, \ldots, f_m)$ is given and consider the relation sheaf

$$\mathcal{R}(f) = \{(a_1, \ldots, a_m) : \textstyle\sum a_j f_j = 0\}.$$

After a linear change of variables, choose one nonzero f_j regular in z_n and replace it by its Weierstrass polynomial. Divide every coefficient a_j by that polynomial. The quotient part is eliminated using the displayed relation; the remainders are polynomials of bounded degree in z_n. Equating their finitely many coefficients gives a finite system of holomorphic equations in z'. By the induction hypothesis, the relations of this lower-dimensional system are generated by finitely many sections. Lifting them and adjoining the elementary division relations generates $\mathcal{R}(f)$ after shrinking the polydisc.

Thus the kernel of every map $\mathcal{O}^m \to \mathcal{O}$ is locally finitely generated. Applying the same argument to the columns of a finite matrix, one column at a time, proves that the kernel of every map $\mathcal{O}^m \to \mathcal{O}^r$ is locally finitely generated. This is precisely coherence of $\mathcal{O}$. Kernels of maps between finite free sheaves are coherent; finite presentations then show that kernels, images, and cokernels of morphisms between coherent sheaves are coherent. Finally an analytic ideal is locally the image of a finite free sheaf and is therefore coherent. The uniform Cauchy estimates in Weierstrass division ensure that the generators work on one neighborhood, rather than only at one stalk. □

Definition 3.13: Analytic set and regular point

A closed set $A \subset X$ is analytic if locally it is the common zero set of finitely many holomorphic functions. A point $x \in A$ is regular if A is a complex submanifold near x; all other points form A_{Sing}.

Full rank of the Jacobian is a sufficient regularity test. If the chosen list of equations is redundant, rank deficiency need not mean that the set is singular: one must inspect the ideal, not merely a presentation.

Theorem 3.14: Local structure of analytic sets

An analytic set has locally finitely many irreducible components. The regular part of each irreducible component is dense and connected; A_{Sing} is a proper lower-dimensional analytic subset. At every point of a pure d-dimensional component, a generic linear projection to $\mathbb{C}^d$ is finite after the source and target are sufficiently shrunk; away from its discriminant it is an unbranched finite covering.

Proof. At a point of an irreducible component, choose a generic linear projection to a d-dimensional coordinate plane. Weierstrass preparation makes the projection finite after shrinking. Its discriminant is analytic; off the discriminant the projection is a finite covering and the source is smooth. Induction on dimension treats the branch and singular loci. Noetherianity gives local finiteness of irreducible components. The connectedness of the regular locus of an irreducible component follows because a separation would extend, by closure and analytic continuation, to a decomposition of the component. Here are the missing details. Noetherianity gives a finite primary decomposition of the local radical ideal; its minimal primes are the finitely many local irreducible components. For one component of dimension d, choose coordinates so that the map to $\mathbb{C}^d$ has discrete fiber over the origin. Weierstrass preparation applied successively to the remaining coordinates makes the local ring finite over $\mathbb{C}\{z_1, \dots, z_d\}$, hence the projection is proper and finite after shrinking. Its discriminant is the zero set of the determinant of the trace pairing. Off this analytic hypersurface the finite algebra is étale, so the source is a disjoint union of graphs of holomorphic maps and is smooth.

The nonsmooth locus is defined intrinsically by the appropriate Fitting ideal of the module of Kähler differentials, so it is analytic. It cannot equal the whole component because the étale locus above is nonempty; its dimension is therefore smaller. Induction applies on that locus. If the regular locus of an irreducible component were disconnected, the Remmert–Stein extension theorem would make the closures of its connected components analytic across the lower-dimensional singular locus. Two nonempty such closures would be proper analytic subsets whose union is the original component, contradicting irreducibility. This proves every assertion. ◻

Example 3.15: The cusp

The curve $A = \{y^2 = x^3\} \subset \mathbb{C}^2$ is singular at the origin, but $t \mapsto (t^2, t^3)$ is its normalization. The ring $\mathbb{C}\{t^2, t^3\} \subset \mathbb{C}\{t\}$ is not integrally closed: t is integral over it.

Warning

Being defined by r equations gives $\operatorname{codim} A \leq r$, not $\operatorname{codim} A \geq r$. Equations may be dependent and some components may have larger dimension. A complete intersection requires equality and, scheme-theoretically, a regular sequence.

3.2.1 A division calculation and its geometric meaning

Let $P(x, y) = y^2 - x$. Weierstrass division in the variable y gives

$$y^4 + x = (y^2 + x)(y^2 - x) + (x^2 + x).$$

The remainder has degree less than two in y, and uniqueness shows that no hidden convergent-series terms can occur. Modulo (P), every germ therefore has a unique representative $a(x) + b(x)y$. Algebraically,

$$\mathbb{C}\{x, y\}/(y^2 - x)$$

is a free module of rank two over $\mathbb{C}\{x\}$. Geometrically, projection to the x-axis is a finite map of degree two near the origin.

For the cusp $y^2 = x^3$, the same projection is finite and the discriminant is supported at $x = 0$. Away from the discriminant the two roots vary holomorphically on simply connected open sets. At the origin the sheets join, and the parametrization $t \mapsto (t^2, t^3)$ separates the hidden parameter. Weierstrass preparation thus converts a local analytic set into a branched finite cover plus lower-dimensional exceptional data.

3.2.2 Tangent space, dimension, and the chosen equations

If the germ $(A, 0) \subset (\mathbb{C}^N, 0)$ has ideal I, its Zariski tangent space is

$$T_0 A = \{v \in \mathbb{C}^N : \mathrm{d}f_0(v) = 0 \text{ for every } f \in I\}.$$

Equivalently,

$$T_0^* A \simeq \mathfrak{m}_A/\mathfrak{m}_A^2.$$

This definition uses the full ideal and is independent of a selected generating list. For $y^2 = x^3$, the defining equation has no linear term, so $T_0 A = \mathbb{C}^2$ although A has dimension one. The tangent-space jump detects singularity.

The local dimension is instead the Krull dimension of $\mathcal{O}_{A,0}$. A finite map to $\mathbb{C}^d$ proves that the local ring is finite over a convergent power-series ring in d variables, making the analytic and algebraic notions of dimension agree. The inequality "r equations give codimension at most r" follows because each new equation can lower dimension by at most one.

Oka coherence upgrades these stalk calculations to neighborhoods. For a finite holomorphic matrix $A : \mathcal{O}^p \to \mathcal{O}^q$, the sheaves $\ker A$, $\operatorname{im} A$, and $\operatorname{coker} A$ are coherent. Rank-drop loci are therefore analytic: they are cut out intrinsically by minors, or more generally by Fitting ideals. This is the basic mechanism for proving that singular loci and degeneracy loci are analytic subsets.

Warning

The dimension of the Zariski tangent space is an upper bound for local dimension, not its definition. It is exact at a smooth point and may be much larger at a singular point. Likewise, one Jacobian matrix built from redundant equations may have smaller rank than the intrinsic conormal space.

3.2.3 Weierstrass preparation and division in examples

Consider

$$f(z, w) = e^w (w^2 - z).$$

Since $f(0,w) = e^w w^2$ has order two, preparation gives a degree-two polynomial. In this case it is visible without a theorem:

$$f = uP, \qquad u = e^w, \qquad P = w^2 - z.$$

To divide $h = w^4 + z$ by P, use

$$w^4 = (w^2 + z)(w^2 - z) + z^2.$$

Hence

$$h = (w^2 + z)P + (z^2 + z),$$

and the remainder has degree zero in w, as required. For division by $f = uP$, the quotient is $(w^2+z)e^{-w}$ and the remainder is unchanged. Uniqueness follows immediately: if $(q - q')P = r' - r$ and the right side has w-degree < 2, it must vanish.

The finite-map content is also explicit. On $X = \{w^2 = z\}$ the projection $(z,w) \mapsto z$ has two points over $z \neq 0$ and one double point over 0. The local algebra

$$\mathcal{O}_X \simeq \mathbb{C}\{z\}[w]/(w^2 - z)$$

is a free $\mathbb{C}\{z\}$-module with basis $1, w$ because Weierstrass division reduces every power series uniquely to $a(z) + b(z)w$.

Regularity, tangent spaces, and tangent cones

For $X = V(f_1, \dots, f_r) \subset \mathbb{C}^n$, differentiating every equation along a holomorphic curve γ in X gives

$$T_x^{1,0}X \subseteq \bigcap_j \ker \mathrm{d}f_j(x).$$

The right-hand side is the Zariski tangent space. At a smooth point, choosing a minimal set of equations and applying the implicit-function theorem gives equality and dimension $\dim_x X$.

For the cusp $C = \{y^2 - x^3 = 0\}$,

$$\mathrm{d}(y^2 - x^3) = 2y\,\mathrm{d}y - 3x^2\,\mathrm{d}x,$$

which vanishes at the origin. Thus the Zariski tangent space is all of $\mathbb{C}^2$ even though the curve has dimension one. The lowest-degree homogeneous term is y^2, so the tangent cone is the double line $y = 0$. The parametrization $t \mapsto (t^2, t^3)$ has derivative zero at 0, another signal that it is not a local embedding there.

For the node $N = \{y^2 - x^2 - x^3 = 0\}$ the differential also vanishes at 0, but the lowest homogeneous part factors:

$$y^2 - x^2 = (y - x)(y + x).$$

The tangent cone is the union of two distinct lines. Locally one can choose a square root of $1 + x$ and factor

$$y^2 - x^2(1 + x) = \bigl(y - x\sqrt{1+x}\bigr)\bigl(y + x\sqrt{1+x}\bigr),$$

so the node has two smooth branches, unlike the irreducible cusp.

Why redundant equations can mislead

The smooth line $L = \{y = 0\} \subset \mathbb{C}^2$ can also be presented as $V(y^2, y^3)$. The Jacobian of this displayed pair vanishes everywhere on L, but L is not singular. The radical ideal is (y), and its one generator has nonzero differential. Therefore the geometric regularity test uses the ideal sheaf (or the module of Kähler differentials), not the rank of an arbitrary redundant list.

For a hypersurface $f = 0$, the singular locus is

$$V\left(f, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right).$$

For the cusp this is only the origin. Away from the origin either x or y is nonzero, one partial derivative is nonzero, and the implicit-function theorem gives a one-dimensional chart. This verifies directly that the regular locus is dense.

3.2.4 Self-study workshop: using Weierstrass division as finite-dimensional algebra

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall convergent power series, units in a local ring, finite modules, and the implicit function theorem. Distinguish a Weierstrass polynomial from an arbitrary polynomial in the distinguished variable.

A model quotient calculated completely. Consider

$$P(z, w) = w^2 - z^3 \in \mathbb{C}\{z, w\}.$$

It is monic of degree two in w. Weierstrass division says that every $f \in \mathbb{C}\{z, w\}$ has a unique expression

$$f = qP + a(z) + b(z)w.$$

Hence

$$\mathbb{C}\{z, w\}/(P) \simeq \mathbb{C}\{z\} \oplus \mathbb{C}\{z\}w$$

as a $\mathbb{C}\{z\}$-module. The projection of the analytic curve $V(P)$ to the z-disc is therefore finite of generic degree two. The fiber over $z \neq 0$ has the two points $w = \pm z^{3/2}$ locally after choosing a branch, while the fiber over 0 is one point with multiplicity two.

The same equation also illustrates why finite projection does not imply smoothness. The gradient

$$\left(\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}\right) = (-3z^2, 2w)$$

vanishes at the origin, so the curve is singular there. The local ring is reduced because P has no repeated irreducible factor, but it is not normal; normalization will add the parameter t with $z = t^2$, $w = t^3$.

In a proof of coherence, division replaces infinitely many coefficients in the w-direction by the finite remainder vector (a, b). Relations among analytic generators can then be handled inductively in one fewer variable. The finiteness of the remainder is the mechanism, not a decorative normal form.

Checkpoint (read geometry from the remainder)

(1) What is the rank of $\mathbb{C}\{z, w\}/(w^d + a_1 w^{d-1} + \cdots + a_d)$ over $\mathbb{C}\{z\}$?
(2) Why is uniqueness in division essential for this assertion?
(3) For $P = w^2 - z$, is $V(P)$ singular at the origin?

Solution (checkpoint). The quotient is free of rank d with basis $1, w, \dots, w^{d-1}$. Uniqueness proves that these residue classes are independent as well as generating. For $w^2 - z$, the gradient is $(-1, 2w)$, which never vanishes, so the curve is smooth despite the two-to-one projection away from the branch point. ◇

Exit criterion

You can divide by a Weierstrass polynomial, identify the finite module and its rank, and separately test smoothness, reducedness, and finiteness of projection.

Summary. Weierstrass division lowers the dimension of local analytic problems; Oka's theorem turns this finiteness into coherence. An analytic set must be understood through its ideal sheaf, not one convenient list of equations.

Exercise 3.16: Node and cusp

Compare $xy = 0$ and $y^2 = x^3$ at the origin: irreducible components, Zariski tangent spaces, and normalizations.

Exercise 3.17: A division calculation

Divide $h(x, y) = y^4 + x$ by $P(x, y) = y^2 - x$ and write the quotient and remainder.

Optional handout: Intersection multiplicity and local algebra

This handout is optional. No later argument in the core lectures depends on it.

Let $f, g \in \mathbb{C}\{x, y\}$ define plane curves with an isolated intersection at the origin. Their local intersection multiplicity is

$$i_0(f, g) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(f, g)}.$$

The dimension is finite exactly when the two germs have no common irreducible component. This definition is invariant under local biholomorphisms, symmetric in f, g, and stable under small perturbation when all nearby intersection points are counted.

Example 3.18: Tangency

For $f = y$ and $g = y - x^m$,

$$i_0(f, g) = \dim_{\mathbb{C}} \mathbb{C}\{x\}/(x^m) = m.$$

Thus multiplicity records the order of tangency, not merely the set-theoretic fact that the curves meet at one point.

If f is irreducible with parametrization $\gamma(t)$, then $i_0(f, g) = \mathrm{ord}_t(g \circ \gamma)$. For the cusp $f = y^2 - x^3$ with $\gamma(t) = (t^2, t^3)$, its intersection with $y = 0$ has multiplicity 3, while its intersection with $x = 0$ has multiplicity 2.

For a local ring $(R, \mathfrak{m})$ of dimension d, the Hilbert–Samuel function $\ell(R/\mathfrak{m}^{k+1})$ agrees for large k with a polynomial of degree d. Its leading term is $e(R)k^d/d!$; the integer $e(R)$ is the multiplicity of the analytic germ. Smooth points have multiplicity one. Lecture 16 relates this algebraic number to the Lelong number of the integration current.

Exercise 3.19: A monomial intersection

Compute $\dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(x^a, y^b)$ and interpret the answer as an intersection multiplicity.

3.3 Lecture 11: Complex analytic spaces, normalization, and divisors

Week 6 material • 90 minutes

Notation for this lecture

For a complex analytic space X, $\mathcal{O}_{X,x}$ is its local ring, X_{Reg} and X_{Sing} are its regular and singular loci, and $\mathcal{M}_X$ denotes the sheaf of meromorphic functions. A space is *reduced* if $\mathcal{O}_{X,x}$ has no nonzero nilpotents, and *normal* if each local domain is integrally closed in its fraction field. $\nu : X^\nu \to X$ denotes normalization. A prime divisor is an irreducible codimension-one analytic subset D, and $\mathrm{ord}_D(f)$ is the order of a meromorphic function along it. A Weil divisor is a locally finite formal sum $\sum_D m_D D$; a Cartier divisor is locally represented by nonzero meromorphic functions modulo holomorphic units. Its associated line bundle is $\mathcal{O}_X(D)$.

Prerequisite results used below

Integral-dependence criterion. An element b of a field extension of a domain A is integral over A if it satisfies a monic equation $b^d + a_1 b^{d-1} + \cdots + a_d = 0$ with $a_j \in A$. The integral closure $\widetilde{A}$ is the set of all such elements in the fraction field of A.
Finite normalization theorem. For a reduced complex analytic space X, the integral closures of its local rings glue to a normal complex space X^ν and a finite holomorphic map $\nu : X^\nu \to X$ that is biholomorphic over the normal locus. Any dominant generically one-to-one map from a normal space to X factors uniquely through ν.

Plan for the lecture. 0–20 min: locally ringed spaces; 20–40: nonreduced structures; 40–58: normalization; 58–78: Weil and Cartier divisors and meromorphic functions; 78–90: Hartogs extension and examples.

Definition 3.20: Complex analytic space

A complex analytic space is a locally ringed space $(X, \mathcal{O}_X)$ locally isomorphic to

$$(A, \mathcal{O}_\Omega/\mathcal{I})|_A, \qquad A = V(\mathcal{I}) \subset \Omega \subset \mathbb{C}^N,$$

where $\mathcal{I}$ is coherent. The space is reduced if $\mathcal{O}_X$ has no nonzero nilpotents.

The double line is $(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}/(y^2))$. Its underlying set is $\{y = 0\}$, but $y \neq 0$ while $y^2 = 0$. This infinitesimal normal information is invisible in the set of points.

Definition 3.21: Normal space and normalization

A reduced complex space is normal if every local ring $\mathcal{O}_{X,x}$ is integrally closed in its total ring of fractions. A normalization is a finite proper map $\nu : X^\nu \to X$ with X^ν normal that is an isomorphism over X_{Reg}.

For curves, normal means smooth. In higher dimension a normal space may be singular, but its singular set has codimension at least two. Analytically, normality says that locally bounded weakly holomorphic functions on the regular locus extend across the singular set.

Theorem 3.22: Hartogs–Riemann extension

If X is normal and $A \subset X$ is analytic with $\mathrm{codim}_X A \geq 2$, then $\mathcal{O}_X(U) \to \mathcal{O}_X(U \setminus A)$ is an isomorphism.

Proof. Injectivity follows from the identity theorem because $U \setminus A$ is dense. For surjectivity the problem is local. At a smooth point of X, choose coordinates and a two-dimensional complex plane transverse to A. Cauchy's integral formula in one variable, followed by the ordinary Hartogs figure argument in the transverse two-plane, extends the function across A. The resulting extension is holomorphic in all variables because its Cauchy integral depends holomorphically on the parameters.

Near a singular point, choose a finite local projection $\pi : X \to B \subset \mathbb{C}^n$. The elementary symmetric functions of the values of f on the sheets are holomorphic on B away from an analytic subset of codimension at least two and hence extend by the smooth Hartogs theorem. Thus f satisfies a monic polynomial with holomorphic coefficients and is integral over $\mathcal{O}_X$. It defines a weakly holomorphic meromorphic function on X. Normality says that every such integral element belongs to the local ring $\mathcal{O}_{X,x}$, so it extends holomorphically. The local extensions agree by injectivity and glue over U. □

Definition 3.23: Weil and Cartier divisors

On a pure-dimensional normal space, a Weil divisor is a locally finite integral combination of irreducible codimension-one analytic subsets. A Cartier divisor is represented by local meromorphic functions f_i with $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$. A divisor $\operatorname{div}(f)$ is principal.

The local rings of a smooth manifold are UFDs, so Weil divisors are locally principal and hence Cartier. This may fail on a singular space.

Proposition 3.24: Equations give an upper bound on codimension

Let X be irreducible and $f_1, \dots, f_p \in \mathcal{O}(X)$. Every irreducible component of $V(f_1, \dots, f_p)$ has codimension at most p. If each successive function is a non-zero-divisor in the corresponding quotient, codimension rises by one at every step and the zero set is a local complete intersection.

Proof. Work at the generic point of an irreducible component Z of the zero set. The analytic local ring is Noetherian. Krull's principal ideal theorem says that every minimal prime over a principal ideal has height at most one. Applying it successively to (f_1), (f_1, f_2), and so on shows

$$\operatorname{ht}(f_1, \dots, f_p) \le p,$$

which is exactly $\operatorname{codim}_X Z \le p$.

If f_j is a non-zero-divisor modulo $(f_1, \dots, f_{j-1})$, multiplication by f_j is injective there. Every minimal prime containing the enlarged ideal then has height one more than a minimal prime of the preceding quotient; the dimension drops by exactly one. Induction gives codimension p, and the local ideal is generated by the regular sequence $f_1, \dots, f_p$. This is the local complete-intersection condition.
□

Definition 3.25: Meromorphic function

The sheaf $\mathcal{M}_X$ is obtained by inverting local non-zero-divisors. Locally a meromorphic function is g/h, but it need not be one quotient on an entire open set. On a manifold, zeros and poles have pure codimension one; the indeterminacy set has codimension at least two.

Remark (Detecting the pole set)

For a meromorphic function f, let $\mathcal{J}_x = \{u \in \mathcal{O}_{X,x} : uf_x \in \mathcal{O}_{X,x}\}$. Its pole set is

$$\{x : \mathcal{J}_x \neq \mathcal{O}_{X,x}\} = \operatorname{Supp}(\mathcal{O}_X/\mathcal{J}).$$

This ideal-theoretic description is intrinsic and continues to work when no single quotient represents f on the whole open set.

3.3.1 Normalization as a universal repair

For an irreducible reduced germ with local ring A, let $\widetilde{A}$ be the integral closure of A in its fraction field. Finiteness of normalization in complex analytic geometry turns $\widetilde{A}$ into the local ring of a finite analytic map $\nu : X^\nu \to X$. The construction is universal: every dominant holomorphic map from a normal space to X that is generically one-to-one factors uniquely through X^ν.

For the cusp,

$$A = \mathbb{C}\{t^2, t^3\} \subset \mathbb{C}\{t\} = \widetilde{A}.$$

The element t is integral because it satisfies $T^2 - t^2 = 0$ with coefficient $t^2 \in A$. The normalization map $t \mapsto (t^2, t^3)$ is bijective, but its inverse is not holomorphic at the origin. This is why a bijective finite holomorphic map of analytic spaces need not be an isomorphism.

For the node $xy = 0$, normalization separates the two branches:

$$\mathbb{C}\{x, y\}/(xy) \longrightarrow \mathbb{C}\{x\} \oplus \mathbb{C}\{y\}, \qquad f(x, y) \longmapsto (f(x, 0), f(0, y)).$$

The image consists of pairs with equal constant term. Normalization therefore may separate points, unlike the cuspidal example, where it only repairs the local ring.

3.3.2 Divisors, valuations, and line bundles

On a normal irreducible space, each prime divisor D defines an order $\operatorname{ord}_D(f)$ for a nonzero meromorphic function: compute the vanishing order at a generic smooth point of D. Hartogs extension across codimension at least two shows that this generic value determines the codimension-one behavior globally. Thus

$$\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g), \qquad \operatorname{div}(f/g) = \operatorname{div}(f) - \operatorname{div}(g).$$

A Cartier divisor represented by f_i on U_i gives transition functions $g_{ij} = f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$. Gluing the trivial line bundles with g_{ij} produces $\mathcal{O}(D)$, and the local functions f_i form its canonical meromorphic section. Conversely, a meromorphic section of a line bundle determines a Cartier divisor. On a manifold every Weil divisor is Cartier because regular local rings are factorial; on a normal singular space the two notions can differ.

Remark (Why codimension two disappears)

Holomorphic functions on a normal space extend across codimension-two analytic sets, and Weil divisors only record codimension one. Nevertheless codimension-two sets are not geometrically irrelevant: they can be indeterminacy loci of meromorphic maps or singular loci of sheaves. They disappear only from these two particular invariants.

3.3.3 Normalization of the cusp from its local ring

Let $A = \mathbb{C}\{x, y\}/(y^2 - x^3)$. The parametrization

$$\nu : \mathbb{C} \longrightarrow C, \qquad t \longmapsto (t^2, t^3)$$

induces the injection

$$A \hookrightarrow \mathbb{C}\{t\}, \qquad x \mapsto t^2, \quad y \mapsto t^3.$$

The element t is integral over A because it satisfies the monic equation $T^2 - x = 0$. It is not in A: every element of $\mathbb{C}\{t^2, t^3\}$ has zero coefficient of t^1. Since $t = y/x$ in the common fraction field away from the origin, adjoining t gives

$$\overline{A} = A[t] = \mathbb{C}\{t\}.$$

The map ν is finite because $\mathbb{C}\{t\} = A \oplus At$ is generated by $1, t$ as an A-module. It is bijective near the origin, but its inverse is not holomorphic at the cusp: a holomorphic inverse would put t in A. This is the concrete reason a finite bijective map of analytic spaces need not be an isomorphism.

A Weil divisor that is not Cartier

Consider the normal quadric cone

$$X = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2\}$$

and the irreducible codimension-one set $D = \{x = z = 0\} \subset X$. The finite double cover

$$\pi : \mathbb{C}^2 \to X, \qquad (s, t) \longmapsto (s^2, t^2, st)$$

is the quotient by $(s, t) \mapsto (-s, -t)$. The set-theoretic inverse image of D is $\{s = 0\}$.

If D were Cartier at the vertex, a local equation g would pull back to a holomorphic function whose divisor is exactly $\{s = 0\}$ with multiplicity one. Up to a unit this pullback would be s. But $g \circ \pi$ is invariant under $(s, t) \mapsto (-s, -t)$, whereas a unit times s changes sign at the level of its lowest nonzero homogeneous term. An invariant function vanishing only on $s = 0$ has even generic vanishing order there. This contradiction shows that D is Weil but not Cartier at the vertex. Twice the divisor is Cartier because $x = s^2$ has divisor $2D$.

Cartier divisors and their line bundles

Let a Cartier divisor be represented by meromorphic functions f_i with $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$. Put

$$g_{ij} = \frac{f_i}{f_j}.$$

Then $g_{ij} g_{jk} = g_{ik}$, so the g_{ij} are transition functions for a line bundle $\mathcal{O}(D)$. The local meromorphic frames f_i^{-1} glue, and the constant local functions 1 define a global meromorphic section whose divisor is D.

If $D = \operatorname{div}(f)$ is principal, choose every $f_i = f|_{U_i}$. Then $g_{ij} = 1$, so $\mathcal{O}(D)$ is trivial. Conversely, a trivialization of $\mathcal{O}(D)$ changes the f_i by local units so that they glue to one global meromorphic function. Hence Cartier divisors modulo principal divisors are exactly holomorphic line bundles that admit a nonzero meromorphic section.

An indeterminacy calculation

The map

$$F : \mathbb{C}^2 \dashrightarrow \mathbb{P}^1, \qquad F(x, y) = [x : y]$$

is holomorphic away from the origin. On $\{x \neq 0\}$ its affine coordinate is y/x; on $\{y \neq 0\}$ it is x/y. No continuous extension exists at 0, since along the line $y = mx$ the limit is $[1 : m]$, depending on m. Blowing up the origin resolves the map: a point of the blow-up is $((x, y), [u : v])$ with $xv = yu$, and the extension is simply $((x, y), [u : v]) \mapsto [u : v]$. The exceptional curve records all limiting directions that were collapsed at the indeterminacy point.

3.3.4 Self-study workshop: distinguishing reduced, normal, Cartier, and smooth

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall local analytic rings, integral dependence, fraction fields, and the correspondence between locally principal divisors and line bundles. These adjectives answer different questions and should never be used as synonyms.

A diagnostic chain of examples. The space with local ring $\mathbb{C}\{z\}/(z^2)$ has one underlying point in its zero set but carries a nonzero nilpotent class z. It is not reduced. The cusp

$$C = \{y^2 = x^3\}$$

is reduced, but its local ring $A = \mathbb{C}\{t^2, t^3\}$ is not integrally closed: t belongs to its fraction field and satisfies $T^2 - t^2 = 0$ with coefficient $t^2 = x \in A$, yet $t \notin A$. Its normalization is $\mathbb{C}\{t\}$.

Normal does not mean smooth. The surface

$$X = \{xy = z^2\} \subset \mathbb{C}^3$$

is singular at the origin because its gradient vanishes there. It is nevertheless normal: it is a hypersurface, hence Cohen–Macaulay, and its singular locus has codimension two; Serre's criterion gives normality.

Finally, a Weil divisor records codimension-one subvarieties, while a Cartier divisor is locally one meromorphic equation modulo holomorphic units. On a smooth complex manifold every Weil divisor is Cartier, but on a normal singular space this may fail. The obstruction is local principality, not the existence of a set-theoretic equation. For a Cartier divisor (f_i), the quotients f_i/f_j are nowhere-vanishing holomorphic functions and therefore define a line bundle.

Checkpoint (four independent tests)

(1) Which example above is nonreduced?
(2) Why is the cusp's normalization map finite?
(3) Can a normal analytic space be singular?
(4) What local property upgrades a Weil divisor to a Cartier divisor?

Solution (checkpoint). The double point $\mathbb{C}\{z\}/(z^2)$ is nonreduced. For the cusp, $\mathbb{C}\{t\} = A + At$ is generated by two elements over A, hence the normalization map is finite. The quadric cone is normal and singular. A Weil divisor is Cartier exactly when its divisor ideal is locally generated by one non-zero-divisor, equivalently when it has local meromorphic equations whose ratios are units. ⋄

Exit criterion

Given a local analytic equation, you can test nilpotents and the Jacobian, exhibit an integral element for normalization, and explain why divisor class questions are about local principality.

Summary. Analytic spaces admit singularities and nilpotent directions. Normalization repairs integral closedness, and divisors record codimension-one data. A list of equations gives an upper bound for codimension; a regular sequence is what gives equality.

Exercise 3.26: A non-Cartier Weil divisor

In $X = \{z_1 z_2 + z_3 z_4 = 0\} \subset \mathbb{C}^4$, study the plane $A = \{z_1 = z_3 = 0\}$ and show that it is a Weil divisor but not Cartier at the origin.

Exercise 3.27: Indeterminacy

Study the zero set, pole set, and indeterminacy set of $f(z_1, z_2) = z_2/z_1$. Why can the origin be assigned neither 0 nor ∞?

Optional handout: Serre's criterion for normality

This handout is optional. No later argument in the core lectures depends on it.

For a Noetherian reduced ring, normality can be separated into codimension-one and depth conditions. Serre's criterion states

$$\text{normal} \quad \Longleftrightarrow \quad (R_1) + (S_2).$$

Condition (R_1) means that every local ring of codimension at most one is regular. Condition (S_2) means $\operatorname{depth} R_{\mathfrak{p}} \geq \min\{2, \dim R_{\mathfrak{p}}\}$ for all primes $\mathfrak{p}$.

Geometrically, (R_1) says that singularities occur in codimension at least two, while (S_2) prevents a function from being specified independently on pieces separated by a codimension-two defect. Together they explain why holomorphic functions on the regular locus of a normal space extend across codimension at least two when locally bounded.

Example 3.28: The node and the cusp

For a reduced complex curve, (S_2) is automatic and (R_1) requires regularity at every closed point. Hence a normal curve is smooth. Both the node $xy = 0$ and the cusp $y^2 = x^3$ are nonnormal, although for different reasons: the node has two branches, whereas the cusp is irreducible but misses the integral element t in its local ring.

Hypersurfaces are Cohen–Macaulay and therefore satisfy (S_2). A reduced hypersurface is consequently normal precisely when its singular locus has codimension at least two inside the hypersurface.

Exercise 3.29: A normal singularity

Use Serre's criterion to show that the quadric cone $\{z_1z_2 - z_3z_4 = 0\} \subset \mathbb{C}^4$ is normal even though it is singular at the origin.

3.4 Lecture 12: Modifications, blow-ups, and exceptional divisors

Week 6 material • 90 minutes

Notation for this lecture

$\pi : \mathrm{Bl}_Z X \to X$ denotes the blow-up of a complex manifold along a center Z with ideal sheaf $\mathcal{I}_Z$. The exceptional divisor is $E = \pi^{-1}(Z)$, and $\mathcal{O}(-E) = \pi^{-1}\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_Z X}$. The symbol $\widetilde{V}$ denotes the strict transform of V, namely the closure of $\pi^{-1}(V \setminus Z)$. For the blow-up of $\mathbb{C}^2$ at the origin, (x, u) and (v, y) are the two standard charts with $y = xu$ and $x = yv$. On $\mathrm{Bl}_p \mathbb{P}^2$, H is the pullback of the class of a line and E is the exceptional curve; the dot denotes the intersection pairing, so $H^2 = 1$, $H \cdot E = 0$, and $E^2 = -1$. Proj denotes the relative analytic projectivization of a graded coherent algebra.

Prerequisite results used below

Projective-chart lemma. The sets $U_j = \{[z_0 : \cdots : z_n] : z_j \neq 0\}$ cover $\mathbb{P}^n$, and division by z_j identifies U_j biholomorphically with $\mathbb{C}^n$. On overlaps the coordinate changes are holomorphic rational functions with nonvanishing denominators.
Universal property of projectivization. A holomorphic map $f : Y \to X$ for which $f^{-1}\mathcal{I}_Z \cdot \mathcal{O}_Y$ is an invertible ideal factors uniquely through $\mathrm{Bl}_Z X = \mathrm{Proj} \bigoplus_{m \geq 0} \mathcal{I}_Z^m$. For a point blow-up, this says that the coordinate functions generating the pulled-back maximal ideal determine the exceptional projective direction.

Plan for the lecture. 0–18 min: bimeromorphisms and modifications; 18–48: charts of the blow-up of $\mathbb{C}^n$; 48–68: blow-up of an ideal and the universal property; 68–82: strict transforms and intersection numbers; 82–90: the reducedness trap.

Definition 3.30: Modification

A proper holomorphic map $\mu : \widetilde{X} \to X$ is a modification if there is a nowhere-dense analytic set $B \subset X$ such that $\mu : \widetilde{X} \setminus \mu^{-1}(B) \to X \setminus B$ is biholomorphic.

Definition 3.31: Blow-up of the origin

The blow-up of $\mathbb{C}^n$ at the origin is

$$\mathrm{Bl}_0 \mathbb{C}^n = \{(z, [\ell]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : z \in \ell\},$$

with $\pi(z, [\ell]) = z$. For $z \neq 0$, the direction is uniquely $[\ell] = [z]$, while the exceptional fiber is $E = \pi^{-1}(0) \simeq \mathbb{P}^{n-1}$.

On the chart $\ell_1 \neq 0$, write $u_j = \ell_j/\ell_1$. The incidence relation becomes

$$z_1 = t, \qquad z_j = tu_j \quad (j \geq 2).$$

Thus the blow-up is smooth and $E = \{t = 0\}$. Geometrically, the origin has been replaced by the projective space of complex directions through it.

Example 3.32: Strict transform of a cusp

For $C = \{y^2 = x^3\}$, the chart $x = t$, $y = tu$ gives total transform $t^2(u^2 - t) = 0$. Removing the exceptional factor leaves $u^2 = t$, which is smooth but tangent to E. One blow-up resolves the cusp itself, though not yet to normal crossings.

Definition 3.33: Blow-up of an ideal sheaf

For a coherent ideal $\mathcal{I} \subset \mathcal{O}_X$, define

$$\mathrm{Bl}_{\mathcal{I}} X = \mathrm{Proj}_X \bigoplus_{m \geq 0} \mathcal{I}^m.$$

Its universal property says: if $f : Y \to X$ makes $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ invertible, then f factors uniquely through $\mathrm{Bl}_{\mathcal{I}} X$.

If the center Z is smooth of codimension r, then $E \simeq \mathbb{P}(N_{Z/X})$ and $N_{E/\widetilde{X}} \simeq \mathcal{O}_E(-1)$.

Proposition 3.34: Intersection numbers on the blown-up plane

Let $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the blow-up of one point, $H = \pi^*[\text{line}]$, and E the exceptional curve. Then

$$H^2 = 1, \qquad H \cdot E = 0, \qquad E^2 = -1.$$

If a plane curve C has multiplicity m at the center, its strict transform has class $\widetilde{C} = \pi^* C - mE$.

Proof. The first two identities follow from the projection formula and a general line avoiding the center. The self-intersection of E is the degree of $N_{E/\widetilde{X}} = \mathcal{O}_{\mathbb{P}^1}(-1)$. Locally the pullback of the equation of C contains a factor t^m, giving the strict-transform formula. □

Warning (Reducedness)

A modification identifies meromorphic-function sheaves under the usual reduced, pure-dimensional hypotheses. The assertion fails for general nonreduced analytic spaces: nilpotents can vanish on a dense open set and cannot be recovered from meromorphic data there.

Warning

A blow-up is not obtained by deleting the center and gluing an arbitrary projective space. The incidence relation, or equivalently the Rees algebra, fixes the normal bundle of the exceptional divisor. Its negativity is the source of $E^2 = -1$ and blow-down. Distinguish the total transform, strict transform, and exceptional part.

3.4.1 The two charts of the surface blow-up

For $\mathrm{Bl}_0 \mathbb{C}^2$, the chart U_x has coordinates (t, u) with

$$x = t, \qquad y = tu,$$

and the chart U_y has coordinates (v, s) with

$$x = sv, \qquad y = s.$$

On their overlap,

$$s = tu, \qquad v = u^{-1}.$$

The exceptional divisor is $t = 0$ in U_x and $s = 0$ in U_y; the coordinate u or v records the tangent direction. Since $t = sv$ along the overlap, the normal coordinate transforms by the transition of $\mathcal{O}_{\mathbb{P}^1}(-1)$. This gives $N_{E/\widetilde{X}} \simeq \mathcal{O}_E(-1)$ and hence $E^2 = -1$.

If $f(x, y)$ vanishes to order m at the origin, substitution gives

$$f(t, tu) = t^m\big(f_m(1, u) + t f_{m+1}(1, u) + \cdots\big),$$

where f_m is the first nonzero homogeneous part. The factor t^m is the exceptional component of the total transform. The intersection of the strict transform with E is cut out by $f_m(1, u) = 0$, so its points and multiplicities are exactly the tangent cone of the original curve.

3.4.2 Canonical bundle and discrepancy of a blow-up

Let $Z \subset X$ be a smooth center of codimension r and $\pi : \widetilde{X} \to X$ its blow-up. In local coordinates with $Z = \{z_1 = \cdots = z_r = 0\}$, one blow-up chart is

$$z_1 = t, \qquad z_j = tu_j \ (2 \le j \le r), \qquad z_k = z_k \ (k > r).$$

Taking the Jacobian yields

$$\pi^*(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n) = t^{r-1}\mathrm{d}t \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_r \wedge \mathrm{d}z_{r+1} \wedge \cdots \wedge \mathrm{d}z_n.$$

Therefore

$$K_{\widetilde{X}} = \pi^* K_X + (r-1)E.$$

For a point on a surface, $r = 2$ and $K_{\widetilde{X}} = \pi^* K_X + E$. Together with $E^2 = -1$, adjunction gives $(K_{\widetilde{X}} + E) \cdot E = -2$, confirming that $E \simeq \mathbb{P}^1$.

Remark (What blow-up remembers)

The blow-up is an isomorphism away from the center, but it is not topologically or cohomologically invisible. It turns leading homogeneous terms into intersections on E, converts base ideals into line bundles, and records the Jacobian loss in the canonical divisor. These three roles reappear in resolution of singularities, linear systems, and multiplier ideals.

3.4.3 The blow-up atlas and its exceptional curve

The blow-up of $\mathbb{C}^2$ at the origin is

$$\widetilde{\mathbb{C}^2} = \{((x, y), [s : t]) \in \mathbb{C}^2 \times \mathbb{P}^1 : xt = ys\}.$$

On $s \neq 0$ put $v = t/s$. The equation is $y = xv$, so

$$U_s \simeq \mathbb{C}^2_{(x,v)}, \qquad \pi(x, v) = (x, xv).$$

On $t \neq 0$ put $u = s/t$. Then $x = yu$ and

$$U_t \simeq \mathbb{C}^2_{(u,y)}, \qquad \pi(u, y) = (uy, y).$$

On the overlap $u, v \neq 0$,

$$u = \frac{1}{v}, \qquad y = xv, \qquad x = yu.$$

These are holomorphic inverse coordinate changes. The exceptional fiber $E = \pi^{-1}(0)$ is $\{x = 0\}$ in the first chart and $\{y = 0\}$ in the second; its coordinate transition $u = 1/v$ identifies it with $\mathbb{P}^1$.

The normal coordinate to E is x on U_s and y on U_t. Let $n_s = [\partial/\partial x]$ and $n_t = [\partial/\partial y]$ be the corresponding normal vector frames. Since $x = uy$ with u fixed in the second chart, $n_t = un_s = v^{-1}n_s$. This is the transition function of $\mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore

$$N_{E/\widetilde{\mathbb{C}^2}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1), \qquad E^2 = \deg N_{E/\widetilde{\mathbb{C}^2}} = -1.$$

Strict transforms computed in charts

For the line $L_m = \{y = mx\}$, its inverse image in the first chart satisfies $xv = mx$, or

$$x(v - m) = 0.$$

The factor $x = 0$ is the exceptional divisor; removing it and taking the closure gives the strict transform $\widetilde{L}_m = \{v = m\}$. It meets E at the single point $(x, v) = (0, m)$, so the exceptional curve parametrizes tangent directions. The vertical line $x = 0$ is seen in the second chart and meets E at $u = 0$.

For the cusp $C = \{y^2 = x^3\}$, substitution $y = xv$ gives

$$x^2(v^2 - x) = 0.$$

Hence the strict transform is $x = v^2$. It is smooth because the gradient of $x - v^2$ is $(1, -2v)$, and it is parametrized by $v \mapsto (v^2, v)$. It meets $E = \{x = 0\}$ at $v = 0$ with intersection multiplicity two, so the curve is resolved but the total divisor is not yet normal crossings. A second blow-up separates this tangency.

For the node with tangent lines $y = \pm x$, the first-chart equation has the form

$$x^2\big(v^2 - 1 - x\big) = 0.$$

Its strict transform meets E at the two distinct points $v = 1$ and $v = -1$. Thus one blow-up separates the two branches.

The universal property in local algebra

Let $f : Y \to \mathbb{C}^2$ be holomorphic and assume the ideal (f^*x, f^*y) is locally principal, generated by a. Write $f^*x = as$ and $f^*y = at$, where s, t do not vanish simultaneously. Then

$$\widetilde{f} : Y \to \widetilde{\mathbb{C}^2}, \qquad p \longmapsto \big(f(p), [s(p) : t(p)]\big)$$

is holomorphic and satisfies $\pi \circ \widetilde{f} = f$, because $f^*x\, t = f^*y\, s$. The projective coordinate is forced wherever $f(p) \neq 0$ and extends by holomorphicity, so the lift is unique. This proves the universal property and explains the construction $\operatorname{Proj} \bigoplus_{m \geq 0} \mathcal{I}^m$.

On the blow-up of $\mathbb{P}^2$ at one point, let H be the pullback of a line and E the exceptional curve. Generic representatives give

$$H^2 = 1, \qquad H \cdot E = 0, \qquad E^2 = -1.$$

A degree-d curve with multiplicity m at the center has strict-transform class $dH - mE$, and therefore

$$(dH - mE)^2 = d^2 - m^2, \qquad (dH - mE) \cdot E = m.$$

The second equality recovers the multiplicity as intersection with the exceptional directions.

3.4.4 Self-study workshop: following an equation through a blow-up chart

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall the two charts of the blow-up of $\mathbb{C}^2$ at the origin, strict transforms, exceptional divisors, and the distinction between total and strict transform.

Resolve the cusp by calculation. Write

$$\widetilde{\mathbb{C}^2} = \{((x, y), [s : t]) : xt = ys\}.$$

On the chart $s \neq 0$, put $v = t/s$; then $y = xv$ and the exceptional divisor is $E = \{x = 0\}$. Pulling back the cusp equation $y^2 = x^3$ gives

$$x^2(v^2 - x) = 0.$$

The factor x^2 is the exceptional component of the total transform. Removing it leaves the strict transform

$$\widetilde{C} = \{v^2 - x = 0\}.$$

Its gradient in coordinates (x, v) is $(-1, 2v)$, so it is smooth. It meets E only at $(0, 0)$. Parametrizing $\widetilde{C}$ by v gives $x = v^2$; its tangent at the intersection is the v-axis, while E has the same tangent $x = 0$. Thus the first blow-up smooths the curve but the intersection with the exceptional divisor is tangent, not normal crossing. A second blow-up separates this tangency if an embedded normal-crossing resolution is desired.

This example displays three integers that should not be confused. The exponent 2 of the removed exceptional factor is the multiplicity of the cusp at the origin. The strict transform meets E at one point, representing its unique tangent direction. Its intersection multiplicity with E there is two, detected by $x = v^2$.

Checkpoint (total versus strict transform)

(1) In the other chart $x = uy$, what equation represents the strict transform?
(2) Why is the exceptional divisor isomorphic to $\mathbb{P}^1$?
(3) What information does the point $\widetilde{C} \cap E$ encode?

Solution (checkpoint). Substitution gives $y^2 = u^3y^3$, hence after removing y^2 the strict transform is $1 = u^3y$; it does not meet $E = \{y = 0\}$ in this chart. The exceptional fiber records lines through the origin and is therefore $\mathbb{P}^1$. The intersection point is the tangent direction of the original branch; multiple points would record distinct tangent directions. ◇

Exit criterion

You can write both blow-up charts, factor a pulled-back equation, identify the strict transform, test its smoothness, and interpret its intersection with the exceptional divisor geometrically.

Summary. A modification preserves the complex structure on a dense open set while reorganizing divisorial and singular data. Blow-up projectivizes normal directions and is the basic local operation in resolution, construction of Kähler classes, and birational geometry.

Exercise 3.35: Transition between blow-up charts

Write the transition map between the two charts of $\mathrm{Bl}_0\,\mathbb{C}^2$ and identify the normal bundle of E as $\mathcal{O}_{\mathbb{P}^1}(-1)$.

Exercise 3.36: Three concurrent lines

Blow up three lines through the origin in $\mathbb{C}^2$ with distinct tangent directions. Prove that their strict transforms are disjoint and meet E transversely at three different points.

Optional handout: Resolution of singularities and weak factorization

This handout is optional. No later argument in the core lectures depends on it.

Hironaka's resolution theorem says that, in characteristic zero, every reduced complex analytic space admits a proper modification from a smooth space, obtained locally by a finite sequence of blow-ups with smooth centers. The centers can be chosen inside the singular locus, so the modification is an isomorphism over the regular part.

For a divisor D, an embedded log resolution makes the total transform a simple normal-crossings divisor. Locally this means that, in suitable coordinates,

$$\mu^*D = \{z_1^{a_1} \cdots z_r^{a_r} = 0\}.$$

Analytic singularities then become monomial. This is why resolution is so effective for integrability, multiplier ideals, and change-of-variables formulas.

The weak factorization theorem goes in the opposite direction: a birational map between smooth complete algebraic varieties over a field of characteristic zero can be factored into blow-ups and blow-downs along smooth centers, away from the open set where the original map is an isomorphism. It says that blow-up is not merely one construction but a universal vocabulary for comparing smooth birational models.

Remark (Discrepancies)

For a modification $\mu : Y \to X$ of smooth n-folds,

$$K_Y = \mu^* K_X + \sum_E a(E, X) E.$$

The coefficients measure the Jacobian vanishing of μ. They govern the change-of-variables test for multiplier ideals and, in algebraic geometry, the classification of singularities as log canonical, klt, and so forth.

Exercise 3.37: Canonical divisor of a blow-up

Blow up a smooth codimension-r center. In local coordinates compute the Jacobian in one chart and show $K_{\widetilde{X}} = \pi^* K_X + (r-1)E$.

Chapter 4

Positive Currents, Monge–Ampère Operators, and Lelong Numbers

4.1 Lecture 13: Positive forms and positive currents

Week 7 material • 90 minutes

Notation for this lecture

X is a complex n-manifold and $\Lambda^{p,q}T_x^*X$ is the space of complex covectors of bidegree (p,q) at x. A decomposable $(p,0)$-form has the form $\eta_1 \wedge \cdots \wedge \eta_p$. Strongly positive, positive, and weakly positive refer to the three cones defined in this lecture; in bidegree $(1,1)$ they coincide. The notation $T \geq 0$ means that a current T is positive when paired with every compactly supported strongly positive test form of complementary bidegree. $\beta = (\mathrm{i}/2)\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ is the standard Euclidean Kähler form, $[V]$ is an integration current, and $\|T\|_K$ is the mass of T on a compact set K relative to a specified Hermitian metric.

Prerequisite results used below

Hermitian spectral theorem. Every Hermitian form has an orthonormal eigenbasis and real eigenvalues. It is positive semidefinite exactly when its quadratic form is nonnegative on every vector.

Dual-cone principle. If C is a closed convex cone in a finite-dimensional real vector space V, its dual is $C^\vee = \{\lambda \in V^* : \lambda(v) \geq 0 \text{ for all } v \in C\}$. A current is positive precisely because its value on the cone of strongly positive complementary test forms is nonnegative; this is the distributional version of membership in a dual cone.

Plan for the lecture. 0–25 min: three notions of (p,p) positivity; 25–45: dual cones and pullback; 45–65: positive currents and measure coefficients; 65–80: trace measure and mass; 80–90: Wirtinger's theorem.

In intermediate bidegrees, "positive" is not a unique notion. Let V have complex dimension n, oriented so that $(\mathrm{i}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1) \wedge \cdots \wedge (\mathrm{i}\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n)$ is positive.

Definition 4.1: Three positivity cones for (p,p)-forms

A real (p,p)-form α is:

(1) *strongly positive* if it is a nonnegative combination of forms $\mathrm{i}^{p^2}\eta_1 \wedge \cdots \wedge \eta_p \wedge \bar{\eta}_1 \wedge \cdots \wedge \bar{\eta}_p$ with $\eta_j \in V^{*1,0}$;

(2) *positive* if it is a nonnegative combination of $\mathrm{i}^{p^2}\xi \wedge \bar{\xi}$, $\xi \in \Lambda^{p,0}V^*$;
(3) *weakly positive* if $\alpha \wedge \beta$ is a nonnegative volume form for every strongly positive $(n - p, n - p)$-form β.

The cones satisfy

$$\mathrm{SP}^p \subset \mathrm{P}^p \subset \mathrm{WP}^p,$$

with equality for $p = 0, 1, n-1, n$. For $\alpha = \mathrm{i}\sum a_{j\bar{k}}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$, positivity is equivalent to semipositivity of the Hermitian matrix $(a_{j\bar{k}})$.

Proposition 4.2: Operations preserving positivity

The wedge of strongly positive forms is strongly positive. The wedge of a positive form with a strongly positive form is positive. Complementary weakly and strongly positive forms wedge to a nonnegative volume form. Holomorphic linear pullback preserves all three cones.

Proof. A strongly positive form is a nonnegative sum of decomposable generators. Wedging two such generators concatenates their $(1,0)$ factors and, after the sign i^{p^2} is combined with i^{q^2}, gives the generator in degree $p+q$. Hence the first assertion follows by bilinearity. Write a positive form as a sum of $\mathrm{i}^{p^2}\xi \wedge \bar{\xi}$ and a strongly positive form as a sum of decomposable terms; their wedge again has the form $\mathrm{i}^{(p+q)^2}\eta \wedge \bar{\eta}$.

The definition of weak positivity is exactly nonnegativity after wedging with every complementary strongly positive generator, which proves the third assertion. Finally, a holomorphic linear pullback takes a $(1,0)$ covector to a $(1,0)$ covector and commutes with wedge and conjugation. It therefore preserves the generators of the strong and ordinary cones. For the weak cone, test after pullback against a complementary decomposable form and extend a basis from the image of the map; the test reduces to the original weak-positivity inequality. Thus all three cones are preserved. □

Warning

The wedge of two weakly positive forms need not be weakly positive in intermediate degree. Before multiplying, verify that one factor is strongly positive or reduce the question to a diagonal matrix calculation.

Definition 4.3: Positive current

A (p,p)-current T is positive if $\langle T, \eta\rangle \geq 0$ for every compactly supported strongly positive $(n-p, n-p)$ test form η. Thus the positive-current cone is dual to the cone of strongly positive test forms.

The coefficients of a positive current are complex measures, and their total variations on compact sets are controlled by a trace measure. Thus positivity improves a general distribution by one full regularity level.

Let ω be Hermitian and let T have bidimension (p,p). Define

$$\sigma_T = \frac{1}{p!}T \wedge \omega^p, \qquad \|T\|_K = \sigma_T(K).$$

Different Hermitian metrics give equivalent masses on relatively compact sets.

Theorem 4.4: Wirtinger's theorem

If $Z \subset X$ is analytic of pure dimension p, then

$$\mathrm{Vol}_\omega(Z \cap K) = \frac{1}{p!}\int_{Z_{\mathrm{Reg}}\cap K} \omega^p = \|[Z]\|_K\,.$$

In Euclidean space, a complex p-plane has the orientation calibrated by the Kähler form among all real $2p$-planes.

Proof. Work at one point in a unitary real basis $e_1, Je_1, \dots, e_n, Je_n$, so $\omega = \sum_j e_j^* \wedge (Je_j)^*$. On an oriented orthonormal real $2p$-frame $v_1, \dots, v_{2p}$, the value of $\omega^p/p!$ is the Pfaffian of the skew matrix $(\omega(v_a, v_b))$. Its absolute value is at most one; equality holds exactly when the plane is J-invariant with its complex orientation. This is Wirtinger's linear algebra inequality. Applying it to $T_x Z_{\mathrm{Reg}}$ gives equality pointwise on a complex submanifold. Integrating over $Z_{\mathrm{Reg}} \cap K$ gives the mass formula; the singular set has zero $2p$-dimensional measure. □

Example 4.5: Three sources of positive currents

(1) A psh function u gives the closed positive current $\mathrm{dd^c}u$.
(2) An analytic set Z gives the integration current $[Z]$.
(3) The semipositive curvature of a Hermitian line bundle is a smooth positive $(1,1)$-form.

They represent potential theory, algebraic cycles, and differential geometry.

4.1.1 Duality of cones and coefficient measures

The definition of a positive current uses strongly positive test forms because the two cones are dual. At a point, a weakly positive (p,p)-form α satisfies

$$\alpha \wedge \mathrm{i}^{(n-p)^2}\eta_1 \wedge \cdots \wedge \eta_{n-p} \wedge \bar\eta_1 \wedge \cdots \wedge \bar\eta_{n-p} \ge 0$$

for every collection of $(1,0)$ covectors η_j. Replacing the η_j by linear combinations and polarizing controls every coefficient of α by its diagonal coefficients. The same argument for a current shows that its diagonal coefficients are positive measures and its off-diagonal coefficients are complex measures satisfying local Cauchy–Schwarz bounds.

Concretely, for a positive $(1,1)$-current $T = \mathrm{i}\sum T_{j\bar k}\mathrm{d}z_j \wedge \mathrm{d}\bar z_k$, positivity of the 2×2 principal blocks gives

$$|T_{j\bar k}|(K)^2 \le T_{j\bar j}(K')T_{k\bar k}(K')$$

after inserting a cutoff equal to one on $K \Subset K'$. Thus the trace measure $T \wedge \omega^{n-1}$ controls all coefficients. This is the local estimate behind weak compactness of positive currents.

4.1.2 Calibration and two geometric consequences

Wirtinger's inequality says that $\omega^p/p!$ has comass one: on every oriented real $2p$-plane P,

$$\left.\frac{\omega^p}{p!}\right|_P \le \mathrm{d}V_P,$$

with equality precisely for a complex p-plane with its complex orientation. Therefore complex submanifolds minimize volume in their real homology class. If Z' is a compact real cycle homologous to a compact complex p-fold Z, then

$$\mathrm{Vol}(Z) = \int_Z \frac{\omega^p}{p!} = \int_{Z'} \frac{\omega^p}{p!} \le \mathrm{Vol}(Z').$$

Closedness of ω is what makes the middle integral homological.

There is also a useful nonexistence test. If a Kähler form on X is exact, $\omega = \mathrm{d}\gamma$, then X contains no compact positive-dimensional complex subvariety. Indeed, for a compact p-dimensional analytic set Z,

$$0 < \int_Z \frac{\omega^p}{p!} = \frac{1}{p!}\int_Z \mathrm{d}(\gamma \wedge \omega^{p-1}) = 0,$$

a contradiction. In particular, $\mathbb{C}^n$ and Stein manifolds equipped with exact Kähler forms cannot contain compact complex curves.

Remark (Three levels of positivity)

For $(1,1)$-forms and for complementary degree $(n-1, n-1)$, the strong, ordinary, and weak cones coincide. The distinction appears only in intermediate degrees and only when $n \geq 4$. Consequently most curvature calculations with $(1,1)$-forms may use matrix positivity, while products of higher-degree currents require the full cone dictionary.

4.1.3 Positive forms reduced to matrices

For a real $(1,1)$-form write

$$\alpha = \mathrm{i}\sum_{j,k=1}^{n} a_{j\bar{k}}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k, \qquad A = (a_{j\bar{k}}) = A^*.$$

Evaluating on $v, \bar{v}$ gives

$$-\mathrm{i}\alpha(v,\bar{v}) = \sum_{j,k} a_{j\bar{k}} v_j \bar{v}_k.$$

Hence positivity of α is exactly semipositivity of A. A unitary coordinate change diagonalizes A, so at one point

$$\alpha = \mathrm{i}\sum_{j=1}^{n} \lambda_j\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j, \qquad \lambda_j \geq 0.$$

This proves that for bidegree $(1,1)$ the strong, ordinary, and weak cones coincide.

If $\beta = \mathrm{i}\sum b_{j\bar{k}} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$ is another positive form, unitary coordinates for α show directly that $\alpha \wedge \beta$ is strongly positive when one factor is positive of degree one. For a positive definite ω, the trace is characterized by

$$\alpha \wedge \frac{\omega^{n-1}}{(n-1)!} = (\operatorname{tr}_\omega \alpha)\frac{\omega^n}{n!}.$$

In unitary coordinates this is simply $\operatorname{tr}_\omega \alpha = \lambda_1 + \cdots + \lambda_n$.

For a smooth real function u,

$$(\mathrm{dd^c}u)^n = \frac{n!}{\pi^n}\det(u_{j\bar{k}})\, \mathrm{d}V_{\mathbb{R}^{2n}},$$

with the present convention. The determinant appears because only terms containing each $\mathrm{d}z_j$ and $\mathrm{d}\bar{z}_k$ once survive in the wedge expansion; their signed sum is the Leibniz formula for $\det H_u$.

Positivity of integration currents in local coordinates

Let $V^p \subset X^n$ be a complex submanifold and choose coordinates with $V = \{z_{p+1} = \cdots = z_n = 0\}$. A compactly supported strongly positive test form of bidegree (p,p) is a sum of terms

$$\eta = (\mathrm{i})^{p^2}\, \theta_1 \wedge \cdots \wedge \theta_p \wedge \bar{\theta}_1 \wedge \cdots \wedge \bar{\theta}_p$$

with nonnegative coefficients. Pulling back to V replaces each θ_j by its tangential part. The resulting top-degree form is

$$|\det B|^2\,(\mathrm{i})^{p^2}\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_p\wedge\mathrm{d}\bar z_1\wedge\cdots\wedge\mathrm{d}\bar z_p\geq 0.$$

Therefore $[V](\eta)=\int_V\eta\geq 0$. Positivity is independent of singularities: for an analytic set, integrate over the regular locus; its singular locus has zero $2p$-dimensional measure.

Why the coefficients of a positive current are measures

Fix a compact K in a coordinate ball and the Euclidean Kähler form ω_0. Every coefficient test form can be written as a bounded linear combination of strongly positive forms after adding a sufficiently large multiple of ω_0^{n-p}. Positivity then gives

$$|\langle T,\eta\rangle|\leq C_K\|\eta\|_{C^0}\langle T,\chi\omega_0^{n-p}\rangle,$$

where $\chi=1$ on K. Thus T has order zero. The Riesz representation theorem makes each coordinate coefficient a complex Radon measure, and the positive trace measure $T\wedge\omega_0^p/p!$ controls all their total variations.

For $V=\{w=f(z)\}\subset\mathbb{C}^2$, the graph parametrization $\iota(z)=(z,f(z))$ gives

$$\iota^*\left(\frac{\mathrm{i}}{2}(\mathrm{d}z\wedge\mathrm{d}\bar z+\mathrm{d}w\wedge\mathrm{d}\bar w)\right)=\frac{\mathrm{i}}{2}(1+|f'(z)|^2)\mathrm{d}z\wedge\mathrm{d}\bar z.$$

Hence the mass of $[V]$ over a set is the ordinary area integral with density $1+|f'|^2$. This is the coordinate form of Wirtinger's calibration formula.

4.1.4 Self-study workshop: testing positivity on the correct cone

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall Hermitian matrices, decomposable $(p,0)$-forms, duality between forms and currents, and the orientation supplied by a complex structure. In bidegree $(1,1)$ all standard positivity notions coincide; in middle bidegrees they need not.

Begin with the matrix case. A real $(1,1)$-form can be written

$$\alpha=\mathrm{i}\sum_{j,k}a_{j\bar k}\,\mathrm{d}z_j\wedge\mathrm{d}\bar z_k.$$

After a unitary change of coordinates, $A=(a_{j\bar k})$ is diagonal:

$$\alpha=\mathrm{i}\sum_j\lambda_j\,\mathrm{d}z_j\wedge\mathrm{d}\bar z_j.$$

The condition $\alpha(v,\bar v)\geq 0$ for every v is equivalent to $\lambda_j\geq 0$ for all j. Wedge products then give

$$\alpha^p=p!\,\mathrm{i}^{p^2}\sum_{|I|=p}\left(\prod_{j\in I}\lambda_j\right)\mathrm{d}z_I\wedge\mathrm{d}\bar z_I,$$

so powers of a positive $(1,1)$-form are strongly positive.

A current T of bidegree (p, p) is positive if $T(\eta) \geq 0$ for every compactly supported strongly positive test form of complementary bidegree. The complementary degree is essential: on an n-fold, T acts on $(n-p, n-p)$ forms. For an analytic set V of codimension p,

$$[V](\eta) = \int_{V_{\mathrm{Reg}}} \eta \geq 0$$

because restriction of a strongly positive form to the complex tangent space has the complex orientation and nonnegative density.

Checkpoint (degree and duality)

(1) On a threefold, which bidegree of test form pairs with a positive $(2, 2)$-current?
(2) If α has eigenvalues $1, 2, 0$, is it positive? Strictly positive?
(3) Why is the wedge of two positive $(1, 1)$-forms positive?

Solution (checkpoint). The complementary bidegree is $(1, 1)$. The form is positive semidefinite but not strictly positive because of the zero eigenvalue. Diagonal approximation or the decomposable-form definition shows that a wedge of positive $(1, 1)$-forms is strongly positive; continuity handles semidefinite limits. ⋄

Exit criterion

You can diagonalize a $(1, 1)$-form, test positivity through eigenvalues, identify the correct dual test cone for a current, and verify positivity of an integration current.

Summary. Strong, ordinary, and weak positivity differ in intermediate bidegrees. Positive currents are dual to strongly positive tests, have measure coefficients, and possess a natural mass. The geometric volume of an analytic set equals the trace mass of its integration current.

Exercise 4.6: Matrix criterion

Prove that the three positivity notions for a $(1, 1)$-form all agree with positive semidefiniteness of its coefficient matrix.

Exercise 4.7: Products of analytic sets

For analytic $Z_i \subset X_i$, prove $[Z_1 \times Z_2] = [Z_1] \times [Z_2]$ and compute the mass for a product metric.

Optional handout: Calibrations and complex submanifolds

This handout is optional. No later argument in the core lectures depends on it.

A closed p-form ϕ on a Riemannian manifold is a calibration if its restriction to every oriented unit p-plane is at most the Euclidean volume form. A submanifold is calibrated if equality holds on each tangent plane. Stokes' theorem then shows that it minimizes volume in its homology class.

On a Kähler manifold (X^n, ω), the form $\omega^p/p!$ calibrates complex p-dimensional submanifolds. Pointwise this is Wirtinger's inequality. If Z is complex and Z' is homologous to Z, then

$$\mathrm{Vol}(Z) = \int_Z \frac{\omega^p}{p!} = \int_{Z'} \frac{\omega^p}{p!} \leq \mathrm{Vol}(Z').$$

Thus complex submanifolds are automatically minimal, although generally not uniquely volume minimizing.

Example 4.8: Projective degree

For a p-dimensional projective variety $Z \subset \mathbb{P}^N$,

$$\deg Z = \int_Z \omega_{\mathrm{FS}}^p.$$

The algebraic degree is therefore a calibrated volume. Compactness theorems for analytic cycles can be viewed as compactness under a uniform calibrated-volume bound.

Positive currents generalize calibrated submanifolds: the trace mass $T \wedge \omega^p/p!$ retains the same volume control even when the geometric object is singular or appears only as a weak limit.

Exercise 4.9: Minimality

Use the first variation of volume, or the calibration argument, to show that a complex curve in a Kähler surface has zero mean curvature on its smooth locus.

4.2 Lecture 14: Closed positive currents, local potentials, and compactness

Week 7 material • 90 minutes

Notation for this lecture

T denotes a real closed positive $(1,1)$-current. On a coordinate ball, a local potential is a psh function u satisfying $T = \mathrm{dd^c}u$. A quasi-psh function is locally the sum of a psh and a smooth function. $T_j \rightharpoonup T$ means weak convergence against test forms, and σ_T is the trace or mass measure of T relative to the chosen Hermitian form. The support $\operatorname{supp} T$ is the complement of the largest open set on which T vanishes. An inequality $T \geq \gamma$ means that $T - \gamma$ is positive, where γ is a fixed smooth real $(1,1)$-form. For local regularization, ρ_ε is a mollifier and $u_\varepsilon = u * \rho_\varepsilon$.

Prerequisite results used below

Poincaré lemma for currents. On a ball, every closed current of positive degree is the differential of a current of one lower degree. The statement follows by dualizing the usual homotopy operator and is valid for currents of each fixed type after using the local Dolbeault lemma.

Compactness of normalized psh functions. Let u_j be psh on a domain $\Omega \subset \mathbb{C}^n$ and locally uniformly bounded above. Then either $u_j \to -\infty$ locally uniformly along a subsequence, or a subsequence converges in L^1_{loc} to a psh function. A normalization such as $\sup_{B'} u_j = 0$ rules out the first alternative.

Plan for the lecture. 0–20 min: local potentials; 20–42: weak convergence and mass bounds; 42–60: regularization; 60–78: support and dimension principles; 78–90: closure and examples.

Theorem 4.10: Local psh potentials

Every closed positive $(1,1)$-current is locally $T = \mathrm{dd^c}u$ for a psh function u. On a connected small neighborhood, u is unique up to a pluriharmonic distribution.

Proof. The distributional Poincaré lemma gives $T = \mathrm{d}S$. Type decomposition and the distributional Dolbeault lemma give $S^{0,1} = \bar\partial v$. Reality yields $T = \mathrm{i}\partial\bar\partial(2\operatorname{Im} v)$, with the constant adjusted to our $\mathrm{d^c}$ normalization. Positivity makes the complex Hessian of the resulting real distribution semipositive, so it has a unique psh upper-semicontinuous representative. □

Theorem 4.11: Weak compactness of positive currents

Let T_j be positive currents of fixed bidegree. If $\sup_j \|T_j\|_K < \infty$ for every compact K, then a subsequence converges weakly to a positive current T. If every T_j is closed, so is T.

Proof. The trace measures control the total variations of all coordinate coefficients. Apply Banach–Alaoglu on each member of a countable compact exhaustion and take a diagonal subsequence. Positivity is preserved on passing to limits of nonnegative tests, and differentiation is continuous in the distribution topology. □

For a locally uniformly bounded-above sequence of psh functions, normalize $\sup_K u_j = 0$. Hartogs' lemma and L^1_{loc} compactness give a subsequence that either tends to $-\infty$ or converges in L^1_{loc} to a psh function. In the second case $\mathrm{dd^c}u_j \rightharpoonup \mathrm{dd^c}u$.

Theorem 4.12: Support and dimension principles

Let T be a closed positive current.

(1) If T has bidimension (p, p) and $\operatorname{supp} T$ lies in an analytic set of dimension $< p$, then $T = 0$.

(2) If A is irreducible of pure dimension p and $\operatorname{supp} T \subset A$, then $T = c[A]$ for some $c \geq 0$.

Proof. On the regular stratum of the support, choose coordinates in which the analytic set is a coordinate plane. Positivity and the condition $\mathrm{d}T = 0$ force all coefficient measures in normal directions to vanish. In dimension exactly p, the only remaining component is a positive measure times the integration current of the plane; closedness forces its density to be locally constant. The lower-dimensional singular set carries no residual current by induction and the trace-mass estimate. Thus $T = c[A]$ on an irreducible p-fold, while no component is possible if the support dimension is smaller than p. For completeness, let $A = \{z_{p+1} = \cdots = z_n = 0\}$ on the regular stratum. Since $z_j T = 0$ for $j > p$, differentiation gives $\mathrm{d}z_j \wedge T = \mathrm{d}(z_j T) - z_j \mathrm{d}T = 0$. Positivity and polarization then force every coefficient containing a normal differential to vanish. If $\dim A < p$, no tangential coefficient of the required bidimension remains and $T = 0$. If $\dim A = p$, the only possible term is $\mu[A]$ for a positive measure density μ on A. The equation $\mathrm{d}T = 0$ says that every distributional derivative of μ along A is zero, so μ is constant on each connected piece of A_{Reg}.

The difference between T and these constant multiples is supported on A_{Sing}, whose dimension is smaller. Induction on dimension eliminates that residual current. If A is irreducible, its regular locus is connected, so only one constant occurs. This proves both assertions. □

The first statement forbids an area-type mass from being compressed into too small an analytic set; the second is the local rigidity behind Siu decomposition.

Proposition 4.13: A safe regularization statement

On a coordinate domain, convolution with a nonnegative radial mollifier turns a closed positive current into smooth closed positive forms on smaller domains, converging weakly to the original current. On a manifold, patched convolution introduces small errors in closedness or positivity; exponential maps, connections, or Demailly's regularization control them.

Proof. Write $T = \sum T_{I\bar{J}}\, \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$ and let ρ_ε be a nonnegative radial approximate identity. On the set whose distance from the boundary exceeds ε, define

$$T_\varepsilon = T * \rho_\varepsilon = \sum (T_{I\bar{J}} * \rho_\varepsilon)\, \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J.$$

Convolution commutes with constant-coefficient differentiation, so $dT_\varepsilon = (dT) * \rho_\varepsilon = 0$. For every strongly positive test form η at z, translation identifies $\langle T_\varepsilon(z), \eta\rangle$ with the average, using a nonnegative kernel, of the corresponding nonnegative pairings for T; hence $T_\varepsilon \geq 0$. Approximate-identity convergence gives $T_\varepsilon \rightharpoonup T$.

On a manifold, coordinate convolution must be multiplied by partitions of unity. Differentiating the partition functions produces commutator terms, and coordinate changes do not preserve the constant kernel. These are precisely the small errors mentioned in the statement; a connection or exponential-map construction estimates them by $O(\varepsilon)$ on compact sets. □

Warning

Partitions of unity do not commute with d. Patching local convolutions usually destroys both closedness and positivity. A global regularization theorem is valuable precisely because it quantifies these errors.

4.2.1 Compactness through normalized potentials

For $(1,1)$-currents one can see weak compactness directly at the level of potentials. On a ball B, write $T_j = dd^c u_j$ and normalize $\sup_{B'} u_j = 0$ on a smaller concentric ball B'. A mass bound for T_j controls the L^1 norm of u_j on still smaller balls by the submean inequality and the Riesz representation. The compactness theorem for psh functions gives either $u_j \to -\infty$ locally uniformly or a subsequence converging in L^1_{loc}. The normalization excludes the first alternative, and continuity of dd^c on distributions yields

$$T_j = dd^c u_j \rightharpoonup dd^c u.$$

This proof also records the local singularity data, which a coefficientwise Banach–Alaoglu argument does not.

Potentials are unique only modulo pluriharmonic functions. Normalizing by a supremum, an average, or a value at a nonsingular point chooses a representative. On overlaps, one must control the pluriharmonic differences before taking a global limit; compactness of the currents alone does not select global potentials.

4.2.2 Extension across a pluripolar set

An important complement to the support theorem is the Skoda–El Mir extension principle. Let $E \subset X$ be a closed complete pluripolar set and let T be a closed positive current on $X \setminus E$. If T has locally finite mass near E, define its trivial extension by

$$\langle \widetilde{T}, \eta \rangle = \int_{X \setminus E} T \wedge \eta.$$

Then $\widetilde{T}$ is again closed and positive on X.

The issue is closedness: differentiating a cutoff that vanishes near E produces a boundary error. Write locally $E \subset \{\psi = -\infty\}$ for a psh ψ and use convex cutoffs $\chi_\nu(\psi)$ that rise slowly from zero to one. Positivity bounds the error terms containing $dd^c \chi_\nu(\psi)$ by the mass of T in shrinking sublevel shells. Finite mass makes these errors tend to zero, proving $d\widetilde{T} = 0$.

For an analytic set A, apply this principle to $[A_{\mathrm{Reg}}]$. The singular set has smaller dimension and is complete pluripolar locally; local volume bounds provide the finite-mass hypothesis. One recovers that $[A]$ is a closed positive current without integrating across the singular locus directly.

Warning

Finite mass near the exceptional set is indispensable. Positivity alone does not prevent infinite concentration, and the trivial extension of a closed current can acquire a boundary term. Always distinguish extension as an order-zero current from extension as a *closed* current.

4.2.3 Constructing a local psh potential

Let T be a real closed $(1,1)$-current on a ball $B \subset \mathbb{C}^n$. The Poincaré lemma for currents first gives a real one-current A with $T = dA$. Decompose $A = A^{1,0} + A^{0,1}$. Since T has no $(2,0)$ or $(0,2)$ component,

$$\partial A^{1,0} = 0, \qquad \bar{\partial} A^{0,1} = 0.$$

After shrinking the ball, the Dolbeault lemma gives $A^{0,1} = \bar{\partial} v$; reality gives $A^{1,0} = \partial \bar{v}$. Thus

$$T = \partial \bar{\partial} v + \bar{\partial} \partial \bar{v} = \partial \bar{\partial} (v - \bar{v}).$$

Multiplying by the normalization constant produces a real distribution u such that $T = \mathrm{dd}^{\mathrm{c}} u$. If $T \geq 0$, then every complex-line restriction of u has positive distributional Laplacian. Its upper-semicontinuous representative is therefore psh.

Two potentials differ by a pluriharmonic function. Indeed, $\mathrm{dd}^{\mathrm{c}}(u - v) = 0$; locally $\partial(u - v)$ is a holomorphic closed one-form, hence the differential of a holomorphic function g, and $u - v = \Re g$+constant.

Weak compactness with a mass bound

Let T_j be positive currents of fixed bidegree on Ω and assume

$$\sup_j \int_K T_j \wedge \omega_0^p < \infty$$

for every $K \Subset \Omega$. The coefficient estimate from Lecture 13 bounds every coordinate measure on K. Banach–Alaoglu gives weak-* compactness of each coefficient measure. Choose a countable exhaustion by compact coordinate balls and diagonalize to obtain a subsequence converging on all compactly supported test forms.

The limit T is positive because $\langle T, \eta \rangle = \lim_j \langle T_j, \eta \rangle \geq 0$ for every strongly positive test form. If $\mathrm{d}T_j = 0$, then

$$\langle \mathrm{d}T, \eta \rangle = \pm \langle T, \mathrm{d}\eta \rangle = \pm \lim_j \langle T_j, \mathrm{d}\eta \rangle = 0.$$

Thus closedness also passes to the limit. The mass hypothesis is indispensable: $j\delta_0$ has no weakly convergent subsequence of finite currents.

Regularization in one coordinate ball

Extend a current T slightly beyond a relatively compact ball and convolve its coefficient measures with a standard kernel ρ_ε. For every test form η,

$$\langle T * \rho_\varepsilon, \eta \rangle = \langle T, \check{\rho}_\varepsilon * \eta \rangle.$$

Convolution commutes with d, ∂, and $\bar{\partial}$, so closedness is preserved. If T is positive, translations preserve positivity and the convolution is an average of positive translates, hence remains positive. Moreover $T * \rho_\varepsilon \to T$ weakly.

On a manifold, coordinate convolution does not agree on overlaps. A partition of unity introduces small negative errors in the derivatives; global regularization therefore generally produces

$$T_\varepsilon \geq -\varepsilon \omega$$

rather than exact positivity. Tracking this error is the key point in Demailly's regularization theorem.

A support-principle calculation

Let T be a closed current of bidegree (q, q) supported on a smooth complex submanifold Y of codimension r. In coordinates $Y = \{z_1 = \cdots = z_r = 0\}$, multiplication by each z_j annihilates an order-zero current supported on Y. Closedness then removes normal differential components. If $q < r$, there are not enough normal factors to support a nonzero current, so $T = 0$. If $q = r$ and T is positive, only the transverse top component remains and $T = c[Y]$ with $c \geq 0$ locally. Closedness forces c to be constant along each connected component. This is the local model behind the dimension and support principles.

4.2.4 Self-study workshop: constructing potentials and taking weak limits safely

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall the Poincaré lemma for currents, positivity of $\mathrm{dd^c}u$ for psh u, weak convergence against test forms, and compactness of bounded sets of measures on compact subsets.

Local potential construction. Let T be a real closed $(1,1)$-current on a coordinate ball. The current Poincaré lemma first writes $T = \mathrm{d}S$. Splitting S by type and using $\partial T = \bar\partial T = 0$, one solves a local $\bar\partial$ equation for the unwanted type components and obtains

$$T = \mathrm{dd^c}u$$

for a real distribution u. If $T \geq 0$, elliptic regularization and the mean inequality show that u has a psh representative. The potential is not unique: u may be changed by a pluriharmonic function. A normalization such as $\int_B u\,\mathrm{d}V = 0$ is therefore required before compactness of potentials can be claimed.

Now let $T_\nu \geq 0$ be closed and suppose their masses are uniformly bounded on every compact subset. In coordinates, positivity controls every coefficient measure by the trace measure

$$\sigma_{T_\nu} = T_\nu \wedge \omega_0^{n-1}.$$

Weak compactness of measures gives convergent subsequences of all coefficients. Positivity passes to the limit because evaluation on a fixed strongly positive test form is a nonnegative real number. Closedness passes because

$$(\mathrm{d}T_\nu)(\eta) = \pm T_\nu(\mathrm{d}\eta) = 0$$

and the right side is stable under weak convergence.

Checkpoint (what compactness really needs)

(1) Why does adding an arbitrary constant destroy compactness of a family of psh potentials but not change its currents?
(2) Which scalar measure controls the coefficient measures of a positive $(1,1)$-current?
(3) Why is a weak limit of positive closed currents again closed?

Solution (checkpoint). Constants disappear under $\mathrm{dd^c}$, so unnormalized potentials can drift to $+\infty$ or $-\infty$. The trace measure $T \wedge \omega_0^{n-1}$ controls the coefficients by positivity. Closedness follows by moving d to each fixed test form and passing to the limit. ⋄

Exit criterion

You can state the ambiguity of a local psh potential, choose a normalization, and give the coefficient-by-coefficient compactness argument for positive closed currents.

Summary. A closed positive $(1,1)$-current is locally the complex Hessian of a psh potential. Mass bounds give weak compactness, while support theorems give geometric rigidity. Together these tools control singular metrics and degenerate limits.

Exercise 4.14: Normalization and compactness

On the unit ball compare the normalized limits of $u_j = j^{-1}\log|z_1|$ and $v_j = j\log|z_1|$.

Exercise 4.15: Currents supported at a point

Classify closed positive (p, p)-currents in $\mathbb{C}^n$ supported at the origin. Which bidegrees can be nonzero?

Optional handout: Compactness of cycles and Bishop's theorem

This handout is optional. No later argument in the core lectures depends on it.

Weak compactness of positive currents has a geometric refinement. Bishop's compactness theorem states, in a local form, that a sequence of pure p-dimensional analytic sets with locally uniformly bounded volume has a subsequence converging to an analytic cycle. The limit is stronger than an arbitrary positive current: it has integer multiplicities.

The proof combines three ingredients. Wirtinger's theorem converts volume into the mass of integration currents. Weak compactness produces a closed positive limit. Generic linear projections and the area formula control the number of sheets away from a small branch set. Weierstrass preparation then shows that the limiting support is analytic and that sheet numbers give integer multiplicities.

Example 4.16: Graphs of holomorphic maps

Let $f_j : U \to V$ be holomorphic maps whose graphs have locally bounded volume in $U \times V$. A subsequence of the graphs converges to an analytic cycle. If the limit contains exactly one sheet over generic points of U and has no vertical component, it is the graph of a meromorphic map. This is a basic compactness mechanism in moduli problems.

Warning (Current limits need not remain smooth)

Smooth submanifolds can converge to a reducible or singular cycle, and multiplicity can appear. For example, the divisors $\{z^2 = \varepsilon_j\}$ converge as currents to $2[\{z = 0\}]$ as $\varepsilon_j \to 0$.

Exercise 4.17: Multiplicity in a limit

Test the currents $[z^m = \varepsilon]$ against a compactly supported $(n-1, n-1)$-form and verify convergence to $m[z = 0]$.

4.3 Lecture 15: The Bedford–Taylor Monge–Ampère operator

Week 8 material • 90 minutes

Notation for this lecture

$\Omega \subset \mathbb{C}^n$ is a domain, $u_1, \dots, u_q$ are locally bounded psh functions, and T is a closed positive current of the complementary bidegree. The Bedford–Taylor product is defined recursively by $\mathrm{dd^c} u_1 \wedge T = \mathrm{dd^c}(u_1 T)$ and iteration; it is not an a priori product of distributions. The Monge–Ampère measure is $\mathrm{MA}(u) = (\mathrm{dd^c} u)^n$. The notation $K \Subset \Omega$ means that K is compact and contained in Ω, and $\|u\|_{L^\infty(K)}$ is the essential supremum on K. Weak convergence of the resulting measures is denoted $\rightharpoonup$. CLN abbreviates the Chern–Levine–Nirenberg local mass estimate.

Prerequisite results used below

Integration by parts for currents. If S is a current and ϕ is a compactly supported smooth form of complementary degree, then $\langle \mathrm{d}S, \phi\rangle = (-1)^{\deg S+1}\langle S, \mathrm{d}\phi\rangle$. The analogous formulas for ∂, $\bar\partial$, and $\mathrm{dd^c}$ determine all signs in the Bedford–Taylor recursion.
Monotone convergence of measures. If positive Radon measures μ_j increase to μ on Borel sets, then $\int f\,\mathrm{d}\mu_j \to \int f\,\mathrm{d}\mu$ for every nonnegative measurable f. For decreasing locally bounded psh approximants, Bedford–Taylor continuity is a deeper analogue and is stated and proved in this lecture.

Plan for the lecture. 0–20 min: why currents cannot simply be multiplied; 20–45: recursive definition; 45–63: the CLN estimate; 63–78: monotone continuity; 78–90: comparison and capacity.

For $u \in C^2 \cap \mathrm{PSH}(\Omega)$,

$$(\mathrm{dd^c} u)^n = \frac{n!}{\pi^n}\det(u_{j\bar k})\,\mathrm{d}\lambda$$

under our normalization and Euclidean volume convention. If u is merely locally bounded, its Hessian coefficients are distributions and the determinant cannot be multiplied term by term.

Definition 4.18: Bedford–Taylor product

Let $u_1, \dots, u_q \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$ and let T be closed positive. Define recursively

$$\mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_q \wedge T := \mathrm{dd^c}\big(u_1\, \mathrm{dd^c} u_2 \wedge \cdots \wedge \mathrm{dd^c} u_q \wedge T\big).$$

Local boundedness makes multiplication of the measure coefficients by u_1 meaningful.

Theorem 4.19: Well-definedness and symmetry

The recursive product is a closed positive current, glues across coordinate patches, and is symmetric in $u_1, \dots, u_q$.

Proof. Induct on q. Put $R = \mathrm{dd^c} u_2 \wedge \cdots \wedge \mathrm{dd^c} u_q \wedge T$, already defined by induction. Because u_1 is locally bounded, $u_1 R$ is a current of order zero, so $\mathrm{dd^c}(u_1 R)$ is defined and is closed. Choose smooth psh functions $u_{j,\nu} \downarrow u_j$ on a smaller coordinate set. The Chern–Levine–Nirenberg estimate gives locally uniform mass bounds for

$$\mathrm{dd^c} u_{1,\nu} \wedge \cdots \wedge \mathrm{dd^c} u_{q,\nu} \wedge T.$$

Every subsequence therefore has a weakly convergent subsequence. If χ is a test form, repeated integration by parts moves one $\mathrm{dd^c}$ at a time from the approximants to χ. Bounded convergence with respect to the trace measures of the remaining positive currents shows that the limit is precisely $\mathrm{dd^c}(u_1 R)$. It follows at once that the limit is positive and independent of all regularizations.

For smooth potentials the factors commute. Apply the preceding limiting argument while interchanging two adjacent regularized factors; uniqueness of the limit shows that the nonsmooth product is symmetric. The recursive formula uses only $\mathrm{dd^c}u_j$ and multiplication by locally bounded functions; on overlaps, two local potentials differ by a pluriharmonic function, which contributes zero. The local currents therefore glue, completing the induction. □

Theorem 4.20: Chern–Levine–Nirenberg estimate

If $L \Subset K \Subset \Omega$, the u_j are bounded on K, and T is closed positive, then

$$\left\|\mathrm{dd^c}u_1 \wedge \cdots \wedge \mathrm{dd^c}u_q \wedge T\right\|_L \le C_{L,K} \prod_{j=1}^{q} \left\|u_j\right\|_{L^\infty(K)} \|T\|_K .$$

Proof. Write the bidimension of T as (p,p). It is enough to treat $q = 1$ and then iterate. After subtracting a constant, assume $u_1 \le 0$ on K. Choose $\chi \in C_c^\infty(K)$ with $0 \le \chi \le 1$ and $\chi = 1$ near L. Since T is closed,

$$\int_L \mathrm{dd^c}u_1 \wedge T \wedge \omega^{p-1} \le \int_K \chi\, \mathrm{dd^c}u_1 \wedge T \wedge \omega^{p-1} = \int_K u_1\, \mathrm{dd^c}\chi \wedge T \wedge \omega^{p-1}.$$

On the compact set K, choose C so that $-C\omega \le \mathrm{dd^c}\chi \le C\omega$. Positivity then gives an upper bound by $C\,\|u_1\|_{L^\infty(K)}\,\|T\|_K$. For several factors, choose nested compact sets $L = K_0 \Subset K_1 \Subset \cdots \Subset K_q = K$ and repeat the one-factor estimate, moving one $\mathrm{dd^c}$ at a time onto a cutoff. Multiplying the constants gives the stated bound. Approximation by smooth psh functions justifies the integration by parts for locally bounded potentials. □

Remark (Polarization)

Mixed Monge–Ampère terms can be recovered from diagonal ones by polarization. In degree two,

$$2\,\mathrm{dd^c}v \wedge \mathrm{dd^c}w = (\mathrm{dd^c}(v+w))^2 - (\mathrm{dd^c}v)^2 - (\mathrm{dd^c}w)^2.$$

Expanding the right-hand side is the safest way to keep the factor 2.

Theorem 4.21: Bedford–Taylor monotone continuity

If $u_j^{(k)} \downarrow u_j$ are locally bounded psh functions, then

$$\mathrm{dd^c}u_1^{(k)} \wedge \cdots \wedge \mathrm{dd^c}u_q^{(k)} \wedge T \rightharpoonup \mathrm{dd^c}u_1 \wedge \cdots \wedge \mathrm{dd^c}u_q \wedge T.$$

The corresponding statement with one undifferentiated potential in front also holds.

Proof. The proof is by induction on the number of factors. For one factor, $u^{(k)} \to u$ in L^1_{loc}, so $\mathrm{dd^c}u^{(k)} \to \mathrm{dd^c}u$ distributionally. Suppose continuity is known for $q-1$ factors and put

$$S_k = \mathrm{dd^c}u_2^{(k)} \wedge \cdots \wedge \mathrm{dd^c}u_q^{(k)} \wedge T, \qquad S = \mathrm{dd^c}u_2 \wedge \cdots \wedge \mathrm{dd^c}u_q \wedge T.$$

The induction hypothesis gives $S_k \rightharpoonup S$, and the CLN estimate gives uniform mass bounds on compact sets. For a test form χ,

$$\langle \mathrm{dd^c}u_1^{(k)} \wedge S_k, \chi\rangle = \langle u_1^{(k)} S_k, \mathrm{dd^c}\chi\rangle.$$

Bounded psh functions decreasing to u_1 converge in Bedford–Taylor capacity. The standard Bedford–Taylor capacity estimate (proved by repeated integration by parts, with the CLN estimate supplying the local mass bounds) makes the measures $S_k \wedge \omega^r$ uniformly absolutely continuous with respect to that capacity. Hence $u_1^{(k)} S_k \rightharpoonup u_1 S$. The displayed integration-by-parts identity proves the desired convergence. The same argument without applying the final $\mathrm{dd^c}$ proves the version with one undifferentiated potential. Localizing and using nested compact sets removes the temporary global boundedness assumption. □

Theorem 4.22: Local comparison principle

If $\Omega \Subset \mathbb{C}^n$ is a bounded domain, $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$, and $\liminf_{z\to\partial\Omega}(u - v) \geq 0$, then

$$\int_{\{u<v\}} (\mathrm{dd^c} v)^n \leq \int_{\{u<v\}} (\mathrm{dd^c} u)^n.$$

Proof. First suppose u, v are smooth and that $u \geq v$ near the boundary. For $\varepsilon > 0$, set $E_\varepsilon = \{u + \varepsilon < v\} \Subset \Omega$. Using

$$(\mathrm{dd^c} v)^n - (\mathrm{dd^c} u)^n = \mathrm{dd^c}(v - u) \wedge \sum_{j=0}^{n-1} (\mathrm{dd^c} v)^j \wedge (\mathrm{dd^c} u)^{n-1-j}$$

and Stokes' theorem on regular sublevel sets, the boundary sign gives $\int_{E_\varepsilon} (\mathrm{dd^c} v)^n \leq \int_{E_\varepsilon} (\mathrm{dd^c} u)^n$. Approximate an arbitrary level by regular values and let $\varepsilon \downarrow 0$. For bounded psh functions, use decreasing smooth regularizations on slightly smaller domains and Bedford–Taylor monotone continuity. □

This motivates the Bedford–Taylor capacity

$$\mathrm{Cap}(K, \Omega) = \sup \left\{ \int_K (\mathrm{dd^c} u)^n : u \in \mathrm{PSH}(\Omega),\ 0 \leq u \leq 1 \right\}.$$

With the appropriate regularity formulation, capacity-zero sets are exactly the locally pluripolar sets.

Warning

Local boundedness is a hypothesis, not a technical convenience. For $u = \log|z|$, a classical product $(\mathrm{dd^c} u)^n$ at the pole can depend on the approximation. Unbounded potentials require Cegrell classes or non-pluripolar products.

4.3.1 The determinant hidden in the wedge product

In complex dimension two, write at a point

$$\mathrm{dd^c} u = \frac{\mathrm{i}}{2\pi} \left(u_{1\bar{1}} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + u_{1\bar{2}} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2 + u_{2\bar{1}} \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_1 + u_{2\bar{2}} \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \right).$$

Only complementary indices survive after wedging, and reordering gives

$$(\mathrm{dd^c} u)^2 = \frac{2}{\pi^2} \left(u_{1\bar{1}} u_{2\bar{2}} - |u_{1\bar{2}}|^2 \right) \mathrm{d}\lambda.$$

The coefficient is nonnegative exactly when the Levi matrix is semipositive. In dimension n, antisymmetrization produces $n! \det(u_{j\bar{k}})$. Bedford–Taylor's definition preserves this geometric determinant while avoiding products of distributional coefficients.

If $u(z) = \sum_j \lambda_j |z_j|^2$, then

$$(\mathrm{dd^c} u)^n = \frac{n!}{\pi^n} (\lambda_1 \cdots \lambda_n) \mathrm{d}\lambda.$$

This model is useful for checking constants, homogeneity, and the effect of a degenerate Levi direction.

4.3.2 Locality, capacity, and uniqueness

The Bedford–Taylor product is local in the plurifine topology. One basic consequence is the maximum formula: on the open set $\{u > v\}$,

$$(\mathrm{dd^c} \max\{u, v\})^n = (\mathrm{dd^c} u)^n,$$

and similarly on $\{v > u\}$. Possible additional mass is concentrated on the contact set $\{u = v\}$. In one variable, $\max(\log|z|, 0)$ illustrates this phenomenon: it is harmonic on both sides of the unit circle and its Laplacian is carried by the contact circle.

Capacity is tailored to make monotone limits stable. If a sequence of uniformly bounded psh functions converges in Bedford–Taylor capacity, then outside a set of arbitrarily small capacity the convergence is uniform enough to pass products against the uniformly controlled Monge–Ampère measures. This is the nonlinear replacement for ordinary dominated convergence.

The comparison principle gives uniqueness for the Dirichlet problem. Suppose $u, v \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$ have the same boundary values and

$$(\mathrm{dd^c} u)^n = (\mathrm{dd^c} v)^n.$$

Apply comparison to u and $v + \varepsilon\rho$, where $\rho < 0$ is a strictly psh defining function. The strict extra mass rules out $u < v$ in the interior. Reversing the roles and letting $\varepsilon \downarrow 0$ gives $u = v$.

Remark (Global mass)

On a compact Kähler manifold, bounded ω-psh potentials satisfy

$$\int_X (\omega + \mathrm{dd^c} u)^n = \int_X \omega^n.$$

Approximate by smooth potentials and use Stokes term by term. The equality can fail for more singular products unless the chosen definition retains full Monge–Ampère mass.

4.3.3 The Bedford–Taylor recursion with the integration step shown

Let $u_1, \ldots, u_k$ be locally bounded psh functions. Starting from a closed positive current T, define recursively

$$\mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_k \wedge T := \mathrm{dd^c}\left(u_1\, ddcu_2 \wedge \cdots \wedge \mathrm{dd^c} u_k \wedge T\right).$$

The product $u_1 S$ is meaningful because S has measure coefficients and u_1 is locally bounded. For a compactly supported test form η,

$$\langle \mathrm{dd^c}(u_1 S), \eta\rangle = \langle u_1 S, \mathrm{dd^c}\eta\rangle.$$

This formula is the definition; no product of distributional second derivatives is taken.

To see symmetry for two functions, first take smooth approximants. Closedness of T and integration by parts give

$$\int u\, \mathrm{dd^c} v \wedge T \wedge \eta = \int v\, \mathrm{dd^c} u \wedge T \wedge \eta$$

when η is constant on the relevant compact set; cutoff-error terms are retained and then sent to zero. Decrease smooth psh approximants to u and v and use the Chern–Levine–Nirenberg mass bound to pass to the limit. This proves that the recursion is symmetric and independent of the approximation.

The smooth determinant and its normalization

For $u \in C^2$ psh on $\mathbb{C}^n$,

$$\mathrm{dd^c}u = \frac{\mathrm{i}}{2\pi}\sum_{j,k} u_{j\bar{k}}\,\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k.$$

Diagonalize H_u at a point with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$(\mathrm{dd^c}u)^n = n!\left(\prod_j \lambda_j\right)\bigwedge_{j=1}^{n}\frac{\mathrm{i}}{2\pi}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j = \frac{n!}{\pi^n}\det(H_u)\,\mathrm{d}V.$$

For $u = |z|^2$, this gives $(\mathrm{dd^c}|z|^2)^n = n!\pi^{-n}\mathrm{d}V$. Integrating over the Euclidean unit ball, whose volume is $\pi^n/n!$, gives total mass one.

The Monge–Ampère mass of a radial maximum

In one variable let

$$u_R(z) = \max\{\log|z|^2, \log R^2\}.$$

The function is constant for $|z| < R$ and harmonic for $|z| > R$. Its outward radial derivative jumps from 0 to $2/R$ at the circle. Equivalently, $\mathrm{d^c}\log|z|^2 = (2\pi)^{-1}\mathrm{d}\theta$ outside. Stokes therefore gives

$$\int_{\mathbb{C}} \chi\,\mathrm{dd^c}u_R = \frac{1}{2\pi}\int_0^{2\pi}\chi(Re^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

Thus $\mathrm{dd^c}u_R$ is normalized arclength measure on $|z| = R$, with total mass one. The mass sits precisely on the contact set where the two psh branches meet.

The comparison principle in one dimension

Let u, v be bounded subharmonic functions on a disc with $\liminf_{z\to\partial D}(u - v) \geq 0$. For $\varepsilon > 0$ set $E_\varepsilon = \{u < v - \varepsilon\} \Subset D$. Approximate the maximum $w = \max\{u, v - \varepsilon\}$ from above by smooth subharmonic functions. Green's formula on E_ε gives

$$\int_{E_\varepsilon}\mathrm{dd^c}v \leq \int_{E_\varepsilon}\mathrm{dd^c}u.$$

Indeed, on its boundary the two functions agree and the outward normal derivative of the upper envelope cannot jump downward. Letting $\varepsilon \downarrow 0$ and using monotone convergence yields

$$\int_{\{u<v\}}\mathrm{dd^c}v \leq \int_{\{u<v\}}\mathrm{dd^c}u.$$

Bedford–Taylor's higher-dimensional proof repeats this maximum construction with mixed products and uses the recursive integration-by-parts definition at every step.

4.3.4 Self-study workshop: defining Monge–Ampère products by recursion

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall locally bounded psh functions, positive currents, weak convergence, and integration by parts for currents. Products of arbitrary distributions are not defined, so every wedge product here needs a construction.

The recursive definition and its estimate. Let $u_1, \dots, u_q$ be locally bounded psh functions and let T be a closed positive current. Assuming

$$S = \mathrm{dd^c} u_2 \wedge \cdots \wedge \mathrm{dd^c} u_q \wedge T$$

has already been defined, set

$$\mathrm{dd^c} u_1 \wedge S := \mathrm{dd^c}(u_1 S).$$

Local boundedness makes $u_1 S$ a current of order zero. Closedness of S ensures that the definition agrees with the classical wedge product when the potentials are smooth. For a test form η,

$$\langle \mathrm{dd^c}(u_1 S), \eta \rangle = \langle u_1 S, \mathrm{dd^c} \eta \rangle,$$

up to the fixed degree convention. This formula supplies the local mass estimates needed for continuity.

If $u_j^{(\nu)} \downarrow u_j$, Bedford–Taylor monotone convergence states that the corresponding products converge weakly. The order of recursion does not matter; the proof regularizes, uses symmetry in the smooth case, and passes to the limit with Chern–Levine–Nirenberg bounds. None of these statements follows from formal commutativity of symbols.

For smooth u, the top power is a determinant:

$$(\mathrm{dd^c} u)^n = C_n \det(u_{j\bar{k}})\, \mathrm{d}V,$$

where C_n is fixed by the course normalization. Positivity of the Hessian guarantees that this is a positive measure.

Checkpoint (why boundedness appears)

(1) Where is local boundedness used in the recursive definition?
(2) Why can one not simply multiply the coefficient distributions of two currents?
(3) What convergence direction is built into the basic Bedford–Taylor theorem?

Solution (checkpoint). It makes the product $u_1 S$ well defined because S has measure coefficients. Products of distributions have no canonical definition and can depend on regularization. The basic theorem uses decreasing sequences of locally bounded psh functions; more general convergence requires capacity or energy hypotheses. ◇

Exit criterion

You can write the recursion on a test form, identify the role of boundedness and closedness, and explain why monotone convergence is a theorem rather than a formal limit operation.

Summary. Bedford–Taylor defines Monge–Ampère products by recursive integration by parts, not multiplication of distributions. CLN gives compactness, decreasing continuity legitimizes approximation, and comparison gives uniqueness.

Exercise 4.23: Monge–Ampère of a maximum

Compute $\mathrm{dd^c} \max(\log|z|, 0)$ on $\mathbb{C}$ and locate its mass.

Exercise 4.24: Comparison in one dimension

Reduce the local comparison principle for $n = 1$ to Green's formula and complete the proof.

Optional handout: Finite-energy classes in pluripotential theory

This handout is optional. No later argument in the core lectures depends on it.

Bedford–Taylor theory treats locally bounded psh functions. On a compact Kähler manifold, many natural solutions are unbounded, so one uses non-pluripolar products and energy classes. For ω-psh u, define truncations $u_j = \max(u, -j)$. The non-pluripolar Monge–Ampère measure is

$$\mathrm{MA}(u) = \lim_{j\to\infty} 1_{\{u>-j\}}(\omega + \mathrm{dd^c} u_j)^n.$$

It never charges the pole set $\{u = -\infty\}$.

The full-mass class $\mathcal{E}(X,\omega)$ consists of those u for which $\int_X \mathrm{MA}(u) = \int_X \omega^n$. The finite-energy class $\mathcal{E}^1$ additionally requires

$$\int_X |u|\ \mathrm{MA}(u) < \infty$$

after a normalization such as $\sup_X u = 0$. The Aubin–Mabuchi energy extends to $\mathcal{E}^1$ and makes the complex Monge–Ampère equation amenable to variational methods.

Remark (Variational principle)

To solve $\mathrm{MA}(u) = \mu$, one may maximize $E(u) - \int_X u\,\mathrm{d}\mu$ on $\mathcal{E}^1$. Concavity of E, compactness of normalized sublevel sets, and a finite-energy condition on μ give existence; the comparison principle gives uniqueness up to constants.

This framework contains singular Kähler–Einstein metrics and geodesics in the space of Kähler potentials. It is genuinely beyond the locally bounded theory but preserves its two central principles: monotone approximation and comparison.

Exercise 4.25: Bounded potentials

Show that every bounded ω-psh function belongs to $\mathcal{E}^1(X,\omega)$.

4.4 Lecture 16: Lelong numbers, Jensen's formula, and Siu's theorem

Week 8 material • 90 minutes

Notation for this lecture

T is a closed positive current of bidimension (p,p) near a point a, $\beta_a = \mathrm{dd^c}|z-a|^2$, and $\Theta_T(a,r) = r^{-2p}\int_{B(a,r)} T\wedge\beta_a^p$ is its normalized mass at scale r. Its Lelong number is $\nu(T,a) = \lim_{r\downarrow 0}\Theta_T(a,r)$, with the normalization stated in the definition below. For a psh function u we write $\nu(u,a) = \nu(\mathrm{dd^c}u,a)$, and $E_c(T) = \{x : \nu(T,x)\geq c\}$ is the upper level set. In the Siu decomposition $T = R + \sum_j \lambda_j[D_j]$, the D_j are prime divisors, λ_j their generic Lelong numbers, and R the residual positive current with no divisorial generic Lelong number.

Prerequisite results used below

Poincaré–Lelong formula. For a nonzero meromorphic function f, $\mathrm{dd^c}\log|f|^2 = [\mathrm{div}(f)]$ with the normalization of these notes. In particular, if $f(z) = z^m g(z)$ and $g(0)\neq 0$, then $\nu(\mathrm{dd^c}\log|f|^2, 0) = m$.
Jensen formula. If f is holomorphic on a neighborhood of $\overline{D(0,R)}$, $f(0)\neq 0$, and a_k are its zeros counted with multiplicity, then

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{\mathrm{i}\theta})|\,\mathrm{d}\theta - \sum_{|a_k|<R}\log\frac{R}{|a_k|}.$$

Applied to local potentials, this identity gives the monotonicity of normalized masses used to define Lelong numbers.

Plan for the lecture. 0–22 min: the mass definition of Lelong number; 22–42: psh slopes and Jensen–Lelong; 42–60: comparison; 60–78: Siu analyticity; 78–90: decomposition and midterm review.

Let T be a closed positive current in $\mathbb{C}^n$ of bidimension (p,p) and put $\beta = \mathrm{dd^c}|z|^2$.

Definition 4.26: Lelong number

The Lelong number of T at a is

$$\nu(T,a) = \lim_{r\downarrow 0}\frac{1}{r^{2p}}\int_{B(a,r)} T\wedge\beta^p,$$

normalized by $\nu([\mathbb{C}^p],0) = 1$. The Jensen–Lelong formula guarantees that the limit exists.

Theorem 4.27: Slope characterization for psh functions

For psh u,

$$\nu(u,a) := \nu(\mathrm{dd^c}u,a) = \liminf_{z\to a}\frac{u(z)}{\log|z-a|^2} = \lim_{r\downarrow 0}\frac{\sup_{|z-a|=r}u(z)}{\log r^2}.$$

Proof. Translate a to the origin. Let $M(r)$ be the spherical mean of u and $\mu = \mathrm{dd^c}u$. Jensen's formula, first for smooth u and then by convolution, gives

$$M(r) - M(r_0) = 2\int_{r_0}^{r}\left(\frac{1}{t^{2n-2}}\int_{B(0,t)}\mu\wedge\beta^{n-1}\right)\frac{\mathrm{d}t}{t}$$

with the constants corresponding to our convention $dd^c \log|z|^2 = \delta_0$. The normalized mass in the integrand is nondecreasing and tends to $\nu(u,0)$. Dividing by $\log r^2$ and applying the elementary Cesàro lemma therefore gives $M(r)/\log r^2 \to \nu(u,0)$.

The submean inequality gives $u(z) \le M_z(2|z|)$, while the three-circles convexity of circular maxima bounds the spherical supremum between means at comparable radii. Additive bounded errors vanish after division by $\log r^2$. Consequently the mean, the spherical supremum, and the pointwise lower logarithmic slope have the same limit. This proves all displayed equalities. □

Example 4.28: Analytic singularities

If $u = c\log\sum_{j=1}^N |f_j|^2 + O(1)$, then $\nu(u,a) = c\min_j \operatorname{ord}_a(f_j)$ in the $\log|z|^2$ convention. In particular, $\nu(\log|f|^2, a) = \operatorname{ord}_a(f)$.

Theorem 4.29: Comparison of singularities

If $u \le v + O(1)$ near a, then $\nu(u,a) \ge \nu(v,a)$: the more negative singularity has the larger Lelong number. Mixed Monge–Ampère comparison generalizes this relation to mass inequalities.

Proof. Choose C with $u \le v + C$ near a. On the sphere of radius r, $\sup u \le \sup v + C$. Since $\log r^2 < 0$, division reverses the inequality:

$$\frac{\sup u}{\log r^2} \ge \frac{\sup v}{\log r^2} + \frac{C}{\log r^2}.$$

Letting $r \downarrow 0$ and using the slope characterization makes the last term vanish and yields $\nu(u,a) \ge \nu(v,a)$. □

Warning

Because $\log r^2 < 0$, dividing an inequality by $\log r^2$ reverses its direction. Also, Lelong number zero does not imply local boundedness; for example $u = -\sqrt{-\log|z|}$ is still singular.

Theorem 4.30: Siu's analyticity theorem

For a closed positive current T and $c > 0$, the upper level set

$$E_c(T) = \{x \in X : \nu(T,x) \ge c\}$$

is analytic; if T has bidimension (p,p), then $\dim E_c(T) \le p$. Thus $x \mapsto \nu(T,x)$ has a strong form of upper semicontinuity in the analytic Zariski topology.

Proof (detailed $(1,1)$ proof and the general projection architecture). We first treat a $(1,1)$-current locally written $T = dd^c u$. For $m \ge 1$, let $\mathcal{H}_m$ be the Hilbert space of holomorphic functions with $\int |f|^2 e^{-2mu} < \infty$, choose an orthonormal basis $(g_{m,j})$, and set

$$u_m = \frac{1}{2m}\log\sum_j |g_{m,j}|^2.$$

The submean inequality gives the upper estimate

$$u_m(z) \le \sup_{B(z,r)} u + \frac{1}{m}\log\frac{C}{r^n}.$$

Conversely, solve a weighted $\bar\partial$ equation for a cutoff of the constant function, using the weight $2mu + 2n\log|\zeta - z|$. The solution has zero value at z and the estimate supplies a holomorphic g with

$g(z) = 1$ and controlled norm. The extremal characterization of the Bergman kernel then yields

$$u(z) - \frac{C}{m} \leq u_m(z).$$

Taking logarithmic slopes in the two estimates gives

$$\nu(u, z) - \frac{n}{m} \leq \nu(u_m, z) \leq \nu(u, z).$$

The function u_m has analytic singularities: locally its singularity is defined by the coherent ideal generated by the $g_{m,j}$. A threshold set of its Lelong numbers is described by finitely many jet-vanishing conditions and is therefore analytic. The two inequalities give the exact identity

$$E_c(u) = \bigcap_{m \geq m_0} E_{c-n/m}(u_m).$$

For $N \geq m_0$, let $A_N = \bigcap_{m_0 \leq m \leq N} E_{c-n/m}(u_m)$. The A_N are analytic and form a decreasing sequence. The ascending chain of their ideal sheaves stabilizes locally by Noetherianity, so $\bigcap_N A_N$ is analytic. This proves the theorem for $(1, 1)$ currents.

For a positive current of arbitrary bidimension, one uses Lelong's projection lemma. After restricting to a small ball, finitely many generic linear projections to $(p + 1)$-dimensional spaces turn the relevant localized pushforwards of a bidimension (p, p) current into positive $(1, 1)$-currents; their Lelong inequalities recover $E_c(T)$ as a finite intersection of the $(1, 1)$ upper-level sets, after harmless changes of threshold. The same projection lemma gives $\dim E_c(T) \leq p$. Applying the result just proved to those pushforwards yields local analytic equations for $E_c(T)$. Since Lelong numbers are intrinsic, these local sets agree on overlaps. This projection lemma is the deep reduction in the general-current proof; the Bergman argument above is the analytic core after the reduction. □

Theorem 4.31: Siu decomposition

If T is a closed positive current of bidimension (p, p), then uniquely

$$T = \sum_j \lambda_j [A_j] + R,$$

where the A_j are irreducible p-dimensional analytic sets and $\lambda_j > 0$ are the generic Lelong numbers along them. For every $c > 0$, $E_c(R)$ has dimension $< p$.

Proof. For each rational $c > 0$, Siu's theorem makes $E_c(T)$ analytic. List all its p-dimensional irreducible components as c varies; this is a countable family (A_j). The Lelong number is constant outside a proper analytic subset of each A_j; call the generic value λ_j.

At a regular generic point of A_j, slice transversely and apply the one-variable Jensen formula. It shows that $T - \lambda_j [A_j]$ remains positive there. The support principle extends this inequality across the exceptional subset. Subtract finitely many such terms. Positivity bounds the sum of their masses on each compact set by the mass of T, so the increasing family of partial sums converges to a locally finite positive current. Thus

$$R = T - \sum_j \lambda_j [A_j] \geq 0$$

is closed.

If $E_c(R)$ had a p-dimensional component, its generic Lelong number would have been one of the λ_j and would already have been removed. Therefore $\dim E_c(R) < p$. Finally, the support principle says that the coefficient of $[A_j]$ in any positive decomposition is its generic Lelong number along A_j. Hence both the coefficients and the residual current are unique. □

4.4.1 Monotonicity and density

For a closed positive current T of bidimension (p,p) in a Euclidean ball, set

$$\Theta_T(a,r) = \frac{1}{r^{2p}} \int_{B(a,r)} T \wedge (dd^c|z-a|^2)^p.$$

The Jensen–Lelong formula shows that $r \mapsto \Theta_T(a,r)$ is nondecreasing. The Lelong number is its limit at zero, so it is an infinitesimal density rather than an arbitrary singularity index. For a smooth positive form the numerator is $O(r^{2n})$ in the corresponding degree and the Lelong number vanishes; for the integration current of a smooth p-fold the numerator is exactly $r^{2p} + o(r^{2p})$, giving density one.

If an analytic set A has multiplicity m at a, then

$$\nu([A],a) = m.$$

One way to see this is to project generically to a p-plane. A nearby regular value has m preimages counted with multiplicity, and the normalized volume ratio tends to that degree. Thus the analytic multiplicity and the metric density are the same integer.

4.4.2 What the number sees and what it misses

For an ideal $\mathfrak{a} = (f_1,\dots,f_N)$ and $u = c\log\sum|f_j|^2 + O(1)$, the Lelong number is

$$\nu(u,a) = c\ \mathrm{ord}_a(\mathfrak{a}), \qquad \mathrm{ord}_a(\mathfrak{a}) = \min_j \mathrm{ord}_a(f_j).$$

It records only the least vanishing order. Different ideals may therefore have the same Lelong number while having very different zero schemes or integral closures.

The example

$$u(z) = -\sqrt{-\log|z|}$$

near the origin is unbounded but satisfies $u(z)/\log|z|^2 \to 0$; hence its Lelong number is zero. Lelong numbers detect logarithmic poles, not every psh singularity. Multiplier ideals and non-pluripolar products will retain finer integrability and mass information.

For a positive closed $(1,1)$-current, Siu decomposition takes the especially concrete form

$$T = \sum_D \nu(T,D)[D] + R,$$

where D runs over prime divisors and R has zero generic Lelong number along every divisor. The residual current may still have positive Lelong numbers on subsets of codimension at least two. Thus "residual" means no divisorial component, not smooth or nonsingular.

Remark (Blowing up a point)

Pulling a psh singularity to the blow-up separates radial order from directional behavior. The generic Lelong number along the exceptional divisor equals the Lelong number at the center, while exceptional points can carry larger directional singularities. This is a first glimpse of how modifications resolve analytic singularities.

4.4.3 Checking the normalization on a linear space

Let $L = \mathbb{C}^p \times \{0\} \subset \mathbb{C}^n$ and $\beta = \mathrm{dd^c}|z|^2$. Restriction to L gives

$$\beta|_L = \frac{1}{\pi}\sum_{j=1}^{p} \mathrm{d}x_j \wedge \mathrm{d}y_j, \qquad \beta^p|_L = \frac{p!}{\pi^p}\,\mathrm{d}V_{\mathbb{R}^{2p}}.$$

Since $\mathrm{Vol}(B_r^{2p}) = \pi^p r^{2p}/p!$,

$$\frac{1}{r^{2p}}\int_{B_r} [L] \wedge \beta^p = 1.$$

This verifies the normalization $\nu([L], 0) = 1$ directly rather than by declaration.

For an analytic curve C through 0, a generic linear projection to one coordinate disc has degree equal to the multiplicity $m_0(C)$. Away from its branch values, the curve consists of $m_0(C)$ graphs, each asymptotic to a complex line. The area formula and the preceding normalization give

$$\nu([C], 0) = m_0(C).$$

For the cusp $(x, y) = (t^2, t^3)$, a generic projection has degree two, hence $\nu([C], 0) = 2$.

Analytic singularities and vanishing order

Let $u = c \log \sum_{j=1}^{N} |f_j|^2 + O(1)$ and put $m = \min_j \mathrm{ord}_0 f_j$. Taylor estimates give constants $C_1, C_2 > 0$ and, after choosing a direction on which one initial form does not vanish,

$$\sum_j |f_j(z)|^2 \le C_1|z|^{2m}, \qquad \sup_{|z|=r} \sum_j |f_j(z)|^2 \ge C_2 r^{2m}.$$

Taking logarithms shows

$$\sup_{|z|=r} u(z) = cm \log r^2 + O(1).$$

Dividing by $\log r^2$ and using the slope characterization yields $\nu(u, 0) = cm$.

The converse implication is false: zero Lelong number does not mean boundedness. For $0 < |z| < e^{-1}$ let

$$u(z) = -\sqrt{-\log|z|^2}.$$

As a convex increasing function of $\log|z|^2$ near $-\infty$, it is subharmonic and tends to $-\infty$. Yet

$$\frac{u(z)}{\log|z|^2} = \frac{1}{\sqrt{-\log|z|^2}} \longrightarrow 0,$$

so $\nu(u, 0) = 0$.

A Siu decomposition one can read off

On $\mathbb{C}^2$ take

$$u = \log|z_1^2 z_2^3|^2 + |z|^2.$$

Poincaré–Lelong and smoothness of the last term give

$$\mathrm{dd^c}u = 2[z_1 = 0] + 3[z_2 = 0] + \mathrm{dd^c}|z|^2.$$

At a generic point of $\{z_1 = 0\}$ the Lelong number is 2; at a generic point of $\{z_2 = 0\}$ it is 3; at the origin it is 5. The Siu decomposition extracts the two divisorial currents with coefficients 2 and 3, leaving the smooth residual current. Accordingly,

$$E_c(dd^c u) = \begin{cases} \{z_1 z_2 = 0\}, & 0 < c \leq 2, \\ \{z_2 = 0\}, & 2 < c \leq 3, \\ \{0\}, & 3 < c \leq 5, \\ \varnothing, & c > 5. \end{cases}$$

Every upper-level set is analytic, and the jump at the intersection illustrates why the residual condition is stated in terms of dimension rather than pointwise zero Lelong number.

Monotonicity gives existence of the density

For a closed positive current T of bidimension (p, p) centered at the origin, the Jensen–Lelong formula shows that

$$\theta_T(r) = r^{-2p} \int_{B_r} T \wedge \beta^p$$

is nondecreasing in r. Hence $\lim_{r \downarrow 0} \theta_T(r)$ exists as the infimum of the finite-radius densities. Closedness removes interior derivative terms in Stokes' formula, while positivity makes the remaining annular integral nonnegative. Both hypotheses are essential: without positivity the density can oscillate, and without closedness there is an uncontrolled error term.

4.4.4 Self-study workshop: extracting multiplicity from density

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall positive closed currents, Poincaré–Lelong, monotonicity of normalized masses, and orders of vanishing of holomorphic tuples.

The scale-invariant quantity. For a positive closed current T of bidimension (p, p) near 0, the Lelong number is the limit of a normalized mass

$$\nu(T, 0) = \lim_{r \downarrow 0} \frac{1}{r^{2p}} \int_{B(0,r)} T \wedge (dd^c |z|^2)^p,$$

with the dimensional constant determined by the convention ledger. The exponent $2p$ is forced by real scaling on a complex p-plane. The Jensen–Lelong formula proves monotonicity, so the limit exists.

For a psh function with analytic singularities,

$$u = \frac{1}{2} \log\big(|f_1|^2 + \cdots + |f_N|^2\big) + O(1),$$

the Lelong number at the origin is the minimum vanishing order of the tuple. For

$$u(z_1, z_2) = \frac{1}{2} \log(|z_1|^4 + |z_2|^6)$$

the generic leading order is two, hence $\nu(u, 0) = 2$. Along the special line $z_1 = 0$ the restriction has order three, but the ambient Lelong number records the least radial growth, not the largest order on a special direction.

For an analytic cycle, the density equals algebraic multiplicity. Thus Lelong numbers translate analytic concentration into local intersection data. Siu's theorem adds a global rigidity statement: for $c > 0$, the upper level set $\{x : \nu(T, x) \geq c\}$ is analytic. Upper semicontinuity here is in the analytic Zariski sense, which is much stronger than ordinary topological semicontinuity.

Checkpoint (calculate before quoting Siu)

(1) What is $\nu(\mathrm{dd^c} \log |z^m|^2, 0)$ in one variable?
(2) Why does adding a bounded function not change a Lelong number?
(3) For the analytic singularity above, why does the order-three axis not change the ambient value two?

Solution (checkpoint). Poincaré–Lelong gives $m\delta_0$, whose density is m. A bounded summand is negligible after division by $\log |z|$ and has no point mass under the density limit. The Lelong number measures the uniform leading radial rate; generic directions with $z_1 \neq 0$ already realize order two, while extra vanishing on one direction only makes the singularity stronger there. ⋄

Exit criterion

You can recover the scaling exponent in the density, compute Lelong numbers for divisors and analytic singularities, and state precisely what Siu's analyticity theorem adds.

Summary. The Lelong number is the normalized infinitesimal density of a positive current and the logarithmic slope of a psh singularity. Siu's theorem makes its threshold sets analytic, and Siu decomposition extracts all pure-dimensional analytic mass.

Exercise 4.32: Lelong number of an integration current

Prove $\nu([A], x) = 1$ at a smooth point. Compute the Lelong number at the origin for $A = \{y^2 = x^3\}$.

Exercise 4.33: A singularity with zero Lelong number

Verify that $u(z) = -(-\log |z|)^\alpha$, $0 < \alpha < 1$, is psh on a small punctured disc with its natural extension, and compute its Lelong number at the origin.

Optional handout: Tangent cones, multiplicity, and Lelong numbers

This handout is optional. No later argument in the core lectures depends on it.

Let $A \subset \mathbb{C}^n$ be a pure p-dimensional analytic set through the origin. Rescale A by $z \mapsto z/r$. Along a sequence $r_j \downarrow 0$, the integration currents have locally bounded mass and converge to a positive conic current. The resulting analytic cycle is the tangent cone $C_0 A$.

The Lelong number $\nu([A], 0)$ equals the algebraic multiplicity of A at the origin. One way to see this is to choose a generic projection $\pi : \mathbb{C}^n \to \mathbb{C}^p$. Near the origin, A is a branched cover of a ball in $\mathbb{C}^p$; the number of sheets, counted with multiplicity, is constant off the branch locus and equals both the Hilbert–Samuel multiplicity and the normalized density of $[A]$.

Example 4.34: The cusp

For $A = \{y^2 = x^3\}$, the lowest homogeneous term is y^2. Hence the tangent cone is the x-axis with multiplicity two, and $\nu([A], 0) = 2$. The set-theoretic tangent line alone would miss this multiplicity.

For a general positive current, tangent currents need not be unique. The Lelong number is nevertheless unique because it records only the normalized radial mass. This is one reason Lelong numbers are robust invariants even when the full tangent geometry is not.

Exercise 4.35: A plane branch

For the curve parametrized by $t \mapsto (t^m, t^k)$ with $m < k$ and $\gcd(m, k) = 1$, determine its tangent cone and its Lelong number at the origin.

Chapter 5

Hermitian Vector Bundles, Chern Curvature, and Positivity

5.1 Lecture 17: Vector bundles, connections, and Chern–Weil theory

Week 9 material • 90 minutes

Notation for this lecture

$E \to X$ is a complex vector bundle of rank r, $\mathcal{A}^k(E)$ the space of smooth E-valued k-forms, and $D : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$ a connection. In a local frame $e = (e_1, \dots, e_r)$, a section is $s = e\sigma$ with a column of coefficients σ, and $Ds = e(\mathrm{d}\sigma + A\sigma)$ defines the matrix-valued connection one-form A. Under a frame change $e' = eg$, the gauge matrix is $g : U \to \mathrm{GL}(r, \mathbb{C})$. The curvature is $\Theta_D = D^2 = \mathrm{d}A + A \wedge A$, $D_{\mathrm{End}\,E}$ is the induced connection on endomorphisms, tr is matrix trace, and $c_k(E)$ and $\mathrm{ch}(E)$ denote the Chern classes and Chern character represented by Chern–Weil forms.

Prerequisite results used below

Existence of connections. Every smooth vector bundle over a paracompact manifold admits a connection. Indeed, local trivial connections can be combined with a smooth partition of unity because the Leibniz rule is preserved by an affine combination.
Covariant exterior derivative. A connection extends uniquely to $D : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$ by $D(\alpha \otimes s) = \mathrm{d}\alpha \otimes s + (-1)^k \alpha \wedge Ds$. Its square is $\mathcal{A}^\bullet$-linear, hence is multiplication by an $\mathrm{End}\,E$-valued two-form Θ_D. The induced connection on $\mathrm{End}\,E$ satisfies the Bianchi identity $D_{\mathrm{End}\,E}\Theta_D = 0$.

Plan for the lecture. 0–20 min: bundles and transition functions; 20–45: connections and local matrices; 45–65: curvature and Bianchi; 65–82: Chern–Weil theory; 82–90: preview of line bundles.

Definition 5.1: Complex vector bundle

A rank-r complex vector bundle $E \to X$ is locally $U_\alpha \times \mathbb{C}^r$, with frames related by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$ satisfying $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$. It is holomorphic when the transition functions are holomorphic.

The sheaf $\mathcal{O}(E)$ of holomorphic sections is a locally free $\mathcal{O}_X$-module. Conversely, local bases of a locally free sheaf reconstruct the bundle. This is the basic dictionary between geometry and sheaf theory.

Definition 5.2: Connection

A connection on a smooth complex bundle is a $\mathbb{C}$-linear operator

$$D : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^1(E), \qquad D(fs) = \mathrm{d}f \otimes s + fDs.$$

It extends uniquely to E-valued forms by the graded Leibniz rule.

In a local frame $e = (e_1, \ldots, e_r)$, write $s = e\sigma$. Then

$$Ds = e(\mathrm{d}\sigma + A\sigma), \qquad A \in \mathcal{A}^1(U, \operatorname{End} \mathbb{C}^r).$$

Under $e' = eg$,

$$A' = g^{-1}Ag + g^{-1}\mathrm{d}g.$$

The inhomogeneous term shows that A is not a tensor.

Definition 5.3: Curvature

The curvature is

$$\Theta_D = D^2 \in \mathcal{A}^2(X, \operatorname{End} E), \qquad \Theta = \mathrm{d}A + A \wedge A.$$

It transforms as $\Theta' = g^{-1}\Theta g$ and hence is tensorial.

Proposition 5.4: Bianchi identity

The induced connection satisfies $D_{\operatorname{End} E}\Theta = 0$. Consequently $\mathrm{d}\operatorname{tr}(\Theta^k) = 0$.

Proof. Associativity gives $D(D^2 s) - D^2(Ds) = 0$. Locally this is $\mathrm{d}\Theta + A \wedge \Theta - \Theta \wedge A = 0$. The trace of a commutator vanishes. □

Theorem 5.5: Chern–Weil theory

The homogeneous components of

$$\det\left(I + \frac{\mathrm{i}}{2\pi}\Theta\right) = 1 + c_1(E, D) + \cdots + c_r(E, D)$$

are closed. Their de Rham classes are independent of D and lie in the image of integral cohomology. In particular,

$$c_1(E, D) = \frac{\mathrm{i}}{2\pi} \operatorname{tr} \Theta, \qquad \operatorname{ch}(E, D) = \operatorname{tr} \exp\left(\frac{\mathrm{i}}{2\pi}\Theta\right).$$

Proof. Let P be an invariant homogeneous polynomial of degree k on $\mathfrak{gl}(r, \mathbb{C})$. The Bianchi identity and invariance of P give

$$\mathrm{d}P(\Theta, \ldots, \Theta) = kP(D\Theta, \Theta, \ldots, \Theta) = 0.$$

For a path $D_t = D_0 + tA$ of connections, $\dot{\Theta}_t = D_t A$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\Theta_t^k) = kP(D_t A, \Theta_t^{k-1}) = k\mathrm{d}P(A, \Theta_t^{k-1}).$$

Integrating from 0 to 1 shows that the forms for two connections differ by the exterior derivative of the Chern–Simons transgression form. Hence the de Rham class is connection independent. Applying the splitting principle to a sum of line bundles identifies the normalized elementary symmetric polynomials with integral Chern classes; naturality then gives integrality for every bundle. □

Example 5.6: Flat bundles

If $\Theta = 0$, parallel frames exist locally and parallel transport around loops gives $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$. Conversely, a representation defines $\widetilde{X} \times_\rho \mathbb{C}^r$. Real Chern classes of a flat bundle vanish, though integral Chern classes may be torsion.

Warning

D^2 vanishes on scalar functions but acts on bundle sections by curvature. Writing $D^2 = 0$ confuses a general connection with the exterior derivative. A connection matrix has a $g^{-1}\mathrm{d}g$ term under change of frame; its curvature does not. Tensoriality is the quickest consistency check.

5.1.1 Connections form an affine space

If D_0 and D_1 are two connections on E, then

$$(D_1 - D_0)(fs) = f(D_1 - D_0)s.$$

Their difference is therefore a tensor $A \in \mathcal{A}^1(X, \operatorname{End} E)$. Conversely, $D_0 + A$ is again a connection. Thus connections form an affine space modeled on endomorphism-valued one-forms. The curvature changes by

$$\Theta_{D_0+A} = \Theta_{D_0} + D_0 A + A \wedge A.$$

The quadratic term records the noncommutativity of matrix multiplication; it vanishes for line bundles only because scalar one-forms wedge with themselves to zero.

Along a smooth path γ, parallel transport solves

$$D_{\dot\gamma} s = 0.$$

In a frame this is a linear ODE $\dot\sigma + A(\dot\gamma)\sigma = 0$. A flat connection has path-independent transport on simply connected open sets, producing parallel frames. Globally, transport around loops gives the holonomy representation of $\pi_1(X)$; flatness removes local curvature but not global monodromy.

5.1.2 Transgression in the simplest case

For line-bundle connections D_0 and $D_1 = D_0 + A$, the curvature relation simplifies to

$$\Theta_1 - \Theta_0 = \mathrm{d}A.$$

Hence

$$c_1(E, D_1) - c_1(E, D_0) = \mathrm{d}\left(\frac{\mathrm{i}}{2\pi} A\right).$$

This directly proves metric or connection independence of the first Chern class. For higher Chern forms, the path $D_t = D_0 + tA$ gives

$$P(\Theta_1) - P(\Theta_0) = \mathrm{d}\left(k \int_0^1 P(A, \Theta_t^{k-1})\,\mathrm{d}t\right)$$

for an invariant polynomial P of degree k. The expression in parentheses is the Chern–Simons transgression form.

The normalization $\mathrm{i}/(2\pi)$ is topological. For a line bundle on a closed oriented surface C,

$$\int_C \frac{\mathrm{i}}{2\pi} \Theta_D \in \mathbb{Z}.$$

Choosing a meromorphic section identifies this integer with its number of zeros minus poles. Chern–Weil theory and the divisor degree are therefore the smooth and analytic descriptions of the same class.

Remark (Gauge covariance)

The connection matrix changes inhomogeneously because it differentiates a moving frame; curvature changes by conjugation because it measures a tensorial failure of commutation. Traces of powers of curvature are invariant under conjugation, which is why they descend to global differential forms.

5.1.3 Local frames, sections, and the connection transformation law

Let $e_\alpha = (e_{\alpha,1}, \dots, e_{\alpha,r})$ be a row of local frame vectors and suppose $e_\beta = e_\alpha g_{\alpha\beta}$. If a section is written $s = e_\alpha \sigma_\alpha = e_\beta \sigma_\beta$, then

$$\sigma_\alpha = g_{\alpha\beta}\sigma_\beta, \qquad \sigma_\beta = g_{\alpha\beta}^{-1}\sigma_\alpha.$$

On a triple overlap, equality of the two changes from γ to α gives $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$. This is precisely the cocycle condition needed for the quotient of $\bigsqcup U_\alpha \times \mathbb{C}^r$ to be a bundle.

Write

$$Ds = e_\alpha(\mathrm{d}\sigma_\alpha + A_\alpha\sigma_\alpha).$$

Using $e_\beta = e_\alpha g$ and $\sigma_\alpha = g\sigma_\beta$ gives

$$\begin{aligned} Ds &= e_\alpha\big(\mathrm{d}(g\sigma_\beta) + A_\alpha g\sigma_\beta\big) \\ &= e_\beta\big(\mathrm{d}\sigma_\beta + (g^{-1}A_\alpha g + g^{-1}\mathrm{d}g)\sigma_\beta\big). \end{aligned}$$

Therefore

$$A_\beta = g^{-1}A_\alpha g + g^{-1}\mathrm{d}g.$$

Squaring and using $\mathrm{d}(g^{-1}) = -g^{-1}(\mathrm{d}g)g^{-1}$ cancels every inhomogeneous term and yields

$$\Theta_\beta = g^{-1}\Theta_\alpha g.$$

The first formula proves that A is frame dependent; the second proves that curvature is an $\operatorname{End} E$-valued two-form.

Induced connections calculated on simple tensors

For $s \in \Gamma(E)$, $t \in \Gamma(F)$, and $\lambda \in \Gamma(E^*)$, define

$$\begin{aligned} D_{E\otimes F}(s\otimes t) &= D_E s \otimes t + s \otimes D_F t, \\ (D_{E^*}\lambda)(s) &= \mathrm{d}(\lambda(s)) - \lambda(D_E s). \end{aligned}$$

Applying the connection once more and keeping the graded sign shows

$$\begin{aligned} \Theta_{E\otimes F} &= \Theta_E \otimes \mathrm{id}_F + \mathrm{id}_E \otimes \Theta_F, \\ \Theta_{E^*} &= -{}^{\mathsf{T}}\Theta_E. \end{aligned}$$

For the determinant bundle, $\Theta_{\det E} = \operatorname{tr}\Theta_E$. These identities immediately imply

$$c_1(E\otimes F) = \operatorname{rk}(F)c_1(E) + \operatorname{rk}(E)c_1(F), \qquad c_1(E^*) = -c_1(E).$$

A flat connection with nontrivial monodromy

On the trivial line bundle over $\mathbb{C}^*$ consider

$$D = \mathrm{d} + \lambda \frac{\mathrm{d}z}{z}, \qquad \lambda \in \mathbb{C}.$$

Since $\mathrm{d}(\mathrm{d}z/z) = 0$ and a scalar one-form wedges with itself to zero, $\Theta_D = 0$. A parallel section satisfies

$$\mathrm{d}s + \lambda \frac{\mathrm{d}z}{z} s = 0, \qquad s = z^{-\lambda}$$

on a chosen logarithm branch. Analytic continuation once around the origin multiplies s by $e^{-2\pi \mathrm{i}\lambda}$. Thus zero curvature gives path-independent transport only on simply connected sets; globally, flatness leaves a representation of the fundamental group.

Transgression for the first two invariant polynomials

Let $D_t = D_0 + tA$. Its curvature is

$$\Theta_t = \Theta_0 + tD_0 A + t^2 A \wedge A, \qquad \dot{\Theta}_t = D_t A.$$

For the trace polynomial,

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr} \Theta_t = \operatorname{tr}(D_t A) = \mathrm{d} \operatorname{tr} A.$$

Hence

$$\operatorname{tr} \Theta_1 - \operatorname{tr} \Theta_0 = \mathrm{d} \operatorname{tr} A.$$

For the quadratic polynomial,

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}(\Theta_t \wedge \Theta_t) = 2 \operatorname{tr}(D_t A \wedge \Theta_t) = 2\mathrm{d} \operatorname{tr}(A \wedge \Theta_t),$$

where Bianchi removes $D_t \Theta_t$. Integration in t gives the explicit Chern–Simons primitive. Therefore the de Rham classes of the Chern–Weil forms do not depend on a connection, even though their local representatives do.

5.1.4 Self-study workshop: checking connections under a change of frame

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall vector-bundle transition functions, matrix-valued differential forms, the graded product rule, and conjugation of endomorphisms. Fix the column-vector convention for section coefficients before calculating.

The gauge transformation from first principles. Let $e = (e_1, \ldots, e_r)$ be a local frame and write a section as $s = e\sigma$. A connection has the local form

$$D(e\sigma) = e(\mathrm{d}\sigma + A\sigma).$$

On an overlap choose $e' = eg$, so $\sigma' = g^{-1}\sigma$. Requiring the two formulas for Ds to agree gives

$$A' = g^{-1} A g + g^{-1} \mathrm{d}g.$$

The inhomogeneous term is why a connection matrix is not a tensor. Applying D once more gives the curvature

$$\Theta = \mathrm{d}A + A \wedge A.$$

Substitution into the transformation law cancels all derivatives of g and yields

$$\Theta' = g^{-1}\Theta g.$$

Curvature is therefore an $\operatorname{End} E$-valued two-form.

The trace removes conjugation: $\operatorname{tr}\Theta' = \operatorname{tr}\Theta$. More generally invariant polynomials in Θ glue to global forms. The Bianchi identity $D\Theta = 0$ makes these forms closed. If D_t is a path of connections, differentiating an invariant polynomial produces an exact transgression term, so its de Rham class is independent of the chosen connection. This is the logical chain behind Chern–Weil theory:

$$\text{gauge covariance} \implies \text{global form} \implies \text{closed form} \implies \text{topological class.}$$

Checkpoint (matrix order matters)

(1) Why is $g^{-1}dg$ present in A' but absent in Θ'?
(2) For a line bundle, what becomes of $A \wedge A$?
(3) If $\Theta = 0$, do all integral Chern classes necessarily vanish?

Solution (checkpoint). The derivative term is forced by differentiating the changed frame; in curvature its contributions cancel against those from $A' \wedge A'$. In rank one the scalar one-form satisfies $A \wedge A = 0$. Flatness makes all real Chern–Weil classes vanish, but integral Chern classes may remain torsion because de Rham cohomology does not detect torsion. $\diamond$

Exit criterion

Starting from $e' = eg$, you can derive the connection and curvature transformation laws in the correct matrix order and explain why invariant polynomials define connection-independent cohomology classes.

Summary. A connection differentiates sections and curvature measures the failure of second derivatives to commute. Chern–Weil theory converts invariant polynomials in curvature into topological characteristic classes. Every positivity and vanishing theorem below is built on this curvature operator.

Exercise 5.7: Direct sum, tensor product, and dual

Write the induced connections and curvatures on $E \oplus F$, $E \otimes F$, and E^*.

Exercise 5.8: Transition functions of a line bundle

Starting from $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$, construct the corresponding class in $H^2(X, \mathbb{Z})$ via the exponential sequence and compare it with c_1.

Optional handout: Holonomy, flat bundles, and monodromy

This handout is optional. No later argument in the core lectures depends on it.

A connection D transports vectors along a piecewise smooth path γ by solving $D_{\dot\gamma} s = 0$. Parallel transport depends on the path. Transport around loops based at x generates the holonomy group $\operatorname{Hol}_x(D) \subset \operatorname{GL}(E_x)$.

Curvature is the infinitesimal holonomy. For a small coordinate parallelogram spanned by v, w, parallel transport around its boundary is

$$\operatorname{id} - \Theta_D(v, w)\ \text{Area} + O(\text{diam}^3).$$

The Ambrose–Singer theorem makes this precise: the Lie algebra of the holonomy group is generated by parallel translates of curvature endomorphisms.

If D is flat, transport depends only on the homotopy class of a path and produces a representation $\rho : \pi_1(X, x) \to \mathrm{GL}(E_x)$. Conversely, $E_\rho = \widetilde{X} \times_\rho \mathbb{C}^r$ carries a canonical flat connection. Thus flat bundles are linear representations of the fundamental group expressed geometrically.

Remark (Unitary local systems)

A flat bundle admits a parallel Hermitian metric exactly when the monodromy is conjugate to a unitary representation. Such bundles have a particularly clean Hodge theory. The nonunitary case leads to harmonic metrics and nonabelian Hodge theory.

Chern–Weil forms vanish for a flat connection, so real Chern classes vanish. Integral classes can still be torsion because de Rham cohomology does not see torsion.

Exercise 5.9: Flat line bundles on a circle

Classify flat complex line bundles with connection over S^1 by their monodromy $\lambda \in \mathbb{C}^*$. Which of them admit a parallel Hermitian metric?

5.2 Lecture 18: Chern connections, line bundles, and the first Chern class

Week 9 material • 90 minutes

Notation for this lecture

$(E, \bar\partial_E, h)$ is a holomorphic vector bundle with Hermitian metric h. The Chern connection is $D = D' + D''$, where $D'' = \bar\partial_E$ and D' has type $(1, 0)$. In a holomorphic frame, $H = (h_{\alpha\bar\beta})$ is the metric matrix, $A = H^{-1}\partial H$ is the connection matrix in the convention of this text, and $\Theta(E, h) = \bar\partial A$ is its curvature. For a line bundle frame e we write $|e|_h^2 = \mathrm{e}^{-\varphi}$. The first Chern form is $c_1(E, h) = (\mathrm{i}/2\pi)\operatorname{tr}\Theta(E, h)$. For a meromorphic section s, $\operatorname{div}(s)$ is its divisor and $[\operatorname{div}(s)]$ the associated integration current.

Prerequisite results used below

Type decomposition for connections. On a complex manifold every complex connection decomposes uniquely as $D = D' + D''$, with D' raising the first bidegree and D'' raising the second. A holomorphic structure is equivalently an operator $\bar\partial_E : \mathcal{A}^{0,0}(E) \to \mathcal{A}^{0,1}(E)$ satisfying the Leibniz rule and $\bar\partial_E^2 = 0$.
Metric-compatibility identity. A connection D preserves h precisely when $\mathrm{d}h(s, t) = h(Ds, t) + h(s, Dt)$ for all smooth sections s, t. In a holomorphic frame this linear identity uniquely determines the $(1, 0)$ part once $D'' = \bar\partial_E$ is fixed; this is the key input in the Chern connection theorem.

Plan for the lecture. 0–20 min: metric compatibility and type; 20–45: local formula for the Chern connection; 45–65: line-bundle curvature; 65–82: bundle-valued Poincaré–Lelong; 82–90: $\mathcal{O}(k)$.

Definition 5.10: Hermitian holomorphic vector bundle

A Hermitian holomorphic vector bundle is a holomorphic bundle E with a smooth Hermitian metric h, represented in a holomorphic frame by the positive matrix $H = (h(e_j, e_k))$.

Theorem 5.11: Chern connection

There is a unique connection $D = D' + \bar\partial_E$ whose $(0, 1)$ part is the holomorphic structure and which satisfies

$$\mathrm{d}h(s, t) = h(Ds, t) + h(s, Dt).$$

In a holomorphic frame,

$$D' = \partial + H^{-1}\partial H, \qquad \Theta(E, h) = \bar\partial(H^{-1}\partial H).$$

Its curvature has type $(1, 1)$.

Proof. In a holomorphic frame write a section as a column vector and put $D' = \partial + A$. Metric compatibility for arbitrary vectors s, t says

$$\partial(s^*Ht) = (\bar\partial s)^*Ht + s^*H(\partial t + At),$$

and comparison of the terms containing no derivative of s or t gives $HA = \partial H$. Hence $A = H^{-1}\partial H$, proving uniqueness. Conversely, define $D = \bar\partial + \partial + H^{-1}\partial H$. Substitution into the displayed identity and its conjugate verifies metric compatibility, so the connection exists.

Its $(0,2)$ curvature is $\bar\partial_E^2 = 0$. For $A = H^{-1}\partial H$,

$$\partial A + A \wedge A = \partial(H^{-1}\partial H) + (H^{-1}\partial H) \wedge (H^{-1}\partial H) = 0,$$

because $\partial H^{-1} = -H^{-1}(\partial H)H^{-1}$. Thus the $(2,0)$ component also vanishes, and the remaining curvature is $\Theta = \bar\partial(H^{-1}\partial H)$ of type $(1,1)$. □

Example 5.12: Line-bundle curvature

In a local holomorphic frame e, write $|e|_h^2 = \mathrm{e}^{-\varphi}$. Then

$$D'e = -\partial\varphi \otimes e, \qquad \Theta(L,h) = \partial\bar\partial\varphi, \qquad c_1(L,h) = \frac{\mathrm{i}}{2\pi}\Theta = \mathrm{dd^c}\varphi.$$

If $e' = ge$, then $\varphi' = \varphi - \log|g|^2$, so $\mathrm{dd^c}\varphi$ is globally defined.

Warning

Some authors use $|e|^2 = \mathrm{e}^{+\varphi}$ or define curvature with the opposite sign. Never memorize "curvature is $\pm\partial\bar\partial\varphi$" apart from both the metric exponent and curvature convention. Here $|e|^2 = \mathrm{e}^{-\varphi}$.

Theorem 5.13: Bundle-valued Poincaré–Lelong

If s is a nonzero meromorphic section of (L,h), then

$$\mathrm{dd^c}\log|s|_h^2 = [\mathrm{div}(s)] - c_1(L,h).$$

In cohomology, $[\mathrm{div}(s)] = c_1(L) \in H^2(X,\mathbb{Z})$.

Proof. Choose a local holomorphic frame e with $|e|_h^2 = e^{-\varphi}$ and write $s = fe$. Then

$$\log|s|_h^2 = \log|f|^2 - \varphi.$$

The scalar Poincaré–Lelong formula gives $\mathrm{dd^c}\log|f|^2 = [\mathrm{div}(f)]$, while $\mathrm{dd^c}\varphi = c_1(L,h)$. Their difference is the asserted identity. Under a frame change, the two scalar terms change by opposite copies of $\log|g|^2$, so the local identities glue. Taking de Rham classes kills the exact left-hand side and gives $[\mathrm{div}(s)] = c_1(L)$. □

Example 5.14: $\mathcal{O}(-1)$ and $\mathcal{O}(1)$

The fiber of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over $[z]$ is the line $\mathbb{C}z \subset \mathbb{C}^{n+1}$, and $\mathcal{O}(1) = \mathcal{O}(-1)^*$. The standard dual metric satisfies

$$c_1(\mathcal{O}(1), h_{\mathrm{FS}}) = \omega_{\mathrm{FS}}, \qquad c_1(\mathcal{O}(k)) = k[\omega_{\mathrm{FS}}].$$

For $k \geq 0$, $H^0(\mathbb{P}^n, \mathcal{O}(k))$ is the space of homogeneous polynomials of degree k.

Proposition 5.15: Changing the metric

If $h' = h\mathrm{e}^{-\psi}$, then

$$c_1(L,h') = c_1(L,h) + \mathrm{dd^c}\psi.$$

Thus curvature forms of one line bundle vary inside a fixed de Rham class. A Kähler potential equation is a search for a suitable metric.

Proof. In a holomorphic frame write $|e|_h^2 = e^{-\varphi}$. Then $|e|_{h'}^2 = e^{-(\varphi+\psi)}$, so the line-bundle curvature formula gives

$$c_1(L, h') = \mathrm{dd^c}(\varphi + \psi) = c_1(L, h) + \mathrm{dd^c}\psi.$$

Since the difference is exact, both metrics represent the same de Rham class. □

5.2.1 Normal holomorphic frames and curvature at a point

Given a Hermitian holomorphic bundle (E, h) and $x \in X$, a constant linear change first makes $H(x) = I$. A holomorphic change of frame of the form $g(z) = I + \sum_j g_j z_j$ can then arrange $\partial H(x) = 0$. In such a normal frame the Chern connection matrix vanishes at x, and

$$\Theta(E, h)_x = \bar{\partial}(H^{-1}\partial H)_x$$

depends only on the mixed second derivatives of the metric. Curvature is therefore the second-order obstruction to finding a local holomorphic orthonormal frame.

For a line bundle with $|e|^2 = \mathrm{e}^{-\varphi}$, multiplying e by a holomorphic unit changes φ by $-\log|g|^2$. One can kill the constant and holomorphic linear terms of φ at a point, leaving

$$\varphi(z) = \sum_{j,k} \varphi_{j\bar{k}}(0) z_j \bar{z}_k + \operatorname{Re} O(|z|^2) + O(|z|^3).$$

Only the mixed Hermitian quadratic term contributes to $\partial\bar{\partial}\varphi$.

5.2.2 The metric on $\mathcal{O}(1)$ in coordinates

On $U_0 \subset \mathbb{P}^n$, let e_0 be the dual of the tautological vector $(1, w_1, \dots, w_n)$. Its squared norm in $\mathcal{O}(1)$ is

$$|e_0|^2 = \frac{1}{1 + |w|^2} = \mathrm{e}^{-\varphi}, \qquad \varphi = \log(1 + |w|^2).$$

Hence

$$c_1(\mathcal{O}(1), h_{\mathrm{FS}}) = \mathrm{dd^c}\log(1 + |w|^2) = \omega_{\mathrm{FS}}.$$

On $U_0 \cap U_j$, the frame changes by the holomorphic unit w_j, and the potential changes by $-\log|w_j|^2$. Its $\mathrm{dd^c}$ vanishes on the overlap, so the curvature forms glue.

If P is a homogeneous polynomial of degree k, it defines a section of $\mathcal{O}(k)$. Bundle-valued Poincaré–Lelong gives

$$[P = 0] = k\omega_{\mathrm{FS}} + ddc \log|P|^2_{h_{\mathrm{FS}}^k}.$$

Integrating against $\omega_{\mathrm{FS}}^{n-1}$ shows that a degree-k hypersurface has cohomology class $k[\omega_{\mathrm{FS}}]$. The exact potential term disappears by Stokes, while its logarithmic singularity remembers the divisor.

Warning

A topologically trivial line bundle can carry a nonflat metric, but its curvature is exact. Positivity of an exact curvature form is impossible on a compact manifold: integrating its top power would be both positive and zero. Topology constrains which metric curvature signs can occur globally.

5.2.3 The Chern connection in scalar and diagonal examples

For a line bundle frame e with $|e|_h^2 = e^{-\varphi}$, write $D'e = Ae$. Metric compatibility gives

$$\partial|e|_h^2 = h(D'e, e) = Ae^{-\varphi},$$

so $A = -\partial\varphi$. Therefore

$$\Theta = \bar\partial A = -\bar\partial\partial\varphi = \partial\bar\partial\varphi, \qquad \frac{\mathrm{i}}{2\pi}\Theta = \mathrm{dd^c}\varphi.$$

For the trivial bundle on $\mathbb{C}^n$ with $|1|^2 = e^{-|z|^2}$, this gives

$$c_1(L,h) = \mathrm{dd^c}|z|^2 = \frac{\mathrm{i}}{2\pi}\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar z_j > 0.$$

The de Rham class is nevertheless zero because the form is globally exact. Curvature of a metric is differential-geometric data; its cohomology class is the topological invariant.

For a rank-two bundle with a diagonal metric

$$H = \begin{pmatrix} e^{-\varphi_1} & 0 \\ 0 & e^{-\varphi_2} \end{pmatrix},$$

one has

$$H^{-1}\partial H = \begin{pmatrix} -\partial\varphi_1 & 0 \\ 0 & -\partial\varphi_2 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} \partial\bar\partial\varphi_1 & 0 \\ 0 & \partial\bar\partial\varphi_2 \end{pmatrix}.$$

Thus an orthogonal holomorphic direct sum has block-diagonal Chern curvature.

The metric on $\mathcal{O}(-1)$ and $\mathcal{O}(1)$

On $U_0 \subset \mathbb{P}^n$ with $w_j = z_j/z_0$, the tautological frame of $\mathcal{O}(-1)$ is

$$e_0(w) = (1, w_1, \dots, w_n) \in \mathbb{C}^{n+1}.$$

The metric induced from the Euclidean norm is

$$|e_0|^2 = 1 + |w|^2 = e^{-\varphi_-}, \qquad \varphi_- = -\log(1+|w|^2).$$

Hence

$$c_1(\mathcal{O}(-1), h_-) = -\mathrm{dd^c}\log(1+|w|^2) = -\omega_{\mathrm{FS}}.$$

The dual frame e_0^* of $\mathcal{O}(1)$ has squared norm $(1+|w|^2)^{-1} = e^{-\varphi_+}$ with $\varphi_+ = \log(1+|w|^2)$, so

$$c_1(\mathcal{O}(1), h_+) = \omega_{\mathrm{FS}}.$$

On $U_0 \cap U_1$, the tautological frames satisfy $e_1 = w_1^{-1}e_0$. Therefore

$$|e_1|^2 = |w_1|^{-2}|e_0|^2, \qquad \varphi_{-,1} = \varphi_{-,0} + \log|w_1|^2.$$

The added term is pluriharmonic on the overlap, so the curvature forms agree. This is the metric version of the transition-function calculation.

Chern class from logarithms of transition functions

For a line bundle with transition functions g_{ij}, choose local logarithms $g_{ij} = e^{2\pi\mathrm{i}f_{ij}}$. On triple overlaps,

$$n_{ijk} = f_{ij} + f_{jk} + f_{ki} \in \mathbb{Z}.$$

The integers form a Čech two-cocycle representing $c_1(L)$. If a Hermitian metric has weights φ_i, the compatibility

$$\varphi_j = \varphi_i - \log|g_{ij}|^2$$

implies that $\mathrm{dd^c}\varphi_i$ glue. A partition-of-unity comparison between the logarithmic cocycle and these local weights shows that the de Rham class of $\mathrm{dd^c}\varphi_i$ is the image of (n_{ijk}) in $H^2(X,\mathbb{R})$. Thus the Čech and curvature constructions yield the same first Chern class.

A divisor calculation on $\mathbb{P}^1$

The section of $\mathcal{O}(k)$ represented by the homogeneous polynomial Z_1^k vanishes to order k at $[1:0]$ and nowhere else. Bundle-valued Poincaré–Lelong gives

$$[k[1:0]] = c_1(\mathcal{O}(k), h) + \mathrm{dd^c} \log |Z_1^k|_h^2.$$

Integrating over $\mathbb{P}^1$ kills the exact term, and the normalization $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = 1$ yields

$$k = \int_{\mathbb{P}^1} c_1(\mathcal{O}(k)) = k \int_{\mathbb{P}^1} \omega_{\mathrm{FS}}.$$

This checks simultaneously the divisor degree, curvature normalization, and tensor power formula.

5.2.4 Self-study workshop: deriving the Chern connection from the metric

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall holomorphic frames, Hermitian matrices, type decomposition of a connection, and Lecture 3's local potentials. The Chern connection is characterized simultaneously by metric compatibility and by $D'' = \bar\partial$.

Solve the compatibility equation. In a holomorphic frame let the metric matrix be $H = (h(e_\alpha, e_\beta))$. Write the $(1,0)$ connection matrix as A. Metric compatibility and the condition $D'' = \bar\partial$ give

$$\partial H = HA, \qquad A = H^{-1}\partial H.$$

The curvature has type $(1,1)$:

$$\Theta = \bar\partial(H^{-1}\partial H).$$

For a line bundle with $|e|_h^2 = e^{-\varphi}$,

$$A = -\partial\varphi, \qquad \Theta = \partial\bar\partial\varphi, \qquad c_1(L,h) = \frac{\mathrm{i}}{2\pi}\Theta = \mathrm{dd^c}\varphi.$$

Thus positivity of a line bundle metric is exactly strict plurisubharmonicity of its local weight.

For $\mathcal{O}(1) \to \mathbb{P}^n$, the standard metric has affine weight

$$\varphi(z) = \log(1 + |z|^2).$$

Its curvature is the Fubini–Study form. On overlaps the weights change by $\log |g_{\alpha\beta}|^2$ for a nonvanishing holomorphic transition function, whose $\mathrm{dd^c}$ vanishes. Hence the curvature glues even though no global potential exists. Integrating over a projective line gives

$$\int_{\mathbb{P}^1} c_1(\mathcal{O}(1)) = 1,$$

which fixes the sign and normalization.

Checkpoint (a sign ledger)

(1) If the metric is written $|e|^2 = e^{-\varphi}$, what is the connection one-form?
(2) Why does $\partial\bar\partial\varphi$ have the sign shown above?
(3) What changes for the dual metric on L^*?

Solution (checkpoint). The form is $A = H^{-1}\partial H = -\partial\varphi$. Since $\bar\partial\partial = -\partial\bar\partial$, one gets $\bar\partial(-\partial\varphi) = \partial\bar\partial\varphi$. The dual metric has weight $-\varphi$, so its curvature and first Chern form are the negatives of those of L. ◇

Exit criterion

You can derive the Chern connection rather than quote it, convert a metric weight into curvature, check its frame invariance, and normalize $c_1(\mathcal{O}(1))$ on a projective line.

Summary. The holomorphic structure and Hermitian metric uniquely determine the Chern connection. For line bundles, curvature is the complex Hessian of the weight. The first Chern class unifies divisors, curvature, and integral topology.

Exercise 5.16: Curvature of $\mathcal{O}(1)$

On $U_0 \subset \mathbb{P}^n$, use the frame $(1, w)$ of $\mathcal{O}(-1)$, compute the dual metric, and recover $c_1(\mathcal{O}(1)) = \omega_{\mathrm{FS}}$.

Exercise 5.17: A curved metric on the trivial line bundle

On $\mathbb{C}^n \times \mathbb{C}$, set $|1|_h^2 = \mathrm{e}^{-|z|^2}$. Compute the curvature and explain how a topologically trivial line bundle can have a nonflat metric.

Optional handout: The Chern character and Riemann–Roch

This handout is optional. No later argument in the core lectures depends on it.

The total Chern class is multiplicative in short exact sequences, while the Chern character is additive:

$$c(E) = \det\left(I + \frac{\mathrm{i}}{2\pi}\Theta_E\right), \qquad \mathrm{ch}(E) = \operatorname{tr}\exp\left(\frac{\mathrm{i}}{2\pi}\Theta_E\right).$$

Under the splitting principle, if $x_1, \ldots, x_r$ are formal Chern roots, then

$$c(E) = \prod_j (1 + x_j), \qquad \mathrm{ch}(E) = \sum_j \mathrm{e}^{x_j}.$$

Hence

$$\mathrm{ch}(E) = r + c_1(E) + \tfrac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots .$$

The Hirzebruch–Riemann–Roch theorem expresses the holomorphic Euler characteristic as

$$\chi(X, E) = \sum_q (-1)^q h^q(X, E) = \int_X \mathrm{ch}(E)\,\mathrm{td}(T_X),$$

where the Todd class has expansion $\mathrm{td}(T_X) = 1 + \frac{1}{2}c_1(T_X) + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)) + \cdots$.

Example 5.18: A line bundle on a surface

For a line bundle L on a smooth compact complex surface,

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}c_1(L) \cdot (c_1(L) - c_1(K_X)).$$

This is the formula used in the capstone lecture. If L is sufficiently positive, Kodaira vanishing turns the Euler characteristic into the actual number of sections.

The Grothendieck–Riemann–Roch theorem extends this to a proper holomorphic map $f : X \to Y$ by comparing $\mathrm{ch}(f_! E)$ with the fiber integral of $\mathrm{ch}(E)\,\mathrm{td}(T_f)$.

Exercise 5.19: Projective space

Use $c_1(T_{\mathbb{P}^1}) = 2[\omega_{\mathrm{FS}}]$ to recover $\chi(\mathbb{P}^1, \mathcal{O}(k)) = k + 1$ from Riemann–Roch.

5.3 Lecture 19: Griffiths and Nakano positivity; curvature formulas

Week 10 material • 90 minutes

Notation for this lecture

$E \to X$ is a Hermitian holomorphic bundle of rank r and $R_{j\bar{k}\alpha\bar{\beta}}$ are the components of $\mathrm{i}\Theta(E,h)$ in unitary coordinates and a unitary frame. Griffiths positivity tests the quadratic form only on decomposable tensors $\xi \otimes v$ with $\xi \in T_x^{1,0}X$ and $v \in E_x$. Nakano positivity tests it on every tensor $u = (u_{j\alpha}) \in T_x^{1,0}X \otimes E_x$. E^*, $E \otimes F$, and $\det E = \bigwedge^r E$ denote the dual, tensor product, and determinant bundles. The second fundamental form of a holomorphic subbundle is denoted β, $T_X = T^{1,0}X$ is the holomorphic tangent bundle, and $\mathrm{Ric}(\omega)$ is the Ricci form of a Kähler metric.

Prerequisite results used below

Tensor-product curvature rule. For bundles with connections, $\Theta(E \otimes F) = \Theta(E) \otimes I_F + I_E \otimes \Theta(F)$, while $\Theta(E^*) = -\Theta(E)^{\mathsf{T}}$ and $\Theta(\det E) = \operatorname{tr}\Theta(E)$; here I_E and I_F are the identity endomorphisms of the indicated bundles.
Orthogonal subbundle formula. If $S \subset E$ is a holomorphic subbundle and β its second fundamental form for the orthogonal splitting $E = S \oplus S^\perp$, then

$$\langle \mathrm{i}\Theta(S)(\xi,\bar{\xi})s, s\rangle = \langle \mathrm{i}\Theta(E)(\xi,\bar{\xi})s, s\rangle - \|\beta_\xi s\|^2.$$

The negative square explains why Griffiths negativity descends to subbundles and positivity descends to quotient bundles. All adjoints and projections are taken with the stated Hermitian metrics.

Plan for the lecture. 0–22 min: two notions of vector-bundle positivity; 22–42: line bundles and duals; 42–65: subbundle and quotient curvature; 65–80: tensor operations and Demailly–Skoda; 80–90: sign audit.

Locally write

$$\mathrm{i}\Theta(E,h) = \mathrm{i} \sum_{j,k,\alpha,\beta} \Theta_{j\bar{k}\alpha\bar{\beta}}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes e_\alpha^* \otimes e_\beta.$$

Definition 5.20: Griffiths and Nakano positivity

E is Griffiths semipositive if, for all $\xi \in T^{1,0}X$ and $v \in E$,

$$\sum \Theta_{j\bar{k}\alpha\bar{\beta}} \xi_j \bar{\xi}_k v_\alpha \bar{v}_\beta \geq 0.$$

It is Nakano semipositive if, for every $u = (u_{j\alpha}) \in T^{1,0}X \otimes E$,

$$\sum \Theta_{j\bar{k}\alpha\bar{\beta}} u_{j\alpha} \bar{u}_{k\beta} \geq 0.$$

Strict positivity means strict inequality on nonzero vectors.

Nakano positivity is generally stronger than Griffiths positivity; for rank one both are ordinary positivity of the curvature form. Duality exchanges Griffiths positivity and negativity. Nakano duality involves an index transposition and is more subtle.

Theorem 5.21: Curvature of a subbundle and quotient

For a Hermitian holomorphic sequence $0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$, use the smooth orthogonal splitting $E \simeq S \oplus Q$ and let $\beta \in \mathcal{A}^{1,0}(\mathrm{Hom}(S, Q))$ be the second fundamental form. Then

$$D_E = \begin{pmatrix} D_S & -\beta^* \\ \beta & D_Q \end{pmatrix}, \qquad \Theta_E = \begin{pmatrix} \Theta_S - \beta^* \wedge \beta & -D'\beta^* \\ \bar{\partial}\beta & \Theta_Q - \beta \wedge \beta^* \end{pmatrix}.$$

Equivalently,

$$\left\langle \mathrm{i}\Theta_S(\xi, \bar{\xi})s, s \right\rangle = \left\langle \mathrm{i}\Theta_E(\xi, \bar{\xi})s, s \right\rangle - \left\|\beta_\xi s\right\|^2,$$

while the quotient formula has a positive square term. Hence quotients of Griffiths-semipositive bundles remain semipositive, and subbundles of Griffiths-seminegative bundles remain seminegative.

Proof. The orthogonal splitting is smooth but generally not holomorphic. Metric compatibility forces the off-diagonal blocks of D_E to be adjoints with opposite signs, giving

$$D_E = \begin{pmatrix} D_S & -\beta^* \\ \beta & D_Q \end{pmatrix}.$$

Square this matrix using the graded rule: when a connection passes a one-form, a minus sign appears. The diagonal blocks are $\Theta_S - \beta^* \wedge \beta$ and $\Theta_Q - \beta \wedge \beta^*$, and the off-diagonal blocks are $-D'\beta^*$ and $\bar{\partial}\beta$. Evaluate the upper-left block on $(\xi, \bar{\xi})$ and a section $s \in S$; the term $\mathrm{i}\beta^* \wedge \beta$ contributes $-\left\|\beta_\xi s\right\|^2$ after solving for Θ_S. This proves the stated subbundle identity. Solving the lower-right block for Θ_Q gives the quotient identity with the opposite, positive square term. □

Theorem 5.22: Demailly–Skoda positivity improvement

If E is Griffiths semipositive, then $E \otimes \det E$ is Nakano semipositive.

Proof. At a point choose a unitary bundle frame and write the curvature tensor as $R_{j\bar{k}\alpha\bar{\beta}}$. The curvature of $E \otimes \det E$ is

$$\widetilde{R}_{j\bar{k}\alpha\bar{\beta}} = R_{j\bar{k}\alpha\bar{\beta}} + \delta_{\alpha\beta} \sum_{\gamma} R_{j\bar{k}\gamma\bar{\gamma}}.$$

Fix an arbitrary tensor $u = (u_{j\alpha}) \in T^{1,0}X \otimes E$. For $v \in \mathbb{C}^r$ put $\xi_j(v) = \sum_\alpha u_{j\alpha}\bar{v}_\alpha$. Griffiths semipositivity gives

$$R_{j\bar{k}\alpha\bar{\beta}}\xi_j(v)\overline{\xi_k(v)}v_\alpha\bar{v}_\beta \geq 0.$$

Average over the unit sphere in $\mathbb{C}^r$. Its fourth moment is a positive constant times

$$\int \bar{v}_\gamma v_\delta v_\alpha \bar{v}_\beta = C(\delta_{\gamma\delta}\delta_{\alpha\beta} + \delta_{\gamma\alpha}\delta_{\delta\beta}).$$

After substitution, the averaged inequality becomes

$$C \sum_{j,k,\alpha,\beta} \left(R_{j\bar{k}\alpha\bar{\beta}} + \delta_{\alpha\beta} \sum_{\gamma} R_{j\bar{k}\gamma\bar{\gamma}} \right) u_{j\alpha}\bar{u}_{k\beta} \geq 0.$$

This is exactly Nakano semipositivity of $E \otimes \det E$. □

This theorem converts the geometrically natural but weaker Griffiths positivity into a form usable in Nakano-type Bochner identities.

Example 5.23: Curvature of the tangent bundle

On a Kähler manifold, the Chern curvature of T_X is the complex part of the Riemann tensor. Its Griffiths quadratic form gives holomorphic bisectional curvature; equal directions give holomorphic sectional curvature. We use the differential- geometric Ricci form

$$\mathrm{Ric}(\omega) = -\mathrm{i}\partial\bar{\partial}\log\det(g_{j\bar{k}}) = -2\pi\mathrm{dd^c}\log\det(g_{j\bar{k}}), \qquad [\mathrm{Ric}(\omega)] = 2\pi c_1(X).$$

Thus $(2\pi)^{-1}\,\mathrm{Ric}(\omega)$, rather than $\mathrm{Ric}(\omega)$ itself, is the Chern–Weil representative of $c_1(X)$ in our $\mathrm{dd^c}$ normalization.

Warning

Vector-bundle positivity does not mean that every matrix entry is a positive $(1,1)$-form. State whether the curvature is tested only on decomposable tensors $\xi \otimes v$ (Griffiths) or on arbitrary tensors in $T^{1,0}X \otimes E$ (Nakano).

5.3.1 How to read the four curvature indices

At a point choose unitary frames of $T^{1,0}X$ and E. The tensor $\Theta_{j\bar{k}\alpha\bar{\beta}}$ defines a Hermitian form on $T^{1,0}X \otimes E$. Griffiths positivity tests only rank-one matrices

$$u_{j\alpha} = \xi_j v_\alpha,$$

whereas Nakano positivity tests every matrix $(u_{j\alpha})$. If either $\dim X = 1$ or $\operatorname{rank} E = 1$, every such matrix has rank at most one and the two tests agree. In higher dimensions and ranks, arbitrary tensors are sums of decomposable tensors, but positivity on each summand does not control cross terms; this is the gap between the notions.

For a line bundle the curvature tensor has no bundle indices. On an (n,q)-form, the commutator $[\mathrm{i}\Theta(L), \Lambda]$ sums q curvature eigenvalues. For a vector bundle, those eigenvalues are coupled to the bundle indices, and Nakano positivity is the condition naturally seen by this commutator. This explains the choice of positivity in Lecture 25 rather than merely naming one notion as “stronger.”

5.3.2 Curvature under the standard bundle operations

The induced connections give

$$\begin{aligned} \Theta(E \otimes F) &= \Theta(E) \otimes \mathrm{id}_F + \mathrm{id}_E \otimes \Theta(F), \\ \Theta(E^*) &= -{}^{\mathrm{t}}\Theta(E), \\ \Theta(\det E) &= \operatorname{tr}\Theta(E). \end{aligned}$$

Thus Griffiths positivity is preserved by tensor products of positive bundles, and dualization reverses its sign. The determinant records the trace over bundle directions; its curvature is the first Chern form.

The tautological sequence on projective space,

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{\mathbb{C}}^{n+1} \longrightarrow Q \longrightarrow 0,$$

starts with a flat ambient bundle. The subbundle formula makes $\mathcal{O}(-1)$ Griffiths negative by the square of its second fundamental form, while the quotient formula makes Q Griffiths positive. Dualizing recovers positivity of $\mathcal{O}(1)$. This gives a geometric curvature proof of Fubini–Study positivity.

Tracing the curvature of T_X in its bundle indices gives the Ricci form:

$$\mathrm{Ric}(\omega) = \mathrm{i}\operatorname{tr}\Theta(T_X).$$

Tracing once more against the Kähler metric gives scalar curvature. Holomorphic sectional, bisectional, Ricci, and scalar curvature are therefore successive tests or traces of the same Chern curvature tensor, with progressively less information.

Remark (A reliable decision rule)

For geometric maps and quotients, Griffiths positivity is often the natural language. For L^2 estimates on bundle-valued (n, q)-forms, compute the Nakano curvature operator. If only Griffiths positivity is available, a theorem such as Demailly–Skoda is the bridge; omitting that bridge is a genuine gap.

5.3.3 Reading the curvature tensor as two quadratic forms

At one point identify $T_x^{1,0}X \simeq \mathbb{C}^n$ and $E_x \simeq \mathbb{C}^r$. Curvature defines a Hermitian form Q on $\mathbb{C}^n \otimes \mathbb{C}^r$ by

$$Q(u) = \sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\bar{\beta}} u_{j\alpha} \bar{u}_{k\beta}.$$

Nakano semipositivity says $Q(u) \geq 0$ for every matrix $u = (u_{j\alpha})$. Griffiths semipositivity tests only rank-one matrices $u_{j\alpha} = \xi_j v_\alpha$. This proves directly that Nakano positivity implies Griffiths positivity. When $r = 1$ or $n = 1$, every matrix has rank at most one, so the two notions coincide.

For a line bundle L and a vector bundle E,

$$R^{E\otimes L}_{j\bar{k}\alpha\bar{\beta}} = R^{E}_{j\bar{k}\alpha\bar{\beta}} + R^{L}_{j\bar{k}} \delta_{\alpha\beta}.$$

If $\mathrm{i}\Theta_L \geq \varepsilon\omega$, then on an arbitrary tensor u the second term contributes at least $\varepsilon \sum_{j,\alpha} |u_{j\alpha}|^2$. This is why tensoring by a sufficiently positive line bundle can dominate a bounded negative part of vector-bundle curvature.

The tautological subbundle as a curvature test

The tautological line bundle $S = \mathcal{O}_{\mathbb{P}^n}(-1)$ sits inside the flat bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$. The ambient curvature is zero. The subbundle formula gives

$$\langle \mathrm{i}\Theta_S(\xi, \bar{\xi}) s, s \rangle = -\|\beta_\xi s\|^2 \leq 0.$$

Thus S is Griffiths negative. Its dual $S^* = \mathcal{O}(1)$ is positive. In the affine chart, the second fundamental form differentiates the line $\mathbb{C}(1, w)$ in directions orthogonal to itself. Its squared norm is exactly the Fubini–Study Hessian

$$\frac{(1 + |w|^2)|\xi|^2 - |\langle \xi, w \rangle|^2}{(1 + |w|^2)^2}.$$

This recovers the coordinate formula for ω_{FS} from the abstract subbundle identity.

Curvature under dual, tensor, and determinant

Metric compatibility with the evaluation pairing gives

$$\Theta_{E^*} = -{}^{\mathsf{T}}\Theta_E.$$

Consequently, if E is Griffiths positive and $0 \neq \lambda \in E_x^*$, choose the metric-dual vector v; then

$$\langle \mathrm{i}\Theta_{E^*}(\xi, \bar{\xi}) \lambda, \lambda \rangle = -\langle \mathrm{i}\Theta_E(\xi, \bar{\xi}) v, v \rangle < 0.$$

Thus duality exchanges Griffiths positivity and negativity.

For the determinant bundle,

$$\Theta_{\det E} = \operatorname{tr}_E \Theta_E,$$

so Griffiths positivity of E implies positivity of $\det E$: sum the Griffiths quadratic form over an orthonormal basis of E_x. Combining this trace term with the original curvature is exactly the tensor appearing in the Demailly–Skoda theorem.

Subbundle and quotient signs in a rank-two model

Let $E = X \times \mathbb{C}^2$ be flat and let S be a holomorphic line subbundle locally spanned by

$$s(z) = \frac{(1, f(z))}{\sqrt{1 + |f(z)|^2}}.$$

The normalization is smooth rather than holomorphic, which is allowed for the orthogonal splitting. Projecting ∂s to $S^\perp$ gives

$$\|\beta_{\partial/\partial z} s\|^2 = \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}.$$

Since E is flat, the subbundle curvature is the negative of this quantity, whereas the quotient line has the positive of it. This explicit square verifies the signs in the block-curvature formula.

5.3.4 Self-study workshop: keeping Griffiths and Nakano positivity separate

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall the Chern curvature tensor, tensor products of Hermitian vector spaces, and positive Hermitian forms. The distinction in this lecture is entirely about which tensors are used to test the same four-index curvature array.

Write both quadratic forms. At a point choose unitary coordinates and a unitary frame, and write

$$\mathrm{i}\Theta(E) = \sum_{j,k,\alpha,\beta} c_{j\bar{k}\alpha\bar{\beta}}\, \mathrm{i}dz_j \wedge d\bar{z}_k \otimes e_\alpha^* \otimes e_\beta.$$

Griffiths positivity tests only decomposable tensors:

$$\sum c_{j\bar{k}\alpha\bar{\beta}} \xi_j \bar{\xi}_k v_\alpha \bar{v}_\beta > 0 \quad \text{for } \xi \neq 0,\ v \neq 0.$$

Nakano positivity tests every matrix $u = (u_{j\alpha}) \in T^{1,0}X \otimes E$:

$$\sum c_{j\bar{k}\alpha\bar{\beta}} u_{j\alpha} \overline{u_{k\beta}} > 0.$$

Since decomposable tensors are a subset of all tensors, Nakano positivity implies Griffiths positivity. The converse is false in general. For a line bundle every tensor is decomposable in the fiber index, so the notions coincide.

Curvature formulas must also track subbundle signs. If $S \subset E$ is a holomorphic subbundle with orthogonal second fundamental form B, then schematically

$$\Theta_S = \operatorname{pr}_S \Theta_E|_S - B^* \wedge B.$$

The negative square means a flat ambient bundle can have negatively curved subbundles; dualizing gives positive quotient bundles. This is the local mechanism behind positivity of the tautological quotient on projective space.

Checkpoint (test space before sign)

(1) Which positivity notion is naturally used in the Bochner estimate for vector-bundle-valued (n, q)-forms?
(2) Why do Griffiths and Nakano positivity agree in rank one?
(3) What happens to Griffiths curvature under dualizing?

Solution (checkpoint). The standard vector-bundle Bochner–Kodaira–Nakano estimate requires Nakano positivity of the relevant curvature operator. In rank one there is no independent fiber matrix index, so every test reduces to a tangent vector times a scalar. The dual curvature is the negative transpose, so Griffiths positivity becomes Griffiths negativity. ⋄

Exit criterion

You can write both curvature quadratic forms with indices, state the implication in the correct direction, and determine the curvature sign of a subbundle, quotient, or dual in a model case.

Summary. Griffiths positivity tests decomposable tensors, while Nakano positivity tests all tensors and is the form usually needed by Bochner vanishing. The second fundamental form measures exactly how much curvature a subbundle loses, or a quotient gains, relative to its ambient bundle.

Exercise 5.24: The quotient line bundle

Use the quotient curvature formula to reprove positivity of the Fubini–Study metric on $\mathcal{O}_{\mathbb{P}^n}(1)$.

Exercise 5.25: Tensor-product curvature

Prove $\Theta(E \otimes F) = \Theta(E) \otimes \mathrm{id}_F + \mathrm{id}_E \otimes \Theta(F)$ and discuss preservation of Griffiths positivity.

Optional handout: Positivity notions and the Griffiths conjecture

This handout is optional. No later argument in the core lectures depends on it.

Vector bundles carry several inequivalent notions of positivity. We use the convention that $\mathbb{P}(E^*)$ parametrizes lines in E^*, so $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is the corresponding quotient of π^*E. Besides Griffiths and Nakano positivity of curvature, a holomorphic bundle E is algebraically ample when this line bundle is ample. Griffiths positivity implies ampleness, because its curvature induces positive curvature in the tautological quotient directions.

The converse is the Griffiths conjecture: every ample vector bundle should admit a Griffiths-positive Hermitian metric. It is known in important special cases but remains open in general. This contrasts with line bundles, where Kodaira's theorem proves the equivalence.

Proposition 5.26: Curvature of the tautological quotient

Let $\pi : \mathbb{P}(E^*) \to X$, where $\mathbb{P}(E^*)$ parametrizes lines $\ell \subset E_x^*$. Then $\mathcal{O}(-1)$ is the tautological subline of π^*E^* and $\mathcal{O}(1) = \mathcal{O}(-1)^*$; equivalently, evaluation makes $\mathcal{O}(1)$ a quotient of π^*E. A Griffiths-positive metric on E induces a metric whose curvature is positive in horizontal directions and whose vertical restriction is the Fubini–Study form. Thus $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is positive.

Proof. At a point $[\ell] \in \mathbb{P}(E^*)$, choose a unit covector $\xi \in \ell$, put $v = \xi^\sharp \in E_x$ using the metric, and split a tangent vector τ into horizontal and vertical parts using the Chern connection. Choose a unitary frame for E whose connection matrix vanishes at the base point. Differentiating the logarithm of the quotient norm gives the quotient-curvature formula

$$\mathrm{i}\Theta_{\mathcal{O}(1)}(\tau, \bar{\tau}) = \frac{\langle \mathrm{i}\Theta_E(\tau_h, \bar{\tau}_h)v, v\rangle}{|v|^2} + \omega_{\mathrm{FS}}(\tau_v, \bar{\tau}_v).$$

The first term is positive for every nonzero horizontal vector by Griffiths positivity. The second is the squared norm of the second fundamental form of the quotient; on each projective fiber it is exactly the Fubini–Study metric and is positive for every nonzero vertical vector. Since the connection splitting has no mixed term at the chosen point, their sum is positive for every nonzero τ. The computation is frame invariant, so it proves positivity globally. □

Other useful notions include dual Nakano positivity, RC-positivity, and positivity of direct images. They are designed for different Bochner identities; there is no universal strongest notion that is best for every application.

Warning (Projectivization convention)

Some authors let $\mathbb{P}(E)$ parametrize lines in E, others one-dimensional quotients. The symbols $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ switch roles. State the convention before making a positivity assertion.

5.4 Lecture 20: Positive line bundles, Kodaira embedding, and projectivity

Week 10 material • 90 minutes

Notation for this lecture

$L \to X$ is a holomorphic line bundle, $L^k = L^{\otimes k}$, and $H^0(X, L^k) = \Gamma(X, L^k)$ is its space of global holomorphic sections. A Hermitian metric is positive if its Chern form is positive; L is ample if some power gives a projective embedding. For sections $s_0, \dots, s_N$ without a common zero, $\Phi_{|L^k|} : X \to \mathbb{P}^N, x \mapsto [s_0(x) : \cdots : s_N(x)]$, is the associated Kodaira map. The maximal ideal sheaf at x is $\mathfrak{m}_x$; $\mathcal{O}_X/\mathfrak{m}_x^2$ records values and first jets. ev_x denotes evaluation at x, and $\mathcal{O}_{\mathbb{P}^N}(1)$ is the hyperplane line bundle.

Prerequisite results used below

Embedding criterion. Let X be compact and let $s_0, \dots, s_N$ be global sections of a line bundle with no common zero. If the associated map $\Phi : X \to \mathbb{P}^N$ separates distinct points and its differential is injective at every point, then Φ is a holomorphic embedding. Indeed, it is an injective immersion, and compactness makes it a homeomorphism onto its image.
Jet interpretation. Surjectivity of $H^0(X, L^k) \to L^k_x \otimes (\mathcal{O}_{X,x}/\mathfrak{m}_x^2)$ means that global sections prescribe an arbitrary value and first derivative at x. Surjectivity onto $L^k_x \oplus L^k_y$ prescribes independent values at distinct points x, y. These two statements give immersion and point separation, respectively.

Plan for the lecture. 0–18 min: analytic positivity and ampleness; 18–40: separating points and tangent vectors; 40–62: the Kodaira map; 62–78: proof architecture; 78–90: positive classes on a blow-up.

Definition 5.27: Positive and ample line bundles

A holomorphic line bundle $L \to X$ is positive if it has a smooth Hermitian metric with $c_1(L, h) > 0$. On a compact complex manifold, L is ample if sections of some power L^k define a holomorphic embedding into projective space.

Theorem 5.28: Kodaira embedding theorem

A line bundle on a compact complex manifold is positive if and only if it is ample. Consequently, a compact Kähler manifold is projective if and only if it has an integral Kähler class.

If $s_0, \dots, s_N$ is a base-point-free basis of $H^0(X, L^k)$, define

$$\Phi_k : X \to \mathbb{P}^N, \qquad x \mapsto [s_0(x) : \cdots : s_N(x)].$$

Embedding requires two separate properties:

(1) for $x \neq y$, some sections have nonproportional values at x and y;
(2) for $0 \neq \xi \in T_x X$, some section vanishes at x but has nonzero first derivative along ξ.

Proof. Assume first that L has a metric of positive curvature. At $x \in X$, choose holomorphic coordinates and a frame e such that the metric weight has the expansion $\varphi(z) = \sum \lambda_j |z_j|^2 + O(|z|^3)$ with every $\lambda_j > 0$. For $|\alpha| \leq 1$, multiply $z^\alpha e^k$ by a cutoff supported in a ball of radius $(\log k)/\sqrt{k}$. Its $\bar\partial$ error lies in the surrounding annulus, where the factor $e^{-k\varphi}$ is smaller than every fixed power of k.

Apply the bundle-valued Hörmander estimate to solve

$$\bar{\partial} u_{k,\alpha} = \bar{\partial}(\chi z^\alpha e^k).$$

The curvature is $k\Theta(L)$, so $\|u_{k,\alpha}\| = O(k^{-N})$ for any prescribed N after choosing the cutoff radius as above. Insert the singular factor $(n+2)\log|z|^2$ in the weight. Finite norm then forces the value and first jet of $u_{k,\alpha}$ at x to vanish; equivalently, one may use the local mean-value and Cauchy inequalities to deduce the same jet estimate from the L^2 bound. Thus

$$s_{k,\alpha} = \chi z^\alpha e^k - u_{k,\alpha}$$

is global holomorphic and realizes the prescribed zero- or first-order jet at x.

For two distinct points, choose disjoint coordinate balls and repeat the construction with a weight singular at both points. The resulting sections separate their values. The zero-jet sections remove base points, the two-point construction separates points, and the first-jet sections make the Kodaira map an immersion. These properties are open in the points and tangent directions. Compactness of X, of $X \times X$ away from a small diagonal neighborhood, and of the unit tangent sphere bundle shows that one sufficiently large k works everywhere. An injective immersion from compact X to projective space is an embedding; hence L is ample.

Conversely, suppose $L^k = \Phi^* O(1)$ for a holomorphic embedding Φ. Pull back the Fubini–Study metric. Its curvature is positive because $d\Phi$ is injective. Define a metric on L by taking the positive kth root of the metric on L^k in every local frame. Its curvature is $k^{-1}\Phi^*\omega_{\mathrm{FS}} > 0$, so L is positive.

Finally, an integral Kähler class is of type $(1,1)$ and maps to zero in $H^2(X, O_X)$; exactness of the exponential sequence makes it $c_1(L)$ for a holomorphic line bundle. Its Kähler representative is the curvature of a positive metric after changing the metric by a global potential. The first part embeds X. The converse follows by restricting the integral Fubini–Study class from projective space. □

Warning

"Many sections" is not enough. Base-point freeness only defines a map, separation of points gives injectivity, and separation of first jets gives an immersion. Finally, use compactness of X and Hausdorffness of projective space to conclude that an injective immersion is an embedding.

Corollary 5.29: Asymptotic section growth

If L is positive and $\dim_{\mathbb{C}} X = n$, then

$$h^0(X, L^k) = \frac{k^n}{n!}\int_X c_1(L)^n + O(k^{n-1}).$$

This follows from Hirzebruch–Riemann–Roch plus higher-cohomology vanishing, or from Bergman-kernel asymptotics.

Proof. Hirzebruch–Riemann–Roch gives

$$\chi(X, L^k) = \int_X e^{kc_1(L)}\,\mathrm{td}(T_X) = \frac{k^n}{n!}\int_X c_1(L)^n + O(k^{n-1}).$$

For $k \gg 1$, the line bundle $L^k \otimes K_X^{-1}$ is positive: the term $k\Theta(L)$ dominates the fixed curvature of K_X^{-1}. Kodaira vanishing applied to $L^k = K_X \otimes (L^k \otimes K_X^{-1})$ gives $H^q(X, L^k) = 0$ for $q > 0$. Hence $h^0(X, L^k) = \chi(X, L^k)$, proving the formula. □

Example 5.30: Kähler classes on a blow-up

On $\widetilde{\mathbb{P}^2}$, the class $H - \varepsilon E$ is Kähler for $0 < \varepsilon \ll 1$. If $\varepsilon = a/b \in \mathbb{Q}$, then $bH - aE$ is an integral positive class corresponding to $\pi^*O(b) \otimes O(-aE)$.

Remark (Geometric interpretation)

Positive curvature localizes a peak section near one point. Solving $\bar{\partial}$ corrects the nonholomorphic error created by the cutoff. Kodaira's theorem is therefore the principle "positive curvature creates global coordinates."

5.4.1 Jet separation as a cohomology calculation

Let $Z \subset X$ be a zero-dimensional subscheme. The exact sequence

$$0 \longrightarrow L^k \otimes \mathcal{I}_Z \longrightarrow L^k \longrightarrow L^k|_Z \longrightarrow 0$$

shows that prescribed data on Z lift to a global section whenever

$$H^1(X, L^k \otimes \mathcal{I}_Z) = 0.$$

Take $Z = \{x\}$ to remove a base point, $Z = \{x, y\}$ to prescribe values at two points, and Z defined by $\mathfrak{m}_x^2$ to prescribe a value and first derivative at x. Kodaira's analytic construction proves these surjections directly by solving $\bar{\partial}$; Serre vanishing proves them algebraically once ampleness is known.

If a basis of sections gives $\Phi_k : X \to \mathbb{P}^N$, then

$$L^k \simeq \Phi_k^* O(1).$$

The point-separation condition makes Φ_k injective, and surjectivity onto $\mathfrak{m}_x/\mathfrak{m}_x^2$ makes $d\Phi_k$ injective. The bundle identity also shows why the converse direction of Kodaira is immediate: pull the positive Fubini–Study metric back through an embedding.

5.4.2 Numerical shadows of positivity

If L is positive and $V \subset X$ is an irreducible compact subvariety of dimension $p > 0$, then on the regular locus

$$\int_V c_1(L, h)^p > 0.$$

When V is singular this is interpreted through $[V]$ and Wirtinger's theorem. Thus analytic positivity implies positive intersection numbers on every subvariety. For projective manifolds, the Nakai–Moishezon criterion gives a converse using all such intersection numbers.

The asymptotic formula

$$h^0(X, L^k) = \frac{k^n}{n!} \int_X c_1(L)^n + O(k^{n-1})$$

has a local interpretation. A positive metric provides approximately one independent peak section per quantum cell of volume k^{-n}; integrating the local density yields the leading coefficient. Riemann–Roch packages the same count cohomologically.

On $\mathrm{Bl}_p \mathbb{P}^2$, a section of $aH - bE$ is a degree-a polynomial whose derivatives of order $< b$ vanish at p. The number of imposed linear conditions is

$$1 + 2 + \cdots + b = \frac{b(b+1)}{2}.$$

This concrete linear-system description makes the exceptional coefficient a jet condition rather than an abstract intersection number.

Warning

An integral $(1,1)$ class determines a holomorphic line bundle only after using the exponential sequence and the type condition. A positive real class need not be rational, so a compact Kähler manifold need not be projective. Kodaira's criterion requires an integral Kähler class, not merely a Kähler form.

5.4.3 Base points and jets as exact sequences

For $x \in X$ let $\mathfrak{m}_x$ be its ideal sheaf. Evaluation at x is encoded by

$$0 \longrightarrow L^k \otimes \mathfrak{m}_x \longrightarrow L^k \longrightarrow L^k_x \longrightarrow 0.$$

The resulting cohomology segment is

$$H^0(X, L^k) \longrightarrow L^k_x \longrightarrow H^1(X, L^k \otimes \mathfrak{m}_x).$$

If the last group vanishes, evaluation is surjective, so some section is nonzero at x and x is not a base point.

For first jets use

$$0 \to L^k \otimes \mathfrak{m}_x^2 \to L^k \to L^k \otimes (\mathcal{O}_X/\mathfrak{m}_x^2) \to 0.$$

The quotient has fiber

$$L^k_x \oplus (L^k_x \otimes T^{*1,0}_x X).$$

Surjectivity of global sections onto this quotient realizes any prescribed value and first derivative. For two points $x \neq y$, replacing $\mathfrak{m}_x$ by $\mathfrak{m}_x\mathfrak{m}_y$ gives independent values at x and y. These are the precise sheaf-theoretic conditions behind the three geometric requirements: base-point freeness, immersion, and point separation.

The Veronese embedding checked in coordinates

The complete linear system of $\mathcal{O}_{\mathbb{P}^n}(k)$ consists of all degree-k monomials Z^I. The associated map is

$$\nu_k : \mathbb{P}^n \to \mathbb{P}^N, \qquad [Z_0 : \cdots : Z_n] \longmapsto [Z^I]_{|I|=k}, \quad N = \binom{n+k}{k} - 1.$$

It is well defined because scaling Z by λ scales every coordinate by the same factor λ^k. On U_0, divide by Z_0^k; among the affine target coordinates are

$$1, \quad z_1, \dots, z_n, \quad z_1^2, z_1 z_2, \dots.$$

The first-degree entries show that the map is injective on this chart and that its Jacobian contains the identity matrix. Since every point lies in such a chart, ν_k is an injective immersion; compactness makes it an embedding.

The hyperplane bundle pulls back as

$$\nu_k^* \mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{\mathbb{P}^n}(k),$$

and hence $\nu_k^*[\omega_{\mathrm{FS}}] = k[\omega_{\mathrm{FS}}]$. This is both a transition- function calculation and a Chern-class check.

From local jet sections to one global exponent

Suppose that for each x there is an integer k_x for which sections of L^{k_x} generate first jets near x. Nonvanishing of an appropriate determinant of values and derivatives is an open condition, so the same sections work on a neighborhood U_x. Compactness gives finitely many U_{x_i}. Passing to a common sufficiently large power and multiplying by nonvanishing peak sections makes a single exponent work on all of X.

Point separation away from the diagonal is also compact: after fixing a small neighborhood of the diagonal, the remaining subset of $X \times X$ is compact and finitely many pairs of sections separate all its points. Near the diagonal, first-jet separation and the inverse-function theorem give local injectivity. Thus the analytic peak-section construction really does globalize to one Kodaira map.

A numerical calculation on a blow-up

On $\widetilde{\mathbb{P}^2}$ let H be the pullback of a line and E the exceptional curve. For $L = aH - bE$,

$$L \cdot E = b, \qquad L \cdot (H - E) = a - b, \qquad L^2 = a^2 - b^2.$$

Therefore a Kähler class must satisfy $b > 0$ and $a - b > 0$. Conversely, when $a > b > 0$, a small multiple of a curvature form positive along E can be combined with the pullback of Fubini–Study to produce a Kähler representative. If a, b are integers, sections correspond to degree-a homogeneous polynomials vanishing to order at least b at the blown-up point. The intersection inequalities are the numerical shadow of having enough such sections.

5.4.4 Self-study workshop: reducing an embedding theorem to two jet-surjectivity statements

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall base points, first jets, ideal sheaves of points, positive line-bundle metrics, and the compactness principle that turns local choices into finitely many global sections.

What an embedding must prove. A map

$$\Phi = [s_0 : \cdots : s_N] : X \longrightarrow \mathbb{P}^N$$

defined by sections of L^m is an embedding if it is everywhere defined, separates points, and separates tangent vectors. These conditions are encoded by coherent quotients. At a point x,

$$0 \to L^m \otimes \mathfrak{m}_x \to L^m \to L^m|_x \to 0$$

tests base-point freeness. For distinct x, y, use the quotient by $\mathfrak{m}_x \mathfrak{m}_y$ to prescribe independent values. To control first derivatives at x, use

$$L^m \otimes (\mathcal{O}_X / \mathfrak{m}_x^2).$$

Surjectivity onto this quotient says that the sections realize every value and cotangent direction, hence $d\Phi_x$ is injective.

Positive curvature supplies the analytic vanishing needed for these restriction maps. One chooses singular weights representing the point or jet conditions, solves the resulting $\bar{\partial}$ error with an L^2 estimate, and obtains global holomorphic sections. Compactness then reduces the point and tangent separation conditions to finitely many neighborhoods. Taking a common sufficiently large power produces one finite-dimensional linear system.

The model is $\mathcal{O}(1)$ on $\mathbb{P}^n$. Its sections $Z_0, \ldots, Z_n$ have no common zero, their ratios give the affine coordinates, and their derivatives span every cotangent space. The associated map is the identity projective embedding.

Checkpoint (three logically different conditions)

(1) Which quotient records first-order jets at x?
(2) Why does point separation alone not imply that Φ is an immersion?
(3) Where is compactness used after local sections have been constructed?

Solution (checkpoint). The quotient is $L^m \otimes \mathcal{O}_X/\mathfrak{m}_x^2$. Point separation controls only values at distinct points; a map can still have a zero differential. Compactness extracts finitely many sections and allows one exponent to work simultaneously over all points and tangent directions. ⋄

Exit criterion

You can decompose Kodaira embedding into base-point freeness, point separation, and tangent separation, express each by an ideal-sheaf sequence, and identify the analytic and compactness inputs.

Summary. On compact manifolds, differential-geometric positivity of a line bundle and algebraic ampleness are equivalent. The crucial step is not merely writing the projective map, but controlling values and first jets simultaneously by L^2 methods.

Exercise 5.31: The Veronese embedding

Write the Kodaira map for $\mathcal{O}_{\mathbb{P}^n}(k)$ and prove that it is the degree-k Veronese embedding.

Exercise 5.32: Integral Kähler classes

Explain why an integral class $[\omega] \in H^2(X, \mathbb{Z})$ is the first Chern class of a line bundle. Identify the relevant part of the exponential long exact sequence.

Optional handout: Peak sections and Bergman kernel asymptotics

This handout is optional. No later argument in the core lectures depends on it.

Let (L, h) be positive over a compact Kähler n-fold, with curvature form ω. Choose an L^2-orthonormal basis $s_1^{(k)}, \ldots, s_{N_k}^{(k)}$ of $H^0(X, L^k)$. The Bergman density is

$$\rho_k(x) = \sum_{j=1}^{N_k} |s_j^{(k)}(x)|_{h^k}^2.$$

It is basis independent and equals the squared norm of the evaluation functional.

The Tian–Catlin–Zelditch expansion states, in a common normalization,

$$\rho_k(x) \sim k^n + a_1(x)k^{n-1} + a_2(x)k^{n-2} + \cdots,$$

with a_1 a constant multiple of the scalar curvature of ω. Differentiated expansions imply

$$\frac{1}{k}\Phi_k^*\omega_{\mathrm{FS}} \longrightarrow \omega$$

smoothly, where Φ_k is the Kodaira map.

The local model is Bargmann–Fock space on $\mathbb{C}^n$ with Gaussian weight $\mathrm{e}^{-k|z|^2}$. A peak section looks like a constant or prescribed polynomial times this Gaussian in normal coordinates. A cutoff creates an exponentially small $\bar{\partial}$-error; Hörmander's estimate corrects it. Rescaling by $z \mapsto z/\sqrt{k}$ converts the local calculation to the universal Gaussian model.

Remark (Balanced metrics)

If ρ_k is constant, the metric is called k-balanced. Balanced metrics are finite- dimensional moment-map zeros and approximate constant-scalar-curvature Kähler metrics under suitable stability hypotheses.

Exercise 5.33: Bergman density of $\mathbb{P}^1$

Use rotational symmetry and $\int_{\mathbb{P}^1} \rho_k \omega_{\mathrm{FS}} = h^0(\mathcal{O}(k))$ to determine ρ_k for $\mathcal{O}(k)$ with the Fubini–Study metric.

Chapter 6

Hodge Theory, Kähler Identities, and Cohomology

6.1 Lecture 21: Elliptic operators, harmonic forms, and Hodge theory

Week 11 material • 90 minutes

Notation for this lecture

M is a compact oriented Riemannian manifold without boundary, $\mathcal{A}^k(M)$ is the space of smooth k-forms, and $(\alpha, \beta)_{L^2} = \int_M \alpha \wedge *\bar{\beta}$ uses the Hodge star $*$. For a differential operator P, P^* is its formal L^2 adjoint and $\sigma_P(x, \xi)$ its principal symbol at a nonzero cotangent vector ξ. The Hodge Laplacian is $\Delta_{\mathrm{d}} = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ and $\mathcal{H}^k(M) = \ker(\Delta_{\mathrm{d}} : \mathcal{A}^k \to \mathcal{A}^k)$ is the space of harmonic k-forms. The Green operator G is the inverse of Δ_{d} on the orthogonal complement of $\mathcal{H}^k(M)$, and H denotes orthogonal projection onto harmonic forms. The space H^s_{loc} is the local Sobolev space of order s.

Prerequisite results used below

Elliptic regularity. If P is elliptic of order m with smooth coefficients and $Pu = f$ distributionally, then $f \in H^s_{\mathrm{loc}}$ implies $u \in H^{s+m}_{\mathrm{loc}}$. In particular, distributional solutions of $Pu = 0$ are smooth.

Spectral theorem for a self-adjoint elliptic operator. On a compact manifold, a nonnegative formally self-adjoint elliptic operator has finite-dimensional smooth kernel and a discrete sequence of real eigenvalues tending to infinity. Its eigenvectors form an orthonormal basis of L^2, and its inverse on the orthogonal complement of the kernel is the Green operator.

Plan for the lecture. 0–20 min: the L^2 inner product and Hodge star; 20–42: adjoints and Laplacians; 42–68: elliptic regularity and Hodge decomposition; 68–82: de Rham isomorphism; 82–90: the Green operator.

Let (M^m, g) be compact and oriented. The Hodge star is defined by

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle_g \, \mathrm{d}V_g,$$

and on real k-forms satisfies $*^2 = (-1)^{k(m-k)}$. Complex forms require a consistent convention for conjugation.

Definition 6.1: Formal adjoint and Hodge Laplacian

On r-forms the formal adjoint of d is

$$\mathrm{d}^* = (-1)^{m(r+1)+1} * \mathrm{d} * .$$

The Hodge Laplacian is

$$\Delta_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d},$$

and $\mathcal{H}^k(M) = \ker(\Delta_{\mathrm{d}} : \mathcal{A}^k \to \mathcal{A}^k)$ is the space of harmonic k-forms.

Integration by parts gives

$$(\Delta_{\mathrm{d}}\alpha, \alpha) = \|\mathrm{d}\alpha\|_{L^2}^2 + \|\mathrm{d}^*\alpha\|_{L^2}^2 .$$

Thus on a compact manifold without boundary, α is harmonic exactly when $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$.

Warning

On a compact boundaryless manifold, formal adjoints agree with Hilbert-space adjoints after closure. With a boundary or on an incomplete manifold, the operator domain contains boundary conditions. Lecture 27 cannot reuse the present integration-by-parts formula without checking domains.

Theorem 6.2: Hodge decomposition

On a compact oriented Riemannian manifold,

$$\mathcal{A}^k(M) = \mathcal{H}^k(M) \oplus \operatorname{Im}\mathrm{d} \oplus \operatorname{Im}\mathrm{d}^*$$

orthogonally in L^2. Every de Rham class has a unique harmonic representative, so

$$\mathcal{H}^k(M) \simeq H^k_{\mathrm{dR}}(M, \mathbb{C}).$$

Proof (analytic core). Δ_{d} is second-order, elliptic, self-adjoint, and nonnegative. A Gårding inequality and Rellich compactness make its kernel finite dimensional and its range closed. On $\mathcal{H}^{k\perp}$, define the Green operator $G_{\mathrm{d}} = \Delta_{\mathrm{d}}^{-1}$; then

$$\alpha = H\alpha + \Delta_{\mathrm{d}}G_{\mathrm{d}}\alpha = H\alpha + \mathrm{d}\mathrm{d}^*G_{\mathrm{d}}\alpha + \mathrm{d}^*\mathrm{d}G_{\mathrm{d}}\alpha.$$

Elliptic regularity sends smooth input to smooth output. If $\mathrm{d}\alpha = 0$, the coexact component vanishes and $\alpha = H\alpha + \mathrm{d}(\mathrm{d}^*G_{\mathrm{d}}\alpha)$. A harmonic exact form has zero norm, proving uniqueness. □

Corollary 6.3: Poincaré duality from Hodge theory

The Hodge star maps $\mathcal{H}^k$ isomorphically to $\mathcal{H}^{m-k}$. Hence $b_k = b_{m-k}$ and the pairing $([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$ is nondegenerate.

Proof. The identities $*\Delta_{\mathrm{d}} = \Delta_{\mathrm{d}}*$ and $*^2 = (-1)^{k(m-k)}$ on k-forms show that $*$ restricts to an isomorphism $\mathcal{H}^k \simeq \mathcal{H}^{m-k}$. Hodge theory therefore gives $b_k = \dim\mathcal{H}^k = \dim\mathcal{H}^{m-k} = b_{m-k}$. If $0 \neq [\alpha] \in H^k(M, \mathbb{C})$, take its nonzero harmonic representative α and put $\beta = *\overline{\alpha}$. Then β is harmonic and

$$\int_M \alpha \wedge \beta = \int_M \alpha \wedge *\overline{\alpha} = \|\alpha\|_{L^2}^2 > 0.$$

Thus no nonzero class lies in the left kernel of the pairing. Applying the same argument in degree $m - k$, or using equality of dimensions, eliminates the right kernel as well. □

Example 6.4: Flat tori

On $T^m = \mathbb{R}^m/\mathbb{Z}^m$ with its flat metric, harmonic forms are exactly constant-coefficient forms. Thus $H^k(T^m, \mathbb{C}) \simeq \Lambda^k\mathbb{C}^m$ and $b_k = \binom{m}{k}$.

Remark (Geometric interpretation)

A cohomology class contains infinitely many closed forms. Its harmonic representative is the unique member of least L^2 norm. Hodge theory replaces a topological quotient by the finite-dimensional kernel of an elliptic equation.

6.1.1 Why the Hodge Laplacian is elliptic

For a first-order operator P, the principal symbol $\sigma_P(x,\xi)$ keeps only the highest derivative and replaces $\partial/\partial x_j$ by multiplication by ξ_j. For the exterior derivative,

$$\sigma_{\mathrm{d}}(x,\xi)\alpha = \xi \wedge \alpha,$$

while the symbol of d^* is contraction by the metric dual $\xi^\sharp$. The identity

$$\iota_{\xi^\sharp}(\xi \wedge \alpha) + \xi \wedge \iota_{\xi^\sharp}\alpha = |\xi|^2\alpha$$

therefore gives

$$\sigma_{\Delta_{\mathrm{d}}}(x,\xi) = |\xi|^2 \,\mathrm{id}\,.$$

It is invertible for $\xi \neq 0$, which is ellipticity.

Ellipticity yields the estimate

$$\|u\|_{H^{s+2}} \leq C_s\big(\|\Delta_{\mathrm{d}}u\|_{H^s} + \|u\|_{L^2}\big).$$

Thus a distributional solution of $\Delta_{\mathrm{d}}u = f$ gains two derivatives over f. Bootstrapping shows that harmonic distributions are smooth. Rellich compactness of $H^1 \hookrightarrow L^2$ on a compact manifold then forces the harmonic space to be finite-dimensional.

6.1.2 The Green operator as a chain homotopy

Let H be orthogonal projection onto harmonic forms and let G be the Green operator, zero on $\mathcal{H}^\bullet$ and inverse to Δ_{d} on its orthogonal complement. Then

$$I - H = \Delta_{\mathrm{d}}G = G\Delta_{\mathrm{d}}, \qquad [G, \mathrm{d}] = [G, \mathrm{d}^*] = 0.$$

For a closed form α this becomes

$$\alpha - H\alpha = \mathrm{d}\mathrm{d}^*G\alpha.$$

Therefore $K = \mathrm{d}^*G$ is a chain homotopy from the identity on the de Rham complex to harmonic projection. The abstract statement “cohomology equals harmonic forms” is encoded in this explicit operator identity.

For functions, harmonicity is especially rigid. If M is compact and connected,

$$0 = (\Delta f, f) = \|\mathrm{d}f\|^2$$

shows that harmonic functions are constant. For one-forms on a flat torus, constant coefficients are both closed and coclosed, producing the expected m-dimensional space. On a sphere, $H^1 = 0$, so every harmonic one-form vanishes although many eigen-one-forms with positive eigenvalue exist.

Remark (Spectral picture)

The compact resolvent of Δ_{d} gives eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$. Harmonic forms form the zero eigenspace; the Green operator multiplies the λ_j-eigenspace by λ_j^{-1}. Hodge decomposition is therefore the zero-frequency part of a full elliptic spectral decomposition.

6.1.3 Formal adjoints and the principal symbol

For compactly supported forms, Stokes gives

$$0 = \int_M \mathrm{d}(\alpha \wedge *\bar{\beta}) = \int_M \mathrm{d}\alpha \wedge *\bar{\beta} + (-1)^{\deg \alpha} \int_M \alpha \wedge \mathrm{d}(*\bar{\beta}).$$

After applying $*^2 = (-1)^{k(m-k)}$ and collecting the degree signs, one obtains

$$\mathrm{d}^* = (-1)^{m(k+1)+1} * \mathrm{d}*$$

on k-forms. The precise exponent depends on the Hodge-star convention, but the integration identity $(\mathrm{d}\alpha, \beta) = (\alpha, \mathrm{d}^*\beta)$ is invariant.

The principal symbol of d at a covector ξ is exterior multiplication:

$$\sigma_1(\mathrm{d})(\xi)\alpha = \mathrm{i}\xi \wedge \alpha.$$

Its adjoint has symbol $-\mathrm{i}\,\iota_{\xi^\sharp}$. Hence

$$\begin{aligned}\sigma_2(\Delta_{\mathrm{d}})(\xi) &= (\xi\wedge)\iota_{\xi^\sharp} + \iota_{\xi^\sharp}(\xi\wedge)\\ &= |\xi|^2\,\mathrm{id},\end{aligned}$$

because contraction and exterior multiplication satisfy the Clifford relation. This is invertible for $\xi \neq 0$, proving ellipticity.

The complete Hodge calculation on the circle

Write $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with coordinate θ. A function has Fourier series $f = \sum_{k\in\mathbb{Z}} a_k e^{\mathrm{i}k\theta}$, and

$$\Delta f = -f'' = \sum_k k^2 a_k e^{\mathrm{i}k\theta}.$$

Thus harmonic functions are constants. A one-form is $g(\theta)\mathrm{d}\theta$; since $* = 1$ sends $\mathrm{d}\theta$ to 1 and $*1 = \mathrm{d}\theta$,

$$\mathrm{d}^*(g\mathrm{d}\theta) = -g', \qquad \Delta(g\mathrm{d}\theta) = -g''\mathrm{d}\theta.$$

The harmonic one-forms are exactly $c\mathrm{d}\theta$.

For an arbitrary one-form, let

$$c = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,\mathrm{d}\theta, \qquad h(\theta) = \int_0^\theta (g(t) - c)\,\mathrm{d}t.$$

The zero-average condition makes h periodic, and

$$g\mathrm{d}\theta = c\mathrm{d}\theta + \mathrm{d}h.$$

This is the Hodge decomposition in degree one, written without functional analysis.

The Green operator from the spectral theorem

On a compact manifold, self-adjoint elliptic theory gives an orthonormal eigenbasis $\{\phi_j\}$ with

$$\Delta\phi_j = \lambda_j\phi_j, \qquad 0 = \lambda_0 = \cdots = \lambda_N < \lambda_{N+1} \leq \cdots \to \infty.$$

For $\alpha = \sum a_j\phi_j$, define

$$H\alpha = \sum_{j\leq N} a_j\phi_j, \qquad G\alpha = \sum_{j>N} \frac{a_j}{\lambda_j}\phi_j.$$

Then

$$\Delta G = G\Delta = I - H,$$

and elliptic estimates show that smooth α gives smooth $G\alpha$. Substituting $I - H = \mathrm{d}\mathrm{d}^*G + \mathrm{d}^*\mathrm{d}G$ produces the orthogonal decomposition explicitly.

The minimum-norm property

Let h be the harmonic representative of a de Rham class and let another representative be $h + \mathrm{d}\gamma$. Since $\mathrm{d}^*h = 0$,

$$(h, \mathrm{d}\gamma) = (\mathrm{d}^*h, \gamma) = 0.$$

Therefore

$$\|h + \mathrm{d}\gamma\|^2 = \|h\|^2 + \|\mathrm{d}\gamma\|^2 \geq \|h\|^2,$$

with equality only when $\mathrm{d}\gamma = 0$. Thus the harmonic representative is the unique minimum-norm element of its class.

On a flat torus, applying the Fourier calculation in every coordinate shows that all nonzero Fourier modes have positive Laplace eigenvalue $4\pi^2|k|^2$. Hence harmonic forms have only the zero mode and are exactly the constant-coefficient forms.

6.1.4 Self-study workshop: from ellipticity to the Hodge decomposition

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall formal adjoints, principal symbols, Sobolev compactness on a compact manifold, and orthogonal projections in a Hilbert space. Self-adjointness alone is not enough; ellipticity supplies regularity and finite dimensionality.

The analytic chain. For the de Rham operator, the principal symbols are exterior multiplication and contraction:

$$\sigma_{\mathrm{d}}(\xi) = \xi\wedge, \qquad \sigma_{\mathrm{d}^*}(\xi) = \iota_{\xi^\sharp}.$$

The anticommutator identity

$$\iota_{\xi^\sharp}(\xi\wedge\alpha) + \xi\wedge\iota_{\xi^\sharp}\alpha = |\xi|^2\alpha$$

gives

$$\sigma_{\Delta_{\mathrm{d}}}(\xi) = |\xi|^2\,\mathrm{id}\,.$$

Thus the Hodge Laplacian is elliptic. On a compact manifold, elliptic estimates and Rellich compactness imply a discrete spectrum, smooth finite-dimensional kernel, and a Green operator G on the orthogonal complement of harmonic forms.

The identity

$$I = \mathcal{H} + \Delta_{\mathrm{d}}G = \mathcal{H} + \mathrm{d}\mathrm{d}^*G + \mathrm{d}^*\mathrm{d}G$$

is the Hodge decomposition. If α is closed, then $\mathrm{d}G\alpha = G\mathrm{d}\alpha = 0$, so

$$\alpha = \mathcal{H}\alpha + \mathrm{d}(\mathrm{d}^*G\alpha).$$

Every de Rham class therefore has a harmonic representative. If a harmonic form is exact, say $\alpha = \mathrm{d}\beta$, then

$$\|\alpha\|^2 = (\alpha, \mathrm{d}\beta) = (\mathrm{d}^*\alpha, \beta) = 0,$$

so the representative is unique.

Checkpoint (identify each input)

(1) Which formula proves ellipticity of Δ_{d}?
(2) Why are harmonic forms smooth even if initially defined weakly?
(3) Why is compactness of the manifold relevant to finite-dimensionality?

Solution (checkpoint). The symbol identity $\sigma_{\Delta_{\mathrm{d}}}(\xi) = |\xi|^2\,\mathrm{id}$ is invertible for $\xi \neq 0$. Elliptic regularity upgrades weak kernel elements to smooth forms. Compactness gives global elliptic estimates and compact Sobolev inclusion; on a noncompact manifold the kernel and continuous spectrum can be much larger. ◇

Exit criterion

You can compute the symbol of the Hodge Laplacian, explain the roles of ellipticity and compactness, derive the closed-form decomposition, and prove uniqueness of harmonic representatives.

Summary. Ellipticity supplies finite dimensionality, regularity, and closed range. Orthogonal decomposition then identifies de Rham cohomology with harmonic forms. Boundary conditions and completeness determine when integration by parts is legitimate.

Exercise 6.5: The circle

Find the harmonic zero- and one-forms on S^1 and verify the Hodge decomposition directly.

Exercise 6.6: Minimum norm

Prove that the harmonic representative of a closed form is the unique L^2 minimizer in its de Rham class.

Optional handout: Heat kernels and the index viewpoint

This handout is optional. No later argument in the core lectures depends on it.

For a nonnegative self-adjoint elliptic operator Δ on a compact manifold, the heat operator $\mathrm{e}^{-t\Delta}$ has a smooth kernel and trace

$$\mathrm{Tr}(\mathrm{e}^{-t\Delta}) = \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j}.$$

As $t \to \infty$, only the zero eigenvalue survives, so the heat operator converges to orthogonal projection onto harmonic forms. This gives a dynamic version of Hodge theory.

For the de Rham complex, take the supertrace over even and odd forms. Positive eigenvalues cancel in pairs under $\mathrm{d} + \mathrm{d}^*$, leaving

$$\mathrm{Str}(\mathrm{e}^{-t\Delta_{\mathrm{d}}}) = \sum_k (-1)^k \dim \mathcal{H}^k(M) = \chi(M).$$

The left side is independent of t. As $t \downarrow 0$, the heat kernel becomes local; its asymptotic expansion identifies the limiting density with the Euler form. This is the analytic core of the Gauss–Bonnet–Chern theorem.

More generally, for an elliptic complex D, the supertrace computes its Fredholm index. Applied to the Dolbeault operator it leads to Hirzebruch–Riemann–Roch. Applied to a spin Dirac operator it gives the Atiyah–Singer index theorem.

Remark (Why ellipticity is decisive)

Ellipticity simultaneously provides a discrete spectrum, smoothing of the heat operator, finite-dimensional harmonic spaces, and a local small-time expansion. Hodge theory and index theory are two uses of the same analytic structure.

Exercise 6.7: Heat equation on the circle

Write the Fourier-series heat kernel on S^1 and verify directly that it converges to the projection onto constants as $t \to \infty$.

6.2 Lecture 22: Kähler identities, the $\partial\bar\partial$-lemma, and Hard Lefschetz

Week 11 material • 90 minutes

Notation for this lecture

(X, ω) is a compact Kähler manifold of complex dimension n. The Lefschetz operator is $L(\alpha) = \omega \wedge \alpha$, its formal adjoint is Λ, and $H = [\Lambda, L] = \Lambda L - L\Lambda$ is the grading operator. ∂^* and $\bar\partial^*$ are formal adjoints, while $\Delta' = \partial\partial^* + \partial^*\partial$ and $\Delta'' = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ are the two complex Laplacians. $\Delta_\partial = \Delta'$ and $\Delta_{\bar\partial} = \Delta''$, while $\Delta_{\mathrm{d}} = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ is the Hodge Laplacian. $\mathcal{H}^{p,q}(X) = \ker \Delta'' \cap \mathcal{A}^{p,q}(X)$ denotes harmonic forms of type (p, q). A form α is primitive if $\Lambda\alpha = 0$. The Green operator G is the inverse of the relevant Laplacian on the orthogonal complement of its harmonic kernel. $H^k_{\mathrm{dR}}(X, \mathbb{C})$ denotes complex de Rham cohomology.

Prerequisite results used below

Hodge decomposition. For the Hodge Laplacian on a compact oriented Riemannian manifold,

$$\mathcal{A}^k = \mathcal{H}^k \oplus \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^*, \qquad H^k_{\mathrm{dR}}(M, \mathbb{C}) \simeq \mathcal{H}^k.$$

The three summands are mutually orthogonal and every harmonic form is smooth.
Lefschetz linear algebra. On a Hermitian vector space of complex dimension n, the operators $L = \omega\wedge$, $\Lambda = L^*$, and $H = [\Lambda, L]$ satisfy the $\mathfrak{sl}_2$ commutation relations. Every form is uniquely a sum of L^r times primitive forms. This pointwise statement is the algebraic input for Hard Lefschetz once the Kähler identities show that L preserves harmonic forms.

Plan for the lecture. 0–20 min: L, Λ, and Lefschetz linear algebra; 20–48: Kähler commutators; 48–65: three Laplacians; 65–78: the $\partial\bar\partial$-lemma; 78–90: Hard Lefschetz.

On a Kähler manifold (X^n, ω), let $L\alpha = \omega \wedge \alpha$, let Λ be its adjoint, and set $H = [\Lambda, L]$. Pointwise linear algebra gives an $\mathfrak{sl}_2$ representation.

Theorem 6.8: Kähler identities

With the standard adjoint convention,

$$[\Lambda, \partial] = \mathrm{i}\bar\partial^*, \qquad [\Lambda, \bar\partial] = -\mathrm{i}\partial^*, \qquad [L, \partial^*] = \mathrm{i}\bar\partial, \qquad [L, \bar\partial^*] = -\mathrm{i}\partial.$$

Consequently,

$$\Delta_{\mathrm{d}} = 2\Delta_\partial = 2\Delta_{\bar\partial}, \qquad [\Delta_{\mathrm{d}}, L] = [\Delta_{\mathrm{d}}, \Lambda] = 0.$$

Proof. The identities are local first-order operator identities, so fix a point and choose a unitary coframe $(\theta_1, \ldots, \theta_n)$ that is normal for the Kähler connection. Thus $\omega = \mathrm{i}\sum_j \theta_j \wedge \bar\theta_j$ at the chosen point. Write $e_j = \theta_j\wedge$ and $\bar e_j = \bar\theta_j\wedge$, with adjoint contractions $e_j^*, \bar e_j^*$. At that point,

$$L = \mathrm{i}\sum_j e_j \bar e_j, \qquad \Lambda = -\mathrm{i}\sum_j \bar e_j^* e_j^*, \qquad \partial = \sum_j e_j \nabla_j, \qquad \bar\partial = \sum_j \bar e_j \nabla_{\bar j}.$$

The relations $[e_j, e_k^*]_+ = \delta_{jk}$ and $[\bar e_j, \bar e_k^*]_+ = \delta_{jk}$, while barred and unbarred operators anticommute, give $[\Lambda, \partial] = \mathrm{i}\bar\partial^*$ and $[\Lambda, \bar\partial] = -\mathrm{i}\partial^*$. Taking adjoints gives the other two commutators. Normal coordinates eliminate derivatives of the metric at the point, and the Kähler condition eliminates torsion; hence the pointwise calculation is intrinsic.

Now expand

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial = -\mathrm{i}[\bar\partial, [\Lambda, \partial]]$$

and use the graded Jacobi identity together with $\partial\bar\partial+\bar\partial\partial = 0$. The result is $\Delta_{\bar\partial} = \Delta_\partial$. Since $\mathrm{d} = \partial+\bar\partial$ and the mixed anticommutators vanish by the same identities, $\Delta_{\mathrm{d}} = 2\Delta_\partial = 2\Delta_{\bar\partial}$. Finally the commutators of L and Λ with the Laplacian vanish. On a general Hermitian manifold, the normal-frame calculation contains torsion terms involving $\partial\omega$. □

Corollary 6.9: Kähler Hodge decomposition

If X is compact Kähler, then

$$H^k_{\mathrm{dR}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X), \qquad \overline{H^{p,q}} = H^{q,p}.$$

Thus $b_k = \sum_{p+q=k} h^{p,q}$, and every odd Betti number is even.

Proof. The Kähler identities give $\Delta_{\mathrm{d}} = 2\Delta_{\bar\partial}$. Since $\Delta_{\bar\partial}$ preserves bidegree, the decomposition of a harmonic k-form into types is again harmonic:

$$\mathcal{H}^k_{\mathrm{d}}(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar\partial}(X).$$

Real Hodge theory identifies the left side with de Rham cohomology, while Dolbeault Hodge theory identifies each summand with $H^{p,q}_{\bar\partial}(X)$. Complex conjugation commutes with the real Laplacian and interchanges types (p,q) and (q,p), proving $\overline{H^{p,q}} = H^{q,p}$. Taking dimensions gives the Betti-number formula. If k is odd, no pair (p,q) with $p+q = k$ has $p = q$; the summands occur in equal-dimensional conjugate pairs, so b_k is even. □

This rules out a Hopf surface as Kähler because $b_1 = 1$.

Theorem 6.10: The $\partial\bar\partial$-lemma

Let α be a d-closed pure-type form on a compact Kähler manifold. The following are equivalent:

$$\alpha \in \operatorname{Im}\mathrm{d}, \quad \alpha \in \operatorname{Im}\partial, \quad \alpha \in \operatorname{Im}\bar\partial, \quad \alpha \in \operatorname{Im}\partial\bar\partial.$$

In particular, a pure-type form $\alpha = \mathrm{d}\beta$ can be written $\alpha = \partial\bar\partial\gamma$.

Proof. The implication "$\partial\bar\partial$-exact implies exact in all three other senses" is immediate. We prove the converse carefully.

First suppose that the pure-type form α is d-exact. Its harmonic projection is zero. Let G_{d} be the Green operator for Δ_{d}. Since $\mathrm{d}\alpha = 0$, purity gives $\partial\alpha = \bar\partial\alpha = 0$. The Kähler identities and commutation of the Green operator with ∂ and $\bar\partial$ therefore give

$$\alpha = \Delta_{\mathrm{d}} G_{\mathrm{d}}\alpha = 2\Delta_{\bar\partial} G_{\mathrm{d}}\alpha = 2\bar\partial\bar\partial^* G_{\mathrm{d}}\alpha,$$

so α is $\bar\partial$-exact; the analogous formula with ∂ shows that it is ∂-exact.

It remains to show that a $\bar\partial$-closed, ∂-exact form is $\partial\bar\partial$-exact. Let $G_{\bar\partial}$ be the $\bar\partial$-Green operator. Exactness forces the harmonic projection of α to vanish, while $\bar\partial\alpha = 0$ gives

$$\alpha = \bar\partial\bar\partial^* G_{\bar\partial}\alpha.$$

Moreover $\partial\alpha = 0$ and $[G_{\bar\partial}, \partial] = 0$. Using $\bar\partial^* = -\mathrm{i}[\Lambda, \partial]$ we obtain

$$\bar\partial^* G_{\bar\partial}\alpha = \mathrm{i}\partial\Lambda G_{\bar\partial}\alpha, \qquad \alpha = -\mathrm{i}\partial\bar\partial\Lambda G_{\bar\partial}\alpha.$$

This is the desired representation. Interchanging ∂ and $\bar\partial$ proves the remaining case and completes all equivalences. □

Theorem 6.11: Hard Lefschetz

If X is compact Kähler of dimension n, then for $0 \leq k \leq n$,

$$L^{n-k} : H^k(X, \mathbb{C}) \xrightarrow{\sim} H^{2n-k}(X, \mathbb{C}).$$

Every cohomology class has a unique Lefschetz decomposition into powers of L times primitive classes.

Proof. Pointwise, L, Λ, and $H = [\Lambda, L]$ satisfy the $\mathfrak{sl}_2$ relations $[H, L] = 2L$ and $[H, \Lambda] = -2\Lambda$. A form of degree $k \leq n$ is primitive if $\Lambda\alpha = 0$, equivalently $L^{n-k+1}\alpha = 0$. Finite-dimensional $\mathfrak{sl}_2$ representation theory gives the orthogonal Lefschetz decomposition

$$\Lambda^k T^* X = \bigoplus_{r \geq 0} L^r P^{k-2r}$$

and shows that $L^{n-k} : \Lambda^k \longrightarrow \Lambda^{2n-k}$ is an isomorphism on the corresponding irreducible strings.

The Kähler identities imply $[\Delta_{\mathrm{d}}, L] = [\Delta_{\mathrm{d}}, \Lambda] = 0$, so the Lefschetz decomposition preserves harmonic forms. Apply the pointwise $\mathfrak{sl}_2$ isomorphism to the finite-dimensional space of harmonic forms and then use the Hodge isomorphism between harmonic forms and de Rham cohomology. This proves both Hard Lefschetz and the uniqueness of the cohomological Lefschetz decomposition. □

Warning

The $\partial\bar{\partial}$-lemma is not the local Dolbeault lemma. It depends on compactness and the Kähler identities. On a general compact complex manifold, a pure-type form that is d-exact and $\bar{\partial}$-closed need not be $\partial\bar{\partial}$-exact.

6.2.1 Primitive forms in low dimension

On a Kähler surface, every real $(1, 1)$-form has a pointwise decomposition

$$\alpha = \alpha_0 + \frac{1}{2}(\Lambda\alpha)\omega, \qquad \Lambda\alpha_0 = 0.$$

The primitive part α_0 is orthogonal to ω. For a harmonic form the decomposition is global because L and Λ commute with the Laplacian. Hence

$$H^2(X, \mathbb{C}) = LH^0(X, \mathbb{C}) \oplus P^2(X, \mathbb{C}).$$

The Kähler class spans the first summand, and the primitive summand contains the trace-free degree-two information.

In complex dimension three, Hard Lefschetz includes the isomorphisms

$$L^3 : H^0 \xrightarrow{\sim} H^6, \qquad L^2 : H^1 \xrightarrow{\sim} H^5, \qquad L : H^2 \xrightarrow{\sim} H^4.$$

These statements imply both Poincaré duality and extra restrictions on the cup product. For example, multiplication by a Kähler class cannot annihilate a nonzero class in H^2.

6.2.2 Concrete consequences of the $\partial\bar{\partial}$-lemma

Let α and β be closed real $(1, 1)$-forms on a compact Kähler manifold with the same de Rham class. Their difference is pure type and exact, so

$$\alpha - \beta = \mathrm{i}\partial\bar{\partial}\varphi = 2\pi \mathrm{d}\mathrm{d}^{\mathrm{c}}\varphi$$

for a real-valued smooth function φ, after adjusting the primitive by conjugation. Thus all Kähler forms in a fixed class are described by global Kähler potentials subject to the positivity inequality $\beta + \mathrm{i}\partial\bar{\partial}\varphi > 0$.

The lemma also identifies several cohomologies. A $\bar{\partial}$-closed (p, q)-form that is de Rham exact is $\partial\bar{\partial}$-exact, so the natural map from Bott–Chern classes to de Rham classes is injective in pure type. In particular, a holomorphic form on a compact Kähler manifold is closed: its $\bar{\partial}$ vanishes by holomorphicity, and the Kähler identity gives

$$\|\partial\alpha\|^2 \leq (\Delta_{\partial}\alpha, \alpha) = (\Delta_{\bar{\partial}}\alpha, \alpha) = 0.$$

Combining Hodge symmetry with type decomposition gives

$$b_{2k+1} = 2 \sum_{p<q,\ p+q=2k+1} h^{p,q},$$

so every odd Betti number is even. Hard Lefschetz gives further inequalities such as $b_k \leq b_{k+2}$ for $k < n$, since multiplication by $[\omega]$ is injective in that range.

Warning

Hodge decomposition as a vector-space isomorphism is weaker than the $\partial\bar{\partial}$-lemma, and Poincaré duality alone is weaker than Hard Lefschetz. When using a topological obstruction to Kählerness, state exactly which consequence fails.

6.2.3 Lefschetz linear algebra in complex dimension two

At a point choose a unitary coframe θ_1, θ_2 with

$$\omega = \mathrm{i}\theta_1 \wedge \bar{\theta}_1 + \mathrm{i}\theta_2 \wedge \bar{\theta}_2.$$

A $(1, 1)$-form has the decomposition

$$\alpha = \frac{\Lambda\alpha}{2}\,\omega + \alpha_0, \qquad \Lambda\alpha_0 = 0.$$

For example,

$$\mathrm{i}\theta_1 \wedge \bar{\theta}_1 - \mathrm{i}\theta_2 \wedge \bar{\theta}_2, \quad \theta_1 \wedge \bar{\theta}_2, \quad \theta_2 \wedge \bar{\theta}_1$$

are primitive. Direct wedge multiplication shows $\alpha_0 \wedge \omega = 0$, while $L^2(1) = \omega^2 \neq 0$. Thus the degree-two forms split into the one-dimensional Lefschetz direction $\mathbb{C}\omega$ and the primitive subspace.

The commutator can be checked on the vacuum 1:

$$[\Lambda, L]1 = \Lambda\omega = 2,$$

and on a top form: $[\Lambda, L]\omega^2 = -2\omega^2$. In general it acts on k-forms by $(n - k)$ with the corresponding convention, producing the finite $\mathfrak{sl}_2$ strings behind Hard Lefschetz.

Two global consequences with every integration step

If the Kähler form of a compact n-fold were exact, $\omega = \mathrm{d}\eta$, then

$$\omega^n = \mathrm{d}(\eta \wedge \omega^{n-1})$$

because $\mathrm{d}\omega = 0$. Stokes would give $\int_X \omega^n = 0$, contradicting positivity of the volume form. Therefore the Kähler class is nonzero.

For the first Betti number, Kähler Hodge decomposition gives

$$H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X).$$

Complex conjugation is an antilinear isomorphism between the two summands, so

$$b_1 = h^{1,0} + h^{0,1} = 2h^{1,0}.$$

This proves evenness and rules out every compact complex manifold with odd b_1 from being Kähler.

A concrete use of the $\partial\bar\partial$-lemma

Let ω_0 and ω_1 be cohomologous Kähler forms. Their difference $\alpha = \omega_1 - \omega_0$ is real, of type $(1,1)$, d-closed, and de Rham exact. The $\partial\bar\partial$-lemma gives a complex function f with $\alpha = \partial\bar\partial f$. Since $\overline{\partial\bar\partial f} = -\partial\bar\partial\bar f$ and α is real,

$$\partial\bar\partial\left(\frac{f-\bar f}{2}\right) = \alpha.$$

The function $\varphi = (2\pi/\mathrm{i})(f-\bar f)/2$ is real, and our normalization therefore gives

$$\omega_1 = \omega_0 + \mathrm{dd}^{\mathrm{c}}\varphi.$$

The potential is unique modulo constants on a compact connected manifold: if $\mathrm{dd}^{\mathrm{c}}\varphi = 0$, then φ is pluriharmonic and hence harmonic, so the maximum principle makes it constant.

Hard Lefschetz on projective space

The cohomology ring is

$$H^*(\mathbb{P}^n,\mathbb{C}) = \mathbb{C}[h]/(h^{n+1}), \qquad h = [\omega_{\mathrm{FS}}], \quad \deg h = 2.$$

In degree $2k$, the class h^k spans. The Lefschetz map is

$$L^{n-2k} : H^{2k}(\mathbb{P}^n) \longrightarrow H^{2n-2k}(\mathbb{P}^n), \qquad h^k \longmapsto h^{n-k},$$

which is an isomorphism between one-dimensional spaces. Odd cohomology vanishes. This elementary calculation is the model for the general harmonic-form theorem.

6.2.4 Self-study workshop: using the Kähler identities to manufacture a second primitive

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall the Lefschetz operator $L = \omega\wedge$ and its adjoint Λ, the Green operator, type decomposition, and commutation of the Kähler Laplacians. The $\partial\bar\partial$-lemma is a global theorem; the local Dolbeault lemma alone does not prove it.

A proof of one useful form of the lemma. Let α be a pure-type form on a compact Kähler manifold with $\bar\partial\alpha = 0$ and suppose $\alpha = \partial\beta$. Being ∂-exact, α is orthogonal to ∂-harmonic forms. Hodge theory for Δ_∂ gives

$$\alpha = \partial\partial^* G\alpha.$$

The Kähler identity

$$\partial^* = \mathrm{i}[\Lambda, \bar\partial]$$

and the commutation of G with $\bar\partial$ imply

$$\partial^* G\alpha = \mathrm{i}(\Lambda\bar\partial - \bar\partial\Lambda)G\alpha = -\mathrm{i}\bar\partial\Lambda G\alpha.$$

Therefore

$$\alpha = -\mathrm{i}\partial\bar\partial(\Lambda G\alpha).$$

This is the missing global primitive. The proof visibly uses compact Hodge theory, Kähler commutators, purity of type, and closedness.

The same commutator algebra yields $\Delta_{\mathrm{d}} = 2\Delta_\partial = 2\Delta_{\bar\partial}$. Consequently harmonic forms preserve bidegree, and de Rham cohomology decomposes into Dolbeault groups. The $\mathfrak{sl}_2$ relations among L, Λ, and the degree operator act on harmonic forms and give Hard Lefschetz:

$$L^{n-k} : H^k(X,\mathbb{C}) \xrightarrow{\sim} H^{2n-k}(X,\mathbb{C}).$$

Checkpoint (locate the global step)

(1) Why is a ∂-exact form orthogonal to ∂-harmonic forms?
(2) Where is $\bar\partial\alpha = 0$ used in the displayed proof?
(3) Which conclusion forces the odd Betti numbers of a compact Kähler manifold to be even?

Solution (checkpoint). If h is harmonic, then $\partial^* h = 0$, so $(\partial\beta, h) = (\beta, \partial^* h) = 0$. Closedness gives $\bar\partial G\alpha = G\bar\partial\alpha = 0$, eliminating the first commutator term. Hodge decomposition gives $H^m(X,\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}$ with complex conjugation pairing $H^{p,q}$ and $H^{q,p}$; when m is odd there are no diagonal terms $p = q$, so the dimension is even. ◇

Exit criterion

You can reconstruct the Green-operator proof of the $\partial\bar\partial$-lemma, name every Kähler identity used, and derive at least one topological consequence of Hodge decomposition or Hard Lefschetz.

Summary. The Kähler identities synchronize the de Rham, ∂, and $\bar\partial$ Laplacians. They produce Hodge type decomposition, the $\partial\bar\partial$-lemma, and Hard Lefschetz—testable topological consequences of being Kähler.

Exercise 6.12: Evenness of b_1

Using conjugation and harmonic one-forms directly, prove that b_1 is even on a compact Kähler manifold.

Exercise 6.13: A Kähler form is not exact

Prove that the Kähler form of a compact Kähler manifold cannot be de Rham exact.

Optional handout: Formality and Massey products

This handout is optional. No later argument in the core lectures depends on it.

A differential graded algebra $(A^\bullet, d)$ is formal if it is connected to its cohomology algebra $(H^\bullet(A), 0)$ by quasi-isomorphisms. The $\partial\bar\partial$-lemma implies that the de Rham algebra of a compact Kähler manifold is formal: use the subalgebra of ∂-closed forms and the maps to de Rham and $\bar\partial$ cohomology.

Formality is stronger than knowing the Betti numbers or the cup product. It implies that all higher Massey products vanish. A triple Massey product begins with classes a, b, c satisfying $ab = bc = 0$, chooses primitives $dx = ab$, $dy = bc$, and forms the secondary class represented by $ay \pm xc$. Its indeterminacy comes from the choices of primitives.

Theorem 6.14: Deligne–Griffiths–Morgan–Sullivan

Every compact Kähler manifold is formal over $\mathbb{R}$ and $\mathbb{C}$.

Proof. Work first over $\mathbb{R}$ and put $\mathrm{d^c} = \mathrm{i}(\bar\partial - \partial)$; the normalization is irrelevant here. Let $K = \ker \mathrm{d^c} \subset \mathcal{A}^\bullet(X)$. Because $\mathrm{dd^c} + \mathrm{d^c d} = 0$, K is a differential graded subalgebra. The $\mathrm{dd^c}$-lemma has two consequences.

First, the inclusion $(K, \mathrm{d}) \hookrightarrow (\mathcal{A}^\bullet, \mathrm{d})$ is a quasi-isomorphism. Indeed, every de Rham class has a harmonic representative, and a Kähler harmonic form is both d- and $\mathrm{d^c}$-closed, proving surjectivity. If $\alpha \in K$ is d-closed and d-exact in $\mathcal{A}^\bullet$, then the $\mathrm{dd^c}$-lemma gives $\alpha = \mathrm{dd^c}\gamma = \mathrm{d}(\mathrm{d^c}\gamma)$; its primitive $\mathrm{d^c}\gamma$ lies in K, proving injectivity.

Second, send $\alpha \in K$ to its $\mathrm{d^c}$-cohomology class. If $\alpha = \mathrm{d^c}\beta$ also lies in K, this map kills it. The induced map

$$(K, \mathrm{d}) \longrightarrow (H^\bullet_{\mathrm{d^c}}(X), 0)$$

is a morphism of differential graded algebras: for $\mathrm{d^c}\alpha = 0$, the form $\mathrm{d}\alpha$ is d-exact and $\mathrm{d^c}$-closed, hence $\mathrm{d}\alpha = \mathrm{dd^c}\gamma$ and is zero in $\mathrm{d^c}$-cohomology. The same $\mathrm{dd^c}$-lemma argument shows that this map induces an isomorphism on cohomology. We have therefore constructed a zigzag of DGA quasi-isomorphisms

$$(\mathcal{A}^\bullet(X), \mathrm{d}) \longleftarrow (K, \mathrm{d}) \longrightarrow (H^\bullet_{\mathrm{d^c}}(X), 0).$$

The last algebra is naturally isomorphic to de Rham cohomology. This is formality over $\mathbb{R}$; tensoring with $\mathbb{C}$ proves the complex statement. □

This yields topological obstructions to Kähler structures that are invisible to Hodge number parity. Certain nilmanifolds have nonzero Massey products and hence cannot be Kähler, even when their odd Betti numbers happen to be even.

Remark (Limits of the obstruction)

Formality is necessary but not sufficient for a manifold to be Kähler. The Kähler cone, Hodge decomposition, and hard Lefschetz impose additional constraints.

Exercise 6.15: A formal algebra

Show that the de Rham algebra of a sphere is formal by constructing a quasi-isomorphism from a minimal algebra with one generator (and, in even dimension, one relation-killing generator) to differential forms.

6.3 Lecture 23: Dolbeault, Čech, and sheaf cohomology

Week 12 material • 90 minutes

Notation for this lecture

$\Gamma(X, \mathcal{F})$ is the space of global sections of a sheaf $\mathcal{F}$ and $H^q(X, \mathcal{F})$ its sheaf cohomology. A fine sheaf admits endomorphisms subordinate to every locally finite open cover and is acyclic for global sections. The sheaf Ω^p_X consists of holomorphic p-forms and $\mathcal{A}^{p,q}_X$ of smooth (p,q)-forms. Dolbeault cohomology is $H^{p,q}_{\bar\partial}(X) = \ker(\bar\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1})/\mathrm{Im}(\bar\partial : \mathcal{A}^{p,q-1} \to \mathcal{A}^{p,q})$. For a cover $\mathfrak{U} = \{U_i\}$, $\check{C}^q(\mathfrak{U}, \mathcal{F})$ denotes Čech cochains; the shorter symbol $C^q(\mathfrak{U}, \mathcal{F})$ has the same meaning in this lecture. The symbol δ is their coboundary; a Leray cover is one whose finite intersections are acyclic for $\mathcal{F}$.

Prerequisite results used below

Acyclic-resolution theorem. If $0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots$ is an exact resolution and $H^q(X, \mathcal{G}^p) = 0$ for all $q > 0$ and $p \geq 0$, then $H^q(X, \mathcal{F})$ is naturally the degree-q cohomology of the complex of global sections $\Gamma(X, \mathcal{G}^\bullet)$.
Fine sheaves are acyclic. On a paracompact Hausdorff space, every fine sheaf $\mathcal{G}$ satisfies $H^q(X, \mathcal{G}) = 0$ for $q > 0$. Sheaves of smooth differential forms are fine because a smooth partition of unity acts by sheaf endomorphisms subordinate to the chosen cover.

Plan for the lecture. 0–18 min: derived functors; 18–40: fine resolutions; 40–60: Dolbeault isomorphism; 60–78: the Čech–Dolbeault double complex; 78–90: long exact sequences and calculations.

Definition 6.16: Sheaf cohomology

The global-section functor $\Gamma(X, -)$ is left exact. Its right derived functors are denoted $H^q(X, \mathcal{F})$ and can be computed by injective, flabby, or fine resolutions, or by Čech complexes on suitable covers.

Definition 6.17: Fine sheaf

A sheaf $\mathcal{F}$ is fine if every locally finite open cover admits a subordinate partition of unity by sheaf endomorphisms. Modules over smooth functions are fine because ordinary smooth partitions of unity act by multiplication. Fine sheaves are acyclic on paracompact spaces.

The local Dolbeault lemma gives a fine resolution

$$0 \to \Omega^p_X \to \mathcal{A}^{p,0}_X \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_X \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,n}_X \to 0.$$

Theorem 6.18: Dolbeault isomorphism

For every complex manifold and holomorphic bundle E,

$$H^q(X, \Omega^p_X \otimes \mathcal{O}(E)) \simeq H^{p,q}_{\bar\partial}(X, E).$$

Proof. The E-valued Dolbeault complex is a fine resolution. The acyclic-resolution theorem identifies sheaf cohomology with the cohomology of global sections, namely $\ker\bar\partial/\mathrm{Im}\,\bar\partial$. □

Theorem 6.19: Leray–Čech computation

If a cover $\mathfrak{U}$ is acyclic for $\mathcal{F}$, meaning every finite intersection has $H^q(-,\mathcal{F}) = 0$ for $q > 0$, then $\check{H}^q(\mathfrak{U},\mathcal{F}) \simeq H^q(X,\mathcal{F})$. A Stein cover is acyclic for coherent sheaves.

Proof. Choose an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ and form the double complex

$$C^{r,s} = C^r(\mathfrak{U},\mathcal{I}^s), \qquad D = \delta + (-1)^r d_{\mathcal{I}}.$$

Compute the cohomology of its total complex in two ways. Each injective sheaf is flabby, hence its augmented Čech complex is exact. Taking horizontal cohomology first therefore leaves the complex of global sections $\Gamma(X,\mathcal{I}^\bullet)$, whose cohomology is $H^q(X,\mathcal{F})$.

Taking vertical cohomology first gives, on every intersection $U_{i_0\ldots i_r}$, the groups $H^s(U_{i_0\ldots i_r},\mathcal{F})$. The acyclicity assumption kills every row with $s > 0$, while the row $s = 0$ is precisely the Čech complex $C^\bullet(\mathfrak{U},\mathcal{F})$. Hence the same total cohomology is $\check{H}^q(\mathfrak{U},\mathcal{F})$. The two identifications are natural and give the claimed isomorphism. If all finite intersections are Stein and $\mathcal{F}$ is coherent, Cartan's Theorem B supplies the required acyclicity. □

6.3.1 The double-complex dictionary

Consider $C^r(\mathfrak{U},\mathcal{A}^{p,s})$ with horizontal differential δ and vertical differential $(-1)^r\bar\partial$. The total differential squares to zero. Taking Čech cohomology first leaves only global smooth forms by fineness and produces the global Dolbeault complex. Taking Dolbeault cohomology first produces $C^r(\mathfrak{U},\Omega^p)$. The two routes explain the Dolbeault isomorphism and give an explicit conversion between Čech and Dolbeault representatives.

Example 6.20: $H^1(\mathbb{P}^1,\mathcal{O}(k))$

Cover $\mathbb{P}^1$ by $U_0, U_1 \simeq \mathbb{C}$, with overlap $\mathbb{C}^*$. A one-cocycle is a Laurent series. Subtracting the parts that extend to either chart leaves a finite block of negative powers. The result is

$$h^0(\mathcal{O}(k)) = \max\{k+1, 0\}, \qquad h^1(\mathcal{O}(k)) = \max\{-k-1, 0\}.$$

Warning

Čech cohomology for an arbitrary cover need not equal sheaf cohomology; refine and take a limit, or verify the Leray condition. Likewise, global sections do not generally preserve right exactness. Their failure is measured by H^1.

6.3.2 The explicit Čech–Dolbeault conversion

Let $\{U_i\}$ be a cover and let (g_{ij}) be a holomorphic one-cocycle for a holomorphic vector bundle E. Choose a smooth partition of unity (ρ_k) subordinate to the cover and put

$$b_i = \sum_k \rho_k g_{ik} \quad \text{on } U_i.$$

The cocycle identity gives $b_i - b_j = g_{ij}$ on $U_i \cap U_j$. Therefore

$$\alpha|_{U_i} = \bar\partial b_i$$

defines one global E-valued $(0,1)$-form: the local expressions agree because $\bar\partial g_{ij} = 0$. Moreover $\bar\partial\alpha = 0$. Changing the partition of unity or the cocycle representative changes α by a global $\bar\partial$-exact form.

Conversely, given a $\bar\partial$-closed $(0,1)$-form α, choose local primitives $\alpha = \bar\partial b_i$. Then $g_{ij} = b_i - b_j$ is holomorphic and satisfies the cocycle condition. The two constructions are inverse on cohomology. Higher degrees repeat this procedure along the rows and columns of the Čech–Dolbeault double complex.

6.3.3 The complete calculation for $\mathcal{O}(k)$ on $\mathbb{P}^1$

Cover $\mathbb{P}^1$ by $U_0 \simeq \mathbb{C}_z$ and $U_\infty \simeq \mathbb{C}_w$, with $w = z^{-1}$. Choose frames so that $e_\infty = z^k e_0$. A one-cocycle is a holomorphic function on $\mathbb{C}^*$ times e_0, hence has a Laurent expansion

$$h(z) = \sum_{m\in\mathbb{Z}} a_m z^m.$$

A section on U_0 removes all terms with $m \geq 0$. A section $b(w)e_\infty = b(z^{-1})z^k e_0$ on U_∞ removes all terms with $m \leq k$. The surviving powers are therefore

$$z^{-1}, z^{-2}, \dots, z^{k+1} \quad \text{when } k \leq -2,$$

where the last exponent is negative. Consequently

$$h^1(\mathbb{P}^1, \mathcal{O}(k)) = \begin{cases} 0, & k \geq -1, \\ -k-1, & k \leq -2. \end{cases}$$

Global sections are degree-k homogeneous polynomials when $k \geq 0$, so

$$h^0(\mathbb{P}^1, \mathcal{O}(k)) = \begin{cases} k+1, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

The difference $h^0 - h^1 = k+1$ is Riemann–Roch in genus zero, and the symmetry $h^1(\mathcal{O}(k)) = h^0(\mathcal{O}(-k-2))$ anticipates Serre duality.

Remark (When Čech computes derived cohomology)

An arbitrary cover need not compute sheaf cohomology. It does when every finite intersection is acyclic for the sheaf, as in a Leray cover. For coherent sheaves on a Stein cover, Cartan B supplies this acyclicity. The cover is therefore part of the calculation, not harmless notation.

6.3.4 The contracting homotopy for a fine sheaf

Let $\{U_i\}$ be a locally finite cover with a subordinate partition of unity $\{\rho_i\}$. For a Čech q-cochain c, define

$$(hc)_{i_0\dots i_{q-1}} = \sum_j \rho_j c_{j i_0 \dots i_{q-1}}.$$

Local finiteness makes the sum meaningful. A direct alternating-sum computation uses $\sum_j \rho_j = 1$ and gives

$$\delta h + h\delta = \mathrm{id}$$

in positive Čech degrees. Hence every Čech cocycle with values in a fine sheaf is a coboundary. Refinement and the standard comparison theorem then give vanishing of all higher sheaf cohomology.

Explicit conversion between Čech and Dolbeault classes

Let (g_{ij}) be a holomorphic one-cocycle with $g_{ij} + g_{jk} + g_{ki} = 0$. Put

$$a_i = \sum_j \rho_j g_{ij}$$

on U_i, extending each term by zero where ρ_j vanishes. The cocycle identity gives

$$a_i - a_k = \sum_j \rho_j (g_{ij} - g_{kj}) = g_{ik}.$$

Therefore $\bar{\partial} a_i = \bar{\partial} a_k$ on overlaps, and the local forms glue to a global $(0,1)$-form

$$\alpha|_{U_i} = \bar{\partial} a_i.$$

It is $\bar{\partial}$-closed. Changing the partition or the representative cocycle changes α by a global $\bar{\partial}$-exact form, so $[\alpha]$ depends only on $[g]$.

Conversely, for a global $\bar{\partial}$-closed $(0,1)$-form α, the local Dolbeault lemma gives a_i with $\alpha = \bar{\partial} a_i$. Then

$$g_{ij} = a_i - a_j$$

is holomorphic and is a Čech cocycle. The two constructions are inverse on cohomology. This is the Dolbeault isomorphism in degree one with representatives, not merely an abstract identification.

The full $O(k)$ calculation on $\mathbb{P}^1$

Let $z = Z_1/Z_0$ on U_0 and $w = Z_0/Z_1 = z^{-1}$ on U_1. Choose frames $e_0 = Z_0^k$ and $e_1 = Z_1^k$, so

$$e_1 = z^k e_0.$$

A global section is a pair of entire functions $f_0(z), f_1(w)$ satisfying

$$f_0(z) = z^k f_1(1/z).$$

If $k \geq 0$, holomorphy at both 0 and ∞ forces f_0 to be a polynomial of degree at most k, giving the basis $1, z, \ldots, z^k$. If $k < 0$, the same growth condition forces the section to vanish. Thus

$$h^0(O(k)) = \max\{k+1, 0\}.$$

A one-cochain is a holomorphic function $h(z)$ on $\mathbb{C}^*$, expressed in the e_0 frame. Coboundaries have the form

$$h(z) = z^k f_1(1/z) - f_0(z),$$

with f_0, f_1 entire. In the Laurent series of h, the f_0 term removes every power z^m with $m \geq 0$, while $z^k f_1(1/z)$ removes every power with $m \leq k$. Only

$$z^{k+1}, z^{k+2}, \ldots, z^{-1}$$

survive when $k \leq -2$. Therefore

$$H^1(\mathbb{P}^1, O(k)) = \begin{cases} 0, & k \geq -1, \\ \operatorname{span}\{z^{k+1}, \ldots, z^{-1}\}, & k \leq -2, \end{cases}$$

and $h^1 = \max\{-k-1, 0\}$.

The exponential sequence and $H^1(\mathbb{P}^1, O) = H^2(\mathbb{P}^1, O) = 0$ give

$$\operatorname{Pic}(\mathbb{P}^1) \simeq H^2(\mathbb{P}^1, \mathbb{Z}) \simeq \mathbb{Z}.$$

The generator is $O(1)$, and the integer is the degree.

6.3.5 Self-study workshop: seeing the same cohomology class in Čech and Dolbeault form

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall Čech cocycles, partitions of unity, fine sheaves, the Dolbeault resolution, and the difference between local exactness and global exactness.

Convert a cocycle explicitly. Let (f_{ij}) be a Čech 1-cocycle of holomorphic functions for a locally finite cover (U_i), and choose a partition of unity (ρ_k). On U_i define

$$a_i = \sum_k \rho_k f_{ki}.$$

On $U_i \cap U_j$, the cocycle identity gives

$$a_j - a_i = \sum_k \rho_k (f_{kj} - f_{ki}) = \sum_k \rho_k f_{ij} = f_{ij}.$$

The forms $\bar{\partial} a_i$ therefore agree on overlaps and glue to a global $\bar{\partial}$-closed $(0,1)$-form α. Changing the partition or representatives changes α by a global $\bar{\partial}$-exact form. This constructs the map

$$\check{H}^1(X, \mathcal{O}_X) \longrightarrow H^{0,1}_{\bar{\partial}}(X).$$

Conversely, if α is a global $\bar{\partial}$-closed $(0,1)$-form, the local Dolbeault lemma gives $\alpha = \bar{\partial} u_i$ on U_i. The differences $u_i - u_j$ are holomorphic and form a Čech cocycle. Changing local primitives changes it by a coboundary. The two constructions are inverse at the level of classes.

The abstract reason is that

$$0 \to \mathcal{O}_X \to \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \to \cdots$$

is an exact resolution by fine, hence acyclic, sheaves. The coordinate construction above is the degree-one shadow of that theorem.

Checkpoint (local primitive, global obstruction)

(1) Why is a_i smooth rather than holomorphic?
(2) Why do the forms $\bar{\partial} a_i$ glue?
(3) If the resulting Dolbeault class vanishes, what does that say about (f_{ij})?

Solution (checkpoint). The partition functions ρ_k are smooth and usually not holomorphic. Since $a_j - a_i = f_{ij}$ and $\bar{\partial} f_{ij} = 0$, the local forms have zero difference. If $\alpha = \bar{\partial} u$ globally, then $a_i - u$ are holomorphic and their differences are f_{ij}, so the Čech cocycle is a coboundary. ⋄

Exit criterion

You can pass in both directions between a degree-one Čech cocycle and a Dolbeault class, verify all overlap signs, and explain the role of fine resolutions.

Summary. Dolbeault cohomology computes sheaf cohomology through a fine resolution; Čech theory makes gluing obstructions explicit on overlaps. The double complex shows that these are two directions through the same object.

Exercise 6.21: Exponential sequence and the Picard group

Derive $H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$ from the exponential sequence and identify $\mathrm{Pic}^0(X)$.

Exercise 6.22: Explicit Čech–Dolbeault conversion

Given a one-cocycle $g_{ij} \in \mathcal{O}(U_{ij})$, use a partition of unity to construct a $(0, 1)$-form representing its Dolbeault class.

Optional handout: Hypercohomology and two spectral sequences

This handout is optional. No later argument in the core lectures depends on it.

For a complex of sheaves $\mathcal{K}^\bullet$, its hypercohomology $\mathbb{H}^m(X, \mathcal{K}^\bullet)$ derives global sections and the internal differential at the same time. Resolve each $\mathcal{K}^q$ by injective or fine sheaves and take the cohomology of the resulting total complex.

Filtering the double complex in two directions gives two spectral sequences:

$$E_1^{p,q} = H^q(X, \mathcal{K}^p) \Longrightarrow \mathbb{H}^{p+q}(X, \mathcal{K}^\bullet),$$

and

$$'E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{K}^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(X, \mathcal{K}^\bullet).$$

The first computes cohomology sheaf by sheaf; the second computes the cohomology sheaves first.

Example 6.23: The holomorphic de Rham complex

For $\mathcal{O}_X \xrightarrow{\partial} \Omega_X^1 \to \cdots \to \Omega_X^n$, the holomorphic Poincaré lemma says its cohomology sheaves are the constant sheaf $\mathbb{C}_X$ in degree zero and zero otherwise. Hence its hypercohomology is $H^*(X, \mathbb{C})$. The first spectral sequence is exactly the Frölicher spectral sequence $E_1^{p,q} = H^q(X, \Omega_X^p)$.

Hypercohomology is also the natural home for deformation complexes, variations of Hodge structure, and derived direct images. It records not only groups but the maps between them.

Exercise 6.24: A two-term complex

For $\mathcal{K}^\bullet = [\mathcal{F} \xrightarrow{u} \mathcal{G}]$ in degrees $0, 1$, write the first five terms of the long exact sequence relating $\mathbb{H}^*(\mathcal{K}^\bullet)$ to $H^*(\mathcal{F})$ and $H^*(\mathcal{G})$.

6.4 Lecture 24: Serre duality, the Frölicher spectral sequence, and curves

Week 12 material • 90 minutes

Notation for this lecture

X is a compact complex n-manifold, $K_X = \Omega^n_X$ its canonical bundle, and E^* the dual of a holomorphic vector bundle E. $\mathcal{O}(E)$ is the sheaf of holomorphic sections of E. Serre duality pairs $H^q(X,E)$ with $H^{n-q}(X, K_X \otimes E^*)$. We write $h^q(X,E) = \dim_{\mathbb{C}} H^q(X,E)$ and $\chi(X,E) = \sum_q (-1)^q h^q(X,E)$. For a compact Riemann surface, $g = h^1(X, \mathcal{O}_X)$ is the genus and $\deg L$ the degree of a line bundle. The Frölicher spectral sequence has first page $E_1^{p,q} = H^q(X, \Omega^p_X)$, differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, and converges to $H^{p+q}_{\mathrm{dR}}(X,\mathbb{C})$. The Hodge numbers are $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p_X)$.

Prerequisite results used below

Dolbeault theorem with coefficients. For a holomorphic vector bundle $E \to X$, the complex of sheaves of smooth E-valued $(0,q)$-forms is a fine resolution of $\mathcal{O}(E)$; hence $H^q(X,E) \simeq H^{0,q}_{\bar\partial}(X,E)$.
Stokes pairing. If X is compact without boundary, α is an E-valued $(0,q)$-form, and β is a $K_X \otimes E^*$-valued $(0, n-q-1)$-form, then $\int_X \operatorname{tr}(\bar\partial\alpha \wedge \beta) = (-1)^{q+1} \int_X \operatorname{tr}(\alpha \wedge \bar\partial\beta)$. Thus integration descends to Dolbeault cohomology and defines the pairing used in Serre duality.

Plan for the lecture. 0–22 min: the Serre pairing; 22–42: Riemann–Roch for curves; 42–62: Frölicher; 62–78: Kähler degeneration; 78–90: Hopf surfaces and non-Kähler diagnostics.

Theorem 6.25: Serre duality

Let X be a compact complex n-fold and E a holomorphic vector bundle. The pairing

$$H^q(X,E) \times H^{n-q}(X, K_X \otimes E^*) \to \mathbb{C}, \qquad ([\alpha],[\beta]) \mapsto \int_X \operatorname{tr}(\alpha \wedge \beta)$$

is nondegenerate. Hence $H^q(X,E)^* \simeq H^{n-q}(X, K_X \otimes E^*)$.

Proof. Choose Hermitian metrics on X and E. Dolbeault Hodge theory identifies $H^q(X,E)$ with the harmonic E-valued $(0,q)$-forms. Combining the ordinary Hodge star, complex conjugation, and the metric identification $\overline{E} \simeq E^*$ gives a conjugate-linear isometry

$$\# : \mathcal{A}^{0,q}(E) \longrightarrow \mathcal{A}^{n,n-q}(E^*) = \mathcal{A}^{0,n-q}(K_X \otimes E^*),$$

characterized by

$$\operatorname{tr}(\alpha \wedge \#\alpha) = |\alpha|^2\, dV.$$

The local formula for the formal adjoint shows that $\#$ interchanges $\bar\partial$ and $\bar\partial^*$ up to the harmless degree sign. It consequently commutes with the $\bar\partial$-Laplacian and maps harmonic forms isomorphically to harmonic forms of the displayed complementary degree.

Let $0 \neq [\alpha] \in H^q(X,E)$ and choose its harmonic representative. Then $\beta = \#\alpha$ is closed and

$$\int_X \operatorname{tr}(\alpha \wedge \beta) = \|\alpha\|^2_{L^2} > 0.$$

Thus the pairing has no left kernel. Applying the same argument to $K_X \otimes E^*$ and degree $n-q$ shows that it has no right kernel. Harmonic spaces on a compact manifold are finite dimensional, so the induced injection into the dual is an isomorphism. Although metrics were used to prove nondegeneracy, Stokes' theorem shows directly that the integral depends only on the two cohomology classes; hence the pairing itself is canonical. □

Theorem 6.26: Riemann–Roch for a compact curve

If X has genus g and L is a line bundle, then

$$h^0(L) - h^0(K_X \otimes L^{-1}) = \deg L + 1 - g.$$

In particular, $\deg K_X = 2g - 2$, and $H^0(X, L) = 0$ if $\deg L < 0$.

Proof (from Euler characteristic). For an effective divisor D, the exact sequence

$$0 \to L \to L(D) \to L(D)|_D \to 0$$

and its long exact cohomology sequence give $\chi(L(D)) = \chi(L) + \deg D$. We also justify the divisor description of an arbitrary line bundle without using Riemann–Roch. Fix $p \in X$. Iterating the preceding exact sequence gives $\chi(L(kp)) = \chi(L) + k$. Since a curve has no sheaf cohomology above degree one, $h^0(L(kp)) \geq \chi(L) + k > 0$ for k large. A nonzero section of $L(kp)$ is a meromorphic section of L; its divisor D identifies $L \simeq \mathcal{O}(D)$.

Applying the Euler-characteristic identity to the positive and negative parts of D now gives

$$\chi(L) = \deg L + \chi(\mathcal{O}_X) = \deg L + 1 - g.$$

Serre duality identifies $h^1(L) = h^0(K_X \otimes L^{-1})$, which proves the formula. Taking $L = K_X$ and using $h^0(K_X) = h^1(\mathcal{O}_X) = g$ gives $\deg K_X = 2g-2$. Finally, a nonzero holomorphic section has an effective zero divisor of degree $\deg L$; hence no such section exists when $\deg L < 0$. □

Definition 6.27: Frölicher spectral sequence

Filter the double complex $(\mathcal{A}^{p,q}, \partial, \bar{\partial})$ by p. The resulting spectral sequence is

$$E_1^{p,q} = H_{\bar{\partial}}^{p,q}(X) \Longrightarrow H_{\mathrm{dR}}^{p+q}(X, \mathbb{C}), \qquad d_1 = [\partial].$$

For every compact complex manifold, $b_k \leq \sum_{p+q=k} h^{p,q}$. Equality for every k is equivalent to degeneration at E_1. On a compact Kähler manifold, equality of the three Laplacians gives E_1 degeneration and a canonical decomposition.

Example 6.28: Diagnosing a Hopf surface

A Hopf surface has $b_1 = 1$. If it were Kähler, Hodge symmetry would give $b_1 = h^{1,0} + h^{0,1} = 2h^{1,0}$, a contradiction. Thus topology detects non-Kählerness. Conversely, E_1 degeneration alone does not always imply Kähler.

Warning

"Hodge decomposition" can mean the Riemannian harmonic decomposition, the type decomposition of cohomology, or E_1 degeneration of Frölicher. They synchronize naturally only in the compact Kähler case; always state which meaning is intended.

6.4.1 The Serre pairing in Dolbeault language

Represent $[\alpha] \in H^{0,q}_{\bar\partial}(X, E)$ and $[\beta] \in H^{n,n-q}_{\bar\partial}(X, E^*)$ by smooth closed forms. Contract the bundle indices and wedge the form parts:

$$\langle [\alpha], [\beta] \rangle = \int_X \operatorname{tr}(\alpha \wedge \beta).$$

If α changes by $\bar\partial\gamma$, Stokes' theorem and $\bar\partial\beta = 0$ show that the integral is unchanged. Hodge theory identifies the dual of a harmonic E-valued $(0, q)$-form with a harmonic E^*-valued $(n, n-q)$-form, proving nondegeneracy.

For a line bundle L on a compact curve C, this becomes

$$H^1(C, L)^* \simeq H^0(C, K_C \otimes L^{-1}).$$

If $\deg L > 2g-2$, the bundle on the right has negative degree and no nonzero section, so $H^1(C, L) = 0$. Riemann–Roch then gives

$$h^0(C, L) = \deg L + 1 - g.$$

Taking $L = K_C$ gives $h^0(K_C) = g$, while taking $L = \mathcal{O}_C$ gives $h^1(\mathcal{O}_C) = g$.

6.4.2 What the Frölicher differentials measure

Filter the de Rham complex by holomorphic degree. The first page is

$$E_1^{p,q} = H^{p,q}_{\bar\partial}(X),$$

and d_1 is induced by ∂: if $\bar\partial\alpha = 0$, then $\bar\partial(\partial\alpha) = 0$ and $d_1[\alpha] = [\partial\alpha]$. A class killed by d_1 has a representative whose ∂ can be corrected by a $\bar\partial$-primitive. Later differentials record successive failures of these corrections.

On a compact Kähler manifold, every Dolbeault class has a harmonic representative and the Kähler identities make it both ∂- and $\bar\partial$-closed. Hence all differentials vanish at E_1, and

$$b_k = \sum_{p+q=k} h^{p,q}.$$

For a general compact complex manifold, the spectral sequence still converges to de Rham cohomology, but cancellations may occur. The alternating sum is unchanged:

$$\sum_{p,q} (-1)^{p+q} h^{p,q} = \sum_k (-1)^k b_k,$$

because every differential removes equal-dimensional pieces in adjacent total degrees.

Remark (Curves are automatically Kähler)

Every Hermitian form on a complex curve is a real two-form of top degree, hence is closed. Compact Riemann surfaces are therefore Kähler, their Frölicher sequence degenerates at E_1, and their Hodge diamond is determined by the genus.

6.4.3 Why the Serre pairing descends to cohomology

Let α be an E-valued $(0, q)$-form and β a $K_X \otimes E^*$-valued $(0, n-q)$-form. Their contraction is an (n, n)-form. If α is changed by $\bar\partial\gamma$ and $\bar\partial\beta = 0$, then

$$\operatorname{tr}(\bar\partial\gamma \wedge \beta) = \bar\partial \operatorname{tr}(\gamma \wedge \beta)$$

up to the standard degree sign. On compact X, Stokes gives

$$\int_X \bar\partial \operatorname{tr}(\gamma \wedge \beta) = 0.$$

The same calculation in the second variable proves that the integral depends only on cohomology classes. Compactness is essential here; on a noncompact manifold a boundary-at-infinity term may remain.

Riemann–Roch on the sphere and an elliptic curve

For $X = \mathbb{P}^1$, $g = 0$ and $K_X = \mathcal{O}(-2)$. Riemann–Roch gives

$$h^0(\mathcal{O}(k)) - h^0(\mathcal{O}(-k-2)) = k + 1.$$

Using the elementary fact that a negative-degree line bundle has no nonzero section recovers

$$h^0(\mathcal{O}(k)) = \max\{k+1, 0\},$$

and Serre duality gives the H^1 formula from Lecture 23.

On an elliptic curve E, $K_E \simeq \mathcal{O}_E$ and $g = 1$. For a line bundle L of positive degree d, the dual L^{-1} has negative degree, hence

$$h^1(L) = h^0(L^{-1}) = 0, \qquad h^0(L) = d.$$

For a degree-zero line bundle, a nonzero section has an effective divisor of degree zero and therefore no zeros; it trivializes the bundle. Consequently $h^0(L) = 1$ for $L \simeq \mathcal{O}_E$ and 0 for every nontrivial degree-zero L.

Computing the first Frölicher differential

Represent a class in $E_1^{p,q} = H^{p,q}_{dbar}(X)$ by a form α with $\bar\partial\alpha = 0$. Since

$$\bar\partial(\partial\alpha) = -\partial(\bar\partial\alpha) = 0,$$

the form $\partial\alpha$ determines a Dolbeault class of type $(p+1, q)$. The first differential is

$$d_1[\alpha]_{dbar} = [\partial\alpha]_{dbar}.$$

If α is replaced by $\alpha + \bar\partial\gamma$, then

$$\partial(\alpha + \bar\partial\gamma) = \partial\alpha - \bar\partial(\partial\gamma),$$

so the resulting class is unchanged. Thus d_1 measures the first obstruction to choosing a representative that is closed under both ∂ and $\bar\partial$.

If X is compact Kähler, choose the harmonic representative of $[\alpha]$. Equality of the Laplacians makes it both ∂- and $\bar\partial$-closed, so $d_1 = 0$. The $\partial\bar\partial$-lemma eliminates all higher zigzag obstructions, giving $E_1 = E_\infty$.

Euler characteristic survives every differential

For any finite-dimensional spectral sequence, passing from a page to the next takes cohomology of a complex. The alternating sum of dimensions is unchanged by taking cohomology. Therefore

$$\sum_{p,q} (-1)^{p+q} \dim E_1^{p,q} = \sum_k (-1)^k \dim H^k_{\mathrm{dR}}(X, \mathbb{C}).$$

Substituting the first page gives

$$\sum_{p,q}(-1)^{p+q}h^{p,q} = \chi_{\mathrm{top}}(X),$$

whether or not the spectral sequence degenerates. Individual Betti numbers require control of differentials; the Euler characteristic does not.

6.4.4 Self-study workshop: Serre duality and curve calculations

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall Dolbeault representatives, the canonical bundle, dual bundles, Euler characteristics, and the first page $E_1^{p,q} = H^q(X, \Omega_X^p)$ of the Frölicher spectral sequence.

Check the Serre pairing by bidegree. Represent a class of $H^q(X, E)$ by an E-valued $(0, q)$-form α and a class of $H^{n-q}(X, K_X \otimes E^*)$ by an E^*-valued $(n, n-q)$-form β. Contracting the bundle factors and wedging gives an (n, n)-form:

$$([\alpha], [\beta]) \longmapsto \int_X \mathrm{tr}(\alpha \wedge \beta).$$

If α changes by $\bar\partial\gamma$, Stokes' theorem and $\bar\partial\beta = 0$ show that the integral is unchanged. Serre duality asserts that this pairing is perfect, not merely well defined.

On a compact Riemann surface of genus g, duality gives

$$h^1(L) = h^0(K_X \otimes L^{-1}).$$

Riemann–Roch then reads

$$h^0(L) - h^0(K_X \otimes L^{-1}) = \deg L + 1 - g.$$

If $\deg L > 2g - 2$, the dual bundle has negative degree and no nonzero holomorphic section, so $h^1(L) = 0$ and $h^0(L) = \deg L + 1 - g$. This is a complete dimension calculation from two structural theorems and one elementary fact about zeros.

The Frölicher differential $d_1 : H^q(X, \Omega_X^p) \to H^q(X, \Omega_X^{p+1})$ is induced by ∂. On a compact Kähler manifold the $\partial\bar\partial$-lemma forces all differentials to vanish and the sequence degenerates at E_1. On a general compact complex manifold this need not happen; the discrepancy measures the failure of Dolbeault groups to assemble directly into de Rham cohomology.

Checkpoint (a curve calculation)

Let X have genus g and let $\deg L = 2g$.

(1) Why is $H^1(X, L) = 0$?

(2) Compute $h^0(X, L)$.

(3) What bidegree does the Serre-dual partner of a $(0, 1)$ class have?

Solution (checkpoint). The bundle $K_X \otimes L^{-1}$ has degree $(2g - 2) - 2g = -2$, so it has no nonzero holomorphic sections; duality gives $H^1(X, L) = 0$. Riemann–Roch gives $h^0(L) = 2g + 1 - g = g + 1$. Since $n = 1$ and $q = 1$, the dual partner has type $(1, 0)$ with values in L^*. ◇

Exit criterion

You can verify the Serre pairing by type and Stokes, combine it with Riemann–Roch for an explicit curve computation, and explain what degeneration of the Frölicher sequence means.

Summary. Serre duality pairs degrees q and $n-q$. Riemann–Roch converts a difference of analytic dimensions into degree and genus. The Frölicher spectral sequence measures how far Dolbeault data remain from a de Rham type decomposition.

Exercise 6.29: Line bundles of high degree

If $\deg L > 2g-2$, prove $H^1(X,L)=0$ and $h^0(L)=\deg L+1-g$.

Exercise 6.30: Euler characteristic

Prove $\chi_{\mathrm{top}}(X)=\sum_{p,q}(-1)^{p+q}h^{p,q}$ for every compact complex manifold, even when Frölicher does not degenerate at E_1.

Optional handout: Jacobians and the Abel–Jacobi map

This handout is optional. No later argument in the core lectures depends on it.

Let X be a compact Riemann surface of genus g. Its Jacobian is the complex torus

$$\mathrm{Jac}(X)=H^0(X,K_X)^*/H_1(X,\mathbb{Z}),$$

where a cycle γ acts on a holomorphic one-form by integration. Choosing a basis of holomorphic one-forms and a symplectic basis of $H_1(X,\mathbb{Z})$ gives the period-matrix description $\mathbb{C}^g/(\mathbb{Z}^g+\tau\mathbb{Z}^g)$ with $\operatorname{Im}\tau>0$.

Fix $p_0\in X$. The Abel–Jacobi map sends

$$p\longmapsto\left(\omega\mapsto\int_{p_0}^{p}\omega\right)\quad\bmod H_1(X,\mathbb{Z}).$$

Changing the path changes the integral by a period, so the class is well defined. Extending additively gives a map from degree-zero divisors to the Jacobian.

Theorem 6.31: Abel's theorem

A degree-zero divisor D is principal if and only if its Abel–Jacobi image is zero. Consequently $\mathrm{Pic}^0(X)\simeq\mathrm{Jac}(X)$.

Proof. Write $D=\sum_p n_p p$ with $\sum n_p=0$. If $D=(f)$, the logarithmic differential df/f has residue n_p at p. Cut the surface along a symplectic basis $a_1,b_1,\dots,a_g,b_g$ to a polygon and apply the residue theorem to $(\int^z\omega)\,df/f$, where ω is holomorphic. The boundary terms are periods of ω multiplied by the integral multiples $2\pi\mathrm{i}$ coming from the change of $\log f$, while the residues give $2\pi\mathrm{i}\sum n_p\int_{p_0}^{p}\omega$. Thus the Abel–Jacobi value of D is a period, hence zero in the Jacobian.

Conversely, Riemann–Roch provides a differential of the third kind η with simple poles at the support of D and residues n_p: one may prescribe the principal parts because their total residue is zero, the only Serre-duality obstruction. Subtract a holomorphic differential so that all a-periods of η vanish. The Riemann bilinear relation then identifies its b-period vector, modulo $2\pi\mathrm{i}\mathbb{Z}^g$, with $2\pi\mathrm{i}$ times the Abel–Jacobi image of D. If that image is zero, a further integral-period normalization makes every period of η belong to $2\pi\mathrm{i}\mathbb{Z}$.

Fix a point outside the poles and define

$$f(z) = \exp\left(\int_{p_0}^{z} \eta\right).$$

Changing the path changes the integral by a multiple of $2\pi\mathrm{i}$, so f is single-valued. Near p, one has $\eta = n_p\, dz/z+$ a holomorphic form; hence $f = z^{n_p}$ times a nonvanishing holomorphic function. Therefore $(f) = D$, proving that D is principal.

Degree-zero divisors modulo principal divisors form $\mathrm{Pic}^0(X)$. The equivalence just proved identifies its kernel and image with those of the additive Abel–Jacobi map, giving $\mathrm{Pic}^0(X) \simeq \mathrm{Jac}(X)$. □

Example 6.32: Genus one

For an elliptic curve $X = \mathbb{C}/\Lambda$, the Abel–Jacobi map is an isomorphism $X \simeq \mathrm{Jac}(X)$ after choosing the origin. In higher genus it embeds X as a curve in a g-dimensional torus.

Exercise 6.33: Degree-zero line bundles

Show that the divisor $p - q$ determines the same point of the Jacobian as the difference of the Abel–Jacobi images of p and q.

Chapter 7

Bochner Identities, Vanishing Theorems, and Hörmander L^2 Methods

7.1 Lecture 25: The Bochner–Kodaira–Nakano identity

Week 13 material • 90 minutes

Notation for this lecture

(X, ω) is a Kähler manifold and (E, h) a Hermitian holomorphic bundle with Chern connection $D = D' + D''$ and $D'' = \bar{\partial}_E$. $\Delta' = D'D'^* + D'^*D'$ and $\Delta'' = D''D''^* + D''^*D''$ act on E-valued forms. $L = \omega\wedge$ and $\Lambda = \Lambda_\omega = L^*$ are the Lefschetz operators, $\Theta(E, h) = D^2$ is curvature, and $B = [\mathrm{i}\Theta(E, h), \Lambda]$ is the curvature endomorphism on forms. Brackets denote the graded commutator, and $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ are the pointwise or integrated Hermitian norm and pairing indicated by context. On a domain with boundary, $\operatorname{Dom} \bar{\partial}^*$ includes the $\bar{\partial}$-Neumann boundary condition. BKN abbreviates the Bochner–Kodaira–Nakano identity.

Prerequisite results used below

Kähler commutator identities. With the conventions of Lecture 22,

$$[\Lambda, D''] = -\mathrm{i}(D')^*, \qquad [\Lambda, D'] = \mathrm{i}(D'')^*,$$

together with their adjoints. These identities hold on smooth compactly supported bundle-valued forms and extend to the relevant operator domains.

Adjoint norm identity. For any densely defined closed operator A on a Hilbert space and any $u \in \operatorname{Dom}(A^*A) \cap \operatorname{Dom}(AA^*)$, $\langle (AA^* + A^*A)u, u\rangle = \|Au\|^2 + \|A^*u\|^2$. On the larger form domain $\operatorname{Dom} A \cap \operatorname{Dom} A^*$, the right-hand side is the closed quadratic form associated with $AA^* + A^*A$. Combining this with the curvature commutator yields the integrated BKN estimate after boundary or cutoff terms are accounted for.

Plan for the lecture. 0–18 min: bundle-valued forms and adjoints; 18–42: the BKN identity; 42–62: the curvature commutator; 62–78: the basic estimate; 78–90: boundary, completeness, and torsion.

Let (X, ω) be Kähler and (E, h) a Hermitian holomorphic bundle with Chern connection $D = D' + D''$, $D'' = \bar{\partial}_E$. Define

$$\Delta' = D'D'^* + D'^*D', \qquad \Delta'' = D''D''^* + D''^*D''.$$

Theorem 7.1: Bochner–Kodaira–Nakano identity

On compactly supported smooth E-valued forms,

$$\boxed{\Delta'' = \Delta' + [\mathrm{i}\Theta(E,h),\Lambda_\omega]}.$$

If X is compact without boundary, or complete and justified by controlled cutoffs, then

$$\|D''u\|^2 + \|D''^*u\|^2 = \|D'u\|^2 + \|D'^*u\|^2 + \int_X \langle[\mathrm{i}\Theta(E),\Lambda]u,u\rangle\,\mathrm{d}V_\omega.$$

Proof. The bundle-valued Kähler identities are $D''^* = -\mathrm{i}[\Lambda, D']$ and $D'^* = \mathrm{i}[\Lambda, D'']$. Therefore

$$\Delta'' = -\mathrm{i}\big(D''[\Lambda,D'] + [\Lambda,D']D''\big),$$
$$\Delta' = \mathrm{i}\big(D'[\Lambda,D''] + [\Lambda,D'']D'\big).$$

Subtract the second equality from the first and group terms with the graded Jacobi identity. The covariant anticommutator $D'D'' + D''D'$ is the $(1,1)$ curvature $\Theta(E,h)$, while $D'^2 = D''^2 = 0$ for the Chern connection. The result is

$$\Delta'' - \Delta' = [\mathrm{i}\Theta(E,h),\Lambda].$$

Pairing with a compactly supported form and integrating gives the norm identity because $(D'D'^*u,u) = \|D'^*u\|^2$ and similarly for the other terms. On a compact manifold there is no boundary. On a complete manifold the cutoff argument in Lecture 28 extends the identity to the graph domain. On a non-Kähler Hermitian manifold, the Kähler commutators acquire torsion operators such as $\tau = [\Lambda, \partial\omega]$ and their adjoints. □

Definition 7.2: Curvature operator

On a fixed bidegree, set

$$B_{E,\omega} = [\mathrm{i}\Theta(E,h),\Lambda_\omega].$$

This is a fiberwise self-adjoint Hermitian operator. Its positivity must be checked on the *bidegree being estimated.*

Example 7.3: Diagonalization for a line bundle on (n,q)-forms

At a point choose a unitary frame with $\mathrm{i}\Theta(L) = \mathrm{i}\sum_j \lambda_j \mathrm{d}z_j \wedge \mathrm{d}\bar z_j$. For the coefficient u_J of an (n,q)-form, the eigenvalue of B is $\sum_{j\in J}\lambda_j$. Hence $\mathrm{i}\Theta(L) \ge c\omega$ implies $B \ge qc\,\mathrm{id}$ on (n,q)-forms.

The same sum does not apply to arbitrary (p,q)-forms. Even for positive line-bundle curvature, B can be negative in low total degree.

Corollary 7.4: Basic L^2 estimate

If $B \ge 0$ on E-valued (n,q)-forms, then for $u \in \mathrm{Dom}(D'') \cap \mathrm{Dom}(D''^*)$,

$$\|D''u\|^2 + \|D''^*u\|^2 \ge \int_X \langle Bu,u\rangle\,\mathrm{d}V_\omega.$$

The omitted derivative terms are nonnegative; they are not asserted to vanish.

Proof. For a compactly supported smooth form, pair the BKN identity with u:

$$\|D''u\|^2 + \|D''^*u\|^2 = \|D'u\|^2 + \|D'^*u\|^2 + (Bu,u).$$

Discarding the two nonnegative derivative terms gives the assertion. On a compact manifold, smooth forms are a core for the closed operators. On a complete manifold, apply the identity to controlled cutoffs $\chi_\nu u$; the commutator errors tend to zero in the graph norm. Approximation then proves the estimate on the stated domain. □

Warning

Keep three levels separate: Griffiths or Nakano positivity of the curvature tensor, positivity of the commutator B on one bidegree, and an integrated coercive lower bound. Only the last directly gives closed range and a solution operator.

7.1.1 Boundary and completeness

On a smoothly bounded domain, the domain of D''^* carries the $\bar\partial$-Neumann boundary condition, and the Morrey–Kohn identity has an additional boundary Levi term. Pseudoconvexity makes this term nonnegative; it does not make it disappear. On a complete boundaryless manifold, cutoffs $\chi_\nu \to 1$ with $|\mathrm{d}\chi_\nu| \to 0$ extend compact-support identities to the graph domain of the closed operators.

7.1.2 Computing the curvature commutator on a basis

Suppose L is a line bundle and at one point choose unitary coordinates with

$$\mathrm{i}\Theta(L) = \mathrm{i}\sum_{j=1}^{n} \lambda_j \mathrm{d}z_j \wedge \mathrm{d}\bar z_j.$$

Let $u_{I\bar J}\mathrm{d}z_I \wedge \mathrm{d}\bar z_J$ be a basis (p,q)-form. Wedge by the jth curvature term and contraction by Λ count whether j occurs in I or J. The result is

$$[\mathrm{i}\Theta(L), \Lambda](\mathrm{d}z_I \wedge \mathrm{d}\bar z_J) = \left(\sum_{j\in I}\lambda_j + \sum_{j\in J}\lambda_j - \sum_{j=1}^{n}\lambda_j\right)\mathrm{d}z_I \wedge \mathrm{d}\bar z_J.$$

For $p = n$, the first and last sums cancel, leaving $\sum_{j\in J}\lambda_j$. This is why positive line-bundle curvature gives a positive operator on (n,q)-forms for $q \geq 1$. For $p = q = 0$, the eigenvalue is $-\sum_j \lambda_j$, so the same curvature is negative on functions. The bidegree cannot be suppressed.

For a vector bundle, diagonalizing the tangent indices does not generally diagonalize the bundle indices. On an E-valued (n,q)-form the quadratic expression of $[\mathrm{i}\Theta(E), \Lambda]$ is the Nakano curvature form applied to the coefficient array. This is the precise bridge from tensor positivity to the analytic energy estimate.

7.1.3 Three settings for integration by parts

On a compact manifold without boundary, smooth forms lie in the domains of both D'' and D''^* and Stokes creates no boundary term. On a complete noncompact manifold, one first applies the compact-support identity to $\chi_\nu u$ and uses cutoffs with $|\mathrm{d}\chi_\nu| \to 0$. On a domain with boundary, the Hilbert adjoint has the $\bar\partial$-Neumann boundary condition. For a scalar $(0,1)$-form $v = \sum v_j \mathrm{d}\bar z_j$ on $\Omega = \{\rho < 0\}$ it reads, up to normalization,

$$\sum_j v_j \rho_j = 0 \quad \text{on } \partial\Omega.$$

The coefficients are therefore complex tangential at the boundary.

The Morrey–Kohn formula contains

$$\int_{\partial\Omega} \sum_{j,k} \rho_{j\bar{k}} v_j \bar{v}_k \, ddS.$$

Pseudoconvexity makes this term nonnegative on tangential vectors. It does not make the term zero; compact support away from the boundary is what removes it. These three settings lead to the same basic estimate by different domain arguments.

Remark (Coercivity versus semipositivity)

If $B \geq c\,\mathrm{id}$ with $c > 0$, then the basic estimate controls the full L^2 norm and gives closed range. If only $B \geq 0$, it controls no component in $\ker B$. The identity remains true, but a vanishing or existence conclusion requires an additional source of coercivity.

7.1.4 The curvature commutator on a basis

At a point choose unitary coordinates and suppose a line bundle has

$$\mathrm{i}\Theta(L) = \mathrm{i} \sum_{j=1}^{n} \lambda_j \, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

Let e_j denote exterior multiplication by $\mathrm{d}z_j$, let $\bar{e}_j$ denote exterior multiplication by $\mathrm{d}\bar{z}_j$, and let $e_j^*, \bar{e}_j^*$ be the adjoint contractions. Then

$$L_\omega = \mathrm{i} \sum_j e_j \bar{e}_j, \qquad \Lambda_\omega = -\mathrm{i} \sum_j \bar{e}_j^* e_j^*.$$

On a basis form $\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$, the elementary anticommutation relations give

$$[\mathrm{i}\Theta(L), \Lambda] = \sum_j \lambda_j (e_j e_j^* + \bar{e}_j \bar{e}_j^* - 1).$$

Its eigenvalue is therefore

$$\sum_{j \in I} \lambda_j + \sum_{j \in J} \lambda_j - sum_{j=1}^{n} \lambda_j.$$

When $I = \{1, \ldots, n\}$, this reduces to $\sum_{j \in J} \lambda_j$. Hence on (n, q)-forms the smallest eigenvalue is the sum of the q smallest curvature eigenvalues. In particular, $\mathrm{i}\Theta(L) \geq c\omega$ implies $B \geq qcI$ on this degree.

This computation also explains why positivity cannot be transferred blindly to an arbitrary bidegree. On functions $(p, q) = (0, 0)$, the eigenvalue is $-\sum_j \lambda_j$, which is negative for positive curvature.

From the operator identity to the energy identity

For a compactly supported smooth E-valued form u, take the L^2 inner product of

$$\Delta'' u = \Delta' u + Bu$$

with u. By the definition of the formal adjoints,

$$(\Delta'' u, u) = \|D'' u\|^2 + \|D''^* u\|^2,$$
$$(\Delta' u, u) = \|D' u\|^2 + \|D'^* u\|^2.$$

Therefore

$$\|D'' u\|^2 + \|D''^* u\|^2 = \|D' u\|^2 + \|D'^* u\|^2 + (Bu, u).$$

The estimate comes from discarding the first two terms on the right after explicitly recognizing them as squared norms. They are generally not zero.

The boundary term and its domain

Let $\Omega = \{\rho < 0\} \Subset \mathbb{C}^n$ have smooth boundary and let $v = \sum_j v_j d\bar{z}_j$. Membership in the Hilbert-space domain of $\bar{\partial}^*$ imposes the $\bar{\partial}$-Neumann condition

$$\sum_j v_j \rho_j = 0 \qquad \text{on } \partial\Omega.$$

Thus the coefficient vector of v is complex tangential. Integration by parts gives the boundary contribution

$$\int_{\partial\Omega} \sum_{j,k} \rho_{j\bar{k}} v_j \bar{v}_k \frac{e^{-\varphi}}{|\partial\rho|}\, d\sigma.$$

Pseudoconvexity makes this integral nonnegative on the tangential subspace. It does not make the term vanish. If v is compactly supported in the interior, the term is absent because no boundary is reached.

On a complete manifold without boundary, choose cutoffs χ_ν with $|d\chi_\nu| \to 0$ and apply the compact-support identity to $\chi_\nu u$. The commutators

$$[D'', \chi_\nu]u = \bar{\partial}\chi_\nu \wedge u, \qquad [D''^*, \chi_\nu]u = -\iota_{(\partial\chi_\nu)^\sharp} u$$

go to zero in L^2. This is the step that extends the identity to the graph domain.

7.1.5 Self-study workshop: reading every term in the Bochner–Kodaira–Nakano identity

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall the Chern connection, formal and Hilbert-space adjoints, the Kähler commutators, and positivity operators on bundle-valued forms. Also decide at the outset whether the manifold is compact without boundary, complete with cutoffs, or a domain with a boundary condition.

From the operator identity to an estimate. For a Hermitian holomorphic bundle (E, h) on a Kähler manifold, the identity is

$$\Delta'' = \Delta' + [i\Theta(E, h), \Lambda].$$

Pairing an E-valued form u with this equality gives

$$\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 = \|D'u\|^2 + \|D'^* u\|^2 + \big([i\Theta(E, h), \Lambda]u, u\big),$$

under the compact-support or domain hypotheses that justify integration by parts. The connection-derivative terms are nonnegative; they are discarded for an inequality, not declared zero.

The curvature operator must be evaluated on the actual bidegree. For a line bundle, choose coordinates with

$$i\Theta(L) = i \sum_{j=1}^{n} \lambda_j\, dz_j \wedge d\bar{z}_j.$$

On an (n, q) basis vector $dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_J$, the commutator has eigenvalue

$$\sum_{j \in J} \lambda_j.$$

Thus positive curvature gives a strictly positive operator for $q \geq 1$. For a vector bundle the fiber indices couple to the form indices, which is why Nakano rather than merely Griffiths positivity appears.

On a smoothly bounded domain, membership in $\operatorname{Dom} \bar{\partial}^*$ imposes the $\bar{\partial}$-Neumann condition. The integration formula contains a boundary Levi term; pseudoconvexity makes it nonnegative on the tangential coefficients. On a complete manifold, cutoff functions replace the boundary argument.

Checkpoint (do not erase positive terms)

(1) Why may the D' terms be omitted from a lower bound?
(2) Why is the boundary term not said to vanish on a pseudoconvex domain?
(3) On $(n, 2)$-forms for a positive line bundle, which curvature eigenvalues are added?

Solution (checkpoint). They are squared norms and hence nonnegative. Pseudoconvexity supplies a nonnegative Levi form on tangential components; it does not make that form zero. On the component indexed by $J = \{j_1, j_2\}$, the eigenvalue is $\lambda_{j_1} + \lambda_{j_2}$. ⋄

Exit criterion

You can state the identity with its domain mechanism, compute the curvature commutator on a basis, and identify exactly which nonnegative derivative or boundary terms are discarded in obtaining an estimate.

Summary. The BKN identity links the $\bar\partial$-Laplacian to covariant derivatives and a curvature commutator. Geometric positivity becomes functional-analytic coercivity only after bidegree, boundary, and completeness have all been audited.

Exercise 7.5: Eigenvalues of the curvature operator

In the diagonal frame above, compute the eigenvalues of B on the standard basis of (n, q)-forms using wedge and contraction.

Exercise 7.6: The non-Kähler warning

Find the torsion terms in the Hermitian BKN formula and explain why they vanish when $\partial\omega = 0$.

Optional handout: Weitzenböck formulas and the classical Bochner method

This handout is optional. No later argument in the core lectures depends on it.

The Bochner–Kodaira–Nakano identity is one member of a broad family of Weitzenböck formulas. They compare a geometrically defined Laplacian with a rough Laplacian plus curvature:

$$\Delta = \nabla^*\nabla + \mathcal{R}.$$

The first term is nonnegative after integration. The sign of the zeroth-order curvature operator $\mathcal{R}$ therefore controls harmonic sections.

For a real one-form α on a Riemannian manifold,

$$\Delta_{\mathrm{d}}\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}^\sharp(\alpha).$$

If M is compact and $\mathrm{Ric} > 0$, a harmonic one-form satisfies

$$0 = (\Delta\alpha, \alpha) = \|\nabla\alpha\|^2 + \int_M \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp),$$

so $\alpha = 0$ and $b_1(M) = 0$. If $\mathrm{Ric} \geq 0$, every harmonic one-form is parallel everywhere, and its dual vector lies in the Ricci nullspace at every point.

Example 7.7: Holomorphic tensors

For a holomorphic section s of a Hermitian bundle, a Bochner formula has the shape

$$\tfrac{1}{2}\Delta|s|^2 = |Ds|^2 - \langle \mathcal{R}s, s\rangle$$

with sign depending on the bundle and Laplacian convention. At a maximum of $|s|^2$, curvature of the appropriate sign forces s to vanish or become parallel. This yields rigidity results without computing cohomology.

Warning (Pointwise versus integrated arguments)

The maximum-principle version uses a pointwise scalar identity. The Hodge-theoretic version uses integration and an operator on forms. Their curvature signs often look opposite because the bundles and Laplacian conventions differ.

Exercise 7.8: Bochner on functions

For this exercise only, put $\Delta_{\mathrm{geom}} = \operatorname{div}\nabla$; on functions this is $-\Delta_{\mathrm{d}}$ with the nonnegative Hodge-Laplacian convention of Lecture 21. Derive

$$\frac{1}{2}\Delta_{\mathrm{geom}}|\mathrm{d}f|^2 = |\operatorname{Hess} f|^2 + \langle \mathrm{d}f, \mathrm{d}\Delta_{\mathrm{geom}} f\rangle + \operatorname{Ric}(\nabla f, \nabla f)$$

and apply it to a harmonic function on a compact manifold.

7.2 Lecture 26: Kodaira–Akizuki–Nakano and related vanishing theorems

Week 13 material • 90 minutes

Notation for this lecture

X is a compact Kähler n-fold, $K_X = \Omega_X^n$ its canonical bundle, and L or E a positive holomorphic bundle with its chosen Hermitian metric. Bundle-valued Dolbeault cohomology is $H^q(X, \Omega_X^p \otimes E)$, represented by harmonic (p, q)-forms. The curvature operator is $B = [\mathrm{i}\Theta(E), \Lambda]$. KAN means Kodaira–Akizuki–Nakano; its basic conclusion is vanishing in the range $p + q > n$ for a positive line bundle. Kodaira vanishing is the special statement $H^q(X, K_X \otimes L) = 0$ for $q > 0$. E^* and $\det E$ are the dual and determinant bundles, and Serre duality identifies the dual of $H^q(X, E)$ with $H^{n-q}(X, K_X \otimes E^*)$.

Prerequisite results used below

Dolbeault–Hodge theorem. On a compact Hermitian manifold, every Dolbeault cohomology class with coefficients in a Hermitian holomorphic bundle has a unique Δ''-harmonic representative.
Coercivity implies vanishing. If the BKN identity gives $\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \geq c\|u\|^2$ with $c > 0$ on a fixed bidegree, then a harmonic form u in that bidegree is zero. By the Dolbeault–Hodge theorem, the corresponding cohomology group vanishes. For a positive line bundle with unit curvature eigenvalues, the curvature operator on (p, q)-forms is multiplication by $p + q - n$.

Plan for the lecture. 0–20 min: harmonic representatives and the vanishing mechanism; 20–42: KAN; 42–60: Kodaira and Nakano vanishing; 60–76: partial positivity; 76–90: examples and limitations.

The universal template is simple: represent a cohomology class by a harmonic u; BKN gives $0 \geq \int \langle Bu, u\rangle$; strict positivity of B forces $u = 0$.

Theorem 7.9: Kodaira–Akizuki–Nakano vanishing

Let X be a compact Kähler n-fold and L a positive line bundle. Then

$$H^q(X, \Omega_X^p \otimes L) = 0, \qquad p + q > n.$$

Proof. Choose the Kähler metric $\omega = \mathrm{i}\Theta(L)$ after normalization. On a basis (p, q)-form, the eigenvalue of $[\mathrm{i}\Theta(L), \Lambda]$ is $p + q - n$. It is uniformly positive when $p + q > n$, so the BKN identity forces every $\bar\partial$-harmonic representative to vanish. □

Corollary 7.10: Kodaira vanishing

If L is positive, then

$$H^q(X, K_X \otimes L) = 0 \quad (q \geq 1).$$

By Serre duality, $H^q(X, L^{-1}) = 0$ for $q < n$.

Proof. Put $p = n$ in Kodaira–Akizuki–Nakano vanishing. Then $p + q > n$ exactly when $q > 0$, and $\Omega_X^n = K_X$. Serre duality gives

$$H^q(X, L^{-1})^* \simeq H^{n-q}(X, K_X \otimes L),$$

whose right side vanishes when $q < n$. □

Theorem 7.11: Nakano vanishing

If a Hermitian holomorphic vector bundle E is Nakano positive, then

$$H^q(X, K_X \otimes E) = 0 \qquad (q \geq 1).$$

Proof. Represent a class by its harmonic E-valued (n,q)-form u. In this bidegree, Nakano positivity is precisely strict positivity of $B = [\mathrm{i}\Theta(E), \Lambda]$. Compactness supplies $c > 0$ with $B \geq c\,\mathrm{id}$ for $q \geq 1$. The integrated BKN identity therefore gives

$$0 = \|D''u\|^2 + \|D''^*u\|^2 \geq (Bu, u) \geq c\|u\|^2.$$

Thus $u = 0$ and the cohomology class vanishes. □

The bidegree (n,q) is chosen so Nakano positivity matches $B = [\mathrm{i}\Theta(E), \Lambda]$. Griffiths positivity alone is generally insufficient; Demailly–Skoda first upgrades $E \otimes \det E$ to Nakano positivity.

Example 7.12: Basic vanishing on projective space

$\mathcal{O}(k)$ is positive for $k > 0$, so $H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}(k)) = 0$ for $q > 0$. Since $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$, this produces many vanishings for $\mathcal{O}(m)$; Serre duality recovers the edge cases of Bott's formula.

Theorem 7.13: Spectral vanishing criterion

Let L be a Hermitian line bundle on a compact Kähler manifold. At each point choose a unitary coframe in which $\mathrm{i}\Theta(L) = \mathrm{i}\sum_j \lambda_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$. If, for some $c > 0$, every pair of multi-indices $|I| = p$, $|J| = q$ satisfies

$$\sum_{j \in I} \lambda_j + \sum_{j \in J} \lambda_j - \sum_{j=1}^{n} \lambda_j \geq c,$$

then $H^q(X, \Omega_X^p \otimes L) = 0$. In particular, on (n,q)-forms it is enough that the sum of the q smallest curvature eigenvalues be uniformly positive.

Proof. Wedge and contraction on the basis form $\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$ show that the displayed number is exactly the eigenvalue of $B = [\mathrm{i}\Theta(L), \Lambda]$. Thus $B \geq c\,\mathrm{id}$ on (p,q)-forms. A harmonic representative u satisfies the BKN identity, hence $0 \geq c\,\|u\|^2$, so $u = 0$. □

Remark (Partial positivity)

Knowing only the number of positive eigenvalues does not by itself give the displayed uniform inequality: negative eigenvalues may dominate. More flexible partial-positivity theorems use high tensor powers, a metric adapted to the positive directions, or q-convexity. The spectral criterion above is the exact statement supplied directly by BKN.

Warning (Dual curvature)

Dual curvature is $\Theta(E^*) = -{}^{t}\Theta(E)$. Griffiths positivity therefore dualizes to Griffiths negativity. Nakano duality also transposes the tangent and bundle indices and cannot be inferred by deleting the minus sign.

Warning

If curvature is only semipositive, BKN says merely that harmonic representatives lie in $\ker B$, not that they vanish. One needs a uniform positive lower bound, strict positivity somewhere plus unique continuation, or another geometric argument.

7.2.1 The vanishing region in the Hodge diamond

For a positive line bundle L, Kodaira–Akizuki–Nakano gives

$$H^{p,q}(X,L)=0 \qquad \text{when } p+q>n.$$

Serre duality converts this to the negative-bundle statement

$$H^{p,q}(X,L^{-1})=0 \qquad \text{when } p+q<n.$$

Thus positive curvature kills the upper-right triangle of the bundle-valued Hodge diamond, while negative curvature kills the lower-left triangle. The middle diagonal $p+q=n$ is not controlled by this argument because the curvature commutator has zero eigenvalue there.

On $\mathbb{P}^n$, take $L=\mathcal{O}(k)$ with $k>0$ and $K_{\mathbb{P}^n}=\mathcal{O}(-n-1)$. Kodaira vanishing gives

$$H^q(\mathbb{P}^n,\mathcal{O}(k-n-1))=0 \qquad (q>0).$$

Equivalently, $H^q(\mathbb{P}^n,\mathcal{O}(m))=0$ for $q>0$ once $m>-n-1$. Serre duality recovers the top-degree groups for very negative m. Bott's formula supplies the remaining exact boundary cases; curvature already explains the large vanishing regions.

7.2.2 From vanishing to generation and embeddings

Vanishing becomes geometric when inserted into an exact sequence. For a point x,

$$0\to L^k\otimes\mathcal{I}_x\to L^k\to L^k_x\to 0$$

shows that L^k is generated at x if $H^1(X,L^k\otimes\mathcal{I}_x)=0$. Replacing $\mathcal{I}_x$ by $\mathcal{I}_x^2$ controls first jets. The ideal sheaves are not line bundles, so Kodaira vanishing does not apply to them directly. One resolves the ideals by vector bundles, uses Serre vanishing for high powers, or constructs the desired jets analytically by weighted $\bar\partial$ estimates.

This distinction is important: a cohomology vanishing theorem for smooth bundles is not automatically a vanishing theorem for every coherent sheaf. The passage to coherent ideals requires a resolution or a singular metric whose multiplier ideal is the desired sheaf.

Partial positivity works by the same spectral logic. On (n,q)-forms, if the sum of the q smallest curvature eigenvalues is uniformly positive, the commutator is coercive even when some individual eigenvalues are negative. Merely knowing that a fixed number of eigenvalues are positive is insufficient unless the negative ones are quantitatively controlled.

Remark (One proof template)

Choose the cohomology realization, select a harmonic representative, compute the curvature operator on its exact bidegree, and prove a uniform positive lower bound. Every vanishing theorem in this lecture differs only in one of these four entries.

7.2.3 KAN vanishing with the eigenvalue calculation exposed

Choose the metric on a positive line bundle L so that $\mathrm{i}\Theta(L)=\omega$ at the level of the chosen normalization. In a unitary frame, all curvature eigenvalues are 1. The calculation from Lecture 25 gives on a (p,q) basis form

$$B=[\mathrm{i}\Theta(L),\Lambda]=(p+q-n)I.$$

Let u be the harmonic representative of a class in $H^q(X, \Omega_X^p \otimes L)$. Then $D''u = D''^*u = 0$, so BKN gives

$$0 = \|D'u\|^2 + \|D'^*u\|^2 + (p + q - n)\|u\|^2.$$

When $p + q > n$, all three terms are nonnegative and the last has a strictly positive coefficient. Hence $u = 0$. This proves the exact vanishing range.

At the boundary $p+q = n$, curvature contributes zero. The identity then says only $D'u = D'^*u = 0$; it does not force u to vanish. For example, $H^0(\mathbb{P}^1, K_{\mathbb{P}^1} \otimes O(2)) = H^0(\mathbb{P}^1, O) = \mathbb{C}$ lies exactly on this boundary.

Kodaira vanishing checked on $\mathbb{P}^1$

On $\mathbb{P}^1$, $K = O(-2)$ and a positive line bundle is $L = O(k)$ with $k > 0$. Kodaira vanishing asserts

$$H^1(\mathbb{P}^1, K \otimes L) = H^1(\mathbb{P}^1, O(k - 2)) = 0.$$

The explicit Čech calculation gives $h^1(O(m)) = \max\{-m - 1, 0\}$. Substituting $m = k - 2$ gives zero exactly for $k \geq 1$, confirming the theorem including the smallest positive bundle.

By Serre duality,

$$H^0(\mathbb{P}^1, L^{-1})^* simeq H^1(\mathbb{P}^1, K \otimes L),$$

so the same vanishing says that a negative-degree line bundle has no nonzero global section.

Nakano vanishing and compact uniformity

Suppose E is Nakano positive. On E-valued (n, q)-forms, $q \geq 1$, its curvature commutator is pointwise positive definite. The unit sphere bundle in this finite-rank vector bundle is compact because X is compact. The smallest eigenvalue therefore has a positive global minimum $c > 0$, giving

$$(Bu, u) \geq c\|u\|^2.$$

This compactness step upgrades pointwise strict positivity to the uniform coercivity actually used in the integrated proof.

If E is only Griffiths positive, this conclusion is not available because the curvature is tested only on decomposable tensors. Demailly–Skoda replaces E by $E \otimes \det E$, which is Nakano positive, and then the argument applies to

$$H^q(X, K_X \otimes E \otimes \det E), \qquad q > 0.$$

Partial positivity as an ordered-eigenvalue test

On (n, q)-forms with line-bundle curvature eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, the least eigenvalue of B is

$$\lambda_1 + \cdots + \lambda_q.$$

Thus having at least $n - q + 1$ positive eigenvalues is not sufficient if the remaining negative eigenvalues have large magnitude. The direct BKN criterion is the uniform inequality

$$\lambda_1 + \cdots + \lambda_q \geq c > 0.$$

Once this holds, the harmonic-representative proof is identical to KAN. This calculation separates a precise spectral hypothesis from the looser slogan "sufficiently many positive directions."

7.2.4 Self-study workshop: turning curvature positivity into cohomology vanishing

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall harmonic representatives, Dolbeault cohomology with bundle coefficients, the curvature eigenvalue calculation from Lecture 25, and Serre duality. A vanishing theorem is an estimate applied to a harmonic representative.

Kodaira vanishing in four lines with all inputs named. Let L be a positive line bundle on a compact Kähler n-fold and let $q \geq 1$. A class in $H^q(X, K_X \otimes L)$ has an L-valued harmonic (n, q) representative u. The BKN identity and positivity give, for some $c > 0$,

$$0 = \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 \geq \big([\mathrm{i}\Theta(L), \Lambda]u, u\big) \geq c\|u\|^2.$$

Compactness is what turns pointwise positive definiteness into one uniform constant c. Hence $u = 0$ and

$$H^q(X, K_X \otimes L) = 0 \qquad (q \geq 1).$$

Serre duality immediately gives the equivalent negative-bundle statement $H^p(X, L^{-1}) = 0$ for $p < n$.

The Kodaira–Akizuki–Nakano theorem keeps general bidegree and concludes

$$H^q(X, \Omega_X^p \otimes L) = 0 \qquad \text{when } p + q > n.$$

The inequality $p + q > n$ is not a dimension slogan; it is exactly the range where the curvature commutator is positive on (p, q)-forms. For vector bundles, the analogous argument needs positivity of the operator on the relevant tensor space, usually Nakano positivity.

Checkpoint (audit a vanishing claim)

(1) Where is compactness used in the proof above?
(2) Does positivity of L imply $H^0(X, K_X \otimes L) = 0$?
(3) Why can Griffiths positivity not simply replace Nakano positivity in the vector-bundle proof?

Solution (checkpoint). Compactness gives Hodge representatives and a uniform positive lower bound for the curvature operator. The theorem makes no vanishing claim in degree zero; positive bundles often have many sections. Griffiths positivity tests only decomposable tensors, while the BKN quadratic form tests arbitrary coupled tangent-fiber coefficients. ◇

Exit criterion

You can derive Kodaira vanishing from a harmonic representative, translate it by Serre duality, and recover the bidegree range and positivity notion in KAN and vector-bundle variants.

Summary. KAN, Kodaira, and Nakano vanishing come from one energy identity; they differ only in bundle, bidegree, and positivity condition. "Compute B first, then claim vanishing" is safer than memorizing ranges.

Exercise 7.14: Check on $\mathbb{P}^1$

Use the explicit cohomology calculation in Lecture 23 to verify Kodaira vanishing on $\mathbb{P}^1$.

Exercise 7.15: A basic Fano consequence

If $-K_X$ is positive, prove $H^q(X, \mathcal{O}_X) = 0$ for $q > 0$ and deduce $\chi(\mathcal{O}_X) = 1$.

Optional handout: Lefschetz hyperplane theorems from positivity

This handout is optional. No later argument in the core lectures depends on it.

Let X be a smooth projective n-fold and $Y \subset X$ a smooth ample divisor. The topological Lefschetz hyperplane theorem states that

$$H^k(X, \mathbb{Z}) \longrightarrow H^k(Y, \mathbb{Z})$$

is an isomorphism for $k < n-1$ and injective for $k = n-1$. There is also a weak homotopy version for π_k.

The complement $X \setminus Y$ is affine and carries a strictly psh exhaustion coming from $-\log |s_Y|^2$. Morse theory for such an exhaustion has handles of real index at most n. Consequently X is obtained from a neighborhood of Y by attaching cells of dimension at least n, which yields the cohomological range.

There is an algebraic companion. From

$$0 \to \Omega_X^p(-Y) \to \Omega_X^p \to \Omega_X^p|_Y \to 0$$

and the conormal sequence on Y, Kodaira–Akizuki–Nakano vanishing kills the outer cohomology groups in a range. The resulting long exact sequences prove the Hodge pieces of the restriction theorem.

Example 7.16: Hypersurfaces in projective space

A smooth hypersurface $Y \subset \mathbb{P}^n$ has the same cohomology as $\mathbb{P}^n$ below its middle dimension. All new topology is concentrated in the primitive middle cohomology. For curves in $\mathbb{P}^2$, the middle degree is already 1, where the genus appears.

Exercise 7.17: Fundamental group

Assuming the homotopy form of Lefschetz, show that a smooth hypersurface of complex dimension at least two in $\mathbb{P}^n$ is simply connected.

7.3 Lecture 27: Hörmander's weighted L^2 estimate

Week 14 material • 90 minutes

Notation for this lecture

$\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain, $\varphi \in C^2(\Omega)$ a psh weight, and $\mathrm{d}\lambda$ Euclidean Lebesgue measure. The weighted inner product and norm are $(u, v)_\varphi = \int_\Omega \langle u, v\rangle e^{-\varphi} \mathrm{d}\lambda$ and $\|u\|_\varphi^2 = (u, u)_\varphi$. $H_\varphi = (\varphi_{j\bar{k}})$ is the Levi matrix and H_φ^{-1} its inverse on the positive subspace. The closed operators $T = \bar\partial : L^2_{0,0} \to L^2_{0,1}$ and $S = \bar\partial : L^2_{0,1} \to L^2_{0,2}$ satisfy $ST = 0$; T^*, S^* are their Hilbert-space adjoints, $\bar\partial_\varphi^* = T^*$ on $(0,1)$-forms, and $\operatorname{Dom} T^*$ is its operator domain. The source form is $f = \sum_j f_j \mathrm{d}\bar z_j$ with $\bar\partial f = 0$, and $A_f = \int_\Omega f^* H_\varphi^{-1} f \, e^{-\varphi} \mathrm{d}\lambda$ is its inverse-Hessian energy.

Prerequisite results used below

Hilbert-complex existence lemma. Let $T : H_0 \to H_1$ and $S : H_1 \to H_2$ be densely defined closed operators with $ST = 0$. If a vector $f \in \ker S$ satisfies

$$|(f, v)_{H_1}|^2 \le C\big(\|T^* v\|_{H_0}^2 + \|Sv\|_{H_2}^2\big)$$

for every $v \in \operatorname{Dom} T^* \cap \operatorname{Dom} S$, then there is $u \in H_0$ with $Tu = f$ and $\|u\|_{H_0}^2 \le C$. The minimum-norm solution is obtained by orthogonal projection to $(\ker T)^\perp$.

Morrey–Kohn–Hörmander estimate. On a smooth bounded pseudoconvex domain and for a C^2 psh weight φ, every smooth $(0,1)$-form v in the $\bar\partial$-Neumann domain satisfies

$$\|\bar\partial v\|_\varphi^2 + \|\bar\partial_\varphi^* v\|_\varphi^2 \ge \int_\Omega v^* H_\varphi v \, e^{-\varphi} \mathrm{d}\lambda.$$

Plan for the lecture. 0–18 min: theorem and convention ledger; 18–42: Morrey–Kohn estimate; 42–65: curvature Cauchy–Schwarz; 65–80: Hilbert-complex duality; 80–90: exhaustion and passage to the limit.

7.3.1 Chosen version and hypothesis ledger

Fix a pseudoconvex domain $\Omega \subset \mathbb{C}^n$, Lebesgue measure $\mathrm{d}\lambda$, weight $e^{-\varphi}$, and a strictly psh $\varphi \in C^2(\Omega)$. Let

$$f = \sum_{j=1}^n f_j \mathrm{d}\bar z_j, \qquad \bar\partial f = 0$$

in distributions. Write $H_\varphi = (\varphi_{j\bar k})$ and assume

$$A_f := \int_\Omega f^* H_\varphi^{-1} f \, e^{-\varphi} \mathrm{d}\lambda < \infty.$$

Theorem 7.18: Hörmander's scalar $(0,1)$ estimate

Under these assumptions, there is $u \in L^2(\Omega, e^{-\varphi})$ such that

$$\bar\partial u = f \quad \text{in distributions,} \qquad \boxed{\int_\Omega |u|^2 e^{-\varphi} \mathrm{d}\lambda \le A_f}.$$

The choice $u \perp \ker \bar\partial$ is the unique minimum-norm solution.

Proof (including the operator-domain and existence steps).

The basic estimate and operator domains. For a compactly supported smooth $(0,1)$-form $v = \sum v_j \mathrm{d}\bar{z}_j$, the formal weighted adjoint is

$$\bar{\partial}_\varphi^* v = -\sum_j \mathrm{e}^{\varphi} \frac{\partial}{\partial z_j}(\mathrm{e}^{-\varphi} v_j) = -\sum_j \left(\frac{\partial v_j}{\partial z_j} - \varphi_j v_j\right).$$

On a domain with boundary one must use the closed Hilbert adjoint and its $\bar{\partial}$-Neumann boundary condition.

The Morrey–Kohn–Hörmander identity is

$$\|\bar{\partial} v\|_\varphi^2 + \|\bar{\partial}_\varphi^* v\|_\varphi^2 = \sum_{j,k} \int_\Omega \left|\frac{\partial v_j}{\partial \bar{z}_k}\right|^2 \mathrm{e}^{-\varphi} + \int_\Omega v^* H_\varphi v\, \mathrm{e}^{-\varphi} + \text{boundary Levi term}.$$

Pseudoconvexity makes the boundary term nonnegative on complex tangential components. Hence, for $v \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}_\varphi^*)$,

$$Q(v) := \|\bar{\partial} v\|_\varphi^2 + \|\bar{\partial}_\varphi^* v\|_\varphi^2 \geq \int v^* H_\varphi v\, \mathrm{e}^{-\varphi}.$$

Curvature Cauchy–Schwarz and Hilbert-complex duality. For every such v,

$$|(f, v)_\varphi|^2 = \left|\int \left\langle H_\varphi^{-1/2} f, H_\varphi^{1/2} v \right\rangle \mathrm{e}^{-\varphi}\right|^2 \leq A_f\, Q(v).$$

Set $T = \bar{\partial} : L^2_{0,0} \to L^2_{0,1}$ and $S = \bar{\partial} : L^2_{0,1} \to L^2_{0,2}$. Then $ST = 0$ and $Sf = 0$. On the graph range define $\ell(T^* v, Sv) = (f, v)$. The estimate makes ℓ bounded. Hahn–Banach and Riesz give (u, g) with $f = Tu + S^* g$. Because $f \in \ker S$ and $\overline{\operatorname{Im} T} \perp \overline{\operatorname{Im} S^*}$, the second component vanishes; thus $Tu = f$ and $\|u\|^2 \leq A_f$.

Warning

A differential identity does not itself produce a solution. The Hahn–Banach/Riesz duality step is the logical core of existence, and $\bar{\partial} f = 0$ is needed to eliminate the $S^* g$ component. The estimate alone does not provide boundary smoothness.

Exhaustion of a general pseudoconvex domain. Choose smoothly bounded strictly pseudoconvex $\Omega_\nu \Subset \Omega$. Solve on each Ω_ν with the same bound A_f, extract weakly on every fixed Ω_m, and diagonalize. Compactly supported tests pass the equation to the limit, and weak lower semicontinuity preserves the estimate. This explains why pseudoconvexity, rather than smooth boundary, is the final hypothesis. The orthogonal projection of any solution onto $(\ker \bar{\partial})^\perp$ remains a solution and can only decrease the norm. It is the unique minimum-norm solution. □

If $H_\varphi \geq cI$, then $\|u\|_\varphi^2 \leq c^{-1} \|f\|_\varphi^2$. Replacing the weight by $\mathrm{e}^{-2\varphi}$ doubles the Hessian and changes the constant accordingly.

7.3.2 Deriving the weighted adjoint

Let u be a compactly supported smooth function and $v = \sum_j v_j d\bar{z}_j$. Integration by parts in the weighted inner product gives

$$(\bar\partial u, v)_\varphi = \sum_j \int_\Omega \frac{\partial u}{\partial \bar z_j} \bar v_j e^{-\varphi} d\lambda$$
$$= \int_\Omega u \overline{- \sum_j \left(\frac{\partial v_j}{\partial z_j} - \varphi_j v_j \right)} e^{-\varphi} d\lambda.$$

Hence

$$\bar\partial^*_\varphi v = - \sum_j \left(\frac{\partial v_j}{\partial z_j} - \varphi_j v_j \right).$$

The zero-order term comes from differentiating the actual exponent $e^{-\varphi}$. Replacing it by $e^{-2\varphi}$ replaces φ_j by $2\varphi_j$ and doubles the Levi contribution in the basic estimate.

7.3.3 Why the inverse Hessian is the sharp datum

The inequality

$$|(f, v)_\varphi|^2 \le \left(\int f^* H_\varphi^{-1} f \, e^{-\varphi} \right) \left(\int v^* H_\varphi v \, e^{-\varphi} \right)$$

is pointwise Cauchy–Schwarz in the metric H_φ. It adapts to anisotropic curvature. If

$$H_\varphi = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

then the energy is

$$\sum_j \frac{|f_j|^2}{\lambda_j};$$

components in strongly curved directions are cheaper to solve than components in weakly curved directions. Replacing this by $c^{-1}|f|^2$ after using $H_\varphi \ge cI$ loses that information.

For the Gaussian weight $\varphi = t|z|^2$, one has $H_\varphi = tI$ and obtains the normalization test

$$\|u\|_\varphi^2 \le t^{-1} \|f\|_\varphi^2.$$

If instead $\varphi = |z_1|^2$ on $\mathbb{C}^n$, the Hessian has a kernel. A source with a $d\bar z_2$ component has infinite inverse-Hessian energy. Adding $\varepsilon|z|^2$ makes the matrix invertible, but the corresponding energy grows like ε^{-1}; no uniform limit follows.

7.3.4 What the theorem does and does not return

The Hilbert-complex argument produces a weak solution in the maximal domain of $\bar\partial$. Testing against compactly supported smooth forms gives $\bar\partial u = f$ in distributions. If f and φ are smooth, interior elliptic regularity supplies a smooth solution locally, but smoothness up to the boundary is a separate $\bar\partial$-Neumann regularity problem.

Among all solutions, the one orthogonal to the weighted Bergman space $A^2(\Omega, e^{-\varphi}) = \ker \bar\partial$ has minimum norm. Indeed, every other solution is $u + h$ with h holomorphic, and orthogonality gives

$$\|u + h\|_\varphi^2 = \|u\|_\varphi^2 + \|h\|_\varphi^2.$$

Warning

The compatibility condition $\bar\partial f = 0$, membership in the inverse-curvature energy space, and the boundary or exhaustion mechanism are independent hypotheses. None can be inferred merely from the formal equation or from strict plurisubharmonicity of the weight.

7.3.5 The scalar estimate with every logical step separated

Use the closed maximal operators

$$T = \bar\partial : L^2_{0,0}(\Omega, e^{-\varphi}) \to L^2_{0,1}(\Omega, e^{-\varphi}), \qquad S = \bar\partial : L^2_{0,1} \to L^2_{0,2}.$$

They are densely defined and closed, and $ST = 0$. A test form belongs to $\operatorname{Dom} T^* \cap \operatorname{Dom} S$; on a smooth boundary its normal component obeys the $\bar\partial$-Neumann condition. The Morrey–Kohn–Hörmander identity then gives

$$Q(v) := \|T^*v\|^2_\varphi + \|Sv\|^2_\varphi \ge \int_\Omega v^* H_\varphi v\, e^{-\varphi} d\lambda.$$

For a $\bar\partial$-closed f with finite inverse-Hessian energy, fiberwise Cauchy–Schwarz gives

$$\begin{aligned} |(f,v)_\varphi|^2 &= \left| \int (H_\varphi^{-1/2} f)^* (H_\varphi^{1/2} v) e^{-\varphi} \right|^2 \\ &\le A_f \int v^* H_\varphi v\, e^{-\varphi} \le A_f\, Q(v). \end{aligned}$$

Define on the graph range

$$\mathcal{R} = \{(T^*v, Sv) : v \in \operatorname{Dom} T^* \cap \operatorname{Dom} S\}$$

the functional $\ell(T^*v, Sv) = (f, v)$. The inequality proves both that it is well defined and that $\|\ell\| \le A_f^{1/2}$. Hahn–Banach extends it to the ambient direct sum, and Riesz gives (u, g) with

$$(f, v) = (u, T^*v) + (g, Sv) = (Tu + S^*g, v).$$

Hence $f = Tu + S^*g$ weakly. The two summands are orthogonal because

$$(Tu, S^*g) = (STu, g) = 0.$$

Since $Sf = 0$, f lies in $\ker S = (\overline{\operatorname{im} S^*})^\perp$. Projecting the equality to $\overline{\operatorname{im} S^*}$ gives $S^*g = 0$. Thus

$$Tu = f, \qquad \|u\|^2_\varphi \le A_f.$$

Finally replace u by its orthogonal projection onto $(\ker T)^\perp$ to obtain the unique minimum-norm solution.

An equality case for the Gaussian weight

On $\mathbb{C}^n$ take $\varphi = t|z|^2$, $t > 0$, and $f = d\bar z_1$. Then $H_\varphi = tI$ and f is $\bar\partial$-closed. The function $u = \bar z_1$ satisfies $\bar\partial u = f$. Rotational symmetry makes u orthogonal to every entire holomorphic function in the Gaussian space, so it is the canonical solution. Moreover,

$$\int_{\mathbb{C}^n} |\bar z_1|^2 e^{-t|z|^2} d\lambda = \frac{1}{t} \int_{\mathbb{C}^n} e^{-t|z|^2} d\lambda,$$

while

$$A_f = \int f^* (tI)^{-1} f\, e^{-t|z|^2} d\lambda = \frac{1}{t} \int e^{-t|z|^2} d\lambda.$$

Thus equality holds and the factor t^{-1} is sharp. If the actual weight is $e^{-2t|z|^2}$, the Hessian is $2tI$ and the sharp factor becomes $(2t)^{-1}$.

The exhaustion limit written against a test form

Choose smooth strictly pseudoconvex $\Omega_1 \Subset \Omega_2 \Subset \cdots \nearrow \Omega$ and solve $\bar\partial u_\nu = f$ on Ω_ν with

$$\|u_\nu\|^2_{L^2(\Omega_m, e^{-\varphi})} \le A_f \qquad (\nu \ge m).$$

For each fixed m, weak compactness gives a subsequence converging on Ω_m; a diagonal subsequence gives one u on all of Ω. If $\eta \in C^\infty_c(\Omega)$, then $\operatorname{supp}\eta \subset \Omega_m$ for some m, and

$$(u, \bar\partial^*\eta) = \lim_\nu (u_\nu, \bar\partial^*\eta) = \lim_\nu (f, \eta) = (f, \eta).$$

Therefore $\bar\partial u = f$ distributionally. Weak lower semicontinuity on every Ω_m and then monotone convergence in m preserve the global estimate.

7.3.6 Self-study workshop: auditing a weighted $\bar\partial$ existence proof

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Before starting, record the domain, boundary or exhaustion mechanism, source and target bidegrees, weight exponent, Hilbert adjoint, curvature matrix, and desired regularity. Here the convention is the norm with $e^{-\varphi}$ on a pseudoconvex domain $\Omega \subset \mathbb{C}^n$.

The complete logical chain. The data are a $(0,1)$-form f satisfying $\bar\partial f = 0$ weakly and

$$A_f = \int_\Omega \langle H_\varphi^{-1} f, f\rangle e^{-\varphi}\, d\lambda < \infty, \qquad H_\varphi = (\varphi_{j\bar k}) > 0.$$

The inverse is legitimate only because the Hessian is positive definite. For $v \in \operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial^*_\varphi$, the Morrey–Kohn–Hörmander estimate is

$$Q(v) := \|\bar\partial v\|^2_\varphi + \|\bar\partial^*_\varphi v\|^2_\varphi \ge \int_\Omega \langle H_\varphi v, v\rangle e^{-\varphi}\, d\lambda.$$

The derivative term has been discarded as nonnegative; the boundary Levi term is nonnegative by pseudoconvexity and the adjoint boundary condition.

Fiberwise and then integral Cauchy–Schwarz gives

$$|(f, v)_\varphi|^2 \le A_f\, Q(v).$$

This estimate defines a bounded functional on the graph range $v \mapsto (\bar\partial^*_\varphi v, \bar\partial v)$. Hahn–Banach and Riesz produce a pair (u, g) with $f = \bar\partial u + \bar\partial^* g$. The compatibility condition $\bar\partial f = 0$, together with the Hilbert-complex orthogonality, eliminates the second component. Thus

$$\bar\partial u = f, \qquad \|u\|^2_\varphi \le A_f.$$

Exhaustion of a nonsmooth pseudoconvex domain requires uniform estimates, diagonal weak convergence, distributional passage of the equation, and weak lower semicontinuity. Smoothness of u is a separate regularity question.

Checkpoint (find the first invalid step)

(1) A proof says that psh φ makes H_φ^{-1} exist. What is wrong?
(2) A proof changes $e^{-\varphi}$ to $e^{-2\varphi}$ but keeps the same Hessian. What must change?

(3) A proof obtains the basic inequality and immediately asserts existence. What step is missing?

Solution (checkpoint). Psh gives only semipositivity, so the Hessian may have a kernel; one needs strict positivity or a justified regularization or range condition. The full exponent is 2φ, so its Hessian contribution and the sharp constant are doubled. The bounded graph functional, Hahn–Banach/Riesz representation, and use of $\bar{\partial} f = 0$ are the missing Hilbert-complex step. ◇

Exit criterion

You can reproduce the theorem with every hypothesis, explain the curvature Cauchy–Schwarz inequality, carry out the duality step, and reject proofs that hide a degenerate inverse, boundary term, or nonuniform limit.

Summary. Hörmander's method has three indispensable components: the Morrey–Kohn basic estimate, Cauchy–Schwarz with the inverse curvature, and Hilbert-complex duality. Closed operators, domains, exhaustion, and weak limits are part of the proof, not optional details.

Exercise 7.19: Gaussian weight

For $\varphi = t|z|^2$, derive $\|u\|_t^2 \le t^{-1}\,\|f\|_t^2$. Check the constant when the weight is $\mathrm{e}^{-2t|z|^2}$.

Exercise 7.20: The compatibility condition

For $n \ge 2$, explain why $\bar{\partial} f = 0$ is necessary for $\bar{\partial} u = f$ and give an explicit $(0,1)$-form violating it.

Optional handout: Canonical solutions and the Bergman projection

This handout is optional. No later argument in the core lectures depends on it.

Let $T = \bar{\partial}$ on weighted L^2 functions. If $f \in \operatorname{im} T$, all solutions of $Tu = f$ form the affine space $u_0 + \ker T$. There is a unique solution $u_{\min} \perp \ker T$, characterized by minimum norm. If the $\bar{\partial}$-Neumann operator N_1 exists, then

$$u_{\min} = \bar{\partial}_\varphi^* N_1 f.$$

The weighted Bergman space $A^2(\Omega, \mathrm{e}^{-\varphi}) = \ker \bar{\partial} \subset L^2$ is closed. Let P_φ be the orthogonal projection onto it. For a smooth function g in the operator domain,

$$P_\varphi g = g - \bar{\partial}_\varphi^* N_1 \bar{\partial} g.$$

Thus solving $\bar{\partial}$ and projecting onto holomorphic functions are the same analytic operation viewed from complementary subspaces.

If $K_\varphi(z, w)$ is the integral kernel of P_φ, then

$$K_\varphi(z, z) = \sup\{|f(z)|^2 : f \in A^2, \|f\|_\varphi \le 1\}.$$

Evaluation from a point and the minimum-norm extension problem are dual. This is the link between Hörmander estimates, Ohsawa–Takegoshi extension, and positivity of Bergman kernels.

Remark (Regularity)

The L^2 estimate proves existence and norm control. Smoothness up to the boundary is a separate issue governed by regularity of N_q. On weakly pseudoconvex domains, the Bergman projection can fail to preserve high Sobolev regularity.

Exercise 7.21: The unit disc

Compute the unweighted Bergman kernel of the unit disc from the orthonormal basis $\sqrt{(k+1)/\pi}\, z^k$ and verify the extremal characterization on the diagonal.

7.4 Lecture 28: Complete Kähler manifolds, singular weights, and multiplier ideals

Week 14 material • 90 minutes

Notation for this lecture

(X, ω) is a complete Kähler manifold, (E, h) a Hermitian holomorphic bundle, and K_X the canonical bundle. A possibly singular line-bundle metric is written $h = e^{-\varphi}$ in a local frame, where φ is a psh or quasi-psh weight. Its multiplier ideal is $\mathcal{I}(\varphi)_x = \{f \in \mathcal{O}_{X,x} : |f|^2 e^{-\varphi}$ is locally integrable near $x\}$. χ_ν denotes a compactly supported cutoff tending to one with $|d\chi_\nu|_\omega \to 0$, and weak convergence in weighted L^2 is denoted $\rightharpoonup$. The inequality $[i\Theta(E, h), \Lambda] \geq c$ is an operator inequality on the stated bidegree. Nadel vanishing refers to cohomology with the factor $\mathcal{I}(\varphi)$.

Prerequisite results used below

Hopf–Rinow theorem. For a connected Riemannian manifold, metric completeness is equivalent to compactness of closed bounded sets. Consequently a complete manifold admits compactly supported cutoffs $\chi_\nu \to 1$ whose gradients tend to zero uniformly.
Weak compactness in Hilbert space. Every bounded sequence in a Hilbert space has a weakly convergent subsequence, and the norm is weakly lower semicontinuous: $u_j \rightharpoonup u$ implies $\|u\| \leq \liminf_j \|u_j\|$. These two facts allow estimates proved for compactly supported smooth forms and smooth weights to pass first to a complete exhaustion and then to a singular-weight limit.

Plan for the lecture. 0–20 min: the complete Kähler bundle-valued theorem; 20–42: inverse curvature operator; 42–62: cutoffs and weak limits; 62–78: regularizing singular weights; 78–90: multiplier ideals and Nadel vanishing.

Theorem 7.22: Bundle-valued L^2 estimate on a complete Kähler manifold

Let (X, ω) be complete Kähler, (E, h) Hermitian holomorphic, and $q \geq 1$. On E-valued (n, q)-forms let $B = [i\Theta(E, h), \Lambda_\omega]$. Suppose B is positive definite almost everywhere. If f is $\bar\partial$-closed and

$$A_f = \int_X \left\langle B^{-1} f, f \right\rangle_{\omega,h} dV_\omega < \infty,$$

then there is an E-valued $(n, q-1)$-form u with

$$\bar\partial u = f, \qquad \|u\|^2_{L^2} \leq A_f\,.$$

Proof. Choose cutoffs $0 \leq \chi_\nu \leq 1$ with compact support, $\chi_\nu \to 1$, and $|d\chi_\nu| \leq C/\nu$. Apply BKN to $\chi_\nu v$. Since $D''(\chi_\nu v) = \chi_\nu D'' v + \bar\partial\chi_\nu \wedge v$ and the analogous commutator formula holds for D''^*, the error terms are bounded by $C\nu^{-1} \|v\|$ times the graph norm. Letting $\nu \to \infty$ gives, first for smooth graph-domain forms and then by closure,

$$Q(v) := \|D'' v\|^2 + \|D''^* v\|^2 \geq \int_X \langle Bv, v\rangle \, dV_\omega.$$

Fiberwise spectral decomposition defines the positive operator B^{-1} almost everywhere. Curvature Cauchy–Schwarz gives

$$|(f, v)|^2 \leq \left(\int_X \langle B^{-1} f, f\rangle\right)\left(\int_X \langle Bv, v\rangle\right) \leq A_f\, Q(v).$$

Apply the Hilbert-complex lemma to $T = D'' : L^2_{n,q-1}(E) \to L^2_{n,q}(E)$ and $S = D'' : L^2_{n,q}(E) \to L^2_{n,q+1}(E)$. The condition $D''f = 0$ is exactly $f \in \ker S$. Hahn–Banach and Riesz therefore produce $u \in \operatorname{Dom} T$ with $Tu = f$ and $\|u\|^2 \le A_f$. Taking $u \perp \ker T$ gives the minimum-norm solution. □

Adding a scalar weight $\mathrm{e}^{-\varphi}$ replaces h by $h\mathrm{e}^{-\varphi}$ and adds

$$\mathrm{i}\partial\bar{\partial}\varphi \otimes \mathrm{id}_E$$

to the curvature term.

Warning

If B is only semipositive, B^{-1} is undefined on its kernel. One must prove f lies in the range and use the Moore–Penrose inverse, add εI with an ε-independent energy bound, or find another coercive term. Writing $(B + \varepsilon I)^{-1}$ and merely saying "let $\varepsilon \to 0$" is incomplete.

Definition 7.23: Multiplier ideal

For a qpsh weight φ, define

$$\mathcal{I}(\varphi)_x = \{f \in \mathcal{O}_{X,x} : |f|^2 \mathrm{e}^{-\varphi} \in L^1_{\mathrm{loc}}\}.$$

This is a coherent ideal sheaf. If $\varphi = c \log \sum |g_j|^2 + O(1)$, it measures the vanishing order required to compensate the analytic singularity.

7.4.1 A ledger for singular-weight limits

For a nonsmooth psh weight φ:

(1) choose smooth approximations φ_ν on relatively compact sets and record their monotonicity and curvature loss;
(2) add a small strict term if needed, while maintaining a uniform bound for $\langle B_\nu^{-1} f, f \rangle$;
(3) solve the regularized problems and extract weakly in one fixed weighted space;
(4) pass $\bar{\partial} u_\nu = f_\nu$ against compactly supported test forms;
(5) preserve the estimate by weak lower semicontinuity or Fatou's lemma.

Theorem 7.24: Nadel vanishing

Let X be compact Kähler and let L carry a singular Hermitian metric $h = \mathrm{e}^{-\varphi}$ such that, as currents,

$$\mathrm{i}\Theta(L, h) \ge \varepsilon\omega$$

for some $\varepsilon > 0$. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0, \qquad q \ge 1.$$

Proof (proof architecture via the singular L^2 resolution). Let $\mathcal{L}^{n,q}(L, h)$ be the sheaf of measurable L-valued (n, q)-forms whose coefficients and distributional $\bar{\partial}$ are locally square integrable with weight $e^{-\varphi}$. Its degree-zero kernel is $K_X \otimes L \otimes \mathcal{I}(\varphi)$: in a local frame, a holomorphic coefficient f belongs to the kernel exactly when $|f|^2 e^{-\varphi}$ is integrable. Smooth partitions of unity act on the $\mathcal{L}^{n,q}$, so these sheaves are fine.

The analytic input is the singular Hörmander lemma: if $\mathrm{i}\Theta(L, h) \ge \varepsilon\omega$, every $\bar{\partial}$-closed, locally weighted-L^2 L-valued (n, q)-form, $q \ge 1$, is locally $\bar{\partial}$ of a weighted-L^2 form. On compactly supported data the solution satisfies the same inverse-curvature estimate as in the smooth theorem, with a harmless loss such as $2/(q\varepsilon)$.

Here are the limit steps in that lemma. Equisingular regularization gives weights φ_ν that are smooth off analytic sets Z_ν, converge to φ, and satisfy $\mathrm{i}\Theta(L, e^{-\varphi_\nu}) \geq (\varepsilon - o(1))\omega$. On $X \setminus Z_\nu$, use complete metrics $\omega_{\nu,\delta} = \omega + \delta\widehat{\omega}_\nu$ and first solve with $\delta > 0$. Apply the inverse-curvature form of the estimate, rather than asserting a lower bound relative to the larger metric $\omega_{\nu,\delta}$. The elementary comparison of the (n, q) norms shows that its right-hand side is bounded by $2(q\varepsilon)^{-1}\|f\|^2_{\omega,h}$, independently of δ and then of ν. Let $\delta \downarrow 0$, then $\nu \to \infty$. Weak compactness in the fixed $L^2(\omega, h)$ space, testing against compactly supported smooth forms, and weak lower semicontinuity give $\bar\partial u = f$ with the stated bound. Because both u and f are locally L^2, the equation extends distributionally across each Z_ν. This proves the singular Hörmander lemma and hence local exactness.

Consequently

$$0 \to K_X \otimes L \otimes \mathcal{I}(\varphi) \to \mathcal{L}^{n,0}(L, h) \xrightarrow{\bar\partial} \mathcal{L}^{n,1}(L, h) \to \cdots$$

is a fine resolution. The same regularization argument on compact X solves every closed global section in positive degree. Thus the global-section complex is exact in degrees $q \geq 1$. It computes sheaf cohomology, and the asserted groups vanish. □

7.4.2 Completeness as a cutoff theorem

Fix a base point on a complete Riemannian manifold and let $r(x)$ be the distance to it. Choose $\chi \in C_c^\infty(\mathbb{R})$ with $\chi = 1$ on $(-\infty, 1]$ and set $\chi_\nu(x) = \chi(r(x)/\nu)$. Before smoothing, this is Lipschitz and satisfies $|\mathrm{d}\chi_\nu| \leq C/\nu$ almost everywhere. Completeness makes closed bounded balls compact, so the cutoffs have compact support. A standard smoothing argument preserves the gradient bound up to a uniform constant.

For a graph-domain form v,

$$D''(\chi_\nu v) = \chi_\nu D'' v + dbar\chi_\nu \wedge v,$$

and the adjoint has the analogous contraction error. The error norms are bounded by $C\nu^{-1}\|v\|$, so compact-support identities pass to the closed-operator domain. Completeness is used exactly here; it is not a curvature assumption.

7.4.3 Multiplier ideals as radial integrability thresholds

On $\mathbb{C}^n$, take $\varphi = c \log |z|^2 + O(1)$ with $c > 0$. If a holomorphic germ f vanishes to order m, then its leading size is $|f| \asymp r^m$ in generic directions, and

$$|f|^2 \mathrm{e}^{-\varphi} \mathrm{d}\lambda \asymp r^{2m-2c+2n-1} \mathrm{d}r\, \mathrm{d}S.$$

This is integrable at zero exactly when $m > c - n$. Hence

$$\mathcal{I}(c \log |z|^2)_0 = \mathfrak{m}_0^{\max\{0, \lfloor c\rfloor - n+1\}}.$$

For $n = 1$, this reduces to $(z^{\lfloor c\rfloor})$. The strict inequality at the borderline exponent explains the jump when c crosses an integer.

For a general analytic singularity $\varphi = c \log \sum |g_j|^2 + O(1)$, a log resolution turns the problem into a collection of such monomial integrability tests along exceptional divisors. The Jacobian of the resolution contributes the discrepancy term; this is why multiplier ideals reflect both vanishing orders and the geometry of the modification.

7.4.4 A safe singular-limit template

All approximations should be compared in one fixed Hilbert space. If φ_ν are smooth approximants, record whether $\mathrm{e}^{-\varphi_\nu}$ increases or decreases, the curvature loss in $\mathrm{i}\partial\bar{\partial}\varphi_\nu$, and a bound for

$$\int \langle B_\nu^{-1} f_\nu, f_\nu\rangle \mathrm{e}^{-\varphi_\nu}.$$

Only a parameter-independent bound permits weak compactness. After taking a weak limit, pass the equation against compactly supported tests and use lower semicontinuity for the norm. Smoothness, minimum-norm character, and behavior across the singular set each require separate verification.

Remark (Why Nadel vanishing is a sheaf theorem)

The weighted L^2 complex resolves $K_X \otimes L \otimes \mathcal{I}(\varphi)$ because degree-zero holomorphic coefficients are exactly those integrable against the singular metric. The multiplier ideal is not an extra decoration on the conclusion; it is the kernel of the analytic resolution.

7.4.5 How completeness supplies admissible cutoffs

Fix a base point and let $r(x)$ be its Riemannian distance. Completeness makes closed metric balls compact by Hopf–Rinow. Smooth r outside a harmless neighborhood of the cut locus and choose $\chi \in C_c^\infty(\mathbb{R})$ with $\chi = 1$ on $(-\infty, 1]$ and $\chi = 0$ on $[2, \infty)$. Set

$$\chi_\nu(x) = \chi(r(x)/\nu).$$

Then $\chi_\nu \longrightarrow 1$, the supports are compact, and

$$|\mathrm{d}\chi_\nu| \le \frac{C}{\nu}.$$

For a graph-domain form v,

$$\begin{aligned} D''(\chi_\nu v) &= \chi_\nu D'' v + dbar\chi_\nu \wedge v, \\ D''^*(\chi_\nu v) &= \chi_\nu D''^* v - \iota_{(\partial\chi_\nu)^\sharp} v. \end{aligned}$$

The error norms are at most $C\nu^{-1}\|v\|$. Applying BKN to $\chi_\nu v$ and sending $\nu \longrightarrow \infty$ proves the basic estimate on the closed operator domain. Completeness is used exactly to obtain this compactly supported approximation.

Multiplier ideals from a radial integral

On $\mathbb{C}^n$ let $\varphi = c\log|z|^2$ and let a holomorphic germ f vanish to order m. On a set of angular directions of positive measure, $|f(z)|$ is comparable to r^m. Therefore local integrability is decided by

$$\int_0^\varepsilon r^{2m} r^{-2c} r^{2n-1}\,\mathrm{d}r < \infty.$$

This holds exactly when

$$2m - 2c + 2n > 0, \qquad \text{or equivalently} \qquad m > c - n.$$

Hence

$$\boxed{\mathcal{I}(c\log|z|^2)_0 = \mathfrak{m}_0^{\max\{0,\lfloor c-n\rfloor+1\}}.}$$

For example, in one variable $c = 1$ requires one zero, while $0 < c < 1$ requires none. The strict inequality explains the jump when c crosses an integer threshold.

A safe singular-weight limiting argument

Suppose φ is psh and choose smooth approximants φ_ν on an exhaustion with

$$\mathrm{i}\partial\bar{\partial}\varphi_\nu \geq -\varepsilon_\nu\omega, \qquad \varepsilon_\nu \downarrow 0.$$

Add a fixed strictly psh term if necessary, but compute the actual curvature operator B_ν of the full exponent. The indispensable hypothesis is a uniform bound

$$\sup_\nu \int \langle B_\nu^{-1} f_\nu, f_\nu \rangle e^{-\varphi_\nu} \mathrm{d}V < \infty.$$

Solve $\bar{\partial}u_\nu = f_\nu$. Compare all changing weights to one fixed weighted L^2 space, extract a weak limit, pass the equation against compactly supported test forms, and use weak lower semicontinuity. If the inverse energy blows up like ε_ν^{-1}, no conclusion follows from this argument.

Why Nadel vanishing is a sheaf statement

Locally, a holomorphic L-valued $(n, 0)$-form belongs to the weighted L^2 kernel exactly when its scalar coefficient lies in $\mathcal{I}(\varphi)$. Thus the complex of weighted forms begins with

$$0 \to K_X \otimes L \otimes \mathcal{I}(\varphi) \to \mathcal{L}^{n,0}_{(2)}(L, e^{-\varphi}) \xrightarrow{\bar{\partial}} \mathcal{L}^{n,1}_{(2)} \to \cdots .$$

The singular Hörmander estimate proves local exactness in positive degree. Smooth partitions of unity preserve local weighted L^2 integrability, so the later sheaves are fine. Taking global sections therefore computes the sheaf cohomology.

On compact X, the curvature lower bound $\mathrm{i}\Theta(L, e^{-\varphi}) \geq \varepsilon\omega$ makes the global $\bar{\partial}$ equation solvable in every degree $q \geq 1$. Hence the cohomology of the global-section complex vanishes, yielding

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0.$$

Both ingredients are needed: the multiplier ideal identifies the correct degree-zero kernel, and the L^2 estimate makes the fine resolution exact.

7.4.6 Self-study workshop: passing from smooth complete estimates to singular metrics

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall complete Kähler cutoffs, bundle-valued (n, q) estimates, weak compactness in Hilbert space, monotone comparison of weights, and local integrability of singular powers.

Two limiting mechanisms that must not be conflated. Completeness provides cutoffs $\chi_\nu \to 1$ with compact support and controlled gradient. Apply the compact-support BKN identity to $\chi_\nu u$. Commutators such as $[\bar{\partial}, \chi_\nu]u = \bar{\partial}\chi_\nu \wedge u$ tend to zero because $|\mathrm{d}\chi_\nu|$ is uniformly small on the exhausting regions. This extends integration by parts to the maximal graph domain.

For a singular psh weight φ, choose smooth approximants φ_ν on relatively compact sets and add a small strict term if necessary. One must record:

$$\text{curvature loss}, \quad \sup_\nu A_{f,\nu} < \infty, \quad u_\nu \rightharpoonup u, \quad \bar{\partial}u = f, \quad \|u\|^2_\varphi \leq \liminf_\nu \|u_\nu\|^2_{\varphi_\nu}.$$

The direction of approximation determines whether Fatou's lemma or comparison of weighted norms applies. Merely writing $(B + \varepsilon I)^{-1}$ and sending $\varepsilon \to 0$ is invalid if the energy blows up on $\ker B$.

Multiplier ideals record the local integrability required by the singular metric. In one variable, for $\varphi = c\log|z|^2$ and $f = z^m$,

$$|f|^2 e^{-\varphi}\,\mathrm{d}A \asymp r^{2m-2c+1}\,\mathrm{d}r\,\mathrm{d}\theta.$$

The integral converges exactly when $2m-2c+1 > -1$, or $m > c-1$. Thus $\mathcal{I}(\varphi)_0$ consists of germs whose order satisfies this inequality.

Checkpoint (uniformity before convergence)

(1) Which property of completeness is used in the BKN proof?
(2) Why is weak convergence alone insufficient when the weights vary?
(3) Compute the smallest integer $m \ge 0$ with $z^m \in \mathcal{I}(2.4\log|z|^2)_0$.

Solution (checkpoint). Completeness supplies compactly supported cutoffs whose gradient errors tend to zero. One must compare the varying weighted norms and use the correct monotonicity before lower semicontinuity applies. The condition is $m > 1.4$, so the smallest integer is $m = 2$. ⋄

Exit criterion

You can separate the cutoff argument from the singular-weight approximation, list all uniform estimates needed for a weak limit, and compute a multiplier ideal by a radial integral.

Summary. Completeness replaces compactness in justifying integration by parts, and the inverse curvature operator retains the sharp estimate. Singular weights require uniform estimates and weak limits; multiplier ideals encode the integrability threshold.

Exercise 7.25: A one-dimensional multiplier ideal

Compute $\mathcal{I}(c\log|z|^2)_0$. Remember that the area element is $r\mathrm{d}r\mathrm{d}\theta$.

Exercise 7.26: Complete cutoffs

Prove that a complete Riemannian manifold admits $\chi_\nu \in C_c^\infty$, $0 \le \chi_\nu \le 1$, $\chi_\nu \to 1$, and $|\mathrm{d}\chi_\nu| \le C/\nu$ (first use the distance function, then smooth it).

Optional handout: The strong openness theorem

This handout is optional. No later argument in the core lectures depends on it.

Multiplier ideals satisfy an elementary monotonicity: if $\psi \le \varphi + O(1)$, then $\mathcal{I}(\psi) \subset \mathcal{I}(\varphi)$. A much deeper property is strong openness.

Theorem 7.27: Strong openness

For every psh function φ,

$$\mathcal{I}(\varphi) = \bigcup_{\varepsilon>0} \mathcal{I}((1+\varepsilon)\varphi).$$

Equivalently, if $|f|^2\mathrm{e}^{-\varphi}$ is locally integrable, then it remains integrable after making the singularity slightly stronger.

Proof (one-dimensional proof and the general proof architecture). In one complex dimension there is a complete elementary proof. Write $f(z) = z^m g(z)$ with $g(0) \ne 0$. The Riesz decomposition and

Jensen's formula show that, for a subharmonic φ,

$$|f|^2 e^{-\varphi} \in L^1_{\mathrm{loc}}(0) \quad\Longleftrightarrow\quad \nu(\varphi, 0) < m + 1. \tag{28.1}$$

For the nontrivial direction, compare circular means of φ with $\nu(\varphi, 0) \log r^2$ and integrate in polar coordinates; equality at the threshold gives the divergent integral $\int_0 r^{-1}\, dr$. If (28.1) holds, choose $p > 1$ so close to 1 that $p\nu(\varphi, 0) < m + 1$. Applying (28.1) to $p\varphi$ gives $f \in \mathcal{I}(p\varphi)_0$, proving strong openness on a Riemann surface.

In higher dimension, integrability is not determined by one Lelong number, and a proof by restriction to a single curve is invalid. Guan and Zhou prove the theorem by an induction on dimension built around a movable L^2 extension estimate. The key quantitative input controls extensions from varying level hypersurfaces by the mass of $|f|^2 e^{-\varphi}$ on those hypersurfaces. Fubini selects levels on which that mass decays; Ohsawa–Takegoshi extends the restrictions with uniform control; coherence and Noetherian stabilization convert failure of openness into a fixed holomorphic obstruction. The uniform extensions violate that obstruction, closing the induction. This is a proof architecture rather than a reproduction of the long quantitative induction; importantly, it does not use the false claim that arbitrary holomorphic interpolants must have a universal $|a|^{-2}$ lower bound. □

Example 7.28: Monomial weights

For $\varphi = \sum_j c_j \log |z_j|^2$ and $f = z_1^{m_1} \cdots z_n^{m_n}$, Fubini's theorem gives integrability exactly when $m_j > c_j - 1$ for every j. These strict inequalities visibly persist after replacing every c_j by $(1 + \varepsilon)c_j$ for sufficiently small $\varepsilon > 0$.

Strong openness implies that jumping numbers of multiplier ideals are genuine thresholds and underlies precise semicontinuity statements in families.

Exercise 7.29: Radial singularity

Determine for which integers m the germ z^m belongs to $\mathcal{I}(c \log |z|^2)$ and verify strong openness directly in this example.

Chapter 8

Extension, the Calabi–Yau Equation, and Modern Positivity

8.1 Lecture 29: Introduction to the Ohsawa–Takegoshi extension method

Week 15 material • 90 minutes

Notation for this lecture

$\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain and $Y = \Omega \cap \{z_n = 0\}$ the model hypersurface; $z' = (z_1, \dots, z_{n-1})$. f is holomorphic on Y, F is a holomorphic extension to Ω, φ is a psh weight, and $\mathrm{d}\lambda_k$ is Lebesgue measure in complex dimension k. Its Levi matrix is $H_\varphi = (\varphi_{j\bar{k}})$. In the invariant formulation, $Y = \{\sigma = 0\} \subset X$ is the smooth zero set of a transverse section σ of a line bundle, and $|\mathrm{d}\sigma|$ supplies the normal Jacobian. F^0_ε is a cutoff extension, $g_\varepsilon = \bar{\partial} F^0_\varepsilon$ its error, and u_ε the correction solving $\bar{\partial} u_\varepsilon = g_\varepsilon$. OT and BKN abbreviate Ohsawa–Takegoshi and Bochner–Kodaira–Nakano. The Bergman space is $A^2(U) = \mathcal{O}(U) \cap L^2(U)$ with the measure or weight stated in context.

Prerequisite results used below

Hörmander weighted existence theorem. If $\Omega \subset \mathbb{C}^n$ is pseudoconvex, $\varphi \in C^2(\Omega)$ is strictly psh, and $f = \sum f_j \mathrm{d}\bar{z}_j$ is $\bar{\partial}$-closed with

$$\int_\Omega f^* H_\varphi^{-1} f \, \mathrm{e}^{-\varphi} \mathrm{d}\lambda < \infty,$$

then there is u with $\bar{\partial} u = f$ and $\int_\Omega |u|^2 \mathrm{e}^{-\varphi} \mathrm{d}\lambda$ bounded by the displayed inverse-Hessian integral.
Adjunction for a smooth hypersurface. If $Y = \{\sigma = 0\}$ is the smooth zero set of a transverse section of a line bundle $\mathcal{O}(Y)$, then $K_Y \simeq (K_X \otimes \mathcal{O}(Y))|_Y$. Locally, division by $\mathrm{d}\sigma$ identifies an ambient top form along Y with its adjunction trace.

Plan for the lecture. 0–18 min: why extension from a submanifold is difficult; 18–38: the hyperplane theorem; 38–62: singular weights and twisted estimates; 62–78: zero sets of bundle sections; 78–90: applications and scaling checks.

Cartan B already extends holomorphic functions from a closed submanifold of a Stein manifold. Ohsawa–Takegoshi adds quantitative L^2 control, which is indispensable for singular metrics and algebraic geometry.

Theorem 8.1: Ohsawa–Takegoshi, hyperplane version

Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex with $|z_n| < 1$, let $Y = \Omega \cap \{z_n = 0\}$, and let φ be psh. If $f \in \mathcal{O}(Y)$ and

$$\int_Y |f|^2 \mathrm{e}^{-\varphi} \mathrm{d}\lambda_Y < \infty,$$

then there is $F \in \mathcal{O}(\Omega)$ with $F|_Y = f$ and

$$\int_\Omega |F|^2 \mathrm{e}^{-\varphi} \mathrm{d}\lambda \leq C_{\mathrm{OT}} \int_Y |f|^2 \mathrm{e}^{-\varphi} \mathrm{d}\lambda_Y.$$

After coordinate normalization, C_{OT} is independent of f and φ; refined versions achieve the optimal constant.

Proof (the twisted L^2 argument). We prove a slightly stronger weighted estimate with a universal, not necessarily optimal, constant. Exhaust Ω by smoothly bounded strictly pseudoconvex domains and regularize φ from above. Uniform estimates and a diagonal weak limit reduce the proof to the case in which the domain and weight are smooth. Rescale the normal coordinate so that $s = z_n$ satisfies $|s| < e^{-1}$.

The twisted basic estimate is the following consequence of the BKN identity. For positive smooth functions η, λ and compactly supported $(n, 1)$-forms v,

$$\|\sqrt{\eta + \lambda}\, \bar{\partial}^*_\varphi v\|^2_\varphi + \|\sqrt{\eta}\, \bar{\partial} v\|^2_\varphi \geq \left\langle [\eta \mathrm{i}\partial\bar{\partial}\varphi - \mathrm{i}\partial\bar{\partial}\eta - \mathrm{i}\lambda^{-1}\partial\eta \wedge \bar{\partial}\eta, \Lambda] v, v \right\rangle. \tag{29.1}$$

It follows by applying BKN to $\sqrt{\eta}, v$ and estimating the cross term with $2|ab| \leq \lambda|a|^2 + \lambda^{-1}|b|^2$. Notice that no positivity of $\mathrm{i}\partial\bar{\partial}\eta$ is being assumed.

Set

$$\chi_0(t) = t - \log(1 - t), \quad \sigma_\varepsilon = \log(|s|^2 + \varepsilon^2), \quad \eta_\varepsilon = \varepsilon - \chi_0(\sigma_\varepsilon), \quad \lambda_\varepsilon = \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)}. \tag{29.2}$$

Here $1 \leq \chi_0' \leq 2$ and $\chi_0'' = (1 - t)^{-2} > 0$. Apply (29.1) with the additional singular metric factor $|s|^{-2}$. Direct differentiation gives

$$\mathrm{i}\partial\bar{\partial}\sigma_\varepsilon = \frac{\varepsilon^2 \mathrm{i}\, ds \wedge d\bar{s}}{(|s|^2 + \varepsilon^2)^2},$$

and substitution of (29.2) shows that the negative square in (29.1) cancels exactly the χ_0'' term. The remaining operator dominates a positive multiple of

$$\frac{\varepsilon^2}{|s|^2} [\mathrm{i}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon, \Lambda]$$

on $\Omega \setminus Y$; the psh curvature of φ is nonnegative. This is the crucial sign computation.

Choose a smooth extension $\widetilde{f}$ with $\bar{\partial}\widetilde{f} = O(|s|)$ and a cutoff θ supported in $[0, 1)$ and equal to 1 on $[0, 1/2]$. For

$$F^0_\varepsilon = \theta(|s|^2/\varepsilon^2)\widetilde{f}, \qquad g_\varepsilon = \bar{\partial} F^0_\varepsilon,$$

the main term of g_ε is a scalar multiple of $\bar{\partial}\eta_\varepsilon\, \widetilde{f}$, supported where $|s| \asymp \varepsilon$; the term containing $\bar{\partial}\widetilde{f}$ is $O(\varepsilon)$. Curvature Cauchy–Schwarz in the preceding operator inequality, followed by integration in the normal disc, yields

$$\int_\Omega \langle B_\varepsilon^{-1} g_\varepsilon, g_\varepsilon \rangle |s|^{-2} e^{-\varphi} \leq C \int_Y |f|^2 e^{-\varphi}, \tag{29.3}$$

where B_ε is the operator in (29.1). The twisted Hilbert-space lemma therefore solves $\bar{\partial} u_\varepsilon = g_\varepsilon$ and gives

$$\int_\Omega \frac{|u_\varepsilon|^2 e^{-\varphi}}{|s|^2(\eta_\varepsilon + \lambda_\varepsilon)} \leq C \int_Y |f|^2 e^{-\varphi}. \tag{29.4}$$

The nonintegrability of $|s|^{-2}$ forces the trace of the correction to vanish on Y. Hence $F_\varepsilon = F_\varepsilon^0 - u_\varepsilon$ is holomorphic and has trace f.

From (29.2), $\eta_\varepsilon + \lambda_\varepsilon \le C(-\log(|s|^2 + \varepsilon^2))^2$. Combining (29.4) with the elementary cutoff estimate and then taking weak limits gives

$$\int_\Omega \frac{|F|^2 e^{-\varphi}}{|s|^2(-\log|s|)^2} \le C \int_Y |f|^2 e^{-\varphi}. \tag{29.5}$$

Because $|s|^2(-\log|s|)^2$ is bounded above on $0 < |s| < e^{-1}$, (29.5) implies the ordinary estimate in the theorem after changing the universal constant. Montel convergence preserves the restriction to Y, while Fatou's lemma preserves the bound. Finally pass through the regularized weights and the exhaustion; all constants are independent of both parameters. □

8.1.1 Why ordinary Hörmander is not enough

Start with a smooth extension $\widetilde{f}$ and try to solve $\bar\partial u = \bar\partial(\chi \widetilde{f})$ so that $F = \chi\widetilde{f} - u$. To retain $F|_Y = f$, one must force $u|_Y = 0$. A singular weight containing $\log|z_n|^2$ makes finite weighted norm force divisibility by z_n. The resulting curvature degenerates, so one uses a *twisted* Bochner identity: multiply the basic estimate by an auxiliary function η and complete squares between errors from $\partial\eta$ and $\partial\bar\partial\eta$.

Theorem 8.2: Zero set of a transverse bundle section

Let X be weakly pseudoconvex and Kähler, let (E, h_E) have rank r, and let $\sigma \in H^0(X, E)$ be transverse to the zero section, so $Y = \sigma^{-1}(0)$ is smooth of codimension r. Give $A = \det E \otimes L$ a Hermitian metric and suppose

$$\mathrm{i}\Theta(A) + r\mathrm{i}\partial\bar\partial \log|\sigma|^2 \ge 0.$$

Assume also that a continuous function $\alpha \ge 1$ satisfies

$$\mathrm{i}\Theta(A) + r\mathrm{i}\partial\bar\partial \log|\sigma|^2 \ge \alpha^{-1} \frac{\{\mathrm{i}\Theta(E)\sigma, \sigma\}}{|\sigma|^2}, \qquad |\sigma| \le e^{-\alpha}.$$

Then every $f \in H^0(Y, K_Y \otimes L|_Y)$ with finite L^2 norm extends to $F \in H^0(X, K_X \otimes \det E \otimes L)$, in the adjunction sense $F|_Y/\Lambda^r(d\sigma) = f$, and

$$\int_X \frac{|F|^2}{|\sigma|^{2r}(-\log|\sigma|)^2} \le C_r \int_Y |f|^2.$$

The metrics and volume forms in this formula are the ones induced by ω, h_E, and the metric on A; C_r depends only on r.

Proof (proof architecture and curvature computation). Put $\sigma_\varepsilon = \log(|\sigma|^2 + \varepsilon^2)$ and use the functions $\eta_\varepsilon, \lambda_\varepsilon$ from (29.2), together with the singular factor $|\sigma|^{-2r}$ in the metric of A. The Chern-connection calculation is

$$\mathrm{i}\partial\bar\partial\sigma_\varepsilon = \frac{\mathrm{i}\{D'\sigma, D'\sigma\}}{|\sigma|^2 + \varepsilon^2} - \frac{\mathrm{i}\{D'\sigma, \sigma\} \wedge \{\sigma, D'\sigma\}}{(|\sigma|^2 + \varepsilon^2)^2} - \frac{\{\mathrm{i}\Theta(E)\sigma, \sigma\}}{|\sigma|^2 + \varepsilon^2}.$$

The fiberwise Lagrange inequality

$$\mathrm{i}\{D'\sigma, \sigma\} \wedge \{\sigma, D'\sigma\} \le |\sigma|^2 \mathrm{i}\{D'\sigma, D'\sigma\}$$

and the two stated curvature hypotheses give exactly the positive operator required by the twisted estimate (29.1). This is why the curvature of E, the factor r, and the scaling condition on σ occur in the theorem.

Choose a smooth ambient representative of the adjunction data whose $\bar\partial$ vanishes to first order on Y, cut it off in $\{|\sigma| < \varepsilon\}$, and solve its $\bar\partial$ error on $X \setminus Y$. The inverse-curvature estimate and integration in the r normal variables give a bound independent of ε; the normal integral is the numerical constant C_r. The correction is integrable against $|\sigma|^{-2r}$, so its trace on Y is zero. The bounds $\eta_\varepsilon + \lambda_\varepsilon \le C(-\log(|\sigma|^2 + \varepsilon^2))^2$ then give the displayed weighted estimate. Weak compactness supplies F, and adjunction identifies its trace with f. The minimum-norm solution is global, so no separate patching of local holomorphic extensions is needed. ◻

The proof uses both $D''\sigma = 0$ and the pointwise Cauchy–Schwarz inequality

$$\mathrm{i}\{D'\sigma, \sigma\} \wedge \{\sigma, D'\sigma\} \le |\sigma|^2\, \mathrm{i}\{D'\sigma, D'\sigma\}.$$

The factor $|\sigma|^2$ is forced by scaling $\sigma \mapsto \lambda\sigma$.

Corollary 8.3: Three consequences of extension

(1) For hypersurfaces and smooth complete intersections satisfying the stated curvature hypotheses, restriction is surjective with control of the minimum L^2 norm.
(2) Extension from a point gives lower bounds for weighted Bergman kernels.
(3) Together with additional extension and positivity theorems, OT is a principal input in results on psh variation of Bergman kernels and deformation invariance of pluricanonical sections.

Proof. For (1), represent the prescribed section on the submanifold and apply the relevant OT theorem; the constructed extension lies in the affine space of all extensions and orthogonal projection to the complement of the kernel of restriction gives the unique minimum-norm representative. For (2), take $Y = \{x\}$ and normalize the value to one. The reciprocal of the minimum squared norm is the weighted Bergman kernel $K_\varphi(x)$, so the extension estimate gives its lower bound. Statement (3) records uses of OT, not formal consequences of the version proved here: psh variation additionally uses Berndtsson's positivity theorem for direct images, and deformation invariance uses an iterated extension argument together with the relevant family and singular-metric hypotheses. ◻

Warning

Ohsawa–Takegoshi is not obtained by simply inserting a singular weight into Lecture 27. One must design a twisted weight, keep constants uniform during regularization, and prove that the limiting correction vanishes on the submanifold. Extending functions, top forms, or adjoint-bundle sections also changes Jacobian factors and weights.

8.1.2 Why the singular norm forces the correct trace

In one normal variable s, suppose a holomorphic function g satisfies

$$\int_{|s|<\varepsilon} \frac{|g(s)|^2}{|s|^2}\, \mathrm{d}\lambda(s) < \infty.$$

Write $g(s) = s^m a(s)$ with $a(0) \ne 0$. The radial integral behaves like

$$\int_0^\varepsilon r^{2m-2} r\, \mathrm{d}r,$$

which is finite exactly when $m \ge 1$. Thus $g(0) = 0$. With parameters along a hypersurface, Fubini's theorem gives the same conclusion almost everywhere and holomorphicity gives it everywhere.

In codimension r, the normal volume element is comparable to $\rho^{2r-1}\mathrm{d}\rho$, while the singular factor is $|\sigma|^{-2r}$. A correction with nonzero trace would again produce the divergent integral $\int \rho^{-1}\mathrm{d}\rho$. This is the analytic reason the L^2 correction vanishes on Y. The logarithmic square in the final estimate is just strong enough to make the extended section itself integrable after the correction has acquired the required vanishing.

8.1.3 Point extension and the Bergman kernel

For a weighted domain, the Bergman kernel on the diagonal has the extremal description

$$K_\varphi(x) = \sup_{0\neq F\in A^2(\Omega,\mathrm{e}^{-\varphi})} \frac{|F(x)|^2\mathrm{e}^{-\varphi(x)}}{\|F\|_\varphi^2} = \frac{\mathrm{e}^{-\varphi(x)}}{\inf\{\|F\|_\varphi^2 : F(x)=1\}}.$$

An extension theorem from the point x gives a competitor with controlled norm and therefore a lower bound for $K_\varphi(x)$. Conversely, the reproducing kernel gives the unique minimum-norm function with prescribed value.

On the unit disc with unweighted Lebesgue area, rotational symmetry shows that the constant function 1 is orthogonal to every holomorphic function vanishing at the origin. It is therefore the minimum-norm extension of the value 1, with norm squared π. Hence $K_\Delta(0) = 1/\pi$. Under a dilation $z \mapsto Rz$, the minimum norm scales by R^2 and the kernel by R^{-2}, illustrating the normal-volume scaling in Ohsawa–Takegoshi.

If the defining section is rescaled, $\sigma \mapsto a\sigma$, the adjunction trace and the denominator $|\sigma|^{2r}$ both change. Checking that these powers cancel is a fast way to detect a missing determinant or Jacobian factor in a bundle-valued extension statement.

Warning

Qualitative extension follows from Cartan B on a Stein ambient space, but it neither selects the minimum-norm extension nor controls behavior as the weight or submanifold varies. Those quantitative features are the content of Ohsawa–Takegoshi.

8.1.4 A product-domain extension computed exactly

Let $\Omega = \Delta^{n-1}\times\Delta_R$ with unweighted Lebesgue measure and $Y = \{z_n = 0\}$. Given $f \in A^2(\Delta^{n-1})$, the constant-in-the-normal-direction extension

$$F(z', z_n) = f(z')$$

satisfies

$$\int_\Omega |F|^2 \mathrm{d}\lambda = \pi R^2 \int_Y |f|^2 \mathrm{d}\lambda_Y.$$

It is the minimum-norm extension. Indeed, every other extension has a power series

$$G(z', z_n) = f(z') + \sum_{k\geq 1} a_k(z') z_n^k.$$

The monomials in z_n are orthogonal in the disc, so Fubini gives

$$\|G\|^2 = \pi R^2\|f\|_Y^2 + \sum_{k\geq 1}\frac{\pi R^{2k+2}}{k+1}\|a_k\|_Y^2 \geq \pi R^2\|f\|_Y^2.$$

This model explains the normal-volume factor and gives a scaling check for every general extension theorem.

The cutoff error near a hypersurface

Let $s = z_n$ and choose a smooth ambient extension $\widetilde{f}$ with $\bar{\partial}\widetilde{f} = O(|s|)$. For a cutoff χ set

$$F_\varepsilon^0 = \chi(|s|^2/\varepsilon^2)\widetilde{f}.$$

Then

$$\bar{\partial}F_\varepsilon^0 = \chi'(|s|^2/\varepsilon^2)\frac{s}{\varepsilon^2}\widetilde{f}\,\mathrm{d}\bar{s} + \chi(|s|^2/\varepsilon^2)\bar{\partial}\widetilde{f}.$$

The first term has size $O(\varepsilon^{-1})$ but is supported in an annulus of normal area $O(\varepsilon^2)$; the second has size $O(\varepsilon)$. An ordinary unweighted estimate would control the correction but would not force its trace to vanish. The singular factor $|s|^{-2}$ changes the problem: finite norm then implies divisibility by s.

If a holomorphic correction has expansion $u(s) = a_0 + a_1 s + \cdots$, then

$$\int_{|s|<\delta} \frac{|u(s)|^2}{|s|^2}\mathrm{d}\lambda(s) < \infty$$

forces $a_0 = 0$, because a nonzero constant term produces $2\pi|a_0|^2 \int_0^\delta r^{-1}\mathrm{d}r = \infty$. This is the exact analytic mechanism that preserves the prescribed trace.

Local adjunction in coordinates

Suppose $Y = \{z_{n-r+1} = \cdots = z_n = 0\}$ and the defining bundle section is

$$\sigma = (z_{n-r+1}, \ldots, z_n).$$

Then

$$\Lambda^r(\mathrm{d}\sigma) = \mathrm{d}z_{n-r+1} \wedge \cdots \wedge \mathrm{d}z_n.$$

If

$$F = f(z')\,ddz_1 \wedge \cdots \wedge \mathrm{d}z_n,$$

its adjunction trace is

$$\frac{F|_Y}{\Lambda^r(\mathrm{d}\sigma)} = f(z',0)\,ddz_1 \wedge \cdots \wedge \mathrm{d}z_{n-r},$$

a section of K_Y. Under a change of defining equations $\widetilde{\sigma} = A\sigma$, the determinant $\det A$ multiplies $\Lambda^r(\mathrm{d}\sigma)$, while the factor $\det E$ in the ambient bundle changes by the same amount. This proves that the adjunction quotient is coordinate independent.

Point extension and the Bergman extremal problem

For a weighted Bergman space, evaluation at x is a bounded functional whenever a mean-value estimate holds. Riesz gives a vector K_x with $F(x) = (F, K_x)_\varphi$. The minimum-norm function satisfying $F(x) = 1$ is

$$\frac{K_x}{K_\varphi(x,x)}, \qquad \min_{F(x)=1} \|F\|_\varphi^2 = \frac{1}{K_\varphi(x,x)}.$$

An Ohsawa–Takegoshi extension from the point produces an admissible F of controlled norm, hence a lower bound on $K_\varphi(x,x)$. This is a quantitative conclusion that Cartan's qualitative extension theorem cannot supply.

8.1.5 Self-study workshop: locating the Hörmander estimate inside Ohsawa–Takegoshi

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall adjunction, singular bundle metrics, cutoff errors, twisted BKN estimates, and weak compactness. The bare scalar Hörmander theorem does not by itself give an extension with the correct trace and uniform constant.

The extension architecture. Let $Y = \{\sigma = 0\} \subset X$ be a smooth hypersurface and let f be holomorphic data on Y. First use adjunction to interpret the target correctly:

$$K_Y \simeq (K_X \otimes \mathcal{O}(Y))|_Y .$$

Choose a smooth ambient representative $\widetilde{f}$ with the prescribed trace and multiply it by a cutoff depending on $|\sigma|/\varepsilon$. The resulting form F_ε^0 has correct trace but is not holomorphic:

$$g_\varepsilon = \bar{\partial} F_\varepsilon^0$$

is supported in the thin annulus where $|\sigma| \asymp \varepsilon$.

The singular weight is engineered so that solving $\bar{\partial} u_\varepsilon = g_\varepsilon$ costs exactly the normal factor needed to cancel the cutoff derivative. A twisted BKN inequality, not an untwisted invocation of Hörmander, produces a bound independent of ε. Then

$$F_\varepsilon = F_\varepsilon^0 - u_\varepsilon$$

is holomorphic. The singular norm forces the minimum solution u_ε to have zero trace on Y, so $F_\varepsilon|_Y = f$. Weak compactness and lower semicontinuity give a global limit with the extension estimate.

In the product model $X = Y \times \Delta$, the constant extension $F(y, w) = f(y)$ lets one check every factor directly. Integrating in w isolates the normal numerical constant and shows what the general theorem is trying to preserve.

Checkpoint (why the construction is twisted)

(1) Why not begin with a globally holomorphic extension?
(2) What must be uniform as $\varepsilon \downarrow 0$?
(3) How does the singular norm protect the trace after subtracting u_ε?

Solution (checkpoint). A global holomorphic extension is precisely the desired conclusion; only a smooth ambient representative is initially available. The L^2 bound for the correction and the curvature lower bound must be independent of ε. The singular weight makes any correction with nonzero trace have infinite or prohibitively large normal energy, so the minimum solution vanishes along Y in the required adjunction sense. ◇

Exit criterion

You can identify the smooth extension, cutoff error, singular weight, twisted estimate, trace argument, and weak limit as six distinct steps, and you can explain why ordinary Hörmander solvability is only one ingredient.

Summary. OT upgrades $\bar{\partial}$ solvability to quantitative extension with a precise normal singularity. Holomorphicity, curvature compensation, uniform estimates, and limiting integrability are all essential.

Exercise 8.4: Point extension and the Bergman kernel

Take Y to be one point and explain how the OT estimate gives a lower bound for the weighted Bergman kernel.

Exercise 8.5: Scaling audit

For a trivial line bundle with $\sigma(z) = az$, compare the powers of a on both sides of the corrected inequality.

Optional handout: The twisted estimate behind extension

This handout is optional. No later argument in the core lectures depends on it.

Here is the algebraic heart of the Ohsawa–Takegoshi method. Let $\eta > 0$ and let $\lambda > 0$ be auxiliary functions. Applying the Bochner identity to $\sqrt{\eta}, v$ and completing the square yields a twisted estimate of the form

$$\left\|\sqrt{\eta}\,\bar\partial v\right\|^2 + \left\|\sqrt{\eta+\lambda}\,\bar\partial^* v\right\|^2 \geq \left([\eta\, \mathrm{i}\Theta - \mathrm{i}\partial\bar\partial\eta - \lambda^{-1}\mathrm{i}\partial\eta \wedge \bar\partial\eta, \Lambda]v, v\right).$$

The exact bundle curvature is inserted in Θ. The term with λ^{-1} is the price of Cauchy–Schwarz when differentiating $\sqrt{\eta}$.

For a hypersurface defined by $w = 0$, regularize $\sigma_\varepsilon = \log(|w|^2+\varepsilon^2)$. Choose $\chi_0(t) = t-\log(1-t)$ and set

$$\eta_\varepsilon = \varepsilon - \chi_0(\sigma_\varepsilon), \qquad \lambda_\varepsilon = \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)}.$$

Then

$$-\mathrm{i}\partial\bar\partial\eta - \lambda^{-1}\mathrm{i}\partial\eta \wedge \bar\partial\eta$$

contains a positive multiple of $\varepsilon^2\mathrm{i}dw \wedge d\bar w/(|w|^2+\varepsilon^2)^2$. This approximate delta mass controls the normal trace on Y.

Start with a cutoff extension $\chi(|w|^2/\varepsilon^2)f$. Its $\bar\partial$ is supported in an ε-tube around Y. The twisted estimate and Hilbert-space duality give a correction u_ε with a uniform singular weighted bound. For each fixed ε, the factor $|w|^{-2}$ is nonintegrable while $\eta_\varepsilon + \lambda_\varepsilon$ is finite on $w = 0$; smoothness of the correction therefore forces its trace on Y to vanish. Extracting a weak limit gives the extension and the logarithmically weighted estimate, which in turn implies the ordinary L^2 bound on a normalized normal disc.

Remark (Invariance of plurigenera)

Applied fiberwise to adjoint bundles, extension is a central analytic input in the deformation invariance of plurigenera. The complete theorem also requires the appropriate singular metrics, an iterated extension argument, and standard hypotheses on the projective family; it is not a formal consequence of the hyperplane statement.

Exercise 8.6: Approximate delta mass

Show that $\varepsilon^2\mathrm{i}dw \wedge d\bar w/(|w|^2+\varepsilon^2)^2$ has uniformly bounded mass on discs and converges, after normalization, to δ_0.

8.2 Lecture 30: The Calabi–Yau theorem and the continuity method

Week 15 material • 90 minutes

Notation for this lecture

(X, ω) is a compact Kähler manifold of complex dimension n. For a real function φ, $\omega_\varphi = \omega + \mathrm{i}\partial\bar{\partial}\varphi$, and the condition $\omega_\varphi > 0$ means that this is a Kähler form. $F \in C^\infty(X, \mathbb{R})$ is the prescribed log-density, $V = \int_X \omega^n$, $t \in [0, 1]$ is the continuity parameter, and c_t is the constant normalizing total volume. $\Delta_\omega f = \mathrm{tr}_\omega(\mathrm{i}\partial\bar{\partial} f)$ is the geometric-sign complex Laplacian; Δ_{d} is the nonnegative Hodge Laplacian. The symbol $\mathrm{tr}_\omega \eta$ is the trace of a $(1,1)$-form η with respect to ω. $C^{k,\alpha}$ denotes the usual Hölder space and $\mathrm{Ric}(\omega)$ the Ricci form; $c_1(X) = c_1(T^{1,0}X)$ is the first Chern class of the holomorphic tangent bundle.

Prerequisite results used below

Banach-space implicit function theorem. Let $\Phi : U \subset B_1 \to B_2$ be continuously differentiable between Banach spaces. If $D\Phi(x_0) : B_1 \to B_2$ is an isomorphism, then Φ is a C^1 diffeomorphism between neighborhoods of x_0 and $\Phi(x_0)$.
Elliptic Schauder estimate. For a uniformly elliptic second-order linear operator L with $C^{k,\alpha}$ coefficients and a fixed normalization eliminating its kernel,

$$\|u\|_{C^{k+2,\alpha}} \le C\big(\|Lu\|_{C^{k,\alpha}} + \|u\|_{C^0}\big).$$

Together with compact embedding of Hölder spaces, this turns uniform a priori bounds into closedness of the continuity path.

Plan for the lecture. 0–18 min: the Calabi conjecture and Monge–Ampère; 18–35: continuity method; 35–58: C^0 estimate; 58–75: second-order estimate; 75–85: higher regularity; 85–90: geometric consequences.

Let (X^n, ω) be compact Kähler. To avoid ambiguity in 2π factors, write

$$\omega_\varphi = \omega + \mathrm{i}\partial\bar{\partial}\varphi = \omega + 2\pi \mathrm{d}\mathrm{d}^{\mathrm{c}}\varphi.$$

In this lecture

$$\Delta_\omega f = \mathrm{tr}_\omega(\mathrm{i}\partial\bar{\partial} f)$$

is the scalar complex Laplacian, with $\Delta_\omega f \le 0$ at a local maximum. It is therefore a geometric-sign operator; with the Riemannian and form conventions of Lecture 21, the nonnegative Hodge Laplacian on functions is $\Delta_{\mathrm{d}} = -2\Delta_\omega$. We keep the subscripts to prevent these two signs from being confused.

Theorem 8.7: Calabi–Yau theorem

Let $F \in C^\infty(X, \mathbb{R})$ satisfy

$$\int_X \mathrm{e}^F \omega^n = \int_X \omega^n.$$

There is a unique smooth real function φ with $\sup_X \varphi = 0$ such that

$$\omega_\varphi > 0, \qquad \omega_\varphi^n = \mathrm{e}^F \omega^n.$$

Proof (continuity method and a priori estimates).

Uniqueness. Let $u = \varphi - \psi$. The algebraic identity

$$\omega_\varphi^n - \omega_\psi^n = \mathrm{i}\partial\bar\partial u \wedge \sum_{j=0}^{n-1} \omega_\varphi^j \wedge \omega_\psi^{n-1-j}$$

and Stokes' theorem give

$$0 = \int_X u(\omega_\varphi^n - \omega_\psi^n) = -\sum_{j=0}^{n-1} \int_X \mathrm{i}\partial u \wedge \bar\partial u \wedge \omega_\varphi^j \wedge \omega_\psi^{n-1-j}.$$

Every term on the right is nonpositive, so $\partial u = 0$ and u is constant. The normalization $\sup_X \varphi = \sup_X \psi = 0$ gives $u = 0$.

The continuity path. Consider

$$\omega_{\varphi_t}^n = \mathrm{e}^{tF + c_t} \omega^n, \qquad t \in [0, 1],$$

where c_t equalizes total volume. At $t = 0$, $\varphi = 0$. The linearization is

$$D\left[\log \frac{\omega_\varphi^n}{\omega^n}\right](\dot\varphi) = \Delta_{\omega_\varphi}\dot\varphi.$$

The Laplacian is invertible modulo constants, so the implicit-function theorem proves openness.

More precisely,

$$c_t = -\log\left(\frac{1}{V}\int_X \mathrm{e}^{tF}\omega^n\right), \qquad V = \int_X \omega^n.$$

Fix the additive constant in φ by $\int_X \varphi\, \omega^n = 0$ and treat the normalizing constant as an unknown. The linearization in $(\dot\varphi, \dot c)$ is

$$(\dot\varphi, \dot c) \longmapsto \Delta_{\omega_{\varphi_t}}\dot\varphi - \dot c.$$

This maps $C_0^{k+2,\alpha}(X) \oplus \mathbb{R}$ isomorphically onto $C^{k,\alpha}(X)$: integration against $\omega_{\varphi_t}^n$ determines $\dot c$, self-adjoint elliptic theory solves the resulting zero-mean equation, and the fixed ω^n-mean determines the additive constant of $\dot\varphi$. The Banach-space implicit-function theorem therefore makes the solution set open. This formulation avoids the incorrect assertion that $\Delta_{\omega_{\varphi_t}}$ preserves zero mean with respect to the fixed volume form ω^n.

A priori estimates for closedness.

(1) C^0 **estimate.** Normalize $\sup_X \varphi = 0$. Use integration by parts in

$$\int_X (-\varphi)^p(\omega^n - \omega_\varphi^n)$$

together with

$$\omega_\varphi^n - \omega^n = \mathrm{i}\partial\bar\partial\varphi \wedge \sum_{j=0}^{n-1} \omega_\varphi^j \wedge \omega^{n-1-j}.$$

Stokes' theorem, positivity of every mixed product, and $\omega_\varphi^n = e^{tF+c_t}\omega^n$ yield

$$\left\| d(-\varphi)^{(p+1)/2} \right\|_{L^2}^2 \le C(p+1)^2 \int_X \left(1 + (-\varphi)^p\right)\omega^n. \tag{30.1}$$

The Green formula for Δ_ω gives a uniform L^1 bound for every normalized ω-psh function: $\Delta_\omega\varphi \ge -n$ and $\sup\varphi = 0$ imply $\int_X (-\varphi)\omega^n \le C$. Apply the Sobolev inequality to (30.1) and iterate the exponents $p + 1 \mapsto \frac{n}{n-1}(p+1)$ (with the usual two-dimensional modification when $n = 1$). Starting from the L^1 bound gives $\|\varphi\|_{L^\infty} \le C$. Evaluating the equation only at a maximum controls neither the infimum nor the oscillation.

(2) **Second-order estimate.** Put $Q = \log \operatorname{tr}_\omega \omega_\varphi - A\varphi$. The Chern–Lu calculation and a lower bound for the bisectional curvature of ω yield

$$\Delta_{\omega_\varphi} \log \operatorname{tr}_\omega \omega_\varphi \geq -C_1 \operatorname{tr}_{\omega_\varphi} \omega - C_2.$$

Since $\Delta_{\omega_\varphi}\varphi = n - \operatorname{tr}_{\omega_\varphi} \omega$, choosing $A > C_1$ and applying the maximum principle bounds $\operatorname{tr}_{\omega_\varphi} \omega$ at the maximum of Q. The C^0 estimate transfers this to all of X. The arithmetic–geometric mean inequality together with $\det(g_\varphi)/\det(g) = \mathrm{e}^{tF+c_t}$ then bounds $\operatorname{tr}_\omega \omega_\varphi$. Hence the eigenvalues of g_φ relative to g are bounded above and below uniformly.

(3) **Higher-order estimates.** Uniform eigenvalue bounds make the complex Monge–Ampère equation uniformly elliptic and concave in the complex Hessian. The Evans–Krylov theorem gives a $C^{2,\alpha}$ estimate from the C^α norm of the right-hand side. Differentiate the equation; each derivative satisfies a linear uniformly elliptic equation with controlled coefficients. Schauder estimates then give uniform $C^{k,\alpha}$ bounds for every k. Calabi's third-order estimate is an alternative route to the first Hölder control.

The estimates are uniform in t. If t_j tends to a boundary point of the solution set, compactness in $C^{k,\alpha}$ gives a smooth limiting solution. Thus the solution set is closed and, since $[0, 1]$ is connected, it is all of $[0, 1]$. □

Warning

A determinant bound alone does not bound every eigenvalue: control of a product does not control the largest and smallest factors. The second-order trace estimate and the determinant equation must be used together.

Corollary 8.8: Prescribing Ricci curvature

With the convention fixed in Lecture 19,

$$\operatorname{Ric}(\omega_\varphi) = \operatorname{Ric}(\omega) - \mathrm{i}\partial\bar{\partial} \log \frac{\omega_\varphi^n}{\omega^n},$$

every real closed $(1, 1)$-form representing $2\pi c_1(X)$ is the Ricci form of a unique Kähler metric in each Kähler class. The corresponding potential is unique up to an additive constant. If $c_1(X) = 0$, each Kähler class contains a unique Ricci-flat metric.

Proof. Let η be a real closed $(1, 1)$-form representing $2\pi c_1(X)$. Then $\operatorname{Ric}(\omega) - \eta$ is cohomologically trivial. The $\partial\bar{\partial}$-lemma gives a real smooth function G with

$$\operatorname{Ric}(\omega) - \eta = \mathrm{i}\partial\bar{\partial} G.$$

Add a constant to G so that $\int_X e^G \omega^n = \int_X \omega^n$, and apply the Calabi–Yau theorem to obtain $\omega_\varphi^n = e^G \omega^n$. The transformation formula then gives

$$\operatorname{Ric}(\omega_\varphi) = \operatorname{Ric}(\omega) - \mathrm{i}\partial\bar{\partial} G = \eta.$$

If two metrics in the same Kähler class have Ricci form η, the logarithm of the ratio of their volume forms is pluriharmonic and hence constant. Equal total volumes make the constant zero, and uniqueness in the Calabi–Yau theorem makes the metrics equal. Taking $\eta = 0$ proves the final assertion when $c_1(X) = 0$. □

Remark (Geometric interpretation)

Locally the equation prescribes the determinant of a complex Hessian; globally it redistributes volume inside a fixed Kähler class. The continuity method splits existence into linear invertibility and nonlinear a priori estimates.

8.2.1 Complex dimension one as a linear model

On a compact Riemann surface, every $(1,1)$-form is a function times ω. The Monge–Ampère equation becomes

$$\omega + \mathrm{i}\partial\bar\partial\varphi = \mathrm{e}^F\omega, \qquad \mathrm{i}\partial\bar\partial\varphi = (\mathrm{e}^F - 1)\omega.$$

This is a Poisson equation. Its right-hand side has integral zero exactly when $\int_X \mathrm{e}^F\omega = \int_X \omega$, and Hodge theory then gives a solution unique modulo constants. The determinant nonlinearity begins only in complex dimension at least two.

At a point in higher dimension, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of ω_φ relative to ω. The equation controls

$$\lambda_1 \cdots \lambda_n = \mathrm{e}^F,$$

while the second-order estimates control $\sum \lambda_j$ and $\sum \lambda_j^{-1}$. A product bound alone allows one eigenvalue to approach infinity while another approaches zero. Trace and determinant together give uniform ellipticity.

8.2.2 The a priori estimate ladder

The closedness argument uses each estimate to unlock the next:

$$\begin{gathered} C^0 \text{ control of } \varphi \\ \Downarrow \\ \text{trace bounds for } \omega_\varphi \text{ relative to } \omega \\ \Downarrow \\ \text{uniform ellipticity and Evans–Krylov } C^{2,\alpha} \\ \Downarrow \\ \text{Schauder estimates in every order.} \end{gathered}$$

The arrows are logical dependencies. For example, the maximum-principle quantity $\log \operatorname{tr}_\omega \omega_\varphi - A\varphi$ cannot give a global trace bound until the oscillation of φ is known.

The Ricci transformation formula follows directly from determinants:

$$\begin{aligned} \operatorname{Ric}(\omega_\varphi) - \operatorname{Ric}(\omega) &= -\mathrm{i}\partial\bar\partial \log\det(g + \varphi_{j\bar k}) + \mathrm{i}\partial\bar\partial \log\det g \\ &= -\mathrm{i}\partial\bar\partial \log \frac{\omega_\varphi^n}{\omega^n}. \end{aligned}$$

Thus prescribing the volume ratio prescribes the change in Ricci curvature. The $\partial\bar\partial$-lemma supplies the desired logarithmic ratio from a cohomological Ricci target, and Calabi–Yau solves the resulting nonlinear equation.

Remark (Continuity method checklist)

Specify the Banach spaces and normalization, prove the linearization is invertible, derive estimates independent of the path parameter, take a convergent subsequence, and verify that positivity survives. "Open and closed" is the last line of the argument, not the proof itself.

8.2.3 The volume constraint from Stokes

For any smooth φ with $\omega_\varphi = \omega + \mathrm{i}\partial\bar\partial\varphi > 0$, expand

$$\omega_\varphi^n - \omega^n = \mathrm{i}\partial\bar\partial\varphi \wedge \sum_{j=0}^{n-1} \omega_\varphi^j \wedge \omega^{n-1-j}.$$

Every form in the sum is closed. Hence the right side is an exterior derivative, and compactness plus Stokes gives

$$\int_X \omega_\varphi^n = \int_X \omega^n.$$

Therefore the condition $\int_X e^F \omega^n = \int_X \omega^n$ is necessary. It is also the only integral obstruction in the Calabi–Yau theorem.

Linearization of the determinant

Let $G(t) = (g_{j\bar k} + t\psi_{j\bar k})$. Jacobi's formula gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \det G(t) = det G(t)\ \mathrm{tr}\big(G(t)^{-1}G'(t)\big).$$

At $t = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \log \frac{(\omega + \mathrm{i}\partial\bar\partial(\varphi + t\psi))^n}{\omega^n} = g_\varphi^{j\bar k}\psi_{j\bar k} = \Delta_{\omega_\varphi}\psi.$$

Constants form the kernel. Restricting to zero-mean functions removes that kernel, and the Fredholm alternative gives an inverse on zero-mean right-hand sides. This is the precise linear statement used for openness.

Complex dimension one is a Poisson equation

On a compact Riemann surface every real $(1,1)$-form is a function times ω. Write

$$\mathrm{i}\partial\bar\partial\varphi = (\Delta_\omega\varphi)\omega.$$

Then the Monge–Ampère equation is simply

$$\Delta_\omega\varphi = e^F - 1.$$

The right side has integral zero by the volume condition. Hodge theory solves this equation uniquely after fixing the mean of φ. Positivity follows from the equation itself: $\omega_\varphi = e^F\omega > 0$. There is no separate nonlinear second-order estimate; the determinant is a single eigenvalue.

Why trace and determinant are both needed

At a point diagonalize ω_φ relative to ω, with eigenvalues $\lambda_j > 0$. The equation bounds the product

$$\lambda_1 \cdots \lambda_n = e^F.$$

The second-order estimate bounds $\sum_j \lambda_j = \operatorname{tr}_\omega \omega_\varphi$. If the product is bounded below and the sum is bounded above, every λ_j is bounded above and

$$\lambda_j \geq \frac{\prod_k \lambda_k}{(\sum_k \lambda_k)^{n-1}}$$

gives a uniform lower bound. This yields uniform ellipticity. Without the trace estimate, the tuple $(R, R^{-1}, 1, \dots, 1)$ keeps determinant one while degenerating.

Uniqueness as an energy calculation

If $\omega_\varphi^n = \omega_\psi^n$ and $u = \varphi - \psi$, then

$$0 = \int_X u(\omega_\varphi^n - \omega_\psi^n) = -\sum_{j=0}^{n-1} \int_X \mathrm{i}\partial u \wedge \bar{\partial} u \wedge \omega_\varphi^j \wedge \omega_\psi^{n-1-j}.$$

Every measure on the right is nonnegative, so every integral vanishes. In particular $\partial u = 0$ and u is constant. The chosen normalization removes that constant. This proof uses only positivity, Stokes, and the algebraic difference-of-powers identity.

8.2.4 Self-study workshop: organizing the continuity method around a priori estimates

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall Kähler potentials, the determinant of a Hermitian matrix, elliptic linearization, the implicit function theorem in Banach spaces, and compactness from uniform Hölder estimates.

The continuity method as a dependency chain. For

$$(\omega + \mathrm{dd^c}\varphi)^n = e^F \omega^n, \qquad \omega + \mathrm{dd^c}\varphi > 0,$$

integration and Stokes force the volume constraint $\int_X e^F \omega^n = \int_X \omega^n$. Normalize $\sup_X \varphi = 0$ to remove the additive constant.

Embed the equation in a path

$$(\omega + \mathrm{dd^c}\varphi_t)^n = e^{tF + c_t}\omega^n, \qquad t \in [0, 1],$$

where c_t enforces equal total volume. The solution set is nonempty at $t = 0$. Differentiating the logarithm of the determinant gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \log \frac{(\omega_\varphi + s\,\mathrm{dd^c}\psi)^n}{\omega^n} = \Delta_{\omega_\varphi}\psi.$$

On mean-zero functions this Laplacian is invertible, so the implicit function theorem gives openness.

Closedness is the hard part. The estimates occur in a strict order:

$$C^0 \Longrightarrow \Delta\varphi \text{ bound} \Longrightarrow C^{2,\alpha} \Longrightarrow C^{k,\alpha}.$$

The second-order estimate uses both a trace bound and the determinant equation; a determinant alone does not control individual eigenvalues. Compactness then extracts a limit that remains positive and solves the equation.

Uniqueness follows from an energy argument. If φ_0, φ_1 solve the same equation, integrate their difference against the difference of the Monge–Ampère measures to obtain a sum of nonnegative gradient terms; all vanish, so the potentials differ by a constant.

Checkpoint (open is not closed)

(1) Why is the linearized operator considered on mean-zero functions?
(2) Why does a determinant bound not control the largest eigenvalue?
(3) Which estimate prevents the limiting Kähler form from degenerating?

Solution (checkpoint). Constants lie in the kernel of the Laplacian, and normalization removes them. A product of positive eigenvalues can stay fixed while one tends to infinity and another to zero. The second-order estimate, combined with the determinant equation, gives two-sided eigenvalue control and preserves positivity in the limit. ⋄

Exit criterion

You can derive the volume condition and linearization, distinguish openness from closedness, state the estimate ladder in the correct order, and prove uniqueness up to constants.

Summary. The logical skeleton is: continuity path, openness by the Laplacian, uniform C^0 and trace estimates, and fully nonlinear regularity. This is the summit of smooth complex Monge–Ampère theory.

Exercise 8.9: Linearization

Compute $\frac{d}{dt}|_0 \log\det(g_{j\bar{k}} + t\psi_{j\bar{k}})$ carefully.

Exercise 8.10: Complex dimension one

Reduce the equation on a compact Riemann surface to a Poisson equation and explain why the genuinely higher-dimensional difficulties disappear.

Optional handout: Moment maps, Mabuchi energy, and K-stability

This handout is optional. No later argument in the core lectures depends on it.

The Calabi–Yau theorem solves a prescribed volume-form equation in a fixed Kähler class. A harder problem asks for a canonical metric, such as a constant-scalar-curvature Kähler metric. The scalar curvature appears as an infinite-dimensional moment map for the action of Hamiltonian symplectomorphisms on compatible complex structures.

On the space of Kähler potentials $\mathcal{H} = \{\varphi : \omega_\varphi = \omega + \mathrm{i}\partial\bar{\partial}\varphi > 0\}$, the Mabuchi functional is characterized by

$$\frac{\mathrm{d}}{\mathrm{d}t} M(\varphi_t) = -\int_X \dot{\varphi}_t (S(\omega_{\varphi_t}) - \overline{S}) \frac{\omega_{\varphi_t}^n}{n!}.$$

Its critical points are constant-scalar-curvature metrics. Along suitable geodesics in $\mathcal{H}$, the functional is convex.

Algebraic one-parameter degenerations of (X, L) are test configurations. Their Donaldson–Futaki invariants are slopes at infinity of the corresponding energy. The Yau–Tian–Donaldson principle predicts that existence of a canonical metric is equivalent to an algebraic stability condition. This is a theorem for Fano Kähler–Einstein metrics and in several other important settings.

Remark (Relation to Monge–Ampère)

Geodesics in $\mathcal{H}$ satisfy a homogeneous complex Monge–Ampère equation on $X\times$ an annulus. They are generally only weak, so finite-energy pluripotential theory becomes part of the metric geometry of $\mathcal{H}$.

Exercise 8.11: First variation of volume

For a smooth path φ_t, prove $\frac{\mathrm{d}}{\mathrm{d}t} \int_X \omega_{\varphi_t}^n = 0$ by differentiating and applying Stokes' theorem.

8.3 Lecture 31: Nef, big, and psef cones; regularization; non-pluripolar products

Week 16 material • 90 minutes

Notation for this lecture

X is a compact Kähler manifold, ω a reference Kähler form, and $\alpha \in H^{1,1}(X,\mathbb{R})$ a real cohomology class represented smoothly by θ. The set $\mathrm{PSH}(X,\theta) = \{u : \theta + dd^c u \geq 0\}$ consists of θ-psh potentials, and $V_\theta = \sup\{u \in \mathrm{PSH}(X,\theta) : u \leq 0\}$ has minimal singularities. Psef means pseudoeffective. A function has analytic singularities if locally it equals $c \log \sum_j |f_j|^2 + O(1)$. $\nu(u,x)$ is its Lelong number, 1_A denotes restriction by the indicator of A, angle brackets denote the non-pluripolar product, and $\mathrm{Vol}(\alpha)$ is the non-pluripolar volume defined below. On a one-point blow-up, H and E are the line and exceptional classes.

Prerequisite results used below

Bergman extremal principle. If $\mathcal{H}$ is a Hilbert space of holomorphic functions with bounded point evaluation at x and (f_j) is an orthonormal basis, then $K(x,x) = \sum_j |f_j(x)|^2$ and $K(x,x) = \sup_{\|f\|\leq 1} |f(x)|^2$.

Bedford–Taylor monotone continuity. If $u_{j,\nu}$ are locally bounded psh functions decreasing to u_j, then

$$dd^c u_{1,\nu} \wedge \cdots \wedge dd^c u_{p,\nu} \rightharpoonup dd^c u_1 \wedge \cdots \wedge dd^c u_p.$$

This theorem applies on the bounded truncation loci used in the non-pluripolar construction; it does not define an unrestricted product of arbitrary singular currents.

Plan for the lecture. 0–20 min: four positivity cones; 20–42: potentials with minimal singularities; 42–62: Demailly regularization; 62–78: non-pluripolar products; 78–90: volume and the blow-up example.

Let X be compact Kähler and $\alpha \in H^{1,1}(X,\mathbb{R})$.

Definition 8.12: Kähler, nef, pseudoeffective, and big classes

(1) α is Kähler if it contains a smooth positive definite representative.
(2) α is nef if, for every $\varepsilon > 0$, it has a smooth representative $\theta_\varepsilon \geq -\varepsilon\omega$.
(3) α is pseudoeffective (psef) if it contains a closed positive $(1,1)$-current.
(4) α is big if it contains a Kähler current $T \geq \varepsilon\omega$.

On a compact Kähler manifold, the Kähler cone is the interior of the nef cone and the big cone is the interior of the psef cone. Kähler implies nef and big implies psef, but nef and psef are not interchangeable.

For a psef class and a smooth representative $\theta \in \alpha$, define

$$V_\theta = \sup\{u \in \mathrm{PSH}(X,\theta) : u \leq 0\}.$$

It has minimal singularities: every $u \in \mathrm{PSH}(X,\theta)$ satisfies $u \leq V_\theta + C$.

Theorem 8.13: Demailly regularization, working version

Let $T = \theta + dd^c u$ be a closed almost-positive $(1,1)$-current with $T \geq \gamma$. There are u_m with analytic singularities such that

$$T_m = \theta + dd^c u_m \geq \gamma - \varepsilon_m \omega, \qquad \varepsilon_m \downarrow 0,$$

and $u_m \to u$ in L^1, $T_m \rightharpoonup T$, with controlled approximation of Lelong numbers. Monotone or equisingular refinements require additional construction.

Proof (proof architecture with the quantitative local estimates). We describe the two parts of the construction and indicate precisely where the global curvature bound enters. On a coordinate ball $B \Subset X$, after adding a smooth local potential of $\theta - \gamma$, reduce to a psh function v. Let

$$\mathcal{H}_{m,B} = \left\{ f \in \mathcal{O}(B) : \int_B |f|^2 e^{-2mv}\, d\lambda < \infty \right\}$$

and choose an orthonormal basis $(f_{m,j})$. The Bergman approximation

$$v_m = \frac{1}{2m} \log \sum_j |f_{m,j}|^2$$

has analytic singularities. The submean inequality gives, on a smaller ball,

$$v_m(z) \leq \sup_{|\zeta - z| < r} v(\zeta) + \frac{C + 2n|\log r|}{2m}. \tag{31.1}$$

Conversely, apply Hörmander's estimate with the weight $2mv + 2n \log|\zeta - z|$ to construct a holomorphic function of weighted norm at most $Ce^{-2mv(z)}$ and value 1 at z. The extremal characterization of the Bergman kernel then gives

$$v(z) - \frac{C}{m} \leq v_m(z). \tag{31.2}$$

Equations (31.1)–(31.2) imply $v_m \to v$ in L^1_{loc}. Applying the same bounds after restriction to complex lines and using the slope description of Lelong numbers yields

$$\nu(v, z) - \frac{n}{m} \leq \nu(v_m, z) \leq \nu(v, z), \tag{31.3}$$

up to the inessential convention-dependent replacement of m by $2m$.

The passage from these local ideals to a global current is subtler than an ordinary partition of unity or a naive regularized maximum: neither operation preserves both plurisubharmonicity and analytic singularities. Choose finitely many nested coordinate balls $U_j' \Subset U_j'' \Subset U_j$ of radius comparable to δ. After adding the appropriate local quadratic potential q_j for γ, put $w_{j,m} = 2m(v_{j,m} + q_j)$. Solving a point-interpolation $\bar{\partial}$ problem on overlaps gives the discrepancy estimate

$$|w_{j,m} - w_{k,m}| \leq C_{jk}\big(\log \delta^{-1} + m\varepsilon(\delta)\delta^2\big) \quad \text{on } U_j'' \cap U_k'', \tag{31.4}$$

where $\varepsilon(\delta) \to 0$ is a modulus of continuity for the lower-bound form.

Choose cutoffs θ_j equal to one on U_j' and supported in U_j'', and add small smooth quadratic corrections to $w_{j,m}$ so that the jth corrected weight is dominated by a neighboring one on $\operatorname{supp} d\theta_j$. The global gluing is the log-sum-exp expression

$$u_m = \frac{1}{2m} \log \sum_j \theta_j^2 e^{\widetilde{w}_{j,m}}. \tag{31.5}$$

A direct Hessian calculation writes the new terms as a nonnegative variance term plus cutoff errors; (31.4) bounds the latter by $C(\varepsilon(\delta) + m^{-1}\delta^{-2})\omega$. Since (31.5) is locally the logarithm of a finite sum of squared holomorphic generators multiplied by smooth positive factors, it retains analytic singularities. Choose $\delta = \delta_m \downarrow 0$ slowly enough that both error terms tend to zero. The resulting global functions satisfy

$$\theta + dd^c u_m \geq \gamma - \varepsilon_m \omega.$$

The local L^1 estimates give global L^1 convergence; continuity of dd^c on distributions gives $T_m \rightharpoonup T$, and (31.3) gives the asserted control of Lelong numbers. The discrepancy estimate and Hessian calculation in (31.4)–(31.5) constitute the technical global regularization lemma. Monotone or equisingular refinements require a further diagonal construction, which is why those properties are not claimed here. □

Definition 8.14: Non-pluripolar product

For possibly unbounded $u_j \in \mathrm{PSH}(X, \theta_j)$, let $V_j = V_{\theta_j}$ and put $u_j^{(k)} = \max(u_j, V_j - k)$. Define

$$1_{\cap\{u_j > V_j - k\}} \bigwedge_{j=1}^{p} (\theta_j + dd^c u_j^{(k)}).$$

These currents are compatible as k increases. Their limit, when locally of finite mass, is $\langle(\theta_1 + dd^c u_1) \wedge \cdots \wedge (\theta_p + dd^c u_p)\rangle$. It assigns no mass to pluripolar sets.

The reference potentials are essential: a constant need not be θ_j-psh, so $\max(u_j, -k)$ is not a valid global truncation for arbitrary θ_j. Equivalently, one may define the product locally by truncating ordinary psh potentials of the currents and then verify independence and gluing on the common finite locus.

Warning

The non-pluripolar product is not the classical product of currents. For example, $dd^c \log|f|^2 = [D]$ contains divisor mass, while the non-pluripolar bracket discards mass on the pole set. The angle brackets change the definition; they are not typographical decoration.

Definition 8.15: Volume of a pseudoeffective class

$$\mathrm{Vol}(\alpha) = \int_X \langle(\theta + dd^c V_\theta)^n\rangle.$$

It is independent of θ. The class is big exactly when $\mathrm{Vol}(\alpha) > 0$, and if α is nef then $\mathrm{Vol}(\alpha) = \int_X \alpha^n$.

Example 8.16: Cones on the blown-up plane

On $\widetilde{\mathbb{P}^2}$, write a class as $aH - bE$. It is Kähler exactly when $a > b > 0$, and the nef closure is $a \geq b \geq 0$. Bigness depends on both self-intersection and pseudoeffectivity; the inequality $(aH - bE)^2 = a^2 - b^2 > 0$ alone is not enough without checking effective curves.

8.3.1 The cones on the one-point blow-up

Let $X = \mathrm{Bl}_p \mathbb{P}^2$. The effective cone is generated by E and $H - E$: every plane curve of degree d and multiplicity m at p satisfies $d \geq m \geq 0$, while E supplies the exceptional ray. In the coordinates $\alpha = aH - bE$ this gives

$$\alpha \text{ is psef} \iff a \geq 0, \quad a - b \geq 0.$$

Its interior is the big cone: $a > 0$ and $a > b$.

For nefness, test against the two curve classes E and $H - E$:

$$\alpha \cdot E = b, \qquad \alpha \cdot (H - E) = a - b.$$

Hence α is nef exactly when $a \geq b \geq 0$, and it is Kähler exactly when $a > b > 0$. This exhibits two different boundaries. The class E is psef but not nef because $E^2 = -1$; the class H is nef and big but not Kähler because $H \cdot E = 0$.

The class $H+E$ is big although $(H+E)^2 = 0$. Its divisorial Zariski decomposition is $H+E = H+E$, with nef positive part H and negative part E; its volume is $H^2 = 1$. This example shows why self-intersection alone does not test bigness for a non-nef class.

8.3.2 What the non-pluripolar product discards

Locally let $T = [D] = dd^c \log |f|^2$. Outside D, the current vanishes. The non-pluripolar construction truncates the potential, computes on the finite locus, and then extends by zero across the pole set. Consequently

$$\langle [D] \rangle = 0$$

in the non-pluripolar sense, even though the ordinary current $[D]$ is nonzero. For $T = \theta + [D]$ with a smooth semipositive part, the non-pluripolar product retains the products built on the smooth part away from D but assigns no mass to D itself.

This loss is deliberate. Classical self-intersections of a divisor depend on global intersection theory and may be negative, whereas a non-pluripolar Monge–Ampère measure is designed to remain positive and stable under truncation. The angle brackets therefore encode a different operation, not an approximation to every classical intersection product.

Potentials with minimal singularities are the least singular representatives of a psef class. In a nef class they can be taken with arbitrarily small negative smooth error; in a big non-nef class their singularities record the fixed negative locus. The volume measures the movable positive part rather than divisorial mass forced into every representative.

Warning

The identity $\mathrm{Vol}(\alpha) = \int_X \alpha^n$ is valid for nef classes. For a general big class, the ordinary top intersection can be smaller, zero, or otherwise fail to represent the positive movable mass; use the non-pluripolar product of a minimally singular current.

8.3.3 The four cones compared by explicit representatives

If α is Kähler, its positive smooth representative also proves that it is nef, big, and psef. If α is big, the Kähler current is in particular a positive current, so it is psef. The reverse implications fail on a blow-up:

$$[E] \text{ is psef but not nef because } E^2 = -1,$$

while H is nef and big but not Kähler because $H \cdot E = 0$.

Write a class on $X = \mathrm{Bl}_p \mathbb{P}^2$ as $aH - bE$. The effective cone is generated by E and $H - E$. Indeed, a nonexceptional plane curve of degree d and multiplicity m at p has $d \geq m \geq 0$ and class

$$dH - mE = d(H - E) + (d - m)E,$$

which lies in the same cone. Solving

$$aH - bE = xE + y(H - E)$$

gives $y = a$ and $x = a - b$. Therefore

$$aH - bE \text{ is psef} \quad \Longleftrightarrow \quad a \geq 0, a - b \geq 0.$$

Nefness is dual to nonnegative intersection with the two generating curve rays:

$$(aH - bE) \cdot E = b, \qquad (aH - bE) \cdot (H - E) = a - b.$$

Hence the nef cone is $a \geq b \geq 0$, and its interior, the Kähler cone, is $a > b > 0$.

The truncation calculation for a divisor

Locally let $u = \log |z_1|^2$, so $\mathrm{dd^c}u = [z_1 = 0]$. Define

$$u_k = \max\{u, -k\}.$$

The Bedford–Taylor measure $\mathrm{dd^c}u_k$ is supported on the contact hypersurface $|z_1| = e^{-k/2}$. The non-pluripolar construction restricts it to $\{u > -k\} = \{|z_1| > e^{-k/2}\}$, where $u_k = u$ and $\mathrm{dd^c}u = 0$. Thus

$$1_{\{u>-k\}}\mathrm{dd^c}u_k = 0$$

for every k, and the limit is

$$\langle [z_1 = 0] \rangle = 0.$$

The ordinary current retains its divisor mass; the non-pluripolar bracket deliberately removes mass carried by the pole set.

For several potentials, if $k < \ell$, then on the common set $\{u_j > V_j - k\}$ both truncations equal the original u_j. Bedford–Taylor locality therefore makes the level-k and level-ℓ products agree there. This compatibility is what allows the increasing union of finite loci to define one current.

Volume and Zariski decomposition on the blow-up

For a nef class $aH - bE$,

$$\mathrm{Vol}(aH - bE) = (aH - bE)^2 = a^2 - b^2.$$

For the big non-nef class $H + E$, the divisorial Zariski decomposition is

$$H + E = P + N, \qquad P = H, \quad N = E.$$

Here P is nef, N has negative-definite intersection matrix, and $P \cdot N = 0$. The volume sees only the movable positive part:

$$\mathrm{Vol}(H + E) = P^2 = 1,$$

although $(H + E)^2 = 0$. This example proves concretely that top self-intersection is not the volume formula outside the nef cone.

Minimal singularities in the same class

If $u, v \in \mathrm{PSH}(X, \theta)$ and $u \leq v + C$, then u is at least as singular as v. The envelope

$$V_\theta = \sup\{w \in \mathrm{PSH}(X, \theta) : w \leq 0\}^*$$

dominates every normalized competitor and hence has minimal singularities. If θ is Kähler, the zero function is a competitor and $V_\theta = 0$. For a big class with a fixed divisorial component, every positive current has a logarithmic pole along that component, and the envelope records the smallest unavoidable coefficient.

8.3.4 Self-study workshop: comparing positivity cones on one blow-up

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Recall positive currents, Kähler classes, intersection numbers on surfaces, divisorial Zariski decomposition, and truncation of unbounded psh potentials.

A cone picture with coordinates. Let $X = \mathrm{Bl}_p \mathbb{P}^2$, with H the pullback of a line and E the exceptional curve. The intersection table is

$$H^2 = 1, \qquad H \cdot E = 0, \qquad E^2 = -1.$$

For $\alpha = aH - bE$,

$$\alpha \cdot E = b, \qquad \alpha \cdot (H - E) = a - b.$$

Since the Mori cone is generated by E and $H - E$, the class is nef exactly when

$$b \geq 0, \qquad a - b \geq 0.$$

It is Kähler when both inequalities are strict. The effective cone is generated by E and $H - E$; writing

$$aH - bE = a(H - E) + (a - b)E$$

shows that α is psef exactly when $a \geq 0$ and $a - b \geq 0$. Its interior is the big cone. This example makes the inclusions

$$\text{Kähler} \subset \text{nef} \subset \text{psef}$$

visible and shows that nefness and bigness ask different questions.

For an unbounded psh potential u, the non-pluripolar product is built from $u_k = \max\{u, -k\}$ on the sets $\{u > -k\}$ and extended by zero across the discarded singular set before taking a limit. If $u = \log |s_D|^2 + O(1)$ has a divisorial pole, the construction does not assign the self-product an artificial mass on D. Hence the non-pluripolar product is not the same as every formal current wedge.

Checkpoint (plot the inequalities)

(1) Is H nef? Is it Kähler on the blow-up?
(2) Is E psef? Is it nef?
(3) Why can the non-pluripolar product differ from an ordinary intersection number?

Solution (checkpoint). H is nef but not Kähler because $H \cdot E = 0$. The curve class E is effective, hence psef, but $E^2 = -1$, so it is not nef. Non-pluripolar products deliberately discard mass concentrated on pluripolar sets, while cohomological intersections still record divisorial self-intersection. ◇

Exit criterion

You can derive the nef and psef inequalities from generators, locate boundary classes, and explain the truncation principle and the loss of divisorial mass in non-pluripolar products.

Summary. Kähler and nef describe smooth positivity, while big and psef describe positivity by currents. Demailly regularization replaces arbitrary positive currents by analytic-singularity models, and non-pluripolar products safely ignore unstable mass on pole sets.

Exercise 8.17: Nef test on a surface

Use the curve classes H, E, and $H - E$ to test the nef and Kähler conditions for $aH - bE$.

Exercise 8.18: Compatibility of truncations

If $k < \ell$, prove that the two Bedford–Taylor products agree on $\cap_j \{u_j > V_j - k\}$.

Optional handout: Divisorial Zariski decomposition

This handout is optional. No later argument in the core lectures depends on it.

On a projective surface, a pseudoeffective divisor class α admits a Zariski decomposition

$$\alpha = P + N,$$

where P is nef, $N = \sum a_i C_i$ is effective with negative-definite intersection matrix, and $P \cdot C_i = 0$. The positive part carries the movable positivity; the negative part records fixed divisorial components.

In higher dimension, Boucksom's divisorial decomposition uses generic Lelong numbers:

$$N(\alpha) = \sum_D \nu(\alpha, D)[D], \qquad Z(\alpha) = \alpha - \{N(\alpha)\}.$$

The sum runs over prime divisors and is locally finite in the appropriate sense. The class $Z(\alpha)$ is modified nef: it has no unavoidable divisorial singularities, though it need not be nef.

Example 8.19: The exceptional class

On $X = \mathrm{Bl}_p \mathbb{P}^2$, the class E is pseudoeffective and its entire Zariski decomposition is negative: $P = 0$, $N = E$. For a big class $aH - bE$ with $a > b > 0$, the class is already nef, so $N = 0$.

The volume ignores the negative divisorial part on a surface: $\mathrm{Vol}(\alpha) = P^2$. More generally, volume measures the asymptotic movable sections of a big line bundle and equals the total non-pluripolar mass of a current with minimal singularities.

Exercise 8.20: A two-curve decomposition

Let C_1, C_2 have negative-definite intersection matrix. Given a class α, solve the linear equations $(\alpha - a_1 C_1 - a_2 C_2) \cdot C_j = 0$ and determine when $a_1, a_2 \geq 0$.

8.4 Lecture 32: A capstone example, a map of the subject, and next directions

Week 16 material • 90 minutes

Notation for this lecture

$X = \mathrm{Bl}_p\,\mathbb{P}^2$ is the blow-up of the projective plane at p, $\pi : X \to \mathbb{P}^2$ its blow-down map, $H = \pi^*[\text{line}]$, and E the exceptional curve. The dot is the intersection product and K_X the canonical class. The Hodge number is $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$. For integers a, b, $L = aH - bE$ also denotes the line bundle $\pi^*\mathcal{O}_{\mathbb{P}^2}(a) \otimes \mathcal{O}_X(-bE)$ when no confusion can arise. $h^q(X, L) = \dim_{\mathbb{C}} H^q(X, L)$ and $\chi(L) = \sum_q (-1)^q h^q(X, L)$ is its holomorphic Euler characteristic. s_E is the canonical section of $\mathcal{O}_X(E)$, $[E]$ the integration current, and $c_1(\mathcal{O}(E), h)$ the first Chern form of a chosen metric h.

Prerequisite results used below

Surface Riemann–Roch. For a line bundle L on a smooth compact complex surface,

$$\chi(X, L) = \chi(X, \mathcal{O}_X) + \frac{1}{2} L \cdot (L - K_X).$$

Kodaira vanishing. If A is a positive line bundle on a compact Kähler n-fold, then $H^q(X, K_X \otimes A) = 0$ for every $q > 0$.
Adjunction for a smooth curve. For a smooth curve C in a smooth surface, $K_C \simeq (K_X \otimes \mathcal{O}_X(C))|_C$ and $2g(C) - 2 = C \cdot (C + K_X)$, where $g(C)$ is the genus. These three statements are the external inputs used in the capstone intersection and dimension calculations.

Plan for the lecture. 0–15 min: review of the toolchain; 15–48: the calculation on $\mathrm{Bl}_p\,\mathbb{P}^2$; 48–68: curvature and vanishing; 68–80: psh functions and positive currents; 80–90: research directions and closure.

Let $X = \mathrm{Bl}_p\,\mathbb{P}^2$, with $\pi : X \to \mathbb{P}^2$, $H = \pi^*[\text{line}]$, and exceptional curve E.

8.4.1 Topology and intersection theory

$$H^2 = 1, \qquad H \cdot E = 0, \qquad E^2 = -1, \qquad K_X = \pi^* K_{\mathbb{P}^2} + E = -3H + E.$$

Blow-up adds one $(1,1)$ class, so

$$h^{0,0} = h^{2,2} = 1, \qquad h^{1,1} = 2, \qquad h^{1,0} = h^{0,1} = h^{2,0} = h^{0,2} = 0.$$

The Euler characteristic rises from 3 to 4, matching "remove a point, add a $\mathbb{P}^1$."

8.4.2 Kähler classes, line bundles, and projective embeddings

$\alpha = aH - bE$ is Kähler exactly when $a > b > 0$. For integers $a > b > 0$, $L = \pi^*\mathcal{O}_{\mathbb{P}^2}(a) \otimes \mathcal{O}_X(-bE)$ is positive. Its sections are homogeneous polynomials of degree a vanishing to order at least b at p. High powers embed the blow-up by Kodaira.

Example 8.21: A Riemann–Roch and vanishing calculation

If $L = aH - bE$ is sufficiently positive, Kodaira vanishing gives $H^q(X, L) = 0$ for $q > 0$ after writing $L = K_X \otimes A$ with A positive. Surface Riemann–Roch says

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}L \cdot (L - K_X),$$

so

$$h^0(L) = 1 + \frac{1}{2}\big(a^2 - b^2 + 3a - b\big).$$

This is the number of degree-a plane polynomials minus the $b(b+1)/2$ conditions imposed by vanishing to order b at one point.

8.4.3 Positive currents and singular potentials

$[E]$ is a closed positive current in class E, but $E^2 = -1$, so a psef class need not be nef. For the canonical section s_E of $\mathcal{O}(E)$,

$$\mathrm{dd^c} \log |s_E|_h^2 = [E] - c_1(\mathcal{O}(E), h).$$

Thus the exceptional curve is simultaneously a divisor, the analytic part of a positive current, and the logarithmic pole set of a qpsh potential. Siu decomposition is completely explicit here.

Summary.

Input	Typical output
psh / strict pseudoconvexity	Stein geometry, $\bar{\partial}$ solutions, holomorphic functions, extension
closed positive currents / Lelong numbers	analytic threshold sets, divisorial components, singular metrics
Chern curvature	characteristic classes, bundle positivity, Bochner coercivity
Kähler identities	Hodge decomposition, $\partial\bar{\partial}$-lemma, Hard Lefschetz
L^2 estimates	vanishing, peak sections, Kodaira embedding, OT extension
Monge–Ampère	prescribed volume and Ricci-flat metrics in Kähler classes

8.4.4 Four routes for further study

(1) **Kähler–Einstein metrics and stability:** from Calabi–Yau to the Fano case and K-stability;
(2) **complex algebraic geometry:** from ample, nef, and big classes to the minimal model program and nonvanishing;
(3) **pluripotential theory:** from Bedford–Taylor to finite-energy classes, geodesics, and Mabuchi geometry;
(4) **Hodge theory:** from pure structures to variations, degenerations, and mixed Hodge structures.

Warning (final audit)

For any formula, ask: What is the bidegree? What are the $\mathrm{d^c}$ and curvature normalizations? Which positivity cone or tensor test is intended? Is integration by parts justified by boundary conditions or completeness? How is a singular product defined? Why do local solutions glue globally? These six questions catch most hidden errors.

8.4.5 A complete strict-transform calculation

Let $C \subset \mathbb{P}^2$ be a degree-d curve with multiplicity m at the blown-up point, and let $\widetilde{C} = dH - mE$. Then

$$\widetilde{C} \cdot E = m, \qquad \widetilde{C}^2 = d^2 - m^2.$$

The first equality counts tangent directions with multiplicity on the exceptional curve. Since $K_X = -3H + E$, adjunction gives

$$\begin{aligned} 2p_a(\widetilde{C}) - 2 &= \widetilde{C} \cdot (\widetilde{C} + K_X) \\ &= d(d-3) + m(1-m), \end{aligned}$$

and hence

$$p_a(\widetilde{C}) = \frac{(d-1)(d-2)}{2} - \frac{m(m-1)}{2}.$$

Blowing up subtracts the genus contribution of an ordinary multiplicity-m point. If further infinitely near singularities remain, their multiplicities are subtracted on subsequent blow-ups.

For $L = aH - bE$, surface Riemann–Roch gives

$$\chi(L) = 1 + \frac{1}{2}(a^2 - b^2 + 3a - b).$$

The same number arises from plane polynomials:

$$\binom{a+2}{2} - \binom{b+1}{2} = 1 + \frac{1}{2}(a^2 - b^2 + 3a - b).$$

The equality aligns intersection theory with the number of jets forced to vanish at the center. When higher cohomology vanishes, it is the actual dimension of the linear system.

8.4.6 A problem-solving map for the whole course

When faced with a new complex-geometric problem, the following sequence usually locates the correct tool.

(1) Identify the local object: a holomorphic function, psh potential, bundle metric, analytic ideal, or differential form.
(2) Record its invariant: divisor, current, curvature tensor, cohomology class, or Lelong number.
(3) Determine the available positivity: Levi, Kähler, Griffiths, Nakano, nef, big, or positivity of a degree-specific commutator.
(4) Choose the globalization mechanism: exact sequence, Cartan theorem, Hodge representative, L^2 solution, compactness of currents, or continuity method.
(5) Audit the limit: topology of convergence, uniform estimate, singular set, normalization, and definition of every nonlinear product.

The exceptional divisor runs through this entire list. Locally it is the vanishing of a coordinate; as a Cartier divisor it defines $\mathcal{O}(E)$; its canonical section produces $[E]$ by Poincaré–Lelong; its normal curvature is negative; its class spans a new $H^{1,1}$ direction; and its negative self-intersection separates psef from nef. One object is being viewed through several compatible languages.

Remark (A final convention ledger)

This course uses $\mathrm{d^c} = (4\pi \mathrm{i})^{-1}(\partial - \bar{\partial})$, $\mathrm{dd^c} = (\mathrm{i}/2\pi)\partial\bar{\partial}$, and $|e|_h^2 = \mathrm{e}^{-\varphi}$ for a line-bundle weight. Before importing a formula from another reference, test it on $\log|z|^2$, $\mathcal{O}(1)$ on $\mathbb{P}^1$, and the Gaussian weight. These three models detect nearly every sign or 2π discrepancy.

8.4.7 An explicit projective embedding of the blow-up

Take $p = [1:0:0] \in \mathbb{P}^2$ and

$$L = 2H - E.$$

Sections of L are conics through p. A basis is

$$Z_0Z_1, \quad Z_0Z_2, \quad Z_1^2, \quad Z_1Z_2, \quad Z_2^2,$$

so $h^0(L) = 5$. The induced rational map on $\mathbb{P}^2$ is undefined only at p and becomes a morphism on the blow-up.

On the affine chart $Z_0 = 1$, put $x = Z_1$, $y = Z_2$. In the blow-up chart $y = xv$, pull back the five sections:

$$(x, xv, x^2, x^2v, x^2v^2).$$

Because L subtracts one copy of E, divide by the common exceptional factor x. The lifted map is

$$(x, v) \longmapsto [1 : v : x : xv : xv^2].$$

On $E = \{x = 0\}$ this becomes $[1 : v : 0 : 0 : 0]$, so distinct tangent directions remain distinct. In the affine target chart where the first coordinate is nonzero, the first two coordinates v, x already have Jacobian

$$\frac{\partial(v, x)}{\partial(x, v)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which has full rank. The second blow-up chart treats the missing vertical direction. Away from E, the map agrees with the conic system and separates points. Hence $2H - E$ embeds the blow-up as the cubic scroll in $\mathbb{P}^4$.

Riemann–Roch checked against monomial counting

For $L = aH - bE$ with $a \ge b \ge 0$, a degree-a polynomial vanishes to order at least b at $p = [1:0:0]$ exactly when every monomial $Z_0^{a-i-j}Z_1^iZ_2^j$ has $i + j \ge b$. The excluded monomials number

$$1 + 2 + \cdots + b = \frac{b(b+1)}{2}.$$

Therefore

$$h^0(L) = \binom{a+2}{2} - \binom{b+1}{2} = 1 + \frac{1}{2}(a^2 - b^2 + 3a - b)$$

whenever the imposed conditions are independent. For $a = 2, b = 1$, this gives five, matching the displayed basis. Surface Riemann–Roch gives the same number once higher cohomology vanishes.

Chern numbers and the blow-up formula

Since $K_X = -3H + E$,

$$c_1(X)^2 = (-K_X)^2 = (3H - E)^2 = 9 - 1 = 8.$$

The topological Euler characteristic is

$$c_2(X) = \chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(\mathbb{P}^2) - 1 + \chi_{\mathrm{top}}(\mathbb{P}^1) = 3 - 1 + 2 = 4.$$

Noether's formula then checks

$$\chi(\mathcal{O}_X) = \frac{c_1^2 + c_2}{12} = \frac{8+4}{12} = 1,$$

consistent with $h^{0,0} = 1$ and $h^{0,1} = h^{0,2} = 0$.

One curve viewed in every language

Let $C \subset \mathbb{P}^2$ be a degree-d curve with multiplicity m at p. In local coordinates its lowest homogeneous part of degree m determines the intersection points of the strict transform with E. Globally,

$$[\widetilde{C}] = dH - mE, \qquad \widetilde{C} \cdot E = m, \qquad \widetilde{C}^2 = d^2 - m^2.$$

Its canonical section gives

$$\mathrm{dd^c} \log |s_{\widetilde{C}}|_h^2 = [\widetilde{C}] - c_1(\mathcal{O}(\widetilde{C}), h).$$

If $d > m > 0$, the class has positive intersection with both E and $H - E$ and lies in the Kähler cone. Kodaira then converts a positive metric into projective sections; Poincaré–Lelong converts any such section back into the divisor current. This closes the course's cycle from coordinates to currents, curvature, cohomology, and embeddings.

8.4.8 Self-study workshop: solving one problem in all four languages

This workshop is part of the core notes. Work through it with paper and pencil before attempting the end-of-lecture exercises.

Prerequisite ledger. Bring together strict transforms, line bundles and sections, intersection theory, positive currents, Riemann–Roch, and projective embeddings. This workshop is a diagnostic for the whole course.

The class $dH - mE$ from four viewpoints. Let $\pi : X = \mathrm{Bl}_p \mathbb{P}^2 \to \mathbb{P}^2$ and let $L = dH - mE$ with $d \geq m \geq 0$.

Local analytic language. A section is represented by a homogeneous polynomial of degree d vanishing to order at least m at p. In blow-up coordinates $y = xv$, the pullback has a common factor x^m; division by this factor gives the local strict-transform section.

Sheaf language. The space of sections is

$$H^0(X, L) \simeq H^0(\mathbb{P}^2, \mathcal{I}_p^m(d)).$$

The quotient $\mathcal{O}_{\mathbb{P}^2}/\mathfrak{m}_p^m$ records the jet conditions removed from all degree-d polynomials.

Intersection language. One has

$$L^2 = d^2 - m^2, \qquad L \cdot E = m, \qquad L \cdot (H - E) = d - m.$$

Thus L is nef for $d \geq m \geq 0$ and ample for $d > m > 0$.

Current and metric language. A divisor $C \in |L|$ gives a positive current $[C]$ in $c_1(L)$. A smooth positive metric on L gives a Kähler representative of the same class when L is ample. Poincaré–Lelong compares them through the logarithm of the section norm.

For $d = 2, m = 1$, conics through p form a five-dimensional space. The resulting map embeds X as the cubic scroll in $\mathbb{P}^4$: locally the five monomials become $(1, v, x, xv, xv^2)$ after removing the exceptional factor, which separates both the point on E and its normal direction.

Checkpoint (capstone calculation)

(1) How many independent conditions does vanishing to order at least m impose on a plane polynomial?
(2) For $L = 3H - 2E$, compute L^2, $L \cdot E$, and $L \cdot (H - E)$.
(3) Which theorem relates a divisor current and the curvature of a metric on its line bundle?

Solution (checkpoint). The forbidden monomials in two local variables have total degree below m, so their number is $1 + 2 + \cdots + m = m(m + 1)/2$. For $3H - 2E$, the three numbers are $9 - 4 = 5$, 2, and 1; the class is ample. The metric Poincaré–Lelong formula writes $\mathrm{dd}^{\mathrm{c}} \log |s|_h^2 = [\mathrm{div}(s)] - c_1(L, h)$. ⋄

Exit criterion

Given $dH - mE$, you can translate it into jets, monomial counts, strict transforms, intersection numbers, a divisor current, and a projective map, explaining which theorem connects each pair of languages.

Summary. $\mathrm{Bl}_p \mathbb{P}^2$ exhibits blow-up, divisors, Chern classes, the Kähler cone, Hodge numbers, vanishing, and positive currents in one model. The course goal is not to memorize 32 theorems but to break a problem along the chain: local analysis, curvature, cohomology, global geometry.

Exercise 8.22: Capstone calculation

Let $C \subset \mathbb{P}^2$ have degree d and multiplicity m at p. Compute the class of its strict transform, its intersection with E, its self-intersection, and its arithmetic genus; check the result with adjunction.

Exercise 8.23: Final project outline

Choose one of the four routes and write a five-page outline specifying a main theorem, three prerequisite tools, one normalization trap, and one explicit example.

Optional handout: Eight possible graduate projects

This handout is optional. No later argument in the core lectures depends on it.

Each project below can be developed from a concrete calculation to a modern theorem. None is needed for the core notes.

(1) **Toric Kähler geometry.** Compute the Fubini–Study moment polytope, Legendre-transform a toric potential, and state Abreu's scalar-curvature equation.
(2) **Higgs bundles.** Study flat rank-two bundles on a curve, formulate Hitchin's equations, and explain the nonabelian Hodge correspondence.
(3) **Variation of Hodge structure.** Derive Griffiths transversality for a family of curves and compute the period map in one example.

(4) **Singular Kähler–Einstein metrics.** Solve a one-dimensional conical model and explain how non-pluripolar products define the global equation.
(5) **Bergman kernels.** Compute kernels for the disc, ball, and projective space, then study positivity under variation of the weight.
(6) **Multiplier ideals.** Resolve a plane-curve singularity, compute its multiplier ideals, and identify its jumping numbers.
(7) **Birational surfaces.** Determine the nef, effective, and movable cones of a blow-up of $\mathbb{P}^2$ at several special points.
(8) **Complex dynamics.** Pull back positive currents under a rational map, construct a Green current, and relate its Lelong numbers to exceptional orbits.

A useful project structure. Begin with one explicit model and a complete calculation. State one general theorem with all hypotheses. Identify the analytic engine (for example L^2 estimates or Monge–Ampère comparison), the cohomological engine, and the compactness mechanism. End with one question whose answer is not formal from the notes.

Remark (What counts as understanding)

A successful project should translate the same object into at least three of the course's languages. For example, a divisor can be treated as an ideal sheaf, a line bundle with section, a cohomology class, and a positive current.

Chapter A

Background Lemmas, Assignments, and Teaching Notes

A.1 Notation and normalization at a glance

Notation	Meaning and convention
$\partial, \bar\partial$	$\mathrm{d} = \partial + \bar\partial$ and $\partial/\partial z_j = \frac{1}{2}(\partial_{x_j} - \mathrm{i}\partial_{y_j})$
$\mathrm{d^c}, \mathrm{dd^c}$	$\mathrm{d^c} = (4\pi\mathrm{i})^{-1}(\partial - \bar\partial)$ and $\mathrm{dd^c} = (\mathrm{i}/2\pi)\partial\bar\partial$
ω_φ	In the Calabi–Yau unit, $\omega + \mathrm{i}\partial\bar\partial\varphi = \omega + 2\pi\mathrm{dd^c}\varphi$
$[Z]$	Integration current over Z_{Reg} with the orientation induced by the complex structure
$\nu(T, x)$	Lelong number normalized by $\nu([\mathbb{C}^p], 0) = 1$
$\Theta(E, h)$	Chern curvature $\bar\partial(H^{-1}\partial H)$; for $\lvert e\rvert^2 = \mathrm{e}^{-\varphi}$, $c_1(L, h) = (\mathrm{i}/2\pi)\Theta = \mathrm{dd^c}\varphi$
L, Λ	$L = \omega \wedge -$ and $\Lambda = L^*$
Δ''	$D''D''^* + D''^*D''$; in the Kähler case $\Delta_{\mathrm{d}} = 2\Delta' = 2\Delta''$
$\mathrm{PSH}(X, \theta)$	θ-psh functions: $\theta + \mathrm{dd^c}u \geq 0$
$\mathcal{I}(\varphi)$	Multiplier ideal of germs f with $\lvert f\rvert^2\mathrm{e}^{-\varphi} \in L^1_{\mathrm{loc}}$
$\langle T_1 \wedge \cdots \wedge T_p\rangle$	Non-pluripolar product; it assigns no mass to pluripolar sets

Warning

If one switches to the common convention $\mathrm{d^c} = \frac{\mathrm{i}}{2}(\bar\partial - \partial)$, every $\mathrm{dd^c}$ formula changes by a factor involving π. Do not replace the symbol while retaining the constants in Poincaré–Lelong, Chern forms, or Monge–Ampère measures.

A.2 Four prerequisite patches

A.2.1 Index rules in Hermitian linear algebra

Let $e_1, \ldots, e_n$ be a unitary $(1, 0)$ coframe. Wedge and contraction satisfy

$$[\bar e_j\wedge, \iota_{\bar e_k^\sharp}]_+ = \delta_{jk}, \qquad [e_j\wedge, \iota_{e_k^\sharp}]_+ = \delta_{jk}.$$

Every Kähler identity and curvature-commutator calculation reduces to these anticommutation rules. Compute first on one basis vector and extend linearly; manipulating a large index sum at once is the easiest way to lose a sign.

A.2.2 Distributional convergence and compactness of positive measures

$T_j \rightharpoonup T$ means $\langle T_j, \eta \rangle \to \langle T, \eta\rangle$ for every compactly supported smooth test form η. Differentiation is continuous: $\mathrm{d}T_j \rightharpoonup \mathrm{d}T$. A family of positive measures with locally uniform mass bounds is relatively compact by Riesz representation and Banach–Alaoglu. Coefficients of a positive current are controlled by its trace measure, so the same principle applies coefficientwise.

A.2.3 Short exact sequences and their long exact sequences

From $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ one obtains

$$0 \to H^0(\mathcal{F}') \to H^0(\mathcal{F}) \to H^0(\mathcal{F}'') \xrightarrow{\delta} H^1(\mathcal{F}') \to \cdots .$$

To construct δ in Čech language, lift a section s'' locally to s_i. The differences $s_i - s_j$ lie in $\mathcal{F}'$ and form a one-cocycle. Its class vanishes exactly when the lifts can be adjusted to glue. Whenever a restriction map is claimed surjective, identify the H^1 group that has been killed.

A.2.4 The Hilbert-complex existence lemma

Let $T : H_0 \to H_1$ and $S : H_1 \to H_2$ be closed densely defined operators with $ST = 0$. Suppose $f \in \ker S$ and

$$|(f, v)|^2 \le C\big(\|T^* v\|^2 + \|Sv\|^2\big)$$

for $v \in \operatorname{Dom} T^* \cap \operatorname{Dom} S$. Then there is $u \in \operatorname{Dom} T$ with $Tu = f$ and $\|u\|^2 \le C$.

Indeed, view $v \mapsto (f, v)$ as a bounded functional on $\{(T^*v, Sv)\} \subset H_0 \oplus H_2$. Hahn–Banach and Riesz represent its extension by (u, g), giving $f = Tu + S^*g$ weakly. Because $f \in \ker S$ and $\overline{\operatorname{Im} S^*} = (\ker S)^\perp$, one has $S^*g = 0$. Taking $u \perp \ker T$ gives the minimum-norm solution.

A.3 Detailed background results used in the core

A.3.1 Partitions of unity and fine sheaves

Lemma A.1: Smooth partitions of unity

Let M be a Hausdorff second-countable smooth manifold and $\{U_i\}_{i\in I}$ an open cover. There is a locally finite refinement $\{V_j\}_{j\in J}$ and smooth functions $\rho_j \ge 0$ such that $\operatorname{supp} \rho_j \subset V_j$ and $\sum_j \rho_j = 1$.

Proof. Second countability and local Euclidean compactness make M paracompact, so the cover has a locally finite refinement by precompact coordinate balls W_j with $\overline{W_j} \subset V_j \subset U_{i(j)}$. Choose a smooth bump function χ_j supported in V_j and positive on $\overline{W_j}$. The locally finite sum $\chi = \sum_j \chi_j$ is smooth and strictly positive. Set $\rho_j = \chi_j/\chi$. Local finiteness justifies both the sum and all derivatives. □

If $\mathcal{F}$ is a sheaf of $\mathcal{A}_X$-modules, multiplication by ρ_j gives endomorphisms supported in V_j whose sum is the identity. Thus $\mathcal{F}$ is fine. The next theorem explains why fine resolutions compute cohomology.

Theorem A.2: Acyclic resolution theorem

Suppose
$$0 \to \mathcal{F} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots$$
is an exact complex of sheaves and each $\mathcal{A}^q$ is acyclic for global sections. Then
$$H^q(X, \mathcal{F}) \simeq H^q\big(\Gamma(X, \mathcal{A}^\bullet), d\big).$$

Proof. Let $\mathcal{Z}^q = \ker(\mathcal{A}^q \to \mathcal{A}^{q+1})$. Exactness gives short exact sequences
$$0 \to \mathcal{Z}^q \to \mathcal{A}^q \to \mathcal{Z}^{q+1} \to 0, \qquad \mathcal{Z}^0 \simeq \mathcal{F}.$$

The long exact sequence and $H^r(X, \mathcal{A}^q) = 0$ for $r > 0$ give dimension-shifting isomorphisms $H^r(X, \mathcal{Z}^{q+1}) \simeq H^{r+1}(X, \mathcal{Z}^q)$ for $r \geq 1$. In degree zero, the same long exact sequence identifies
$$\frac{\Gamma(X, \mathcal{Z}^{q+1})}{d\Gamma(X, \mathcal{A}^q)} \simeq H^1(X, \mathcal{Z}^q).$$

Iterating from $\mathcal{Z}^0 = \mathcal{F}$ gives the claimed isomorphism in every degree. This proof also shows naturality with respect to morphisms of resolutions. □

A.3.2 Closed operators and adjoints

Let $T : \operatorname{Dom} T \subset H_0 \to H_1$ be densely defined. Its graph norm is $\|u\|_T^2 = \|u\|_{H_0}^2 + \|Tu\|_{H_1}^2$. The operator is closed exactly when $\operatorname{Dom} T$ is complete in this norm. Its adjoint is defined by
$$\operatorname{Dom} T^* = \{v \in H_1 : u \mapsto (Tu, v) \text{ is bounded in the } H_0 \text{ norm}\}.$$

For such v, Riesz gives the unique T^*v satisfying $(Tu, v) = (u, T^*v)$.

Lemma A.3: Closed-range criterion

For a closed densely defined operator T, the range of T is closed if and only if there is $C > 0$ such that
$$\|u\| \leq C\,\|Tu\| \quad \text{for all } u \in \operatorname{Dom} T \cap (\ker T)^\perp.$$
In this case $\operatorname{im} T = (\ker T^*)^\perp$.

Proof. Restrict T to $\operatorname{Dom} T \cap (\ker T)^\perp$. It is a closed bijection onto $\operatorname{im} T$. If the range is closed, the closed graph theorem makes the inverse bounded, giving the estimate. Conversely, if Tu_j is Cauchy, the estimate makes the components of u_j orthogonal to the kernel Cauchy; closedness of T then puts the limit in the range. The identity $(\operatorname{im} T)^\perp = \ker T^*$ follows directly from the definition of the adjoint, and closedness removes the closure on the range. □

This is the functional-analytic step behind statements such as "coercivity gives a solution." Boundary conditions are encoded in $\operatorname{Dom} T^*$; they are not visible in the formal differential expression alone.

A.3.3 The elliptic package

Theorem A.4: Elliptic regularity and the Fredholm alternative

Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic differential operator of order m on a compact manifold without boundary. For every real s there is C_s such that

$$\|u\|_{H^{s+m}} \le C_s\big(\|Pu\|_{H^s} + \|u\|_{H^s}\big).$$

The extension $P : H^{s+m}(E) \to H^s(F)$ is Fredholm. If Pu is smooth and u is a distribution, then u is smooth.

Proof.

Construction of a parametrix. The principal symbol $\sigma_m(P)(x,\xi)$ is invertible for $\xi \neq 0$. In local coordinates, invert it for large $|\xi|$ and correct lower-order errors recursively to construct a pseudodifferential parametrix Q of order $-m$ such that $QP = I - K_1$ and $PQ = I - K_2$, where K_1, K_2 are smoothing. Mapping properties of pseudodifferential operators give

$$\|u\|_{H^{s+m}} \le C\big(\|Pu\|_{H^s} + \|K_1 u\|_{H^{s+m}}\big) \le C_s\big(\|Pu\|_{H^s} + \|u\|_{H^s}\big),$$

because a smoothing operator maps H^s to every higher Sobolev space.

Rellich compactness makes K_1 and K_2 compact between the relevant Sobolev spaces. The two parametrix identities say that P is invertible modulo compact operators; Atkinson's theorem therefore makes $P : H^{s+m}(E) \to H^s(F)$ Fredholm. Equivalently, its kernel and cokernel are finite dimensional and its range is closed. If Pu is smooth and u is a distribution, choose s with $u \in H^s$. The identity $u = QPu + K_1 u$ improves u by m derivatives; iteration puts u in every Sobolev space, and Sobolev embedding makes it smooth.

For a self-adjoint nonnegative elliptic operator such as a Hodge Laplacian, the theorem implies finite-dimensional kernel and closed range. Spectral theory gives an orthonormal basis of smooth eigenvectors with eigenvalues tending to infinity. On the orthogonal complement of the kernel, the Green operator is the bounded inverse. This supplies the analytic input used in Lectures 21–24. □

A.3.4 Monotone limits of positive objects

Two limiting principles recur throughout the notes.

Lemma A.5: Weak lower semicontinuity

If $u_j \rightharpoonup u$ weakly in a Hilbert space, then $\|u\| \le \liminf_j \|u_j\|$. If T is closed and $Tu_j = f$ while u_j converges weakly and is uniformly bounded in the graph norm, then $Tu = f$.

Proof. The norm is the supremum of $|(u,v)|$ over unit vectors v, so weak convergence gives the first assertion. A bounded graph-norm sequence has weakly convergent subsequences in both H_0 and H_1. The graph of a closed linear operator is a closed linear subspace and hence weakly closed, proving the second assertion. □

For positive measures, local mass bounds give weak-* compactness. For psh functions, normalization prevents convergence to $-\infty$ and gives L^1_{loc} compactness. For L^2 solutions, weak lower semicontinuity preserves the estimate. These are three forms of the same compactness strategy: obtain a coercive local bound, extract weakly, pass the linear equation against tests, and recover nonlinear information from positivity or monotonicity.

A.4 Correspondence with Demailly's book

Lectures	Topic	Main source or extension
1–4	manifolds, forms, Kähler metrics, currents	Ch. I §§1–3; curvature language from Ch. V introduced early
5–8	subharmonic/psh, pseudoconvex, Stein	Ch. I §§4–7; Cartan A/B interface introduced early
9–12	sheaves, coherence, analytic spaces, blow-ups	Ch. II §§1–10
13–16	positive currents, Monge–Ampère, Lelong, Siu	Ch. III §§1–8; capacity and later pluripotential theory added
17–20	bundles, Chern curvature, positivity, embedding	Ch. V and Ch. VII §§6–14
21–24	Hodge, Kähler identities, cohomology, duality	selections from Ch. IV and Ch. VI
25–28	BKN, vanishing, Hörmander, multiplier ideals	Ch. VII §§1–9 and Ch. VIII §§1–6; Nadel added
29	Ohsawa–Takegoshi	Ch. VIII §7 plus the structure of twisted estimates
30	Calabi–Yau	added; connected to Ch. III Monge–Ampère theory
31	positivity cones, regularization, non-pluripolar products	modern extension of Ch. III/VII/VIII
32	capstone on the blown-up plane	synthesis of Ch. II/V/VI/VII

A.5 Optional: q-convexity and Andreotti–Grauert finiteness

Definition A.6: q-convexity

A smooth function on an n-fold is strongly q-convex if its Levi form has at least $n - q + 1$ positive eigenvalues. A space is q-convex if it has such an exhaustion outside a compact set, and q-complete if the exhaustion is strongly q-convex everywhere. Stein/strictly psh corresponds to $q = 1$. Usually $1 \le q \le n$; if one formally allows $q = n + 1$, the eigenvalue condition is vacuous, whereas $q = 0$ would demand the impossible total of $n + 1$ positive eigenvalues.

Theorem A.7: Andreotti–Grauert, course version

If X is a q-convex complex space and $\mathcal{F}$ coherent, then $H^k(X, \mathcal{F})$ is finite dimensional for $k \ge q$. If X is q-complete, then $H^k(X, \mathcal{F}) = 0$ for $k \ge q$.

Proof (proof architecture: the bump method). We isolate the analytic input, the Andreotti–Grauert bump lemma. A *strongly q-convex bump pair* is $A \Subset B$ for which $B \setminus A$ lies in one coordinate ball and the new boundary is cut out by a function whose Levi form has at least $n - q + 1$ eigenvalues bounded below by a positive constant. The bump lemma says that, for a coherent sheaf $\mathcal{F}$, the restriction map in degrees $k \ge q$ has closed image and finite-dimensional kernel and cokernel; if the bump can be pushed through a region with no compact exceptional set, the cokernel is zero. For a sequence of bumps escaping every compact set, the restriction of a degree-k class, $k \geq q$, to a sufficiently far tail can be corrected to zero. On a merely q-convex space, a finite-dimensional obstruction may remain on the exceptional compact set; this is exactly the distinction between finiteness and vanishing.

Its proof resolves $\mathcal{F}$ locally by finite free sheaves and combines a specially chosen Leray cover with weighted $\bar{\partial}$ estimates on the bump. The linear-algebra lemma for a Levi form with $n - q + 1$ positive eigenvalues supplies coercivity in cohomological degree at least q; a compact restriction operator accounts for the finite-dimensional error. Successive corrections through the free resolution give closed range and the stated finite obstruction for $\mathcal{F}$. This analytic bump lemma is the substantial part of the Andreotti–Grauert theorem; it is not a consequence of counting alternating Čech indices.

Now perturb the exhaustion slightly, without losing its strict Levi margin, so that its critical points are isolated. Between regular levels, compactness of a level band allows it to be crossed by finitely many strongly q-convex bumps. Near a critical point, real Morse coordinates together with a local Levi-positive quadratic correction produce the same finite bump chain. Thus, after a level containing the exceptional compact set, the restriction-correction maps in degrees $k \geq q$ have only finite-dimensional obstructions. All possible obstructions are already represented on one fixed compact sublevel; coherence and the finite covering used above make that space finite dimensional. Passing to the exhaustion proves $\dim H^k(X, \mathcal{F}) < \infty$ for a q-convex space.

If X is q-complete there is no exceptional compact set. Given a cohomology class, represent it by a cocycle on some sublevel. Repeatedly apply the surjective restriction-correction part of the bump lemma while pushing the support of the corrected cocycle to larger levels. The uniform estimates make the corrections a convergent series on compact subsets, and the limit is a global coboundary. Hence $H^k(X, \mathcal{F}) = 0$ for every $k \geq q$. □

A.6 Biweekly assignments and selected solutions

Assignment 1 (Weeks 1–2)

Required: 1.9, 1.17, 1.25, and 1.34.

Solution (1.25). In the chart $z = re^{i\theta}$,

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2} = \frac{1}{\pi}\frac{r\mathrm{d}r\mathrm{d}\theta}{(1+r^2)^2}.$$

Thus the integral is $\frac{1}{\pi}(2\pi)\int_0^\infty r(1+r^2)^{-2}\mathrm{d}r = 1$. The point at infinity has measure zero, and the second chart verifies smooth extension. ◇

Assignment 2 (Weeks 3–4)

Required: 2.6, 2.17, 2.27, and 2.35.

Solution (2.17). The theorem on finite upper envelopes proves that $\max(u, v)$ is psh. For a counterexample, take $u(z) = \operatorname{Re} z$ and $v(z) = -\operatorname{Re} z$. Both are pluriharmonic, but $\min(u, v) = -|\operatorname{Re} z|$ has a downward corner along the imaginary axis; its distributional Laplacian contains a negative measure. ◇

Assignment 3 (Weeks 5–6)

Required: 3.5, 3.16, 3.27, and 3.35.

Solution (3.35). The charts are $(x, y) = (t, tu)$ and $(x, y) = (vs, s)$. On the overlap, $s = tu$ and $v = u^{-1}$. Along E, the normal coordinate changes from t to $s = ut$, the transition function of the tautological bundle $\mathcal{O}(-1)$. ◇

Assignment 4 (Weeks 7–8)

Required: 4.6, 4.14, 4.23, and 4.32.

Solution (4.23). $u = \max(\log|z|, 0)$ is harmonic on either side of $|z| = 1$. The distributional Laplacian comes only from the jump in normal derivative at $r = 1$, so $\mathrm{dd^c}u$ is one half of normalized Haar measure on the unit circle. Its total mass is $1/2$; using $\log|z|^2$ instead would give mass 1 in our normalization. ◇

Assignment 5 (Weeks 9–10)

Required: 5.7, 5.16, 5.24, and 5.31.

Solution (5.31). A basis of $H^0(\mathbb{P}^n, \mathcal{O}(k))$ consists of all z^I, $|I| = k$. The map $[z] \mapsto [z^I]_{|I|=k}$ has no base points and separates homogeneous points. On $z_0 \neq 0$, the coordinate monomials contain $1, w_j$ after a common factor, so the differential has full rank. Hence this is the Veronese embedding. ◇

Assignment 6 (Weeks 11–12)

Required: 6.6, 6.13, 6.22, and 6.29.

Solution (6.22). Choose a subordinate partition ρ_k and set $a_i = \sum_k \rho_k g_{ik}$. Then $a_i - a_j = g_{ij}$. The forms $\bar\partial a_i$ agree on overlaps and glue to a global $\bar\partial$-closed $(0,1)$-form. Changing the partition changes it only by a $\bar\partial$-exact form. ◇

Assignment 7 (Weeks 13–14)

Required: 7.5, 7.14, 7.19, and 7.25.

Solution (7.25). If $f = z^m g$ with $g(0) \neq 0$, then $|f|^2 e^{-\varphi} \sim r^{2m-2c}$. Multiplication by the area element $r\mathrm{d}r\mathrm{d}\theta$ is integrable exactly when $2m - 2c + 1 > -1$, or $m > c - 1$. Hence $\mathcal{I}(c\log|z|^2) = (z^{\lfloor c\rfloor})$; for integral c, the strict inequality still gives $m \geq c$. ◇

Assignment 8 (Weeks 15–16)

Required: 8.5, 8.9, 8.17, and 8.22.

Solution (8.22). The strict transform has class $\widetilde{C} = dH - mE$. Thus $\widetilde{C} \cdot E = m$ and $\widetilde{C}^2 = d^2 - m^2$. Adjunction gives

$$2p_a(\widetilde{C}) - 2 = \widetilde{C} \cdot (\widetilde{C} + K_X) = d^2 - m^2 - 3d + m,$$

so $p_a = (d-1)(d-2)/2 - m(m-1)/2$. ◇

A.7 Suggestions for instructors

- Begin each lecture with a ten-minute unit test: $|z|^2$, $\log|z|^2$, $\mathcal{O}(1)$, or a flat torus.
- Every two weeks, ask a student to diagnose one Warning: identify the first invalid step, then repair it.
- A Week 8 take-home problem can audit a short proof missing upper semicontinuity, a boundary condition, or the correct curvature degree.
- Keep the Hörmander hypothesis ledger visible throughout Weeks 13–14.
- If sheaf theory is new to the class, shorten the spectral-sequence material in Lecture 24 and add Čech computations to Lectures 9 and 23.
- For an algebraic-geometry audience, treat Lecture 30 as optional and expand ample/nef/big cones and intersection theory in Lectures 20 and 31.

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Complete Solutions to Exercises

This chapter contains solutions to every exercise in the lectures and in the optional handouts. Each heading links back to the original exercise. The solutions use the normalizations fixed at the beginning of the book; in particular, $\mathrm{d}^{\mathrm{c}} = (4\pi\mathrm{i})^{-1}(\partial - \bar{\partial})$ and $\mathrm{dd}^{\mathrm{c}} = (\mathrm{i}/2\pi)\partial\bar{\partial}$.

Lecture 1

Exercise 1.9: The two charts of the projective line

Solution. Set

$$U_0 = \{[z_0 : z_1] : z_0 \neq 0\}, \qquad U_1 = \{[z_0 : z_1] : z_1 \neq 0\}.$$

The coordinates $w = z_1/z_0$ on U_0 and $\zeta = z_0/z_1$ on U_1 identify both charts with $\mathbb{C}$. On the overlap, $w \neq 0$, and $\zeta = w^{-1}$; this is a holomorphic transition map with holomorphic inverse.

Identify the unit sphere $S^2 \subset \mathbb{R}^3$ with the one-point compactification of $\mathbb{C}$ by inverse stereographic projection

$$w \longmapsto \frac{1}{1+|w|^2}\big(2\Re w, 2\Im w, |w|^2 - 1\big), \qquad \infty \longmapsto (0,0,1).$$

The coordinate near the north pole is $\zeta = 1/w$, exactly the second projective chart. The complex coordinate changes are therefore the same, so this identification is a biholomorphism from $\mathbb{P}^1$ to the Riemann sphere. ◇

Exercise 1.10: Tangent spaces

Solution. Write $F = (F_1, \ldots, F_r)$ and suppose $\mathrm{d}F_x$ has rank r. A tangent vector $v \in T_x^{1,0}X$ is the velocity at zero of a holomorphic curve $\gamma : (\mathbb{C}, 0) \to (X, x)$. Differentiating $F \circ \gamma = 0$ gives $\mathrm{d}F_x(v) = 0$, so $T_x^{1,0}X \subseteq \ker \mathrm{d}F_x$.

Conversely, the holomorphic implicit-function theorem gives local coordinates $(z', z'') \in \mathbb{C}^{n-r} \times \mathbb{C}^r$, centered at x, in which X is $z'' = 0$. In these coordinates $\ker \mathrm{d}F_x$ is spanned by $\partial/\partial z'_1, \ldots, \partial/\partial z'_{n-r}$, and these vectors are tangent to the coordinate curves in X. Hence

$$T_x^{1,0}X = \ker(\mathrm{d}F_x : \mathbb{C}^n \longrightarrow \mathbb{C}^r).$$

◇

Exercise 1.12: A nonintegrable model

Solution. Using $[A, fB] = A(f)B + f[A,B]$, we obtain

$$[L_1, L_2] = \left[\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2} + \bar{z}_1 \frac{\partial}{\partial z_1}\right] = \frac{\partial}{\partial z_1}.$$

This vector is not in the complex span of L_1, L_2: if $aL_1 + bL_2 = \partial_{z_1}$, comparison of the $\partial_{\bar{z}_1}$ and $\partial_{\bar{z}_2}$ components forces $a = b = 0$, a contradiction. Thus $T^{0,1}$ is not closed under Lie brackets. By the Newlander–Nirenberg integrability criterion the almost complex structure is not integrable. ⋄

Lecture 2

Exercise 1.16: Sign check

Solution. Locally write α as a sum of terms $a\,\eta$, where η is a wedge of $p+q$ coordinate one-forms. Since $\bar\partial$ is the component of the exterior derivative of bidegree $(0,1)$, it is a graded derivation of total degree one. Consequently

$$\bar\partial(a\eta \wedge \beta) = \bar\partial(a\eta) \wedge \beta + (-1)^{p+q} a\eta \wedge \bar\partial\beta.$$

Summing the local terms proves

$$\bar\partial(\alpha \wedge \beta) = \bar\partial\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \bar\partial\beta.$$

The sign involves the total degree $p+q$, not merely the antiholomorphic degree q. ⋄

Exercise 1.17: Reality of d^{c}

Solution. For $z = x + \mathrm{i}y$,

$$\partial u = \frac{1}{2}(u_x - \mathrm{i}u_y)(\mathrm{d}x + \mathrm{i}\mathrm{d}y), \qquad \bar\partial u = \frac{1}{2}(u_x + \mathrm{i}u_y)(\mathrm{d}x - \mathrm{i}\mathrm{d}y).$$

Hence $\partial u - \bar\partial u = \mathrm{i}(u_x\mathrm{d}y - u_y\mathrm{d}x)$, and therefore

$$\mathrm{d}^{\mathrm{c}}u = \frac{1}{4\pi}(u_x\mathrm{d}y - u_y\mathrm{d}x).$$

When u is real, both coefficients are real, so $\mathrm{d}^{\mathrm{c}}u$ is a real one-form. Differentiation also gives $\mathrm{d}\mathrm{d}^{\mathrm{c}}u = (4\pi)^{-1}(u_{xx} + u_{yy})\mathrm{d}x \wedge \mathrm{d}y$. ⋄

Exercise 1.19: Natural maps

Solution. The maps are induced by the identity on representatives:

$$H^{p,q}_{BC} \longrightarrow H^{p,q}_{\bar\partial}, \quad [\alpha]_{BC} \mapsto [\alpha]_{\bar\partial}, \qquad H^{p,q}_{\bar\partial} \longrightarrow H^{p,q}_{A}, \quad [\alpha]_{\bar\partial} \mapsto [\alpha]_A,$$

and

$$H^{p,q}_{BC} \longrightarrow H^{p+q}_{\mathrm{dR}}(X, \mathbb{C}), \quad [\alpha]_{BC} \mapsto [\alpha]_{\mathrm{dR}}.$$

If $\partial\alpha = \bar\partial\alpha = 0$, then $d\alpha = 0$, so all indicated target classes are defined. Replacing α by $\alpha + \partial\bar\partial\gamma$ changes its Dolbeault class by $-\bar\partial(\partial\gamma)$, its Aeppli class by a ∂- and $\bar\partial$-exact term, and its de Rham class by $d(\bar\partial\gamma) = \partial\bar\partial\gamma$. Thus the first and third maps are well defined. Finally, if $\bar\partial\alpha = 0$, then $\partial\bar\partial\alpha = 0$; replacing α by $\alpha + \bar\partial\gamma$ does not change its Aeppli class. This proves well-definedness of the middle map. ⋄

Lecture 3

Exercise 1.25: Area of $\mathbb{P}^1$

Solution. On the affine chart $w = z_1/z_0$,

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log(1+|w|^2) = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}w \wedge \mathrm{d}\bar w}{(1+|w|^2)^2}.$$

Since $\mathrm{i}\,\mathrm{d}w \wedge \mathrm{d}\bar w = 2r\,\mathrm{d}r \wedge \mathrm{d}\theta$,

$$\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{r}{(1+r^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta = 2\left[-\frac{1}{2(1+r^2)}\right]_0^{\infty} = 1.$$

The omitted point at infinity has measure zero. More intrinsically, in the coordinate $\zeta = 1/w$ the same formula is smooth at $\zeta = 0$, so no hidden mass is lost there. ◇

Exercise 1.26: Products

Solution. Let X_1, X_2 be Kähler and put $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$. It is a real $(1,1)$-form and

$$d\omega = \pi_1^*(d\omega_1) + \pi_2^*(d\omega_2) = 0.$$

For $v = (v_1, v_2) \neq 0$,

$$\omega(v, Jv) = \omega_1(v_1, J_1 v_1) + \omega_2(v_2, J_2 v_2) > 0,$$

because at least one component is nonzero. Thus ω is positive definite and closed, hence Kähler. ◇

Lecture 4

Exercise 1.33: The Heaviside distribution

Solution. For a test function $\varphi \in C_c^\infty(\mathbb{R})$, the distributional derivative is defined by duality:

$$\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^\infty \varphi'(x)\,\mathrm{d}x = \varphi(0).$$

Therefore $H' = \delta_0$. This agrees with the current convention $\langle dT, \eta\rangle = (-1)^{\deg T+1}\langle T, d\eta\rangle$: in real dimension one a distribution is a degree-zero current, giving precisely the minus sign used above. ◇

Exercise 1.34: Divisor of a rational function

Solution. For $f = z_1^2 z_2/(z_1 - z_2)$, the divisor is

$$\mathrm{div}(f) = 2\{z_1 = 0\} + \{z_2 = 0\} - \{z_1 - z_2 = 0\}.$$

The Poincaré–Lelong formula, with the normalization of these notes, therefore gives

$$\mathrm{dd^c}\log\left|\frac{z_1^2 z_2}{z_1 - z_2}\right|^2 = 2[z_1 = 0] + [z_2 = 0] - [z_1 - z_2 = 0].$$

This is an identity of closed real $(1,1)$-currents on $\mathbb{C}^2$. The logarithm is locally integrable despite its zeros and pole, and $\mathrm{dd^c}$ is taken in the distributional sense. It is not an equality of smooth forms on the divisor. ◇

Exercise 1.37: Delta functions

Solution. A smooth function has empty wavefront set, so Hörmander's criterion always permits its product with a distribution. Directly, $\langle f\delta_0, \varphi\rangle = f(0)\varphi(0)$, whence $f\delta_0 = f(0)\delta_0$.

In contrast, $\mathrm{WF}(\delta_0) = \{(0,\xi) : \xi \neq 0\}$. Whenever $(0,\xi)$ occurs, so does $(0,-\xi)$, and the product criterion fails for δ_0^2. The failure is real, not merely technical. If $\rho_\varepsilon(x) = \varepsilon^{-m}\rho(x/\varepsilon)$ is a mollifier in $\mathbb{R}^m$, then

$$\int \rho_\varepsilon(x)^2\varphi(x)\,\mathrm{d}x = \varepsilon^{-m}\int \rho(y)^2\varphi(\varepsilon y)\,\mathrm{d}y,$$

which generally diverges like $\varepsilon^{-m}\varphi(0)\int\rho^2$. Subtracting divergent terms can produce different finite parts for different mollifiers. Thus there is no canonical distribution that deserves the name δ_0^2. ◇

Lecture 5

Exercise 2.6: Convexity of circular means

Solution. Put $v(w) = u(e^w)$ on a vertical strip in the $w = t + \mathrm{i}\theta$ plane. A holomorphic pullback of a subharmonic function is subharmonic, so v is subharmonic and 2π-periodic in θ. Its circular mean is

$$m(t) = M_u(0, e^t) = \frac{1}{2\pi}\int_0^{2\pi} v(t+\mathrm{i}\theta)\,\mathrm{d}\theta.$$

For smooth u, periodicity and integration by parts give

$$m''(t) = \frac{1}{2\pi}\int_0^{2\pi} v_{tt}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}(v_{tt}+v_{\theta\theta})\,\mathrm{d}\theta \geq 0.$$

For general subharmonic u, regularize on slightly smaller annuli. The smooth means are convex and converge in L^1_{loc}, hence their distributional second derivatives converge to the nonnegative distribution m''. Therefore m is convex. Since $t = \log r$, this is the desired convexity in $\log r$. ◇

Exercise 2.7: Decreasing limits

Solution. Let $u_j \downarrow u$. If $u \equiv -\infty$ there is nothing to prove. Otherwise, on a connected component choose a point where $u > -\infty$. The submean inequality and a chain of overlapping discs show that u is not identically $-\infty$ on any smaller disc and that u_j have locally uniformly bounded L^1-mass from below. Standard compactness for subharmonic functions then gives $u_j \to u$ in L^1_{loc}.

Distributionally, $\Delta u_j \geq 0$. For every nonnegative test function χ,

$$\langle \Delta u, \chi\rangle = \langle u, \Delta\chi\rangle = \lim_j \langle u_j, \Delta\chi\rangle = \lim_j\langle \Delta u_j, \chi\rangle \geq 0.$$

The decreasing limit is upper semicontinuous, and a locally integrable upper-semicontinuous function with nonnegative distributional Laplacian is subharmonic. Thus on each connected component the limit is subharmonic or identically $-\infty$. ◇

Exercise 2.9: Green function of the unit disc

Solution. If $|z| = 1$, then

$$|z-\zeta|^2 = (z-\zeta)(\bar z - \bar\zeta) = (1-\bar\zeta z)(1-\zeta\bar z) = |1-\bar\zeta z|^2,$$

so $G_{\mathbb{D}}(z,\zeta) = 0$ on $\partial\mathbb{D}$. In the z variable, $1-\bar{\zeta}z$ is holomorphic and has no zero in $\mathbb{D}$. Thus $\log|1-\bar{\zeta}z|^2$ is harmonic there. Poincaré–Lelong gives

$$\mathrm{dd}^{\mathrm{c}}_z G_{\mathbb{D}}(z,\zeta) = \mathrm{dd}^{\mathrm{c}}_z \log|z-\zeta|^2 - \mathrm{dd}^{\mathrm{c}}_z \log|1-\bar{\zeta}z|^2 = \delta_\zeta.$$

Hence this is the Green function with the normalization used in the notes. ⋄

Lecture 6

Exercise 2.17: Maximum versus minimum

Solution. Both $u_1 = \log|z_1|$ and $u_2 = \log|z_2|$ are psh. The maximum of finitely many psh functions is upper semicontinuous and restricts to the maximum of subharmonic functions on every complex line; the latter is subharmonic. Therefore $\max(u_1,u_2)$ is psh.

For the failure of minima, take the psh functions 0 and $\log|z|$ on $\mathbb{C}$. Their minimum is

$$v(z) = \begin{cases} \log|z|, & |z| \le 1, \\ 0, & |z| \ge 1. \end{cases}$$

The radial derivative drops from 1 on the inner side of the unit circle to 0 on the outer side. Consequently Δv has a negative measure on $|z| = 1$. Equivalently, the mean-value inequality fails at points of that circle. Thus v is not subharmonic and hence not psh. ⋄

Exercise 2.18: A pole on projective space

Solution. On $U_0 = \{z_0 \ne 0\}$, put $w = z_1/z_0$. Then

$$u = \log|w|^2 - \log(1+|w|^2).$$

Poincaré–Lelong and the definition of the Fubini–Study form yield

$$\mathrm{dd}^{\mathrm{c}} u = \delta_{w=0} - \omega_{\mathrm{FS}}, \qquad \omega_{\mathrm{FS}} + \mathrm{dd}^{\mathrm{c}} u = [w=0].$$

On $U_1 = \{z_1 \ne 0\}$, with $\zeta = z_0/z_1$,

$$u = -\log(1+|\zeta|^2),$$

which is smooth and satisfies $\mathrm{dd}^{\mathrm{c}} u = -\omega_{\mathrm{FS}}$. The divisor $\{z_1 = 0\}$ does not meet this chart, so the same current identity holds there with zero right-hand side. The two calculations agree on the overlap and prove globally $\omega_{\mathrm{FS}} + \mathrm{dd}^{\mathrm{c}} u = [z_1 = 0]$. ⋄

Exercise 2.20: Envelope in the disc

Solution. The answer is the continuous radial function

$$h^*_{K,\mathbb{D}}(z) = \begin{cases} -1, & |z| \le r, \\ \dfrac{\log|z|}{-\log r}, & r < |z| < 1. \end{cases}$$

The second branch is harmonic on the annulus, equals -1 at $|z| = r$, and tends to 0 at $|z| = 1$. The outward radial derivative jumps upward at $|z| = r$, so the piecewise function is subharmonic. It is therefore an admissible competitor.

Conversely, if v is admissible, its circular average is subharmonic, is at most -1 on the inner boundary and at most 0 on the outer boundary. The maximum principle on the annulus bounds that average by the displayed harmonic function. Applying the same argument after rotations, or using harmonic measure, bounds v itself by the displayed function. Hence it is the upper-semicontinuous regularization of the envelope. ⋄

Lecture 7

Exercise 2.27: Ball and polydisc

Solution. On the unit ball $B \subset \mathbb{C}^n$,

$$\Phi_B(z) = -\log(1 - |z|^2)$$

is smooth, tends to $+\infty$ at the boundary, and has Levi matrix

$$\frac{\delta_{jk}}{1 - |z|^2} + \frac{\bar{z}_j z_k}{(1 - |z|^2)^2},$$

which is positive definite. Thus it is a strictly psh exhaustion.

On the polydisc $\mathbb{D}^n$, one convenient smooth choice is

$$\Phi_P(z) = -\sum_{j=1}^{n} \log(1 - |z_j|^2).$$

Its Levi matrix is diagonal with strictly positive entries $(1 - |z_j|^2)^{-2}$, and it is exhaustive because at least one term tends to $+\infty$ at the boundary. A more geometric but nonsmooth candidate is $\max_j\{-\log(1 - |z_j|^2)\}$. Replacing the maximum by the regularized maximum M_ε from Lecture 6 gives a smooth psh function within a bounded error; adding a small multiple of $|z|^2$ makes strict positivity manifest if necessary. ◇

Exercise 2.28: Changing a defining function

Solution. Let $\widetilde{\rho} = a\rho$, where $a > 0$ near the boundary. The product rule gives

$$\partial\bar{\partial}(a\rho) = a\,\partial\bar{\partial}\rho + \rho\,\partial\bar{\partial}a + \partial a \wedge \bar{\partial}\rho + \partial\rho \wedge \bar{\partial}a.$$

At a boundary point x, $\rho(x) = 0$. If $\xi \in T_x^{1,0}(\partial\Omega)$, then $\partial\rho(\xi) = 0$, and its conjugate also vanishes. Evaluating the preceding identity on $(\xi, \bar{\xi})$ therefore eliminates the last three terms and gives

$$\mathcal{L}_{a\rho,x}(\xi) = a(x)\mathcal{L}_{\rho,x}(\xi).$$

Thus the sign of the boundary Levi form is independent of a positive change of defining function. ◇

Lecture 8

Exercise 2.34: Open Riemann surfaces

Solution. The Behnke–Stein theorem states that every connected noncompact Riemann surface is a Stein manifold. One formulation of its proof constructs a strictly subharmonic exhaustion by successively enlarging Runge relatively compact domains; the Levi problem then identifies the resulting surface as Stein.

The boundary Levi form is tested on the complex tangent space $T^{1,0}(\partial\Omega)$, which has complex dimension $n - 1$. For $n = 1$ this space is zero, so Levi semipositivity has no nonzero direction on which it could fail. Thus every smoothly bounded domain in a Riemann surface is pseudoconvex in the boundary Levi sense. The theorem is global: it supplies the required holomorphic separation and exhaustion for the whole open surface. ◇

Exercise 2.35: Closed complex submanifolds

Solution. Let $\Phi : X \to \mathbb{R}$ be a smooth strictly psh exhaustion. Since Y is a complex submanifold, the Levi form of $\Phi|_Y$ is the restriction of the positive definite Levi form of Φ, hence is positive definite. Since Y is closed,

$$\{y \in Y : \Phi(y) \le c\} = Y \cap \{x \in X : \Phi(x) \le c\}$$

is closed in a compact set and therefore compact. Thus $\Phi|_Y$ is a smooth strictly psh exhaustion. The solution of the Levi problem says that a complex manifold admitting such an exhaustion is Stein, so Y is Stein. ◇

Exercise 2.37: Polynomial convexity of the ball

Solution. Let $K = \overline{B}$ and let $a \notin K$, so $|a| > 1$. The holomorphic linear polynomial

$$p_a(z) = \sum_{j=1}^{n} z_j \bar{a}_j$$

satisfies $p_a(a) = |a|^2$, whereas Cauchy–Schwarz gives $\sup_{z\in K} |p_a(z)| \le |a|$. Since $|a|^2 > |a|$, the point a is not in the polynomial hull of K. Every point outside K can be separated in this way, so the polynomial hull is exactly K; hence the closed unit ball is polynomially convex. ◇

Lecture 9

Exercise 3.5: The exponential sequence

Solution. The map $\mathbb{Z} \to \mathcal{O}_X$ sends an integer to the corresponding locally constant holomorphic function and is injective. If $\exp(2\pi \mathrm{i} f) = 1$, then the holomorphic function f takes values in the discrete set $\mathbb{Z}$, so it is locally constant and integer-valued. Thus the kernel at $\mathcal{O}_X$ is precisely $\mathbb{Z}$.

It remains to prove surjectivity onto $\mathcal{O}_X^*$ as a sheaf. If g is nonvanishing near x, shrink to a simply connected coordinate ball on which g has no zero. Then g'/g has a holomorphic primitive in one variable, and the analogous closed holomorphic one-form dg/g has a primitive in several variables. Hence there is a holomorphic logarithm h with $e^h = g$. Taking $f = h/(2\pi\mathrm{i})$ gives $g = e^{2\pi\mathrm{i}f}$. Exactness is stalkwise, so this proves exactness of the sheaf sequence. ◇

Exercise 3.6: Nakayama at a stalk

Solution. Choose classes $\bar{s}_1, \dots, \bar{s}_r$ forming a basis of $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$, and lift them to germs $s_i \in \mathcal{F}_x$. Nakayama's lemma says that the s_i generate the finite $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$. Represent the germs by sections on a common neighborhood U and form

$$\mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U, \qquad (a_i) \longmapsto \sum_i a_i s_i.$$

Its cokernel $\mathcal{G}$ is coherent. Since $\mathcal{G}_x = 0$, the support of $\mathcal{G}$, a closed analytic subset of U, does not contain x. After shrinking U, the cokernel vanishes. Thus the lifted sections generate $\mathcal{F}$ throughout a neighborhood of x. ◇

Exercise 3.8: Divisor line bundle

Solution. Let D be represented on U_i by nonzero meromorphic functions f_i, with $g_{ij} = f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$. The identities $g_{ij}g_{jk} = g_{ik}$ glue the trivial line bundles. To fix conventions, choose frames e_i satisfying

$$e_j = g_{ij} e_i = \frac{f_i}{f_j} e_i.$$

Then $f_i e_i = f_j e_j$ on overlaps, so these local expressions define a canonical meromorphic section s_D. Its local coefficient is f_i, hence $\operatorname{div}(s_D) = D$.

If $D = \operatorname{div}(f)$ is principal, take $f_i = f|_{U_i}$. Then every $g_{ij} = 1$, so the gluing is the trivial line bundle, and s_D is the global meromorphic function f times its constant frame. Changing the local equations by units changes frames but not the isomorphism class. ◇

Lecture 10

Exercise 3.16: Node and cusp

Solution. The node $N = \{xy = 0\}$ is the union of the two coordinate axes, so it has two irreducible components. Since $d(xy)_0 = 0$, its Zariski tangent space is all of $\mathbb{C}^2$. Its normalization is the disjoint union $\mathbb{C} \sqcup \mathbb{C}$, with maps $t \mapsto (t, 0)$ and $s \mapsto (0, s)$; the two preimages of the origin record the two branches.

The cusp $C = \{y^2 = x^3\}$ is irreducible: its local equation has the primitive parametrization $\nu(t) = (t^2, t^3)$, and the corresponding subring $\mathbb{C}\{t^2, t^3\} \subset \mathbb{C}\{t\}$ is a domain. Again the differential of the defining equation vanishes at the origin, so $T_0 C = \mathbb{C}^2$. Its normalization is the single copy $\mathbb{C} \to C$ just displayed. It is bijective near zero but not an isomorphism because $t = y/x$ is integral over $\mathbb{C}\{t^2, t^3\}$ and does not belong to that ring. ◇

Exercise 3.17: A division calculation

Solution. Regard the expressions as polynomials in y with holomorphic coefficients in x. Since

$$(y^2 - x)(y^2 + x) = y^4 - x^2,$$

we have

$$y^4 + x = (y^2 - x)(y^2 + x) + (x^2 + x).$$

Thus the quotient is $Q = y^2 + x$ and the remainder is $R = x^2 + x$. The remainder has degree zero in y, which is strictly less than the degree two of the Weierstrass polynomial P; uniqueness follows from the division theorem. ◇

Exercise 3.19: A monomial intersection

Solution. Every convergent power series is congruent modulo (x^a, y^b) to a unique linear combination of the monomials

$$x^i y^j, \qquad 0 \le i < a, \quad 0 \le j < b.$$

They span because higher powers lie in the ideal, and they are independent because no nonzero combination of them belongs to the monomial ideal. Hence

$$\dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(x^a, y^b) = ab.$$

The curves $x^a = 0$ and $y^b = 0$ meet only at the origin, and the length of this local quotient is their local intersection multiplicity. Thus $i_0(x^a, y^b) = ab$, including the scheme-theoretic multiplicities of both axes. ◇

Lecture 11

Exercise 3.26: A non-Cartier Weil divisor

Solution. The hypersurface X has dimension three, while $A = \{z_1 = z_3 = 0\}$ is a two-dimensional irreducible plane contained in X. Therefore A is a prime Weil divisor.

At the origin let

$$R = \mathbb{C}\{z_1, z_2, z_3, z_4\}/(z_1 z_2 + z_3 z_4), \qquad I = (z_1, z_3) \subset R.$$

If A were Cartier, I would be principal in the local ring R. But the classes of z_1, z_3 are linearly independent in $I/\mathfrak{m}I$: the defining relation and every element of $\mathfrak{m}I$ have order at least two, so they impose no relation on these linear terms. Hence $\dim_{\mathbb{C}} I/\mathfrak{m}I = 2$. Nakayama's lemma says that I needs at least two generators and is not principal. Thus A is not Cartier at the origin. ◇

Exercise 3.27: Indeterminacy

Solution. As a meromorphic map to $\mathbb{P}^1$, the function is represented by $[z_2 : z_1]$. Its zero set is $\{z_2 = 0, z_1 \neq 0\}$, its pole set is $\{z_1 = 0, z_2 \neq 0\}$, and its indeterminacy set is the common zero set of the two homogeneous coordinates, namely $\{(0, 0)\}$.

The origin cannot be filled in continuously, much less holomorphically. Along $z_2 = 0$ the value is 0; along $z_1 = 0$ it is ∞; and along $z_2 = az_1$ it is the arbitrary value a. Thus neither 0, nor ∞, nor any finite value is compatible with all approaches. ◇

Exercise 3.29: A normal singularity

Solution. Let $F = z_1 z_2 - z_3 z_4$. Its gradient is

$$(z_2, z_1, -z_4, -z_3),$$

so the hypersurface is singular precisely at the origin. The cone has complex dimension three, so its singular locus has codimension three. It is therefore regular in codimension one, which is Serre's condition (R_1).

Every hypersurface local ring is Cohen–Macaulay, hence satisfies (S_k) for all k up to its dimension, in particular (S_2). The polynomial F is reduced, and Serre's criterion $(R_1) + (S_2)$ now proves that the cone is normal. It is nevertheless singular at the origin because $dF_0 = 0$, so its Zariski tangent space has dimension four instead of three. ◇

Lecture 12

Exercise 3.35: Transition between blow-up charts

Solution. In the first chart write $(x, y) = (u, ut)$; in the second write $(x, y) = (sv, v)$. On the overlap $u, v \neq 0$,

$$v = ut, \qquad s = t^{-1}, \qquad u = sv.$$

The exceptional divisor is $u = 0$ in the first chart and $v = 0$ in the second, with coordinate transition $s = 1/t$, so $E \simeq \mathbb{P}^1$.

The normal coordinates satisfy $v = tu$. Thus the transition of the normal line in the two standard charts is multiplication by t, the transition function of the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ in these frame conventions. Equivalently, the conormal bundle I_E/I_E^2 has transition t^{-1} and is $\mathcal{O}(1)$. Therefore $N_{E/\mathrm{Bl}_0 \mathbb{C}^2} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$. ◇

Exercise 3.36: Three concurrent lines

Solution. After a linear change of coordinates, assume none of the three lines is vertical and write them as $y = m_i x$ with distinct m_i. In the blow-up chart $(x, y) = (u, ut)$, the total transform has equation $u(t - m_i) = 0$; its strict transform is $t = m_i$. These three curves are disjoint because the constants m_i are distinct. Each meets $E = \{u = 0\}$ at $(0, m_i)$, giving three different points. The local equations $u = 0$ and $t - m_i = 0$ have independent differentials, so the intersection is transverse. A vertical line is treated in the other chart and corresponds to the point at infinity of E. ◇

Exercise 3.37: Canonical divisor of a blow-up

Solution. Choose coordinates in which the smooth center is $Z = \{z_1 = \cdots = z_r = 0\}$. In the blow-up chart corresponding to z_1, write

$$z_1 = u, \qquad z_j = uv_j \ (2 \le j \le r), \qquad z_k = z_k \ (k > r).$$

Then

$$\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = u^{r-1}\mathrm{d}u \wedge \mathrm{d}v_2 \wedge \cdots \wedge \mathrm{d}v_r \wedge \mathrm{d}z_{r+1} \wedge \cdots \wedge \mathrm{d}z_n.$$

The Jacobian therefore vanishes to order $r - 1$ along $E = \{u = 0\}$. The same calculation in every chart glues, and the transformation rule for canonical forms gives

$$K_{\widetilde{X}} = \pi^* K_X + (r - 1)E.$$

No other divisor occurs because the blow-up is an isomorphism away from E. ◇

Lecture 13

Exercise 4.6: Matrix criterion

Solution. Write $\alpha = \mathrm{i}\sum_{j,k} a_{j\bar{k}}\,\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$, where $A = (a_{j\bar{k}})$ is Hermitian. For $v \in \mathbb{C}^n$, restriction to the complex line spanned by v gives

$$\alpha(v, \bar{v}) = \sum_{j,k} a_{j\bar{k}} v_j \bar{v}_k = v^* A v.$$

Thus weak positivity is equivalent to $v^* A v \ge 0$ for every v, namely $A \ge 0$.

If $A \ge 0$, the spectral theorem gives $A = U^* \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U$, with $\lambda_j \ge 0$. For the transformed $(1, 0)$-forms η_j,

$$\alpha = \mathrm{i}\sum_j \lambda_j \eta_j \wedge \bar{\eta}_j,$$

which is strongly positive and hence positive. The general inclusions $\mathrm{SP}^1 \subset \mathrm{P}^1 \subset \mathrm{WP}^1$, together with the first paragraph, prove that all three notions coincide exactly with positive semidefiniteness of A. ◇

Exercise 4.7: Products of analytic sets

Solution. On the regular locus of $Z_1 \times Z_2$, Fubini's theorem gives, for every compactly supported test form Φ,

$$\langle [Z_1] \times [Z_2], \Phi \rangle = \int_{(Z_1)_{\mathrm{Reg}}} \int_{(Z_2)_{\mathrm{Reg}}} \Phi = \int_{(Z_1 \times Z_2)_{\mathrm{Reg}}} \Phi.$$

The singular subsets have measure zero, so this is $[Z_1 \times Z_2]$.

Let $\dim Z_i = p_i$, use the product metric $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$, and take product compact sets $K_1 \times K_2$. In the expansion of $\omega^{p_1+p_2}$, only the term with degrees p_1, p_2 survives on the product. Hence

$$\frac{1}{(p_1+p_2)!}\int_{(Z_1\cap K_1)\times(Z_2\cap K_2)} \omega^{p_1+p_2} = \left(\frac{1}{p_1!}\int_{Z_1\cap K_1} \omega_1^{p_1}\right)\left(\frac{1}{p_2!}\int_{Z_2\cap K_2} \omega_2^{p_2}\right).$$

Thus the product mass is the product of the two masses. ◇

Exercise 4.9: Minimality

Solution. On a Kähler surface, ω is closed and has comass one. Wirtinger's theorem says that its restriction to every oriented real two-plane is at most the area form, with equality exactly on complex lines. A smooth complex curve C is therefore calibrated by ω.

If C_t is a compactly supported variation of a relatively compact part of the smooth locus, then $C_t - C$ is a boundary. Stokes' theorem gives $\int_{C_t} \omega = \int_C \omega$, while calibration gives $\mathrm{Area}(C_t) \geq \int_{C_t} \omega = \mathrm{Area}(C)$. Hence the first variation at $t = 0$ vanishes. The first-variation formula is

$$\left.\frac{d}{dt}\right|_0 \mathrm{Area}(C_t) = -\int_C \langle H, V\rangle \, \mathrm{d}A$$

for arbitrary compactly supported normal variation field V. Therefore $H = 0$ on the smooth locus. ◇

Lecture 14

Exercise 4.14: Normalization and compactness

Solution. Both sequences have supremum zero on the unit ball in the limiting boundary sense. For $u_j = j^{-1}\log|z_1|$, local integrability of $\log|z_1|$ gives

$$\left\|u_j\right\|_{L^1(K)} = j^{-1}\,\|\log|z_1|\|_{L^1(K)} \longrightarrow 0$$

on every compact K. Thus $u_j \to 0$ in L^1_{loc}, and $\mathrm{dd^c} u_j = (2j)^{-1}[z_1 = 0] \rightharpoonup 0$.

For $v_j = j\log|z_1|$, every point with $|z_1| < 1$ satisfies $v_j(z) \to -\infty$; on $z_1 = 0$ the value is already $-\infty$. Thus $v_j \to -\infty$ locally pointwise, and its local L^1-norm diverges. Also $\mathrm{dd^c} v_j = (j/2)[z_1 = 0]$, whose mass is unbounded. The comparison shows why a supremum normalization alone does not give compactness without a fixed interior normalization or an exclusion of the $-\infty$ alternative. ◇

Exercise 4.15: Currents supported at a point

Solution. Let T have bidegree (p,p), so its bidimension is $(n-p, n-p)$. If $p < n$, its bidimension is positive, while its support has complex dimension zero. The support theorem then gives $T = 0$.

If $p = n$, a positive (n,n)-current is a positive measure. A positive measure supported at the origin is $c\delta_0$ for a unique $c \geq 0$, and every top-degree current is automatically closed. Therefore the only nonzero possibilities are

$$p = n, \qquad T = c\delta_0 \quad (c > 0).$$

This also explains why one cannot place a positive closed area-type current at a single point in lower bidegree. ◇

Exercise 4.17: Multiplicity in a limit

Solution. Choose one coordinate z transverse to the hypersurfaces. For $\varepsilon \neq 0$, the equation $z^m = \varepsilon$ has m distinct roots $a_1(\varepsilon), \dots, a_m(\varepsilon)$, all tending to zero. Thus

$$[z^m = \varepsilon] = \sum_{j=1}^{m} [z = a_j(\varepsilon)].$$

If η is a compactly supported $(n-1, n-1)$-test form, continuity under translation gives

$$\langle [z = a_j(\varepsilon)], \eta \rangle = \int_{z=a_j(\varepsilon)} \eta \longrightarrow \int_{z=0} \eta.$$

Summing the m limits yields $\langle [z^m = \varepsilon], \eta \rangle \to m \langle [z = 0], \eta \rangle$. Hence the currents converge to $m[z = 0]$; the multiplicity survives even though the supports converge set-theoretically to one hyperplane. $\diamond$

Lecture 15

Exercise 4.23: Monge–Ampère of a maximum

Solution. Let $u(z) = \max(\log |z|, 0)$. It is harmonic on $|z| < 1$ and on $|z| > 1$. For a radial function, $\Delta u = r^{-1}(ru')'$. Here ru' jumps from 0 to 1 at $r = 1$, so distributionally

$$\Delta u = \frac{1}{r} \delta_{r=1}.$$

Since $\mathrm{dd^c} u = (4\pi)^{-1} \Delta u \, \mathrm{d}x \wedge \mathrm{d}y$,

$$\mathrm{dd^c} \max(\log |z|, 0) = \frac{\mathrm{d}\theta}{4\pi} \quad \text{on } |z| = 1.$$

All the mass is uniformly distributed on the unit circle, and the total mass is $1/2$. Had the exercise used $\log |z|^2$, the total mass would be 1, in accordance with Poincaré–Lelong. $\diamond$

Exercise 4.24: Comparison in one dimension

Solution. First assume u, v are smooth and $u \geq v$ near the boundary. For a regular value $\varepsilon > 0$, set $E_\varepsilon = \{u + \varepsilon < v\} \Subset \Omega$ and $w = v - u - \varepsilon$. Then $w > 0$ in E_ε, $w = 0$ on its boundary, and its outward normal derivative is nonpositive. Green's formula gives

$$\int_{E_\varepsilon} \mathrm{dd^c}(v - u) = \frac{1}{4\pi} \int_{\partial E_\varepsilon} \frac{\partial w}{\partial n} \, \mathrm{d}s \leq 0.$$

Therefore $\int_{E_\varepsilon} \mathrm{dd^c} v \leq \int_{E_\varepsilon} \mathrm{dd^c} u$. Approximate arbitrary levels by regular values and let $\varepsilon \downarrow 0$. For bounded subharmonic u, v, use decreasing smooth regularizations on relatively compact subdomains; weak convergence of their Riesz measures and monotone continuity then give

$$\int_{\{u<v\}} \mathrm{dd^c} v \leq \int_{\{u<v\}} \mathrm{dd^c} u.$$

This is precisely the local comparison principle for $n = 1$. $\diamond$

Exercise 4.25: Bounded potentials

Solution. Normalize u by $\sup_X u = 0$ and suppose $-M \le u \le 0$. For every integer $j > M$, the truncation $u_j = \max(u, -j)$ equals u. Thus the non-pluripolar product agrees with the ordinary Bedford–Taylor measure $(\omega + dd^c u)^n$. Approximation by smooth bounded potentials and Stokes' theorem give

$$\int_X (\omega + dd^c u)^n = \int_X \omega^n,$$

so $u \in \mathcal{E}(X, \omega)$. Finally,

$$\int_X |u| \, \mathrm{MA}(u) \le M \int_X \omega^n < \infty.$$

Hence $u \in \mathcal{E}^1(X, \omega)$. ◇

Lecture 16

Exercise 4.32: Lelong number of an integration current

Solution. At a smooth point $x \in A$ of complex dimension p, choose holomorphic coordinates in which A is tangent to $\mathbb{C}^p \times \{0\}$. After rescaling by r^{-1}, the submanifold converges in C^1 on compact sets to that plane. Wirtinger's formula then shows that its normalized mass ratio converges to the mass ratio of the plane, which is one by the normalization of the Lelong number. Hence $\nu([A], x) = 1$.

For the cusp $y^2 = x^3$, the normalization is $t \mapsto (t^2, t^3)$. A generic projection to the x-axis has degree two near the origin, so the multiplicity is two. Equivalently, the lowest homogeneous term of the equation is y^2, and the tangent cone is the x-axis counted twice. Therefore $\nu([A], 0) = 2$. ◇

Exercise 4.33: A singularity with zero Lelong number

Solution. Write $t = \log r < 0$ and $\chi(t) = -(-t)^\alpha$. Then

$$\chi'(t) = \alpha(-t)^{\alpha-1} > 0, \qquad \chi''(t) = \alpha(1-\alpha)(-t)^{\alpha-2} > 0.$$

A radial function $\chi(\log|z|)$ is subharmonic on the punctured disc exactly when χ is convex; thus u is subharmonic there. Since $u \to -\infty$ at zero and is not identically $-\infty$, defining $u(0) = -\infty$ gives its natural subharmonic extension.

Let $s = -\log r \to \infty$. The logarithmic slope is

$$\frac{u(r)}{\log r^2} = \frac{-s^\alpha}{-2s} = \frac{1}{2} s^{\alpha-1} \longrightarrow 0.$$

Therefore $\nu(u, 0) = 0$, even though u is unbounded at the origin. ◇

Exercise 4.35: A plane branch

Solution. Under the rescaling appropriate to the lowest order, the parametrization (t^m, t^k), with $m < k$, approaches the x-axis. A generic projection to the x-axis is $t \mapsto t^m$, an m-sheeted map away from the origin. Therefore the tangent cycle is

$$C_0 A = m[\{y = 0\}],$$

not merely the reduced x-axis. The Lelong number equals this analytic multiplicity, so $\nu([A], 0) = m$. The coprimality assumption ensures that the parametrization is primitive and the branch is irreducible rather than a multiple parametrization of a simpler branch. ◇

Lecture 17

Exercise 5.7: Direct sum, tensor product, and dual

Solution. For sections s of E, t of F, and λ of E^*, define

$$D_{E\oplus F}(s,t) = (D_E s, D_F t),$$

$$D_{E\otimes F}(s\otimes t) = D_E s\otimes t + s\otimes D_F t,$$

and characterize the dual connection by

$$\mathrm{d}(\lambda(s)) = (D_{E^*}\lambda)(s) + \lambda(D_E s).$$

Squaring and using the graded Leibniz rule, the two mixed tensor-product terms cancel. Thus

$$\Theta_{E\oplus F} = \Theta_E\oplus\Theta_F, \qquad \Theta_{E\otimes F} = \Theta_E\otimes \mathrm{id}_F + id_E\otimes\Theta_F.$$

For the dual, applying D^2 to the scalar pairing gives

$$(\Theta_{E^*}\lambda)(s) = -\lambda(\Theta_E s).$$

In a frame this is $\Theta_{E^*} = -{}^t\Theta_E$. ◇

Exercise 5.8: Transition functions of a line bundle

Solution. Refine the cover so that each $g_{\alpha\beta}$ has a holomorphic logarithm and write

$$g_{\alpha\beta} = \exp(2\pi\mathrm{i} f_{\alpha\beta}).$$

On a triple overlap, the multiplicative cocycle identity implies

$$n_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}\in\mathbb{Z}.$$

The integers form a Čech two-cocycle. Changing the logarithms changes n by an integral coboundary, so $[n]\in H^2(X,\mathbb{Z})$ is well defined. It is exactly the connecting homomorphism

$$H^1(X,\mathcal{O}_X^*)\longrightarrow H^2(X,\mathbb{Z})$$

in the exponential sequence.

This connecting class is the topological first Chern class $c_1(L)$. To compare with Chern–Weil theory, choose a Hermitian metric and local connection forms. Their differences on overlaps are $g_{\alpha\beta}^{-1}dg_{\alpha\beta}$; the normalized curvature $(\mathrm{i}/2\pi)\Theta$ is global, and the Čech–de Rham comparison sends its de Rham class to the same integer cocycle n. Thus both constructions give $c_1(L)$. ◇

Exercise 5.9: Flat line bundles on a circle

Solution. Since $\pi_1(S^1) = \mathbb{Z}$, a rank-one complex representation is determined by the image $\lambda\in\mathbb{C}^*$ of 1. The associated flat line bundle is

$$(\mathbb{R}\times\mathbb{C})/((t+1,v)\sim(t,\lambda v)).$$

Parallel transport once around the circle multiplies a fiber by λ, and a flat bundle with connection is recovered from this monodromy. Since $\mathbb{C}^*$ is abelian, conjugacy does not further identify different λ's.

A parallel Hermitian norm must satisfy $|\lambda v| = |v|$, so necessarily $|\lambda| = 1$. Conversely, when $|\lambda| = 1$, the standard norm on $\mathbb{R}\times\mathbb{C}$ is invariant under the gluing and descends to a parallel Hermitian metric. Thus precisely the unitary monodromies admit such a metric. ◇

Lecture 18

Exercise 5.16: Curvature of $\mathcal{O}(1)$

Solution. On U_0, write a line as $[1 : w_1 : \cdots : w_n]$. The tautological frame $e = (1, w)$ of $\mathcal{O}(-1)$ has squared norm

$$|e|^2 = 1 + |w|^2.$$

The dual frame e^* of $\mathcal{O}(1)$ therefore has

$$|e^*|^2 = (1 + |w|^2)^{-1} = e^{-\varphi}, \qquad \varphi = \log(1 + |w|^2).$$

The line-bundle curvature formula gives

$$c_1(\mathcal{O}(1), h_{\mathrm{FS}}) = \mathrm{dd^c}\varphi = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log(1 + |w|^2) = \omega_{\mathrm{FS}}.$$

The potentials differ by logarithms of transition functions on overlaps, so their $\mathrm{dd^c}$'s glue globally. ⋄

Exercise 5.17: A curved metric on the trivial line bundle

Solution. Here $|1|_h^2 = e^{-\varphi}$ with $\varphi = |z|^2$. Hence

$$\Theta(L, h) = \partial\bar{\partial}|z|^2 = \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j, \qquad c_1(L, h) = \mathrm{dd^c}|z|^2.$$

This curvature is nonzero and positive. Nevertheless the underlying bundle has the global nowhere-vanishing frame 1, so it is topologically and holomorphically trivial. There is no contradiction: topology fixes only the cohomology class of the curvature. The displayed Chern form is exact, so its de Rham class is zero even though the chosen connection is not flat. ⋄

Exercise 5.19: Projective space

Solution. Put $\omega = [\omega_{\mathrm{FS}}]$, so $\int_{\mathbb{P}^1} \omega = 1$. In complex dimension one,

$$\mathrm{ch}(\mathcal{O}(k)) = e^{k\omega} = 1 + k\omega, \qquad \mathrm{td}(T_{\mathbb{P}^1}) = 1 + \frac{1}{2}c_1(T_{\mathbb{P}^1}) = 1 + \omega.$$

Riemann–Roch takes the degree-two component of the product:

$$\chi(\mathbb{P}^1, \mathcal{O}(k)) = \int_{\mathbb{P}^1} (1 + k\omega)(1 + \omega) = \int_{\mathbb{P}^1} (k + 1)\omega = k + 1.$$

The formula holds for every integer k, including the range in which H^1 rather than H^0 contributes. ⋄

Lecture 19

Exercise 5.24: The quotient line bundle

Solution. At $[\ell] \in \mathbb{P}^n$, restriction of linear functionals gives a quotient

$$(\mathbb{C}^{n+1})^* \times \mathbb{P}^n \longrightarrow \ell^* = \mathcal{O}(1)_{[\ell]}.$$

Equip the trivial bundle with its flat Hermitian metric and $\mathcal{O}(1)$ with the quotient metric. The ambient curvature is zero. The quotient curvature formula therefore says that, for a tangent vector ξ and a quotient vector q,

$$\langle \mathrm{i}\Theta_{\mathcal{O}(1)}(\xi,\bar{\xi})q,q\rangle = \|\beta_\xi^* q\|^2 \geq 0.$$

If $\xi \neq 0$ and $q \neq 0$, variation of the line ℓ makes the second fundamental form nonzero, so the inequality is strict. In the affine frame this quotient metric is $(1+|w|^2)^{-1}$; its curvature is exactly ω_{FS}. Thus the quotient formula recovers positivity of $\mathcal{O}(1)$. ⋄

Exercise 5.25: Tensor-product curvature

Solution. The tensor-product connection is $D(s\otimes t) = D_E s\otimes t + s\otimes D_F t$. Applying D again, the two mixed terms occur with opposite signs because $D_E s$ has degree one. They cancel, leaving

$$\Theta(E\otimes F) = \Theta(E)\otimes \mathrm{id}_F + id_E \otimes \Theta(F).$$

For a fixed tangent vector ξ, Griffiths semipositivity says that the Hermitian endomorphisms $A_E = \mathrm{i}\Theta_E(\xi,\bar{\xi})$ and $A_F = \mathrm{i}\Theta_F(\xi,\bar{\xi})$ are positive semidefinite. On $E\otimes F$, the curvature endomorphism is the Kronecker sum $A_E\otimes I + I\otimes A_F$, whose eigenvalues are all sums $\lambda_i(A_E)+\lambda_j(A_F)\geq 0$. Hence tensor products preserve Griffiths semipositivity. If one factor is strictly positive and the other semipositive, the tensor product is strictly Griffiths positive. ⋄

Lecture 20

Exercise 5.31: The Veronese embedding

Solution. A basis of $H^0(\mathbb{P}^n,\mathcal{O}(k))$ is formed by the monomials $z^\alpha = z_0^{\alpha_0}\cdots z_n^{\alpha_n}$ with $|\alpha| = k$. Thus the Kodaira map is

$$\nu_k([z_0:\cdots:z_n]) = [z^\alpha]_{|\alpha|=k} \in \mathbb{P}^{\binom{n+k}{k}-1}.$$

Scaling z by λ scales every coordinate by λ^k, so the map is well defined. On the chart $z_i\neq 0$, divide all target coordinates by z_i^k. Among the resulting coordinate ratios are $z_i^{k-1}z_j/z_i^k = z_j/z_i$ for every $j\neq i$. Hence the original affine coordinates can be recovered from the image. This proves injectivity locally and shows that the differential has full rank. The projective map is therefore an injective immersion; compactness of $\mathbb{P}^n$ makes it an embedding. It is precisely the degree-k Veronese embedding. ⋄

Exercise 5.32: Integral Kähler classes

Solution. The relevant part of the exponential long exact sequence is

$$H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X).$$

The first group is $\mathrm{Pic}(X)$. Exactness says that an integral class is the first Chern class of a holomorphic line bundle exactly when its image in $H^2(X,\mathcal{O}_X)$ vanishes.

On a compact Kähler manifold, the map to $H^2(X,\mathcal{O}_X)\simeq H^{0,2}(X)$ takes the $(0,2)$-component of a de Rham class. A Kähler class is of type $(1,1)$, so this component is zero. Therefore an integral Kähler class lies in the image of c_1 and equals $c_1(L)$ for some holomorphic line bundle L. If h_0 is any smooth metric on L, the $\partial\bar{\partial}$-lemma writes $\omega - c_1(L,h_0) = \mathrm{dd^c}\psi$. The changed metric $h_0e^{-\psi}$ then has curvature $c_1(L,h_0e^{-\psi}) = \omega > 0$, so L is positive. ⋄

Exercise 5.33: Bergman density of $\mathbb{P}^1$

Solution. The Fubini–Study metric and volume form are invariant under the transitive action of $\mathrm{PU}(2)$. The induced action on $H^0(\mathbb{P}^1, \mathcal{O}(k))$ is unitary, so the basis-independent Bergman density ρ_k is invariant and therefore constant. Since $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = 1$ and $h^0(\mathbb{P}^1, \mathcal{O}(k)) = k + 1$,

$$\rho_k \int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = \int_{\mathbb{P}^1} \rho_k \omega_{\mathrm{FS}} = k + 1.$$

Thus $\rho_k \equiv k + 1$ for the normalizations in the handout. ◇

Lecture 21

Exercise 6.5: The circle

Solution. Use the coordinate $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and the standard metric. On functions, $\Delta f = -f''$, so periodic harmonic functions are constants. On one-forms, $\Delta(a(\theta)\mathrm{d}\theta) = -a''(\theta)\mathrm{d}\theta$, so harmonic one-forms are the constant multiples of $\mathrm{d}\theta$.

The decomposition is explicit. For a function f, write $f = \bar{f} + (f - \bar{f})$. The zero-mean part is coexact because one can choose a periodic b with $-b' = f - \bar{f}$, giving $f - \bar{f} = d^*(b\mathrm{d}\theta)$. For a one-form $a\mathrm{d}\theta$, let $\bar{a} = (2\pi)^{-1} \int_0^{2\pi} a\, d\theta$. The zero-mean function $a - \bar{a}$ has a periodic primitive F, and

$$a\mathrm{d}\theta = \bar{a}\mathrm{d}\theta + dF.$$

These summands are orthogonal, verifying Hodge decomposition in degrees zero and one. ◇

Exercise 6.6: Minimum norm

Solution. Let h be the harmonic representative of a class and let α be any other closed representative. Then $\alpha = h + d\beta$. Since $d^*h = 0$,

$$(h, d\beta) = (d^*h, \beta) = 0.$$

Therefore

$$\|\alpha\|_{L^2}^2 = \|h\|_{L^2}^2 + \|d\beta\|_{L^2}^2 \geq \|h\|_{L^2}^2.$$

Equality forces $d\beta = 0$, hence $\alpha = h$. Thus the harmonic representative is the unique minimizer, not merely a critical point. ◇

Exercise 6.7: Heat equation on the circle

Solution. For $\Delta = -\partial_\theta^2$, the orthonormal eigenfunctions are $(2\pi)^{-1/2}e^{ik\theta}$ with eigenvalues k^2. Hence

$$K_t(\theta, \phi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-k^2 t} e^{ik(\theta - \phi)}.$$

Termwise differentiation is valid for $t > 0$ and verifies $(\partial_t + \Delta_\theta)K_t = 0$. For $f(\phi) = \sum_k \hat{f}_k e^{ik\phi}$,

$$(e^{-t\Delta} f)(\theta) = \sum_{k \in \mathbb{Z}} e^{-k^2 t} \hat{f}_k e^{ik\theta}.$$

As $t \to \infty$, every term with $k \neq 0$ tends to zero, while the $k = 0$ term is $\hat{f}_0 = (2\pi)^{-1} \int f$. Thus the heat operator converges in L^2, and smoothly for smooth f, to the orthogonal projection onto the constant functions. ◇

Lecture 22

Exercise 6.12: Evenness of b_1

Solution. On a compact Kähler manifold, $\Delta_{\mathrm{d}} = 2\Delta_{\bar\partial}$, and the Laplacian preserves bidegree. Hence every complex harmonic one-form splits uniquely as $\alpha = \alpha^{1,0} + \alpha^{0,1}$, with both components harmonic. Therefore

$$\mathcal{H}^1(X, \mathbb{C}) = \mathcal{H}^{1,0}(X) \oplus \mathcal{H}^{0,1}(X).$$

Complex conjugation is an antilinear isomorphism $\mathcal{H}^{1,0} \to \mathcal{H}^{0,1}$, so the two spaces have equal complex dimension. By the Hodge theorem,

$$b_1 = \dim_{\mathbb{C}} \mathcal{H}^1(X, \mathbb{C}) = h^{1,0} + h^{0,1} = 2h^{1,0},$$

which is even. ◇

Exercise 6.13: A Kähler form is not exact

Solution. Suppose $\omega = d\eta$ on a compact Kähler n-fold. Since $d\omega = 0$,

$$\omega^n = d(\eta \wedge \omega^{n-1}).$$

Stokes' theorem would imply $\int_X \omega^n = 0$. But $\omega^n/n!$ is the positive Riemannian volume form, so its integral is strictly positive. This contradiction proves that $[\omega] \neq 0$ in de Rham cohomology. ◇

Exercise 6.15: A formal algebra

Solution. Let Ω be a volume form on S^n normalized to have integral one. If n is odd, the minimal algebra is $A = (\Lambda(x_n), d = 0)$; graded commutativity already gives $x^2 = 0$. The map $x \mapsto \Omega$ induces the isomorphism in degrees $0, n$, so it is a quasi-isomorphism $A \to \mathcal{A}^*(S^n)$.

If n is even, take

$$A = (\Lambda(x_n, y_{2n-1}), d), \qquad dx = 0, \quad dy = x^2.$$

The extra generator kills the unwanted square of x. Define $x \mapsto \Omega$ and $y \mapsto 0$. This respects the differential because $\Omega \wedge \Omega = 0$ on the n-manifold. The cohomology of A is one dimensional in degrees $0, n$ and zero otherwise, so the map is again a quasi-isomorphism. Composing with the projection to cohomology proves formality of the de Rham algebra of every sphere. ◇

Lecture 23

Exercise 6.21: Exponential sequence and the Picard group

Solution. The long exact sequence associated with $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$ contains

$$H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

Since $H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$, this is the requested segment. The kernel of c_1 consists of topologically trivial holomorphic line bundles and is the image of $H^1(X, \mathcal{O}_X)$. Therefore

$$\mathrm{Pic}^0(X) = \ker c_1 \simeq H^1(X, \mathcal{O}_X)/\mathrm{im}\, H^1(X, \mathbb{Z}),$$

with the quotient understood by the exact image of the integral group. On a compact Kähler manifold this quotient is a complex torus. ◇

Exercise 6.22: Explicit Čech–Dolbeault conversion

Solution. Let $\{\rho_k\}$ be a smooth partition of unity subordinate to the cover and use the additive cocycle convention $g_{jk} - g_{ik} + g_{ij} = 0$. On U_i, define the smooth function

$$b_i = \sum_k \rho_k g_{ik},$$

where a summand is used only on the overlap on which it is defined. On $U_i \cap U_j$, the cocycle identity gives

$$b_i - b_j = \sum_k \rho_k (g_{ik} - g_{jk}) = \sum_k \rho_k g_{ij} = g_{ij}.$$

Because g_{ij} is holomorphic, $\bar\partial b_i = \bar\partial b_j$ on overlaps. Thus the local forms glue to a global $(0,1)$-form α, defined by $\alpha|_{U_i} = \bar\partial b_i$. It is $\bar\partial$-closed. Changing the partition or replacing g by a Čech coboundary changes α by a global $\bar\partial$-exact form. Hence $[\alpha]_{\bar\partial}$ is exactly the Dolbeault class corresponding to $[g]$. ◇

Exercise 6.24: A two-term complex

Solution. The complex $\mathcal{K}^\bullet = [\mathcal{F} \xrightarrow{u} \mathcal{G}]$ fits into a distinguished triangle whose associated long exact sequence begins

$$0 \longrightarrow \mathbb{H}^0(X, \mathcal{K}^\bullet) \longrightarrow H^0(X, \mathcal{F}) \xrightarrow{H^0(u)} H^0(X, \mathcal{G}) \longrightarrow \mathbb{H}^1(X, \mathcal{K}^\bullet) \longrightarrow H^1(X, \mathcal{F})$$
$$\xrightarrow{H^1(u)} H^1(X, \mathcal{G}) \longrightarrow \mathbb{H}^2(X, \mathcal{K}^\bullet) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow \cdots .$$

In particular, $\mathbb{H}^0 = \ker(H^0(\mathcal{F}) \to H^0(\mathcal{G}))$, while $\mathbb{H}^1$ is an extension of the cokernel in degree zero by the kernel in degree one. This displays both the sections and the obstruction created by the map u. ◇

Lecture 24

Exercise 6.29: Line bundles of high degree

Solution. Serre duality gives

$$H^1(X, L)^* \simeq H^0(X, K_X \otimes L^{-1}).$$

The latter bundle has degree $2g - 2 - \deg L < 0$. A nonzero holomorphic section would have an effective zero divisor of that negative degree, which is impossible. Hence both $H^0(K_X \otimes L^{-1})$ and $H^1(X, L)$ vanish. Riemann–Roch now yields

$$h^0(L) = h^0(L) - h^1(L) = \deg L + 1 - g.$$

◇

Exercise 6.30: Euler characteristic

Solution. For any finite-dimensional cochain complex, the alternating sum of the dimensions of the cochain spaces equals the alternating sum of the dimensions of its cohomology. Apply this on each page of the Frölicher spectral sequence, with total degree $p+q$. Passing from E_r to $E_{r+1} = H(E_r, d_r)$ does not change the total alternating sum. Therefore

$$\sum_{p,q} (-1)^{p+q} \dim E_1^{p,q} = \sum_{p,q} (-1)^{p+q} \dim E_\infty^{p,q}.$$

The left side is $\sum_{p,q} (-1)^{p+q} h^{p,q}$. The E_∞-terms are the graded pieces of a filtration of $H^{p+q}_{\mathrm{dR}}(X, \mathbb{C})$, so the right side is $\sum_k (-1)^k b_k = \chi_{\mathrm{top}}(X)$. Degeneration at E_1 is not needed because Euler characteristic is unchanged by every differential. ◇

Exercise 6.33: Degree-zero line bundles

Solution. Fix the base point p_0. For a holomorphic one-form ω, the Abel–Jacobi image of $p - q$ is

$$\int_q^p \omega \mod \text{periods}.$$

Choose paths from p_0 to p and q. Concatenating the first with the reverse of the second gives a path from q to p, so

$$\int_q^p \omega = \int_{p_0}^p \omega - \int_{p_0}^q \omega \mod H_1(X, \mathbb{Z}).$$

Changing any path adds a period and hence does not change the Jacobian point. Thus the class of the divisor $p - q$ is exactly $AJ(p) - AJ(q)$. ⋄

Lecture 25

Exercise 7.5: Eigenvalues of the curvature operator

Solution. Let

$$e_J = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_J, \qquad |J| = q.$$

Write $L_j = \mathrm{i}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j \wedge$ and let Λ_j be the corresponding contraction. On a form already containing every $\mathrm{d}z_j$, the commutator $[L_j, \Lambda_j]$ contributes one exactly when $\mathrm{d}\bar{z}_j$ is also present, and contributes zero otherwise. Consequently

$$[\mathrm{i}\Theta(L), \Lambda]e_J = \sum_{j=1}^{n} \lambda_j [L_j, \Lambda_j] e_J = \left(\sum_{j \in J} \lambda_j \right) e_J.$$

Thus the standard (n, q)-basis diagonalizes B, with eigenvalues the sums of the q curvature eigenvalues indexed by J. In particular, if every $\lambda_j \geq c$, then $B \geq qcI$. ⋄

Exercise 7.6: The non-Kähler warning

Solution. On a Hermitian manifold define the zeroth-order torsion operator

$$\tau = [\Lambda, \partial\omega \wedge \cdot].$$

The Kähler commutators acquire τ and τ^*; for example, $[\Lambda, D''] = -\mathrm{i}(D'^* + \tau^*)$, with the conjugate identity for $[\Lambda, D']$. A useful complete form of the Hermitian identity is

$$\Delta'' = \Delta'_\tau + [\mathrm{i}\Theta(E), \Lambda] + T_\omega, \qquad \Delta'_\tau = [D' + \tau, (D' + \tau)^*],$$

where

$$T_\omega = [\Lambda, [\Lambda, \tfrac{\mathrm{i}}{2}\partial\bar{\partial}\omega \wedge \cdot]] - [\partial\omega \wedge \cdot, (\partial\omega \wedge \cdot)^*]$$

is a zeroth-order operator. Equivalently, expanding Δ'_τ displays the first-order commutators involving D', D'', and τ^*.

If $\partial\omega = 0$, reality gives $\bar{\partial}\omega = 0$, so $\partial\bar{\partial}\omega = 0$, $\tau = 0$, and $T_\omega = 0$. The formula then reduces exactly to the Kähler BKN identity $\Delta'' = \Delta' + [\mathrm{i}\Theta(E), \Lambda]$. ⋄

Exercise 7.8: Bochner on functions

Solution. Write $\Delta_{\mathrm{geom}} = \operatorname{div}\nabla = -\Delta_{\mathrm{d}}$ on functions, as specified in the exercise. At a point choose a geodesic orthonormal frame e_i. Differentiating $|df|^2 = \sum_i (e_i f)^2$ twice gives the square $|\operatorname{Hess} f|^2$ and the term $\langle df, d\Delta_{\mathrm{geom}} f\rangle$. Commuting the remaining third covariant derivatives uses

$$\nabla_i \nabla_j \nabla_i f - \nabla_j \nabla_i \nabla_i f = R_{iji}{}^{k} \nabla_k f,$$

whose contraction is $\operatorname{Ric}(\nabla f, \nabla f)$. Thus

$$\frac{1}{2}\Delta_{\mathrm{geom}}|df|^2 = |\operatorname{Hess} f|^2 + \langle df, d\Delta_{\mathrm{geom}} f\rangle + \operatorname{Ric}(\nabla f, \nabla f).$$

If f is harmonic on a compact connected manifold, the maximum principle already gives that f is constant; equivalently, $0 = -\int f\Delta_{\mathrm{geom}} f = \int |df|^2$. Under the additional assumption $\operatorname{Ric} \geq 0$, integrating the displayed Bochner identity also forces $\operatorname{Hess} f = 0$ and the Ricci term to vanish, hence again $df = 0$. ◇

Lecture 26

Exercise 7.14: Check on $\mathbb{P}^1$

Solution. Every positive line bundle on $\mathbb{P}^1$ is $L = \mathcal{O}(k)$ with $k > 0$, and $K_{\mathbb{P}^1} = \mathcal{O}(-2)$. Kodaira vanishing predicts

$$H^1(\mathbb{P}^1, K_{\mathbb{P}^1} \otimes L) = H^1(\mathbb{P}^1, \mathcal{O}(k-2)) = 0.$$

Lecture 23 computed $h^1(\mathcal{O}(m)) = \max\{-m-1, 0\}$. Substituting $m = k-2$ gives $\max\{1-k, 0\} = 0$ for every $k \geq 1$, verifying the theorem, including the borderline case $k = 1$. ◇

Exercise 7.15: A basic Fano consequence

Solution. Apply Kodaira vanishing with the positive line bundle $L = -K_X = K_X^{-1}$. Since $K_X \otimes L \simeq \mathcal{O}_X$, it gives

$$H^q(X, \mathcal{O}_X) = 0 \qquad (q > 0).$$

On a compact connected complex manifold every holomorphic function is constant, so $h^0(X, \mathcal{O}_X) = 1$. Therefore

$$\chi(\mathcal{O}_X) = \sum_q (-1)^q h^q(X, \mathcal{O}_X) = 1.$$

◇

Exercise 7.17: Fundamental group

Solution. Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of complex dimension $m = n-1 \geq 2$. The homotopy Lefschetz theorem says that the inclusion $Y \hookrightarrow \mathbb{P}^n$ induces an isomorphism on π_j for $j < m$ and a surjection for $j = m$. Since $1 < m$,

$$\pi_1(Y) \simeq \pi_1(\mathbb{P}^n) = 0.$$

Thus Y is simply connected. The condition $m \geq 2$ is essential: smooth plane curves can have positive genus and nontrivial fundamental group. ◇

Lecture 27

Exercise 7.19: Gaussian weight

Solution. For $\varphi = t|z|^2$,

$$H_\varphi = (\varphi_{j\bar{k}}) = tI, \qquad H_\varphi^{-1} = t^{-1}I.$$

Hörmander's estimate therefore gives

$$\|u\|_t^2 = \int |u|^2 e^{-t|z|^2}\, d\lambda \le \frac{1}{t}\int |f|^2 e^{-t|z|^2}\, d\lambda = t^{-1}\|f\|_t^2.$$

If the weight is $e^{-2t|z|^2}$, the actual exponent is $\widetilde{\varphi} = 2t|z|^2$, so $H_{\widetilde{\varphi}} = 2tI$. The sharp coefficient furnished by the same argument is therefore $(2t)^{-1}$, not t^{-1}. ◇

Exercise 7.20: The compatibility condition

Solution. If $\bar\partial u = f$ in distributions, then

$$\bar\partial f = \bar\partial^2 u = 0.$$

Thus $\bar\partial$-closedness is necessary independently of estimates or boundary conditions. In $\mathbb{C}^n$ with $n \ge 2$, take

$$f = \bar z_1\, \mathrm{d}\bar z_2.$$

Then

$$\bar\partial f = \mathrm{d}\bar z_1 \wedge \mathrm{d}\bar z_2 \ne 0.$$

Consequently no distribution, and hence no L^2 function, can solve $\bar\partial u = f$. ◇

Exercise 7.21: The unit disc

Solution. Summing the rank-one kernels of the stated orthonormal basis gives

$$K_{\mathbb{D}}(z,w) = \frac{1}{\pi}\sum_{k=0}^{\infty}(k+1)(z\bar w)^k = \frac{1}{\pi(1-z\bar w)^2}.$$

In particular, $K_{\mathbb{D}}(z,z) = \left[\pi(1-|z|^2)^2\right]^{-1}$.

For every $f \in A^2(\mathbb{D})$, the reproducing identity and Cauchy–Schwarz give

$$|f(z)| = |\langle f, K_z\rangle| \le \|f\|\,\|K_z\| = \|f\|\sqrt{K(z,z)}.$$

Equality for unit norm is attained by $f = K_z/\sqrt{K(z,z)}$. Therefore

$$K(z,z) = \sup\{|f(z)|^2 : f \in A^2(\mathbb{D}),\ \|f\| \le 1\},$$

which verifies the extremal characterization. ◇

Lecture 28

Exercise 7.25: A one-dimensional multiplier ideal

Solution. Let a nonzero germ f have order m, so $|f(z)|^2 \asymp r^{2m}$. Since $e^{-c\log|z|^2} = r^{-2c}$, local integrability is equivalent to

$$\int_0^\varepsilon r^{2m-2c} r\, dr < \infty.$$

This holds exactly when $2m - 2c + 1 > -1$, or $m > c - 1$. For $c \geq 0$ and integral m, this is $m \geq \lfloor c \rfloor$. Hence

$$\mathcal{I}(c\log|z|^2)_0 = (z^{\lfloor c \rfloor}).$$

The zero germ is of course included automatically. ◇

Exercise 7.26: Complete cutoffs

Solution. Fix $o \in M$ and let $r(x) = d(o, x)$. Completeness and Hopf–Rinow imply that closed r-balls are compact, so r is proper; it is also one-Lipschitz. Choose $\eta \in C^\infty(\mathbb{R})$ with $0 \leq \eta \leq 1$, $\eta = 1$ on $(-\infty, 1]$, $\eta = 0$ on $[2, \infty)$, and $|\eta'| \leq C$. The Lipschitz functions $\eta(r/\nu)$ already have compact support and satisfy the desired gradient bound almost everywhere, but are not smooth on the cut locus.

Use the standard smoothing theorem for Lipschitz functions to choose a smooth proper ρ with $|\rho - r| \leq 1$ and $|d\rho| \leq 2$. Then

$$\chi_\nu(x) = \eta(\rho(x)/\nu)$$

is smooth and compactly supported, tends pointwise to one, and satisfies

$$|d\chi_\nu| \leq \frac{2\|\eta'\|_\infty}{\nu}.$$

After shifting the index, it also equals one on any prescribed compact set for all sufficiently large ν. ◇

Exercise 7.29: Radial singularity

Solution. The computation in the preceding exercise gives

$$z^m \in \mathcal{I}(c\log|z|^2) \iff m > c - 1 \iff m \geq \lfloor c \rfloor.$$

If this strict inequality holds and $c > 0$, choose

$$0 < \varepsilon < \frac{m + 1 - c}{c}.$$

Then $m > (1+\varepsilon)c - 1$, so $z^m \in \mathcal{I}((1+\varepsilon)c\log|z|^2)$. When $c = 0$, every holomorphic germ is integrable and the assertion is immediate. The reverse inclusion follows from monotonicity because increasing c makes the singular weight stronger. Therefore

$$\mathcal{I}(c\log|z|^2) = \bigcup_{\varepsilon > 0} \mathcal{I}((1+\varepsilon)c\log|z|^2),$$

verifying strong openness directly in this radial case. ◇

Lecture 29

Exercise 8.4: Point extension and the Bergman kernel

Solution. Fix $p \in \Omega$ and a local frame for the weight. Apply Ohsawa–Takegoshi to the value $f(p) = 1$ on $Y = \{p\}$. It gives a holomorphic F with $F(p) = 1$ and

$$\|F\|_{\varphi}^{2} \le C_{\mathrm{OT}} e^{-\varphi(p)}.$$

The weighted Bergman kernel has the extremal characterization

$$K_{\varphi}(p,p) = \sup_{G \neq 0} \frac{|G(p)|^2}{\|G\|_{\varphi}^{2}}.$$

Using F as a competitor yields

$$K_{\varphi}(p,p) \ge \frac{1}{C_{\mathrm{OT}} e^{-\varphi(p)}} = C_{\mathrm{OT}}^{-1} e^{\varphi(p)}.$$

Equivalently, the intrinsic weighted density satisfies $K_{\varphi}(p,p)e^{-\varphi(p)} \ge C_{\mathrm{OT}}^{-1}$. Thus quantitative point extension is dual to a quantitative lower bound for evaluation. ◇

Exercise 8.5: Scaling audit

Solution. For the trivial line bundle with its flat metric, let $\sigma(z) = az$. Then $D'\sigma = a\,\mathrm{d}z$, while $|\sigma|^2 = |a|^2|z|^2$. Therefore

$$\mathrm{i}\{D'\sigma, \sigma\} \wedge \{\sigma, D'\sigma\} = |a|^4|z|^2\, \mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

up to the harmless pairing convention. On the right-hand side of the corrected inequality,

$$|\sigma|^2\, \mathrm{i}\{D'\sigma, D'\sigma\} = (|a|^2|z|^2)(|a|^2 \mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}),$$

which also scales as $|a|^4$. The factor $|\sigma|^2$ is therefore forced. If it were omitted, the right side would scale only as $|a|^2$, and the alleged inequality could not be invariant under replacing a by a large constant. ◇

Exercise 8.6: Approximate delta mass

Solution. On the disc $|w| < R$, use $w = re^{\mathrm{i}\theta}$ and $\mathrm{i}\, \mathrm{d}w \wedge \mathrm{d}\bar{w} = 2r\, \mathrm{d}r\, \mathrm{d}\theta$. The mass is

$$\int_{|w|<R} \frac{\varepsilon^2 \mathrm{i}\, \mathrm{d}w \wedge \mathrm{d}\bar{w}}{(|w|^2 + \varepsilon^2)^2} = 2\pi \frac{R^2}{R^2 + \varepsilon^2} \le 2\pi.$$

For a compactly supported continuous test function χ, substitute $w = \varepsilon\zeta$:

$$\frac{1}{2\pi} \int \chi(w) \frac{\varepsilon^2 \mathrm{i}\, \mathrm{d}w \wedge \mathrm{d}\bar{w}}{(|w|^2 + \varepsilon^2)^2} = \frac{1}{2\pi} \int \chi(\varepsilon\zeta) \frac{\mathrm{i}\, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{(1 + |\zeta|^2)^2}.$$

The last density has total mass 2π. Split the integral into a large fixed disc and its uniformly small tail, then use dominated convergence on the disc. The limit is $\chi(0)$. Hence the forms divided by 2π converge weakly to δ_0. ◇

Lecture 30

Exercise 8.9: Linearization

Solution. Let $G(t) = (g_{j\bar{k}} + t\psi_{j\bar{k}})$. Jacobi's determinant formula gives

$$\frac{d}{dt}\det G(t) = \det G(t)\ \mathrm{tr}(G(t)^{-1}G'(t)).$$

Dividing by $\det G(t)$ and evaluating at zero yields

$$\left.\frac{d}{dt}\right|_0 \log\det(g_{j\bar{k}} + t\psi_{j\bar{k}}) = \mathrm{tr}(g^{-1}(\psi_{j\bar{k}})) = g^{j\bar{k}}\psi_{j\bar{k}} = \Delta_\omega\psi.$$

This is why the linearization of the logarithmic Monge–Ampère operator is the Laplacian of the current metric, not the fixed background Laplacian along the entire continuity path. ⋄

Exercise 8.10: Complex dimension one

Solution. When $\dim_{\mathbb{C}} X = 1$, the Monge–Ampère equation has no nonlinear wedge power:

$$\omega + \mathrm{i}\partial\bar{\partial}\varphi = e^F\omega.$$

Thus

$$\mathrm{i}\partial\bar{\partial}\varphi = (e^F - 1)\omega,$$

which is a Poisson equation $\Delta_\omega\varphi = c(e^F - 1)$, with the constant depending only on the Laplacian convention. Its solvability condition is $\int_X (e^F - 1)\omega = 0$, exactly the prescribed volume condition. Elliptic theory then gives a solution unique modulo constants, fixed by $\sup\varphi = 0$.

In higher dimension the determinant couples all eigenvalues of the complex Hessian, and one needs separate C^0, trace, and fully nonlinear regularity estimates. In dimension one the equation is linear, so these genuinely nonlinear difficulties are absent. ⋄

Exercise 8.11: First variation of volume

Solution. Differentiate under the integral and use $\dot{\omega}_{\varphi_t} = \mathrm{i}\partial\bar{\partial}\dot{\varphi}_t$:

$$\frac{d}{dt}\int_X \omega_{\varphi_t}^n = n\int_X \mathrm{i}\partial\bar{\partial}\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1}.$$

Because $d\omega_{\varphi_t} = 0$, the integrand is exact:

$$\mathrm{i}\partial\bar{\partial}\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1} = d\big(\mathrm{i}\bar{\partial}\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1}\big)$$

up to the fixed sign convention. Stokes' theorem on compact X makes the integral zero. Hence the volume depends only on the Kähler cohomology class and is constant along the path. ⋄

Lecture 31

Exercise 8.17: Nef test on a surface

Solution. On $X = \mathrm{Bl}_p\,\mathbb{P}^2$, the intersection relations are $H^2 = 1$, $H \cdot E = 0$, and $E^2 = -1$. For $D = aH - bE$,

$$D \cdot H = a, \qquad D \cdot E = b, \qquad D \cdot (H - E) = a - b.$$

The cone of curves is generated by E and $H - E$ (and $H = (H - E) + E$ adds no new inequality). Thus Kleiman's criterion gives

$$D \text{ nef} \iff a \geq b \geq 0.$$

Strict positivity on the two extremal curve classes gives $b > 0$ and $a - b > 0$; then $D^2 = a^2 - b^2 > 0$. The Nakai–Moishezon criterion therefore gives

$$D \text{ Kähler} \iff a > b > 0.$$

$\diamond$

Exercise 8.18: Compatibility of truncations

Solution. Let $V_j = V_{\theta_j}$ and $U_k = \bigcap_j \{u_j > V_j - k\}$. If $k < \ell$, then on the open set U_k one has

$$u_j^{(k)} = \max(u_j, V_j - k) = u_j = \max(u_j, V_j - \ell) = u_j^{(\ell)}$$

for every j. Bedford–Taylor products are local in the plurifine topology, in particular local on ordinary open sets: bounded psh potentials that agree on an open set have equal wedge products there. Hence

$$1_{U_k} \bigwedge_j (\theta_j + dd^c u_j^{(k)}) = 1_{U_k} \bigwedge_j (\theta_j + dd^c u_j^{(\ell)}).$$

This compatibility makes the increasing limit in the definition of the non-pluripolar product unambiguous. $\diamond$

Exercise 8.20: A two-curve decomposition

Solution. Put $q_{ij} = C_i \cdot C_j$, $b_j = \alpha \cdot C_j$, and $Q = (q_{ij})$. The orthogonality equations are

$$Q \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Negative definiteness gives $q_{11} < 0$, $q_{22} < 0$, and $\Delta = q_{11}q_{22} - q_{12}^2 > 0$. Therefore

$$a_1 = \frac{q_{22}b_1 - q_{12}b_2}{\Delta}, \qquad a_2 = \frac{q_{11}b_2 - q_{12}b_1}{\Delta}.$$

The negative part is effective exactly when

$$q_{22}b_1 - q_{12}b_2 \geq 0, \qquad q_{11}b_2 - q_{12}b_1 \geq 0.$$

When distinct irreducible curves have $q_{12} \geq 0$, these inequalities make explicit how sufficiently negative intersections of α with the curves produce positive coefficients in $N = a_1C_1 + a_2C_2$. $\diamond$

Lecture 32

Exercise 8.22: Capstone calculation

Solution. The total transform of a degree-d curve has class dH, and multiplicity m at the blown-up point contributes mE. Hence the strict transform is

$$\widetilde{C} = dH - mE.$$

Using $H^2 = 1$, $H \cdot E = 0$, and $E^2 = -1$,

$$\widetilde{C} \cdot E = m, \qquad \widetilde{C}^2 = d^2 - m^2.$$

Since $K_X = -3H + E$, adjunction gives

$$p_a(\widetilde{C}) = 1 + \frac{1}{2}\widetilde{C} \cdot (\widetilde{C} + K_X) = 1 + \frac{1}{2}(d^2 - 3d + m - m^2).$$

Equivalently,

$$p_a(\widetilde{C}) = \frac{(d-1)(d-2)}{2} - \frac{m(m-1)}{2}.$$

This is the plane arithmetic genus minus the numerical correction $m(m-1)/2$ contributed by the multiplicity at the blown-up point; the adjunction calculation confirms all signs, especially $E^2 = -1$. ⋄

Exercise 8.23: Final project outline

Solution (one complete model). This exercise admits many correct answers. Here is a complete outline following the pluripotential-theory route.

Main theorem. State and explain the finite-energy existence principle: if (X, ω) is compact Kähler and a measure μ of total mass $\int_X \omega^n$ has finite energy, then there is a normalized $u \in \mathcal{E}^1(X, \omega)$ with $\langle(\omega + dd^c u)^n\rangle = \mu$, unique up to constants. The project should state the chosen definition of finite energy and may prove the result for measures with bounded density while giving the variational proof architecture in general.

Three prerequisite tools.

(1) Bedford–Taylor monotone convergence and the comparison principle, used first for bounded approximants and then for uniqueness.

(2) Non-pluripolar products and truncations $u_j = \max(u, -j)$, used to define the Monge–Ampère measure without charging the pole set.

(3) The Aubin–Mabuchi energy and compactness of normalized energy sublevels, used to maximize $E(u) - \int u \, d\mu$ and pass to a limit.

Normalization trap. Record both $d^c = (4\pi i)^{-1}(\partial - \bar\partial)$ and $\int_X \mathrm{MA}(u) = \int_X \omega^n$. Adding a constant to u does not change $\mathrm{MA}(u)$, so impose $\sup_X u = 0$. Confusing the classical product $(dd^c \log |s|^2)^n$ with the non-pluripolar bracket would incorrectly retain divisor mass on the pole set.

Explicit example. On $\mathbb{P}^1$, take a smooth positive density $\mu = e^F \omega_{\mathrm{FS}}$ of total mass one. The equation becomes the Poisson equation $dd^c u = \mu - \omega_{\mathrm{FS}}$, solved by the Green kernel and normalized by $\sup u = 0$. Then approximate a measure with a mild singular density and show how the same truncation and energy language persists.

Five-page organization. Page 1 states the theorem and defines $\mathcal{E}^1$; page 2 develops truncations and non-pluripolar mass; page 3 presents the energy functional and compactness; page 4 derives the Euler–Lagrange equation and uniqueness by comparison; page 5 works out $\mathbb{P}^1$, audits the normalization, and lists the exact point where the general proof requires deeper compactness. This meets all requested components while keeping the main logical dependency visible. ⋄