ANALYTIC SINGULARITIES ARE NOT BIRATIONALLY INVARIANT

Recall that a closed positive $(1,1)$-current T on a complex manifold X is said to have *analytic singularities* if locally around each point on X, T admits a potential like

$$c\log\Big(|f_1|^2+\cdots+|f_N|^2\Big)+\mathcal{O}(1),$$

where $c\geq 0$ is a rational number, and $f_1,\ldots,f_N$ are holomorphic functions.

By definition, pull-backs with respect to birational morphisms preserve analytic singularities. But the converse fails. Namely analytic singularities do not descend. I have explained this phenomenon in several talks. This short note provides an explicit example.

Let $X=\mathbb{P}^3$. Blow up a point $p\in X$. The exceptional divisor is $E_0\simeq\mathbb{P}^2$. Choose a smooth cubic curve $C\subseteq E_0$ (note that C is an elliptic curve) and choose nine points $p_1,\ldots,p_9\in C$ such that

$$\mathcal{O}_C(3H-p_1-\cdots-p_9)$$

is non-torsion in $\operatorname{Pic}^0(C)$, where H is a hyperplane on E_0. Blow up the nine points $p_1,\ldots,p_9$ inside $\operatorname{Bl}_p X$, and denote the resulting morphism by $\pi\colon Y\to X$. Let $S\subseteq Y$ be the strict transform of E_0, and let $F_1,\ldots,F_9$ be the exceptional divisors of the second blow-ups. Then

$$S\simeq\operatorname{Bl}_{p_1,\ldots,p_9}\mathbb{P}^2.$$

On S, let h denote the pullback of the hyperplane class on $\mathbb{P}^2$, and let e_i be the exceptional curves. Put

$$L\coloneqq 3h-e_1-\cdots-e_9.$$

Then L is nef but not semiample.

Indeed, L is represented by the strict transform C' of the cubic C. Moreover

$$L|_{C'}\simeq\mathcal{O}_C(3H-p_1-\cdots-p_9),$$

which is non-torsion of degree 0. Hence for every $m\geq 1$,

$$H^0(C',mL|_{C'})=0.$$

From the exact sequence

$$0\longrightarrow\mathcal{O}_S((m-1)L)\longrightarrow\mathcal{O}_S(mL)\longrightarrow\mathcal{O}_{C'}(mL|_{C'})\longrightarrow 0$$

one obtains inductively

$$h^0(S,mL)=1$$

for every $m\geq 1$. Thus

$$|mL|=\{mC'\}$$

for every $m\geq 1$, and consequently L is not semiample.

Now define an effective exceptional divisor on Y by

$$D\coloneqq 3S+4(F_1+\cdots+F_9).$$

A direct restriction computation gives

$$\mathcal{O}_Y(-D)|_S\simeq\mathcal{O}_S(3h-e_1-\cdots-e_9)=\mathcal{O}_S(L),$$

and

$$\mathcal{O}_Y(-D)|_{F_i}\simeq\mathcal{O}_{F_i}(1).$$

Therefore $\mathcal{O}_Y(-D)$ is π-nef. However it is not π-semiample, because its restriction to S is the non-semiample line bundle $\mathcal{O}_S(L)$.

Let ω be a Kähler form on X. We can choose a smooth Hermitian form η in $\{-D\}$, and a sufficiently small rational number $\epsilon>0$ so that

$$R\coloneqq\pi^*\omega+\varepsilon\eta$$

is a Kähler form on Y. Define $T\coloneqq\pi_*R$.

Computing the cohomology class, we have

$$\pi^*T = R + \varepsilon[D].$$

In particular, π^*T has analytic singularities.

Now we claim that T itself does not have analytic singularities. Suppose, for contradiction, that T had analytic singularities. Then, there would exist a coherent ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,p}$ and an integer $m > 0$ such that a local potential of T near p is of the form

$$\frac{1}{m} \log \left(\sum_j |f_j|^2 \right) + \mathcal{O}(1),$$

where $\mathfrak{a} = (f_j)$. We may assume that $m\epsilon$ is an integer by changing the local presentation.

Pulling back to Y, we would get

$$\log \left(\sum_j |\pi^* f_j|^2 \right) = m\varepsilon \log |s_D|^2 + \mathcal{O}(1).$$

Therefore,

$$\pi^{-1}\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-m\varepsilon D)$$

near the exceptional fibre. Hence $\mathcal{O}_Y(-m\varepsilon D)$ is generated by pullbacks of functions from X. In other words, $\mathcal{O}_Y(-D)$ is π-semiample. This is a contradiction.