Relative pluripotential theory [1]

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[1] First three chapters of an unfinished book.

Contents

Symbols and Abbreviations

psh: pluri-subharmonic. 9

Conventions and special notations

(1) Let X be a manifold. Let $f : X \to [-\infty, \infty]$ be a function, we write f^* for the usc regularization of f, namely,

$$f^*(x) := \varlimsup_{y \to x} f(y), \quad x \in X.$$

(2) Let X be a manifold. Let $f_j : X \to [-\infty, \infty]$ ($j \in J$) be a family of functions, we write

$$\sup_{j \in J}{}^* f_j := \left(\sup_{j \in J} f_j \right)^*.$$

(3) For $n \in \mathbb{Z}_{>0}$, $\mathbb{B}^n$ denotes the unit ball in $\mathbb{C}^n$. Δ^n denotes the unit polydisk in $\mathbb{C}^n$.

(4) $\xrightarrow{C}$ denotes convergence in capacity.

(5) Let X be a complex manifold, $p \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{D}'^{p,p}(X)$ denotes the space of (p,p)-currents on X, $\mathcal{D}'^{p,p}_{+}(X)$ denotes the space of *closed* positive (p,p)-currents on X.

(6) Let X be a compact Kähler manifold. Let $[\theta]$ be a big cohomology class, where θ is a closed smooth representative form. We write V_θ for the supreme of elements in $\mathrm{PSH}(X, \theta)$ that are less than 0. For $\varphi, \psi \in \mathrm{PSH}(X, \theta)$, we write $\varphi \wedge \psi$ for the rooftop operator instead of the more common $P_\theta(\varphi, \psi)$, we write $\varphi \vee \psi$ for the maximum of φ and ψ. We write $[\varphi] \wedge \psi$ instead of the more common $P[\varphi](\psi)$.

(7) Let X be a locally compact Hausdorff space. We write $\mathcal{M}(X)$ for the space of signed Radon measures on X. We write $\mathcal{M}_+(X)$ for the space of positive Radon measures on X. They are equipped with the weak star topology. We denote the weak star convergence of measures by $\rightharpoonup$. Be careful, the same notation is used for the weak star convergence of currents. For measures, they are not equivalent if X is not compact. We make distinction by writing weak convergence *as currents* for the latter notion when necessary.

(8) Let X be a compact Kähler manifold. Let $[\theta]$ be a big cohomology class, where θ is a closed smooth representative form. Let $\varphi_j \in \mathrm{PSH}(X, \theta)$ ($j \in \mathbb{Z}_{>0}$). The notations $\varlimsup \varphi_j$, $\varliminf \varphi_j$ are not the usual limsup and liminf. They are defined in Definition I.1.9. When we need the latter, we write $\limsup \varphi_j$, $\liminf \varphi_j$ instead.

(9) Kähler locus, ample locus of a big cohomology class mean the same thing.

(10) We always follow the convention:

$$\mathrm{dd^c} = \frac{\mathrm{i}}{2\pi} \partial \bar{\partial}.$$

(11) For two singular types $[\varphi]$, $[\psi]$, we write $[\varphi] \preceq [\psi]$ for the relation: φ is more singular than ψ instead of the converse.
(12) Let M be a compact Kähler manifold. Then $\mathring{\mathcal{M}}_+(X) \subseteq \mathcal{M}_+(X)$ denotes the set of non-pluripolar measures on X.

CHAPTER 1

Preliminaries

I.1. The space of pluri-subharmonic functions

In this section, we let X be a Kähler manifold of dimension n. Let θ be a smooth closed real $(1,1)$-form on X.

I.1.1. The space of quasi-psh functions. Recall the following standard notations, $\bar{\partial}$ and ∂ denote the standard Dolbeault operators on X, induced by the given complex structure on X. In terms of local holomorphic coordinates $(z_j = x_j + \mathrm{i}y_j)_{j=1}^n$, we have

$$\bar{\partial} = \frac{1}{2}\mathrm{d}\bar{z}_j \wedge \left(\frac{\partial}{\partial x_j} - \mathrm{i}\frac{\partial}{\partial y_j}\right), \quad \partial = \frac{1}{2}\mathrm{d}z_j \wedge \left(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial y_j}\right).$$

The operator $\mathrm{dd^c}$ is defined as

$$\mathrm{dd^c} := \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}.$$

We write $\mathcal{D}'^{p,p}(X)$ for the space of (p,p)-currents on X and $\mathcal{D}'^{p,p}_+(X)$ for the space of *closed* positive (p,p)-currents on X. We refer to [GZ17, Section 2.2] for their definitions.

Definition I.1.1. A θ-*pluri-subharmonic function* (or θ-psh function for short) on X is a quasi-plurisubharmonic function $\varphi : X \to [-\infty, \infty)$, such that

$$\theta + \mathrm{dd^c}\varphi \in \mathcal{D}'^{1,1}_+(X).$$

The set of θ-psh functions on X is denoted as $\mathrm{PSH}(X,\theta)$. Write

$$\overline{\mathrm{PSH}}(X,\theta) = \mathrm{PSH}(X,\theta) \cup \{-\infty\}.$$

When $\theta = 0$, we omit it from the notations and write $\mathrm{PSH}(X)$ and $\overline{\mathrm{PSH}}(X)$.

Proposition I.1.1. *Let* $\varphi, \psi \in \mathrm{PSH}(X,\theta)$.

(1)
$$\limsup_{y\to x} \varphi(y) = \varphi(x), \quad \forall x \in X.$$
(2) $\varphi \in L^p_{\mathrm{loc}}(X)$ *for any* $p \in [1,\infty)$.
(3) If $\varphi = \psi$ *a.e., then* $\varphi = \psi$.
(4) The subspace topology on $\mathrm{PSH}(X,\theta)$ *induced by the following embeddings are the same:*
$$\mathrm{PSH}(X,\theta) \subseteq L^p_{\mathrm{loc}}(X), \quad \mathrm{PSH}(X,\theta) \subseteq \mathcal{D}'(X),$$
where $p \in [1,\infty)$ *is arbitrary.*

Here $L^p_{\mathrm{loc}}(X)$ means the L^p_{loc}-space with respect to the Hausdorff measure $\mathcal{H}^n$.

Remark I.1.1. It is important to remember that a θ-psh function is not an equivalence class of functions up to values on a null set, but is indeed a definite function. The natural map $\mathrm{PSH}(X,\theta) \to L^p_{\mathrm{loc}}(X)$ is injective by Proposition I.1.1 (3), so we can write

$$\mathrm{PSH}(X,\theta) \subseteq L^p_{\mathrm{loc}}(X)\,.$$

For the proof, see [GZ17, Corollary 1.38, Proposition 1.40, Theorem 1.46, Theorem 1.48].

We shall always endow $\mathrm{PSH}(X,\theta)$ with the topology defined in Proposition I.1.1 (4).

Theorem I.1.2. *When X is compact, the following set is compact in $\mathrm{PSH}(X,\theta)$:*

$$\left\{ \varphi \in \mathrm{PSH}(X,\theta) : C_1 \leq \sup_X \varphi \leq C_2 \right\},$$

where $C_1, C_2 \in \mathbb{R}$ are constants with $C_1 \leq C_2$.

For a proof, see [GZ17, Theorem 1.46].

I.1.2. Pluripolar sets.

Definition I.1.2. A subset $E \subseteq X$ is called *complete pluripolar* in X if there is $\varphi \in \mathrm{PSH}(X)$, such that

$$E = \{\varphi = -\infty\}\,.$$

A subset $E \subseteq X$ is called *pluripolar* if for any $x \in E$, there is a neighbourhood $U \subseteq X$ of x and $\varphi \in \mathrm{PSH}(U)$, such that

$$E \cap U \subseteq \{\varphi = -\infty\}\,.$$

Definition I.1.3. A function $\varphi \in \mathrm{PSH}(X,\theta)$ is said to have *small unbounded locus* if there is a pluripolar closed set $A \subseteq X$, such that $\varphi \in L^\infty_{\mathrm{loc}}(X - A)$.

I.1.3. Singular types. Now assume that X is compact Kähler and that the cohomology class $\alpha := [\theta] \in H^{1,1}(X,\mathbb{R})$ is big. In particular, $\mathrm{PSH}(X,\theta)$ is non-empty. Let Z be the non-Kähler locus of α ([Bou04]), then Z is a proper analytic subset of X.

Definition I.1.4. Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we say that φ is *more singular* than ψ if $\varphi \leq \psi + C$ for some constant $C \in \mathbb{R}$. We write $\varphi \preceq \psi$.

If

$$\varphi \preceq \psi\,, \quad \psi \preceq \varphi\,,$$

we say that φ and ψ have *equivalent singularities* and write $[\varphi] = [\psi]$. This defines an equivalence relation on $\mathrm{PSH}(X,\theta)$. The equivalence classes containing φ is called the *singularity type* of φ and is denoted as $[\varphi]$. The relation $\preceq$ induces a partial order (still denoted by $\preceq$) on the set of singularity types $\mathrm{ST}(X,\theta)$ (or ST for short). Write

$$\overline{\mathrm{ST}} := \mathrm{ST} \cup \{[-\infty]\}\,.$$

The partial order extends to $\overline{\mathrm{ST}}$ by setting $[-\infty]$ as a least element.

Define $V_\theta \in \mathrm{PSH}(X,\theta)$:

$$V_\theta := \sup \{ \psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0 \}\,.$$

Then $[V_\theta]$ is the greatest element in $\overline{\mathrm{ST}}$. In fact, V_θ^* is obviously θ-psh and $V_\theta^* \leq 0$, so $V_\theta = V_\theta^*$ is θ-psh.

Definition I.1.5. We write

$$\mathcal{E}^\infty(X,\theta) := \{\, \varphi \in \mathrm{PSH}(X,\theta) : [\varphi] = [V_\theta] \,\}\ .$$

We say an element $\varphi \in \mathrm{PSH}(X,\theta)$ has *minimal singularities* if $\varphi \in \mathcal{E}^\infty(X,\theta)$.

Proposition I.1.3.

(1) V_θ is locally bounded on $X - Z$.
(2) Let $\varphi \in \mathcal{E}^\infty(X,\theta)$, then φ has small unbounded locus.

Proof. (1) As shown in [Bou04, Theorem 3.17], there exists $\psi \in \mathrm{PSH}(X,\theta)$ with analytic singularities, $\psi \leq 0$, such that the polar set $\{\psi = -\infty\}$ is exactly Z, so we conclude that V_θ is locally bounded from below on $X - Z$.

(2) This follows immediately from (1). □

I.1.4. Operators. Assume that X is a compact Kähler manifold of dimension n and that the cohomology class $\alpha := [\theta] \in H^{1,1}(X,\mathbb{R})$ is big.

There are several natural operations on $\overline{\mathrm{PSH}}(X,\theta)$.

Definition I.1.6. Let $\varphi_1, \varphi_2 \in \overline{\mathrm{PSH}}(X,\theta)$. Define

(1)
$$\varphi_1 \wedge \varphi_2 := \sup{}^* \left\{ \psi \in \overline{\mathrm{PSH}}(X,\theta) : \psi \leq \varphi_1, \psi \leq \varphi_2 \right\}\ .$$

(2)
$$\varphi_1 \vee \varphi_2 := \max\{\varphi_1, \varphi_2\} \in \overline{\mathrm{PSH}}(X,\theta)\,.$$

The first one is known as the *rooftop envelope*.

Remark I.1.2. It is easy to see

$$\varphi_1 \wedge \varphi_2 := \sup\{\psi \in \overline{\mathrm{PSH}}(X,\theta) : \psi \leq \varphi_1, \psi \leq \varphi_2\}.$$

It can happen that for $\varphi_1, \varphi_2 \in \mathrm{PSH}(X,\theta)$, we have $\varphi_1 \wedge \varphi_2 = -\infty$.

Definition I.1.7. Let $[\psi] \in \mathrm{ST}(X,\theta)$, $\varphi \in \overline{\mathrm{PSH}}(X,\theta)$. Define $[\psi] \wedge \varphi \in \overline{\mathrm{PSH}}(X,\theta)$ as

$$[\psi] \wedge \varphi = \sup_{C \geq 0}{}^*(\psi + C) \wedge \varphi.$$

It is easy to see that this definition does not depend on the choice of ψ. We extend the operator to the case where $[\psi] = [-\infty]$ by setting

$$[-\infty] \wedge \varphi = -\infty, \quad \forall \varphi \in \overline{\mathrm{PSH}}(X,\theta).$$

Definition I.1.8. A potential $\psi \in \mathrm{PSH}(X,\theta)$ is called a *model potential* if

$$[\psi] \wedge V_\theta = \psi.$$

A singularity type $[\psi] \in \mathrm{ST}(X,\theta)$ is called a *model singularity type* if

$$[[\psi] \wedge V_\theta] = [\psi].$$

Proposition I.1.4. *Let $\varphi, \varphi_1, \varphi_2, \varphi_3, \psi \in \overline{\mathrm{PSH}}(X,\theta)$.*

(1) $\wedge$ and $\vee$ are both associative, idempotent and commutative.

(2)
$$(\varphi \vee \psi) \wedge \psi = \psi.$$

(3)
$$(\varphi_1 \wedge \varphi_2) \vee \psi \leq (\varphi_1 \vee \psi) \wedge (\varphi_2 \vee \psi).$$

(4)
$$(\varphi_1 \vee \varphi_2) \wedge \psi \geq (\varphi_1 \wedge \psi) \vee (\varphi_2 \wedge \psi).$$

Definition I.1.9. Let $\varphi_j \in \mathrm{PSH}(X,\theta)$ ($j \in \mathbb{Z}_{>0}$). Let $\phi \in \mathrm{PSH}(X,\theta)$. Assume that

$$\phi \leq \varphi_j \leq C \tag{1.1}$$

for some constant $C > 0$ independent of j.

Then we define

(1)
$$\varliminf_{j\to\infty} \varphi_j := \sup_{j\in\mathbb{Z}_{>0}}{}^{*} \inf_{k\in\mathbb{Z}_{>0}} \varphi_j \wedge \varphi_{j+1} \wedge \cdots \wedge \varphi_{j+k}.$$

(2)
$$\varlimsup_{j\to\infty} \varphi_j := \inf_{j} \sup_{k}{}^{*} \varphi_j \vee \cdots \vee \varphi_{j+k}.$$

The condition (1.1) just makes sure that $\varliminf \varphi_j, \varlimsup \varphi_j \in \mathrm{PSH}(X,\theta)$ are well-defined. These definitions are independent of the choice of ϕ.

I.2. Local theory on hyperconvex domains

I.2.1. Bedford–Taylor capacity. Let $\Omega \subseteq \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary.

Definition I.2.10. Let $E \subseteq \Omega$ be a Borel subset. The *Bedford–Taylor capacity* of E relative to Ω is defined as

$$\mathrm{Cap}(E,\Omega) := \sup\left\{ \int_E (\mathrm{dd^c}\varphi)^n : \varphi \in \mathrm{PSH}(\Omega)\,, -1 \leq \varphi \leq 0 \right\}. \tag{1.2}$$

The theory of Bedford–Taylor capacity is studied in detail in [GZ17, Section 4.1.3].

Definition I.2.11. Let $f : \Omega \to [-\infty,\infty]$ be a function. The function f is said to be *quasi-continuous* if for any $\varepsilon > 0$ and all compact subset $K \subseteq \Omega$, there is an open set $G \subseteq \Omega$ with $\mathrm{Cap}(G,\Omega) < \varepsilon$, such that $f|_{K-G}$ is continuous.

By [GZ17, Proposition 4.18], quasi-continuity is a local property, hence we can define this notion on a general manifold.

Definition I.2.12. Let X be a complex manifold. A function $f : X \to [-\infty,\infty]$ is said to be *quasi-continuous* if for any strictly pseudoconvex open subset $\Omega \subseteq X$ with smooth boundary, the restriction of f to Ω is quasi-continuous.

Definition I.2.13. Let $f_j, f : \Omega \to [-\infty,\infty]$ ($j \in \mathbb{N}$) be Borel measurable functions. Assume that $f_j - f$ is well-defined outside a set of zero capacity. We say f_j converges to f *in capacity* if for any $\delta > 0$ and any compact set $K \subseteq \Omega$,

$$\lim_{j\to\infty} \mathrm{Cap}\left(K \cap \{|f_j - f| > \delta\}, \Omega\right) = 0\,.$$

We write $f_j \xrightarrow{C} f$ in this case.

Definition I.2.14. Let X be a complex manifold. Let $f_j, f : X \to [-\infty,\infty]$ ($j \in \mathbb{N}$) be Borel measurable functions. Assume that $f_j - f$ is well-defined outside a set of zero capacity (namely, of zero capacity on each strictly pseudoconvex domain with smooth boundary). We say f_j converges to f *in capacity* if for any strictly pseudoconvex open subset $\Omega \subseteq X$ with smooth boundary, $f_j|_\Omega \xrightarrow{C} f|_\Omega$. In this case, we write $f_j \xrightarrow{C} f$.

THEOREM I.2.5. *Let $f^k, f : \mathbb{B}^n \to \mathbb{R}$ be uniformly bounded and quasi-continuous functions. Let φ_j^k, $\varphi_j \in \mathrm{PSH}(\mathbb{B}^n)$ be uniformly bounded functions. Assume that*

$$f^k \xrightarrow{C} f, \quad \varphi_j^k \xrightarrow{C} \varphi_j$$

as $k \to \infty$. Then

$$f^k \,\mathrm{dd^c}\varphi_1^k \wedge \cdots \wedge \mathrm{dd^c}\varphi_n^k \rightharpoonup f\,\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_n\,, \quad k \to \infty$$

as currents.

See [GZ17, Theorem 4.26] for a proof.

I.2.2. Canonical approximations. Let $\Omega \subseteq \mathbb{C}^n$ be an open bounded set. Let $\mathrm{d}\lambda$ be the standard Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. Let $\mu \in \mathcal{M}_+(\Omega)$. Take a cube I containing $\bar{\Omega}$. For each $k \geq 1$, divide I into 3^{2kn} congruent semiopen cubes I_k^j ($j = 1, \dots, 3^{2kn}$). We may assume that μ puts no mass on the boundary of each I_k^j up to a small perturbating of the partition. Let

$$\mu_k := \sum_{j \in I_k} \frac{\mu(I_k^j)}{\lambda(I_k^j)} \mathbb{1}_{I_k^j}\,\lambda\,,$$

where I_k is the set of indices j such that $I_k^j \subseteq \Omega$. We call μ_k the *canonical approximations* of μ. [1]

PROPOSITION I.2.6. *When Ω has smooth boundary and $\mu(\partial\Omega) = 0$, we have $\mu_k \rightharpoonup \mu$ as $k \to \infty$.*

PROOF. Let f be a non-negative bounded Lipschitz continuous function on Ω, then

$$\left|\int_\Omega f\,\mathrm{d}\mu_k - \int_\Omega f\,\mathrm{d}\mu\right| \leq \sum_{j \in I_k} \left|\int_{I_k^j} f \frac{\mu(I_k^j)}{\lambda(I_k^j)}\,\mathrm{d}\lambda - \int_{I_k^j} f\,\mathrm{d}\mu\right| + \sum_{j \in J_k} \int_{I_k^j \cap \Omega} |f|\,\mathrm{d}\mu\,, \tag{1.3}$$

where J_k is the set of indices j such that $I_k^j \cap \partial\Omega$ is non-empty. Since f is uniformly continuous, for each $\varepsilon > 0$, we can take k large enough, such that for all j,

$$\sup_{I_k^j} f - \inf_{I_k^j} f < \varepsilon\,.$$

Then it is easy to see that the first term on right-hand side of (1.3) is bounded by $\varepsilon\mu(\Omega)$.

As for the second term, since f is bounded, it suffices to estimate

$$\sum_{j \in J_k} \mu(I_k^j).$$

This term is obviously bounded by the μ-measure of the set of points with a distance at most $C3^{-k}$ to $\partial\Omega$. As $\partial\Omega$ is smooth, as $k \to \infty$, the intersection of all these sets is just the boundary of Ω. By assumption, $\mu(\partial\Omega) = 0$, we conclude. □

REMARK I.2.3. The assumptions of Proposition I.2.6 are satisfied if $\Omega = \mathbb{B}^n$ and if μ is non-pluripolar.

[1] Here and in the sequel we follow the unfortunate terminology of Kołodziej and Dinew, although μ_k's are by no means canonical.

Now given $\varphi \in \mathrm{PSH}(\bar{\mathbb{B}}^n) \cap L^\infty(\mathbb{B}^n)$. Let f_k be a decreasing sequence of smooth functions on $\partial\mathbb{B}^n$, converging to $\varphi|_{\partial\mathbb{B}^n}$. We solve the following Dirichlet problem for any $k \geq 1$:

$$\begin{cases} \varphi^k \in \mathrm{PSH}(\mathbb{B}^n) \cap C^0(\bar{\mathbb{B}}^n), \\ (\mathrm{dd^c}\varphi^k)^n = (\mathrm{dd^c}\varphi)^n_k, \\ \varphi^k|_{\partial\mathbb{B}^n} = f_k. \end{cases} \tag{1.4}$$

Here we have denoted the canonical approximations of $(\mathrm{dd^c}\varphi)^n$ by $(\mathrm{dd^c}\varphi)^n_j$. We call φ^k the *canonical approximations* of φ.

Proposition I.2.7. *Let $\varphi_1, \dots, \varphi_n \in \mathrm{PSH}(\bar{\mathbb{B}}^n) \cap L^\infty(\mathbb{B}^n)$. Let φ_j^k ($k \geq 1$) be canonical approximations of φ_j, then*

$$\varphi_j = \left(\limsup_{k\to\infty} \varphi_j^k \right)^* .$$

Then $\varphi_j^k \to \varphi_j$ in L^1. Moreover,

$$\mathrm{dd^c}\varphi_1^k \wedge \cdots \mathrm{dd^c}\varphi_n^k \rightharpoonup \mathrm{dd^c}\varphi_1 \wedge \cdots \mathrm{dd^c}\varphi_n, \quad k \to \infty$$

in $\mathbb{B}^n$.

For a proof, see [Din09, Proposition 3.1].

I.3. Special currents

Let X be a compact Kähler manifold of dimension n. Let $\Theta \in \mathcal{D}'^{n-1,n-1}_+(X)$. Let $U \subseteq X$ be an open subset.

Proposition I.3.8. *Let $\varphi, \psi \in \mathrm{PSH}(U) \cap L^\infty_{\mathrm{loc}}(U)$. Then the following currents on U are of order 0:*

(1) $\mathrm{d}\varphi \wedge \mathrm{d^c}\varphi \wedge \Theta$.
(2) $\mathrm{d^c}\varphi \wedge \Theta := \mathrm{d^c}(\varphi\Theta)$.
(3) $\mathrm{d}\psi \wedge \mathrm{d^c}\varphi \wedge \Theta := \frac{1}{2}\left(\mathrm{d}(\varphi+\psi) \wedge \mathrm{d^c}(\varphi+\psi) \wedge \Theta - \mathrm{d}\varphi \wedge \mathrm{d^c}\varphi \wedge \Theta - \mathrm{d}\psi \wedge \mathrm{d^c}\psi \wedge \Theta\right)$.
(4) $\mathrm{d}\left(\psi\, \mathrm{d^c}\varphi \wedge \Theta\right)$.

Moreover,

$$\mathrm{d}\left(\psi\, \mathrm{d^c}\varphi \wedge \Theta\right) = \mathrm{d}\psi \wedge \mathrm{d^c}\varphi \wedge \Theta + \psi\, \mathrm{dd^c}\varphi \wedge \Theta \tag{1.5}$$

as currents on U.

Proof. Since the problem is local, we may shrink U if necessary. In particular, we may assume that $\varphi, \psi \in L^\infty(U)$. We can add a constant to φ so that $\varphi \geq 0$.

(1) Recall that φ^2 is also psh, it follows from the definition itself ([GZ17, Definition 3.2])

$$\mathrm{d}\varphi \wedge \mathrm{d^c}\varphi \wedge \Theta := \frac{1}{2}\mathrm{dd^c}\varphi^2 \wedge \Theta - \frac{1}{2}\varphi\, \mathrm{dd^c}\varphi \wedge \Theta$$

and Bedford–Taylor's theorem ([GZ17, Proposition 3.3]) that $\mathrm{d}\varphi \wedge \mathrm{d^c}\varphi \wedge \Theta$ is of order 0.

(2) We want to show that for each compact set $K \subseteq U$, there is a constant $C = C(K)$ such that for each smooth 1-form T on X with support in K, we have

$$\left| \int_U T \wedge \mathrm{d^c}\varphi \wedge \Theta \right| \leq C \|T\|_{0,K}, \tag{1.6}$$

where the semi-norm $\|\cdot\|_{0,K}$ is the zeroth order seminorm of a positive current, defined as in [GZ17, Proposition 2.18]. Let $\chi : U \to [0,1]$ be a smooth function on U with compact support, $\chi = 1$ on K. We have the following Cauchy–Schwarz inequality

$$\left|\int_U T \wedge \mathrm{d}^{\mathrm{c}}\varphi \wedge \Theta\right| \leq \left(\int_U \chi T \wedge \bar{T} \wedge \Theta\right)^{1/2} \left(\int_U \chi\, \mathrm{d}\varphi \wedge \mathrm{d}^{\mathrm{c}}\varphi \wedge \Theta\right)^{1/2}. \tag{1.7}$$

This is the usual Cauchy–Schwarz inequality when φ is smooth. The general case follows from Demailly approximation on φ. The convergence or right-hand side along Demailly approximations follow from [GZ17, Proposition 3.3]. The second bracket on the right-hand side of (1.7) is finite by (1), hence (1.6) follows.

(3) This follows from (1).

(4) This follows from (1.5) and (3). Let us prove (1.5). First notice that we may always assume that ψ is smooth. In fact, let ψ_k be smooth psh functions on U decreasing to ψ. Then $\psi_k\, \mathrm{d}^{\mathrm{c}}\varphi \wedge \Theta$ converges to $\psi\, \mathrm{d}^{\mathrm{c}}\varphi \wedge \Theta$ as currents by (2) and the dominated convergence theorem. So

$$\mathrm{d}\left(\psi_k\, \mathrm{d}^{\mathrm{c}}\varphi \wedge \Theta\right) \to \mathrm{d}\left(\psi\, \mathrm{d}^{\mathrm{c}}\varphi \wedge \Theta\right), \quad k \to \infty.$$

The right-hand side of (1.5) is also continuous along ψ_k by [GZ17, Proposition 3.3].

Similarly, one may assume that φ is smooth. In this case, (1.5) is obvious. □

Remark I.3.4. The proof of (1.7) explains how to apply the Cauchy–Schwarz type inequality in general. In the sequel, we usually omit the detailed arguments of this type and just refer to the Cauchy–Schwarz inequality.

Lemma I.3.9. *Let φ_1, φ_2 be qpsh functions on X. Assume that $u := \varphi_1 - \varphi_2 \in L^\infty(X)$. Let $U \subseteq X$ be an open subset such that $\varphi_1 \in L^\infty_{\mathrm{loc}}(U)$. Then*

$$\int_U \mathrm{d}u \wedge \mathrm{d}^{\mathrm{c}}u \wedge \Theta < \infty. \tag{1.8}$$

Here $\mathrm{d}u \wedge \mathrm{d}^{\mathrm{c}}u \wedge \Theta$ is defined in the obvious way by linearity. For a proof see [BEGZ10, Lemma 1.15].

Proposition I.3.10. *Let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be qpsh functions on X, assume that $u := \varphi_1 - \varphi_2, v = \psi_1 - \psi_2 \in L^\infty(X)$. Let $U \subseteq X$ be an open set on which φ_1 is locally bounded. The the following currents on U are of order 0:*

(1) $\mathrm{d}^{\mathrm{c}}u \wedge \Theta$.
(2) $\mathrm{d}v \wedge \mathrm{d}^{\mathrm{c}}u \wedge \Theta$.
(3) $\mathrm{d}\left(v\, \mathrm{d}^{\mathrm{c}}u \wedge \Theta\right)$.

Moreover on U, we have

$$\mathrm{d}\left(v\, \mathrm{d}^{\mathrm{c}}u \wedge \Theta\right) = \mathrm{d}v \wedge \mathrm{d}^{\mathrm{c}}u \wedge \Theta + v\, \mathrm{d}\mathrm{d}^{\mathrm{c}}u \wedge \Theta. \tag{1.9}$$

Proof. (1) It follows from Proposition I.3.8 that $\mathrm{d}^{\mathrm{c}}u \wedge \Theta$ is a current of order 0.

(2) This follows from Cauchy–Schwarz inequality and Lemma I.3.9.

(3) This follows from (1.9), which itself follows from (1.5). □

I.4. Non-pluripolar measures

Definition I.4.15. Let $\mu \in \mathcal{M}_+(X)$. We say that μ is *non-pluripolar* if for any pluripolar set $A \subseteq X$, we have $\mu(A) = 0$[2]. In this case, we write $\mu \in \mathring{\mathcal{M}}_+(X)$.

Let μ be a non-pluripolar measure on X. We define a sequence of good measures that converges to μ as follows: Cover X by open sets Ω_j, each being biholomorphic to the unit ball in $\mathbb{C}^n$. Let χ_k be a Friedrichs kernel. Let ρ_j be a partition of unity subordinate to Ω_j. We define

$$\mu_k = c_k \sum_j \rho_j \chi_k * \mu|_{\Omega_j}, \tag{1.10}$$

where c_k is a constant making sure that $\mu_k(X) = \mu(X)$. Obviously $\mu_k \rightharpoonup \mu$. Note that μ_k has L^∞ density.

We shall refer to μ_k as a *Friedrichs approximation* of μ.

I.5. The envelope operator

Definition I.5.16. Let $u \in \mathrm{USC}(X)$. We define

$$\mathrm{P}(u) := \sup{}^* \{\varphi \in \mathrm{PSH}(X, \theta) : \varphi \leq u\} . \tag{1.11}$$

Remark I.5.5. In fact,

$$\mathrm{P}(u) = \sup \{ \varphi \in \mathrm{PSH}(X, \theta) : \varphi \leq u \} .$$

This is because $\mathrm{P}(u)$ is itself a candidate in (1.11).

Proposition I.5.11.

(1) P *is concave, increasing on* $\mathrm{USC}(X)$.
(2) Let $u_j, u \in \mathrm{USC}(X)$. *Assume that* u_j *decreases to* u *pointwisely. Then* $\mathrm{P}(u_j)$ *also decreases to* $\mathrm{P}(u)$ *pointwisely.*
(3) For any $C \in \mathbb{R}$, $u \in \mathrm{USC}(X)$,

$$\mathrm{P}(u + C) = \mathrm{P}(u) + C .$$

(4) For $u, v \in \mathrm{USC}(X)$, *then*

$$\sup_X |\mathrm{P}(u) - \mathrm{P}(v)| \leq \sup_X |u - v| .$$

(5) Let $u \in C^0(X)$, *then*

$$\int_X \mathbb{1}_{\{\mathrm{P}(u)<u\}} \, \theta^n_{\mathrm{P}(u)} = 0 .$$

Proof. (1) This is obvious.

(2) From (1), we know that $\mathrm{P}(u_j)$ is decreasing. Let v be the pointwise limit of $\mathrm{P}(u_j)$. Then $v \in \mathrm{PSH}(X, \theta)$. It follows from (1) that $v \geq \mathrm{P}(u)$. On the other hand,

$$v = \lim_{j\to\infty} \mathrm{P}(u_j) \leq \lim_{j\to\infty} u_j = u.$$

So $v \leq \mathrm{P}(u)$. Hence $v = \mathrm{P}(u)$.

(3) This follows from definition.

(4) We may assume that $\sup_X |u-v| < \infty$. Then this follows immediately from (3) and (1).

[2]Strictly speaking, A is not necessarily measurable. We are in fact identifying μ with its completion measure.

(5) This follows from the standard *balayage* argument. □

I.5.1. Miscellaneous. Let X be a compact Kähler manifold of dimension n. Let α be a big cohomology class with a smooth representative θ.

THEOREM I.5.12. *We have*

$$\theta_{V_\theta}^n \le \mathbb{1}_{\{V_\theta=0\}}\,\theta^n\,. \tag{1.12}$$

For a proof, see [DDNL18, Theorem 2.6].

CHAPTER 2

Non-pluripolar products

II.1. Definition and basic properties of the non-pluripolar products

Let X be a complex manifold of dimension n, not necessarily compact. Let $p \leq n$ be a non-negative integer. Let $u_1, \dots, u_p \in \mathrm{PSH}(X)$. We want to define a closed positive (p,p)-current

$$\langle \mathrm{dd^c} u_1 \wedge \dots \wedge \mathrm{dd^c} u_p \rangle \in \mathcal{D}'^{p,p}_+(X)$$

satisfying with the following extra assumptions:

(1) When $u_1, \dots, u_p \in L^\infty_{\mathrm{loc}}(X)$, the product coincides with the Bedford–Taylor product.
(2) The product is local in the plurifine topology.
(3) The product puts no weight on pluripolar sets.

Note that these conditions fix the definition of $\langle \mathrm{dd^c} u_1 \wedge \dots \wedge \mathrm{dd^c} u_p \rangle$ completely, if it ever exists. In fact, let

$$O_k := \bigcap_{j=1}^{p} \{u_j > -k\}, \quad k \in \mathbb{Z}_{>0}.$$

By (1) and (2),

$$\mathbb{1}_{O_k} \langle \mathrm{dd^c} u_1 \wedge \dots \wedge \mathrm{dd^c} u_p \rangle = \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{u_j, -k\}. \tag{2.1}$$

Note that

$$X - \bigcup_{k \geq 0} O_k = \bigcup_{j=1}^{p} \{u_j = -\infty\}$$

is a pluripolar set, so the definition of $\langle \mathrm{dd^c} u_1 \wedge \dots \wedge \mathrm{dd^c} u_p \rangle$ is completely fixed.

Definition II.1.1. Let $u_1, \dots, u_p \in \mathrm{PSH}(X)$. We say that $\langle \mathrm{dd^c} u_1 \wedge \dots \wedge \mathrm{dd^c} u_p \rangle$ is *well-defined* if for each open subset $U \subseteq X$ such that there is a Kähler form ω on U, each compact subset $K \subseteq U$, we have

$$\sup_{k \geq 0} \int_{K \cap O_k} \left(\bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{u_j, -k\} \right) \Bigg|_U \wedge \omega^{n-p} < \infty. \tag{2.2}$$

In this case, we define[1] $\langle \mathrm{dd^c} u_1 \wedge \dots \wedge \mathrm{dd^c} u_p \rangle$ by (2.1) on $\bigcup_{k \geq 0} O_k$ and make a zero-extension to X.

Remark II.1.1. The condition (2.2) is clearly independent of the choice of U and ω.

[1] Here we use implicitly the fact the Bedford–Taylor product is local in the plurifine topology

REMARK II.1.2. Let $u_1,\dots,u_p\in \mathrm{PSH}(X)$. Let $\sigma\in\mathcal{S}_p$. By definition, $\langle \mathrm{dd^c} u_1\wedge\dots\wedge \mathrm{dd^c} u_p\rangle$ is well-defined iff $\langle \mathrm{dd^c} u_{\sigma(1)}\wedge\dots\wedge \mathrm{dd^c} u_{\sigma(p)}\rangle$ is. Moreover, in this case,

$$\langle \mathrm{dd^c} u_1\wedge\dots\wedge \mathrm{dd^c} u_p\rangle=\langle \mathrm{dd^c} u_{\sigma(1)}\wedge\dots\wedge \mathrm{dd^c} u_{\sigma(p)}\rangle.$$

In particular, we may use the following notation for either product:

$$\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j\right\rangle.$$

Let us verify that our product indeed satisfies all requirements. We need a few lemmata.

LEMMA II.1.1. *Let $u_1,\dots,u_p\in \mathrm{PSH}(X)$. Assume that $\langle \mathrm{dd^c} u_1\wedge\dots\wedge \mathrm{dd^c} u_p\rangle$ is well-defined. Let $E_k\subseteq O_k$ ($k\geq 0$) be Borel sets such that $X-\cup_k E_k$ is pluripolar. Let Ω be a $(n-p,n-p)$-form with measurable coefficients. Assume that the following conditions are satisfied:*

1. $\operatorname{Supp}\Omega$ *is compact.*
2. *For each open subset $U\subseteq X$, each Kähler form ω on U, there is a constant $C>0$ such that*
$$-C\omega^{n-p}\leq\Omega\leq C\omega^{n-p}$$
holds on $\operatorname{Supp}\Omega\cap U$.

Then

$$\lim_{k\to\infty}\int_X \mathbb{1}_{E_k}\bigwedge_{j=1}^p \mathrm{dd^c}\max\{u_j,-k\}\wedge\Omega=\int_X\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j\right\rangle\wedge\Omega.$$

In particular,

$$\mathbb{1}_{E_k}\bigwedge_{j=1}^p \mathrm{dd^c}\max\{u_j,-k\}\rightharpoonup\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j\right\rangle,\quad k\to\infty$$

as currents and the convergence is strong on each compact subset of X.

PROOF. Since the problem is local, we may assume that $\operatorname{Supp}\Omega\subseteq U$, where $U\subseteq X$ is an open subset with a Kähler form ω. Take $C>0$ so that

$$-C\omega^{n-p}\leq\Omega\leq C\omega^{n-p}.$$

Then observe that

$$\begin{aligned}0\leq\int_X \mathbb{1}_{O_k}\bigwedge_{j=1}^p \mathrm{dd^c}\max\{u_j,-k\}\wedge\Omega-\int_X \mathbb{1}_{E_k}\bigwedge_{j=1}^p \mathrm{dd^c}\max\{u_j,-k\}\wedge\Omega\\ \leq\int_{\operatorname{Supp}\Omega}(1-\mathbb{1}_{E_k})\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j\right\rangle\wedge\Omega.\end{aligned}$$

The RHS tends to 0 by dominated convergence theorem. So it suffices to prove the theorem for $E_k=O_k$. In this case, the theorem again follows from dominated convergence theorem. □

LEMMA II.1.2. *Let $u\in\mathrm{PSH}(X)$, $u\leq 0$. Let $\chi:\mathbb{R}\to\mathbb{R}$ be a smooth, convex, increasing function satisfying*

$$\chi(1)=1,\quad\chi(t)=0,\quad t\leq\frac{1}{2}.$$

Let $\vartheta : \mathbb{R} \to [0,1]$ be a smooth increasing function such that

$$\vartheta(0) = 0, \quad \vartheta(t) = 1, \quad t \geq \frac{1}{2}.$$

For each $k \geq 1$, let $w_k : X \to \mathbb{R}$ be defined by

$$w_k := \chi\left(e^{u/k}\right).$$

Note that w_k are uniformly bounded positive psh functions.

Then as $k \to \infty$,

(1) $\vartheta(w_k)$ is increasing and

$$\vartheta(w_k) \leq \mathbb{1}_{\{u>-k\}},$$

Moreover,

$$\vartheta(w_k) \to 1$$

outside $\{u = -\infty\}$ pointwisely.

(2) $\vartheta'(w_k)$ vanishes outside $\{u > -k\}$ and

$$\vartheta'(w_k) \to 0$$

outside $\{u = -\infty\}$ pointwisely.

The lemma follows directly by writing down all definitions.

Proposition II.1.3. *Let $u_1, \dots, u_p \in \mathrm{PSH}(X)$.*

(1) The product $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ is local in plurifine topology. In the following sense: let $O \subseteq X$ be a plurifine open subset, let $v_1, \dots, v_p \in \mathrm{PSH}(X)$, assume that

$$u_j|_O = v_j|_O, \quad j = 1, \dots, p.$$

Assume that

$$\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j \right\rangle, \quad \left\langle \bigwedge_{j=1}^p \mathrm{dd^c} v_j \right\rangle$$

are both well-defined, then

$$\left.\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j \right\rangle\right|_O = \left.\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} v_j \right\rangle\right|_O. \tag{2.3}$$

If O is open in the usual topology, then the product

$$\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j|_O \right\rangle$$

on O is well-defined and

$$\left.\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j \right\rangle\right|_O = \left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j|_O \right\rangle. \tag{2.4}$$

Let $\mathcal{U}$ be an open covering of X. Then $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ is well-defined iff each of the following product is well-defined

$$\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j|_U \right\rangle, \quad U \in \mathcal{U}.$$

(2) *The current* $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ *and the fact that it is well-defined depend only on the currents* $\mathrm{dd^c} u_j$*, not on specific* u_j*.*
(3) *When* $u_1, \dots, u_p \in L^\infty_{\mathrm{loc}}(X)$*,* $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ *is well-defined and is equal to the Bedford–Taylor product.*
(4) *Assume that* $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ *is well-defined, then* $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ *puts not mass on pluripolar sets.*
(5) *Assume that* $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ *is well-defined, then*

$$\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j \right\rangle \in \mathcal{D}'^{p,p}_+(X).$$

(6) *The product is multi-linear: let* $v_1 \in \mathrm{PSH}(X)$*, then*

$$\left\langle \mathrm{dd^c}(u_1+v_1) \wedge \bigwedge_{j=2}^p \mathrm{dd^c} u_j \right\rangle = \left\langle \mathrm{dd^c} u_1 \wedge \bigwedge_{j=2}^p \mathrm{dd^c} u_j \right\rangle + \left\langle \mathrm{dd^c} v_1 \wedge \bigwedge_{j=2}^p \mathrm{dd^c} u_j \right\rangle \tag{2.5}$$

in the sense that LHS is well-defined iff both terms on RHS are well-defined, and the equality holds in that case.

PROOF. (1) For any $k \geq 0$, let

$$E_k := \bigcap_{j=1}^p \{u_j > -k, v_j > -k\}.$$

By plurilocality of the Bedford–Taylor product,

$$\mathbb{1}_{O \cap E_k} \bigwedge_{j=1}^p \mathrm{dd^c} \max\{u_j, -k\} = \mathbb{1}_{O \cap E_k} \bigwedge_{j=1}^p \mathrm{dd^c} \max\{v_j, -k\}.$$

By Lemma II.1.1, let $k \to \infty$, (2.3) follows.

When O is open in the usual topology, (2.4) follows from the corresponding property of the Bedford–Taylor product.

The last statement is obvious.

(2) By (1), we may assume that there is a Kähler form ω on X. Let w_j ($j = 1, \dots, p$) be pluriharmonic functions on X. Assume that $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ is well-defined. We want to prove that $\langle \mathrm{dd^c}(w_1 + u_1) \wedge \cdots \wedge \mathrm{dd^c}(w_p + u_p) \rangle$ is also well-defined and

$$\left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j \right\rangle = \left\langle \bigwedge_{j=1}^p \mathrm{dd^c} (w_j + u_j) \right\rangle. \tag{2.6}$$

By further shrinking X, we may assume that w_j are bounded from above on X, say

$$w_j \leq C, \quad j = 1, \dots, p.$$

Then for any $k \geq 0$, on the pluriopen set

$$V_k := \bigcap_{j=1}^p \{u_j + w_j > -k\},$$

we have $u_j > -k - C$, so by (1),

$$\max\{u_j + w_j, -k\} = \max\{u_j, -k - C\} + w_j.$$

Let $K \subseteq X$ be a compact subset, then

$$\int_K \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{u_j+w_j, -k\} \wedge \omega^{n-p} = \int_K \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{u_j, -k-C\} \wedge \omega^{n-p}.$$

The RHS is bounded by assumption. So the RHS of (2.6) is well-defined and (2.6) follows.

(3) By (1) and the locality of the Bedford–Taylor product, the problem is local, so we may assume that u_j are bounded on X. In this case, (3) follows directly from definition.

(4) The problem is again local, by reduction to the local setting, it follows from Lemma II.1.1.

(5) It follows directly from definition that $\langle \mathrm{dd^c} u_1 \wedge \cdots \wedge \mathrm{dd^c} u_p \rangle$ is positive, so it suffices to prove that it is closed. We may assume that $p < n$. Since the problem is local, we may assume that X is the unit polydisk in $\mathbb{C}^n$. Take a closed positive $(n-p-1, n-p-1)$-form ρ on X with constant coefficients. By (3), we may assume that $u_j \leq 0$. Let

$$\Theta_k := \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{u_j, -k\} \wedge \rho.$$

By Lemma II.1.1, we have

$$\mathbb{1}_{O_k} \Theta_k \rightharpoonup \left\langle \bigwedge_{j=1}^{p} \mathrm{dd^c} u_j \right\rangle \wedge \rho, \quad k \to \infty.$$

Hence

$$\mathrm{d}\left(\mathbb{1}_{O_k} \Theta_k\right) \rightharpoonup \mathrm{d} \left\langle \bigwedge_{j=1}^{p} \mathrm{dd^c} u_j \right\rangle \wedge \rho, \quad k \to \infty.$$

So it suffices to prove

$$\mathrm{d}\left(\mathbb{1}_{O_k} \Theta_k\right) \rightharpoonup 0, \quad k \to \infty.$$

We now apply the construction of Lemma II.1.2 with $u = \sum_{j=1}^{p} u_j$, we shall use the same notations. Then

$$0 \leq \left(\mathbb{1}_{O_k} - \vartheta(w_k)\right) \Theta_k \leq \left(1 - \vartheta(w_k)\right) \left\langle \bigwedge_{j=1}^{p} \mathrm{dd^c} u_j \right\rangle \wedge \rho.$$

RHS converges weakly to 0 as $k \to \infty$ by dominated convergence theorem and by Lemma II.1.2. So it remains to prove

$$\mathrm{d}\left(\vartheta(w_k)\Theta_k\right) \rightharpoonup 0, \quad k \to \infty.$$

It follows from the chain rule ([BEGZ10] Lemma 1.9) that

$$\mathrm{d}\left(\vartheta(w_k)\Theta_k\right) = \vartheta'(w_k)\, \mathrm{d}w_k \wedge \Theta_k,$$

where by definition

$$\mathrm{d}w_k \wedge \Theta_k := \mathrm{d}\left(w_k \Theta_k\right)$$

is a closed current of order 0 by Bedford–Taylor theory.

Now take an arbitrary smooth 1-form ψ on X with compact support, we need to prove

$$\lim_{k\to\infty} \int_X \vartheta'(w_k)\, \mathrm{d}w_k \wedge \Theta_k \wedge \psi = 0. \tag{2.7}$$

Let $\tau : X \to [0,1]$ be a smooth function with compact support, $\tau = 1$ in a neighbourhood of $\operatorname{Supp}\psi$. Then we have the following Cauchy–Schwarz inequality:

$$\left|\int_X \vartheta'(w_k)\,\mathrm{d}w_k \wedge \Theta_k \wedge \psi\right|^2 \leq 2\pi \left(\int_X \vartheta'(w_k)^2 \Theta_k \wedge \psi \wedge \bar{\psi}\right)\left(\int_X \tau\,\mathrm{d}w_k \wedge \mathrm{d^c} w_k \wedge \Theta_k\right).$$

In fact, when w_k is smooth, this follows from the standard Cauchy–Schwarz inequality, it holds even without τ on RHS, for a general w_k, it suffices to apply the Demailly approximation, the extra τ ensures the convergence of RHS under Demailly approximation.

For the first bracket, by Lemma II.1.1,

$$0 \leq \int_X \vartheta'(w_k)^2 \Theta_k \wedge \psi \wedge \bar{\psi} \leq \int_X \vartheta'(w_k)^2 \left\langle \bigwedge_{j=1}^p \mathrm{dd^c} u_j \right\rangle \wedge \rho \wedge \psi \wedge \bar{\psi}.$$

Again by dominated convergence theorem and by Lemma II.1.1, the right-most term tends to 0.

As for the second bracket,

$$2\int_X \tau\,\mathrm{d}w_k \wedge \mathrm{d^c} w_k \wedge \Theta_k \leq \int_X \tau\,\mathrm{dd^c} w_k^2 \wedge \Theta_k = \int_X w_k^2\,\mathrm{dd^c}\tau \wedge \Theta_k = \int_{O_k} w_k^2\,\mathrm{dd^c}\tau \wedge \Theta_k.$$

Note that w_k are uniformly bounded. Also, it follows from Lemma II.1.1 that the masses of $\mathrm{dd^c}\tau \wedge \Theta_k$ are uniformly bounded. So the second bracket is bounded. This concludes the proof of (2.7).

(6) The problem is local, so we may assume that there is a global Kähler form ω on X. Moreover, by (2) we may assume that $u_1 \leq 0, v_1 \leq 0$ after possibly shrinking X. Note that for any $k \geq 0$, on $\{u_1 + v_1 > -k\}$, we have

$$\mathrm{dd^c} \max\{u_1 + v_1, -k\} = \mathrm{dd^c} \max\{u_1, -k\} + \mathrm{dd^c} \max\{v_1, -k\}. \tag{2.8}$$

Also note that

$$\{u_1 + v_1 > -k\} \subseteq \{u_1 > -k\} \cup \{v_1 > -k\}.$$

Assume that both terms on the RHS of (2.5) are well-defined. Let $K \subseteq X$ be a compact subset, then for any $k \geq 0$,

$$\begin{aligned}
&\int_K \mathbb{1}_{\{u_1+v_1>-k\}} \mathbb{1}_{\cap_{j=2}^p \{u_j>-k\}} \mathrm{dd^c} \max\{u_1 + v_1, -k\} \wedge \bigwedge_{j=2}^p \max\{u_j, -k\} \wedge \omega^{n-p} \\
= &\int_K \mathbb{1}_{\{u_1+v_1>-k\}} \mathbb{1}_{\cap_{j=2}^p \{u_j>-k\}} \mathrm{dd^c} \max\{u_1, -k\} \wedge \bigwedge_{j=2}^p \max\{u_j, -k\} \wedge \omega^{n-p} \\
&+ \int_K \mathbb{1}_{\{u_1+v_1>-k\}} \mathbb{1}_{\cap_{j=2}^p \{u_j>-k\}} \mathrm{dd^c} \max\{u_1, -k\} \wedge \bigwedge_{j=2}^p \max\{u_j, -k\} \wedge \omega^{n-p} \\
\leq &\int_K \mathbb{1}_{\{u_1>-k\}} \mathbb{1}_{\cap_{j=2}^p \{u_j>-k\}} \mathrm{dd^c} \max\{u_1, -k\} \wedge \bigwedge_{j=2}^p \max\{u_j, -k\} \wedge \omega^{n-p} \\
&+ \int_K \mathbb{1}_{\{v_1>-k\}} \mathbb{1}_{\cap_{j=2}^p \{u_j>-k\}} \mathrm{dd^c} \max\{u_1, -k\} \wedge \bigwedge_{j=2}^p \max\{u_j, -k\} \wedge \omega^{n-p}.
\end{aligned}$$

The RHS is bounded for all k by assumption. So the LHS of (2.5) is well-defined.

Conversely, assume that the LHS of (2.5) is well-defined. Let $K \subseteq X$ be a compact subset, then for any $k \geq 0$,

$$
\begin{aligned}
&\int_K \mathbb{1}_{\{u_1>-k\}}\mathbb{1}_{\cap_{j=2}^p\{u_j>-k\}}\mathrm{dd^c}\max\{u_1,-k\}\wedge\bigwedge_{j=2}^p\max\{u_j,-k\}\wedge\omega^{n-p}\\
\leq&\int_K \mathbb{1}_{\{u_1>-k\}}\mathbb{1}_{\cap_{j=2}^p\{u_j>-k\}}\mathrm{dd^c}\max\{u_1+v_1,-k\}\wedge\bigwedge_{j=2}^p\max\{u_j,-k\}\wedge\omega^{n-p}\\
\leq&\int_K \mathbb{1}_{\{u_1+v_1>-k-1\}}\mathbb{1}_{\cap_{j=2}^p\{u_j>-k\}}\mathrm{dd^c}\max\{u_1+v_1,-k\}\wedge\bigwedge_{j=2}^p\max\{u_j,-k\}\wedge\omega^{n-p},
\end{aligned}
$$

where the third line follows from the fact that on $\{u_1+v_1<-k\}$, we have $\mathrm{dd^c}\max\{u_1+v_1,-k\}=0$. Now the RHS is bounded for various k by assumption. So the first term on RHS of (2.5) is well-defined. By symmetry, the same is true for the other term.

Now assume that both sides of (2.5) are well-defined, we prove (2.5). Let

$$E_k := \{u_1>-k/2\}\cap\{v_1>-k/2\}\cap\bigcap_{j=2}^p\{u_j>-k\}.$$

Then

$$E_k \subseteq \{u_1+v_1>-k\}\cap\bigcap_{j=2}^p\{u_j>-k\}.$$

Moreover,

$$X-\bigcup_{k=1}^\infty E_k$$

is pluripolar. So by Lemma II.1.1,

$$\mathbb{1}_{E_k}\mathrm{dd^c}\max\{u_1+v_1,-k\}\wedge\bigwedge_{j=2}^p\mathrm{dd^c}\max\{u_j,-k\}\rightharpoonup\left\langle\bigwedge_{j=1}^p\mathrm{dd^c}u_j\right\rangle.$$

By (2.8),

$$
\begin{aligned}
&\mathbb{1}_{E_k}\mathrm{dd^c}\max\{u_1+v_1,-k\}\wedge\bigwedge_{j=2}^p\mathrm{dd^c}\max\{u_j,-k\}\\
=&\mathbb{1}_{E_k}\mathrm{dd^c}\max\{u_1,-k\}\wedge\bigwedge_{j=2}^p\mathrm{dd^c}\max\{u_j,-k\}\\
&+\mathbb{1}_{E_k}\mathrm{dd^c}\max\{v_1,-k\}\wedge\bigwedge_{j=2}^p\mathrm{dd^c}\max\{u_j,-k\}.
\end{aligned}
$$

Again by Lemma II.1.1, we find that the RHS converges weakly to the RHS of (2.5) as $k\to\infty$. This concludes the proof. □

DEFINITION II.1.2. Let $T_1,\dots,T_p\in\mathcal{D}'^{1,1}_+(X)$. We say that $\langle T_1\wedge\cdots\wedge T_p\rangle$ is *well-defined* if there exists an open covering $\mathcal{U}$ of X, such that on each $U\in\mathcal{U}$, we can find $u_j^U\in\mathrm{PSH}(U)$ $(j=1,\dots,p)$ such that

$$\mathrm{dd^c}u_j^U=T_j,\quad j=1,\dots,p$$

and such that $\langle \mathrm{dd^c} u_1^U \wedge \cdots \wedge \mathrm{dd^c} u_p^U \rangle$ is well-defined. In this case, we define $\langle T_1 \wedge \cdots \wedge T_p \rangle \in \mathcal{D}'^{p,p}(X)$ by

$$\langle T_1 \wedge \cdots \wedge T_p \rangle|_U = \langle \mathrm{dd^c} u_1^U \wedge \cdots \wedge \mathrm{dd^c} u_p^U \rangle, \quad U \in \mathcal{U}. \tag{2.9}$$

Remark II.1.3. It follows from Proposition II.1.3 that (2.9) defines a unique well-defined current in $\mathcal{D}'^{p,p}(X)$ and that the well-defineness of $\langle T_1 \wedge \cdots \wedge T_p \rangle$ and its exact value are bot independent of the choice of $\mathcal{U}$ and u_j^U.

Proposition II.1.4. *Let $T_1, \ldots, T_p \in \mathcal{D}'^{1,1}_+(X)$.*

1. *The product $\langle T_1 \wedge \cdots \wedge T_p \rangle$ is local in plurifine topology. In the following sense: let $O \subseteq X$ be a plurifine open subset, let $S_1, \ldots, S_p \in \mathcal{D}'^{1,1}_+(X)$, assume that*
$$T_j|_O = S_j|_O, \quad j = 1, \ldots, p.$$
Assume that
$$\langle T_1 \wedge \cdots \wedge T_p \rangle, \quad \langle S_1 \wedge \cdots \wedge S_p \rangle.$$
are both well-defined, then
$$\langle T_1 \wedge \cdots \wedge T_p \rangle|_O = \langle S_1 \wedge \cdots \wedge S_p \rangle|_O. \tag{2.10}$$
If O is open in the usual topology, then the product
$$\langle T_1 \wedge \cdots \wedge T_p|_O \rangle$$
on O is well-defined and
$$\langle T_1 \wedge \cdots \wedge T_p \rangle|_O = \langle T_1 \wedge \cdots \wedge T_p|_O \rangle. \tag{2.11}$$
Let $\mathcal{U}$ be an open covering of X. Then $\langle T_1 \wedge \cdots \wedge T_p \rangle$ is well-defined iff each of the following product is well-defined
$$\langle T_1 \wedge \cdots \wedge T_p|_U \rangle, \quad U \in \mathcal{U}.$$
2. *Assume that $\langle T_1 \wedge \cdots \wedge T_p \rangle$ is well-defined, then the product $\langle T_1 \wedge \cdots \wedge T_p \rangle$ puts not mass on pluripolar sets.*
3. *Assume that $\langle T_1 \wedge \cdots \wedge T_p \rangle$ is well-defined, then*
$$\langle T_1 \wedge \cdots \wedge T_p \rangle \in \mathcal{D}'^{p,p}_+(X).$$
4. *The product $\langle T_1 \wedge \cdots \wedge T_p \rangle$ is symmetric (the meaning is as in Remark II.1.2).*
5. *The product is multi-linear: let $T_1' \in \mathcal{D}'^{1,1}_+(X)$, then*
$$\langle (T_1 + T_1') \wedge T_2 \wedge \cdots \wedge T_p \rangle = \langle T_1 \wedge T_2 \wedge \cdots \wedge T_p \rangle + \langle T_1' \wedge T_2 \wedge \cdots \wedge T_p \rangle$$
in the sense that LHS is well-defined iff both terms on RHS are well-defined, and the equality holds in that case.

Proof. All statements follow immediately from the corresponding statements in Proposition II.1.3. □

Let us observe that we have the following log concavity property.

Theorem II.1.5. *Let $T_1, \ldots, T_n \in \mathcal{D}'^{1,1}_+(X)$. Let $\mu \in \mathcal{M}_+(X)$ be a non-pluripolar measure. Let f_j $(j = 1, \ldots, n)$ be non-negative measurable functions on X. Assume that the following currents are well-defined:*
$$\langle T_j^n \rangle, \quad \langle T_1 \wedge \cdots \wedge T_n \rangle, \quad j = 1, \ldots, n.$$

Assume that

$$\langle T_j^n \rangle \geq f_j \mu, \quad j = 1, \dots, n. \tag{2.12}$$

Then

$$\langle T_1 \wedge \cdots \wedge T_n \rangle \geq (f_1 \cdots f_n)^{1/n} \mu. \tag{2.13}$$

Proof. This result has a local nature, so we may assume that X is the unit ball in $\mathbb{B}^n \subseteq \mathbb{C}^n$. Then we write $T_j = \mathrm{dd^c}\varphi_j$, where $\varphi_j \in \mathrm{PSH}(\mathbb{B}^n)$. By possibly shrinking X, we may assume that $\varphi_j \in \mathrm{PSH}(\bar{\mathbb{B}}^n)$.

Step 1. We show that it suffices to prove (2.13) under the assumption that φ_j are all bounded.

In fact, by Proposition II.1.4 and (2.12), we have for any $k \geq 0$,

$$(\mathrm{dd^c}(\varphi_1 \vee (-k)))^n \geq \mathbb{1}_{O_k} f_j \mu.$$

Hence if the theorem holds in the bounded case, we have

$$\mathrm{dd^c}(\varphi_1 \vee (-k)) \wedge \cdots \wedge \mathrm{dd^c}(\varphi_n \vee (-k)) \geq \mathbb{1}_{O_k} (f_1 \cdots f_n)^{1/n} \mu.$$

Again by Proposition II.1.4, we get

$$\langle \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_n \rangle \geq \mathbb{1}_{O_k} (f_1 \cdots f_n)^{1/n} \mu.$$

Let $k \to \infty$, we conclude (2.13) since μ does not charge the pluripolar set $X - \bigcup_k O_k$.

Step 2. Reduce to the case where $\mu = \lambda$. Here λ denotes the Lebesgue measure. Assume that the theorem holds for Lebesgue measure.

Let φ_j^k ($k \geq 1$) be the canonical approximations of φ_j constructed in Section I.2.2. We shall use the notations in Section I.2.2. Then

$$\begin{aligned}
\mathrm{dd^c}\varphi_1^k \wedge \cdots \wedge \mathrm{dd^c}\varphi_n^k &\geq \sum_{a \in I_k} \frac{\left(\prod_{j=1}^n \int_{I_k^a} (\mathrm{dd^c}\varphi_j^k)^n\right)^{1/n}}{\lambda(I_k^a)} \mathbb{1}_{I_k^a} \lambda \\
&\geq \sum_{a \in I_k} \frac{\left(\prod_{j=1}^n \int_{I_k^a} f_j \,\mathrm{d}\mu\right)^{1/n}}{\lambda(I_k^a)} \mathbb{1}_{I_k^a} \lambda \\
&\geq \sum_{a \in I_k} \frac{\int_{I_k^a} \left(\prod_{j=1}^n f_j\right)^{1/n} \mathrm{d}\mu}{\lambda(I_k^a)} \mathbb{1}_{I_k^a} \lambda,
\end{aligned}$$

where the first inequality follows from our assumption, the second follows from (2.12), the third is just the Hölder inequality.

Let $k \to \infty$, it follows from Proposition I.2.6, Remark I.2.3 and Proposition I.2.7 that (2.13) holds.

Step 3. Reduce to smooth φ_j. Assume that the theorem is known when φ_j are smooth. We may assume that φ_j are defined and is psh in a neighbourhood of $\bar{\mathbb{B}}^n$.

Let χ_ε be the Friedrichs kernels.

A direct calculation shows that

$$(\mathrm{dd^c}(\varphi_j * \chi_\varepsilon))^n \geq f_j * \chi_\varepsilon \lambda.$$

So

$$\mathrm{dd^c}(\varphi_1 * \chi_\varepsilon) \wedge \cdots \wedge \mathrm{dd^c}(\varphi_n * \chi_\varepsilon) \geq \left(\prod_{j=1}^{n} f_j * \chi_\varepsilon \right)^{1/n} \lambda$$

Let $\varepsilon \to 0$ and use Theorem I.2.5, we are done.

Step 4 When φ_j are smooth, $\mu = \lambda$. The result follows from the concavity of $H \mapsto \log\det H$, where H is an $n \times n$ positive Hermitian matrix. □

Finally we concentrate on the most important case.

PROPOSITION II.1.6. *Let X be a compact Kähler manifold. Let $T_1, \ldots, T_p \in \mathcal{D}'^{1,1}_{+}(X)$. Then*

$$\langle T_1 \wedge \cdots \wedge T_p \rangle$$

is well-defined.

PROOF. Fix a Kähler form ω on X. In this case, write $T_j = (T_j + C\omega) - C\omega$ for $C > 0$ large enough and apply Proposition II.1.4 (5), we may assume that T_j is in a Kähler class. So we can write

$$T_j = \omega_j + \mathrm{dd^c}\varphi_j,$$

where ω_j is a Kähler form and φ_j is ω_j-psh. Let U be an open subset on which we can write

$$\omega_j = \mathrm{dd^c}\psi_j$$

with psh functions $\psi_j \leq 0$ on U. Now on U, for each $k \geq 0$,

$$\{\psi_j + \varphi_j > -k\} \subseteq \{\varphi_j > -k\},$$

so for each compact subset $K \subseteq U$,

$$\begin{aligned}
&\int_K \mathbb{1}_{\cap_{j=1}^p \{\psi_j + \varphi_j > -k\}} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\psi_j + \varphi_j, -k\} \wedge \omega^{n-p} \\
=& \int_K \mathbb{1}_{\cap_{j=1}^p \{\psi_j + \varphi_j > -k\}} \bigwedge_{j=1}^{p} (\omega_j + \mathrm{dd^c} \max\{\varphi_j, -k\}) \wedge \omega^{n-p} \\
\leq& \int_X \bigwedge_{j=1}^{p} (\omega_j + \mathrm{dd^c} \max\{\varphi_j, -k\}) \wedge \omega^{n-p} \\
=& \int_X \bigwedge_{j=1}^{p} \omega_j \wedge \omega^{n-p}.
\end{aligned}$$

□

From now on, we will omit the angle brackets from our notations.

II.2. Semicontinuity of non-pluripolar products

Let X be a compact Kähler manifold of dimension n. Let $m \in \mathbb{N}$. Let $\alpha_1, \ldots, \alpha_n \in H^{1,1}(X, \mathbb{R})$ be big cohomology classes. Let $\theta_j \in \alpha_j$ ($i = 1, \ldots m$, $j = 1, \ldots, n$) be smooth representatives.

LEMMA II.2.7. *Let $U \subseteq X$ be an open set. Let θ^i ($i = 1, \dots, m$) be smooth $(1,1)$-forms on U. Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X, \theta_j)$ ($k \in \mathbb{N}$, $j = 1, \dots, n$). Let $\varphi^{i,k}, \psi^{i,k}, \varphi^i, \psi^i \in \mathrm{PSH}(U, \theta^i)$ ($k \in \mathbb{N}$, $i = 1, \dots, m$). Let $\chi \geq 0$ be a bounded quasi-continuous function on X with $\operatorname{Supp}\chi \subseteq U$. Assume the following:*

1. *There is a closed pluripolar set $S \subseteq X$ such that $\varphi_j^k, \varphi_j \in L^\infty_{\mathrm{loc}}(X - S)$ ($k \in \mathbb{N}$, $j = 1, \dots, n$).*
2. *For any $j = 1, \dots, n$, $i = 1, \dots, m$, as $k \to \infty$,*
$$\varphi_j^k \xrightarrow{C} \varphi_j\,, \quad \text{on } X$$
and
$$\varphi^{i,k} \xrightarrow{C} \varphi^i\,, \quad \psi^{i,k} \xrightarrow{C} \psi^i\,, \quad \text{on } U\,.$$

Then we have
(2.14)
$$\varliminf_{k\to\infty} \int_{\bigcap_{i=1}^m \{\varphi^{i,k} > \psi^{i,k}\}} \chi\, \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k} \geq \int_{\bigcap_{i=1}^m \{\varphi^i > \psi^i\}} \chi\, \theta_{1,\varphi_1} \wedge \dots \wedge \theta_{n,\varphi_n}\,.$$

PROOF. For each $\varepsilon > 0$, $i = 1, \dots, m$, $k \in \mathbb{N}$, define the following functions on U,
$$f_\varepsilon^{i,k} := \frac{(\varphi^{i,k} - \psi^{i,k}) \vee 0}{(\varphi^{i,k} - \psi^{i,k}) \vee 0 + \varepsilon}\,, \quad f_\varepsilon^i := \frac{(\varphi^i - \psi^i) \vee 0}{(\varphi^i - \psi^i) \vee 0 + \varepsilon}\,.$$
Then $f_\varepsilon^{i,k}$ and f_ε^i are quasi-continuous and take value in $[0,1]$. We then have as $k \to \infty$,
$$f_\varepsilon^{i,k} \xrightarrow{C} f_\varepsilon^i\,.$$
By Theorem I.2.5, as $k \to \infty$, we have
$$\chi \prod_{i=1}^m f_\varepsilon^{i,k}\, \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k} \rightharpoonup \chi \prod_{i=1}^m f_\varepsilon^i\, \theta_{1,\varphi_1} \wedge \dots \wedge \theta_{n,\varphi_n}$$
as currents on $U - S$. Since S is pluripolar, we get
$$\begin{aligned}\int_U \chi \prod_{i=1}^m f_\varepsilon^i\, \theta_{1,\varphi_1} \wedge \dots \wedge \theta_{n,\varphi_n} &\leq \varliminf_{k\to\infty} \int_U \chi \prod_{i=1}^m f_\varepsilon^{i,k}\, \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k} \\ &\leq \varliminf_{k\to\infty} \int_{\bigcap_{i=1}^m \{\varphi^{i,k} > \psi^{i,k}\}} \chi\, \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k}\,,\end{aligned}$$
where the second inequality follows the following inequality on U
$$f_\varepsilon^{i,k} \leq \mathbb{1}_{\{\varphi^{i,k} > \psi^{i,k}\}}$$
and the fact that $\operatorname{Supp}\chi \subseteq U$.

Observe that as $\varepsilon \to 0+$, f_ε^i increases pointwisely to $\mathbb{1}_{\varphi^i > \psi^i}$. Let $\varepsilon \to 0+$ and apply the monotone convergence theorem, we conclude (2.14). □

REMARK II.2.4. Here convergence in capacity (resp. quasi-continuity) means convergence (resp. quasi-continuity) in local Bedford–Taylor capacity as in Definition I.2.14 (resp. Definition I.2.13). As we will show later in Theorem III.2.46 and Theorem III.2.48, this is equivalent to convergence (resp. quasi-continuity) in global Monge–Ampère capacity.

Theorem II.2.8. *Let $U \subseteq X$ be an open set. Let θ^i $(i = 1, \dots, m)$ be smooth $(1,1)$-forms on U. Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X, \theta_j)$ $(k \in \mathbb{Z}_{>0}$, $j = 1, \dots, n)$. Let $\varphi^{i,k}, \psi^{i,k}, \varphi^i, \psi^i \in \mathrm{PSH}(U, \theta^i)$ $(k \in \mathbb{N}$, $i = 1, \dots, m)$. Let $\chi \geq 0$ be a bounded quasi-continuous function on X with $\operatorname{Supp} \chi \subseteq U$. Assume that for any $j = 1, \dots, n$, $i = 1, \dots, m$, as $k \to \infty$,*

$$\varphi_j^k \xrightarrow{C} \varphi_j \,, \quad \text{on } X$$

and

$$\varphi^{i,k} \xrightarrow{C} \varphi^i \,, \quad \psi^{i,k} \xrightarrow{C} \psi^i \,, \quad \text{on } U \,.$$

Then we have

(2.15)

$$\varliminf_{k \to \infty} \int_{\bigcap_{i=1}^m \{\varphi^{i,k} > \psi^{i,k}\}} \chi \, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_{\bigcap_{i=1}^m \{\varphi^i > \psi^i\}} \chi \, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \,.$$

Proof. Let

$$\Omega = \bigcap_{j=1}^n \operatorname{Amp}(\theta_j) \,.$$

Then by definition, V_{θ_j} $(j = 1, \dots, n)$ is locally bounded on Ω. For each $C > 0$, let

$$\varphi_j^{k,C} := \varphi_j^k \vee (V_\theta - C) \,, \quad \varphi_j^{;C} := \varphi_j \vee (V_\theta - C) \,.$$

Then as $k \to \infty$

$$\varphi_j^{k,C} \xrightarrow{C} \varphi_j^{;C} \,.$$

Then by Lemma II.2.7,

$$\begin{aligned}
&\int_{\bigcap_{i=1}^m \{\varphi^i > \psi^i\} \cap \bigcap_{j=1}^n \{\varphi_j > V_{\theta_j} - C\}} \chi \, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \\
\leq &\int_{\bigcap_{i=1}^m \{\varphi^i > \psi^i\} \cap \bigcap_{j=1}^n \{\varphi_j > V_{\theta_j} - C\}} \chi \, \theta_{1,\varphi_1^{;C}} \wedge \cdots \wedge \theta_{n,\varphi_n^{;C}} \\
\leq &\varliminf_{k \to \infty} \int_{\bigcap_{i=1}^m \{\varphi^{i,k} > \psi^{i,k}\} \cap \bigcap_{j=1}^n \{\varphi_j^{;k} > V_{\theta_j} - C\}} \chi \, \theta_{1,\varphi_1^{k,C}} \wedge \cdots \wedge \theta_{n,\varphi_n^{k,C}} \\
= &\varliminf_{k \to \infty} \int_{\bigcap_{i=1}^m \{\varphi^{i,k} > \psi^{i,k}\} \cap \bigcap_{j=1}^n \{\varphi_j^{;k} > V_{\theta_j} - C\}} \chi \, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \\
\leq &\varliminf_{k \to \infty} \int_{\bigcap_{i=1}^m \{\varphi^{i,k} > \psi^{i,k}\}} \chi \, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \,.
\end{aligned}$$

Let $C \to \infty$, we conclude by monotone convergence theorem. $\square$

Corollary II.2.9. *Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X, \theta_j)$ $(k \in \mathbb{Z}_{>0}$, $j = 1, \dots, n)$. Let $\chi \geq 0$ be a bounded quasi-continuous function on X. Assume that for any $j = 1, \dots, n$, $i = 1, \dots, m$, as $k \to \infty$,*

$$\varphi_j^k \xrightarrow{C} \varphi_j \,.$$

Then for any open set $U \subseteq X$, we have

$$\varliminf_{k \to \infty} \int_U \chi \, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_U \chi \, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \,. \tag{2.16}$$

Corollary II.2.10. *Assume in addition that*

$$\varlimsup_{k\to\infty}\int_X \theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}\,. \tag{2.17}$$

Then

$$\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k} \rightharpoonup \theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}\,.$$

Remark II.2.5. (2.17) is automatically satisfied if any of the following case is true. 1. $\varphi_j^k \nearrow \varphi_j$, a.e. as $k\to\infty$. (Corollary II.3.23) 2. $\varphi_j^k,\varphi_j\in\mathcal{E}(X,\theta_j)$.

Theorem II.2.11. *Let $p\leq n$. Let $\alpha_0,\dots,\alpha_p$ be big classes on X with smooth representatives $\theta_0,\dots,\theta_p$.*

Let $W\subseteq X$ be an open subset. Let $\chi\in C^0_c(W)$, $\chi\geq 0$. Let $\Theta\in\mathcal{D}'^{n-p,n-p}_+(W)$.
Let $\varphi_j^k,\varphi_j\in\mathrm{PSH}(W,\theta_j)$ ($j=0,\dots,p$ and $k\in\mathbb{Z}_{>0}$). Let $\psi^k,\psi\in\mathrm{PSH}(W,\theta_0)$.
Assume that there is a closed pluripolar set S, such that

(1) φ_j^k,ψ^k are uniformly bounded on each compact subset of $\operatorname{Supp}\chi - S$.
(2) φ_j^k (resp. ψ^k) decrease to φ_j (resp. ψ) for any j.
(3) $\varphi_0^k-\psi^k$ are uniformly bounded on $\operatorname{Supp}\chi - S$.
(4)

$$\chi\mathbb{1}_{X-S}\,\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{p,\varphi_p^k}\wedge\Theta \rightharpoonup \chi\mathbb{1}_{X-S}\,\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{p,\varphi_p}\wedge\Theta,\quad k\to\infty.$$

Then as $k\to\infty$,

$$\chi(\varphi_0^k-\psi^k)\mathbb{1}_{X-S}\,\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{p,\varphi_p^k}\wedge\Theta \rightharpoonup \chi(\varphi_0-\psi)\mathbb{1}_{X-S}\,\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{p,\varphi_p}\wedge\Theta.$$

Proof. Let

$$\mu^k := \chi\mathbb{1}_{X-S}\,\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{p,\varphi_p^k}\wedge\Theta,\quad \mu := \chi\mathbb{1}_{X-S}\,\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{p,\varphi_p}\wedge\Theta.$$

Let $\rho\in C^0_c(W)$. It suffices to prove

$$\lim_{k\to\infty}\int_W \rho(\varphi_0^k-\psi^k)\,\mu^k = \int_X \rho(\varphi_0-\psi)\,\mu.$$

By assumption, $\mu^k\rightharpoonup\mu$, so

$$\lim_{k\to\infty}\mu^k(W)=\mu(W).$$

Fix $\varepsilon>0$. Take open sets $U\Subset V\Subset W-S$ such that

$$\mu(W-U)<\varepsilon.$$

Let $\tau\in C^0_c(V)$, $1\geq\tau\geq 0$, $\tau=1$ on U.
Notice that

$$\varliminf_{k\to\infty}\mu^k(U)\geq\mu(U).$$

So

$$\varlimsup_{k\to\infty}\mu^k(W-U)\leq\mu(W-U)<\varepsilon.$$

By assumption, on V, $\chi(\varphi_0^k-\psi^k)$, φ_j^k are all bounded, so we can apply the local result in Bedford–Taylor theory ([GZ17] Theorem 3.18) to get

$$\chi(\varphi_0^k-\psi^k)\mu_k \rightharpoonup \chi(\varphi_0-\psi)\mu.$$

So

$$\int_V \tau\rho\chi(\varphi_0-\psi)\,\mu = \lim_{k\to\infty}\int_V \tau\rho\chi(\varphi_0^k-\psi^k)\,\mu_k.$$

So
$$\left|\int_W \rho(\varphi_0^k-\psi^k)\,\mu^k-\int_W \rho(\varphi_0-\psi)\,\mu\right|\leq C\varepsilon+\left|\int_V \rho\tau(\varphi_0^k-\psi^k)\,\mu^k-\int_V \rho\tau(\varphi_0-\psi)\,\mu\right|.$$
Hence
$$\varlimsup_{k\to\infty}\left|\int_W \rho(\varphi_0^k-\psi^k)\,\mu^k-\int_W \rho(\varphi_0-\psi)\,\mu\right|\leq C\varepsilon.$$
Let $\varepsilon\to 0+$, we conclude. □

THEOREM II.2.12. *Let $\alpha_1,\dots,\alpha_n$ be big classes on X with smooth representatives $\theta_1,\dots,\theta_n$. Let $\varphi_j^k,\varphi_j\in\mathrm{PSH}(X,\theta_j)$ ($j=1,\dots,n$ and $k\in\mathbb{Z}_{>0}$). Let f^k,f are bounded quasi-continuous function on X. Assume that f^k are uniformly bounded and that $f^k\xrightarrow{C} f$. Assume that there is a closed pluripolar set S, such that*

(1) φ_j^k are uniformly bounded on each compact subset of $X-S$.
(2) $\varphi_j^k\xrightarrow{C}\varphi_j$ as $k\to\infty$.
(3)
$$\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}\rightharpoonup\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n},\quad k\to\infty.$$

Then
$$f^k\,\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}\rightharpoonup f\,\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n},\quad k\to\infty.$$

PROOF. The proof is almost identical to that of Theorem II.2.11. □

REMARK II.2.6. One can also state Theorem II.2.12 in a local way as Theorem II.2.11.

II.3. Monotonicity of Monge–Ampère masses

II.3.1. Notations. Let X be a compact Kähler manifold of dimension n. Let α be a big class with smooth representative θ. Let Z be the complement of the ample locus of α. For each $N\geq 1$, define
$$\Sigma_N:=\{\alpha\in\mathbb{R}^N_{\geq 0}:|\alpha|\leq 1\},$$
where $|\alpha|$ is the sum of components of α.

For each $N\geq 1$, we fix a basis $Z_0,\dots,Z_N$ of $H^0(\mathbb{P}^N,\mathcal{O}(1))$. Let
$$H=H_N:=\{Z_0=0\}\subseteq\mathbb{P}^N.$$

On $\mathbb{P}^N-H$, define
$$z_a:=\frac{Z_a}{Z_0}\in\Gamma(\mathbb{P}^N-H,\mathcal{O}),\quad a=1,\dots,N.$$
We will identify $\mathbb{P}^N-H$ with $\mathbb{C}^N$ via $(z_1,\dots,z_N)$.

Let ω_N be the Fubini–Study form on $\mathbb{P}^N$, normalized so that
$$\int_{\mathbb{P}^N}\omega_N^N=1.$$
By abuse of notation, we denote the metric induced by ω_N on $\mathcal{O}(1)$ by ω_N. Observe that on $\mathbb{P}^N-H$,
$$\omega_N=-\mathrm{dd^c}\log|Z_0|^2_{\omega_N}.\tag{2.18}$$

For each $N\geq 1$, let
$$X_N:=X\times\mathbb{P}^N.$$

Let π_1^N, π_2^N be the natural projections:

$$\begin{array}{ccc} X_N & \xrightarrow{\pi_2^N} & \mathbb{P}^N \\ \big\downarrow{\scriptstyle \pi_1^N} & & \\ X & & \end{array}$$

For simplicity, we denote $\pi_2^{N*} Z_A$ by Z_A $(A = 0, \dots, N)$, similar convention is used for $z_1, \dots, z_N$. Similarly, we omit π_1^{N*} from our notations from time to time.

Let
$$\theta_N = \left(\pi_1^N\right)^* \theta + \left(\pi_2^N\right)^* \omega_N.$$
Note that $[\theta_N]$ is a big class on X_N.

Fix $\eta \in \mathrm{PSH}(X, \theta)$ such that

(1) $\eta \in C^\infty(X - Z)$.
(2) $\eta \leq 0$.

We may even assume that η has analytic singularity by [Bou04] Theorem 3.17.

II.3.2. Quadratic optimization. Let $N \geq 1$. We study the following function $f_N : \mathbb{R}^N \to \mathbb{R}$:
$$f_N(x) := \min_{\alpha \in \Sigma_N} (x - \alpha)^2.$$
Let $\Pi : \mathbb{R}^N \to \Sigma_N$ be the closest point projection. It is well-defined since Σ_N is convex and closed. Let $e = (1, 1, \dots, 1) \in \mathbb{R}^N$.

Let $\mathcal{F}$ be the set of faces of Σ_N as a simplex. By a face, we mean the interior of the face. The extremal points of Σ_N are also considered as faces in $\mathcal{F}$. So
$$\Sigma_N = \coprod_{F \in \mathcal{F}} F.$$
Observe that if $\Pi(x) \in F \in \mathcal{F}$, then so is $\Pi(x + \varepsilon e)$ for small enough $\varepsilon > 0$. Let $A_F = \Pi^{-1} F$, then
$$\mathbb{R}^N = \coprod_{F \in \mathcal{F}} A_F.$$
Now observe that on each A_F, Π is affine.

Define $g_N : \mathbb{R}^N \to \mathbb{R}$:
$$g_N(x) = f_N(x) - x^2.$$
Then we have
$$g_N(x) = (x - \Pi x)^2 - x^2.$$

Proposition II.3.13. *For $x \in \mathbb{R}^N$,*
$$g_N(x + te) - g_N(x) = tL_N(x) + \mathcal{O}(t^2), \quad t \to 0+,$$
where the $\mathcal{O}$-constant depends only on N, $L_N(x)$ is a bounded continuous piecewise linear function whose coefficients depend only on N.

When $x \in \mathring{\Sigma}_N$, $L_N(x) = -2|x|$.

Proof. All statements are obvious except that $L_N(x)$ is bounded and continuous. To see that L_N is bounded, it suffices to show that $g_N(x + te) - g_N(x)$ is bounded for a fixed $t > 0$. More generally, let $x, y \in \mathbb{R}^N$, then
$$g_N(x) - g_N(y) \geq \min_{\alpha \in \Sigma_N} \left((x - \alpha)^2 - x^2 - (y - \alpha)^2 + y^2\right) = \min_{\alpha \in \Sigma_N} 2\alpha \cdot (y - x).$$

A similar inequality hold if we interchange x and y. So

$$|g_N(x) - g_N(y)| \leq C.$$

To see L_N is continuous, observe that

$$g_N(x+te) - g_N(x)$$

is a quadratic function in t for any x. And since $L_N(x)$ is nothing but the coefficient of t, it suffices to show that $g_N(x+te) - g_N(x)$ is continuous in x for three value of t. So the result follows from the obvious continuity of g_N. $\square$

PROPOSITION II.3.14. *The function $g_N \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$.*

PROOF. It follows from general facts that

$$x \mapsto (x - \Pi x)^2$$

is in $C^{1,1}$. See [BL10] Section 3.3 Exercise 12(d) and Section 2.1 Exercise 8(c.iii). $\square$

Now we extend the domain of definition of g_N, we will get a symmetric function $g_N : [-\infty,\infty)^N \to \mathbb{R}$. The definition is by induction on N, when $N=1$, we simply define

$$g_1(-\infty) = 0.$$

For $N > 1$, define

$$g_N(x_1,\ldots,x_M,x_{M+1},\ldots,x_N) = g_M(x_1,\ldots,x_M),$$

where $x_{M+1},\ldots,x_N = -\infty$ and $x_1,\ldots,x_M \in \mathbb{R}$. We formally set $g_0 = 0$. We get a full definition of g_N by requiring that it is symmetric in the N-arguments. It is not hard to see that g_N is continuous.

PROPOSITION II.3.15. *The function $g_N : [-\infty,\infty)^N \to \mathbb{R}$ is decreasing in each of its arguments.*

PROOF. It suffices to prove this on $\mathbb{R}^N$. By definition, it suffices to show that for each $\alpha \in \Sigma_N$, the function

$$(x-\alpha)^2 - x^2$$

is decreasing in each argument. This reduces immediately to the case $N=1$ and the result is obvious. $\square$

II.3.3. Witt Nyström construction. Let $W \subseteq X$ be an open subset. Let φ be a θ-psh function on W. We define $\Phi_N[\varphi] \in \mathrm{PSH}(W \times \mathbb{P}^N, \theta_N)$ by

(2.19)
$$\Phi_N[\varphi] := \sup_{\alpha\in\Sigma_N}{}^{*} \left((1-|\alpha|)(\eta + \log|Z_0|^2_{\omega_N}) + |\alpha|\varphi + \sum_{a=1}^{N} \alpha_a \log|Z_a|^2_{\omega_N} - \alpha^2 \right).$$

Define $\hat{\alpha}_a = \hat{\alpha}_a[\varphi] : (W - Z) \times \mathbb{C}^N \to [-\infty,\infty)$ $(a = 1,\ldots,N)$ by

$$\hat{\alpha}_a := \frac{\log|z_a|^2 + \varphi - \eta}{2}. \tag{2.20}$$

Observe that $\hat{\alpha}_a$ is usc.

We define $\hat{\alpha} : (W-Z) \times \mathbb{C}^N \to [-\infty,\infty)^N$ by

$$\hat{\alpha} = (\hat{\alpha}_1,\ldots,\hat{\alpha}_N). \tag{2.21}$$

PROPOSITION II.3.16. *Let $W \subseteq X$ be an open subset. Let $\varphi \in \mathrm{PSH}(W,\theta)$. Then*

(1) $\Phi_N[\varphi]$ *is increasing in* φ.

(2)

(2.22)
$$\Phi_N[\varphi] = \sup_{\alpha\in\Sigma_N}\left((1-|\alpha|)(\eta+\log|Z_0|^2_{\omega_N}) + |\alpha|\varphi + \sum_{a=1}^{N}\alpha_a\log|Z_a|^2_{\omega_N} - \alpha^2\right)$$

on $(W-Z)\times(\mathbb{P}^N-H)$. *Moreover, on this set,*

$$\Phi_N[\varphi] = \log|Z_0|^2_{\omega_N} + \eta - g_N\circ\hat{\alpha}[\varphi], \tag{2.23}$$

where g_N *is the function defined in Section II.3.2.*

Proof. (1) follows directly from definition.

(2) In order to prove (2.22), it suffices to show that the RHS of (2.22) is usc on $(W-Z)\times(\mathbb{P}^N-H)$.

Since $\log|Z_0|^2_{\omega_N}$ is obviously continuous, it suffices to prove that the following function is usc on $(W-Z)\times\mathbb{C}^N$:

$$\begin{aligned} I :=& \sup_{\alpha\in\Sigma_N}\left((1-|\alpha|)\eta + |\alpha|\varphi + \sum_{a=1}^{N}\alpha_a\log|z_a|^2 - \sum_{a=1}^{N}\alpha_a^2\right)\\ =&\eta - g_N\circ\hat{\alpha}\end{aligned}$$

by completing the square.

Since $\hat{\alpha}_a$ is usc and g_N is continuous and decreasing (Proposition II.3.15), we conclude that I is usc. Moreover, (2.23) is implied by our calculation. □

Corollary II.3.17. *Let* φ_j,φ $(j\in\mathbb{Z}_{>0})$ *be* θ*-psh functions on* W. *If* φ_j *converges to* φ *outside a pluripolar set, then* $\Phi_N[\varphi_j]$ *also converges to* $\Phi_N[\varphi]$ *outside a pluripolar set. The sequence* $\Phi_N[\varphi_j]$ *is decreasing if* φ_j *is.*

Corollary II.3.18. *Let* $\varphi\in\mathrm{PSH}(X,\theta)$, *then*

$$\Phi_N[\varphi]\geq\eta+\log|Z_0|^2_{\omega_N}.$$

II.3.4. Witt Nyström's theorem.

Theorem II.3.19. *Let* $W\subseteq X$ *be an open subset. Let* $\varphi\in\mathrm{PSH}(W,\theta)$, $N\geq 1$. *Assume that both* θ^n_φ *and* $\theta^{N+n}_{N,\Phi_N[\varphi]}$ *are well-defined. Then we have*

$$\pi^N_{1*}\theta^{N+n}_{N,\Phi_N[\varphi]} = \binom{N+n}{n}N\int_0^1\theta^n_{(1-t)\eta+t\varphi}t^{N-1}\,\mathrm{d}t. \tag{2.24}$$

Remark II.3.7. We can evaluate the integral on the RHS, this is an example of the so called β-integral, the result is

$$\binom{N+n}{n}N\int_0^1\theta^n_{(1-t)\eta+t\varphi}t^{N-1}\,\mathrm{d}t = \sum_{j=0}^{n}\binom{N+n-1-j}{n-j}\frac{1}{j!}\,\theta^j_\eta\wedge\theta^{n-j}_\varphi. \tag{2.25}$$

Proof. Observe that $\theta_{(1-t)\eta+t\varphi}$ is well-defined for any $t\in[0,1]$ by Proposition II.1.4.

Since the problem is local in nature, we may assume that W is the unit disk in $\mathbb{C}^n$. Moreover, since locally we may absorb θ into the potentials and set $\theta=0$, so that φ is psh. In the following, we keep θ only to make the notations less messy.

Step 1. Let us show that we can always reduce to the case where φ is bounded. In fact, for each $C>0$, let $\varphi^C:=\varphi\vee(-C)$. If we have proved the special case, we get

$$\pi_{1*}^N\theta^{N+n}_{N,\Phi_N[\varphi^C]}=\binom{N+n}{n}N\int_0^1\theta^n_{(1-t)\eta+t\varphi^C}t^{N-1}\,\mathrm{d}t.$$

Hence

$$\mathbb{1}_{\{\varphi>-C\}}\pi_{1*}^N\theta^{N+n}_{N,\Phi_N[\varphi^C]}=\mathbb{1}_{\{\varphi>-C\}}\binom{N+n}{n}N\int_0^1\theta^n_{(1-t)\eta+t\varphi^C}t^{N-1}\,\mathrm{d}t.$$

Observe that

$$\mathbb{1}_{\{\varphi>-C\}}\pi_{1*}^N\theta^{N+n}_{N,\Phi_N[\varphi^C]}=\pi_{1*}^N\left(\mathbb{1}_{\{\varphi>-C\}}\theta^{N+n}_{N,\Phi_N[\varphi^C]}\right),$$

since $\Phi_N[\varphi^C]=\Phi_N[\varphi]$ on the plurifine open set $\{\varphi>-C\}\subseteq W\times\mathbb{P}^N$.

So we get

$$\pi_{1*}^N\left(\mathbb{1}_{\{\varphi>-C\}}\theta^{N+n}_{N,\Phi_N[\varphi]}\right)=\mathbb{1}_{\{\varphi>-C\}}\binom{N+n}{n}N\int_0^1\theta^n_{(1-t)\eta+t\varphi}t^{N-1}\,\mathrm{d}t. \tag{2.26}$$

When $C\to\infty$, by dominated convergence theorem, we have

$$\mathbb{1}_{\{\varphi>-C\}}\binom{N+n}{n}N\int_0^1\theta^n_{(1-t)\eta+t\varphi}t^{N-1}\,\mathrm{d}t\rightharpoonup\binom{N+n}{n}N\int_0^1\theta^n_{(1-t)\eta+t\varphi}t^{N-1}\,\mathrm{d}t,$$

Similarly, as $C\to\infty$,

$$\mathbb{1}_{\{\varphi>-C\}}\theta^{N+n}_{N,\Phi_N[\varphi]}\rightharpoonup\theta^{N+n}_{N,\Phi_N[\varphi]}.$$

Since π_1^N is continuous,

$$\pi_{1*}^N\left(\mathbb{1}_{\{\varphi>-C\}}\theta^{N+n}_{N,\Phi_N[\varphi]}\right)\rightharpoonup\pi_{1*}^N\theta^{N+n}_{N,\Phi_N[\varphi]}.$$

Hence let $C\to\infty$ in (2.26), we conclude (2.24).

Step 2. Let us show that we can further reduce to the case where φ is smooth. Assume that the theorem holds when φ is smooth. Let φ^k ($k\in\mathbb{Z}_{>0}$) be a decreasing sequence of smooth psh functions on W converging to φ.

Then we get

$$\pi_{1*}^N\theta^{N+n}_{N,\Phi_N[\varphi^k]}=\binom{N+n}{n}N\int_0^1\theta^n_{(1-t)\eta+t\varphi^k}t^{N-1}\,\mathrm{d}t. \tag{2.27}$$

By Corollary II.3.17, $\Phi_N[\varphi^k]$ decrease to $\Phi_N[\varphi]$ outside a pluripolar set, hence to $\Phi_N[\varphi]$ everywhere ([GZ17] Corollary 1.38). In particular, by Bedford–Taylor theory

$$\theta^{N+n}_{N,\Phi_N[\varphi^k]}\rightharpoonup\theta^{N+n}_{N,\Phi_N[\varphi]},\quad k\to\infty$$

on $W\times\mathbb{C}^N$. As follows from Step 3, the support of $\theta^{N+n}_{N,\Phi_N[\varphi^k]}$ is contained in W times a fixed bounded subset of $\mathbb{C}^N$, so

$$\pi_{1*}^N\theta^{N+n}_{N,\Phi_N[\varphi^k]}\rightharpoonup\pi_{1*}^N\theta^{N+n}_{N,\Phi_N[\varphi]},\quad k\to\infty.$$

The RHS of (2.24) can be written as

$$\sum_{j=0}^n\binom{N+n-1-j}{n-j}\frac{1}{j!}\,\theta_\eta^j\wedge\theta_\varphi^{n-j}. \tag{2.28}$$

See (2.25).

But we know that

$$\theta_\eta^j \wedge \theta_{\varphi^k}^{n-j} \rightharpoonup \theta_\eta^j \wedge \theta_\varphi^{n-j}, \quad k \to \infty$$

from Bedford–Taylor's theory. Hence

$$\binom{N+n}{n} N \int_0^1 \theta^n_{(1-t)\eta + t\varphi^k} t^{N-1} \, \mathrm{d}t \rightharpoonup \binom{N+n}{n} N \int_0^1 \theta^n_{(1-t)\eta + t\varphi} t^{N-1} \, \mathrm{d}t, \quad k \to \infty.$$

Let $k \to \infty$ in (2.27), we get (2.24).

Step 3. Let us show that we can further reduce to the case where η is smooth as well. First observe that neither side of (2.24) charges the closed set Z, so we may assume that $W \cap Z = \emptyset$.

Replacing W be a smaller set, we may assume that η is bounded on W.

Take a Demailly approximation as in Step 2, the remaining argument is similar as that in Step 2.

Step 4. We prove the theorem under the assumption that φ, η are both smooth. Locally, we may set $\theta = 0$ as before. We still keep θ in our notations, but with $\theta = 0$ understood.

Since $\theta^{N+n}_{N,\Phi_N[\varphi]}$ does not charge

$$W \times \bigcup_{a=0}^N \{Z_a = 0\},$$

we may restrict $\Phi_N[\varphi]$ to $W \times \mathbb{C}^{*N}$ when proving (2.24). In this case, by (2.23) and Proposition II.3.14, we have $\Phi_N[\varphi] \in C^{1,1}_{\mathrm{loc}}(W \times \mathbb{C}^{*N})$.

Let $\mathrm{Log} : \mathbb{C}^{*N} \to \mathbb{R}^N$ defined by

$$(z_1, \dots, z_N) \mapsto (\log |z_1|^2, \dots, \log |z_N|^2).$$

By abuse of notation, we also write Log for the map $W \times \mathbb{C}^N \to W \times \mathbb{R}^N$ defined by $(\mathrm{Id}, \mathrm{Log})$.

We identify $\Phi_N[\varphi]$ with a map on $W \times \mathbb{R}^N$:

$$\Phi_N[\varphi](x,y) = \sup_{\alpha \in \Sigma_N} \left((1 - |\alpha|)\eta(x) + |\alpha|\varphi(x) + \sum_{a=1}^N \alpha_a y_a - \sum_{a=1}^N \alpha_a^2 \right). \tag{2.29}$$

Let

$$V_N := \hat{\alpha}[\varphi]^{-1}(\mathring{\Sigma}_N) \subseteq W \times \mathbb{C}^N.$$

For each $x \in W$, let

$$V_{N,x} = V_N \cap \left(\{x\} \times \mathbb{C}^N\right).$$

Let $\mu_N : \mathrm{Log}[V_N] \to W \times \mathring{\Sigma}_N$ be the following map:

$$(x,y) \mapsto (x, \nabla_y \Phi_N[\varphi](x,y)) = (x, \hat{\alpha}[\varphi]),$$

where by abuse of notation, we have denoted the function on $\mathrm{Log}[V_N]$ induced by $\hat{\alpha}$ by the same notation.

Now we claim that ∂V_N is has zero Lebesgue measure. In fact, by definition,

$$\partial V_N \subseteq \hat{\alpha}[\varphi]^{-1}(\partial \Sigma_N).$$

It suffices to show that the inverse image of each face of Σ_N has zero measure. In particular, by Fubini theorem, it suffices to prove the following: Let $(b_a) \in$

$[0,1]^N - \{(0,\dots,0)\}$. Then for any constant $C \geq 0$,

$$\left\{(x,y)\in W\times\mathbb{R}^N : \sum_{a=1}^N b_a(y_a+\varphi(x)-\eta(x))=C\right\}$$

has zero measure. Again by Fubini theorem, it suffices to prove that for almost all $x \in W$, the set

$$\left\{y\in\mathbb{R}^N \sum_{a=1}^N b_a(y_a+\varphi(x)-\eta(x))=C\right\}$$

is null, which is obvious.

As $\Phi_N[\varphi]$ has $C^{1,1}$-regularity, $\theta^{N+n}_{N,\Phi_N[\varphi]}$ is absolutely continuous, hence does not charge V_N. So

$$\theta^{N+n}_{N,\Phi_N[\varphi]} = \mathbb{1}_{V_N}\theta^{N+n}_{N,\Phi_N[\varphi]} + \mathbb{1}_{W\times\mathbb{C}^N-\overline{V_N}}\theta^{N+n}_{N,\Phi_N[\varphi]}.$$

Note that then

$$\mathrm{Log}_*\,\theta^{N+n}_{N,\Phi_N[\varphi]} = \mathrm{MA}_{\mathbb{R}}(\Phi_N[\varphi]),$$

where $\Phi_N[\varphi]$ on RHS is understood as a function on $W\times\mathbb{R}^N$. The convention here for the real Monge–Ampère operator is the same as in [CGSZ19, Lemma 2.2].

We also notice that by (2.29), $\Phi_N[\varphi](x,\cdot)$ is the Legendre transform of the following function $\Psi:\mathbb{R}^N\to\mathbb{R}$:

$$\Psi(\alpha)=\begin{cases}\alpha^2-(1-|\alpha|)\eta(x)-|\alpha|\varphi(x), & \alpha\in\Sigma_N,\\ \infty, & \alpha\notin\Sigma_N.\end{cases}$$

So $\Phi_N[\varphi]$ is not strictly convex outside V_N, we find

$$\mathbb{1}_{W\times\mathbb{C}^N-\overline{V_N}}\theta^{N+n}_{N,\Phi_N[\varphi]}=0.$$

On the other hand, on V_N, by Proposition II.3.16 (3) and the explicit expression of g_N, we have

$$\Phi_N[\varphi]=\log|Z_0|^2_{\omega_N}+\eta+\hat{\alpha}[\varphi]^2.$$

In particular,

$$\omega_N+\mathrm{dd^c}\Phi_N[\varphi]=(1-|\hat{\alpha}[\varphi]|)\,\mathrm{dd^c}\eta+|\hat{\alpha}[\varphi]|\mathrm{dd^c}\varphi+\sum_{a=1}^N \mathrm{d}\hat{\alpha}_a[\varphi]\wedge\mathrm{d^c}\hat{\alpha}_a[\varphi].$$

Obviously,

$$\left(\sum_{a=1}^N \mathrm{d}\hat{\alpha}_a[\varphi]\wedge\mathrm{d^c}\hat{\alpha}_a[\varphi]\right)^{N+1}=0.$$

So we get

(2.30)

$$\theta^{N+n}_{N,\Phi_N[\varphi]}=\binom{N+n}{n}\mathbb{1}_{V_N}\left((1-|\hat{\alpha}[\varphi]|)\,\mathrm{dd^c}\eta+|\hat{\alpha}[\varphi]|\mathrm{dd^c}\varphi\right)^n\wedge\left(\sum_{a=1}^N \mathrm{d}\hat{\alpha}_a[\varphi]\wedge\mathrm{d^c}\hat{\alpha}_a[\varphi]\right)^N.$$

Let us evaluate the RHS. It is obvious that in evaluating $\mathrm{d}\hat{\alpha}_a[\varphi]$ and $\mathrm{d^c}\hat{\alpha}_a[\varphi]$, we only have to consider differentials in variables in $\mathbb{C}^N$. Then

$$\theta^{N+n}_{N,\Phi_N[\varphi]}=\binom{N+n}{n}\mathbb{1}_{V_N}\left((1-|\hat{\alpha}[\varphi]|)\,\mathrm{dd^c}\eta+|\hat{\alpha}[\varphi]|\mathrm{dd^c}\varphi\right)^n\wedge\left(\sum_{a=1}^N \mathrm{d}\log|z_a|\wedge\mathrm{d^c}\log|z_a|\right)^N.$$

Pushing forward both sides by $\mu_N\circ\mathrm{Log}$, we get

$$(\mu_N\circ\mathrm{Log})_*\theta^{N+n}_{N,\Phi_N[\varphi]}=N!\binom{N+n}{n}\left((1-|\alpha|)\,\mathrm{dd^c}\eta+|\alpha|\mathrm{dd^c}\varphi\right)^n\otimes\mathrm{d}\lambda(\alpha),$$

as measures on $W\times\mathring{\Sigma}_N$, where $\mathrm{d}\lambda$ is the standard Lebesgue measure on $\mathring{\Sigma}_N$.

Pushing-forward both sides to W, we get

$$\begin{aligned}\pi^N_{1,*}\theta^{N+n}_{N,\Phi_N[\varphi]}&=N!\binom{N+n}{n}\int_{\mathring{\Sigma}_N}\left((1-|\alpha|)\,\mathrm{dd^c}\eta+|\alpha|\mathrm{dd^c}\varphi\right)^n\mathrm{d}\lambda(\alpha)\\&=N\binom{N+n}{n}\int_0^1\left((1-t)\mathrm{dd^c}\eta+t\mathrm{dd^c}\varphi\right)^n t^{N-1}\,\mathrm{d}t.\end{aligned}$$

□

COROLLARY II.3.20. *Let $W\subseteq X$ be an open subset. Let $\varphi\in\mathrm{PSH}(W,\theta)$, assume that θ^n_φ and $\theta^{N+n}_{N,\Phi_N[\varphi]}$ are well-defined for all $N\geq 1$, then*

$$\frac{n!}{N^n}\pi^N_{1*}\theta^{N+n}_{N,\Phi_N[\varphi]}\to\theta^n_\varphi,\quad N\to\infty \tag{2.31}$$

in total variation.

II.3.5. Witt Nyström's monotonicity theorem. Let X be a fixed compact Kähler manifold of dimension n. Let $\alpha_1,\dots,\alpha_n\in H^{1,1}(X,\mathbb{R})$ be big cohomology classes. Let $\theta_j\in\alpha_j$ be a smooth representative.

THEOREM II.3.21. *Let $\varphi_j,\psi_j\in\mathrm{PSH}(X,\theta_j)$. Assume that $[\varphi_j]\succeq[\psi_j]$ for every j, then*

$$\int_X\langle\theta_{1,\varphi_1}\wedge\cdots\theta_{n,\varphi_n}\rangle\geq\int_X\langle\theta_{1,\psi_1}\wedge\cdots\theta_{n,\psi_n}\rangle.$$

We begin with a special case.

LEMMA II.3.22. *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. Assume that $[\varphi]=[\psi]$, then*

$$\int_X\theta^n_\varphi=\int_X\theta^n_\psi.$$

PROOF. **Step 1**. We shall prove this lemma under the additional assumption that φ and ψ both have small unbounded loci. In this case, it is more convenient to prove more generally for $\varphi_j,\psi_j\in\mathrm{PSH}(X,\theta)$ with $[\varphi_j]=[\psi_j]$ that

$$\int_X\langle\theta_{\varphi_1}\wedge\cdots\wedge\theta_{\varphi_n}\rangle=\int_X\langle\theta_{\psi_1}\wedge\cdots\wedge\theta_{\psi_n}\rangle.$$

In turn, it suffices to prove the following: let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$, $[\varphi]\preceq[\psi]$, let $\Theta\in\mathcal{D}^{n-1,n-1}_+(X)$, let A be a closed pluripolar set outside which φ is locally bounded, then

$$\int_{X-A}\theta_\varphi\wedge\Theta\leq\int_{X-A}\theta_\psi\wedge\Theta,$$

We may assume that $\psi\geq\varphi$. Let η be a qpsh function that equals $-\infty$ exactly on A. Adding $\varepsilon\eta$ to φ and let $\varepsilon\to 0+$ in the end, we may assume that $\varphi-\psi\to-\infty$. Define $\psi_k=\varphi\vee(\psi-k)$ for $k>0$. Then ψ_k coincides with $\psi-k$ in a neighbourhood of A. Then by Stokes theorem

$$\int_{X-A}\mathrm{dd^c}\psi_k\wedge\Theta=\int_{X-A}\mathrm{dd^c}\psi\wedge\Theta$$

On the other hand, as ψ_k is decreasing, on $X - A$, we have

$$\mathrm{dd^c}\psi_k \wedge \Theta \rightharpoonup \mathrm{dd^c}\varphi \wedge \Theta, \quad k \to \infty$$

as currents by Bedford–Taylor theory. Taking the integral on $X - A$, the result follows.

Step 2. We shall reduce the general case the case in Step 1.

By Step 1,

$$\int_{X_N} \theta_{N,\Phi_N[\varphi]}^{N+n} = \int_{X_N} \theta_{N,\Phi_N[\psi]}^{N+n}.$$

The desired result follows from Corollary II.3.20. □

Proof of Theorem II.3.21. For each $t > 0$, set

$$\psi_j^t := (\varphi_j - t) \vee \psi_j.$$

Then $\psi_j^t \xrightarrow{C} \psi_j$. So by Theorem II.2.8, it suffices to prove that

$$\int_X \langle \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \rangle = \int_X \langle \theta_{1,\psi_1^t} \wedge \cdots \wedge \theta_{n,\psi_n^t} \rangle.$$

Obviously, $[\psi_j^t] = [\varphi_j]$. So we reduce to prove the theorem in case $[\varphi_j] = [\psi_j]$. This follows from Lemma II.3.22 by polarization. More precisely, for each $t \in \mathbb{R}^n_{>0}$, define

$$\varphi_t = \sum_j t_j \varphi_j, \quad \psi_t = \sum_j t_j \psi_j, \quad \theta_t = \sum_j t_j \theta_j.$$

Then both $\int_X \theta^n_{t,\varphi_t}$ and $\int_X \theta^n_{t,\psi_t}$ are homogeneous polynomials in t of degree n. Lemma II.3.22 identifies their coefficients. □

Corollary II.3.23. *Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X, \theta_j)$ for $k \in \mathbb{Z}_{>0}$. Assume that $\varphi_j^k \nearrow \varphi_j$, a.e. as $k \to \infty$. Let χ be a bounded quasi-continuous function on X. Then*

$$\chi \langle \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \rangle \rightharpoonup \chi \langle \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \rangle.$$

II.3.6. Full mass class.

Definition II.3.3. We say $\varphi \in \mathrm{PSH}(X, \theta)$ has *full mass* if

$$\int_X \theta_\varphi^n = \int_X \theta_{V_\theta}^n.$$

The set of $\varphi \in \mathrm{PSH}(X, \theta)$ having full masses is denoted by $\mathcal{E}(X, \theta)$.

Notice that by Theorem II.3.21, we always have the following inequality:

$$\int_X \theta_\varphi^n \leq \int_X \theta_{V_\theta}^n$$

Again by Theorem II.3.21, we have

$$\mathcal{E}^\infty(X, \theta) \subseteq \mathcal{E}(X, \theta). \tag{2.32}$$

Proposition II.3.24. *Let $\varphi \in \mathrm{PSH}(X, \theta)$. Assume that $\frac{1}{2}(\varphi + V_\theta) \in \mathcal{E}(X, \theta)$, then $\varphi \in \mathcal{E}(X, \theta)$.*

PROOF. Let
$$u=\frac{1}{2}\left(\varphi+V_\theta\right).$$
For any $j\geq 0$, let
$$u_j=u\vee\left(V_\theta-j\right),\quad \varphi_j=\varphi\vee(V_\theta-j).$$
Then
$$u_j=\frac{1}{2}\left(\varphi_{2j}+V_\theta\right).$$
Hence
$$\theta_{u_j}\geq\frac{1}{2}\theta_{\varphi_{2j}}.$$
So
$$\int_X \mathbb{1}_{\{\varphi\leq V_\theta-2j\}}\,\theta^n_{\varphi_{2j}}\leq 2^n\int_X \mathbb{1}_{\{u\leq V_\theta-j\}}\,\theta^n_{u_j}.$$
We conclude that
$$\lim_{j\to\infty}\int_X \mathbb{1}_{\{\varphi\leq V_\theta-2j\}}\,\theta^n_{\varphi_{2j}}\to 0.$$
□

PROPOSITION II.3.25. *Let $\varphi\in\mathrm{PSH}(X,\theta)$. Then*
$$\int_X\theta^n_\varphi=\int_X\theta^n_{[\varphi]\wedge V_\theta}.$$

PROOF. By definition, $[\varphi]\preceq[\varphi]\wedge V_\theta$, so Theorem II.3.21 implies that
$$\int_X\theta^n_\varphi\leq\int_X\theta^n_{[\varphi]\wedge V_\theta}.$$
On the other hand,
$$[\varphi]=[(\varphi+C)\wedge V_\theta],$$
so again by Theorem II.3.21,
$$\int_X\theta^n_\varphi=\int_X\theta^n_{(\varphi+C)\wedge V_\theta}.$$
Take the limit $C\to\infty$, according to Theorem II.2.8,
$$\int_X\theta^n_\varphi\geq\int_X\theta^n_{[\varphi]\wedge V_\theta}.$$
We conclude. □

COROLLARY II.3.26. *Let $\varphi\in\mathrm{PSH}(X,\theta)$. Then $\varphi\in\mathcal{E}(X,\theta)$ if $[\varphi]\wedge V_\theta=V_\theta$.*

REMARK II.3.8. We will see in Theorem III.4.66 that the converse is also true.

II.4. Comparison principles

Let X be a compact Kähler manifold of dimension n. Let $\alpha, \alpha_1, \dots, \alpha_n \in H^{1,1}(X, \mathbb{R})$ be big cohomology classes. Let $\theta \in \alpha$, $\theta_j \in \alpha_j$ ($j = 1, \dots, n$) be smooth representatives.

THEOREM II.4.27 (Comparison principle. I). *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Assume that $[\psi] \preceq [\varphi] \wedge V_\theta$. Then*

$$\int_{\{\varphi<\psi\}} \theta_\psi^n \leq \int_{\{\varphi<\psi\}} \theta_\varphi^n. \tag{2.33}$$

PROOF. **Step 1**. We reduce to the case where $\varphi \leq \psi$.

Let $\eta = \varphi \vee \psi$.

$$\int_{\{\varphi<\psi\}} \theta_\psi^n = \int_{\{\varphi<\eta\}} \theta_\eta^n, \quad \int_{\{\varphi<\eta\}} \theta_\varphi^n = \int_{\{\varphi<\psi\}} \theta_\varphi^n.$$

So (2.33) is equivalent to the corresponding statement with ψ replaced by η.

Step 2. We assume that $\varphi \leq \psi$.

Now Proposition II.3.25 and Theorem II.3.21 imply

$$\int_X \theta_\varphi^n = \int_X \theta_\psi^n = \int_X \theta_{[\varphi]\wedge V_\theta}^n.$$

For any $\varepsilon > 0$, let

$$\psi_\varepsilon = (\varphi + \varepsilon) \vee \psi.$$

Again by Proposition II.3.25 and Theorem II.3.21,

$$\int_X \theta_\varphi^n = \int_X \theta_{\psi_\varepsilon}^n.$$

So

$$\begin{aligned} \int_X \theta_{\psi_\varepsilon}^n &\geq \int_{\{\varphi+\varepsilon>\psi\}} \theta_\varphi^n + \int_{\{\varphi+\varepsilon<\psi\}} \theta_\psi^n \\ &= \int_X \theta_\varphi^n - \int_{\{\varphi+\varepsilon\leq\psi\}} \theta_\varphi^n + \int_{\{\varphi+\varepsilon<\psi\}} \theta_\psi^n. \end{aligned}$$

Hence

$$\int_{\{\varphi+\varepsilon\leq\psi\}} \theta_\varphi^n \geq \int_{\{\varphi+\varepsilon<\psi\}} \theta_\psi^n.$$

Let $\varepsilon \to 0+$, we conclude by monotone convergence theorem. □

THEOREM II.4.28 (Comparison principle II.). *Assume that $\varphi, \psi \in \mathcal{E}(X, \theta)$, then*

$$\int_{\{\varphi<\psi\}} \theta_\psi^n \leq \int_{\{\varphi<\psi\}} \theta_\varphi^n.$$

The proof is almost identical to that of Theorem II.4.27 and is left to the reader.

REMARK II.4.9. As we will see later, Theorem II.4.28 follows directly from Theorem II.4.27.

COROLLARY II.4.29. *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$.*

(1) Assume that $\varphi \leq \psi$ and that $\varphi \in \mathcal{E}(X, \theta)$, then $\psi \in \mathcal{E}(X, \theta)$. Moreover, in this case, for any $C > 0$,

$$\int_X \mathbb{1}_{\{\psi<V_\theta-C\}}\, \theta_\psi^n \leq 2^n \int_X \mathbb{1}_{\{\varphi<V_\theta-C/2\}}\, \theta_\varphi^n\,. \tag{2.34}$$

(2) Assume that $\varphi, \psi \in \mathcal{E}(X,\theta)$, then $\frac{1}{2}(\varphi+\psi) \in \mathcal{E}(X,\theta)$. In particular, $\mathcal{E}(X,\theta)$ is convex.

Proof. We may assume that $\varphi, \psi \leq -2$.

(1) By Proposition II.3.24, it suffices to show that

$$v := \frac{1}{2}(\psi + V_\theta) \in \mathcal{E}(X,\theta).$$

For each $j \geq 1$, let

$$v_j := v \vee (V_\theta - j), \quad \varphi_j = \varphi \vee (V_\theta - j).$$

Now we have

$$\{v \leq V_\theta - j\} \subseteq \{\varphi_{2j} < v_j - j + 1\} \subseteq \{\varphi \leq V_\theta - j\}.$$

So by Theorem II.4.28,

$$\int_X \mathbb{1}_{\{v \leq V_\theta - j\}}\, \theta^n_{v_j} \leq \int_X \mathbb{1}_{\{\varphi_{2j} < v_j - j + 1\}}\, \theta^n_{v_j} \leq \int_X \mathbb{1}_{\{\varphi_{2j} < v_j - j+1\}}\, \theta^n_{\varphi_{2j}} \leq \int_X \mathbb{1}_{\{\varphi \leq V_\theta - j\}}\, \theta^n_{\varphi_{2j}}.$$

Hence

$$\int_X \mathbb{1}_{\{\varphi > V_\theta - j\}}\, \theta^n_\varphi \leq \int_X \mathbb{1}_{\{v > -j\}} \theta^n_v\,.$$

Let $j \to \infty$, we conclude that $v \in \mathcal{E}(X,\theta)$.

As for (2.34), observe that

$$\{\psi < V_\theta - C\} \subseteq \{\varphi < v + C/2\} \subseteq \{\varphi < V_\theta - C/2\}\,.$$

So by Theorem II.4.28,

$$\begin{aligned}\int_X \mathbb{1}_{\{\psi < V_\theta - C\}}\, \theta^n_\psi &\leq \int_X \mathbb{1}_{\{\varphi < v + C/2\}}\, \theta^n_\psi \leq 2^n \int_X \mathbb{1}_{\{\varphi < v + C/2\}}\, \theta^n_v \\ &\leq 2^n \int_X \mathbb{1}_{\{\varphi < v + C/2\}}\, \theta^n_\varphi \leq 2^n \int_X \mathbb{1}_{\{\varphi < V_\theta - C/2\}}\, \theta^n_\varphi\,.\end{aligned}$$

(2) By Proposition II.3.24, it suffices to prove that

$$w := \frac{1}{4}\,(\varphi + \psi + 2V_\theta) \in \mathcal{E}(X,\theta).$$

We may assume that $\varphi \leq -2$, $\psi \leq -2$. For any $j \geq 1$, let

$$w_j = w \vee (V_\theta - j), \quad \varphi_j = \varphi \vee (V_\theta - j), \quad \psi_j = \psi \vee (V_\theta - j).$$

Observe that

$$\{\varphi \leq V_\theta - 2j\} \subseteq \{\varphi_{2j} < w_j - j + 1\} \subseteq \{\varphi \leq V_\theta - j\}.$$

Hence

$$\int_X \mathbb{1}_{\{\varphi \leq V_\theta - 2j\}}\, \theta^n_{v_j} \leq \int_X \mathbb{1}_{\{\varphi \leq V_\theta - j\}}\, \theta^n_{\varphi_{2j}} \leq \int_X \mathbb{1}_{\{\varphi \leq V_\theta - j\}}\, \theta^n_{\varphi_j}.$$

Let $j \to \infty$, we find

$$\int_X \mathbb{1}_{\{\varphi \leq V_\theta - 2j\}}\, \theta^n_{v_j} \to 0.$$

By symmetry, the same holds for ψ in place of φ. As

$$\{v \leq V_\theta - j\} \subseteq \{\varphi \leq V_\theta - 2j\} \cup \{\psi \leq V_\theta - 2j\},$$

we conclude

$$\int_X \mathbb{1}_{\{v \leq V_\theta - j\}}\, \theta^n_{v_j} \to 0.$$

Hence $v \in \mathcal{E}(X,\theta)$. $\square$

Corollary II.4.30. *Let $\varphi_1,\dots,\varphi_n \in \mathcal{E}(X,\theta)$. Then*

$$\int_X \theta_{\varphi_1}\wedge\cdots\wedge\theta_{\varphi_n} = \int_X \theta_{V_\theta}^n.$$

Proof. Let $t_1,\dots,t_n \in \mathbb{R}_{\geq 0}$ be such that $\sum_{j=1}^n t_j = 1$. Then by Corollary II.4.29,

$$\sum_{j=1}^n t_j\varphi_j \in \mathcal{E}(X,\theta).$$

Hence

$$\int_X \left(\sum_{j=1}^n t_j\theta_{\varphi_j}\right)^n = \int_X \theta_{V_\theta}^n.$$

Comparing the coefficients for various t_j, we conclude. $\square$

For later use, we also need a polarized version.

Theorem II.4.31 (Comparison principle III.). *Let $\varphi_k,\psi_k \in \mathrm{PSH}(X,\theta_k)$ for $k=1,\dots,j$, where $j\leq n$. Let $u,v,\varphi\in\mathrm{PSH}(X,\theta)$. Assume*

(1)

$$[u]\preceq[\varphi],\quad [\psi_k]\preceq[\varphi_k],\quad [v]\preceq[\varphi].$$

(2)

$$\int_X \theta_u^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} = \int_X \theta_v^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} = \int_X \theta_\varphi^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}$$

Then

$$\int_{\{u<v\}} \theta_v^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} \leq \int_{\{u<v\}} \theta_u^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}.$$

Proof. By Theorem II.3.21,

$$\begin{aligned}\int_X \theta_\varphi^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} &= \int_X \theta_u^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}\\ \leq \int_X \theta_{u\vee v}^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} &\leq \int_X \theta_\varphi^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}.\end{aligned}$$

Hence equality holds everywhere. Now by Proposition II.1.4,

$$\begin{aligned}&\int_X \theta_{u\vee v}^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}\\ \geq&\int_{\{u>v\}} \theta_u^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} + \int_{\{v>u\}} \theta_v^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}\\ \geq&\int_X \theta_u^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} + \int_{\{v>u\}} \theta_v^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} - \int_{\{u\leq v\}} \theta_u^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}\\ \geq&\int_X \theta_{u\vee v}^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} + \int_{\{v>u\}} \theta_v^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} - \int_{\{u\leq v\}} \theta_u^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}\end{aligned}$$

So

$$\int_{\{u<v\}} \theta_v^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j} \leq \int_{\{u\leq v\}} \theta_u^{n-j}\wedge\theta_{1,\psi_1}\wedge\cdots\wedge\theta_{j,\psi_j}.$$

Replace u by $u+\varepsilon$ and let $\varepsilon \to 0+$, we conclude by monotone convergence theorem. $\square$

COROLLARY II.4.32. *Let $\varphi, \psi, \varphi_k, \psi_k \in \mathcal{E}(X,\theta)$ for $k = 1, \ldots, j$, where $j \leq n$. Then*
$$\int_{\{\varphi<\psi\}} \theta_\psi^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} \leq \int_{\{\varphi<\psi\}} \theta_\varphi^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j}.$$

PROOF. This follows directly from Corollary II.4.30 and Theorem II.4.31. $\square$

COROLLARY II.4.33. *Let $\varphi, \psi \in \mathcal{E}(X,\theta)$. Assume that $\varphi \leq \psi$, then*
$$j \mapsto \int_X (\psi - \varphi)\, \theta_\psi^j \wedge \theta_\varphi^{n-j}$$
is decreasing in $j = 0, \ldots, n$.

PROOF. We write
$$\int_X (\psi - \varphi)\, \theta_\psi^j \wedge \theta_\varphi^{n-j} = \int_0^\infty \mathrm{d}t \int_{\{\psi-\varphi>t\}} \theta_\psi^j \wedge \theta_\varphi^{n-j}.$$
It follows from Corollary II.4.32 that
$$j \mapsto \int_{\{\psi-\varphi>t\}} \theta_\psi^j \wedge \theta_\varphi^{n-j}$$
is decreasing in $j = 0, \ldots, n$. $\square$

LEMMA II.4.34. *Let $\varphi, \psi, \eta \in \mathrm{PSH}(X,\theta)$. Assume that*
$$[\psi] \preceq [\varphi]\,, \quad [\eta] \preceq [\varphi]\,.$$
Assume that
$$\psi \leq \varphi\,, \quad \theta_\varphi^n - a.e.\,,$$
then $\psi \leq \varphi$ θ_η-a.e..

PROOF. We may assume that $\eta \leq \varphi$. For $\varepsilon \in (0,1)$, we have
$$\{\varphi < (1-\varepsilon)\psi + \varepsilon\eta\} \subseteq \{\varphi < \psi\}\,.$$
So by Theorem II.4.27,
$$\begin{aligned}
\varepsilon^n \int_{\{\varphi<(1-\varepsilon)\psi+\varepsilon\eta\}} \omega^n &\leq \varepsilon^n \int_{\{\varphi<(1-\varepsilon)\psi+\varepsilon\eta\}} \theta_\rho^n \\
&\leq \int_{\{\varphi<(1-\varepsilon)\psi+\varepsilon\eta\}} \theta_{(1-\varepsilon)\psi+\varepsilon\eta}^n \\
&\leq \int_{\{\varphi<(1-\varepsilon)\psi+\varepsilon\eta\}} \theta_\varphi^n \\
&\leq \int_{\{\varphi<\psi\}} \theta_\varphi^n = 0\,.
\end{aligned}$$
Let $\varepsilon \to 0+$, we conclude. $\square$

THEOREM II.4.35 (Domination principle I.). *Let $\varphi \in \mathcal{E}^\infty(X,\theta)$, $\psi \in \mathrm{PSH}(X,\theta)$. Assume that*
$$\psi \leq \varphi\,, \quad \theta_\varphi^n - a.e.\,,$$
then $\psi \leq \varphi$.

PROOF. Take $\eta \in \mathrm{PSH}(X,\theta)$ such that $\theta_\rho \geq \omega$ for a Kähler form ω on X. By Lemma II.4.34, $\psi \leq \varphi$ ω^n-a.e., hence by Proposition I.1.1, $\psi \leq \varphi$. $\square$

We note that the proof actually implies the following general lemma.

II.5. Integration by parts

In this section, we prove two different versions of integration by parts formulae.

Let X be a fixed compact Kähler manifold of dimension n. Let $\alpha, \alpha_0, \alpha_1, \dots, \alpha_n$ be big cohomology classes. We fix a smooth representative of each, say $\theta, \theta_0, \theta_1, \dots, \theta_n$.

II.5.1. Potentials with small unbounded loci.

THEOREM II.5.36. *Let* $\Theta \in \mathcal{D}'^{n-1,n-1}_{+}(X)$*. Let* $\varphi_1, \varphi_2 \in \mathrm{PSH}(X,\theta_0)$*,* $\psi_1, \psi_2 \in \mathrm{PSH}(X,\theta_1)$*. Assume that*

(1)

$$[\varphi_1] = [\varphi_2], \quad [\psi_1] = [\psi_2].$$

(2) φ_1, ψ_1 *have small unbounded loci.*

Write

$$u = \varphi_1 - \varphi_2, \quad v = \psi_1 - \psi_2.$$

Then

$$\int_X u \,\mathrm{dd^c} v \wedge \Theta = \int_X v \,\mathrm{dd^c} u \wedge \Theta = -\int_X \mathrm{d}v \wedge \mathrm{d^c} u \wedge \Theta. \tag{2.35}$$

REMARK II.5.10. Here we explain the meanings of first two integrals in (2.35).

Let $A \subseteq X$ be a closed pluripolar set such that $\varphi_1, \psi_1 \in L^\infty_{\mathrm{loc}}(X - A)$. Then on $X - A$, the Bedford–Taylor products $\theta_{\psi_j} \wedge \Theta$ $(j = 1, 2)$ are well-defined *finite* measures on $X - A$, which we identify with their zero extension to X. So we can define

$$\mathrm{dd^c} v \wedge \Theta := \theta_{\psi_1} \wedge \Theta - \theta_{\psi_2} \wedge \Theta$$

as a signed measure on X. Hence the first integral in (2.35) makes sense. It is not hard to see that the integral is independent of the choice of ψ_j and φ_j once u, v are held fixed. Since the Bedford–Taylor product puts no mass on pluripolar sets, the definition is independent of the choice of A, hence explaining our notations. This explains the first two integrals in (2.35).

We refer to Section I.3 for the definitions of various currents appearing in the proof.

PROOF. Let $A \subseteq X$ be a closed pluripolar set such that $\varphi_1, \psi_1 \in L^\infty_{\mathrm{loc}}(X - A)$. We claim that

$$\mathrm{d}\left(\mathbb{1}_{X-A} v \,\mathrm{d^c} u \wedge \Theta\right) = \mathbb{1}_{X-A} \mathrm{d}\left(v \,\mathrm{d^c} u \wedge \Theta\right). \tag{2.36}$$

Assume that (2.36) holds for the time being. It follows from Stokes theorem and (1.9) that

$$0 = \int_{X-A} \mathrm{d}\left(v \,\mathrm{d^c} u \wedge \Theta\right) = \int_{X-A} \mathrm{d}v \wedge \mathrm{d^c} u \wedge \Theta + \int_{X-A} v \mathrm{dd^c} u \wedge \Theta.$$

Hence (2.35) follows by symmetry.

Now we prove (2.36). Note that (2.36) holds on $X - A$, so it suffices to prove it in a small neighbourhood U of a fixed point $a \in A$. Take a psh function $\tau \leq 0$ on U such that $A \subseteq \{\tau = -\infty\}$. Fix a smooth increasing convex function χ on $\mathbb{R}$

such that $\chi(t) = 0$ for $t \leq 1/2$ and $\chi(1) = 1$. Let $w_k(t) := \chi(e^{\tau/k})$. Take a smooth increasing function $\theta : \mathbb{R} \to [0,1]$ such that $\theta = 0$ near 0 and $\theta = 1$ near 1. Then $\theta(w_k) = 0$ near A and $\theta(w_k)$ increases pointwisely to 1 on $U - A$. Hence

$$\theta(w_k) v \,\mathrm{d}^{\mathrm{c}} u \wedge \Theta \to \mathbb{1}_{X-A} v \,\mathrm{d}^{\mathrm{c}} u \wedge \Theta, \quad \theta(w_k) \mathrm{d}\left(v \,\mathrm{d}^{\mathrm{c}} u \wedge \Theta\right) \to \mathbb{1}_{X-A} \mathrm{d}\left(v \,\mathrm{d}^{\mathrm{c}} u \wedge \Theta\right).$$

By [BEGZ10] Lemma 1.9, it remains to prove

$$\theta'(w_k) v \,\mathrm{d}^{\mathrm{c}} u \wedge \Theta \to 0.$$

In fact, let χ be a smooth positive function on U with compact support, then

$$\left| \int_U \chi \theta'(w_k) v \,\mathrm{d}^{\mathrm{c}} u \wedge \Theta \right|^2 \leq \left(\int_U \chi \,\mathrm{d} w_k \wedge \mathrm{d}^{\mathrm{c}} w_k \wedge \Theta \right) \left(\int_U \chi \theta'(w_k)^2 v^2 \mathrm{d} u \wedge \mathrm{d}^{\mathrm{c}} u \wedge \Theta \right).$$

Now

$$2 \int_U \chi \,\mathrm{d} w_k \wedge \mathrm{d}^{\mathrm{c}} w_k \wedge \Theta \leq \int_U \chi \,\mathrm{dd}^{\mathrm{c}} w_k^2 \wedge \Theta = \int_U w_k^2 \,\mathrm{dd}^{\mathrm{c}} \psi \wedge \Theta$$

is bounded. Hence it suffices to prove

$$\int_U \chi \theta'(w_k)^2 v^2 \mathrm{d} u \wedge \mathrm{d}^{\mathrm{c}} u \wedge \Theta \to 0.$$

Observe that $\theta'(w_k) \to 0$. So this follows from dominated convergence theorem. □

II.5.2. Notations. In the remaining of this section, we use the notations of Section II.3.1. In addition, we introduce a few other notations.

We introduce two variables $a, b \in [0,1]$ with $b = 1-a$. For an expression $f(a,b)$, we write

$$[f(a,b)]_1 = \partial_a|_{a=0} f(a, 1-a),$$

where the derivative means the right derivative. When writing such an expression, we mean implicitly that the derivative exists.

Let $W \subseteq X - Z$ be an open subset. Let ψ_1, ψ_2, γ be θ-psh functions on W. For each $N \geq 1$, define

$$A^N[a,b] := \Phi_N[a\psi_1 + b\gamma] - \Phi_N[a\psi_2 + b\gamma].$$

We do not mention $\psi_1, \psi_2, \gamma, W$ in the notation explicitly, but they will always be clear from the context.

Proposition II.5.37. *Let $W \subseteq X - Z$ be an open subset. Let ψ_1, ψ_2, γ be θ-psh functions on W. Assume that*

$$v := \psi_1 - \psi_2 \in L^\infty_{\mathrm{loc}}(W), \quad \gamma \leq \psi_1, \quad \gamma \leq \psi_2.$$

(1) On $W \times \mathbb{C}^N$,

$$A^N[a,b] = -g_N \circ \hat{\alpha}[a\psi_1 + b\gamma] + g_N \circ \hat{\alpha}[a\psi_2 + b\gamma].$$

(2) For $a \in [0,1)$, $1 - a > \varepsilon > 0$, on $\hat{\alpha}^{-1}(\mathbb{R}^N)$, we have

$$A^N[a+\varepsilon, b-\varepsilon] - A^N[a,b] = \frac{\varepsilon}{2}(\gamma - \psi_1) L \circ \hat{\alpha}[a\psi_1 + b\gamma] - \frac{\varepsilon}{2}(\gamma - \psi_2) L \circ \hat{\alpha}[a\psi_2 + b\gamma] + \mathcal{O}(\varepsilon^2),$$

where $L : \mathbb{R}^N \to \mathbb{R}$ is the piecewise linear bounded function defined in Appendix II.3.2. The $\mathcal{O}$-constant depends only on N.

(3) For $a \in [0,1]$, on $\hat{\alpha}^{-1}(\mathbb{R}^N)$,

$$A^N[a,b] = -\frac{va}{2} L \circ \hat{\alpha}[\gamma] + \mathcal{O}(a^2), \tag{2.37}$$

where the $\mathcal{O}$-constant depends only on N.

(4)
$$\left|A^N[a,b]\right| \leq a|v|.$$

PROOF. (1) This follows from Proposition II.3.16.

(2) Observe that
$$\hat{\alpha}[a\psi_1 + b\gamma] - \hat{\alpha}[(a+\varepsilon)\psi_1 + (b-\varepsilon)\gamma] = \frac{\varepsilon}{2}(\gamma - \psi_1)e,$$
where $e = (1,\dots,1)$. By assumption, $\gamma - \psi_1 \leq 0$, so (2) follows from Proposition II.3.13.

(3) Note that (2.37) is a special case of (2).

(4) This follows directly from definition. □

COROLLARY II.5.38. *Let $\psi_1, \psi_2, \gamma \in \mathrm{PSH}(X,\theta)$. Assume that*
$$[\psi_1] = [\psi_2], \quad \gamma \leq \psi_1, \quad \gamma \leq \psi_2.$$
As $a \to 0+$, $A^N[a,b]$ converges to 0 in capacity.

PROOF. Let $v = \psi_1 - \psi_2$.

We need to show that for each $\varepsilon > 0$,
$$\lim_{a\to 0+} \operatorname{Cap}\left\{\left|A^N[a,b]\right| > \varepsilon\right\} = 0.$$
By Proposition II.5.37, we can take $C = C(N)$ such that
$$\left|A^N[a,b] + \frac{va}{2} L \circ \hat{\alpha}[\gamma]\right| \leq Ca^2.$$
Take a small enough, we can thus assume that $Ca^2 < \varepsilon/2$, then
$$\left\{\left|A^N[a,b] > \varepsilon\right|\right\} \subseteq \left\{\left|\frac{va}{2} L \circ \hat{\alpha}[\gamma]\right| > \frac{\varepsilon}{2}\right\}.$$
Take a constant C_1 so that $|L| \leq C_1$, then
$$\left\{\left|A^N[a,b] > \varepsilon\right|\right\} \subseteq \left\{a|v| > \frac{\varepsilon}{C_1}\right\}.$$
But since v is the difference of two θ-psh functions,
$$\lim_{a\to 0+} \operatorname{Cap}\left\{|v| > \frac{\varepsilon}{C_1 a}\right\} = 0.$$
Here the capacity is still the capacity on X_N instead of on X, we have omitted the pull-back notations. □

II.5.3. Integral estimates.

LEMMA II.5.39. *Let $W \subseteq X - Z$ be an open set. Let $\gamma, \psi, \varphi, \psi_j \in \mathrm{PSH}(X,\theta)$ $(j = 1,2)$. Assume that*
$$0 \leq \psi - \gamma \in L^\infty_{\mathrm{loc}}(W), \quad v := \psi_1 - \psi_2 \in L^\infty_{\mathrm{loc}}(W).$$
Take $\chi \in C^0_c(W)$, $\chi \geq 0$.

Define
$$I_{W,N}[a,b] := \int_{W\times\mathbb{C}^N} \chi A^N[a,b]\, \pi_1^{N*}\theta_\varphi \wedge \theta^{N+n-1}_{N,\Phi_N[a\psi+b\gamma]}.$$

Then

$$I_{W,N}[a,b] = a\binom{N+n-1}{n-1}N\int_0^1 t^N \int_W \chi v\,\theta_\varphi \wedge ((1-t)\theta_\eta + t\theta_\gamma)^{n-1}\,\mathrm{d}t + \mathcal{O}(a^2). \tag{2.38}$$

Notice that γ, ψ_1, ψ_2 appear in the definition of $A^N[a,b]$. Also notice that the coefficient of a in (2.38) is independent of the choice of ψ.

Remark II.5.11. An easy calculation shows that the coefficient of a in (2.38) can be written as

$$\sum_{r=0}^{n-1}\binom{N+r-1}{r}\frac{N+r}{N+n}\int_W \chi v\,\theta_\varphi \wedge \theta_\gamma^r \wedge \theta_\eta^{n-1-r}. \tag{2.39}$$

Proof. Since the problem is local, we may shrink W when necessary. Let

$$\gamma' = \gamma'[a,b] = a\psi + b\gamma.$$

Then

$$\gamma' - \gamma = a(\psi - \gamma) \geq 0. \tag{2.40}$$

Step 1. We claim that we may assume that $\psi_1, \psi_2, \gamma, \psi$ are smooth.

To be more precise, take an open subset $W' \Subset W$ containing $\operatorname{Supp}\chi$.

We start with ψ_1, ψ_2. Take sequences of smooth θ-psh functions on W, say ψ_j^k ($k \geq 1, j = 1,2$) that decreases to ψ_j as $k \to \infty$, we may assume that

$$|\psi_1^k - \psi_2^k|$$

are uniformly bounded on W' as well.

Let

$$A_k^N[a,b] := \Phi_N[a\psi_1^k + b\gamma] - \Phi_N[a\psi_2^k + b\gamma]$$

According to Corollary II.3.17, $\Phi_N[a\psi_j^k + b\gamma]$ decreases to $\Phi_N[a\psi_j + b\gamma]$ outside a pluripolar set, hence everywhere. By Proposition II.5.37,

$$\left|A_k^N[a,b]\right| \leq Ca.$$

By dominated convergence theorem, we have

$$I_{W,N}^k[a,b] := \int_{W\times\mathbb{C}^N} \chi A_k^N[a,b]\,\pi_1^{N*}\theta_\varphi \wedge \theta_{N,\Phi_N[\gamma']}^{N+n-1}$$

converges to $I_{W,N}[a,b]$. Similar reasoning applies to the coefficient of a in (2.38). The $\mathcal{O}$-constant in (2.38) can be taken to be independent of k as we will see in Step 3, so we conclude that we may assume that both ψ_1, ψ_2 are smooth.

Now we deal with ψ, γ. At first, we notice that we can reduce to the case where ψ, γ are bounded exactly as in the proof of Theorem II.3.19 Step 1. Then we can take smooth θ-psh functions ψ^k (resp. γ^k) decreasing to ψ, γ, keeping $|\psi^k - \gamma^k|$ uniformly bounded on W. As will follow from Step 2, the currents $\pi_1^{N*}\theta_\varphi \wedge \theta_{N,\Phi_N[a\psi^k + b\gamma^k]}^{N+n-1}$ are supported on $W \times B$, where $B \subseteq \mathbb{C}^N$ is a bounded set independent of k. It follows that

$$\int_{W\times\mathbb{C}^N} \chi A^N[a,b]\,\pi_1^{N*}\theta_\varphi \wedge \theta_{N,\Phi_N[a\psi^k + b\gamma^k]}^{N+n-1} \to I_{W,N}[a,b].$$

Similarly,

$$\int_0^1 t^N \int_W \chi v\, \theta_\varphi \wedge \big((1-t)\theta_\eta + t\theta_{\gamma^k}\big)^{n-1}\,\mathrm{d}t \to \int_0^1 t^N \int_W \chi v\, \theta_\varphi \wedge ((1-t)\theta_\eta + t\theta_\gamma)^{n-1}\,\mathrm{d}t.$$

As will be proved in Step 3, the big $\mathcal{O}$-constant in (2.38) does not depend on k, so let $k \to \infty$, we are done.

Now $\Phi_N[\gamma']$ is $C^{1,1}$ on $W \times \mathbb{C}^N$ by Proposition II.3.14 and (2.23).

Step 2. We claim that the measure

$$\theta_{N,\Phi} \wedge \theta^{N+n-1}_{N,\Phi_N[\gamma']}$$

is supported on V_N for any local θ_N-psh function Φ on $W \times \mathbb{C}^N$.

Here

$$V_N := \hat{\alpha}[\gamma']^{-1}\mathring{\Sigma}_N \subseteq W \times \mathbb{C}^N.$$

Note that V_N depends on a, b.

Since the problem is local on $W \times \mathbb{C}^N$, we may take $\theta_N = 0$ by adding to $\Phi_N[\gamma']$ and Φ a smooth function. We may focus on an open subset $A \subseteq W \times \mathbb{C}^N$ on which $\Phi_N[\gamma']$ is bounded.

For $k \geq 0$ large enough, let $O_k = \{\Phi > -k\}$. Then by definition of the non-pluripolar product, it suffices to prove that

$$\mathbb{1}_{O_k}\mathrm{dd^c}(\Phi \vee (-k)) \wedge (\mathrm{dd^c}\Phi_N[\gamma']|_{O_k})^{N+n-1}$$

supports on V_N. Hence we may assume that Φ is bounded. By continuity of the Bedford–Taylor product, we may then assume that Φ is smooth.

In this case, it is well-known that

$$(N+n)\mathrm{dd^c}\Phi \wedge (\mathrm{dd^c}\Phi_N[\gamma'])^{N+n-1} = \big(\Delta_{\mathrm{dd^c}\Phi_N[\gamma']}\Phi\big)\,(\mathrm{dd^c}\Phi_N[\gamma'])^{N+n}.$$

As shown in the proof of Theorem II.3.19, $(\mathrm{dd^c}\Phi_N[\gamma'])^{N+n}$ is supported on V_N. This proves our claim.

Step 3. By Step 2,

$$I_{W,N}[a,b] = \int_{V_N \cap (W \times \mathbb{C}^N)} \chi A^N[a,b]\, \theta_\varphi \wedge \theta^{N+n-1}_{N,\Phi_N[\gamma']}.$$

We have omitted π_1^{N*} from our notation.

We calculate its value now. Note that

$$\hat{\alpha}[\gamma'] = \hat{\alpha}[\gamma] + \frac{a}{2}(\psi - \gamma)e,$$

where $e = (1, \ldots, 1) \in \mathbb{R}^N$. By Section II.3.3, the piecewise linear function L has the same coefficients at $\hat{\alpha}[\gamma]$ and $\hat{\alpha}[\gamma']$. So

$$|L \circ \hat{\alpha}[\gamma'] - L \circ \hat{\alpha}[\gamma]| \leq Ca|\psi - \gamma|,$$

where C depends only on N.

It follows that

$$\int_{V_N \cap (W \times \mathbb{C}^N)} \chi v\, |L \circ \hat{\alpha}[\gamma'] - L \circ \hat{\alpha}[\gamma]|\, \theta_\varphi \wedge \theta^{N+n-1}_{N,\Phi_N[\gamma']} \leq Ca \tag{2.41}$$

for a constant C independent of a.

So by Proposition II.5.37,

$$\begin{aligned} I_{W,N}[a,b] &= -\frac{a}{2}\int_{V_N\cap(W\times\mathbb{C}^N)} \chi v L\circ\hat{\alpha}[\gamma]\,\theta_\varphi\wedge\theta^{N+n-1}_{N,\Phi_N[\gamma']}+\mathcal{O}(a^2) \\ &= -\frac{a}{2}\int_{V_N\cap(W\times\mathbb{C}^N)} \chi v L\circ\hat{\alpha}[\gamma']\,\theta_\varphi\wedge\theta^{N+n-1}_{N,\Phi_N[\gamma']}+\mathcal{O}(a^2). \end{aligned} \tag{2.42}$$

By Proposition II.3.13, on V_N, we have

$$-\frac{1}{2}L\circ\hat{\alpha}[\gamma']=|\hat{\alpha}[\gamma']|\,.$$

Hence

$$I_{W,N}[a,b]=a\int_{V_N\cap(W\times\mathbb{C}^N)}\chi v\,|\hat{\alpha}[\gamma']|\;\theta_\varphi\wedge\theta^{N+n-1}_{N,\Phi_N[\gamma']}+\mathcal{O}(a^2).$$

Now one can calculate the RHS exactly as in the proof of Theorem II.3.19, one gets

$$\begin{aligned} I_{W,N}[a,b]=a\binom{N+n-1}{n-1}\int_W \chi(x)v(x)\int_{V_x}\omega^N_{N,\Phi_N[\gamma']|_{V_x}}(z)|\hat{\alpha}|(x,z) \\ \left(\theta_\varphi\wedge((1-|\hat{\alpha}|)\theta_\eta+|\hat{\alpha}|\theta_{\gamma'})^{n-1}\right)(x)+\mathcal{O}(a^2). \end{aligned}$$

We can push-forward the integral to $\{x\}\times\mathbb{R}^N$ by the log map and pushing forward further to $\{x\}\times\Sigma_N$ by the gradient of $\Phi_N[\gamma'](x,k)$ as a function of $k\in\mathbb{R}^N$ as in [WN19], we get

$$\begin{aligned} I_{W,N}[a,b] =&a\binom{N+n-1}{n-1}N!\int_W\chi v\int_{\Sigma_N}|\hat{\alpha}|\mathrm{d}\hat{\alpha}\,\theta_\varphi\wedge((1-|\hat{\alpha}|)\theta_\eta+|\hat{\alpha}|\theta_{\gamma'})^{n-1}+\mathcal{O}(a^2) \\ =&a\binom{N+n-1}{n-1}N\int_0^1 t^N\int_W\chi v\,\theta_\varphi\wedge((1-t)\theta_\eta+t\theta_{\gamma'})^{n-1}\,\mathrm{d}t+\mathcal{O}(a^2) \\ =&a\binom{N+n-1}{n-1}N\int_0^1 t^N\int_W\chi v\,\theta_\varphi\wedge((1-t)\theta_\eta+t\theta_{\gamma})^{n-1}\,\mathrm{d}t+\mathcal{O}(a^2), \end{aligned}$$

where the last line follows from (2.40). □

Lemma II.5.40. *Let $W\subseteq X-Z$ be an open set. Let $\gamma,\varphi_j,\psi_j\in\mathrm{PSH}(X,\theta)$ ($j=1,2$). Let*

$$u:=\varphi_1-\varphi_2,\quad v:=\psi_1-\psi_2.$$

Assume that

$$v\in L^\infty_{\mathrm{loc}}(W).$$

Take $\chi\in C^0_c(W)$, $\chi\geq 0$. Define

$$I_W:=\lim_{N\to\infty}\frac{(n-1)!}{N^{n-1}}\left[\int_{W\times\mathbb{C}^N}\chi A^N[a,b]\,\mathrm{dd^c}u\wedge\theta^{N+n-1}_{N,\Phi_N[\gamma]}\right]_1.$$

Then $[\cdot]_1$ here exists and the limit exists and

$$I_W=\int_W\chi v\,\mathrm{dd^c}u\wedge\theta_\gamma^{n-1}.$$

Proof. We apply Lemma II.5.39 with $\psi=\gamma$. The result follows from (2.39). □

Lemma II.5.41. *Let $\gamma,\varphi_j,\psi_j\in\mathrm{PSH}(X,\theta)$ ($j=1,2$). Let $u=\varphi_1-\varphi_2$, $v=\psi_1-\psi_2$. Assume the following:*

(1)

$$[\varphi_1] = [\varphi_2], \quad [\psi_1] = [\psi_2] = [\gamma].$$

(2)

$$\gamma \leq \psi_2 \leq \psi_1.$$

(3) φ_1 *has small unbounded locus.*

Then

$$\int_X u\, \mathrm{dd^c} v \wedge \theta_\gamma^{n-1} = \lim_{N\to\infty} \frac{(n-1)!}{N^{n-1}} \left[\int_{X_N} A^N[a,b]\, \pi_1^{N*} \mathrm{dd^c} u \wedge \theta^{N+n-1}_{N,\Phi_N[\gamma]} \right]_1 . \tag{2.43}$$

Remark II.5.12. As we will see in the proof of Theorem II.5.42, assumption (3) can be omitted.

Proof. By Lemma II.5.40, the limit on the RHS of (2.43) exists.

Notice that

$$\Phi[a\psi_1 + b\gamma] \geq \Phi[a\psi_2 + b\gamma],$$

so $A^N[a,b] \geq 0$.

Define

$$I = \int_X u\, \mathrm{dd^c} v \wedge \theta_\gamma^{n-1}.$$

Then

$$I = \frac{1}{n} \left[\int_X u \left((a\theta_{\psi_1} + b\theta_\gamma)^n - (a\theta_{\psi_2} + b\theta_\gamma)^n \right) \right]_1 .$$

By Corollary II.3.20,

$$I = \frac{1}{n} \lim_{N\to\infty} \binom{N+n}{n}^{-1} \left[\int_{X_N} \pi_1^{N*} u \left(\theta^{N+n}_{N,\Phi_N[a\psi_1+b\gamma]} - \theta^{N+n}_{N,\Phi_N[a\psi_2+b\gamma]} \right) \right]_1 .$$

Here we have made use of the fact that the integral on RHS is polynomial in a and b of bounded degree to change the order of limit and $[\cdot]_1$. Then

(2.44)

$$\begin{aligned} nI &= \lim_{N\to\infty} \binom{N+n}{n}^{-1} \left[\sum_{r=0}^{N+n-1} \int_{X_N} \pi_1^{N*} u\, \mathrm{dd^c} A^N[a,b] \wedge \left(\theta^r_{N,\Phi_N[a\psi_1+b\gamma]} \wedge \theta^{N+n-1-r}_{N,\Phi_N[a\psi_2+b\gamma]} \right) \right]_1 \\ &= \lim_{N\to\infty} \binom{N+n}{n}^{-1} \left[\sum_{r=0}^{N+n-1} \int_{X_N} A^N[a,b]\, \mathrm{dd^c} \pi_1^{N*} u \wedge \left(\theta^r_{N,\Phi_N[a\psi_1+b\gamma]} \wedge \theta^{N+n-1-r}_{N,\Phi_N[a\psi_2+b\gamma]} \right) \right]_1 , \end{aligned}$$

where on the second line, we perform the integration by parts. This is allowed by our assumption and by Theorem II.5.36.

For $r = 0, \ldots, N+n-1$, define

$$J_r[a,b] := \int_{X_N} A^N[a,b]\, \pi_1^{N*} \theta_{\varphi_1} \wedge \theta^r_{N,\Phi_N[a\psi_1+b\gamma]} \wedge \theta^{N+n-1-r}_{N,\Phi_N[a\psi_2+b\gamma]}.$$

Observe that J_r is decreasing with respect to r. In fact,

$$J_r[a,b] = \int_0^\infty \mathrm{d}t \int_{\{A^N[a,b]>t\}} \pi_1^{N*} \theta_{\varphi_1} \wedge \theta^r_{N,\Phi_N[a\psi_1+b\gamma]} \wedge \theta^{N+n-1-r}_{N,\Phi_N[a\psi_2+b\gamma]}.$$

So it suffices to prove that the inner integral is decreasing with respect to r. Then since $\Phi_N[a\psi_j + b\gamma]$ $(j = 1, 2)$ have the same singularity type, we can apply Theorem II.4.31 to conclude.

We claim that

$$J_r[a,b] - \int_{X_N} A^N[a,b]\, \pi_1^{N*}\theta_{\varphi_1} \wedge \theta^{N+n-1}_{N,\Phi_N[\gamma]} = o(a), \quad a \to 0+\,. \tag{2.45}$$

By monotonicity in r, it suffices to prove this for $r=0$ and $r=n+N-1$. Since the two cases are parallel, we can assume $r=0$. In fact, by Lemma II.5.39,

$$\int_{X_N} A^N[a,b]\, \pi_1^{N*}\theta_{\varphi_1} \wedge \theta^{N+n-1}_{N,\Phi_N[a\psi_2+b\gamma]} - \int_{X_N} A^N[a,b]\, \pi_1^{N*}\theta_{\varphi_1} \wedge \theta^{N+n-1}_{N,\Phi_N[\gamma]} = \mathcal{O}(a^2). \tag{2.46}$$

So our claim holds. Hence

$$[J_r[a,b]]_1 = \left[\int_{X_N} A^N[a,b]\, \pi_1^{N*}\theta_{\varphi_1} \wedge \theta^{N+n-1}_{N,\Phi_N[\gamma]}\right]_1.$$

The same argument holds with φ_1 replaced by φ_2, so (2.44) implies that

$$nI = \lim_{N\to\infty} \binom{N+n}{n}^{-1} (N+n) \left[\int_{X_N} A^N[a,b]\, \mathrm{dd^c}\pi_1^{N*}u \wedge \theta^{N+n-1}_{N,\Phi_N[\gamma]}\right]_1$$

and (2.43) follows. □

II.5.4. Integration by parts for non-pluripolar products.

THEOREM II.5.42. *Let α_j ($j=0,\dots,n$) be big cohomology classes on X. Let θ_j ($j=0,\dots,n$) be smooth representatives in α_j. Let $\gamma_j \in \mathrm{PSH}(X,\theta_j)$ ($j=2,\dots,n$). Let $\varphi_1,\varphi_2 \in \mathrm{PSH}(X,\theta_0)$, $\psi_1,\psi_2 \in \mathrm{PSH}(X,\theta_1)$. Let $u=\varphi_1-\varphi_2$, $v=\psi_1-\psi_2$. Assume that*

$$[\varphi_1]=[\varphi_2], \quad [\psi_1]=[\psi_2].$$

Then

$$\int_X u\, \mathrm{dd^c} v \wedge \theta_{2,\gamma_2} \wedge \cdots \wedge \cdots \wedge \theta_{n,\gamma_n} = \int_X v\, \mathrm{dd^c} u \wedge \theta_{2,\gamma_2} \wedge \cdots \wedge \cdots \wedge \theta_{n,\gamma_n}. \tag{2.47}$$

PROOF. **Step 1**. By polarization, we may assume that $\theta_1 = \ldots = \theta_n = \theta$. Similarly, by another polarization, we may assume that $\gamma_1 = \cdots = \gamma_{n-1} = \gamma$. Then we want to prove

$$\int_X u\, \mathrm{dd^c} v \wedge \theta_\gamma^{n-1} = \int_X v\, \mathrm{dd^c} u \wedge \theta_\gamma^{n-1}. \tag{2.48}$$

By a further polarization, we may assume that $[\psi_1]=[\gamma]$. In fact, if the theorem holds in this case, for any $a,b\in[0,1]$, $a+b=1$, we have

$$\int_X u\, \mathrm{dd^c}\left((a\psi_1+b\gamma)-(a\psi_2+b\gamma)\right)\wedge\theta^{n-1}_{a\psi_2+b\gamma} = \int_X \left((a\psi_1+b\gamma)-(a\psi_2+b\gamma)\right)\mathrm{dd^c}u\wedge\theta^{n-1}_{a\psi_2+b\gamma}.$$

Hence for $a>0$,

$$\int_X u\, \mathrm{dd^c} v \wedge \theta^{n-1}_{a\psi_2+b\gamma} = \int_X v\, \mathrm{dd^c} u \wedge \theta^{n-1}_{a\psi_2+b\gamma}.$$

Since both sides are polynomials in a, equality for all $a>0$ implies immediately equality at $a=0$. That is,

$$\int_X u\, \mathrm{dd^c} v \wedge \theta_\gamma^{n-1} = \int_X v\, \mathrm{dd^c} u \wedge \theta_\gamma^{n-1}.$$

We may assume that

$$\gamma \le \psi_2 \le \psi_1.$$

Step 2. Let us prove (2.48) under the additional assumption that φ_1 has small unbounded locus.

In this case, we can apply Lemma II.5.41 and Lemma II.5.40 to conclude.

Step 3. We prove (2.48) holds in general. It suffices to show that Lemma II.5.41 holds without assumption (3). In this case, we repeat the same proof of Lemma II.5.41 with the following differences:

(1) Integration by parts in (2.44) is now due to Step 2.
(2) The RHS of (2.46) is replaced by $o(a)$.
 To prove this, by Proposition II.5.37, it suffices to prove

$$\int_{X_N} vL\circ\hat{\alpha}[\gamma]\,\theta_{\varphi_1}\wedge\theta^{N+n-1}_{N,\Phi_N[a\psi_2+b\gamma]}-\int_{X_N} vL\circ\hat{\alpha}[\gamma]\,\theta_{\varphi_1}\wedge\theta^{N+n-1}_{N,\Phi_N[\gamma]}=o(1),\quad a\to 0+. \tag{2.49}$$

 Note that $vL\circ\hat{\alpha}[\gamma]$ is quasi-continuous outside a closed pluripolar set: v and $\hat{\alpha}[\gamma]$ are quasi-continuous (outside a closed pluripolar set). Since L is continuous, $L\circ\hat{\alpha}[\gamma]$ is quasi-continuous as well. Now (2.49) follows from Theorem II.2.12 and Corollary II.5.38.

□

II.6. Inequalities of the Monge–Ampère operators

Let X be a compact Kähler manifold of dimension n. Let $\alpha\in H^{1,1}(X,\mathbb{R})$ be a big cohomology class. Let $\theta\in\alpha$ be a smooth representative.

Lemma II.6.43. *Let $\varphi,\psi\in \mathrm{PSH}(X,\theta)$. Let $\mu\in\mathcal{M}_+(X)$. Assume that*

$$\theta^n_\varphi\geq\mu,\quad \theta^n_\psi\geq\mu,$$

then

$$\theta^n_{\varphi\vee\psi}\geq\mu.$$

Proof. By the locality of non-pluripolar product

$$\theta^n_{\varphi\vee\psi}\geq \mathbb{1}_{\{\varphi\neq\psi\}}\mu.$$

So this lemma is true in case $\mu\{\varphi=\psi\}=0$.

We claim that the set

$$I:=\{t\in\mathbb{R}:\mu\{\varphi=\psi+t\}>0\}.$$

In fact, this set of t is exactly the set of discontinuity of

$$t\mapsto\mu\{\varphi<\psi+t\},$$

which is an increasing function.

Now we take t_i increasing to 0 such that $t_i\in I$, so we have

$$\theta^n_{(\varphi+t_i)\vee\psi}\geq\mu.$$

By Corollary II.2.10 and Remark II.2.5, as $i\to\infty$,

$$\theta^n_{(\varphi+t_i)\vee\psi)}\to\theta^n_{\varphi\vee\psi}.$$

We conclude. □

PROPOSITION II.6.44. *Let $\varphi_j \in \mathrm{PSH}(X,\theta)$ for $j \in \mathbb{Z}_{>0}$. Assume that $\varphi_j \leq 0$. Let $\mu \in \mathcal{M}_+(X)$ so that*

$$\theta^n_{\varphi_j} \geq \mu.$$

Then

$$\theta^n_{\sup^* \varphi_j} \geq \mu.$$

PROOF. For each $j \in \mathbb{Z}_{>0}$, let

$$\psi_j := \varphi_1 \vee \cdots \vee \varphi_j.$$

It follows from Lemma II.6.43 that

$$\theta^n_{\psi_j} \geq \mu.$$

So we may assume that φ_j is increasing.

Now the desired result follows from Corollary II.2.10 and Remark II.2.5. □

THEOREM II.6.45. *Let $\mu \in \mathcal{M}_+(X)$ be a non-pluripolar measure. Let $\varphi_j \in \mathrm{PSH}(X,\theta)$ $(j > 0)$, such that*

$$\theta^n_{\varphi_j} \geq f_j\mu, \quad 0 \leq f_j \in L^1(X,\mu).$$

Assume that $f_j \to f \in L^1(X,\mu)$ and that $\varphi_j \to \varphi \in \mathrm{PSH}(X,\theta)$ in L^1. Then

$$\theta^n_{\varphi} \geq f\mu.$$

PROOF. We may assume that $f_j \to f$, μ-a.e. by taking a subsequence. Let

$$\psi_j := \sup_{k\geq j}{}^* \varphi_k.$$

Then ψ_j decreases to φ. By Proposition II.6.44,

$$\theta^n_{\psi_j} \geq \left(\inf_{j\geq k} f_j\right)\mu.$$

In particular, for each $t > 0$,

$$\theta^n_{\psi_j \vee (V_\theta - t)} \geq \mathbb{1}_{\{\varphi > V_\theta - t\}}\left(\inf_{j\geq k} f_j\right)\mu.$$

Now by Corollary II.2.10 and Remark II.2.5, let $k \to \infty$, we find

$$\theta^n_{\varphi \vee (V_\theta - t)} \geq \mathbb{1}_{\{\varphi > V_\theta - t\}} f\mu.$$

Hence

$$\mathbb{1}_{\{\varphi > V_\theta - t\}}\theta^n_{\varphi} \geq \mathbb{1}_{\{\varphi > V_\theta - t\}} f\mu.$$

Let $t \to \infty$ and use the fact that μ is non-pluripolar, we are done. □

COROLLARY II.6.46. *Let $\varphi_j, \psi \in \mathrm{PSH}(X,\theta)$ $(j \in \mathbb{Z}_{>0})$. Let $\mu \in \mathcal{M}_+(X)$. Assume that*

$$0 \geq \varphi_j \geq \psi, \quad \theta_{\varphi_j} \geq \mu, \quad j \in \mathbb{Z}_{>0}.$$

Then

$$\theta^n_{\underline{\lim}\, \varphi_j} \geq \mu.$$

PROPOSITION II.6.47 (Demailly's estimate). *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Then*

$$\theta^n_{\varphi\vee\psi} \geq \mathbb{1}_{\{\varphi\geq\psi\}}\theta^n_{\varphi} + \mathbb{1}_{\{\psi>\varphi\}}\theta^n_{\psi}. \tag{2.50}$$

Proof. **Step 1**. We reduce to the case where φ, ψ both have minimial singularities. Assume that the proposition holds in this case.

For each $k \geq 0$, let

$$\psi_k := \psi \vee (V_\theta - k), \quad \varphi_k := \varphi \vee (V_\theta - k).$$

Then

$$\theta^n_{\varphi_k \vee \psi_k} \geq \mathbb{1}_{\{\varphi_k \geq \psi_k\}} \theta^n_{\varphi_k} + \mathbb{1}_{\{\psi_k > \varphi_k\}} \theta^n_{\psi_k}.$$

Hence by plurilocality,

$$\mathbb{1}_{\{\varphi > V_\theta - k, \psi > V_\theta - k\}} \theta^n_{\varphi \vee \psi} \geq \mathbb{1}_{\{\varphi > V_\theta - k, \psi > V_\theta - k, \varphi \geq \psi\}} \theta^n_{\varphi} + \mathbb{1}_{\{\varphi > V_\theta - k, \psi > V_\theta - k, \psi > \varphi\}} \theta^n_{\psi}.$$

Let $k \to \infty$, we conclude (2.50).

Step 2. We assume that φ and ψ have minimal singularities. Note that the non-Kähler locus is pluripolar, so it suffices to prove (2.50) in the ample locus.

Now the problem is local, so it suffices to prove the following: if $\varphi, \psi \in \mathrm{PSH}(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n)$, then

$$(\mathrm{dd^c}(\varphi \vee \psi))^n \geq \mathbb{1}_{\{\varphi \geq \psi\}} (\mathrm{dd^c}\varphi)^n + \mathbb{1}_{\{\psi > \varphi\}} (\mathrm{dd^c}\psi)^n.$$

We may assume that φ, ψ are defined and is psh in a neighbourhood of $\bar{\mathbb{B}}^n$,

$$-1 \leq \varphi \leq 0, \quad -1 \leq \psi \leq 0.$$

By symmetry, it suffices to prove this inequality on the set $\{\varphi \geq \psi\}$, namely, it suffices to prove

$$(\mathrm{dd^c}(\varphi \vee \psi))^n \geq \mathbb{1}_{\{\varphi \geq \psi\}} (\mathrm{dd^c}\varphi)^n \tag{2.51}$$

Fix a compact subset $K \subseteq \{\varphi \geq \psi\}$. Let χ_ε be the Friedrichs kernels. Let

$$\varphi_\varepsilon = \varphi * \chi_\varepsilon, \quad \psi_\varepsilon = \psi * \chi_\varepsilon.$$

Recall that φ, ψ are quasi-continuous. So given $\delta > 0$, we can take an open set $G \subseteq \mathbb{B}^n$ with capacity less than δ, such that φ, ψ are both continuous on $\mathbb{B}^n - G$. Then we know that as $\varepsilon \to 0$,

$$\varphi_\varepsilon \to \varphi, \quad \psi_\varepsilon \to \psi$$

uniformly on compact subsets of $\mathbb{B}^n - G$. Thus for any $\delta' > 0$, we can take a neighborhood U of K such that

$$\varphi_\varepsilon + \delta' \geq \psi_\varepsilon$$

on $U - G$ for ε small enough. Hence

$$\int_K (\mathrm{dd^c}\varphi)^n \leq \varliminf_{\varepsilon \to 0+} \int_U (\mathrm{dd^c}\varphi_\varepsilon)^n \leq \delta + \varliminf_{\varepsilon \to 0+} \int_{U-G} (\mathrm{dd^c}\varphi_\varepsilon)^n = \delta + \varliminf_{\varepsilon \to 0+} \int_{U-G} (\mathrm{dd^c}((\varphi_\varepsilon + \delta') \vee \psi_\varepsilon))^n$$

Let $\delta \to 0+$, we find

$$\int_K (\mathrm{dd^c}\varphi)^n \leq \int_{\bar{U}} (\mathrm{dd^c}((\varphi + \delta') \vee \psi))^n.$$

We can take U arbitrarily small, so

$$\int_K (\mathrm{dd^c}\varphi)^n \leq \int_K (\mathrm{dd^c}((\varphi + \delta') \vee \psi))^n.$$

Let $\delta' \to 0+$, we get

$$\int_K (\mathrm{dd^c}\varphi)^n \leq \int_K (\mathrm{dd^c}(\varphi \vee \psi))^n.$$

$\square$

CHAPTER 3

Absolute pluripotential theory

In this chapter, X is a compact Kähler manifold of dimension n. Let α be a big cohomology class on X. Let θ be a smooth representative of α.

III.1. Basic energy functionals

In this section, we define and study several energy functionals. Recall that we have defined the full mass class $\mathcal{E}(X,\theta)$ and the class $\mathcal{E}^\infty(X,\theta)$ in Definition II.3.3 and Definition I.1.5.

III.1.1. Definitions and first properties.

Definition III.1.1. Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Assume that $[\varphi] \preceq [\psi]$. We define $E_1(\varphi,\psi) \in (-\infty,\infty]$ as follows:

(1) When $[\varphi] = [\psi]$, define

$$E_1(\varphi,\psi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X (\psi-\varphi)\,\theta_\psi^j \wedge \theta_\varphi^{n-j} \in \mathbb{R}. \tag{3.1}$$

(2) In general, define

$$E_1(\varphi,\psi) = \sup_\eta E_1(\eta,\psi), \tag{3.2}$$

where the sup is taken over $\eta \in \mathrm{PSH}(X,\theta)$ such that $\varphi \le \eta$ and $[\eta] = [\psi]$.

When $\psi = V_\theta$, we write

$$E(\varphi) = -E_1(\varphi, V_\theta).$$

The functional $E : \mathrm{PSH}(X,\theta) \to [-\infty,\infty)$ is known as the *Monge–Ampère energy*.

Remark III.1.1. $E_1(\varphi,\psi) > -\infty$. To see, it suffices to observe that for any $C \in \mathbb{R}$, the function $\eta_C := (\eta - C) \vee \varphi$ is a candidate for the sup in (3.2).

Definition III.1.2. The space $\mathcal{E}^1(X,\theta)$ is defined as

$$\mathcal{E}^1(X,\theta) := \{\varphi \in \mathrm{PSH}(X,\theta) : E(\varphi) > -\infty\}. \tag{3.3}$$

The topology on $\mathcal{E}^1(X,\theta)$ is the coarsest refinement of the subspace topology induced from $\mathrm{PSH}(X,\theta)$ that makes $E : \mathcal{E}^1(X,\theta) \to \mathbb{R}$ continuous.

For each $C \in \mathbb{R}$, let

$$\mathcal{E}^1_C(X,\theta) := \left\{\varphi \in \mathrm{PSH}(X,\theta) : \sup_X \varphi \le 0\,, E(\varphi) \ge -C\right\}. \tag{3.4}$$

Observe that

$$\mathcal{E}^1_C(X,\theta) \subseteq \mathcal{E}^1(X,\theta)\,.$$

PROPOSITION III.1.1. *Let $\varphi,\psi,\gamma\in\mathrm{PSH}(X,\theta)$. Assume that $[\varphi]=[\psi]=[\gamma]$, then*

$$E_1(\varphi,\psi)+E_1(\psi,\gamma)=E_1(\varphi,\gamma), \tag{3.5}$$

where by E_1, we mean the functional defined in (3.1).

PROOF. We have

$$\begin{aligned}
&(n+1)\left(E_1(\varphi,\gamma)-E_1(\varphi,\psi)-E_1(\psi,\gamma)\right)\\
=&\sum_{j=0}^{n}\int_X(\gamma-\psi)\left(\theta_\gamma^j\wedge\theta_\varphi^{n-j}-\theta_\gamma^j\wedge\theta_\psi^{n-j}\right)+\sum_{j=0}^{n}\int_X(\psi-\varphi)\left(\theta_\gamma^j\wedge\theta_\varphi^{n-j}-\theta_\psi^j\wedge\theta_\varphi^{n-j}\right)\\
=&\sum_{\substack{j+a+b=n-1\\ j,a,b\geq 0}}(n-j)\int_X(\psi-\varphi)\left(\theta_\varphi^a\wedge\theta_\psi^{b+1}\wedge\theta_\gamma^j-\theta_\varphi^a\wedge\theta_\psi^b\wedge\theta_\gamma^{j+1}\right)\\
&+\sum_{\substack{j+a+b=n-1\\ j,a,b\geq 0}}(n-j)\int_X(\psi-\varphi)\left(\theta_\varphi^j\wedge\theta_\gamma^{a+1}\wedge\theta_\psi^b-\theta_\varphi^j\wedge\theta_\gamma^a\wedge\theta_\psi^{b+1}\right)\\
=&0,
\end{aligned}$$

where the second equality follows from Theorem II.5.42. □

COROLLARY III.1.2. *Let $\varphi,\psi,\gamma\in\mathrm{PSH}(X,\theta)$. Assume that $[\varphi]=[\psi]=[\gamma]$.*

(1) If $\varphi\leq\psi$, then
$$E_1(\varphi,\gamma)\geq E_1(\psi,\gamma).$$
Here by E_1, we mean E_1 defined in (3.1).

(2) If $\psi\leq\gamma$, then
$$E_1(\varphi,\gamma)\geq E_1(\varphi,\psi).$$
Here by E_1, we mean E_1 defined in (3.1).

In particular, when $[\varphi]=[\psi]$, $E_1(\varphi,\psi)$ defined in (3.1) *and in* (3.2) *coincide.*

PROPOSITION III.1.3. *Let $\varphi,\psi,\gamma\in\mathrm{PSH}(X,\theta)$. Assume that $\varphi\leq\psi$, $[\psi]\preceq[\gamma]$, then*
$$E_1(\varphi,\gamma)\geq E_1(\psi,\gamma).$$

PROPOSITION III.1.4. *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$, $[\varphi]\preceq[\psi]$. For any $C\in\mathbb{R}$,*

$$E_1(\varphi,\psi+C)=E_1(\varphi,\psi)+C\int_X\theta_\psi^n\,,\quad E_1(\varphi+C,\psi)=E_1(\varphi,\psi)-C\int_X\theta_\psi^n\,. \tag{3.6}$$

PROOF. When $[\varphi]=[\psi]$, (3.6) follows from Proposition III.1.1. The general case follows by definition. □

PROPOSITION III.1.5. *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$.*

(1) Let $\varphi_j\in\mathrm{PSH}(X,\theta)$ ($j\in\mathbb{Z}_{>0}$). Assume that φ_j is decreasing with limit φ, $[\varphi_1]\preceq[\psi]$, then
$$\lim_{j\to\infty}E_1(\varphi_j,\psi)=E_1(\varphi,\psi). \tag{3.7}$$

(2) Let $\psi_j\in\mathrm{PSH}(X,\theta)$ ($j\in\mathbb{Z}_{>0}$). Assume that ψ_j is increasing with a.e. limit ψ, $[\varphi]=[\psi_1]=[\psi]$, then
$$\lim_{j\to\infty}E_1(\varphi,\psi_j)=E_1(\varphi,\psi). \tag{3.8}$$

PROOF. (1) Note that the limit on LHS of (3.7) exists since $E_1(\varphi_j,\psi)$ is increasing by Proposition III.1.3. Moreover,

$$\lim_{j\to\infty} E_1(\varphi_j,\psi) \leq E_1(\varphi,\psi). \tag{3.9}$$

It suffices to prove the reverse inequality. We may assume that $\varphi_1 \leq \psi$ by Proposition III.1.4.

When $[\varphi]=[\psi]$. The reverse inequality follows from Theorem II.2.8. In general, let $\eta \in \mathrm{PSH}(X,\theta)$ be such that $[\eta]=[\psi]$ and that $\varphi \leq \eta$, then by Proposition III.1.3,

$$E_1(\eta,\psi) = \lim_{j\to\infty} E_1(\varphi_j \vee \eta,\psi) \leq \lim_{j\to\infty} E_1(\varphi_j,\psi).$$

Take sup with respect to η, we conclude (3.7).

(2) By Corollary III.1.2, the limit in (3.8) exists and

$$\lim_{j\to\infty} E_1(\varphi,\psi_j) \leq E_1(\varphi,\psi).$$

For the reverse inequality, when $\varphi \leq \psi_1$, it follows from Theorem II.2.8. In general, there is $C>0$ such that $\varphi \leq \psi_1 + C$, then we have

$$\lim_{j\to\infty} E_1(\varphi,\psi_j + C) = E_1(\varphi,\psi+C).$$

By Proposition III.1.4, we then have

$$\lim_{j\to\infty}\left(E_1(\varphi,\psi_j) + C\int_X \theta^n_{\psi_j}\right) = E_1(\varphi,\psi) + C\int_X \theta^n_{\psi}.$$

But it follows from Theorem II.2.8 that

$$\lim_{j\to\infty}\int_X \theta^n_{\psi_j} = \int_X \theta^n_{\psi}.$$

So we conclude (3.8). □

COROLLARY III.1.6. *Let $\varphi,\psi,\gamma \in \mathrm{PSH}(X,\theta)$. Assume that $[\varphi]\preceq[\psi]=[\gamma]$, then*

$$E_1(\varphi,\psi) + E_1(\psi,\gamma) = E_1(\varphi,\gamma). \tag{3.10}$$

PROOF. This follows from Proposition III.1.1 and Proposition III.1.5. □

PROPOSITION III.1.7. *Let $\varphi,\psi,\gamma \in \mathrm{PSH}(X,\theta)$. Assume that $\varphi \leq \psi$, $[\psi]\preceq[\gamma]$. Then*

$$E_1(\varphi,\psi) + E_1(\psi,\gamma) \leq E_1(\varphi,\gamma). \tag{3.11}$$

In particular, if $\gamma \geq \psi \geq \varphi$, we have

$$E_1(\varphi,\psi) \leq E_1(\varphi,\gamma).$$

PROOF. By Proposition III.1.5, we may assume that $[\varphi]=[\psi]$.
Then

$$\begin{aligned}
E_1(\varphi,\gamma) &\geq \lim_{C\to\infty} E_1(\varphi\vee(\gamma-C),\gamma)\\
&= \lim_{C\to\infty}\left(E_1(\varphi\vee(\gamma-C),\psi\vee(\gamma-C)) + E_1(\psi\vee(\gamma-C),\gamma)\right)\\
&= \lim_{C\to\infty} E_1(\varphi\vee(\gamma-C),\psi\vee(\gamma-C)) + E_1(\psi,\gamma),
\end{aligned}$$

where the first line follows from definition, the second follows from Corollary III.1.6, the third follows from Proposition III.1.5.

We claim that

$$\lim_{C\to\infty} E_1(\varphi \vee (\gamma - C), \psi \vee (\gamma - C)) \geq E_1(\varphi, \psi)\,.$$

To see, by monotonicity of E_1 in the first variable, we may assume that $[\varphi] = [\psi]$. Then this follows Theorem II.2.8.

Hence (3.11) follows. □

Proposition III.1.8. *Let $\varphi_0, \varphi_1, \psi \in \mathrm{PSH}(X, \theta)$. Assume that $[\varphi_0] = [\varphi_1] = [\psi]$. Then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+} E_1(t\varphi_1 + (1-t)\varphi_0, \psi) = -\int_X (\varphi_1 - \varphi_0)\, \theta_{\varphi_0}^n. \tag{3.12}$$

Proof. For $0 < t < 1$,

$$\begin{aligned}
&(n+1)\left(E_1(t\varphi_1 + (1-t)\varphi_0, \psi) - E_1(\varphi_0, \psi)\right)\\
=&\sum_{j=0}^n \left(\int_X (\psi - t\varphi_1 - (1-t)\varphi_0)\, \theta_\psi^j \wedge \theta_{t\varphi_1 + (1-t)\varphi_0}^{n-j} - \int_X (\psi - \varphi_0)\, \theta_\psi^j \wedge \theta_{\varphi_0}^{n-j} \right)\\
=&t\sum_{j=0}^n I_j + \mathcal{O}(t^2),
\end{aligned}$$

where

$$I_j := (n-j) \int_X (\psi - \varphi_0)\, (\theta_{\varphi_1} - \theta_{\varphi_0}) \wedge \theta_\psi^j \wedge \theta_{\varphi_0}^{n-j-1} + \int_X (\varphi_0 - \varphi_1)\, \theta_\psi^j \wedge \theta_{\varphi_0}^{n-j}.$$

Here the first term is understood as 0 when $j = n$.

By Theorem II.5.42,

$$I_j = (n-j) \int_X (\varphi_1 - \varphi_0) \theta_\psi^{j+1} \wedge \theta_{\varphi_0}^{n-j-1} - (n-j+1) \int_X (\varphi_1 - \varphi_0)\, \theta_\psi^j \wedge \theta_{\varphi_0}^{n-j}.$$

Here the first term is understood as 0 when $j = n$.

Hence

$$\sum_{j=0}^n I_j = -(n+1) \int_X (\varphi_1 - \varphi_0)\, \theta_{\varphi_0}^n.$$

Hence (3.12) follows. □

Definition III.1.3. Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Assume that $\varphi \leq \psi$. We define

$$\begin{aligned}
F_1(\varphi, \psi) &:= \int_X (\psi - \varphi)\, \theta_\varphi^n\,,\\
G_1(\varphi, \psi) &:= \int_X (\psi - \varphi)\, \theta_\psi^n\,.
\end{aligned} \tag{3.13}$$

Lemma III.1.9. *Let $\varphi, \psi \in \mathcal{E}(X, \theta)$, $\varphi \leq \psi$, then for any $k > 0$,*

$$F_1(\varphi \vee (V_\theta - k), \psi) \leq F_1(\varphi, \psi).$$

Proof. For $k > 0$, let

$$\varphi_k := \varphi \vee (V_\theta - k)\,.$$

Then

$$\begin{aligned}\int_X (\psi-\varphi_k)\,\theta_{\varphi_k}^n =& \int_{\{\varphi<\psi-k\}} (\psi-\varphi_k)\,\theta_{\varphi_k}^n + \int_{\{\varphi>\psi-k\}} (\psi-\varphi_k)\,\theta_{\varphi}^n\\ =&k^p \int_{\{\varphi<\varphi_k\}} \theta_{\varphi_k}^n + \int_{\{\varphi>\psi-k\}} (\psi-\varphi_k)\,\theta_{\varphi}^n\\ \leq&k^p \int_{\{\varphi<\psi-k\}} \theta_{\varphi}^n + \int_{\{\varphi>\psi-k\}} (\psi-\varphi)\,\theta_{\varphi}^n\\ \leq& \int_X (\psi-\varphi)\,\theta_{\varphi}^n,\end{aligned}$$

where on the third line, we have applied Theorem II.4.28. □

Proposition III.1.10. *Let $\varphi,\psi\in\mathcal{E}^1(X,\theta)$. Assume that $\varphi\leq\psi$. Then*

(3.14)

$$G_1(\varphi,\psi)\leq E_1(\varphi,\psi)\leq F_1(\varphi,\psi)\leq F_1(\varphi,\psi)+nG_1(\varphi,\psi)\leq (n+1)E_1(\varphi,\psi)<\infty.$$

Proof. When $[\varphi]=[\psi]$, (3.14) is a direct consequences of Corollary II.4.33. In general, for each $j\geq 1$, let $\varphi_j=(\psi-j)\vee\varphi$.

By Lemma III.1.9,

$$F_1(\varphi_k,\psi)\leq F_1(\varphi,\psi).$$

So

$$E_1(\varphi,\psi)=\lim_{k\to\infty} E_1(\varphi_k,\psi)\leq \varlimsup_{k\to\infty} F_1(\varphi_k,\psi)\leq F_1(\varphi,\psi).$$

By Fatou's lemma,

$$G_1(\varphi,\psi)\leq \varliminf_{k\to\infty} G_1(\varphi_k,\psi)\leq \lim_{k\to\infty} E_1(\varphi_k,\psi)=E_1(\varphi,\psi).$$

For any $j\geq 1$,

$$\begin{aligned}\int_X (\psi-\varphi_j)\,\theta_{\varphi}^n+nG_1(\varphi,\psi)=&\lim_{k\to\infty}\int_{\{\varphi>V_\theta-k\}} (\psi-\varphi_j)\,\theta_{\varphi_k}^n+nG_1(\varphi,\psi)\\ \leq& \varliminf_{k\to\infty}\left(\int_{\{\varphi>V_\theta-k\}} (\psi-\varphi_k)\,\theta_{\varphi_k}^n+nG_1(\varphi_k,\psi)\right)\\ \leq&(n+1)\lim_{k\to\infty} E_1(\varphi_k,\psi)\\ =&(n+1)E_1(\varphi,\psi).\end{aligned}$$

Let $j\to\infty$, by monotone convergence theorem,

$$F_1(\varphi,\psi)+G_1(\varphi,\psi)\leq (n+1)E_1(\varphi,\psi).$$

Finally, let us prove that $F_1(\varphi,\psi)<\infty$. Take a constant $C_1>0$ such that

$$\psi<V_\theta+C_1.$$

In fact,

$$\int_X (\psi-\varphi)\,\theta_{\varphi}^n\leq \int_X (C_1+V_\theta-\varphi)\,\theta_{\varphi}^n\leq C+C\int_X (V_\theta-\varphi)\,\theta_{\varphi}^n=C+CF_1(\varphi-C_1,V_\theta).$$

But from what we have established,

$$F_1(\varphi-C_1,V_\theta)\leq -(n+1)E(\varphi-C_1)<\infty.$$

□

Corollary III.1.11. *Let* $\varphi_0,\dots,\varphi_n\in\mathcal{E}^1(X,\theta)$. *Then*

$$\int_X (V_\theta-\varphi_0)\,\theta_{\varphi_1}\wedge\cdots\wedge\theta_{\varphi_n}<\infty\,. \tag{3.15}$$

Proof. Let

$$\psi:=\frac{1}{n}\sum_{j=1}^n\varphi_j\,.$$

Then there is a constant $\varepsilon>0$ such that

$$\theta_\psi^n\geq\varepsilon\theta_{\varphi_1}\wedge\cdots\wedge\theta_{\varphi_n}\,.$$

So we may assume that $\varphi_1=\cdots=\varphi_n=\varphi$ and we need to show

$$\int_X(V_\theta-\varphi_0)\,\theta_\varphi^n<\infty\,.$$

It suffices to write

$$\int_X(V_\theta-\varphi_0)\,\theta_\varphi^n=\int_X(V_\theta-\varphi)\,\theta_\varphi^n+\int_X(\varphi-\varphi_0)\,\theta_\varphi^n\,.$$

Both terms on RHS are finite by Proposition III.1.10. □

III.1.2. Monge–Ampère energy.

Proposition III.1.12. *Let* $\varphi_0,\varphi_1\in\mathcal{E}^\infty(X,\theta)$. *Then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+}E(t\varphi_1+(1-t)\varphi_0)=\int_X(\varphi_1-\varphi_0)\,\theta_{\varphi_0}^n. \tag{3.16}$$

Proof. This is a special case of Proposition III.1.8. □

Proposition III.1.13. *The Monge–Ampère energy* $E:\mathrm{PSH}(X,\theta)\to[-\infty,\infty)$ *satisfies the following:*

(1) For any $C\in\mathbb{R}$, $\varphi\in\mathrm{PSH}(X,\theta)$,

$$E(\varphi+C)=E(\varphi)+C\int_X\mathrm{vol}\,\alpha.$$

(2) E *is increasing, concave and usc.*

(3) Let $\varphi,\psi\in\mathcal{E}^1(X,\theta)$. *Assume that* $\varphi\leq\psi$, *then we have*

$$E(\psi)-E(\varphi)\geq E_1(\varphi,\psi). \tag{3.17}$$

(4) For $\varphi\in\mathrm{PSH}(X,\theta)$,

$$E(\varphi)\leq\mathrm{vol}\,\alpha\sup_X\varphi\,.$$

Proof. (1) This follows from Proposition III.1.4.

(2) E is increasing by definition. For the concavity of E, it suffices to prove that E is concave on $\mathcal{E}^\infty(X,\theta)$. By (1), we may assume $\varphi_0\leq\varphi_1$. Let $0\leq t\leq s\leq 1$, by Proposition III.1.12, it suffices to prove

$$\int_X(\varphi_1-\varphi_0)\,\theta^n_{t\varphi_1+(1-t)\varphi_0}\geq\int_X(\varphi_1-\varphi_0)\,\theta^n_{s\varphi_1+(1-s)\varphi_0}\,.$$

We may assume that $t=0$, $s=1$, then we need to prove

$$\int_X(\varphi_1-\varphi_0)\,\theta^n_{\varphi_0}\leq\int_X(\varphi_1-\varphi_0)\,\theta^n_{\varphi_1}.$$

This follows from Theorem II.4.27.

Now we prove that E is usc. Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ ($j \in \mathbb{Z}_{>0}$). Assume that $\varphi_j \to \varphi$ in L^1, then

$$\psi_j := \sup_{k\geq j}{}^* \varphi_k$$

decreases to φ. Hence by Proposition III.1.5 and Proposition III.1.3,

$$E(\varphi) = \lim_{j\to\infty} E(\psi_j) \geq \varlimsup_{j\to\infty} E(\varphi_j).$$

(3) This follows from Proposition III.1.7.

(4) By (1) and (2),

$$E(\varphi) \leq E(\varphi - \sup_X \varphi) + \operatorname{vol}\alpha \sup_X \varphi \leq \operatorname{vol}\alpha \sup_X \varphi\,.$$

□

Corollary III.1.14. *For each $C \in \mathbb{R}$, the set $\mathcal{E}^1_C(X,\theta) \subseteq \mathrm{PSH}(X,\theta)$ is convex and compact in the subspace topology.*

Proof. The set $\mathcal{E}^1_C(X,\theta)$ is convex as E is convex (Proposition III.1.13).

By Proposition III.1.13, E is usc, hence $\mathcal{E}^1_C(X,\theta) \subseteq \mathrm{PSH}(X,\theta)$ is a closed set.

As $E(\varphi) \leq \sup_X \varphi$ by definition, we have

$$\mathcal{E}^1_C(X,\theta) \subseteq \{\varphi \in \mathrm{PSH}(X,\theta) : -C \leq \sup_X \varphi \leq 0\}\,.$$

Hence it is compact by Theorem I.1.2 □

Corollary III.1.15. *Let $\varphi, \psi \in \mathcal{E}^1(X,\theta)$. Then*

$$E(\psi) - E(\varphi) \leq \int_X (\psi - \varphi)\,\theta_\varphi^n. \tag{3.18}$$

Proof. **Step 1**. We prove (3.18) under the additional assumption that $\psi \in \mathcal{E}^\infty(X,\theta)$.

In this case, by Corollary III.1.6,

$$E(\psi) - E(\varphi) = E_1(\varphi, \psi).$$

But by Proposition III.1.10,

$$E_1(\varphi,\psi) \leq \int_X (\psi - \varphi)\,\theta_\varphi^n\,.$$

Step 2. In general, for any $C > 0$, let

$$\psi^C := \psi \vee (V_\theta - C)\,.$$

Then by Step 1,

$$E(\psi^C) - E(\varphi) \leq \int_X (\psi^C - \varphi)\,\theta_\varphi^n\,.$$

Let $C \to \infty$, by Proposition III.1.5, Proposition III.1.10 and dominated convergence theorem, we conclude (3.18). □

Proposition III.1.16. *Let $\varphi_j, \varphi \in \mathcal{E}(X,\theta)$ ($j \in \mathbb{N}$). Assume one of the following conditions is satisfied:*

(1) φ_j decreases to φ.
(2) φ_j increases a.e. to φ.
(3) φ_j converges uniformly φ.

Then

$$\lim_{j\to\infty} E(\varphi_j) = E(\varphi)\,. \tag{3.19}$$

Proof. (1) This follows from Proposition III.1.5.

(2) We may assume that $\varphi \leq V_\theta$.

When $\varphi_1 \in \mathcal{E}^\infty(X,\theta)$, (3.19) follows from Theorem II.2.12. In general, for any $C > 0$, let

$$\varphi_j^C := \varphi_j \vee (V_\theta - C)\,.$$

by Corollary III.1.15, for any $C > 0$,

$$\begin{aligned} 0 \leq E(\varphi_j^C) - E(\varphi_j) \leq \int_X \left(\varphi_j^C - \varphi_j\right)\, \theta^n_{\varphi_j} &= \int_C^\infty \mathrm{d}t \int_X \mathbb{1}_{\{\varphi_j < V_\theta - t\}} \theta^n_{\varphi_j} \\ \leq 2^n \int_C^\infty \mathrm{d}t \int_X \mathbb{1}_{\{\varphi_1 < V_\theta - t/2\}} \theta^n_{\varphi_1} &= 2^{n+1} \int_X (\varphi_1^{C/2} - \varphi_1)\, \theta^n_{\varphi_1}\,, \end{aligned}$$

where the last but one step follows from Corollary II.4.29. By Proposition III.1.10,

$$\int_X (V_\theta - \varphi_1)\, \theta^n_{\varphi_1} < \infty,$$

so by dominated convergence theorem, we find that $E(\varphi_j^C) \to E(\varphi_j)$ as $C \to \infty$ uniformly in j. Hence (3.19) follows.

(3) This follows from Theorem II.2.8. □

Corollary III.1.17. *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}^1(X,\theta)$.

(2) $\varphi \in \mathcal{E}(X,\theta)$ and

$$\int_X (V_\theta - \varphi)\, \theta^n_\varphi < \infty\,. \tag{3.20}$$

(3) The following holds:

$$\int_0^\infty \mathrm{d}t \int_{\{\varphi = V_\theta - t\}} \theta^n_{\varphi \vee (V_\theta - t)} < \infty\,. \tag{3.21}$$

Proof. (1) implies (2). This follows from Proposition III.1.10.

(2) implies (1). We may further assume that $\varphi \leq V_\theta$. Then by Lemma III.1.9, $F_1(\varphi \vee (V_{\theta - C}))$ is bounded for all $C > 0$. Hence by Proposition III.1.10, $E(\varphi \vee (V_{\theta - C}))$ is also bounded. By Proposition III.1.16, $E(\varphi)$ is thus finite.

Now in order to relate (2) and (3), observe the following general relation:

(3.22)
$$\int_0^\infty \mathrm{d}t \int_{\{\varphi = V_\theta - t\}} \theta^n_{\varphi \vee (V_\theta - t)} = \int_0^\infty \mathrm{d}t \left(\mathrm{vol}\,\alpha - \int_{\{\varphi < V_\theta - t\}} \theta^n_{V_\theta} - \int_{\{\varphi > V_\theta - t\}} \theta^n_\varphi\right)\,.$$

(2) implies (3). By (3.22), since φ has full mass,

$$\begin{aligned} \int_0^\infty \mathrm{d}t \int_{\{\varphi = V_\theta - t\}} \theta^n_{\varphi \vee (V_\theta - t)} &= \int_0^\infty \mathrm{d}t \left(\int_{\{\varphi \leq V_\theta - t\}} \theta^n_\varphi - \int_{\{\varphi < V_\theta - t\}} \theta^n_{V_\theta}\right) \\ &= \int_X (V_\theta - \varphi)_+\, \theta^n_\varphi - \int_X (V_\theta - \varphi)_+\, \theta^n_{V_\theta}\,. \end{aligned}$$

The first term is finite by assumption. To see the second is also finite, we may assume that $\varphi \leq V_\theta$, then it follows from Corollary II.4.33 that it is also finite.

(3) implies (2). By assumption, the integral in (3.22) is finite. Hence we can take $t_j \to \infty$ ($j \in \mathbb{N}$), such that

$$\operatorname{vol}\alpha - \int_{\{\varphi < V_\theta - t_j\}} \theta_{V_\theta}^n - \int_{\{\varphi > V_\theta - t_j\}} \theta_\varphi^n \to 0\,.$$

Namely,

$$\int_X \theta_\varphi^n = \operatorname{vol}(\alpha)\,.$$

Insert this back to (3.22), we find (3.20). □

III.1.3. Berman–Boucksom differentiablity theorem.

THEOREM III.1.18. *Let $\varphi \in \mathcal{E}^1(X,\theta)$. Let $v \in C^0(X)$. Then $E(\mathrm{P}(\varphi+tv))$ is differentiable at $t=0$ and*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E\left(\mathrm{P}(\varphi+tv)\right) = \int_X v\,\theta_\varphi^n\,. \tag{3.23}$$

PROOF. By Proposition III.1.13 and Proposition I.5.11, for any $t \in [0,1]$,

$$E(\mathrm{P}(\varphi+tv)) \geq E(\varphi) - t\|v\|_{C^0(X)} > -\infty\,.$$

Observe that (3.23) for all $v \in C^0(X)$ is equivalent to

$$E\left(\mathrm{P}(\varphi+v)\right) - E(\varphi) = \int_0^1 \int_X v\,\theta_{\mathrm{P}(\varphi+tv)}^n\,\mathrm{d}t\,, \quad v \in C^0(X)\,. \tag{3.24}$$

We prove the following more general result: for any $u \in \mathrm{USC}(X)$,

$$E\left(\mathrm{P}(u+v)\right) - E(\mathrm{P}(u)) = \int_0^1 \int_X v\,\theta_{\mathrm{P}(u+tv)}^n\,\mathrm{d}t\,, \quad v \in C^0(X)\,. \tag{3.25}$$

By similar argument as above, for any $t \in [0,1]$,

$$E(\mathrm{P}(u+tv)) > -\infty\,.$$

Step 1. Assume that (3.25) holds when $u \in C^0(X)$, we prove (3.25) for general u.

Let $u_j \in C^0(X)$ ($j \in \mathbb{N}$) be a sequence decreasing to u. Then $\mathrm{P}(u_j+tv)$ decreases to $\mathrm{P}(u+tv)$ for any $t \in [0,1]$ by Proposition I.5.11. Hence by Proposition III.1.16,

$$E(\mathrm{P}(u+tv)) = \lim_{j\to\infty} E(\mathrm{P}(u_j+tv))\,, \quad t \in [0,1]\,. \tag{3.26}$$

Recall that by Proposition I.5.11, $\mathrm{P}(u+tv) \in \mathcal{E}(X,\theta)$, so by Theorem II.2.8,

$$\int_X v\,\theta_{\mathrm{P}(u+tv)}^n = \lim_{j\to\infty} \int_X v\,\theta_{\mathrm{P}(u_j+tv)}^n\,, \quad t \in [0,1]\,.$$

Observe that

$$\left|\int_X v\,\theta_{\mathrm{P}(u_j+tv)}^n\right| \leq \|v\|_{C^0(X)} \int_X \theta_{V_\theta}^n\,,$$

so by dominated convergence theorem,

$$\int_0^1 \int_X v\,\theta_{\mathrm{P}(u+tv)}^n\,\mathrm{d}t = \lim_{j\to\infty} \int_0^1 \int_X v\,\theta_{\mathrm{P}(u_j+tv)}^n\,\mathrm{d}t\,. \tag{3.27}$$

By assumption, we know that (3.25) holds for u_j in place of u. Let $j \to \infty$ and apply (3.26) and (3.27), we conclude (3.25).

From now on, we assume that $u \in C^0(X)$.

Step 2. Assume that (3.25) holds when $v \in C^\infty(X)$, we prove (3.25) in general.

Let $v_j \in C^\infty(X)$ ($j \in \mathbb{N}$) be a sequence converging uniformly to v. By Proposition I.5.11, $\mathrm{P}(u+tv_j) \to \mathrm{P}(u+tv)$ ($t \in [0,1]$) uniformly. So by Proposition III.1.16,

$$\lim_{j\to\infty} E(\mathrm{P}(u+tv_j)) = E(\mathrm{P}(u+tv))\,, \quad t \in [0,1]\,. \tag{3.28}$$

On the other hand,

$$\int_X v_j\,\theta^n_{\mathrm{P}(u+tv_j)} - \int_X v\,\theta^n_{\mathrm{P}(u+tv)} = \int_X (v_j - v)\,\theta^n_{\mathrm{P}(u+tv_j)} + \int_X v\left(\theta^n_{\mathrm{P}(u+tv_j)} - \theta^n_{\mathrm{P}(u+tv)}\right)\,.$$

The first term tends to 0 as $j \to \infty$ since $v_j \to v$ uniformly. By Theorem II.2.8, the second term tends to 0 as $j \to \infty$. So for each t,

$$\lim_{j\to\infty} \int_X v_j\,\theta^n_{\mathrm{P}(u+tv_j)} = \int_X v\,\theta^n_{\mathrm{P}(u+tv)}.$$

By dominated convergence theorem,

$$\lim_{j\to\infty} \int_0^1 \int_X v_j\,\theta^n_{\mathrm{P}(u+tv_j)}\,\mathrm{d}t = \int_0^1 \int_X v\,\theta^n_{\mathrm{P}(u+tv)}\,\mathrm{d}t. \tag{3.29}$$

By assumption, (3.25) holds for v_j in place of v. Let $j \to \infty$ and apply (3.28) and (3.29), we conclude (3.25).

From now on we assume that $v \in C^\infty(X)$. It remains to prove (3.25), which in turn is equivalent to the following:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+} E\left(\mathrm{P}(u+tv)\right) = \int_X v\,\theta^n_{\mathrm{P}(u)}. \tag{3.30}$$

Step 3. We claim that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+} E\left(\mathrm{P}(u+tv)\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+} \int_X \left(\mathrm{P}(u+tv) - \mathrm{P}(u)\right)\,\theta^n_{\mathrm{P}(u)}\,. \tag{3.31}$$

Observe that RHS makes sense since $\mathrm{P}(u+tv)$ is concave in t by Proposition I.5.11. By Proposition III.1.13 and Proposition III.1.12,

$$E(\mathrm{P}(u+tv)) \leq E(\mathrm{P}(u)) + \int_X \left(\mathrm{P}(u+tv) - \mathrm{P}(u)\right)\,\theta^n_{\mathrm{P}(u)}\,.$$

Hence in (3.31), LHS is not greater than RHS. We prove the reverse inequality. For each $\varepsilon > 0$ small enough, we can take $\delta > 0$ small enough, so that

$$\int_X \left(\mathrm{P}(u+\delta v) - \mathrm{P}(u)\right)\,\theta^n_{\mathrm{P}(u)} \geq \delta\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+} \int_X \left(\mathrm{P}(u+tv) - \mathrm{P}(u)\right)\,\theta^n_{\mathrm{P}(u)} - \varepsilon\right). \tag{3.32}$$

By Proposition III.1.12,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+} E\left((1-t)\mathrm{P}(u) + t\mathrm{P}(u+\delta v)\right) = \int_X \left(\mathrm{P}(u+\delta v) - \mathrm{P}(u)\right)\,\theta^n_{\mathrm{P}(u)}\,.$$

So for small enough $t > 0$ (depending on δ and ε),

$$E\left((1-t)\mathrm{P}(u) + t\mathrm{P}(u+\delta v)\right) - E(\mathrm{P}(u)) \geq t\int_X \left(\mathrm{P}(u+\delta v) - \mathrm{P}(u)\right)\,\theta^n_{\mathrm{P}(u)} - t\delta\varepsilon \tag{3.33}$$

By Proposition I.5.11 and Proposition III.1.13, we have

$$E\left(\mathrm{P}(u+t\delta v)\right) \geq E\left((1-t)\mathrm{P}(u) + t\mathrm{P}(u+\delta v)\right)\,. \tag{3.34}$$

From (3.32), (3.33) and (3.34), we get

$$E\left(\mathrm{P}(u+t\delta v)\right)-E(\mathrm{P}(u))\geq t\delta\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+}\int_X\left(\mathrm{P}(u+tv)-\mathrm{P}(u)\right)\,\theta^n_{\mathrm{P}(u)}-2t\varepsilon\delta.$$

Let $t\to 0+$, then let $\varepsilon\to 0+$, (3.31) follows.

Step 4. It remains to prove

$$I_t:=\int_X\left(\mathrm{P}(u+tv)-\mathrm{P}(u)\right)\,\theta^n_{\mathrm{P}(u)}-t\int_X v\,\theta^n_{\mathrm{P}(u)}=o(t)\,,\quad t\to 0+\ . \tag{3.35}$$

By Proposition I.5.11,

$$\mathrm{P}(u+tv)\leq u+tv=\mathrm{P}(u)+tv\,,\quad \theta^n_{\mathrm{P}(u)}-a.e.$$

Hence

$$I_t=\int_{\{\mathrm{P}(u+tv)<\mathrm{P}(u)+tv\}}\left(\mathrm{P}(u+tv)-\mathrm{P}(u)-tv\right)\,\theta^n_{\mathrm{P}(u)}.$$

By Proposition I.5.11,

$$\sup_X\left|\mathrm{P}(u+tv)-\mathrm{P}(u)-tv\right|=\mathcal{O}(t)\,.$$

Hence

$$|I_t|=\mathcal{O}(t)\int_{\{\mathrm{P}(u+tv)<\mathrm{P}(u)+tv\}}\theta^n_{\mathrm{P}(u)}.$$

So it suffices to prove

$$\lim_{t\to 0+}\int_{\{\mathrm{P}(u+tv)<\mathrm{P}(u)+tv\}}\theta^n_{\mathrm{P}(u)}=0. \tag{3.36}$$

Since the class α is big, we can take $\varphi_0\in\mathrm{PSH}(X,\theta)$ such that θ_{φ_0} is strictly positive. Then $\varphi_0+\varepsilon v\in\mathrm{PSH}(X,\theta)$ for $\varepsilon>0$ small enough.

For $t>0$ small enough (depending on ε),

$$\mathrm{P}(u)+tv+t\varphi_0\,,\mathrm{P}(u+tv)+t\varphi_0\in\mathrm{PSH}(X,(1+t)\theta).$$

By Proposition I.5.11,

$$\mathrm{P}(u)+tv+t\varphi_0\geq\mathrm{P}(u+tv)+t\varphi_0-2t\|v\|_{C^0(X)}\,.$$

By Theorem II.4.27,

$$\int_{\{\mathrm{P}(u+tv)<\mathrm{P}(u)+tv\}}\left(\theta_{\mathrm{P}(u)}+t\theta_{v+\varphi_0}\right)^n\leq\int_{\{\mathrm{P}(u+tv)<\mathrm{P}(u)+tv\}}\left(\theta_{\mathrm{P}(u+tv)}+t\theta_{\varphi_0}\right)^n.$$

On the other hand, by Proposition I.5.11,

$$\int_{\{\mathrm{P}(u+tv)<\mathrm{P}(u)+tv\}}\left(\theta_{\mathrm{P}(u+tv)}+t\theta_{\varphi_0}\right)^n\leq\int_{\{\mathrm{P}(u+tv)<u+tv\}}\theta^n_{\mathrm{P}(u+tv)}+\mathcal{O}(t)=\mathcal{O}(t)\,.$$

Hence (3.36) follows. $\square$

Corollary III.1.19. *The functional $E\circ\mathrm{P}:C^0(X)\to\mathbb{R}$ is Fréchet differntiable and*

$$\delta_u\left(E\circ\mathrm{P}\right)=\theta^n_{\mathrm{P}(u)}\,.$$

Proof. By (3.25), $E\circ\mathrm{P}$ is Gateaux differentiable with Gateaux derivative $\theta^n_{\mathrm{P}(u)}$. Notice that

$$u\mapsto\theta^n_{\mathrm{P}(u)}\,,C^0(X)\to\mathcal{M}_+(X)$$

is continuous. Hence $E\circ\mathrm{P}$ is Fréchet differentiable. $\square$

III.1.4. Free energy type functionals.

DEFINITION III.1.4. Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Define $L_\mu : \mathrm{PSH}(X,\theta) \to [-\infty,\infty)$ by

$$L_\mu(\varphi) = \int_X (\varphi - V_\theta)\,\mathrm{d}\mu. \tag{3.37}$$

LEMMA III.1.20. *Let $\varphi, \psi \in \mathcal{E}^\infty(X,\theta)$. Then*

$$\left| \int_X (V_\theta - \psi^C)\,\theta_\varphi^n - \int_X (V_\theta - \psi^C)\,\theta_{V_\theta}^n \right| \leq 2n \sup_X |\varphi - V_\theta| \operatorname{vol}\alpha\,. \tag{3.38}$$

PROOF.

$$\begin{aligned}\int_X (V_\theta - \psi)\,\theta_\varphi^n = \int_X (V_\theta - \psi)\,\theta_{V_\theta} \wedge \theta_\varphi^{n-1} &+ \int_X (\varphi - V_\theta)\,\theta_{V_\theta} \wedge \theta_\varphi^{n-1} \\ &- \int_X (\varphi - V_\theta)\,\theta_\psi \wedge \theta_\varphi^{n-1}\,.\end{aligned}$$

Hence

$$\left| \int_X (V_\theta - \psi)\,\theta_\varphi^n - \int_X (V_\theta - \psi)\,\theta_{V_\theta} \wedge \theta_\varphi^{n-1} \right| \leq 2 \sup_X |\varphi - V_\theta| \operatorname{vol}\alpha\,.$$

We iterate this procedure n-times and find

$$\left| \int_X (V_\theta - \psi)\,\theta_\varphi^n - \int_X (V_\theta - \psi)\,\theta_{V_\theta}^n \right| \leq 2n \sup_X |\varphi - V_\theta| \operatorname{vol}\alpha\,. \tag{3.39}$$

□

PROPOSITION III.1.21. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Then*

1. *For any $C \in \mathbb{R}$, $\varphi \in \mathrm{PSH}(X,\theta)$, then $L_\mu(\varphi + C) = L_\mu(\varphi) + C\mu(X)$.*
2. *$L_\mu : \mathrm{PSH}(X,\theta) \to [-\infty,\infty)$ is usc.*
3. *Assume that L_μ is finite on $\mathrm{PSH}(X,\theta)$, $\mu \neq 0$. There is a constant $C > 0$, such that for any $\varphi \in \mathrm{PSH}(X,\theta)$,*

$$\sup_X \varphi - C \leq \frac{1}{\mu(X)} L_\mu(\varphi) \leq \sup_X \varphi\,. \tag{3.40}$$

4. *When there exists $\varphi \in \mathcal{E}^1(X,\theta)$ such that $\mu = \theta_\varphi^n$, then L_μ is finite on $\mathcal{E}^1(X,\theta)$.*

PROOF. (1) This is obvious.
(2) This is a direct consequence of Hartogs' lemma.
(3) This follows from Proposition III.1.23.
(4) This follows from Corollary III.1.11. □

PROPOSITION III.1.22. *Assume that $\varphi \in \mathcal{E}^\infty(X,\theta)$, then $L_{\theta_\varphi^n}$ is finite on $\mathrm{PSH}(X,\theta)$.*

PROOF. Let $\psi \in \mathrm{PSH}(X,\theta)$, we want to show that

$$L_{\theta_\varphi^n}(\psi) = \int_X (V_\theta - \psi)\,\theta_\varphi^n < \infty\,. \tag{3.41}$$

By Proposition III.1.21, we may assume that $\psi \leq 0$.

Step 1. We prove (3.41) for $\varphi = V_\theta$. In this case, $\theta_{V_\theta}^n$ is absolutely continuous by Theorem I.5.12. Hence (3.41) follows.

Step 2. We prove (3.41) in general.

For a general ψ, $C \geq 0$, let $\psi^C := \psi \vee (V_\theta - C)$ be the canonical approximations of ψ. By Lemma III.1.20,

$$\int_X (V_\theta - \psi^C)\, \theta_\varphi^n \leq 2n \sup_X |\varphi - V_\theta| \operatorname{vol} \alpha + \int_X (V_\theta - \psi^C)\, \theta_{V_\theta}^n \,.$$

Let $C \to \infty$ and use monotone convergence theorem, we conclude

$$\int_X (V_\theta - \psi)\, \theta_\varphi^n \leq 2n \sup_X |\varphi - V_\theta| \operatorname{vol} \alpha + \int_X (V_\theta - \psi)\, \theta_{V_\theta}^n \,.$$

Hence we conclude (3.41) by Step 1. □

Proposition III.1.23. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$, $\mu(X) > 0$. Assume that L_μ is finite on* $\mathrm{PSH}(X,\theta)$. *Then*

$$\left\{ \varphi \in \mathrm{PSH}(X,\theta) : \int_X (\varphi - V_\theta)\, \mathrm{d}\mu = 0 \right\} \subseteq \mathrm{PSH}(X,\theta)$$

is relatively compact.

In particular, there is a constant $C = C(\mu) > 0$, such that

$$-C + \sup_X \varphi \leq \frac{1}{\mu(X)} \int_X (\varphi - V_\theta)\, \mathrm{d}\mu \leq \sup_X \varphi \,. \tag{3.42}$$

Proof. Let $\varphi_j \in \mathrm{PSH}(X,\theta)$ $(j \in \mathbb{N})$,

$$\int_X (\varphi_j - V_\theta)\, \mathrm{d}\mu = 0 \,.$$

Let

$$\psi_j := \varphi_j - \sup_X \varphi_j \,.$$

It suffices to prove that there is a constant $C > 0$, such that

$$\int_X (V_\theta - \psi_j)\, \mathrm{d}\mu \leq C \,, \tag{3.43}$$

since then

$$\mu(X) \sup_X \varphi_j = \int_X (V_\theta - \varphi)\, \mathrm{d}\mu + \mu(X) \sup_X \varphi_j = \int_X (V_\theta - \psi_j)\, \mathrm{d}\mu$$

is also bounded hence we conclude by Theorem I.1.2.

Assume that (3.43) fails, after extracting a subsequence, we may assume that

$$\int_X (V_\theta - \psi_j)\, \mathrm{d}\mu \geq 2^j \,.$$

Let

$$\psi = \sum_{j=1}^{\infty} 2^{-j} \psi_j \in \mathrm{PSH}(X,\theta) \,.$$

By monotone convergence theorem,

$$\infty > \int_X (V_\theta - \psi)\, \mathrm{d}\mu = \sum_{j=1}^{\infty} 2^{-j} \int_X (V_\theta - \psi_j)\, \mathrm{d}\mu = \infty \,.$$

□

Theorem III.1.24. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Let $\mathcal{K} \subseteq \mathrm{PSH}(X,\theta)$ be a compact convex subset such that L_μ is finite on $\mathcal{K}$. Then (1) implies (2) implies (3):*

(1) The map $T : \mathcal{K} \to L^1(\mu)$ given by $\varphi \mapsto \varphi - V_\theta$ is continuous.
(2) The set $\mathcal{K}$ is uniformly integrable with respect to μ:

$$\lim_{k\to\infty} \sup_{\varphi\in\mathcal{K}} \int_k^\infty \mu\{\varphi \le V_\theta - t\}\,\mathrm{d}t = 0. \tag{3.44}$$

(3) $L_\mu : \mathcal{K} \to \mathbb{R}$ is continuous.

Remark III.1.2. The converse is not true. ADD A COUNTEREXAMPLE LATER.

Proof. (1) implies (2): We claim that $T(\mathcal{K}) \subseteq L^1(\mu)$ is closed and convex. In fact, it is obviously convex. To see this set is closed, let $\varphi_j \in \mathcal{K}$ ($j \in \mathbb{N}$) be a sequence such that $T(\varphi_j) \to f \in L^1(\mu)$. By compactness of $\mathcal{K}$, after substracting a subsequence, we may assume that $\varphi_j \to \varphi \in \mathrm{PSH}(X,\theta)$. Since $\mathcal{K}$ is closed by assumption, $\varphi \in \mathcal{K}$. Now as T is continuous, we find $T(\varphi) = f$. This proves that $T(\mathcal{K})$ is closed. Now by Hahn–Banach theorem, $T(\mathcal{K})$ is closed in weak star topology. Hence by Dunford–Pettis theorem, $\mathcal{K}$ is uniformly integrable.

(2) implies (3): Assume that $\mathcal{K}$ is uniformly integrable. Let $\varphi_j, \varphi \in \mathcal{K}$ ($j \in \mathbb{N}$). Assume that $\varphi_j \to \varphi$ in L^1-topology. We want to show that

$$\lim_{j\to\infty} L_\mu(\varphi_j) = L_\mu(\varphi)\,.$$

By Dunford–Pettis theorem, $T(\mathcal{K}) \subseteq L^1(\mu)$ is weakly compact, hence bounded. Hence after extracting a subsequence, we may assume that $L_\mu(\varphi_j)$ converges with limit $L \in \mathbb{R}$. Let K_j be the closed convex hull of φ_k ($k \ge j$). Then $T(K_j)$ is weakly compact and decreasing in j. Take

$$f \in \bigcap_{j=1}^\infty K_j\,.$$

In particular, we can take a finite convex combination ψ_j of φ_k ($k \ge j$), such that $T(\psi_j) \to f$ in $L^1(\mu)$. It is easy to see that $f = T(\varphi)$. By construction, $L_\mu(\psi_k) \to L$. Hence $L_\mu(f) = L$. □

Definition III.1.5. Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Define the *free energy* functional $F_\mu : \mathcal{E}^1(X,\theta) \to (-\infty,\infty]$ by

$$F_\mu := E - L_\mu\,.$$

Define the pluricomplex energy $E^* : \mathring{\mathcal{M}}_+(X) \to (-\infty,\infty]$ by

$$E^*(\mu) := \sup_{\mathcal{E}^1(X,\theta)} F_\mu\,.$$

Proposition III.1.25. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Assume that $\mu(X) = \operatorname{vol}\alpha$. Then for any $C \in \mathbb{R}$, $\varphi \in \mathcal{E}^1(X,\theta)$,*

$$F_\mu(\varphi + C) = F_\mu(\varphi)\,.$$

Proposition III.1.26. *Let $\varphi \in \mathcal{E}^1(X,\theta)$. Let $\mu \in \mathring{\mathcal{M}}_+(X)$, $\mu(X) = \operatorname{vol}\alpha$. The following are equivalent:*

(1) $F_\mu(\varphi) = E^*(\mu)$.
(2) $\mu = \theta_\varphi^n$.

Proof. (2) implies (1). Assume that $\mu = \theta_\varphi^n$. Then by Corollary III.1.15,

$$E(\varphi) + \int_X (\psi - V_\theta)\,\theta_\varphi^n \geq E(\psi) + \int_X (\varphi - V_\theta)\,\theta_\varphi^n\,, \quad \psi \in \mathcal{E}^1(X,\theta)\,.$$

In other words,

$$F_\mu(\psi) \leq F_\mu(\varphi).$$

(1) implies (2). Assume that φ is a maximizer of F_μ. Let $v \in C^0(X)$. Consider the following function

$$g(t) := E(\mathrm{P}(\varphi + tv)) - L_\mu(\varphi) - t\int_X v\,\mathrm{d}\mu\,.$$

where $t \in \mathbb{R}$. As $\mathrm{P}(\varphi + tv) \leq \varphi + tv$, we find

$$L_\mu(\mathrm{P}(\varphi + tv)) \leq L_\mu(\varphi) + t\int_X v\,\mathrm{d}\mu\,.$$

So

$$g(t) \leq E(\mathrm{P}(\varphi + tv)) - L_\mu(\mathrm{P}(\varphi + tv)) = F_\mu(\mathrm{P}(\varphi + tv)) \leq g(0)\,.$$

By Theorem III.1.18,

$$0 = g'(0) = \int_X v\,\theta_\varphi^n - \int_X v\,\mathrm{d}\mu\,.$$

Hence (2) holds. □

Definition III.1.6. We define $J : \mathcal{E}^1(X,\theta) \to [0,\infty)$ as

$$J := L_{\theta^n_{V_\theta}} - E\,. \tag{3.45}$$

Define $I : \mathcal{E}^1(X,\theta) \to [0,\infty)$ as

$$I(\varphi) := \int_X (\varphi - V_\theta)(\theta^n_{V_\theta} - \theta^n_\varphi)\,. \tag{3.46}$$

Proposition III.1.27. *Let $\varphi \in \mathcal{E}^1(X,\theta)$.*

(1) $J(\varphi + C) = J(\varphi)$ for any constant $C \in \mathbb{R}$.
(2) The image of J lies in $[0,\infty)$.
(3) For each $C \geq 0$, the set

$$\{\varphi \in \mathcal{E}^1(X,\theta) : \sup_X \varphi = 0\,, J(\varphi) \leq C\} \subseteq \mathcal{E}^1(X,\theta) \tag{3.47}$$

is relatively compact in the subspace topology induced from $\mathrm{PSH}(X,\theta)$.

Proof. (1) This is obvious.

(2) By (1), we may assume that $\varphi \leq V_\theta$. Then this follows from Proposition III.1.10.

(3) By Proposition III.1.21 and Proposition III.1.22, there is a constant $C_1 > 0$, such that for any $\varphi \in \mathcal{E}^1(X,\theta)$,

$$\sup_X \varphi - C_1 \leq \frac{1}{\mu(X)} L_\mu(\varphi) \leq \sup_X \varphi\,.$$

Hence

$$\{\varphi \in \mathcal{E}^1(X,\theta) : \sup_X \varphi = 0\,, J(\varphi) \leq C\} \subseteq \{\varphi \in \mathcal{E}^1(X,\theta) : \sup_X \varphi = 0\,, E(\varphi) \leq -\mu(X)C_1 - C\}\,.$$

Hence the result follows from Corollary III.1.14. □

Proposition III.1.28. *Let $\varphi \in \mathcal{E}^1(X,\theta)$. Then*

(1) For any $C \in \mathbb{R}$,

$$I(\varphi + C) = I(\varphi)\,.$$

(2)

$$\frac{1}{n+1} I(\varphi) \leq J(\varphi) \leq I(\varphi)\,. \tag{3.48}$$

PROOF. (1) This is obvious.

(2) By (1) and Proposition III.1.27, we may assume that $\varphi \leq V_\theta$. Then by definition,

$$J(\varphi) = -G_1(\varphi, V_\theta) + E_1(\varphi, V_\theta)\,, \quad I(\varphi) = -G_1(\varphi, V_\theta) + F_1(\varphi, V_\theta)\,.$$

Hence the inequality follows from Proposition III.1.10. □

PROPOSITION III.1.29. *Let $\varphi \in \mathcal{E}^1(X, \theta)$. Let $t \in [0,1]$. Then*

$$I(t\varphi + (1-t)V_\theta) \leq nt^2 I(\varphi)\,. \tag{3.49}$$

PROOF. We may assume that $\varphi \leq V_\theta$.

By Corollary II.4.32

$$\begin{aligned}
\int_X (\varphi - V_\theta)\, \theta^n_{t\varphi + (1-t)V_\theta} =& (1-t)^n \int_X (\varphi - V_\theta)\, \theta^n_{V_\theta} + \sum_{j=1}^n \binom{n}{j} t^j (1-t)^{n-j} \int_X (\varphi - V_\theta)\, \theta^j_\varphi \wedge \theta^{n-j}_\psi \\
\geq& (1-t)^n \int_X (\varphi - V_\theta)\, \theta^n_{V_\theta} + \sum_{j=1}^n \binom{n}{j} t^j (1-t)^{n-j} \int_X (\varphi - V_\theta)\, \theta^n_\varphi \\
=& (1-t)^n \int_X (\varphi - V_\theta)\, \theta^n_{V_\theta} + (1 - (1-t)^n) \int_X (\varphi - V_\theta)\, \theta^n_\varphi\,.
\end{aligned}$$

Consequently,

$$I(t\varphi + (1-t)V_\theta) \leq t(1 - (1-t)^n) I(\varphi)\,.$$

Hence (3.49) follows. □

PROPOSITION III.1.30. *Let $L : \mathrm{PSH}(X, \theta) \to [-\infty, \infty)$ be a convex, increasing function satisfying $L(\varphi + c) = L(\varphi) + c \operatorname{vol} \alpha$ for any $c \in \mathbb{R}$, $\varphi \in \mathrm{PSH}(X, \theta)$.*

(1) Let $\mathcal{K} \subseteq \mathrm{PSH}(X, \theta)$ be a compact convex set. Assume that L is finite on $\mathcal{K}$, then L is bounded on $\mathcal{K}$.

(2) If L is finite on $\mathcal{E}^1(X, \theta)$, then

$$\sup_{\mathcal{E}^1_C} |L| = \mathcal{O}(C^{1/2})\,, \quad C \to \infty\,. \tag{3.50}$$

PROOF. (1) An upper bound of L is immediate: there is a constant $C_1 > 0$ such that

$$\sup_X (\varphi - V_\theta) \leq C_1\,, \quad \varphi \in \mathcal{K}\,.$$

So

$$L(\varphi) \leq L(V_\theta) + C_1 < \infty\,.$$

For the lower bound, assume that there ia a sequence $\varphi_j \in \mathcal{K}$ so that

$$L(\varphi_j) \leq -2^j.$$

Let

$$\varphi := \sum_{j=1}^\infty 2^{-j} \varphi_j\,.$$

By assumption, $\varphi \in \mathcal{K}$. Note that for each $N \in \mathbb{N}$,

$$\varphi \leq \sum_{j=1}^{N} 2^{-j} \varphi_j + 2^{-N} V_\theta .$$

So

$$L(\varphi) \leq \sum_{j=1}^{N} 2^{-j} L(\varphi_j) + 2^{-N} L(V_\theta) \leq -N + 2^{-N} L_{V_\theta} .$$

Let $N \to \infty$, we find a contradiction.

(2) Assume that

$$\sup_{\mathcal{E}^1_C} |L| = \mathcal{O}(C^{1/2})$$

fails, we may take $\varphi_j \in \mathcal{E}^1(X,\theta)$, so that

$$\sup_X \varphi_j = 0 \,, \quad t_j := |E(\varphi_j)|^{-1/2} \to 0 \,, \quad t_j L(\varphi_j) \to -\infty \,.$$

On the other hand, we claim that there is $C_0 > 0$, so that

$$E(t\varphi_j + (1-t)V_\theta) \geq -C_0. \tag{3.51}$$

In particular,

$$t_j \varphi_j + (1-t_j) V_\theta \in \mathcal{E}^1_{C_0} \,.$$

So

$$t_j L(\varphi_j) + (1-t_j) L(V_\theta) \geq \inf_{\mathcal{E}^1_{C_0}} L.$$

This is a contradiction.

It remains to prove the claim. By (1) and Proposition III.1.22, there is a constant $C > 0$ such that for all $\varphi \in \mathrm{PSH}(X,\theta)$, $\sup_X \varphi = 0$, we have

$$\int_X (\varphi - V_\theta)\, \theta_{V_\theta}^n \geq -C \,.$$

In particular, this applies to $t\varphi_j + (1-t)V_\theta$ and φ_j, hence we find

$$\begin{aligned}
E(t\varphi + (1-t)V_\theta) =& - J(t\varphi + (1-t)V_\theta) + \mathcal{O}(1) \\
\geq& - (n+1) I(t\varphi + (1-t)V_\theta) + \mathcal{O}(1) \\
\geq& - (n+1) n t^2 I(\varphi_j) + \mathcal{O}(1) \\
\geq& - (n+1) n t^2 J(\varphi_j) + \mathcal{O}(1) \\
\geq& (n+1) n t^2 E(\varphi_j) + \mathcal{O}(1) \\
\geq& \mathcal{O}(1) \,,
\end{aligned}$$

where the first and the third inequalities follow from Proposition III.1.28, the second follows from Proposition III.1.29. □

Corollary III.1.31. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$, $\mu(X) = \operatorname{vol} \alpha$. Assume that L_μ is finite on $\mathcal{E}^1(X,\theta)$. Then there are positive constants $\varepsilon > 0$, $A > 0$ such that*

$$F_\mu \leq -\varepsilon J + A \,. \tag{3.52}$$

Proof. By Proposition III.1.25 and Proposition III.1.27, it suffices to prove (3.52) for potentials $\varphi\in\mathcal{E}^1(X,\theta)$ with $\sup_X\varphi=0$.

In this case, by Proposition III.1.21 and Proposition III.1.22,

$$L_{\theta^n_{V_\theta}}=\mathcal{O}(1)\,.$$

Hence (3.52) follows from

$$\inf_{\mathcal{E}^1_C} L_\mu\geq -(1-\varepsilon)C+\mathcal{O}(1)\,,\quad C\to\infty\,.$$

This follows from Proposition III.1.30. □

Proposition III.1.32. *Let $\varphi\in\mathcal{E}^1(X,\theta)$. Then for any $C\in\mathbb{R}$, the functional $L_{\theta^n_\varphi}$ is continuous on $\varphi\in\mathcal{E}^1(X,\theta)$ with respect to the subspace topology inherited from* $\mathrm{PSH}(X,\theta)$.

Proof. TO be add later □

Corollary III.1.33 (Global uniform Skoda theorem). *Let $\varphi\in\mathcal{E}^1(X,\theta)$. Then for any $C\in\mathbb{R}$,*

$$\sup_{\mathcal{E}^1_C} L_\mu<\infty\,.$$

Proof. This follows from Corollary III.1.14 and Proposition III.1.32. □

Definition III.1.7. Let $\mu\in\mathring{\mathcal{M}}_+(X)$, $\lambda>0$. Define the λ-*free energy* functional $F_{\mu,\lambda}:\mathcal{E}^1(X,\theta)\to\mathbb{R}$ by

$$F_{\mu,\lambda}(\varphi):=E(\varphi)-\int_X e^{\lambda\varphi}\,\mathrm{d}\mu\,.$$

Remark III.1.3. $F_{\mu,\lambda}$ is finite on $\mathcal{E}^1(X,\theta)$. In fact,

$$0\leq\int_X e^{\lambda\varphi}\,\mathrm{d}\mu\leq e^{\lambda\sup_X\varphi}\mu(X)\,.$$

Proposition III.1.34. *Let $\mu\in\mathring{\mathcal{M}}_+(X)$, $\lambda>0$. Let $\varphi\in\mathcal{E}^1(X,\theta)$. Assume that*

$$F_{\mu,\lambda}(\varphi)=\sup_{\mathcal{E}^1(X,\theta)}F_{\mu,\lambda}\,.\tag{3.53}$$

Then

$$\theta_\varphi^n=e^{\lambda\varphi}\,\mathrm{d}\mu\,.\tag{3.54}$$

Proof. Let $v\in C^0(X)$. For $t\in\mathbb{R}$, let

$$g(t):=E(\mathrm{P}(\varphi+tv))-\int_X e^{\lambda(\varphi+tv)}\,\mathrm{d}\mu\,.$$

Observe that

$$g(t)\leq E(\mathrm{P}(\varphi+tv))-\int_X e^{\lambda\mathrm{P}(\varphi+tv)}\,\mathrm{d}\mu=F_{\mu,\lambda}(\mathrm{P}(\varphi+tv))\leq F_{\mu,\lambda}(\varphi)=g(0)\,.$$

By Theorem III.1.18, $g(t)$ is differentiable at $t=0$, hence

$$\int_X v\,\theta_\varphi^n=\int_X ve^{\lambda\varphi}\,\mathrm{d}\mu\,.$$

Hence (3.54) follows. □

III.2. Monge–Ampère capacity

III.2.1. Extremal functions.

DEFINITION III.2.8. Let $E \subseteq X$ be a Borel set.

(1) The *global extremal function* of E is defined as

$$V^*_{E,\theta} := \sup{}^*\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi|_E \leq 0\}\,. \tag{3.55}$$

(2) The *extremal function* of E is defined as

$$h^*_{E,\theta} := \sup{}^*\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0\,, \varphi|_E \leq -1\}\,. \tag{3.56}$$

PROPOSITION III.2.35. *Let $E \subseteq X$ be a non-pluripolar Borel set. Then*

(1) $V^*_{E,\theta} \in \mathcal{E}^\infty(X,\theta)$.

(2) $\theta^n_{V^*_{E,\theta}}$ is supported on $\overline{E}$.

PROOF. (1) To see that $V^*_{E,\theta} \in \mathrm{PSH}(X,\theta)$, it suffices to prove that

$$\sup_X V^*_{E,\theta} \leq \infty\,.$$

If this is not true, by Choquet's lemma, we can take an increasing sequence $\varphi_j \in \mathrm{PSH}(X,\theta)$ $(j \in \mathbb{N})$, such that

$$\varphi_j|_E \leq 0\,, \quad V^*_{E,\theta} = \sup_j{}^* \varphi_j\,, \quad \sup_X \varphi_j \geq 2^j\,.$$

Let

$$\psi_j := \varphi_j - \sup_X \varphi_j\,, \quad \psi = \sum_{j=1}^\infty 2^{-j}\psi_j\,.$$

Then by Theorem I.1.2, $\psi \in \mathrm{PSH}(X,\theta)$. Moreover,

$$K \subseteq \{\psi = -\infty\}\,.$$

Hence K is pluripolar, this is a contradiction.

Now by definition,

$$V_\theta \leq V^*_{E,\theta}\,. \tag{3.57}$$

Hence $V_{E,\theta}$ has minimal singularity.

(2) It suffices to prove that for any $x \in \mathrm{Amp}(\alpha) \cap (X - \overline{E})$, there is a neighbourhood B of x, such that

$$\theta^n_{V^*_{E,\theta}}|_B = 0\,.$$

By Choquet's lemma, take an increasing sequence $\varphi_j \in \mathrm{PSH}(X,\theta)$ $(j \in \mathbb{N})$, such that

$$V^*_{E,\theta} = \sup_j{}^* \varphi_j\,, \quad \varphi_j|_E \leq 0\,.$$

We take the neighbourhood $B \subseteq \mathrm{Amp}(\alpha) \cap (X - \overline{E})$ of x as a small enough ball in a coordinate neighbourhood. We solve the following Dirichlet problems of $\psi_j \in \mathrm{PSH}(\overline{B},\theta)$:

$$\begin{cases} \theta^n_{\psi_j} = 0\,, & \text{in } B\,, \\ \psi_j = \varphi_j\,, & \text{on } \partial B\,. \end{cases}$$

(add ref)

Then we get an increasing sequence $\psi_j \geq \varphi_j$ converging to $V^*_{K,\theta}$ a.e.. So

$$\theta^n_{V^*_{K,\theta}}|_B = 0\,.$$

□

PROPOSITION III.2.36. *Let $E \subseteq X$ be a Borel set. Then $h^*_{E,\theta} \in \mathrm{PSH}(X,\theta)$ and $\theta^n_{h^*_{E,\theta}}$ is supported on $\{h^*_{E,\theta} = 0\} \cup \overline{E}$. Moreover,*

$$V_\theta - 1 \leq h^*_{E,\theta} \leq V_\theta \,.$$

PROOF. The proof is similar to that of Proposition III.2.35 and we leave it to the readers. □

III.2.2. Monge–Ampère capacity.

DEFINITION III.2.9. Let $E \subseteq X$ be a Borel set. We define the *Monge–Ampère capacity* of E as

$$\mathrm{Cap}_\theta(E) := \sup\left\{\int_E \theta^n_\varphi : \varphi \in \mathrm{PSH}(X,\theta)\,, V_\theta - 1 \leq \varphi \leq V_\theta\right\}\,. \tag{3.58}$$

LEMMA III.2.37. *Let $E \subseteq X$ be a Borel subset. Then*

$$\mathrm{Cap}_\theta(E) = \sup\left\{\mathrm{Cap}_\theta(K) : K \subseteq E\,, K \textit{ is compact}\right\}\,. \tag{3.59}$$

PROOF. By definition,

$$\mathrm{Cap}_\theta(E) \geq \mathrm{Cap}_\theta(K)$$

for any compact subset $K \subseteq E$. Conversely, for any $\varepsilon > 0$, we take $\varphi \in \mathrm{PSH}(X,\theta)$, $V_\theta - 1 \leq \varphi \leq V_\theta$, such that

$$\int_E \theta^n_\varphi \geq \mathrm{Cap}_\theta(E) - \varepsilon\,.$$

Take a compact subset $K \subseteq E$, such that

$$\int_K \theta^n_\varphi \geq \int_E \theta^n_\varphi - \varepsilon \geq \mathrm{Cap}_\theta(E) - 2\varepsilon\,.$$

Take sup with respect to φ, then $\mathrm{Cap}_\theta(K) \geq \mathrm{Cap}_\theta(E) - 2\varepsilon$. Let $\varepsilon \to 0+$ to conclude. □

PROPOSITION III.2.38. *Let $E \subseteq X$ be a Borel set. Then*

$$\mathrm{Cap}_\theta(E) = \int_E \theta^n_{h^*_{E,\theta}} \leq \int_X (V_\theta - h^*_{E,\theta})\,\theta^n_{h^*_{E,\theta}}\,. \tag{3.60}$$

Moreover. when E is either compact or open, we have

$$\mathrm{Cap}_\theta(E) = \int_X (V_\theta - h^*_{E,\theta})\,\theta^n_{h^*_{E,\theta}}\,. \tag{3.61}$$

PROOF. Since $h^*_{E,\theta}$ is a candidate in the sup of (3.58), we have

$$\mathrm{Cap}_\theta(E) \geq \int_E \theta^n_{h^*_{E,\theta}}\,. \tag{3.62}$$

Let $A = \{h_{E,\theta} < h^*_{E,\theta}\}$. Then A is a pluripolar set by Bedford–Taylor's theorem. By Proposition III.2.36,

$$h^*_{E,\theta}|_{E-A} = V_\theta - 1\,. \tag{3.63}$$

Let $\varphi \in \mathrm{PSH}(X,\theta)$, $V_\theta - 1 \leq \varphi \leq V_\theta$. For any $\varepsilon \in (0,1)$, let

$$\varphi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon V_\theta\,.$$

Then by (3.63),

$$E - A \subseteq \{h^*_{E,\theta} < \varphi_\varepsilon\}\,.$$

Hence by Theorem II.4.28 and Proposition III.2.36,

$$(1-\varepsilon)^n \int_E \theta_\varphi^n \leq \int_{\{h^*_{E,\theta}<\varphi_\varepsilon\}} \theta^n_{\varphi_\varepsilon} \leq \int_{\{h^*_{E,\theta}<\varphi_\varepsilon\}} \theta^n_{h^*_{E,\theta}} = \int_E \theta^n_{h^*_{E,\theta}}\,.$$

Let $\varepsilon \to 0+$, we find

$$\int_X \theta_\varphi^n \leq \int_E \theta^n_{h^*_{E,\theta}}\,.$$

Take sup with respect to φ, we find

$$\mathrm{Cap}_\theta(E) \leq \int_E \theta^n_{h^*_{E,\theta}}\,.$$

Hence together with (3.62), we conclude the equality part of (3.60). The inequality follows immediately from Proposition III.2.36 and (3.63).

When E is compact, (3.61) follows immediately from the first part, Proposition III.2.36 and (3.63).

Now assume that E is an open set. Let K_j ($j \in \mathbb{N}$) be an increasing sequences of compact sets with

$$E = \bigcup_{j=1}^\infty K_j$$

and such that $K_j \subseteq \mathring{K}_{j+1}$.

Then we claim that $h^*_{K_j,\theta}$ decreases to $h^*_{E,\theta}$. It is immediate that $h^*_{K_j,\theta}$ is decreasing and is always greater than $h^*_{E,\theta}$. Hence it suffices to show that $h^*_{K_j,\theta}$ decreases to $h^*_{E,\theta}$ a.e.. Again, by Bedford–Taylor's theorem, it suffices to show that

$$\sup{}^*\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0\,, \varphi|_{K_j} \leq -1\}$$

decreases a.e. to

$$\sup{}^*\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0\,, \varphi|_{E} \leq -1\}\,,$$

which is obvious.

We already know that

$$\mathrm{Cap}_\theta(K_j) = \int_X (V_\theta - h^*_{K_j,\theta})\,\theta^n_{h^*_{K_j,\theta}}\,.$$

By Lemma III.2.37 and our assumption on K_j, we know that

$$\mathrm{Cap}_\theta(E) = \lim_{j\to\infty} \mathrm{Cap}_\theta(K_j)\,.$$

On the other hand, by Theorem II.2.12,

$$\lim_{j\to\infty} \int_X (V_\theta - h^*_{K_j,\theta})\,\theta^n_{h^*_{K_j,\theta}} = \int_X (V_\theta - h^*_{E,\theta})\,\theta^n_{h^*_{E,\theta}}\,.$$

Hence we conclude (3.61) in the case where E is open. □

Theorem III.2.39. Cap_θ *is a regular Choquet capacity, namely an increasing, subadditive map from the Borel algebra to* $[0,\infty)$*, such that*

(1) $\mathrm{Cap}_\theta(\emptyset) = 0$.

(2) Let $E_j \subseteq X$ ($j \in \mathbb{N}$) be an increasing sequence of Borel sets. Let $E = \cup_j E_j$, then

$$\operatorname{Cap}_\theta(E) = \lim_{j\to\infty} \operatorname{Cap}_\theta(E_j)\,. \tag{3.64}$$

(3) Let $K_j \subseteq X$ be a decreasing sequence of compact sets. Let $K = \cap_j K_j$, then

$$\operatorname{Cap}_\theta(K) = \lim_{j\to\infty} \operatorname{Cap}_\theta(K_j)\,. \tag{3.65}$$

In particular, for each compact $K \subseteq X$, we have

$$\operatorname{Cap}_\theta(K) = \inf\left\{\operatorname{Cap}_\theta(U) : K \subseteq U\,, U \subseteq X \text{ is open}\right\}\,. \tag{3.66}$$

Proof. By definition, $\operatorname{Cap}_\theta$ is increasing, subadditive and (1) is satisfied. Now we verify (2). We only have to prove that

$$\operatorname{Cap}_\theta(E) \leq \lim_{j\to\infty} \operatorname{Cap}_\theta(E_j)\,.$$

In fact, for any $\varepsilon > 0$, take $\varphi \in \operatorname{PSH}(X,\theta)$, $V_\theta - 1 \leq \varphi \leq V_\theta$, such that

$$\operatorname{Cap}_\theta(E) \leq \int_E \theta_\varphi^n + \varepsilon\,.$$

By dominated convergence theorem, we can take j_0 large enough, so that

$$\int_E \theta_\varphi^n \leq \int_{E_{j_0}} \theta_\varphi^n + \varepsilon\,.$$

Then for any $j \geq j_0$,

$$\operatorname{Cap}_\theta(E) \leq \operatorname{Cap}_\theta(E_j) + 2\varepsilon\,.$$

Let $j \to \infty$ then let $\varepsilon \to 0+$, we conclude (3.64).

Let us verify (3). In fact, it is not hard to verify that $h^*_{K_j,\theta}$ increases to $h^*_{K,\theta}$ a.e.. (add details) By Proposition III.2.38,

$$\operatorname{Cap}_\theta(K_j) = \int_X (V_\theta - h^*_{K_j,\theta})\,\theta^n_{h^*_{K_j,\theta}}\,.$$

Let $j \to \infty$ and apply Theorem II.2.12, we find

$$\lim_{j\to\infty} \operatorname{Cap}_\theta(K_j) = \int_X (V_\theta - h^*_{K,\theta})\,\theta^n_{h^*_{K,\theta}}\,.$$

Again by Proposition III.2.38, we conclude (3.65).

Finally, (3.66) holds for general regular capacity. □

III.2.3. Capacity and finite energy class.

Proposition III.2.40. *Let $\varphi \in \operatorname{PSH}(X,\theta)$. Assume that*

$$\int_0^\infty t^n \operatorname{Cap}_\theta\{\varphi < V_\theta - t\}\,\mathrm{d}t < \infty\,. \tag{3.67}$$

Then $\varphi \in \mathcal{E}^1(X,\theta)$.

Proof. For each $C \geq 1$, let

$$\varphi^C := \varphi \vee (V_\theta - C)$$

be the canonical approximations of φ. Then

$$\psi_C := C^{-1}\varphi^C + (1 - C^{-1})V_\theta$$

is a candidate for the sup term in (3.58), hence for any Borel set $E \subseteq X$,

$$C^{-n}\theta^n_{\varphi^C}(E) \leq \theta^n_{\psi_C}(E) \leq \mathrm{Cap}_\theta(E)\,.$$

Hence by (3.67),

$$\int_0^\infty \mathrm{d}t \int_{\{\varphi < V_\theta - t\}} \theta^n_{\varphi^C} < \infty\,.$$

It follows from Corollary III.1.17 that $\varphi \in \mathcal{E}^1(X,\theta)$. $\square$

Proposition III.2.41. *For each $C \geq 0$,*

$$\sup_{\varphi \in \mathcal{E}^1_C(X,\theta)} \int_0^\infty t\,\mathrm{Cap}_\theta\{\varphi < V_\theta - t\}\,\mathrm{d}t < \infty\,.$$

Proof. Fix $\varphi \in \mathcal{E}^1_C(X,\theta)$.

Pick $\psi \in \mathrm{PSH}(X,\theta)$, so that $V_\theta - 1 \leq \psi \leq V_\theta$. For $t \geq 1$,

$$\{\varphi < V_\theta - 2t\} \subseteq \{t^{-1}\varphi + (1-t^{-1})V_\theta < \psi - 1\} \subseteq \{\varphi < V_\theta - t\}\,.$$

So by Theorem II.4.28, we have

$$\begin{aligned}\int_{\{\varphi < V_\theta - 2t\}} \theta^n_\psi \leq & \int_{\{\varphi < V_\theta - t\}} \theta^n_{t^{-1}\varphi + (1-t^{-1})V_\theta} \\ \leq & (1-t^{-1}) \int_{\{\varphi < V_\theta - t\}} \theta^n_{V_\theta} + C_1 t^{-1} \sum_{j=1}^n \int_{\{\varphi < V_\theta - t\}} \theta^j_\varphi \wedge \theta^{n-j}_{V_\theta}\,,\end{aligned}$$

where C_1 is a constant independent of the choice of φ. Hence

$$t\,\mathrm{Cap}_\theta(\{\varphi < V_\theta - 2t\}) \leq t \int_{\{\varphi < V_\theta - t\}} \theta^n_{V_\theta} + C_1 \sum_{j=1}^n \int_{\{\varphi < V_\theta - t\}} \theta^j_\varphi \wedge \theta^{n-j}_{V_\theta}\,.$$

So

$$\int_0^\infty t\,\mathrm{Cap}_\theta\{\varphi < V_\theta - t\}\,\mathrm{d}t \leq C_1 \sum_{j=1}^\infty (V_\theta - \varphi)\,\theta^j_\varphi \wedge \theta^{n-j}_{V_\theta} + \frac{1}{2}\int_X (V_\theta - \varphi)^2\,\theta^n_{V_\theta}\,.$$

Since $E(\varphi) \geq -C$, we know that there is a constant $C_2 > 0$ independent of φ, such that

$$\int_0^\infty t\,\mathrm{Cap}_\theta\{\varphi < V_\theta - t\}\,\mathrm{d}t \leq C_2 + \frac{1}{2}\int_X (V_\theta - \varphi)^2\,\theta^n_{V_\theta}\,.$$

By Theorem I.5.12, $\theta^n_{V_\theta}$ has bounded density. So it follows from uniform version of Skoda's integrability theorem that RHS is uniformly bounded. $\square$

Proposition III.2.42. *Let $\varphi \in \mathcal{E}(X,\theta)$;. Let $t > 0$, $\delta \in (0,1)$, we have*

$$\mathrm{Cap}_\theta\{\varphi < V_\theta - t - \delta\} \leq \frac{1}{\delta^n} \int_{\{\varphi < V_\theta - t\}} \theta^n_\varphi\,. \tag{3.68}$$

Proof. Take $\psi \in \mathrm{PSH}(X,\theta)$, $V_\theta - 1 \leq \psi \leq V_\theta$, then

$$\{\varphi < V_\theta - t - \delta\} \subset \{\varphi < \delta\psi + (1-\delta)V_\theta - t\} \subset \{\varphi < V_\theta - t\}.$$

So by Theorem II.4.28,

$$\begin{aligned}\delta^n \int_{\{\varphi<V_\theta-t-\delta\}} \theta_\psi^n &\leq \int_{\{\varphi<\delta\psi+(1-\delta)V_\theta-t\}} \theta^n_{\delta\psi+(1-\delta)V_\theta} \\ &\leq \int_{\{\varphi<\delta\psi+(1-\delta)V_\theta-t\}} \theta^n_\varphi \\ &\leq \int_{\{\varphi<V_\theta-t\}} \theta^n_\varphi\,.\end{aligned}$$

Take sup with respect to ψ, we conclude (3.68). □

PROPOSITION III.2.43. *Let $\mu\in\mathcal{M}_+(X)$ be a non-pluripolar measure. Then there exists a probability measure ν, such that*

$$\mu \ll \nu, \quad \nu\leq \mathrm{Cap}\,.$$

PROOF. Consider the following set

$$L:=\{\nu\in\mathcal{P}(X):\nu\leq\mathrm{Cap}\}.$$

We shall prove that L is closed.

By regularity of both $\nu\in\mathcal{P}(X)$ and Cap, it suffices to prove that if $\nu_n\in L$, $\nu_n\rightharpoonup\nu\in\mathcal{P}(X)$, the for any open set $U\subset X$,

$$\nu(U)\leq\mathrm{Cap}(U).$$

This is the so called Portmanteau theorem.

Now L is closed and is obviously convex, we could apply Theorem ??? so decompose

$$\mu=f\nu+\tau,$$

where $f\in L^1(\nu)$, $\tau\perp L$. Since μ is non-pluripolar, we conclude that $\tau=0$. □

III.2.4. Alexander–Taylor capacity.

DEFINITION III.2.10. Let $E\subseteq X$ be a Borel set. The *Alexander–Taylor capacity* of E is defined as $e^{-M_\theta(E)}$, where

$$M_\theta(E):=\sup_X V^*_{E,\theta}\,. \tag{3.69}$$

PROPOSITION III.2.44. *There is a constant $A>0$, such that for any Borel set $E\subseteq X$, when $M_\theta(E)>1$,*

$$\left(\frac{\mathrm{vol}\,\alpha}{\mathrm{Cap}_\theta(E)}\right)^{1/n}\leq M_\theta(E)\leq\frac{A}{\mathrm{Cap}_\theta(E)}\,. \tag{3.70}$$

If $M_\theta(E)\leq 1$, then

$$\mathrm{Cap}_\theta(E)=\mathrm{vol}\,\alpha\,. \tag{3.71}$$

PROOF. By Lemma III.2.37, it suffices to consider the case where E is compact. Let $M:=M_\theta(E)$.

Assume that $M\leq 1$. Then $V^*_{E,\theta}-1$ is a candidate in defining $\mathrm{Cap}_\theta(E)$ in (3.58). So

$$\mathrm{Cap}_\theta(E)\geq\int_E\theta^n_{V^*_{E,\theta}}=\int_X\theta^n_{V^*_{E,\theta}}=\mathrm{vol}\,\alpha\,,$$

where the second equality follows from Proposition III.2.35.

Assume that $M > 1$. Then

$$V_\theta - 1 \leq \frac{1}{M} V^*_{E,\theta} + \left(1 - \frac{1}{M}\right) V_\theta - 1 \leq V_\theta \,,$$

So by Proposition III.2.35,

$$\mathrm{Cap}_\theta(E) \geq \int_E \theta^n_{\frac{1}{M}V^*_{E,\theta} + (1-\frac{1}{M})V_\theta} \geq \frac{1}{M^n} \int_E \theta^n_{V^*_{E,\theta}} = \frac{1}{M^n} \int_X \theta^n_{V^*_{E,\theta}} = \frac{1}{M^n} \operatorname{vol} \alpha \,.$$

The left-hand part of (3.70) follows.

Now we prove the other part. By Proposition III.2.36,

$$\frac{1}{M} V^*_{E,\theta} + \left(1 - \frac{1}{M}\right) V_\theta - 1 \leq V_\theta$$

and

$$\left(\frac{1}{M} V^*_{E,\theta} + \left(1 - \frac{1}{M}\right) V_\theta - 1\right)|_E \leq -1$$

on $E - A$, where $A \subseteq X$ is a pluripolar set. So

$$h^*_E \geq \frac{1}{M} V^*_{E,\theta} + \left(1 - \frac{1}{M}\right) V_\theta - 1.$$

Hence by Proposition III.2.38,

$$\mathrm{Cap}_\theta(E) = \int_X (V_\theta - h^*_E)\, \theta^n_{h^*_E} \leq \frac{1}{M} \int_X (V_\theta - V^*_{E,\theta} + M)\, \theta^n_{h^*_E} \,.$$

Hence by Lemma III.1.20 and Theorem I.5.12, there is a constant $A > 0$, such that

$$M \,\mathrm{Cap}_\theta(E) \leq \int_X (V_\theta - V^*_{E,\theta} + M)\, \theta^n_{V_\theta} + A \leq \int_X (V_\theta - V^*_{E,\theta} + M)\omega^n + A \,,$$

Now

$$\sup_X \left(-V_\theta + V^*_{E,\theta} - M\right) \leq 0 \,,$$

So

$$M \,\mathrm{Cap}_\theta(E) \leq A \,.$$

□

PROPOSITION III.2.45. *Let $E \subseteq X$ be a Borel set. Then the following are equivalent:*

(1) E is a pluripolar set.
(2) $\mathrm{Cap}_\theta(E) = 0$.
(3) $M_\theta(E) = \infty$.

PROOF. (1) implies (2). This follows immediately from definition.

(2) implies (1). Assume that $\mathrm{Cap}_\theta(E) = 0$.

Step 1. Assume that $\theta = \omega$ is a Kähler form. This result is a classical theorem in Bedford–Taylor theory. To be added later.

Step 2. In the general case, Fix a Kähler form $\omega \geq \theta$ on X. If E is non-pluripolar, by Step 1, $\mathrm{Cap}_\omega(E) > 0$. By Lemma III.2.37, there is a compact set $K \subseteq E$, such that $\mathrm{Cap}_\omega(K) > 0$. In particular, K is non-pluripolar.

Since $\theta \leq \omega$, we have

$$\mathrm{PSH}(X, \theta) \subseteq \mathrm{PSH}(X, \omega) \,.$$

So there is a constant $C>1$ such that

$$V_\theta \leq V^*_{K,\theta} \leq V^*_{K,\omega} \leq V_\theta + C\,.$$

Then

$$\psi := \left(1-\frac{1}{C}\right) V_\theta + \frac{1}{C} V^*_{K,\theta} - 1 \in \mathrm{PSH}(X,\theta)$$

is a candidate in the sup term of (3.58). Hence by Proposition III.2.35,

$$\mathrm{Cap}_\theta(E) \geq \mathrm{Cap}_\theta(K) \geq \int_K \theta^n_\psi \geq \frac{1}{C^n}\int_K \theta^n_{V^*_{K,\theta}} = \frac{1}{C^n}\int_X \theta^n_{V^*_{K,\theta}} = \frac{1}{C^n}\operatorname{vol}\alpha > 0.$$

This is a contradiction.

(1) implies (3). This is obvious.

(3) implies (2). This follows from Proposition III.2.44. □

THEOREM III.2.46. *Let α_1,α_2 be two big classes with smooth representatives θ_1,θ_2. Then there is a constant $C(\theta_1,\theta_2)>0$, such that*

$$C^{-1}\,\mathrm{Cap}^n_{\theta_1} \leq \mathrm{Cap}_{\theta_2} \leq C\,\mathrm{Cap}^{1/n}_{\theta_1}\,. \tag{3.72}$$

PROOF. Take a Kähler form $\omega \geq \theta_1$. Then

$$\mathrm{PSH}(X,\theta_1) \subseteq \mathrm{PSH}(X,\omega)\,.$$

By Lemma III.2.37, it suffices to prove (3.72) for a compact set $K \subseteq X$. Take $\psi \in \mathrm{PSH}(X,\theta_2)$ with analytic singularity with

$$X - \mathrm{Amp}\,\alpha_2 = \{\psi = -\infty\}$$

and such that

$$\theta_{2,\psi} \geq \varepsilon\omega\,, \quad \sup_X \psi = 0$$

for some $\varepsilon>0$, the existence is guaranteed by Boucksom's theorem (ADD REF).

Let $U := \{\psi > -1\}$. Then U is non-empty and open. Define

$$\varphi = \psi + \varepsilon V^*_{\theta_1,K} \in \mathrm{PSH}(X,\theta_2).$$

By Proposition III.2.35, $\varphi \leq 0$ on $E-A$, where $A \subseteq X$ is a pluripolar set. So by (3.55),

$$\varphi \leq V^*_{\theta_2,E}\,.$$

Taking sup of this expression on U, we find

$$M_{\theta_2}(E) \geq \sup_U \varphi \geq \varepsilon \sup_U V^*_{\theta_1,E} - 1\,.$$

On the other hand, $V^*_{\theta_1,E} - \sup_U V^*_{\theta_1,E} \leq 0$ on U, so

$$V^*_{\theta_1,E} - \sup_U V^*_{\theta_1,E} \leq V^*_{\theta_1,U}\,.$$

So

$$M_{\theta_2}(E) \geq \varepsilon(M_{\theta_1}(E) - M_{\theta_1,U}) - 1\,.$$

It follows from Proposition III.2.44 that

$$\mathrm{Cap}_{\theta_2}(E) \leq C\,\mathrm{Cap}_{\theta_1}(E)^{1/n}.$$

This proves one part of (3.72), the other follows by symmetry. □

PROPOSITION III.2.47. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Assume that one of the following condition holds:*

(1) μ is absolutely continuous with bounded density.

(2) $\mu = \theta_\psi^n$ *for some* $\psi \in \mathcal{E}^1(X,\theta)$.

Assume that L_μ *is finite on* $\mathrm{PSH}(X,\theta)$. *Let* $0 \leq f \in L^p(X,\mu)$ *(*$p>1$*). Then*

$$f\mu \leq C\,\mathrm{Cap}_\theta^2\,, \tag{3.73}$$

where

$$C = (p-1)^{-2n} C_1 \|f\|_{L^p(X,\mu)}\,,$$

where $C_1 = C_1(\theta,\mu) > 0$ *is a constant.*

Proof. By Lemma III.2.37, it suffices to prove that for each non-pluripolar compact set $K \subseteq X$,

$$\int_K f\,\mathrm{d}\mu \leq C\,\mathrm{Cap}_\theta(K)^2\,. \tag{3.74}$$

We claim that there is a constant $\nu = \nu(\theta,\mu), C = C(\theta,\mu) > 0$, independent of K, such that

$$\int_X \exp(-\nu^{-1} V^*_{K,\theta})\,\mu \leq C\exp(-\nu^{-1} M_\theta(K))\,.$$

In Case (1), we apply the uniform Skoda theorem. In Case (2), we apply Corollary III.1.33, just notice that $V^*_{K,\theta} \geq V_{K,\theta}$, hence $E(V^*_{K,\theta}) \geq 0$ by Proposition III.1.13.

It follows from Proposition III.2.35 and Bedford–Taylor's theorem that $V^*_{K,\theta} = 0$ on K up to a pluripolar set, so

$$\mu(K) \leq C\exp(-\nu^{-1} M_\theta(K))\,.$$

So by Hölder's inequality,

$$\int_K f\,\mathrm{d}\mu \leq \exp\left(-\frac{p-1}{\nu p} M_\theta(K)\right)\,.$$

We may assume that $M_\theta(K) \geq 1$. Otherwise the proof of Proposition III.2.44 implies that $\mathrm{Cap}_\theta(K) = \mathrm{vol}\,\alpha$. The result is trivial. By Proposition III.2.44, (3.74) follows. □

III.2.5. Comparison with Bedford–Taylor capacity. Let $(U_j)_{j=1,\dots,N}$ be a finite open covering of X by strictly pseudoconvex domains with smooth boundaries contained in a coordinate chart. Let ρ_j be a smooth strictly psh function defined in a neighbourhood of $\overline{U}_j$ with $U_j = \{\rho < 0\}$. Fix $\delta > 0$. Let

$$U_j^\delta = \{\rho < -\delta\}\,.$$

Let χ_j be a partition of unity subordinate to U_j. For each Borel subset $E \subset X$, we define

$$\mathrm{Cap}(E) = \sum_{j=1}^N \mathrm{Cap}(E \cap U_j^\delta, U_j)\,.$$

By [GZ17] Proposition 4.18, for two different choices of δ and U_j, the resulting Cap will bound each other by a constant multiple.

Theorem III.2.48. *Let* ω *be a Kähler form on* X. *There is a constant* $C \geq 1$ *such that*

$$C^{-1}\,\mathrm{Cap}(E) \leq \mathrm{Cap}_\omega(E) \leq C\,\mathrm{Cap}(E)\,.$$

Proof. It suffices to prove that for any $j = 1, \dots, N$ and for each Borel set $E \subseteq U_j^\delta$,

$$C^{-1}\operatorname{Cap}(E,U) \leq \operatorname{Cap}_\omega(E) \leq C\operatorname{Cap}(E,U)\,.$$

Since we work in a fixed U_j, we omit j from the subindex. After passing to a finer covering, we may assume that $\omega = \mathrm{dd^c}\psi$ for a psh function ψ defined in a neighbourhood of $\bar{U}$. Let $C_1 > 1$ be a constant such that

$$-C_1 \leq \psi|_{\bar{U}} \leq C_1\,.$$

Now take $\varphi \in \operatorname{PSH}(X,\omega)$, $-1 \leq \varphi \leq 0$. Then

$$\tilde{\varphi} := (2C_1)^{-1}(\varphi + \psi - C_1) \in \operatorname{PSH}(U)$$

and $-1 \leq \tilde{\psi} \leq 0$. So

$$\operatorname{Cap}_\omega(E) \leq (2C_1)^n \operatorname{Cap}(E,U)\,.$$

For the reverse inequality, let $\chi \in C^\infty(X) \cap \operatorname{PSH}(X,\omega)$ be a non-positive function that vanishes outside U and strictly negative on U. Fix $\varepsilon > 0$ so that $\chi < -\varepsilon$ on U^δ. Let $\eta \in \operatorname{PSH}(U)$, $-1 \leq \eta \leq 0$. Define

$$\varphi(x) = \begin{cases} \dfrac{\eta(x) + 1 - \psi(x) + C_1}{2 + 2C_1} - 1\,, & x \in U^\delta\,, \\ \left(\dfrac{\eta(x) + 1 - \psi(x) + C_1}{2 + 2C_1} - 1\right) \vee \left(\dfrac{2}{\varepsilon}\chi(x)\right), & x \in U - U^\delta\,, \\ 0\,, & x \in X - U\,. \end{cases}$$

One can verify that $\varphi \in \operatorname{PSH}(X, 2\varepsilon^{-1}\omega)$. Hence

$$\frac{1}{(2C_1+2)^n}\int_E (\mathrm{dd^c}\eta)^n \leq \int_E \left((2C_1+2)^{-1}\omega + \mathrm{dd^c}\varphi\right)^n \leq \int_E \left(2\varepsilon^{-1}\omega + \mathrm{dd^c}\varphi\right)^n \leq 2^n\varepsilon^{-n}\operatorname{Cap}_\omega(E)\,.$$

Take sup with respect to η, we find

$$\operatorname{Cap}(E,U) \leq 2^n(2C_1+2)^n\varepsilon^{-n}\operatorname{Cap}_\omega(E)\,.$$

□

III.3. Monge–Ampère equation I. Existence and regularity

Theorem III.3.49. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$ with $\mu(X) = \operatorname{vol}\alpha$. Then the followings are equivalent:*

(1) $\mu = \theta_\varphi^n$ for some $\varphi \in \mathcal{E}^1(X,\theta)$.
(2) $E^*(\mu) < \infty$.

Proof. (1) implies (2). By Proposition III.1.26,

$$E^*(\mu) = E(\varphi) - \int_X (\varphi - V_\theta)\,\theta_\varphi^n\,.$$

The RHS is finite by Proposition III.1.10.

(2) implies (1).

Step 1. We prove the theorem under the assumption that $\mu \leq A\operatorname{Cap}_\theta$ for some $A > 0$.

In this case, we claim that $L_\mu : \mathcal{E}^1(X,\theta) \to \mathbb{R}$ and L_μ is continuous.

In fact, by our assumption and Proposition III.2.41, for any $C > 0$,

$$\sup_{\varphi \in \mathcal{E}^1_C(X,\theta)} \int_0^\infty t\mu\{\varphi < V_\theta - t\}\,\mathrm{d}t < \infty\,.$$

This implies immediately that L_μ is finite on $\mathcal{E}^1_C(X,\theta)$. It also follows that $\mathcal{E}^1_C(X,\theta)$ is uniformly integrable in the sense of Theorem III.1.24 and hence L_μ is continuous on $\mathcal{E}^1_C(X,\theta)$.

Let $\varphi_j \in \mathcal{E}^1(X,\theta)$ ($j\in\mathbb{N}$) be a sequence such that

$$F_\mu(\varphi_j)\to E^*(\mu)\,,\quad \sup_X \varphi_j = 0\,.$$

We may assume that

$$\varphi_j \to \varphi \in \mathrm{PSH}(X,\theta)\,.$$

Since L_μ is continuous and since E is usc (Proposition III.1.13), we find $F_\mu(\varphi) = E^*(\mu)$. Hence $\mu = \theta_\varphi^n$ by Proposition III.1.26.

Step 2. Now we deal with the general case.

By Proposition III.2.43, we may write $\mu = f\nu$, where $\nu\in\mathring{\mathcal{M}}_+(X)$, $\nu\leq \mathrm{Cap}_\theta$, $f\in L^1(\nu)$.

For any $k\in\mathbb{Z}_{>0}$, set

$$\mu_k := c_k\min\{f,k\}\nu\,,$$

where $c_k>1$ is a normalization constant so that RHS has total mass $\operatorname{vol}\alpha$. Note that for k large, we may assume that $c_k\leq 2$, then

$$\mu_k \leq 2k\,\mathrm{Cap}_\theta\ .$$

By the Step 1, there exists $\varphi_k\in\mathcal{E}^1(X,\theta)$ such that

$$\mu_k = \theta^n_{\varphi_k}\,,\quad \sup_X\varphi_k = 0\,.$$

Hence L_{μ_k} is finite on $\mathcal{E}^1(X,\theta)$ by Proposition III.1.21. By Proposition III.1.30, there is a constant $A>0$, such that

$$\sup_{\mathcal{E}^1_C(X,\theta)} |L_{\mu_k}| \leq A + AC^{1/2}$$

for any $C>0$. Since $\mu_k\leq 2\mu$, we thus find

$$\sup_{\mathcal{E}^1_C(X,\theta)} |L_{\mu}| \leq 2A + 2AC^{1/2}\,.$$

Hence there is a constant $C>0$, such that

$$E^*(\mu_k)\leq C\,.$$

Now we claim that there is a constant $C>0$, such that

$$J(\mu_k)\leq C\,.$$

In fact, by Proposition III.1.26,

$$E^*(\mu_k) = E(\varphi_k) - \int_X (\varphi_k - V_\theta)\,\theta^n_{\varphi_k}\,.$$

It follows from Proposition III.1.10 that

$$nE^*(\mu_k)\geq J(\varphi_k)\,.$$

Now by Proposition III.1.27, after extracting a subsequence, we may assume that $\varphi_k\to\varphi$ in L^1-topology with $\varphi\in\mathcal{E}^1$.

It follows from Theorem II.6.45 and the mass condition that $\mu=\theta^n_\varphi$. $\square$

THEOREM III.3.50. *Let $\mu\in\mathring{\mathcal{M}}_+(X)$, $\mu(X) = \operatorname{vol}\alpha$. Then there exists $\varphi\in\mathcal{E}(X,\theta)$, such that $\mu=\theta^n_\varphi$.*

PROOF. By Proposition III.2.43, we may write $\mu = f\nu$, where $\nu \in \mathring{\mathcal{M}}_+(X)$, $\nu \leq \mathrm{Cap}_\theta$, $f \in L^1(\nu)$. For each $j > 0$, let c_j be the constant such that

$$\int_X c_j \min\{f, -j\}\, \nu = \mathrm{vol}\, \alpha \,.$$

For j large enough, we may assume that $1 \leq c_j \leq 2$. Note that $c_j \to 1$ as $j \to \infty$. By Theorem III.3.49 Step 2, we can take $\varphi_j \in \mathcal{E}^1(X, \theta)$ such that

$$\theta^n_{\varphi_j} = c_j \min\{f, -j\}\, \nu \,, \quad \sup_X \varphi_j = 0 \,.$$

By extracting a subsequence, we may assume that $\varphi_j \to \varphi \in \mathrm{PSH}(X, \theta)$. By Theorem II.6.45 and the mass condition, we conclude that $\mu = \theta^n_\varphi$. $\square$

LEMMA III.3.51. *Let $f : (0, \infty) \to (0, \infty)$ be a decreasing, right continuous function. Assume that there is a constant $C > 0$ such that*

$$f(t + \delta) \leq \frac{C}{\delta} f(t)^2 \,,$$

for any $t > 0$, $\delta \in (0, 1)$. Assume furthermore that

$$f(t_0) < \frac{1}{2C}$$

for some $t_0 > 0$. Then

$$f(t_0 + 4C) = 0 \,.$$

For a proof, see [EGZ09] Lemma 2.4, Remark 2.5.

THEOREM III.3.52 (Kołodziej estimate). *Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Let $a \in [0, 1)$, $A > 0$. Let $0 \leq f \in L^p(X, \mu)$ ($1 < p \leq \infty$). Let $\varphi \in \mathcal{E}(X, \theta)$. Assume that*

$$\theta^n_\varphi \leq f\mu + a\, \theta^n_{V_\theta} \,. \tag{3.75}$$

Assume one of the following conditions holds

(1) *$p = \infty$ and $\mu = \theta^n_\eta$ for some $\eta \in \mathcal{E}^\infty(X, \theta)$.*
(2) *μ is absolutely continuous with bounded density.*

Then there is a constant $C = C(\theta, \mu, a)$, such that

$$\varphi - \sup_X \varphi \geq V_\theta - C\|f\|_{L^p(X,\mu)} \,. \tag{3.76}$$

PROOF. We may assume that $\sup_X \varphi = 0$. For $t > 0$, let

$$g(t) := \left(\mathrm{Cap}_\theta\{\varphi < V_\theta - t\}\right)^{1/n} \,.$$

It suffices to prove that $g(t) = 0$ for $t \leq -C$ for some C as in the statement of the theorem. Since it will then follow from Proposition III.2.45 that (3.76) holds outside a pluripolar set. Hence it holds everywhere.

By (3.75) and Theorem II.4.28, for any $t > 0$,

$$\begin{aligned}\int_{\{\varphi < V_\theta - t\}} \theta^n_\varphi &\leq \int_{\{\varphi < V_\theta - t\}} f\, \mathrm{d}\mu + a \int_{\{\varphi < V_\theta - t\}} \theta^n_{V_\theta} \\ &\leq \int_{\{\varphi < V_\theta - t\}} f\, \mathrm{d}\mu + a \int_{\{\varphi < V_\theta - t\}} \theta^n_\varphi \,.\end{aligned}$$

So

$$\int_{\{\varphi < V_\theta - t\}} \theta^n_\varphi \leq \frac{1}{1-a} \int_{\{\varphi < V_\theta - t\}} f\, \mathrm{d}\mu \,.$$

By Proposition III.2.42,

$$g(t+\delta) \leq \delta^{-1} \int_{\{\varphi < V_\theta - t\}} \theta_\varphi^n \leq \frac{1}{\delta(1-a)} \int_{\{\varphi < V_\theta - t\}} f \,\mathrm{d}\mu \,. \tag{3.77}$$

By Proposition III.2.47, there is a constant $C_0(\theta, \mu, a) > 0$ such that

$$g(t+\delta) \leq \frac{C_0}{\delta} g(t)^2 \,, \quad \delta \in (0,1) \,.$$

For $t_0 > 0$,

$$\int_{\{\varphi < V_\theta - t_0\}} f \,\mathrm{d}\mu \leq \int_{\{\varphi < V_\theta - t_0\}} \frac{V_\theta - \varphi}{t_0} f \,\mathrm{d}\mu \leq \frac{1}{t_0} \left(\int_X f^p \,\mathrm{d}\mu \right)^{1/p} \left(\int_X (V_\theta - \varphi)^q \,\mathrm{d}\mu \right)^{1/q} ,$$

where q is the conjugate index of p. We take t_0 large enough, such that

$$\int_{\{\varphi < V_\theta - t_0\}} f \,\mathrm{d}V \leq \frac{1-a}{(2C_0)^n} \,.$$

Note that t_0 can be taken to be of the form

$$t_0 = C_0 \|f\|_{L^p(X,\mu)} \,,$$

where C_0 depends only on θ, μ, a. It suffices to control

$$\int_X (V_\theta - \varphi)^q \,\mathrm{d}\mu$$

from above in terms of θ, μ, a. In case (1), we have $q = 1$, so this follows from Proposition III.1.23 and Proposition III.1.22. In case (2), this follows from uniform Skoda theorem (ADD DETAILS). By (3.77),

$$g(t_0 + 1) \leq (2C_0)^{-1} \,.$$

By Lemma III.3.51, (3.76) follows. □

Corollary III.3.53. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Let $0 \leq f \in L^p(X, \mu)$ $(p > 1)$. Assume that*

$$\int_X f \,\mathrm{d}\mu = \operatorname{vol} \alpha.$$

Let $\varphi \in \mathcal{E}(X, \theta)$ be a solution to

$$\theta_\varphi^n = f\mu \,. \tag{3.78}$$

Assume one of the following conditions holds

(1) $p = \infty$ and $\mu = \theta_\eta^n$ for some $\eta \in \mathcal{E}^\infty(X, \theta)$.
(2) μ is absolutely continuous with bounded density.

Then

$$\varphi - \sup_X \varphi \geq V_\theta - C \|f\|_{L^p}^{1/n} \,, \tag{3.79}$$

where $C = C(\theta, \mu, p) > 0$ is a constant.

In particular, $\varphi \in \mathcal{E}^\infty(X, \theta)$.

Now we consider the Aubin–Yau equation.

Theorem III.3.54. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$, $\mu(X) > 0$. Let $\lambda > 0$. Then there is a unique $\varphi \in \mathcal{E}^1(X, \theta)$ such that*

$$\theta_\varphi^n = e^{\lambda\varphi}\mu \,. \tag{3.80}$$

Moreover, if μ is absolutely continuous with L^p-density ($p > 1$), then there is a constant $C = C(\theta, \mathrm{d}\mu, p) > 0$ such that

$$\varphi - \sup_X \varphi \geq V_\theta - C \,.$$

Remark III.3.4. If (3.80) holds for $\varphi \in \mathcal{E}(X, \theta)$. Then

$$\int_X (V_\theta - \varphi)\, \theta_\varphi^n \leq -\int_X \varphi e^{\lambda\varphi} \,\mathrm{d}\mu \,.$$

As $-xe^{\lambda x}$ is bounded from above when x is bounded from above, we conclude that

$$\int_X (V_\theta - \varphi)\, \theta_\varphi^n < \infty \,.$$

Hence by Corollary III.1.17, $\varphi \in \mathcal{E}^1(X, \theta)$.

Proof. Let $\varphi_j \in \mathcal{E}^1(X, \theta)$ ($j \in \mathbb{N}$) be a sequence such that

$$\lim_{j\to\infty} F_{\mu,\lambda}(\varphi_j) = \sup_{\mathcal{E}^1(X,\theta)} F_{\mu,\lambda} \,. \tag{3.81}$$

The value lies in $(-\infty, \infty]$ by Remark III.1.3.

We claim that $\sup_X \varphi_j$ is bounded from above. Otherwise, by extracting a subsequence, we may assume that

$$\sup_X \varphi_j \to \infty \,.$$

By further extracting a subsequence, by Theorem I.1.2, we may assume that $\varphi_j - \sup_X \varphi_j \to \psi \in \mathrm{PSH}(X, \theta)$. Since μ is non-pluripolar and has positive mass, we find

$$\varepsilon := \int_X e^{\lambda\psi} \,\mathrm{d}\mu > 0 \,.$$

Hence

$$\int_X e^{\lambda\varphi_j} \,\mathrm{d}\mu \geq \varepsilon e^{\lambda \sup_X \varphi_j} \,.$$

On the other hand, by Proposition III.1.13,

$$E(\varphi_j) \leq \operatorname{vol}\alpha \sup_X \varphi_j \,.$$

Hence

$$F_{\mu,\lambda}(\varphi_j) \leq \operatorname{vol}\alpha \sup_X \varphi_j - e^{\lambda \sup_X \varphi_j} \to -\infty \,.$$

This is a contradiction.

Now by extracting a subsequence, we may assume that $\varphi_j \to \varphi \in \mathrm{PSH}(X, \theta)$. Since $\varphi \mapsto \int_X e^{\lambda\varphi} \,\mathrm{d}\mu$ is continuous (ADD PROOF) and E is usc (Proposition III.1.13), we conclude that $\varphi \in \mathcal{E}^1(X, \theta)$ and that

$$F_{\mu,\lambda}(\varphi) = \sup_{\mathcal{E}^1(X,\theta)} F_{\mu,\lambda} \,.$$

We conclude (3.80) by Proposition III.1.34.

The finial claim follows from Corollary III.3.53. ∎

PROPOSITION III.3.55. *Let* $\mu \in \mathring{\mathcal{M}}_+(X)$. *Let* $\lambda > 0$. *Let* $\varphi \in \mathcal{E}^\infty(X,\theta)$, $\psi \in \mathcal{E}(X,\theta)$. *Assume that*

$$\theta_\varphi^n \leq e^{\lambda\varphi}\mu\,, \quad \theta_\psi^n \geq e^{\lambda\psi}\mu\,. \tag{3.82}$$

Then

$$\varphi \geq \psi\,.$$

REMARK III.3.5. In Corollary III.4.65, we will prove that one can weaken the assumption $\varphi \in \mathcal{E}^\infty(X,\theta)$ with $\varphi \in \mathcal{E}(X,\theta)$.

PROOF. By Corollary II.4.32 and (3.82),

$$\int_{\{\varphi<\psi\}} e^{\lambda\psi}\,\mathrm{d}\mu \leq \int_{\{\varphi<\psi\}} \theta_\psi^n \leq \int_{\{\varphi<\psi\}} \theta_\varphi^n \leq \int_{\{\varphi<\psi\}} e^{\lambda\varphi}\,\mathrm{d}\mu \leq \int_{\{\varphi<\psi\}} e^{\lambda\psi}\,\mathrm{d}\mu\,.$$

So equality holds. In particular, from the last two terms, we find $\varphi \geq \psi$ μ-a.e., hence θ_φ^n-a.e.. Hence $\varphi \geq \psi$ by Theorem II.4.35. □

LEMMA III.3.56. *Let* $\lambda > 0$ *be a constant. Let* $\varphi, \psi \in \mathcal{E}^\infty(X,\theta)$. *Then there is* $\gamma \in \mathcal{E}^\infty(X,\theta)$, *such that*

$$\theta_\gamma^n = e^{\lambda(\gamma-\varphi)}\,\theta_\varphi^n + e^{\lambda(\gamma-\psi)}\,\theta_\psi^n\,. \tag{3.83}$$

PROOF. For each $j \geq 1$, consider the canonical approximations

$$\varphi_j := \max\{\varphi, -j\}\,, \quad \psi_j := \max\{\psi, -j\}\,.$$

Let

$$\mu_j := e^{-\lambda\varphi_j}\,\theta_\varphi^n + e^{-\lambda\psi_j}\,\theta_\psi^n\,.$$

Then $\mu_j(X) > 0$. By Theorem III.3.54, there exists $\gamma_j \in \mathcal{E}^1(X,\theta)$, such that

$$\theta_{\gamma_j}^n = e^{\lambda\gamma_j}\mu_j\,. \tag{3.84}$$

Note that by Proposition III.1.21,

$$\mathrm{PSH}(X,\theta) \subseteq L^1(X,\mu_j)\,.$$

Hence by Corollary III.3.53, $\varphi_j \in \mathcal{E}^\infty(X)$. Take a constant $C > 0$ so that

$$|\varphi - \psi| \leq 2C\,.$$

Let

$$\eta = \frac{\varphi+\psi}{2} - C - n\log 2\,.$$

Then $\eta \in \mathcal{E}^\infty(X,\theta)$ and

$$\theta_\eta^n \geq e^{\lambda\eta}\mu_j\,.$$

by Proposition III.3.55, $\gamma_j \geq \eta$ and γ_j is decreasing in j, let

$$\gamma := \lim_{j\to\infty}\gamma_j\,.$$

Then $\gamma \geq \eta$, hence $\gamma \in \mathcal{E}^\infty(X,\theta)$. Now (3.83) follows from (3.84) by letting $j \to \infty$ using Theorem II.2.8 and the dominated convergence theorem. □

III.4. The rooftop operators

THEOREM III.4.57. *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Assume that $\varphi \wedge \psi \neq -\infty$, then*

$$\theta^n_{\varphi\wedge\psi} \leq \mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\theta^n_{\varphi} + \mathbb{1}_{\{\varphi\wedge\psi=\psi\}}\theta^n_{\psi}\,. \tag{3.85}$$

PROOF. **Step 1**. Assume that (3.85) holds when $\varphi, \psi \in \mathcal{E}^\infty(X, \theta)$.

For each $k > 0$, define

$$\varphi_k = \varphi \vee (V_\theta - k)\,, \quad \psi_k = \psi \vee (V_\theta - k)\,,$$
$$\eta_k = \varphi_k \wedge \psi_k\,, \quad \eta = \varphi \wedge \psi\,.$$

Then

$$\theta^n_{\eta_k} \leq \mathbb{1}_{\{\eta_k=\varphi_k\}}\,\theta^n_{\varphi_k} + \mathbb{1}_{\{\eta_k=\psi_k\}}\,\theta^n_{\psi_k}\,.$$

Then for $k > C > 0$,

$$\begin{aligned}
\mathbb{1}_{\{\eta>V_\theta-C\}}\,\theta^n_{\eta_k\vee(V_\theta-C)} =& \mathbb{1}_{\{\eta>V_\theta-C\}}\,\theta^n_{\eta_k} \\
\leq& \mathbb{1}_{\{\eta>V_\theta-k\}\cap\{\eta_k=\varphi_k\}}\,\theta^n_{\varphi_k} + \mathbb{1}_{\{\eta>V_\theta-k\}\cap\{\eta_k=\psi_k\}}\,\theta^n_{\psi_k} \\
=& \mathbb{1}_{\{\eta>V_\theta-k\}\cap\{\eta_k=\varphi_k\}}\,\theta^n_{\varphi} + \mathbb{1}_{\{\eta>V_\theta-k\}\cap\{\eta_k=\psi_k\}}\,\theta^n_{\psi} \\
\leq& e^{A(\eta_k-\varphi_k)}\,\theta^n_{\varphi} + e^{A(\eta_k-\psi_k)}\,\theta^n_{\psi}\,,
\end{aligned}$$

where $A > 0$ is an arbitrary constant. Note that η_k decreases to η. Let $k \to \infty$ and apply Theorem II.2.8 and dominated convergence theorem,

$$\mathbb{1}_{\{\eta>V_\theta-C\}}\,\theta^n_{\eta\vee(V_\theta-C)} \leq e^{A(\eta-\varphi)}\,\theta^n_{\varphi} + e^{A(\eta-\psi)}\,\theta^n_{\psi}\,.$$

Let $C \to \infty$, we find

$$\theta^n_{\eta} \leq e^{A(\eta-\varphi)}\,\theta^n_{\varphi} + e^{A(\eta-\psi)}\,\theta^n_{\psi}\,.$$

Let $A \to 0$, we conclude (3.85) by monotone convergence theorem.

Step 2. We prove the theorem assuming that $\varphi, \psi \in \mathcal{E}^\infty(X, \theta)$.

We may assume that $\varphi, \psi \leq 0$.

By Lemma III.3.56, there exists $\eta_j \in \mathcal{E}^\infty(X, \theta)$ ($j \in \mathbb{N}$), such that

$$\theta^n_{\eta_j} = e^{j(\eta_j-\varphi)}\,\theta^n_{\varphi} + e^{j(\eta_j-\psi)}\,\theta^n_{\psi}\,. \tag{3.86}$$

It follows from Proposition III.3.55 that η_j is increasing and $\eta_j \leq \varphi$, $\eta_j \leq \psi$. Let $\eta_\infty \in \mathrm{PSH}(X, \theta)$ the a.e. limit of η_j, then $\eta_\infty \in \mathcal{E}(X, \theta)$. Hence

$$\eta_\infty \leq \varphi \wedge \psi\,.$$

We claim that equality indeed holds. Fix $\varepsilon > 0$, then by (3.86),

$$\int_{\{\eta_\infty<\varphi\wedge\psi-\varepsilon\}} \theta^n_{\eta_j} \leq \int_{\{\eta_j<\varphi\wedge\psi-\varepsilon\}} \theta^n_{\eta_j} \leq 2\,\mathrm{vol}\,\alpha\, e^{-j\varepsilon}\,.$$

Let $j \to \infty$, by Corollary II.2.9,

$$\int_{\{\eta_\infty<(\varphi\wedge\psi)-\varepsilon\}} \theta^n_{\eta_\infty} = 0\,.$$

Let $\varepsilon \to 0+$, we find

$$\int_{\{\eta_\infty<\varphi\wedge\psi\}} \theta^n_{\eta_\infty} = 0\,.$$

Hence $\eta_\infty = \varphi \wedge \psi$ by Theorem II.4.35.

Now (3.85) follows by letting $j \to \infty$ in (3.86) and applying Theorem II.2.8 and monotone convergence theorem. ☐

COROLLARY III.4.58. *Let $U \subseteq X$ be a plurifine open subset. Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Assume that $\varphi \wedge \psi \neq -\infty$. Let $\mu \in \mathcal{M}_+(X)$. Assume that*

$$\mathbb{1}_U \theta_\varphi^n \leq \mu\,, \quad \mathbb{1}_U \theta_\psi^n \leq \mu\,.$$

Then

$$\mathbb{1}_U \theta_{\varphi\wedge\psi}^n \leq \mu\,.$$

PROOF. Replacing μ by $\mathbb{1}_{X-P}\mu$, where $P = \{\varphi = \psi = -\infty\}$, we may assume that $\mu(P) = 0$. Now the function $r \mapsto \mu\{\varphi \leq \psi + r\}$ is increasing and bounded, so there are at most countable r so that

$$\mu\{\varphi = \psi + r\} > 0\,.$$

So we can take a sequence $\varepsilon_i > 0$ ($i \in \mathbb{N}$) decreasing to 0, such that

$$\mu\{\varphi = \psi - \varepsilon_i\} = 0\,.$$

By Theorem III.4.57,

$$\mathbb{1}_U \theta_{\varphi\wedge(\psi-\varepsilon_i)} \leq \mu\,.$$

Let $i \to \infty$, by Corollary II.2.9,

$$\mathbb{1}_U \theta_{\varphi\wedge\psi}^n \leq \mu\,.$$

$\square$

COROLLARY III.4.59. *Let $U \subseteq X$ be a plurifine open subset. Let $\phi, \varphi_j \in \mathrm{PSH}(X,\theta)$ ($j \in \mathbb{N}$). Assume that there is $C > 0$ so that $|\phi - \varphi_j| \leq C$. Let $\mu \in \mathcal{M}_+(X)$. Assume that*

$$\mathbb{1}_U \theta_{\varphi_j}^n \leq \mu\,. \tag{3.87}$$

Then

$$\mathbb{1}_U\, \theta_{\underline{\lim}\,\varphi_j}^n \leq \mathbb{1}_U\, \mu\,.$$

PROOF. For $j, k \in \mathbb{N}$, let

$$\varphi_{j,k} := \varphi_j \wedge \cdots \wedge \varphi_{j+k}\,.$$

Then by Corollary III.4.58,

$$\mathbb{1}_U \theta_{\varphi_{j,k}}^n \leq \mu\,.$$

Let $\varphi_j = \inf_k \varphi_{j,k}$. Then by Corollary II.2.9,

$$\mathbb{1}_U \theta_{\varphi_j}^n \leq \mu\,.$$

Let $\varphi = \sup^* \varphi_j$, again by Corollary II.2.9,

$$\mathbb{1}_U \theta_\varphi^n \leq \mu\,.$$

$\square$

THEOREM III.4.60. *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Assume that $[\varphi] \preceq [\psi]$. Then*

$$\theta_{[\varphi]\wedge\psi}^n \leq \mathbb{1}_{\{[\varphi]\wedge\psi=\psi\}}\, \theta_\psi^n\,.$$

PROOF. We may assume that $\varphi, \psi \leq 0$. By Theorem III.4.57, for any $t > 0$,

$$\theta_{(\varphi+t)\wedge\psi}^n \leq \mathbb{1}_{\{(\varphi+t)\wedge\psi=\varphi+t\}}\, \theta_\varphi^n + \mathbb{1}_{\{(\varphi+t)\wedge\psi=\psi\}}\, \theta_\psi^n\,.$$

Observe that

$$\{(\varphi+t)\wedge\psi = \varphi+t\} \subset \{\varphi + t \leq V_\theta\}\,.$$

So as $t \to \infty$, the first term vanishes. Now we can apply Corollary II.2.9 and dominated convergence theorem to conclude. $\square$

Corollary III.4.61. *Let $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$\theta^n_{[\varphi]\wedge V_\theta} \leq \mathbb{1}_{\{[\varphi]\wedge V_\theta=0\}}\,\theta^n\,. \tag{3.88}$$

Proof. This follows from Theorem III.4.60 and Theorem I.5.12. $\square$

Lemma III.4.62. *Let $\varphi \in \mathrm{PSH}(X,\theta)$, $\int_X \theta^n_\varphi > 0$. Let $B \subseteq X$ be a Borel set with positive Lebesgue measure. Then there is $\psi \in \mathrm{PSH}(X,\theta)$, such that*

$$[\psi]=[\varphi]\,,\quad \int_B \theta^n_\phi > 0\,.$$

Proof. Let ω be a Kähler form on X. By Theorem III.3.50 and Corollary III.3.53, there exists $\eta \in \mathcal{E}^\infty(X,\theta)$, such that

$$\theta^n_\eta = \frac{1}{\int_B \omega^n}\mathbb{1}_B\omega^n\,.$$

For each $C>0$, let

$$\varphi^C := \varphi \vee (V_\theta - C)\,.$$

Note that $[\varphi^C]=[\varphi]$. By Theorem III.4.57,

$$\begin{aligned}\theta^n_{\varphi^C} \leq& \mathbb{1}_{\{\varphi^C=\phi+C\}}\,\theta^n_\varphi + \mathbb{1}_{\{\varphi^C=V_\theta-C\}}\,\theta^n_\eta\\ \leq& \mathbb{1}_{\{\phi+C\leq\eta\}}\,\theta^n_\varphi + \frac{1}{\int_B\omega^n}\mathbb{1}_{\{\varphi^C=V_\theta-C\}\cap B}\omega^n\,.\end{aligned}$$

Hence

$$\int_{X-B}\theta^n_{\varphi^C} \leq \int_{\{\phi\leq\eta-C\}}\theta^n_\varphi\,.$$

As $C\to\infty$, the RHS tends to 0. In particular, we can achieve that

$$\int_{X-B}\theta^n_{\varphi^C} < \int_X \theta^n_{\varphi^C}\,,$$

since by Theorem II.3.21 and our assumption, RHS is positive. Thus

$$\int_B \theta^n_{\varphi^C} > 0\,.$$

$\square$

Theorem III.4.63 (Domination principle. II). *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)$. Assume that $[\psi]\preceq[\varphi]$ and that $\int_X\theta^n_\psi>0$. Assume that*

$$\psi\leq\varphi\,,\quad \theta^n_\varphi-a.e.\,, \tag{3.89}$$

then $\psi\leq\varphi$.

Proof. It suffices to prove $A:=\{\varphi<\psi\}$ has zero Lebesgue measure by Proposition I.1.1.

Assume that A has positive Lebesgue measure. Then by Lemma III.4.62, there exists $\eta\in\mathrm{PSH}(X,\theta)$, such that

$$[\eta]=[\psi]\,,\quad \int_A\theta^n_\eta>0\,.$$

It follows from Lemma II.4.34 and (3.89) that $\int_A\theta^n_\eta=0$. This is a contradiction. $\square$

THEOREM III.4.64 (Domination principle. III). *Let $\varphi, \psi \in \mathcal{E}(X,\theta)$. Assume that*

$$\psi \leq \varphi\,, \quad \theta_\varphi^n - a.e.\,, \tag{3.90}$$

then $\psi \leq \varphi$.

PROOF. **Step 1**. We prove the theorem under the additional assumption that $[\varphi] \preceq [\psi]$.

Let $A = \{\psi > \varphi\}$. Let $\eta \in \mathrm{PSH}(X,\theta)$, $[\eta] = [\varphi]$. By Lemma III.4.62, it suffices to prove

$$\int_A \theta_\eta^n = 0\,. \tag{3.91}$$

We may assume that $\eta \leq \varphi$, $\eta \leq \psi$. We claim that for any $\varepsilon \in (0,1)$, $(1-\varepsilon)\psi + \varepsilon\eta \in \mathcal{E}(X,\theta)$. In fact, by assumption in this step,

$$[\varphi] \preceq [(1-\varepsilon)\psi + \varepsilon\eta]\,.$$

We have

$$\{\varphi < (1-\varepsilon)\psi + \varepsilon\eta\} \subseteq A\,.$$

By Theorem II.4.28,

$$\varepsilon^n \int_{\{\varphi < (1-\varepsilon)\psi + \varepsilon\eta\}} \theta_\eta^n \leq \int_{\{\varphi < (1-\varepsilon)\psi + \varepsilon\eta\}} \theta^n_{\{\varphi < (1-\varepsilon)\psi + \varepsilon\eta\}} \leq \int_{\{\varphi < (1-\varepsilon)\psi + \varepsilon\eta\}} \theta_\varphi^n \leq \int_A \theta_\varphi^n = 0\,.$$

Let $\varepsilon \to 0+$, we conclude (3.91).

Step 2. In general,

$$\int_{\{\varphi < \varphi \vee \psi\}} \theta_\varphi^n = \int_{\{\varphi < \psi\}} \theta_\varphi^n = 0\,.$$

Hence by Step 1, $\varphi = \varphi \vee \psi$. Similarly, $\psi = \varphi \vee \psi$. We conclude $\varphi = \psi$. $\square$

COROLLARY III.4.65. *Let $\mu \in \mathring{\mathcal{M}}_+(X)$. Let $\lambda > 0$. Let $\varphi, \psi \in \mathcal{E}(X,\theta)$. Assume that*

$$\theta_\varphi^n \leq e^{\lambda\varphi}\mu\,, \quad \theta_\psi^n \geq e^{\lambda\psi}\mu\,. \tag{3.92}$$

Then

$$\varphi \geq \psi\,.$$

PROOF. The proof is the same as that of Proposition III.3.55. One just has to replace Theorem II.4.35 by Theorem III.4.64. $\square$

THEOREM III.4.66. *Take $\varphi \in \mathrm{PSH}(X,\theta)$. Then the followings are equivalent:*

(1) $\varphi \in \mathcal{E}(X,\theta)$.
(2) $[\varphi] \wedge V_\theta = V_\theta$.

Assume that these conditions holds, then for any $\psi \in \mathcal{E}(X,\theta)$,

$$[\varphi] \wedge \psi = \psi\,. \tag{3.93}$$

PROOF. (1) implies (2). Let $\psi \in \mathrm{PSH}(X,\theta)$, $\psi \leq 0$. By Corollary III.4.61,

$$\int_{\{[\varphi]\wedge V_\theta < \psi\}} \theta^n_{[\varphi]\wedge V_\theta} \leq \int_{\{[\varphi]\wedge V_\theta = 0\} \cap \{[\varphi]\wedge V_\theta < \psi\}} \theta^n = 0\,.$$

By Theorem III.4.63,

$$\psi \leq [\varphi] \wedge V_\theta\,.$$

So $[\varphi] \wedge V_\theta = V_\theta$.

(2) implies (1). This follows from Corollary II.3.26.

It remains to prove (3.93). By definition

$$[\varphi] \wedge \psi \leq \psi\,.$$

For the reverse inequality, note that by Theorem III.4.60,

$$\int_{[\varphi]\wedge\psi<\psi} \theta^n_{[\varphi]\wedge\psi} = 0\,.$$

By Theorem III.4.64, it follows that $[\varphi] \wedge \psi \geq \psi$. □

III.5. Monge–Ampère equation II. Uniqueness

THEOREM III.5.67. *Let $\varphi, \psi \in \mathcal{E}(X, \theta)$. Assume that*

$$\theta^n_\varphi = \theta^n_\psi\,. \tag{3.94}$$

Then $\varphi - \psi$ is a constant.

PROOF. Let $\mu = \theta^n_\varphi$.

We claim that there is $t \in \mathbb{R}$, such that $\varphi = \psi + t$, μ-a.e.. Then we can apply Theorem III.4.64 to conclude.

Assume that the claim were not true. Then for any $t \in \mathbb{R}$, $\mu\{\varphi = \psi + t\} < 1$. Hence there is $t_0 \in \mathbb{R}$, so that

$$0 < \mu\{\varphi < \psi + t_0\} < 1.$$

The set of $t \in \mathbb{R}$ such that $\mu\{\varphi = \psi + t\} > 0$ is exactly the set of discontinuity of $t \mapsto \mu\{\varphi < \psi + t\}$, hence countable. So we may assume after a small perturbation that

$$\mu\{\varphi = \psi + t_0\} = 0\,.$$

Replacing ψ by $\psi + t_0$, we may set $t_0 = 0$. Take $c > 1$ so that

$$c^n \mu(U) = 1\,,$$

where $U = \{\varphi < \psi\}$.

According to Theorem III.3.50, we may take $\eta \in \mathcal{E}(X, \theta)$, so that

$$\theta^n_\eta = c^n \mathbb{1}_U \mu\,, \quad \sup_X \eta = 1\,.$$

For $t \in (0, 1)$, set

$$U_t := \{(1-t)\varphi + tV_\theta < (1-t)\psi + t\eta\}\,.$$

Observe that U_t increases to $U - \{\eta = -\infty\}$ as $t \to 0+$. By Theorem II.1.5,

$$\theta^{n-1}_\varphi \wedge \theta_\eta \geq c\mathbb{1}_U\,\mu\,, \quad \theta^k_\varphi \wedge \theta^{n-k}_\psi \geq \mu\,, \tag{3.95}$$

for any $k = 0, \ldots, n$. By Theorem II.3.21, we indeed have

$$\theta^k_\varphi \wedge \theta^{n-k}_\psi = \mu\,.$$

By Corollary II.4.32,

$$\int_{U_t} \theta^{n-1}_\varphi \wedge \theta_{(1-t)\psi+t\eta} \leq \int_{U_t} \theta^{n-1}_\varphi \wedge \theta_{(1-t)\varphi+tV_\theta}\,.$$

This holds for all $t \in (0, 1)$, hence comparing the coefficients of t, we find

$$\int_{U_t} \theta^{n-1}_\varphi \wedge \theta_\eta \leq \int_{U_t} \theta^{n-1}_\varphi \wedge \theta_{V_\theta}\,.$$

By (3.95),
$$c\mu(U_t) \leq \int_{U_t} \theta_\varphi^{n-1} \wedge \theta_{V_\theta}\,.$$
Let $t \to 0+$,
$$c\mu(U) \leq \int_{U} \theta_\varphi^{n-1} \wedge \theta_{V_\theta}\,.$$
Similar argument applies to $V = \{\varphi > \psi\}$, we find
$$b\mu(V) \leq \int_{V} \theta_\varphi^{n-1} \wedge \theta_{V_\theta}\,,$$
where
$$b^n\mu(V) = 1\,.$$
So
$$0 < \min\{b, c\} \leq \int_{X} \theta_\varphi^{n-1} \wedge \theta_{V_\theta} = 1\,.$$
But $b, c > 1$, this is a contradiction. □

III.6. Compactness in $\mathcal{E}^1$

In this section, let X be a compact Kähler manifold of dimension n. Let α be a big cohomology class on X with smooth representative θ.

III.7. Finite energy classes

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