A PROOF OF SATZ VON H. CARTAN

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In this short note, we prove the celebrated *Satz von H. Cartan.*

Theorem 0.1. *Let S be a Stein space and $A = \Gamma(S, \mathcal{O}_S)$ be the corresponding Stein algebra. Then for any ideal $I \subseteq A$, we have*

$$\overline{I} = \Gamma(S, I\mathcal{O}_S). \tag{0.1}$$

Here A is endowed with the *canonical Fréchet topology* as in [GR77, Kapitel V]. Recall that by construction, it is the unique Fréchet topology on A so that for all $s \in S$, the following map is continuous:

$$A \to \mathcal{O}_{S,s}, \quad f \mapsto f_s.$$

When S is reduced, this topology is nothing but the compact-open topology. The left-hand side of (0.1) refers to the closure of I in A with respect to this topology.

On the right-hand side $I\mathcal{O}_S$ refers to the coherent ideal sheaf on S generated by I.

This theorem is usually referred to as *Satz von H. Cartan*, and people refer to [Car50] for its proof. But Henri Cartan only handled the case of pseudoconvex domains, and his proof can only be generalized directly to *reduced* Stein spaces. As for the general case, in the standard textbook [GR77, Kapitel V], Grauert and Remmert promised to give a proof in [CAS], but the proof never appeared.

As far as I know, a proof is not explicitly written down anywhere. So I decide to fill in this gap by myself. This note is the result of a discussion with Ning Guo.

1. Runge sets

Let S be a Stein space. An open subset U is *Runge* if U is Stein and $\Gamma(S, \mathcal{O}_S) \to \Gamma(U, \mathcal{O}_U)$ has dense image. We need the following basic fact proved in [Ben25]: $U \subseteq S$ is Runge if and only if $U^{\mathrm{red}} \subseteq S^{\mathrm{red}}$ is Runge. The proof is elementary and does not rely on any consequences of Theorem 0.1.

In particular, any Stein space S admits a Runge exhaustion:

$$U_1 \Subset U_2 \Subset \cdots .$$

In other words, each U_i is a Runge open subset of S and $\bigcup_i U_i = S$. The reduced case is well-known (see [Nar61, Theorem 3]) and the general case follows from the above-mentioned result.

2. The proof

Proof. The inclusion $\subseteq$ in (0.1) is trivial, as the right-hand side is always closed, see [GR77, Kapitel V.6.4]. It suffices to prove the reverse inclusion.

Step 1. We first assume that $S = \mathbb{C}^n$.

Let $f \in \Gamma(S, I\mathcal{O}_S)$. We prove that $f \in \overline{I}$.

Since S is reduced, the topology on A is the compact-open topology. Let $K \subseteq S$ be compact and let $\epsilon > 0$. It suffices to find $g \in I$ such that

$$\sup_K |f - g| < \epsilon. \tag{2.1}$$

Without loss of generality, we may replace K by a larger convex subset and assume that K is a closed polydisk centered at $0 \in \mathbb{C}^n$.

For each $x \in K$, we can find holomorphic functions $g_1, \ldots, g_m \in I$ generating $I\mathcal{O}_S$ in an open neighborhood of x. By the compactness of K, we may then assume that $g_1, \ldots, g_m \in I$ generate $I\mathcal{O}_S$ on an open polydisk U containing K.

Then we have a surjective homomorphism of coherent $\mathcal{O}_U$-modules:

$$\mathcal{O}_S|_U^m \xrightarrow{\cdot(g_1,\ldots,g_m)} (I\mathcal{O}_S)\,|_U.$$

By Cartan's Theorem B, we have an induced surjective homomorphism

$$\Gamma(U, \mathcal{O}_S)^m \to \Gamma(U, I\mathcal{O}_S), \quad (f_1, \ldots, f_m) \mapsto \sum_{i=1}^m f_i g_i|_U.$$

In particular, we can find $f_1, \ldots, f_m \in \Gamma(U, \mathcal{O}_S)$ so that

$$f|_U = \sum_{i=1}^m f_i g_i|_U.$$

By the Oka–Weil approximation theorem, for each $i = 1, \ldots, m$, we can find $f_i' \in A$ with

$$\sup_K |f_i - f_i'| \leq m^{-1}\epsilon \cdot \left(1 + \sup_K |g_i|\right)^{-1}.$$

Then (2.1) follows with

$$g = \sum_{i=1}^m f_i' g_i.$$

Step 2. We assume that S admits a closed immersion $i \colon S \hookrightarrow \mathbb{C}^n$.

Let A' be the Stein algebra of $\mathbb{C}^n$. Then we have a natural continuous surjective map $F \colon A' \to A$. Note that F is also an open map by the open mapping theorem. Due to Theorem 2.1, we find that

$$\overline{I} = F\left(\overline{F^{-1}(I)}\right).$$

By Step 1, we have

$$\overline{F^{-1}(I)} = \Gamma\left(\mathbb{C}^n, F^{-1}(I)\mathcal{O}_{\mathbb{C}^n}\right).$$

So it remains to argue that any $f \in \Gamma(S, I\mathcal{O}_S)$ can be written as $F(g)$ for some $g \in \Gamma\left(\mathbb{C}^n, F^{-1}(I)\mathcal{O}_{\mathbb{C}^n}\right)$. For this purpose, observe that we have a natural surjective homomorphism of coherent $\mathcal{O}_{\mathbb{C}^n}$-modules

$$F^{-1}(I)\mathcal{O}_{\mathbb{C}^n} \to i_*(I\mathcal{O}_S).$$

By Cartan's Theorem B again, our assertion follows.

Step 3. We handle the general case. Let $f \in \Gamma(S, I\mathcal{O}_S)$. We prove that $f \in \overline{I}$.

Choose a Runge Stein exhaustion of S:

$$U_1 \Subset U_2 \Subset \cdots.$$

Write A_i for the Stein algebra of U_i. Write the restriction map as $F_i \colon A \to A_i$.

Note that A is the inverse limit of A_i in the category of locally convex topological vector spaces. This can be seen using the uniqueness of the Fréchet topology on A mentioned above. Therefore, in view of Theorem 2.2[1], it suffices to verify that for each i,

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$$f|_{U_i} \in \overline{F_i(I)}. \tag{2.2}$$

Fix i. Note that each U_i admits a closed immersion into some $\mathbb{C}^n$ and hence Step 2 is applicable. Step 2 gives

$$\overline{IA_i} = \Gamma\left(U_i, (IA_i\mathcal{O}_S)|_{U_i}\right).$$

Therefore,

$$f|_{U_i} \in \overline{IA_i}.$$

Hence in order to prove (2.2), it suffices to prove

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$$\overline{F_i(I)} = \overline{IA_i}. \tag{2.3}$$

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[1] Recall that the underlying topological space of an inverse limit of locally convex topological vector spaces is the inverse limit of the underlying topological spaces.

The inclusion $F_i(I) \subseteq IA_i$ is clear, so it suffices to prove that every element of IA_i belongs to $\overline{F_i(I)}$.

Let $g \in IA_i$. Then

$$g = \sum_{j=1}^{N} a_j F_i(b_j)$$

for some $a_j \in A_i$ and $b_j \in I$. Since $F_i(A)$ is dense in A_i, we can approximate each coefficient a_j in A_i by elements $a_{j,k} \in A$. Then

$$g_k := \sum_{j=1}^{N} a_{j,k} b_j \in I,$$

and

$$F_i(g_k) = \sum_{j=1}^{N} F_i(a_{j,k}) F_i(b_j) \to \sum_{j=1}^{N} a_j F_i(b_j) = g$$

in A_i as $k \to \infty$. Hence $g \in \overline{F_i(I)}$. This proves (2.3). ☐

`pen_closure` **Lemma 2.1.** *Let $f\colon X \to Y$ be a continuous, surjective and open map of topological spaces. Then for every subset $S \subseteq Y$, one has*

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$$f\big(\overline{f^{-1}(S)}\big) = \overline{S}. \tag{2.4}$$

Proof. We first prove the $\subseteq$ direction in (2.4). Since f is continuous, we have

$$\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S}).$$

Applying f and use the surjectivity we get $\subseteq$.

Conversely, let $y \in \overline{S}$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. We claim that $x \in \overline{f^{-1}(S)}$.

Let $U \subseteq X$ be an open neighborhood of x. Since f is open, $f(U)$ is an open neighborhood of y in Y. Since $y \in \overline{S}$, we have

$$f(U) \cap S \neq \varnothing.$$

Choose $s \in f(U) \cap S$. Choose $u \in U$ such that $f(u) = s$. Then $u \in f^{-1}(S)$. Thus

$$U \cap f^{-1}(S) \neq \varnothing.$$

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`nv_limt_clo` **Lemma 2.2.** *Let $(X_i, \varphi_{ij})_{i \le j,\ i,j \in \mathbb{N}}$ be an inverse system of topological spaces. Let*

$$X = \varprojlim_i X_i$$

be the inverse limit in the category of topological spaces, and denote by $p_i\colon X \to X_i$ the canonical projections. Then for every subset $S \subseteq X$, and $x \in X$ the following are equivalent:

(1) $x \in \overline{S}$;

(2)

$$p_i(x) \in \overline{p_i(S)}$$

for every $i \in \mathbb{N}$.

Proof. Since each projection $p_i\colon X \to X_i$ is continuous, we have $p_i(\overline{S}) \subseteq \overline{p_i(S)}$. Therefore,

$$\overline{S} \subseteq \bigcap_{i \in \mathbb{N}} p_i^{-1}\big(\overline{p_i(S)}\big).$$

Conversely, let

$$x \in \bigcap_{i \in \mathbb{N}} p_i^{-1}\big(\overline{p_i(S)}\big).$$

We show that $x \in \overline{S}$. Let $U \subseteq X$ be an open neighborhoods of x. By the definition of the inverse limit topology, there exist indices

$$i_1, \ldots, i_m \in \mathbb{N}$$

and open neighborhoods $U_{i_k} \subseteq X_{i_k}$ of $p_{i_k}(x)$ such that

$$x \in \bigcap_{k=1}^{m} p_{i_k}^{-1}(U_{i_k}) \subseteq U.$$

Let

$$N = \max\{i_1, \dots, i_m\}.$$

For each k, we have

$$p_{i_k} = \varphi_{i_k N} \circ p_N.$$

Define the open neighborhoods

$$V_N = \bigcap_{k=1}^{m} \varphi_{i_k N}^{-1}(U_{i_k}) \subseteq X_N$$

of $p_N(x)$. Since $p_N(x) \in \overline{p_N(S)}$, there exists $s \in S$ such that $p_N(s) \in V_N$.

It follows that for every k,

$$p_{i_k}(s) = \varphi_{i_k N}(p_N(s)) \in U_{i_k}.$$

Hence,

$$s \in \bigcap_{k=1}^{m} p_{i_k}^{-1}(U_{i_k}) \subseteq U.$$

Thus $U \cap S \neq \varnothing$. Since U was arbitrary, $x \in \overline{S}$. $\square$

References

Ben25 [Ben25] O. Benoist. Étale cohomology of algebraic varieties over Stein compacta. *Invent. Math.* 240.2 (2025), pp. 497–536. URL: `https://doi.org/10.1007/s00222-025-01323-2`.

Car50 [Car50] H. Cartan. Idéaux et modules de fonctions analytiques de variables complexes. *Bull. Soc. Math. France* 78 (1950), pp. 29–64. URL: `http://www.numdam.org/item?id=BSMF_1950__78__29_0`.

CAS [CAS] H. Grauert and R. Remmert. Coherent analytic sheaves. Vol. 265. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1984, pp. xviii+249. URL: `https://doi.org/10.1007/978-3-642-69582-7`.

GR77 [GR77] H. Grauert and R. Remmert. Theorie der Steinschen Räume. Vol. 227. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin-New York, 1977, pp. xx+249.

Nar61 [Nar61] R. Narasimhan. The Levi problem for complex spaces. *Math. Ann.* 142 (1960/61), pp. 355–365. URL: `https://doi.org/10.1007/BF01451029`.

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