

Mingchen Xia

Stein spaces

The current version is not polished enough. Use it at your own risk!
Updated on June 24, 2026.

Preface

*En mathématiques l'indispensable processus d'assimilation est
un processus de trivialisaton.*
— Pierre Lelong, 1981

These notes were written during the author's attempt to understand the recent works of O. Benoist on the étale cohomology of varieties over Stein algebras. Benoist's papers rely on a number of older papers written in the 1960s and 1970s. These older papers are usually difficult for a modern reader, and I have tried to reformulate the proofs and fill in the omitted details in these notes.

All non-trivial results are essentially known to experts, despite the fact that the proofs of some of them have never been written down or correctly written down. I do not claim any originality.

The reader is expected to be familiar with the classical theory of Stein spaces at the level of [GR77], and the theory of complex analytic spaces as in [CAS].

Mingchen Xia
in Hefei, June 2026

Contact information:

Mingchen Xia, Institute of Geometry and Physics, USTC

Email address: xiamingchen2008@gmail.com

Homepage: <https://mingchenxia.github.io/>.

Acknowledgments

*What I gained by being in France was, learning to be better
satisfied with my own country.
— Samuel Johnson (1709–1784)*

I would like to thank Olivier Benoist and Ning Guo for discussions.

The book was written with the help of ChatGPT and Claude. In most cases, I sketched the proofs and ChatGPT filled in the details.

I am extremely disappointed by Anthropic's extreme hatred toward my homeland and by Dario Amodei's decision to block all users from mainland China, including myself.

I am supported by the National Key R&D Program of China 2025YFA1018200.

Contents

1	Preliminaries	1
1.1	General topology	1
1.2	Commutative algebra	9
1.3	Locally ringed spaces	20
1.4	Locally convex topological vector spaces	21
1.5	Topological algebras	23
1.5.1	The Gelfand dual	24
1.5.2	The local completions	26
1.5.3	The associated locally \mathbb{C} -ringed space	27
1.5.4	Properties	30
1.5.5	Fréchet algebras	34
2	Complex spaces	37
2.1	Local analytic algebras	37
2.2	The section ring of complex spaces	40
2.2.1	The canonical Fréchet topology	40
2.2.2	Envelopes	48
2.3	Auxiliary results about complex spaces	51
2.4	Holomorphically convex hull	59
3	Stein algebras	61
3.1	Preliminary results about Stein spaces	61
3.2	Runge subsets	64
3.3	Stein algebras	69
3.3.1	0-dimensional case	69
3.3.2	Satz von H. Cartan	71
3.3.3	The equivalence of categories	76
3.4	Stein modules	85
3.5	The entire function ring	89
3.5.1	Divisors of entire functions	90
3.5.2	Divisor filters	91

3.5.3	Prime divisor filters and the spectrum	92
3.5.4	Fixed and free ideals	93
3.5.5	Finitely generated ideals	94
3.5.6	Maximal ideals	95
3.5.7	Residue fields of maximal ideals	97
3.5.8	Summary of the spectrum	99
3.5.9	Zariski topology	100
4	Stein compacta	101
4.1	The algebra of subsets	101
4.1.1	The algebra of germs along general subsets	101
4.2	Germs along subspaces	103
4.2.1	Generalities	103
4.2.2	Properties of germs	107
4.3	Stein compactum	109
4.3.1	Stein compact subsets	109
4.3.2	Morphisms to Stein compacta	114
4.4	The section rings of Stein compacta	118
4.4.1	The ideals	119
4.4.2	Excellence	126
5	Analytifications	135
5.1	List of properties	135
5.1.1	Fpqc local properties	135
5.1.2	Properties of modules	136
5.1.3	Properties of ideals	138
5.1.4	Properties of morphisms of schemes and complex spaces	140
5.2	Interior properties	142
5.3	Analytification functor	144
5.4	Properties of the analytification functor	159
5.4.1	Analytification of modules	159
5.4.2	Analytification of morphisms	163
5.4.3	Cohomology comparison	169
	Comments	201
	Index	203
	References	205

Conventions

Throughout the book, we adopt the following conventions:

- A complex space is not necessarily Hausdorff or paracompact/ σ -compact;
- Given a complex space X , $C^0(X)$ is the space of continuous functions $X \rightarrow \mathbb{C}$ endowed with the compact-open topology.

Chapter 1

Preliminaries

Wir wissen nicht, was Mathematik ist. Aber was gute Mathematik ist, das wissen wir.
— Reinhold Remmert^a

^a Reinhold Remmert (1930–2016) was one of the central figures of the Münster school in several complex variables. A student of Heinrich Behnke, he worked at the meeting point of function theory, Stein theory, and the new language of complex analytic spaces. Together with Hans Grauert, he helped establish complex spaces as a natural category for analytic geometry; the Grauert–Remmert theory made local analytic algebras, coherent sheaves, and finite analytic maps part of the basic vocabulary of the subject.

1.1 General topology

Lemma 1.1.1 *Let $f: X \rightarrow Y$ be a quasi-finite, open and continuous map between compact Hausdorff spaces. Assume that Y is connected. Then X has only finitely many connected components.*

Recall that a map of sets $f: X \rightarrow Y$ is *quasi-finite* if for any $y \in Y$, the inverse image $f^{-1}(y)$ is a finite set.

Proof If $X = \emptyset$, there is nothing to prove. Let $C \subseteq X$ be a connected component. We claim that

$$f(C) = Y.$$

Assume this for the moment. Then for any $y \in Y$, each connected component C of X meets the finite set $f^{-1}(y)$, and hence X has at most $\#f^{-1}(y)$ connected components.

It remains to prove the claim. Choose $c \in C$. Suppose, by contradiction, that there exists $y \in Y \setminus f(C)$. Since $f^{-1}(y)$ is finite, write

$$f^{-1}(y) = \{x_1, \dots, x_n\},$$

where each $x_i \notin C$.

For each $i = 1, \dots, n$, since X is compact Hausdorff, the connected component of c coincides with the quasi-component of c . Hence there exists a clopen subset $A_i \subseteq X$ such that

$$c \in A_i, \quad x_i \notin A_i.$$

Since A_i is clopen and contains c , it contains the whole connected component C . Set

$$A := \bigcap_{i=1}^n A_i.$$

Then A is clopen, $C \subseteq A$, and $y \notin f(A)$. Since A is compact and Y is Hausdorff, $f(A)$ is closed in Y . Since f is open, $f(A)$ is also open in Y . Moreover $f(A)$ is non-empty. Thus $f(A)$ is a non-empty clopen subset of the connected space Y , so $f(A) = Y$, a contradiction. \square

Lemma 1.1.2 *Let E be a compact Hausdorff space, let $U \subseteq E$ be open, and let $K \subseteq U$ be a connected component of U . Assume that K is compact. Then K is a connected component of E .*

Proof Let C be the connected component of E containing K . We show that $C = K$.

Since E is compact Hausdorff, it is normal. As K is compact and U is an open neighborhood of K , there exists an open subset $V \subseteq E$ such that

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

Put $L = \bar{V}$. Then L is compact Hausdorff.

We first claim that K is a connected component of L . Indeed, if $T \subseteq L$ is a connected subset meeting K , then

$$T \subseteq L \subseteq U.$$

Since K is a connected component of U , we must have $T \subseteq K$. Thus K is a connected component of L .

In a compact Hausdorff space, connected components coincide with quasi-components. Hence, for every point $x \in L \setminus K$, there exists a clopen subset $W_x \subseteq L$ such that

$$K \subseteq W_x, \quad x \notin W_x.$$

Apply this to the compact subset $L \setminus V \subseteq L \setminus K$. The open subsets $L \setminus W_x$ cover $L \setminus V$, so we may choose $x_1, \dots, x_m \in L \setminus V$ such that

$$L \setminus V \subseteq \bigcup_{v=1}^m (L \setminus W_{x_v}).$$

Set

$$W := \bigcap_{v=1}^m W_{x_v}.$$

Then W is clopen in L , and

$$K \subseteq W \subseteq V.$$

Since L is closed in E , the subset W is closed in E . Moreover, as $V \subseteq L$ and $W \subseteq V$, we have

$$W = V \cap (E \setminus (L \setminus W)),$$

so W is also open in E . Hence W is clopen in E .

The connected component C of E meets W , since $K \subseteq C \cap W$. As W is clopen in E , connectedness of C gives

$$C \subseteq W \subseteq U.$$

But C is a connected subset of U meeting the connected component K of U . Hence $C \subseteq K$. Since $K \subseteq C$ by definition of C , we get $C = K$. \square

Lemma 1.1.3 *Let $\{f_i: X_i \rightarrow Y_i\}_{i \in I}$ be a family of continuous maps between topological spaces, and assume that $X_i \neq \emptyset$ for all $i \in I$. Consider the product map*

$$f = \prod_{i \in I} f_i: \prod_{i \in I} X_i \longrightarrow \prod_{i \in I} Y_i.$$

Then the following are equivalent:

- (1) *The image $f(\prod_{i \in I} X_i)$ is dense in $\prod_{i \in I} Y_i$.*
- (2) *For every $i \in I$, the image $f_i(X_i)$ is dense in Y_i .*

Proof (1) \implies (2). Fix $i \in I$, and let $p_i: \prod_{j \in I} Y_j \rightarrow Y_i$ be the projection. Then

$$p_i(f(\prod_j X_j)) = f_i(X_i).$$

Since p_i is continuous, we have

$$\overline{f_i(X_i)} = \overline{p_i(f(\prod_j X_j))} \supseteq p_i(\overline{f(\prod_j X_j)}) = p_i(\overline{\prod_j Y_j}) = Y_i,$$

hence $f_i(X_i)$ is dense in Y_i .

(2) \implies (1). Let $U \subseteq \prod_{i \in I} Y_i$ be a nonempty basic open set. Then

$$U = \prod_{i \in I} U_i,$$

where each $U_i \subseteq Y_i$ is open and $U_i = Y_i$ for all but finitely many i . Let $F \subseteq I$ be the finite set of indices such that $U_i \neq Y_i$.

For each $i \in F$, since $f_i(X_i)$ is dense in Y_i , we can choose $x_i \in X_i$ such that $f_i(x_i) \in U_i$. For $i \notin F$, choose any $x_i \in X_i$ (possible since $X_i \neq \emptyset$). Then

$$f((x_i)_{i \in I}) \in U.$$

Thus every nonempty basic open set intersects $f(\prod_i X_i)$, which proves that $f(\prod_i X_i)$ is dense in $\prod_i Y_i$. \square

Lemma 1.1.4 *Let $f: X \rightarrow Y$ be a continuous, surjective and open map of topological spaces. Then for every subset $S \subseteq Y$, one has*

$$f(\overline{f^{-1}(S)}) = \overline{S}. \quad (1.1)$$

Proof We first prove the \subseteq direction in (1.1). Since f is continuous, we have

$$\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S}).$$

Applying f and using the surjectivity, we get \subseteq .

Conversely, let $y \in \overline{S}$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. We claim that $x \in \overline{f^{-1}(S)}$.

Let $U \subseteq X$ be an open neighborhood of x . Since f is open, $f(U)$ is an open neighborhood of y in Y . Since $y \in \overline{S}$, we have

$$f(U) \cap S \neq \emptyset.$$

Choose $s \in f(U) \cap S$. Choose $u \in U$ such that $f(u) = s$. Then $u \in f^{-1}(S)$. Thus

$$U \cap f^{-1}(S) \neq \emptyset.$$

We conclude the proof. \square

Lemma 1.1.5 Let $(X_i, \varphi_{ij})_{i \leq j, i, j \in \mathbb{N}}$ be a cofiltered inverse system of topological spaces. Let

$$X = \varprojlim_i X_i$$

be the inverse limit in the category of topological spaces, and denote by $p_i: X \rightarrow X_i$ the canonical projections. Then for every subset $S \subseteq X$ and every $x \in X$, the following are equivalent:

- (1) $x \in \overline{S}$;
- (2) $p_i(x) \in \overline{p_i(S)}$ for every $i \in \mathbb{N}$.

Proof Since each projection $p_i: X \rightarrow X_i$ is continuous, we have $p_i(\overline{S}) \subseteq \overline{p_i(S)}$. Therefore,

$$\overline{S} \subseteq \bigcap_{i \in \mathbb{N}} p_i^{-1}(\overline{p_i(S)}).$$

Conversely, let

$$x \in \bigcap_{i \in \mathbb{N}} p_i^{-1}(\overline{p_i(S)}).$$

We show that $x \in \overline{S}$. Let $U \subseteq X$ be an open neighborhood of x . By the definition of the inverse limit topology, there exist indices

$$i_1, \dots, i_m \in \mathbb{N}$$

and open neighborhoods $U_{i_k} \subseteq X_{i_k}$ of $p_{i_k}(x)$ such that

$$x \in \bigcap_{k=1}^m p_{i_k}^{-1}(U_{i_k}) \subseteq U.$$

Let

$$N = \max\{i_1, \dots, i_m\}.$$

For each k , we have

$$p_{i_k} = \varphi_{i_k N} \circ p_N.$$

Define the open neighborhoods

$$V_N = \bigcap_{k=1}^m \varphi_{i_k N}^{-1}(U_{i_k}) \subseteq X_N$$

of $p_N(x)$. Since $p_N(x) \in \overline{p_N(S)}$, there exists $s \in S$ such that $p_N(s) \in V_N$.

It follows that for every k ,

$$p_{i_k}(s) = \varphi_{i_k N}(p_N(s)) \in U_{i_k}.$$

Hence,

$$s \in \bigcap_{k=1}^m p_{i_k}^{-1}(U_{i_k}) \subseteq U.$$

Thus $U \cap S \neq \emptyset$. Since U was arbitrary, $x \in \overline{S}$. \square

Lemma 1.1.6 *Let $(Y_i, \pi_{ji})_{i \in I}$ be a cofiltered inverse system of topological spaces, and let Y be the inverse limit of the system. Let X be a topological space, and suppose that $f_i: X \rightarrow Y_i$ is a compatible family of continuous maps, inducing $f: X \rightarrow Y$. Assume that for every $i \in I$, the image $f_i(X)$ is dense in Y_i . Then $f(X)$ is dense in Y .*

Proof It suffices to show that every nonempty basic open subset of Y meets $f(X)$. A basic open subset of Y is of the form

$$Y \cap \bigcap_{k=1}^n p_{i_k}^{-1}(U_k),$$

where $i_1, \dots, i_n \in I$, each $U_k \subseteq Y_{i_k}$ is open, and $p_i: Y \rightarrow Y_i$ denotes the projection.

Since I is filtered, choose $j \in I$ with $j \geq i_k$ for all k . Then

$$Y \cap \bigcap_{k=1}^n p_{i_k}^{-1}(U_k) = p_j^{-1}(V),$$

where

$$V := \bigcap_{k=1}^n \pi_{ji_k}^{-1}(U_k) \subseteq Y_j.$$

If the original basic open set is nonempty, then V is a nonempty open subset of Y_j . Since $f_j(X)$ is dense in Y_j , there exists $x \in X$ such that $f_j(x) \in V$. By compatibility, this implies $f_{i_k}(x) \in U_k$ for all $k = 1, \dots, n$. Hence $f(x)$ lies in the given basic open subset of Y . Therefore $f(X)$ is dense in Y . \square

Lemma 1.1.7 *Let $(X_i)_{i \in I}$ be a cofiltered inverse system of topological spaces, and let*

$$X = \varprojlim_{i \in I} X_i$$

with projections $p_i: X \rightarrow X_i$. Let $Y \subseteq X$ be a closed subset, and set

$$Y_i := p_i(Y) \subseteq X_i$$

with the subspace topology. Then the natural map

$$Y \rightarrow \varprojlim_{i \in I} Y_i$$

is a homeomorphism.

Proof The transition maps of the system (X_i) send Y_j into Y_i , so the Y_i form an inverse system. The projections $p_i|_Y: Y \rightarrow Y_i$ therefore induce a continuous map

$$\Phi: Y \rightarrow \varprojlim_i Y_i.$$

The map Φ is injective, since the projections $p_i: X \rightarrow X_i$ separate points of the inverse limit X .

We prove surjectivity. Let

$$(y_i)_{i \in I} \in \varprojlim_i Y_i.$$

Since each Y_i is a subset of X_i , the compatible family (y_i) also defines a point

$$x \in \varprojlim_i X_i = X.$$

We claim that $x \in Y$.

Let $W \subseteq X$ be an open neighborhood of x . Since the topology on X is the inverse limit topology, after shrinking W we may assume that

$$W = \bigcap_{\nu=1}^n p_{i_\nu}^{-1}(V_\nu),$$

where $V_\nu \subseteq X_{i_\nu}$ is open and $y_{i_\nu} \in V_\nu$.

Because I is cofiltered, there exists an index k mapping to all i_ν . Since $y_k \in Y_k = p_k(Y)$, there exists a point $z \in Y$ such that $p_k(z) = y_k$.

By compatibility of the family (y_i) , it follows that $p_{i_\nu}(z) = y_{i_\nu}$ for all ν . Hence $z \in W \cap Y$. Thus every open neighborhood of x meets Y , so $x \in \bar{Y}$. Since Y is closed, $x \in Y$. Therefore Φ is surjective.

It remains to check that Φ is a homeomorphism. The topology on Y is the initial topology with respect to the maps

$$p_i|_Y: Y \rightarrow Y_i.$$

The topology on $\varprojlim_i Y_i$ is the initial topology with respect to its projections

$$q_i: \varprojlim_i Y_i \rightarrow Y_i.$$

Under the bijection Φ , these maps agree:

$$q_i \circ \Phi = p_i|_Y.$$

Therefore Φ and Φ^{-1} are both continuous. Hence Φ is a homeomorphism. \square

Lemma 1.1.8 *Let $(X_i)_{i \in I}$ be a family of topological spaces, and let*

$$X = \prod_{i \in I} X_i$$

be endowed with the product topology. For $Y \subset X$, one has

$$\bar{Y} = \bigcap_{\substack{J \subset I \\ J \text{ finite}}} p_J^{-1}(\overline{p_J(Y)}),$$

where

$$p_J : X \rightarrow \prod_{j \in J} X_j$$

is the natural projection.

Proof A basis of neighborhoods of a point $x = (x_i)_{i \in I} \in X$ is given by sets of the form

$$\prod_{j \in J} U_j \times \prod_{i \notin J} X_i,$$

where $J \subset I$ is finite and each $U_j \subset X_j$ is an open neighborhood of x_j .

Thus $x \in \bar{Y}$ if and only if every such basic neighborhood meets Y . This is equivalent to saying that, for every finite $J \subset I$, every neighborhood of $p_J(x)$ in $\prod_{j \in J} X_j$ meets $p_J(Y)$. In other words,

$$p_J(x) \in \overline{p_J(Y)}$$

for every finite $J \subset I$. This gives the claimed formula. \square

Lemma 1.1.9 *Let*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C' \longrightarrow 0 \end{array}$$

be a commutative diagram of Fréchet spaces and continuous linear maps, with exact rows. Suppose that $\alpha(A)$ is dense in A' and $\gamma(C)$ is dense in C' . Then $\beta(B)$ is dense in B' .

Proof Let $b' \in B'$ and let $V \subseteq B'$ be an open neighborhood of b' . We must show that $V \cap \beta(B) \neq \emptyset$.

Since $p': B' \rightarrow C'$ is a continuous surjective linear map between Fréchet spaces, it is open by the open mapping theorem. Hence $p'(V)$ is an open neighborhood of $p'(b')$ in C' . Since $\gamma(C)$ is dense in C' , there exists $c \in C$ such that $\gamma(c) \in p'(V)$. Choose $b \in B$ with $p(b) = c$. Then

$$p'(\beta(b)) = \gamma(p(b)) = \gamma(c) \in p'(V).$$

Thus there exists $v \in V$ such that

$$p'(v) = p'(\beta(b)).$$

Therefore

$$v - \beta(b) \in \ker p' = \operatorname{Im} i'.$$

Thus there exists $a' \in A'$ with

$$i'(a') = v - \beta(b).$$

Since $\alpha(A)$ is dense in A' and i' is continuous, we can choose $a \in A$ such that

$$\beta(b + i(a)) = \beta(b) + i'(\alpha(a))$$

is arbitrarily close to v . In particular, choosing a sufficiently well, we get

$$\beta(b + i(a)) \in V.$$

Thus $V \cap \beta(B) \neq \emptyset$. Hence $\beta(B)$ is dense in B' . □

Lemma 1.1.10 *Let K be a compact Hausdorff space. Let $(\mathcal{F}_i)_{i \in I}$ be a filtered inductive system of sheaves of abelian groups on K . Then, for every $p \geq 0$, the canonical morphism*

$$\varinjlim_i H^p(K, \mathcal{F}_i) \longrightarrow H^p\left(K, \varinjlim_i \mathcal{F}_i\right)$$

is an isomorphism.

Proof Recall that there is a site LC_{qc} defined in [Stacks, Tag 09WY]. Let

$$a_K: \mathrm{Sh}(\mathrm{LC}_{qc}/K) \longrightarrow \mathrm{Sh}(K)$$

be the comparison morphism from the site LC_{qc}/K to the usual topological space K . By the comparison theorem for cohomology on a Hausdorff locally quasi-compact space, we have, for every sheaf \mathcal{G} of abelian groups on K ,

$$H^p(K, \mathcal{G}) \cong H_{qc}^p(K, a_K^{-1}\mathcal{G}),$$

see [Stacks, Tag 09WY].

Apply this first to each \mathcal{F}_i and then to $\varinjlim_i \mathcal{F}_i$. We get

$$\varinjlim H^p(K, \mathcal{F}_i) \cong \varinjlim H_{qc}^p(K, a_K^{-1}\mathcal{F}_i).$$

Since K is compact Hausdorff, it is an object of LC_{qc} and is quasi-compact. On the site LC_{qc}/K , the object K admits a cofinal system of finite qc-coverings by quasi-compact objects, and finite fibre products of quasi-compact objects are again quasi-compact. Hence the hypotheses of the filtered-colimit criterion for cohomology are satisfied. Therefore

$$\varinjlim H_{qc}^p(K, a_K^{-1}\mathcal{F}_i) \cong H_{qc}^p\left(K, \varinjlim a_K^{-1}\mathcal{F}_i\right),$$

by [Stacks, Tag 0737].

Finally, since inverse image functors commute with colimits, we have

$$\varinjlim a_K^{-1}\mathcal{F}_i \cong a_K^{-1}\left(\varinjlim \mathcal{F}_i\right).$$

Hence

$$\begin{aligned} \varinjlim H^p(K, \mathcal{F}_i) &\cong H_{qc}^p\left(K, a_K^{-1}\left(\varinjlim \mathcal{F}_i\right)\right) \\ &\cong H^p\left(K, \varinjlim \mathcal{F}_i\right), \end{aligned}$$

where the last isomorphism again follows from [Stacks, Tag 09WY]. This proves the claim. \square

1.2 Commutative algebra

Unless otherwise specified, all rings are commutative and unital.

Lemma 1.2.1 *Let A be a ring, let $\mathfrak{m} \subseteq A$ be a maximal ideal, and let $n > 0$ be an integer. Then \mathfrak{m}^n is an \mathfrak{m} -primary ideal.*

Proof First observe that

$$\sqrt{\mathfrak{m}^n} = \mathfrak{m}.$$

Indeed, since $\mathfrak{m}^n \subseteq \mathfrak{m}$, we have

$$\sqrt{\mathfrak{m}^n} \subseteq \sqrt{\mathfrak{m}} = \mathfrak{m}.$$

Conversely, if $x \in \mathfrak{m}$, then $x^n \in \mathfrak{m}^n$, hence

$$x \in \sqrt{\mathfrak{m}^n}.$$

Thus $\sqrt{\mathfrak{m}^n} = \mathfrak{m}$.

It remains to check that \mathfrak{m}^n is primary. Suppose that

$$ab \in \mathfrak{m}^n, \quad a \notin \mathfrak{m}^n.$$

We claim that $b \in \mathfrak{m}$. Indeed, if $b \notin \mathfrak{m}$, then the image of b in the field A/\mathfrak{m} is nonzero, hence invertible. Therefore there exists $c \in A$ such that

$$bc = 1 + u$$

for some $u \in \mathfrak{m}$. In A/\mathfrak{m}^n , the element $1 + u$ is invertible, with inverse

$$1 - u + u^2 - \cdots + (-u)^{n-1},$$

because $u^n \in \mathfrak{m}^n$. Hence b is invertible in A/\mathfrak{m}^n .

Since $ab \in \mathfrak{m}^n$, this implies

$$a \in \mathfrak{m}^n,$$

contradicting the assumption. Therefore $b \in \mathfrak{m}$. Consequently,

$$b^n \in \mathfrak{m}^n.$$

This proves that \mathfrak{m}^n is primary. □

Theorem 1.2.1 (Cohen) *A ring R is noetherian if and only if any prime ideal in R is finitely generated.*

See [Stacks, Tag 05KG].

Definition 1.2.1 A ring R is *stalkwise noetherian* if the localization of R at any prime ideal is noetherian.

Equivalently, it suffices to check the localization at all maximal ideals.

Remark 1.2.1 It is very tempting to call these rings locally noetherian, but $\text{Spec } R$ is not locally noetherian in general. This terminology would then be very confusing.

Definition 1.2.2 A ring R is *stalkwise regular* if the localization of R at any prime ideal is regular.

Equivalently, it suffices to check the localization at all maximal ideals.

Recall that a ring R is regular if $\text{Spec } R$ is regular, equivalently if R is stalkwise regular and noetherian, cf. [Stacks, Tag 02IR].

Example 1.2.1 A stalkwise noetherian ring is not noetherian in general. For example, $R = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$.

Proposition 1.2.1 *Let $f: R \rightarrow S$ be a ring homomorphism of finite type. If R is stalkwise noetherian, then so is S . In particular, if I is an ideal in R and R is stalkwise noetherian, then so is R/I .*

Proof Let \mathfrak{p} be a prime ideal in S . Write $\mathfrak{q} = f^{-1}(\mathfrak{p})$. Then by localization, we get a ring homomorphism

$$R_{\mathfrak{q}} \rightarrow S_{\mathfrak{p}}$$

of finite type. It follows that $S_{\mathfrak{p}}$ is noetherian. \square

Proposition 1.2.2 *Let R be a ring and $I \subseteq R$ a finitely generated nilpotent ideal. If R/I is stalkwise noetherian, then R is stalkwise noetherian.*

Proof Let $\mathfrak{p} \subseteq R$ be a prime ideal. Since I is nilpotent, we have $I \subseteq \mathfrak{p}$. Hence $I_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}}$, and there is a canonical isomorphism

$$(R/I)_{\mathfrak{p}/I} \cong R_{\mathfrak{p}}/I_{\mathfrak{p}}.$$

Because R/I is stalkwise noetherian, the ring $(R/I)_{\mathfrak{p}/I}$ is noetherian. Therefore, $R_{\mathfrak{p}}/I_{\mathfrak{p}}$ is a noetherian local ring. We wish to show that $R_{\mathfrak{p}}$ is noetherian. By [Theorem 1.2.1](#), it suffices to show that each prime ideal \mathfrak{q} in $R_{\mathfrak{p}}$ is finitely generated. But since $I_{\mathfrak{p}}$ is nilpotent, $\mathfrak{q} \supseteq I_{\mathfrak{p}}$. Then $\mathfrak{q}/I_{\mathfrak{p}}$ is finitely generated, as $R_{\mathfrak{p}}/I_{\mathfrak{p}}$ is noetherian. Since I is finitely generated, we conclude that \mathfrak{q} is finitely generated. \square

We say an R -module M is *torsion-free* if for any non-zero divisor $r \in R$ and any $m \in M$, if $rm = 0$, then $m = 0$. For a noetherian ring R and a finite R -module N , by [\[Stacks, Tag 00LD\]](#), N is torsion-free if and only if $\text{Ass}_R(N) \subseteq \text{Ass}_R(R)$.

Lemma 1.2.2 *Let A be a noetherian ring, and let M be a finite A -module. Then*

$$U := \{\mathfrak{p} \in \text{Spec } A : M_{\mathfrak{p}} \text{ is torsion-free over } A_{\mathfrak{p}}\}$$

is a Zariski open subset of $\text{Spec } A$.

Proof Note that

$$M_{\mathfrak{p}} \text{ is torsion-free over } A_{\mathfrak{p}} \iff \text{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \subseteq \text{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}),$$

By [\[Stacks, Tag 05BZ\]](#), for any $\mathfrak{p} \in \text{Spec } A$ we have

$$\text{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \{\mathfrak{q}A_{\mathfrak{p}} \mid \mathfrak{q} \in \text{Ass}_A(M), \mathfrak{q} \subseteq \mathfrak{p}\}, \quad \text{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \{\mathfrak{r}A_{\mathfrak{p}} \mid \mathfrak{r} \in \text{Ass}_A(A), \mathfrak{r} \subseteq \mathfrak{p}\}.$$

We conclude that

$$\text{Spec } A \setminus U = \bigcup_{\mathfrak{q} \in \text{Ass}_A(M) \setminus \text{Ass}_A(A)} V(\mathfrak{q}).$$

Since $\text{Ass}_A(M)$ is finite, this is a closed subset of $\text{Spec } A$. Therefore U is open. \square

Proposition 1.2.3 *Let X be a quasi-excellent scheme, and let \mathcal{F} be a coherent \mathcal{O}_X -module. Then the locus of $x \in X$ at which \mathcal{F}_x as an $\mathcal{O}_{X,x}$ -module satisfies Serre's condition (S_n) is open.*

This is a special case of [\[EGA IV₂, Proposition 6.11.6\]](#). More generally, the same holds for any CM-quasi-excellent scheme as studied in [\[Čes21\]](#).

Proposition 1.2.4 *Let $f: R \rightarrow R'$ be a flat local homomorphism between noetherian local rings. Assume that I is a proper ideal in R . Then*

$$\text{ht}(IR') = \text{ht}(I).$$

Recall that the height $\text{ht}(I)$ of I is the minimal height of the prime ideals containing I . Equivalently, it is the codimension of $V(I)$ in $\text{Spec } R$.

Proof Since f is flat and local, it is faithfully flat.

We first prove

$$\text{ht}(IR') \geq \text{ht}(I).$$

Let $\mathfrak{q} \subseteq R'$ be a prime ideal containing IR' , and put $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Then $\mathfrak{p} \supseteq I$. Since $R \rightarrow R'$ is flat, going-down holds. Therefore every chain of prime ideals in R ending at \mathfrak{p} lifts to a chain of prime ideals in R' ending at \mathfrak{q} . Hence

$$\text{ht}(\mathfrak{q}) \geq \text{ht}(\mathfrak{p}) \geq \text{ht}(I).$$

Taking the infimum over all $\mathfrak{q} \supseteq IR'$, we get the first inequality.

Conversely, choose a prime ideal $\mathfrak{p} \supseteq I$ such that

$$\text{ht}(\mathfrak{p}) = \text{ht}(I).$$

Since $R \rightarrow R'$ is faithfully flat, the ideal $\mathfrak{p}R'$ is proper. Choose a prime ideal $\mathfrak{q} \subseteq R'$ minimal over $\mathfrak{p}R'$. Then $\mathfrak{q} \supseteq IR'$. By the height inequality for flat morphisms at primes minimal over the extended ideal,

$$\text{ht}(\mathfrak{q}) \leq \text{ht}(\mathfrak{p}).$$

Therefore

$$\text{ht}(IR') \leq \text{ht}(\mathfrak{q}) \leq \text{ht}(\mathfrak{p}) = \text{ht}(I).$$

The two inequalities give the claim. \square

Proposition 1.2.5 *Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of nonzero finite-dimensional \mathbb{C} -algebras. Then every \mathbb{C} -algebra homomorphism*

$$\chi: \prod_{n \in \mathbb{N}} A_n \longrightarrow \mathbb{C}$$

factors through one of the projections $\prod_{n \in \mathbb{N}} A_n \rightarrow A_{n_0}$.

Proof Write $A = \prod_{n \in \mathbb{N}} A_n$. For each subset $S \subseteq \mathbb{N}$, define an idempotent

$$e_S = (\varepsilon_n)_{n \in \mathbb{N}} \in A$$

by

$$\varepsilon_n = \begin{cases} 1_{A_n}, & n \in S, \\ 0, & n \notin S. \end{cases}$$

Since $e_S^2 = e_S$, we have $\chi(e_S)^2 = \chi(e_S)$ and therefore $\chi(e_S) \in \{0, 1\}$. Therefore

$$\mathcal{U}_\chi := \{S \subseteq \mathbb{N} : \chi(e_S) = 1\}$$

is an ultrafilter on \mathbb{N} .

We now show that this ultrafilter is principal. Consider the central element

$$t := (1, 2, \dots) \in A.$$

Put $c := \chi(t) \in \mathbb{C}$. Then $\chi(t - c) = 0$. Let

$$Z_c := \{n \in \mathbb{N} : n = c\}.$$

For $n \notin Z_c$, the element $(n - c)1_{A_n} \in A_n$ is invertible. Hence there exists an element $g = (g_n)_{n \in \mathbb{N}} \in A$ such that

$$(t - c)g = e_{\mathbb{N} \setminus Z_c}.$$

Indeed, one can take

$$g_n = \begin{cases} (n - c)^{-1}1_{A_n}, & n \notin Z_c, \\ 0, & n \in Z_c. \end{cases}$$

Applying χ , we obtain

$$\chi(e_{\mathbb{N} \setminus Z_c}) = \chi(t - c)\chi(g) = 0.$$

Since

$$e_{Z_c} + e_{\mathbb{N} \setminus Z_c} = 1,$$

it follows that $\chi(e_{Z_c}) = 1$. Hence, $Z_c \in \mathcal{U}_\chi$.

But Z_c is either empty, if $c \notin \mathbb{N}$, or a singleton $\{n_0\}$, if $c = n_0 \in \mathbb{N}$. Since an ultrafilter cannot contain the empty set, we must have $c = n_0 \in \mathbb{N}$. Therefore $\{n_0\} \in \mathcal{U}_\chi$. Hence \mathcal{U}_χ is the principal ultrafilter at n_0 .

In particular,

$$\chi(e_{\mathbb{N} \setminus \{n_0\}}) = 0, \quad \chi(e_{\{n_0\}}) = 1.$$

Now let $a = (a_n)_{n \in \mathbb{N}} \in A$. We can decompose

$$a = e_{\{n_0\}}a + e_{\mathbb{N} \setminus \{n_0\}}a.$$

Applying χ , we get

$$\chi(a) = \chi(e_{\{n_0\}}a) + \chi(e_{\mathbb{N} \setminus \{n_0\}}a).$$

Since

$$\chi(e_{\mathbb{N} \setminus \{n_0\}}a) = \chi(e_{\mathbb{N} \setminus \{n_0\}})\chi(a) = 0,$$

we have

$$\chi(a) = \chi(e_{\{n_0\}}a).$$

Thus $\chi(a)$ depends only on the n_0 -th component a_{n_0} . □

For the notion of derivations, we refer to [Stacks, Tag 00RN].

Definition 1.2.3 Let R be a ring, and let S be an R -algebra. An R -derivation $S \rightarrow M$ is

- (1) *finite* if M is finite as an S -module;
- (2) *universally finite* if it is finite, and for any finite R -derivation $S \rightarrow N$, there is a unique S -module homomorphism $M \rightarrow N$ making the following diagram commutative:

$$\begin{array}{ccc} S & \longrightarrow & M \\ & \searrow & \downarrow \\ & & N. \end{array}$$

Lemma 1.2.3 Let R be a ring, let S be an R -algebra, and let $d: S \rightarrow M$ be a universally finite R -derivation. Then the S -submodule of M generated by $d(S)$ is just M .

Proof Let N be the S -submodule of M generated by $d(S)$. Since M is finite, M/N is a finite S -module. Let

$$\pi: M \rightarrow M/N$$

be the quotient map. Then

$$\pi \circ d = 0.$$

Hence both π and the zero map $M \rightarrow M/N$ make the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{d} & M \\ & \searrow & \downarrow \\ & & M/N. \end{array}$$

By the uniqueness in the universal property of d , we get $\pi = 0$. Thus $M = N$. \square

Definition 1.2.4 Let R be a ring. An R -module M is called *prefinite* if for every $x \in M \setminus \{0\}$, there exists a finite R -module N and an R -linear map $\varphi: M \rightarrow N$ such that

$$\varphi(x) \neq 0.$$

Proposition 1.2.6 Let R be a ring, let S be an R -algebra, and let $d: S \rightarrow M$ be a universally finite R -derivation. Then for any R -derivation $\delta: S \rightarrow N$ with N a prefinite S -module, there is a unique S -module homomorphism $M \rightarrow N$ making the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{d} & M \\ & \searrow & \downarrow \\ & & N. \end{array} \tag{1.2}$$

Proof The uniqueness of the map $M \rightarrow N$ follows immediately from [Lemma 1.2.3](#). It remains to prove its existence.

Since N is prefinite, the natural map

$$\iota: N \longrightarrow \prod_{\lambda \in \Lambda} N_\lambda$$

obtained by taking all homomorphisms from N to finite S -modules is injective. Hence, the composite

$$\iota \circ \delta: S \rightarrow \prod_{\lambda \in \Lambda} N_\lambda$$

is an R -derivation.

Because d is universally finite, for each $\lambda \in \Lambda$, we can find a unique S -module homomorphism $M \rightarrow N_\lambda$ making the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{d} & M \\ & \searrow & \downarrow \\ & & N_\lambda. \end{array}$$

Therefore, there exists an S -linear map

$$h: M \rightarrow \prod_{\lambda \in \Lambda} N_\lambda$$

such that the following diagram is commutative:

$$\begin{array}{ccc} S & \xrightarrow{d} & M \\ & \searrow \iota \circ \delta & \downarrow h \\ & & \prod_{\lambda \in \Lambda} N_\lambda. \end{array} \tag{1.3}$$

Observe that

$$h(M) \subseteq \iota(N).$$

In fact, by [Lemma 1.2.3](#), it suffices to verify that if $s \in S$, then $h(d(s)) \in \iota(N)$. But this follows immediately from (1.3).

Thus, h factors uniquely through ι , and the resulting map $M \rightarrow N$ is exactly the desired homomorphism in (1.2). \square

Theorem 1.2.2 *Let R be a ring. Let S be a noetherian R -algebra and let $d: S \rightarrow M$ be a finite R -derivation. Then the following are equivalent:*

- (1) d is universally finite;
- (2) for every maximal ideal $\mathfrak{m} \subseteq S$, the localized R -derivation

$$d_{\mathfrak{m}}: S_{\mathfrak{m}} \rightarrow M_{\mathfrak{m}}$$

is universally finite.

Proof (1) \implies (2): Assume (1). Fix a maximal ideal $\mathfrak{m} \subseteq S$. We will show that $d_{\mathfrak{m}}$ is universally finite.

Let $\delta: S_{\mathfrak{m}} \rightarrow N$ be a finite R -derivation, where N is a finite $S_{\mathfrak{m}}$ -module. We shall construct an S -module homomorphism $M \rightarrow N$ making the diagram commutative:

$$\begin{array}{ccc} S_{\mathfrak{m}} & \xrightarrow{d_{\mathfrak{m}}} & M_{\mathfrak{m}} \\ & \searrow \delta & \vdots \\ & & N. \end{array} \quad (1.4)$$

Since $S_{\mathfrak{m}}$ is local and N is finite over $S_{\mathfrak{m}}$, the natural map

$$N \rightarrow \prod_{i=1}^{\infty} N/\mathfrak{m}^i N$$

is injective by Krull's intersection theorem. Each quotient $N/\mathfrak{m}^i N$ is a finite $S_{\mathfrak{m}}/\mathfrak{m}^i S_{\mathfrak{m}} \cong S/\mathfrak{m}^i$ -module, hence in particular a finite S -module. Therefore N , viewed as an S -module, is prefinite. By [Proposition 1.2.6](#), we can find a unique S -module homomorphism $h: M \rightarrow N$ making the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{d} & M \\ \downarrow & & \vdots \\ S_{\mathfrak{m}} & \xrightarrow{\delta} & N. \end{array} \quad (1.5)$$

Localizing at \mathfrak{m} , we obtain an $S_{\mathfrak{m}}$ -linear map

$$h_{\mathfrak{m}}: M_{\mathfrak{m}} \rightarrow N,$$

which is the desired map in (1.4). Note that any dotted map in (1.4) induces a map $M \rightarrow N$ making the diagram (1.5) commutative, and is therefore unique.

(2) \implies (1): Assume (2).

First, observe that the S -submodule generated by $d(S)$ is precisely M . This can be checked after localizing to each maximal ideal of S , in which case, the result follows from [Lemma 1.2.3](#).

Now let $\delta: S \rightarrow N$ be any finite R -derivation. We want to show that there is a unique S -linear map $h: M \rightarrow N$ making the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{d} & M \\ & \searrow \delta & \vdots \\ & & N. \end{array} \quad (1.6)$$

The uniqueness of h is immediate as $M = Sd(S)$. It suffices to show the existence.

Because $M = \text{Sd}(S)$, it is enough to show that whenever

$$\sum_{i=1}^r b_i d(a_i) = 0 \quad (a_i, b_i \in S),$$

one also has

$$\sum_{i=1}^r b_i \delta(a_i) = 0. \quad (1.7)$$

Assuming this,

$$h \left(\sum_{i=1}^r b_i d(a_i) \right) := \sum_{i=1}^r b_i \delta(a_i)$$

defines the arrow h in (1.6).

It remains to establish (1.7). Fix a maximal ideal $\mathfrak{m} \subseteq S$. After localizing, we have

$$\sum_{i=1}^r \frac{b_i}{1} d_{\mathfrak{m}} \left(\frac{a_i}{1} \right) = 0 \quad \text{in } M_{\mathfrak{m}}.$$

Since $d_{\mathfrak{m}}$ is universally finite, the localized derivation $\delta_{\mathfrak{m}}: S_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ factors through $d_{\mathfrak{m}}$. Hence,

$$\sum_{i=1}^r \frac{b_i}{1} \delta_{\mathfrak{m}} \left(\frac{a_i}{1} \right) = 0 \quad \text{in } N_{\mathfrak{m}},$$

that is,

$$\left(\sum_{i=1}^r b_i \delta(a_i) \right)_{\mathfrak{m}} = 0 \quad \text{in } N_{\mathfrak{m}}.$$

Therefore, (1.7) follows. \square

Theorem 1.2.3 (Scheja–Storch) *Let k be a field of characteristic 0, and let R be a noetherian k -algebra. If R admits a universally finite k -derivation, then R is excellent.*

See [SS72, Satz 8.10].

Proposition 1.2.7 *Let R be a ring, let (A, \mathfrak{m}) be a noetherian local R -algebra, and let $d: A \rightarrow M$ be a finite R -derivation. Let*

$$\widehat{d}: \widehat{A} \rightarrow \widehat{M}$$

be the \mathfrak{m} -adic completion of d . Then the following are equivalent:

- (1) d is universally finite;
- (2) \widehat{d} is universally finite.

Proof We first recall that every R -derivation

$$\delta: A \rightarrow N$$

with N a finite A -module is continuous for the \mathfrak{m} -adic topologies. Indeed, for every $n \geq 1$,

$$\delta(\mathfrak{m}^n) \subseteq \mathfrak{m}^{n-1}N.$$

Thus δ extends uniquely to an R -derivation

$$\widehat{\delta}: \widehat{A} \rightarrow \widehat{N}.$$

Assume first that d is universally finite. Let

$$\eta: \widehat{A} \rightarrow P$$

be a finite R -derivation, where P is a finite \widehat{A} -module. We regard P as an A -module. Then P is prefinite as an A -module: the natural map

$$P \longrightarrow \prod_{n \geq 1} P/\mathfrak{m}^n P$$

is injective by Krull's intersection theorem, and each quotient $P/\mathfrak{m}^n P$ is a finite A -module. Hence, by [Proposition 1.2.6](#), the restricted derivation

$$\eta|_A: A \rightarrow P$$

factors uniquely through an A -linear map

$$h: M \rightarrow P.$$

Since M is finite and P is complete and separated for the \mathfrak{m} -adic topology, h extends uniquely to an \widehat{A} -linear map

$$\widehat{h}: \widehat{M} \rightarrow P.$$

The two R -derivations $\widehat{h} \circ \widehat{d}$ and η from \widehat{A} to P agree on A , hence agree on \widehat{A} by continuity and separatedness of P . This proves the existence of the desired factorization. The uniqueness follows similarly: two \widehat{A} -linear maps $\widehat{M} \rightarrow P$ inducing η restrict to two A -linear maps $M \rightarrow P$ inducing $\eta|_A$, and these are equal by the uniqueness for d ; by separatedness, their completions are equal. Thus \widehat{d} is universally finite.

Conversely, assume that \widehat{d} is universally finite. We first prove that

$$M = A d(A).$$

Let $M_0 := A d(A)$ and $Q := M/M_0$. The composite derivation

$$A \xrightarrow{d} M \longrightarrow Q$$

is zero. Passing to completions, the composite

$$\widehat{A} \xrightarrow{\widehat{d}} \widehat{M} \longrightarrow \widehat{Q}$$

is still zero. Since \widehat{d} is universally finite, [Lemma 1.2.3](#) gives

$$\widehat{M} = \widehat{A} \widehat{d}(\widehat{A}).$$

It follows that the map $\widehat{M} \rightarrow \widehat{Q}$ is zero, hence $\widehat{Q} = 0$. Since $A \rightarrow \widehat{A}$ is faithfully flat and Q is finite, we get $Q = 0$. Therefore $M = M_0$.

Now let

$$\delta: A \rightarrow N$$

be a finite R -derivation, with N a finite A -module. Extending to completions gives

$$\widehat{\delta}: \widehat{A} \rightarrow \widehat{N}.$$

By the universal finiteness of \widehat{d} , there exists a unique \widehat{A} -linear map

$$\widehat{h}: \widehat{M} \rightarrow \widehat{N}$$

such that

$$\widehat{h} \circ \widehat{d} = \widehat{\delta}.$$

Since $M = A \, d(A)$, it is enough to define h on generators by

$$h \left(\sum_{i=1}^r a_i \, d(x_i) \right) := \sum_{i=1}^r a_i \, \delta(x_i), \quad a_i, x_i \in A,$$

and to check that this is well-defined.

Suppose

$$\sum_{i=1}^r a_i \, d(x_i) = 0 \quad \text{in } M.$$

After completion, we have

$$\sum_{i=1}^r a_i \, \widehat{d}(x_i) = 0 \quad \text{in } \widehat{M}.$$

Applying \widehat{h} , we obtain

$$\sum_{i=1}^r a_i \, \widehat{\delta}(x_i) = 0 \quad \text{in } \widehat{N}.$$

This is the image of

$$\sum_{i=1}^r a_i \, \delta(x_i) \in N$$

under the natural map $N \rightarrow \widehat{N}$. Since N is finite over the noetherian local ring A , this map is injective by Krull's intersection theorem. Hence

$$\sum_{i=1}^r a_i \delta(x_i) = 0 \quad \text{in } N.$$

Thus h is well-defined, A -linear, and satisfies $h \circ d = \delta$. The uniqueness follows from $M = A d(A)$. Therefore d is universally finite. \square

Definition 1.2.5 Given a subset S , we let

$$\mathbb{C}\{S\} = \mathbb{C}\{T_a : a \in S\}$$

be the \mathbb{C} -algebra consisting of all power series as follows:

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha T_{\mathbf{a}}^\alpha, \quad (1.8)$$

where $\mathbf{a} = (a_1, \dots, a_n) \in S^n$, $c_\alpha \in \mathbb{C}$, the series is assumed to be convergent, namely it has a positive radius of convergence¹.

Note that $\mathbb{C}\{S\}$ can be equivalently described as

$$\mathbb{C}\{S\} = \varinjlim_{T \subseteq S \text{ finite}} \mathbb{C}\{T\}, \quad (1.9)$$

in the category of \mathbb{C} -algebras and hence it is indeed a local \mathbb{C} -algebra.

We have the following obvious functoriality: Suppose that $f: S \rightarrow S'$ is a map between sets. Then we get a local \mathbb{C} -algebra homomorphism

$$\tilde{f}: \mathbb{C}\{S\} \rightarrow \mathbb{C}\{S'\}, \quad \sum_{\alpha \in \mathbb{N}^n} c_\alpha T_{\mathbf{a}}^\alpha \mapsto \sum_{\alpha \in \mathbb{N}^n} c_\alpha T_{f(\mathbf{a})}^\alpha. \quad (1.10)$$

Here $f(\mathbf{a})$ is the tuple with f applied to each component of \mathbf{a} .

1.3 Locally ringed spaces

Let $\mathcal{L}ocRing$ be the category of locally ringed spaces.

Definition 1.3.1 Given a ring A , we define the category $\mathcal{L}ocRing_{/A}$ of *locally A -ringed spaces* as the slice category $\mathcal{L}ocRing_{/Spec A}$.

In concrete terms, the objects of $\mathcal{L}ocRing_{/A}$ are morphisms of locally ringed spaces $X \rightarrow Spec A$, and a morphism between two such objects is a morphism of locally ringed spaces over $Spec A$.

In view of [Stacks, Tag 01I1], an object in $\mathcal{L}ocRing_{/A}$ can be equivalently described as a locally ringed space X together with a ring homomorphism $A \rightarrow$

¹ More precisely, we regard the T_{a_i} 's as different formal variables even when there are repetitions in the a_i 's.

$\Gamma(X, \mathcal{O}_X)$, and a morphism from X to Y is a morphism $X \rightarrow Y$ as locally ringed spaces such that the following diagram is commutative:

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \Gamma(X, \mathcal{O}_X). \end{array}$$

Definition 1.3.2 A *complex space* is a locally \mathbb{C} -ringed space (X, \mathcal{O}_X) such that for each $x \in X$, there is an open neighborhood U of x in X , a natural number $n \geq 0$, and a closed analytic subspace Y of Δ^n such that $(U, \mathcal{O}_X|_U)$ is isomorphic to (Y, \mathcal{O}_Y) as locally \mathbb{C} -ringed spaces.

Here Δ^n denotes the unit polydisc in \mathbb{C}^n . Note that a complex space is not necessarily Hausdorff or paracompact in our definition.

Let CompSpace denote the category of complex spaces, which is a full subcategory of $\text{LocRing}_{/\mathbb{C}}$.

Proposition 1.3.1 *Fiber products exist in $\text{LocRing}_{/A}$, and can be described as follows: suppose that X, Y, Z are locally A -ringed spaces and that $X \rightarrow Z$ and $Y \rightarrow Z$ are two morphisms. Then $X \times_Z Y$ in the category of $\text{LocRing}_{/A}$ is just $X \times_Z Y$ in the category LocRing endowed with the natural morphism to $\text{Spec } A$.*

Proof The existence of fiber products in LocRing is a well-known fact, see [Gil11, Section 3.2]. The remaining statements are just formal consequences.

For later use, let us recall the construction of the underlying set of $X \times_Z Y$. A point in $X \times_Z Y$ is a tuple (x, y, \mathfrak{p}) , where $x \in X$ and $y \in Y$ are points having the same image $z \in Z$, and

$$\mathfrak{p} \in \text{Spec}(\kappa(x) \otimes_{\kappa(z)} \kappa(y)).$$

Proposition 1.3.2 *Fiber products exist in the category CompSpace of complex spaces. More precisely, if $X \rightarrow Z, Y \rightarrow Z$ are morphisms of complex spaces, then the fiber product $X \times_Z Y$ exists in CompSpace , and its underlying topological space is canonically identified with the topological fiber product $|X| \times_{|Z|} |Y|$.*

See, for example, [Fis76, Corollary 0.32].

Corollary 1.3.1 *Let $f: X \rightarrow Y$ be an injective morphism of complex spaces. Then every base change of f in CompSpace is injective.*

Proof This follows immediately from **Proposition 1.3.2**. □

1.4 Locally convex topological vector spaces

In this section, a vector space is understood as a vector space over \mathbb{C} , not necessarily of finite dimension.

Definition 1.4.1 A *topological vector space* is a vector space V together with a topology on V such that

- (1) addition $V \times V \rightarrow V$ is continuous;
- (2) scalar multiplication $\mathbb{C} \times V \rightarrow V$ is continuous.

By abuse of notation, we denote the topological vector space by V , when the topology is clear from the context.

A topological vector space V is *locally convex* if $0 \in V$ admits a fundamental system of convex open neighborhoods.

A *morphism* between locally convex topological vector spaces is a continuous linear map. The category of locally convex topological vector spaces will be denoted by \mathcal{LCS} .

We remind the reader that a locally convex topological vector space is not necessarily Hausdorff. Given a locally convex topological vector space V , throughout the book, V^* will denote the space of continuous linear maps $V \rightarrow \mathbb{C}$, endowed with the weak topology. It is a Hausdorff locally convex topological vector space, see [SW99, § II.5].

Given a locally convex topological vector space V , the subspace topology on any linear subspace $W \subseteq V$ makes W a locally convex topological vector space. Similarly, for any linear subspace $Z \subseteq V$, the quotient V/Z with the quotient topology is again locally convex. If V is Hausdorff, then V/Z is Hausdorff if and only if Z is closed.

Given a family of locally convex topological vector spaces $\{V_i\}_{i \in I}$, the space $V = \bigoplus_{i \in I} V_i$ with the final topology induced by the family of maps $V_i \rightarrow V$ is again a locally convex topological vector space.

Theorem 1.4.1 *Small limits exist in \mathcal{LCS} . Moreover, the small limits commute with the forgetful functor to the category of topological spaces.*

Filtered colimits exist in \mathcal{LCS} , and commute with the forgetful functor to the category of vector spaces.

See [SW99, § II.5, II.6]. In particular, the underlying vector space of the (small) limit of a family of locally convex topological spaces is the same as the limit in the category of vector spaces. In general, filtered colimits do not commute with the forgetful functor to the category of topological spaces.

Theorem 1.4.2 *Let I be a directed set, and let $\{f_i: E_i \rightarrow F_i\}_{i \in I}$ be an inductive system of morphisms in \mathcal{LCS} . Assume that for each $i \in I$, the morphism f_i is an admissible epimorphism in the sense that f_i is surjective, and F_i carries the quotient topology induced by f_i . Then*

$$\lim_{\rightarrow i \in I} f_i: \lim_{\rightarrow i \in I} E_i \rightarrow \lim_{\rightarrow i \in I} F_i$$

is also an admissible epimorphism.

Proof The only non-trivial point is to show that $\lim_{\rightarrow i \in I} f_i$ is open.

Let G denote the vector space $\varinjlim_{i \in I} F_i$ with the quotient topology induced by $\varinjlim_{i \in I} f_i$. Then it suffices to show that the identity map

$$\varinjlim_{i \in I} F_i \rightarrow G$$

is continuous. Fix $i \in I$. We need to show that $F_i \rightarrow G$ is continuous. Since f_i is an admissible epimorphism, this reduces to showing that the composition

$$E_i \xrightarrow{f_i} F_i \rightarrow G$$

is continuous. But this composition can be equivalently written as

$$E_i \rightarrow \varinjlim_{i \in I} E_i \rightarrow G,$$

where both maps are continuous. □

Theorem 1.4.3 *Let I be a directed set. Let $(A_i)_{i \in I}$ be an inductive system in \mathcal{LCS} . Then the natural map*

$$\left(\varinjlim_{i \in I} A_i \right)^* \cong \varprojlim_{i \in I} A_i^*$$

is an isomorphism in \mathcal{LCS} .

See [SW99, § IV.4.5]. The Hausdorff assumption in [SW99] is not necessary for this theorem.

Theorem 1.4.4 *Let E, F be barrelled DF-spaces (e.g. DFS spaces or Fréchet spaces), and let G be a locally convex topological vector space. Then every separately continuous bilinear map*

$$E \times F \rightarrow G$$

is continuous.

See [Jar81, Proposition 15.6.7]. We refer to [Jar81] for the definitions of barrelled spaces, DF-spaces, and DFS spaces.

1.5 Topological algebras

Throughout the book, unless otherwise specified, algebras are assumed to be commutative, unital, and over \mathbb{C} .

Definition 1.5.1 A *topological (\mathbb{C} -)algebra* is a commutative \mathbb{C} -algebra A with a topology on the underlying \mathbb{C} -vector space, making A a locally convex topological vector space and such that the multiplication

$$A \times A \rightarrow A$$

is continuous.

A *morphism* $F: A \rightarrow B$ between topological \mathbb{C} -algebras is a \mathbb{C} -algebra homomorphism $F: A \rightarrow B$ which is continuous.

Let $\mathcal{T}\text{opAlg}$ be the category of topological \mathbb{C} -algebras.

Definition 1.5.2 Let A be a topological algebra. A *topological A -module* is an A -module M with a topology on the underlying \mathbb{C} -vector space, making M a locally convex topological vector space and such that the action map

$$A \times M \rightarrow M$$

is continuous.

A *morphism* $f: M \rightarrow N$ between two topological A -modules is an A -linear map which is continuous.

1.5.1 The Gelfand dual

Definition 1.5.3 Let A be a topological algebra. We define the *spectrum* or the *Gelfand dual* $\text{Sp } A$ of A as the following topological space:

- (1) The underlying set of $\text{Sp } A$ consists of all continuous characters $A \rightarrow \mathbb{C}$. Here a character refers to a \mathbb{C} -algebra homomorphism to \mathbb{C} ;
- (2) The topology (referred to as the *weak topology* or the *Gelfand topology*) on $\text{Sp } A$ is the weakest one so that for each $a \in A$, its *Gelfand transform*

$$\hat{a}: \text{Sp } A \rightarrow \mathbb{C}, \quad f \mapsto f(a)$$

is continuous.

Note that $\text{Sp } A$ is automatically Hausdorff. For each $a \in \text{Sp } A$, we write

$$\mathfrak{m}(a) := \ker A \xrightarrow{a} \mathbb{C}, \tag{1.11}$$

which is automatically a closed maximal ideal in A .

As a consequence, we obtain an algebra homomorphism

$$\text{Gel}: A \rightarrow C^0(\text{Sp } A), \quad a \mapsto \hat{a}. \tag{1.12}$$

We call this map the *Gelfand map*.

Remark 1.5.1 When A is a Banach algebra, $\text{Sp } A$ is the usual Gelfand spectrum of A . To see this, it suffices to recall that all characters of a commutative Banach algebra are automatically continuous.

We can enhance Sp to a contravariant functor from the category of topological algebras to that of topological spaces. We shall consider a slightly more general functoriality.

Definition 1.5.4 Let A, B be topological algebras. A \mathbb{C} -algebra homomorphism $F: A \rightarrow B$ is *spectral* if for each $y \in \text{Sp } B$, the composition $\text{Sp } F(y) = F^*y := y \circ F$ is continuous.

In this case, $\text{Sp } F: \text{Sp } B \rightarrow \text{Sp } A$ is well-defined, and if $y \in \text{Sp } B$, and $x = \text{Sp } F(y)$, then

$$F(\mathfrak{m}(x)) \subseteq \mathfrak{m}(y). \quad (1.13)$$

Note that a continuous homomorphism is spectral. The composition of spectral homomorphisms is spectral.

Definition 1.5.5 The category $\mathcal{T}\text{op}\mathcal{A}\text{lg}^s$ is the following category:

- (1) The objects are topological \mathbb{C} -algebras;
- (2) the morphisms are spectral homomorphisms;
- (3) the compositions are the usual compositions.

Then $\mathcal{T}\text{op}\mathcal{A}\text{lg}$ is a subcategory of $\mathcal{T}\text{op}\mathcal{A}\text{lg}^s$.

Definition 1.5.6 Let $\alpha: A \rightarrow B$ be a spectral homomorphism of topological algebras. We define

$$\text{Sp } \alpha: \text{Sp } B \rightarrow \text{Sp } A$$

as follows: Given a continuous character $f \in \text{Sp } B$, we define $\text{Sp } f$ as the continuous character given by the composition:

$$A \xrightarrow{\alpha} B \xrightarrow{f} \mathbb{C}.$$

Note that $\text{Sp } \alpha$ is automatically continuous. In fact, by the definition of the topology on $\text{Sp } A$, it suffices to show that for each $a \in A$, the composition

$$\text{Sp } B \xrightarrow{\text{Sp } \alpha} \text{Sp } A \xrightarrow{\hat{a}} \mathbb{C}$$

is continuous. But this composition is nothing but $\widehat{\alpha(a)}$.

The Gelfand map is also functorial:

Proposition 1.5.1 Let $\alpha: A \rightarrow B$ be a spectral homomorphism of topological algebras. Then we have a commutative diagram in the category of topological spaces:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow \text{Gel} & & \downarrow \text{Gel} \\ C^0(\text{Sp } A) & \xrightarrow{(\text{Sp } \alpha)^*} & C^0(\text{Sp } B). \end{array}$$

Proof This follows immediately by unwinding the definitions. □

1.5.2 The local completions

Definition 1.5.7 Let A be a topological algebra. For any $x \in \operatorname{Sp} A$, we define

$$\widehat{A}_x := \widehat{A_{\mathfrak{m}(x)}} = \varprojlim_{n>0} A/\mathfrak{m}(x)^n,$$

where $\mathfrak{m}(x)$ is the closed maximal ideal of A introduced in (1.11).

Given a spectral homomorphism of topological algebras $F: A \rightarrow B$, let $y \in \operatorname{Sp} B$ and set $x = \operatorname{Sp} F(y)$. Then by (1.13), F induces a functorial local homomorphism of \mathbb{C} -algebras:

$$\widehat{A}_x \rightarrow \widehat{B}_y. \quad (1.14)$$

Since A and $A_{\mathfrak{m}(x)}$ need not be noetherian in general, the "completion" \widehat{A}_x can behave rather wildly, cf. [Stacks, Tag 05JA].

We shall use the algebra defined in Definition 1.2.5.

Definition 1.5.8 Let A be a topological algebra, and let $x \in \operatorname{Sp} A$. We define a \mathbb{C} -algebra homomorphism

$$\theta_x: \mathbb{C}\{\mathfrak{m}(x)\} \rightarrow \widehat{A}_x$$

as follows:

- (1) Take $m \geq 0$, $\mathbf{f} \in \mathfrak{m}(x)^m$ and $n > 0$, we define a \mathbb{C} -algebra homomorphism

$$\mathbb{C}\{T_{\mathbf{f}}\} \rightarrow A/\mathfrak{m}(x)^n, \quad \sum_{\alpha \in \mathbb{N}^m} c_{\alpha} T_{\mathbf{f}}^{\alpha} \mapsto \sum_{|\alpha| < n} c_{\alpha} \mathbf{f}^{\alpha} \pmod{\mathfrak{m}(x)^n}.$$

- (2) Take $m \geq 0$ and $\mathbf{f} \in \mathfrak{m}(x)^m$, and define a \mathbb{C} -algebra homomorphism

$$\mathbb{C}\{T_{\mathbf{f}}\} \rightarrow \widehat{A}_x$$

by patching together the homomorphisms in the previous step.

- (3) Define θ_x by patching together the homomorphisms in the previous step, using (1.9).

We define

$$I_x(A) := \ker \theta_x = \left\{ \sum_{\alpha} c_{\alpha} T_{\mathbf{f}}^{\alpha} : \sum_{|\alpha| < m} c_{\alpha} \mathbf{f}^{\alpha} \in \mathfrak{m}(x)^m \text{ for all } m > 0 \right\}. \quad (1.15)$$

The construction has the following functoriality: If $F: A \rightarrow B$ is a spectral homomorphism of topological algebras, $y \in \operatorname{Sp} B$ and $x = \operatorname{Sp} F(y)$, then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}\{\mathfrak{m}(x)\} & \xrightarrow{\theta_x} & \widehat{A}_x \\ \bar{F} \downarrow & & \downarrow \\ \mathbb{C}\{\mathfrak{m}(y)\} & \xrightarrow{\theta_y} & \widehat{B}_y, \end{array} \quad (1.16)$$

where \tilde{F} is defined in (1.10), the right vertical map is (1.14). In particular, \tilde{F} restricts to a linear map

$$\tilde{F}: I_x(A) \rightarrow I_y(B). \quad (1.17)$$

Therefore, \tilde{F} also induces a local homomorphism of \mathbb{C} -algebras:

$$\text{Im } \theta_x \rightarrow \text{Im } \theta_y. \quad (1.18)$$

1.5.3 The associated locally \mathbb{C} -ringed space

In this section, we shall enhance the spectrum studied in Section 1.5.1 to a locally \mathbb{C} -ringed space.

Theorem 1.5.1 *The functor Sp in Definition 1.5.6 can be canonically enhanced to a contravariant functor*

$$\text{Sp}: \mathcal{T}\text{opAlg}^s \rightarrow \mathcal{L}\text{ocRing}/\mathbb{C}. \quad (1.19)$$

By abuse of language, we also denote the composition as Sp :

$$\mathcal{T}\text{opAlg} \rightarrow \mathcal{T}\text{opAlg}^s \xrightarrow{\text{Sp}} \mathcal{L}\text{ocRing}/\mathbb{C}.$$

1.5.3.1 The objects

Given a topological algebra A , we shall construct a corresponding locally \mathbb{C} -ringed space $\text{Sp } A$ in this section.

The underlying topological space of $\text{Sp } A$ is already constructed in Definition 1.5.3.

In order to construct the structure sheaf, we shall use the ring of convergent power series introduced in Definition 1.2.5.

Definition 1.5.9 We define the presheaf of \mathbb{C} -algebras $\mathcal{E}(A)$ on $\text{Sp } A$ as follows:

- (1) For any open subset $U \subseteq \text{Sp } A$, we let $\Gamma(U, \mathcal{E}(A))$ be the set of all assignments $(\alpha_x)_{x \in U}$, with each $\alpha_x \in \mathbb{C}\{\mathfrak{m}(x)\}$ satisfying the following condition: For any $x \in U$, if we write

$$\alpha_x = \sum_{\alpha \in \mathbb{N}^n} c_\alpha T_{\mathbf{f}}^\alpha \quad (1.20)$$

for some $n \geq 0$ and $\mathbf{f} \in \mathfrak{m}(x)^n$, then there is $\epsilon > 0$ smaller than the radius of convergence of (1.20) such that any $y \in \text{Sp } A$ with $|y(f_i)| < \epsilon$ for all $i = 1, \dots, n$ lies in U and α_y is the series obtained by formally expanding

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha (y(\mathbf{f}) + T_{\mathbf{f}-y(\mathbf{f})})^\alpha.$$

More precisely,

$$\alpha_y = \sum_{\beta \in \mathbb{N}^n} \left(\sum_{\substack{\alpha \in \mathbb{N}^n, \\ \alpha \geq \beta}} c_\alpha \binom{\alpha}{\beta} y(\mathbf{f})^{\alpha-\beta} \right) T_{\mathbf{f}-y(\mathbf{f})}^\beta.$$

The \mathbb{C} -algebra structure on $\Gamma(U, \mathcal{E}(A))$ is induced by the fiberwise \mathbb{C} -algebra structure.

(2) Given two open subsets $V \subseteq U \subseteq \text{Sp } A$, the restriction is given by the map

$$(\alpha_x)_{x \in U} \mapsto (\alpha_x)_{x \in V}.$$

Since the conditions in (1) are all local, we see easily that $\mathcal{E}(A)$ is indeed a sheaf. Furthermore, for each $x \in \text{Sp } A$, we have a natural isomorphism

$$\mathcal{O}_{\mathcal{E}(A), x} \cong \mathbb{C}\{\mathfrak{m}(x)\} \quad (1.21)$$

of \mathbb{C} -algebras.

Definition 1.5.10 We define a sheaf of ideals $\mathcal{I}(A) \subseteq \mathcal{E}(A)$ by

$$\Gamma(U, \mathcal{I}(A)) = \{\alpha \in \Gamma(U, \mathcal{E}(A)) : \alpha_x \in I_x(A) \text{ for every } x \in U\}.$$

Since the condition is a fiberwise one, it clearly defines a subsheaf.

Finally, we get the following definition:

Definition 1.5.11 The structure sheaf on $\text{Sp } A$ is the sheaf of local \mathbb{C} -algebras defined by

$$\mathcal{O}_{\text{Sp } A} := \mathcal{E}(A)/\mathcal{I}(A).$$

Taking $x \in \text{Sp } A$, from the description (1.21) and **Definition 1.5.10**, we find

$$\mathcal{O}_{\text{Sp } A, x} \cong \mathbb{C}\{\mathfrak{m}(x)\}/I_x(A) = \text{Im } \theta_x. \quad (1.22)$$

In particular, $(\text{Sp } A, \mathcal{O}_{\text{Sp } A})$ is a locally \mathbb{C} -ringed space.

1.5.3.2 The morphisms

Suppose that $F: A \rightarrow B$ is a spectral homomorphism of topological algebras, see **Definition 1.5.4** for the definition.

We shall define a morphism $\text{Sp } F: \text{Sp } B \rightarrow \text{Sp } A$ in the category of locally \mathbb{C} -ringed spaces. The underlying continuous map has already been defined in **Definition 1.5.6**. It remains to construct a homomorphism

$$F^\sharp: (\text{Sp } F)^{-1} \mathcal{O}_{\text{Sp } A} \rightarrow \mathcal{O}_{\text{Sp } B} \quad (1.23)$$

of sheaves of \mathbb{C} -algebras which is fiberwise local.

We shall first construct a homomorphism

$$(\mathrm{Sp} F)^{-1} \mathcal{E}(A) \rightarrow \mathcal{E}(B). \quad (1.24)$$

For this purpose, we take open sets $U \subseteq \mathrm{Sp} B$ and $V \subseteq \mathrm{Sp} A$ with $V \supseteq \mathrm{Sp} F(U)$. Define

$$\Gamma(V, \mathcal{E}(A)) \rightarrow \Gamma(U, \mathcal{E}(B)), \quad (\alpha_x)_{x \in V} \mapsto (\tilde{F}(\alpha_{\mathrm{Sp} F(y)}))_{y \in U}, \quad (1.25)$$

where \tilde{F} is defined in (1.10). We need to verify that the image indeed lies in $\Gamma(U, \mathcal{E}(B))$. Take $y \in U$, and set $x = \mathrm{Sp} F(y)$. We expand α_x as in (1.20). Then

$$\tilde{F}(\alpha_x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha T_{F(\mathbf{f})}^\alpha.$$

Then we can take $\epsilon > 0$ small enough as in **Definition 1.5.9**. Then note that ϵ is less than the radius of convergence of $\tilde{F}(\alpha_x)$. Now take $z \in \mathrm{Sp} B$ with

$$|z(F(f_i))| < \epsilon \quad \text{for all } i = 1, \dots, n.$$

Then for $w = \mathrm{Sp} F(z)$, we have

$$|w(f_i)| < \epsilon \quad \text{for all } i = 1, \dots, n.$$

Hence,

$$\alpha_w = \sum_{\beta \in \mathbb{N}^n} \left(\sum_{\substack{\alpha \in \mathbb{N}^n, \\ \alpha \geq \beta}} c_\alpha \binom{\alpha}{\beta} w(\mathbf{f})^{\alpha - \beta} \right) T_{\mathbf{f} - w(\mathbf{f})}^\beta.$$

Applying \tilde{F} , we find

$$\tilde{F}(\alpha_w) = \sum_{\beta \in \mathbb{N}^n} \left(\sum_{\substack{\alpha \in \mathbb{N}^n, \\ \alpha \geq \beta}} c_\alpha \binom{\alpha}{\beta} z(F(\mathbf{f}))^{\alpha - \beta} \right) T_{F(\mathbf{f}) - z(F(\mathbf{f}))}^\beta.$$

In other words, (1.25) is indeed well-defined and is a \mathbb{C} -algebra homomorphism by construction.

The compatibilities of (1.25) with respect to the restrictions of V and U show that they patch together to give a homomorphism (1.24) of sheaves of rings. Take $y \in \mathrm{Sp} B$ and put $x = (\mathrm{Sp} F)(y)$. Then the fiberwise map (1.24) corresponds precisely to \tilde{F} in (1.16). Therefore, the relevant ideal subsheaf is respected fiberwise. Hence (1.24) descends to a homomorphism (1.23). Note that fiberwise F^\sharp is indeed local.

The functoriality of Sp is now obvious and **Theorem 1.5.1** is therefore proved.

1.5.4 Properties

Proposition 1.5.2 *Let A be a topological algebra. Then there is a canonical homomorphism of \mathbb{C} -algebras*

$$A \rightarrow \Gamma(\mathrm{Sp} A, \mathcal{O}_{\mathrm{Sp} A}). \quad (1.26)$$

Here *canonical* refers to the fact that if $A \rightarrow B$ is a spectral homomorphism of topological algebras, then the following diagram is commutative:

$$\begin{array}{ccc} A & \longrightarrow & \Gamma(\mathrm{Sp} A, \mathcal{O}_{\mathrm{Sp} A}) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \Gamma(\mathrm{Sp} B, \mathcal{O}_{\mathrm{Sp} B}). \end{array}$$

Proof For $a \in A$, define a section $s_a \in \Gamma(\mathrm{Sp} A, \mathcal{E}(A))$ by

$$(s_a)_x = x(a) + T_{a-x(a)} \in \mathbb{C}\{\mathfrak{m}(x)\}.$$

The defining local compatibility condition of $\mathcal{E}(A)$ is immediate from the identity

$$x(a) + T_{a-x(a)} = y(a) + T_{a-y(a)}$$

after Taylor expansion near y .

Let \bar{s}_a be the image of s_a in $\Gamma(\mathrm{Sp} A, \mathcal{O}_{\mathrm{Sp} A})$. We claim that

$$a \mapsto \bar{s}_a$$

is a \mathbb{C} -algebra homomorphism. This can be checked on stalks. For every $x \in \mathrm{Sp} A$, the composition

$$\mathbb{C}\{\mathfrak{m}(x)\} \longrightarrow \mathcal{O}_{\mathrm{Sp} A, x} \hookrightarrow \widehat{A}_x$$

sends $x(a) + T_{a-x(a)}$ to the image of a in \widehat{A}_x . Hence the differences

$$s_{a+b} - s_a - s_b, \quad s_{ab} - s_a s_b, \quad s_1 - 1$$

have all their stalks in $\mathcal{I}(A)$. Therefore their images in $\mathcal{O}_{\mathrm{Sp} A}$ vanish. This proves that

$$A \rightarrow \Gamma(\mathrm{Sp} A, \mathcal{O}_{\mathrm{Sp} A}), \quad a \mapsto \bar{s}_a$$

is a canonical homomorphism of \mathbb{C} -algebras. \square

Lemma 1.5.1 *Let $(A_i)_{i \in I}$ be a non-empty family of topological algebras. Let B be a Hausdorff topological algebra with no nontrivial idempotents (i.e. the only idempotents are 0 and 1). Then every continuous homomorphism $\varphi: \prod_{i \in I} A_i \rightarrow B$ factors through one of the coordinate projections.*

Proof We define

$$A := \prod_{i \in I} A_i.$$

For any subset $J \subseteq I$, let $e_J \in A$ be the idempotent defined by

$$(e_J)_i = \begin{cases} 1, & i \in J, \\ 0, & i \notin J. \end{cases} \quad (1.27)$$

Then $\varphi(e_J)$ is an idempotent of B . Hence $\varphi(e_J) \in \{0, 1\}$.

Since B is Hausdorff, choose a neighborhood V of 0 in B such that $1 \notin V$.

By continuity of φ at 0, there exists an open neighborhood $W = \prod_{i \in I} W_i$ of 0 in A such that $\varphi(W) \subseteq V$, where $W_i = A_i$ for all but finitely many i . Let

$$J := \{i \in I : W_i \neq A_i\}.$$

Then J is finite, and $e_{I \setminus J} \in W$. Therefore $\varphi(e_{I \setminus J}) \in V$. But $\varphi(e_{I \setminus J})$ is an idempotent of B , so it is either 0 or 1. Since $1 \notin V$, we get $\varphi(e_{I \setminus J}) = 0$. Hence

$$\varphi(e_J) = \varphi(1 - e_{I \setminus J}) = 1.$$

For any $a \in A$, write

$$a = e_J a + e_{I \setminus J} a.$$

Then

$$\varphi(e_{I \setminus J} a) = \varphi(e_{I \setminus J}) \varphi(a) = 0.$$

Thus

$$\varphi(a) = \varphi(e_J a),$$

so φ factors through the finite product

$$\prod_{j \in J} A_j.$$

Write $J = \{j_1, \dots, j_m\}$. The idempotents e_{j_1}, \dots, e_{j_m} are pairwise orthogonal and satisfy

$$e_{j_1} + \dots + e_{j_m} = e_J.$$

Applying φ , we get pairwise orthogonal idempotents

$$\varphi(e_{j_1}), \dots, \varphi(e_{j_m})$$

whose sum is $\varphi(e_J) = 1$.

Since B has no nontrivial idempotents, each $\varphi(e_{j_k})$ is either 0 or 1. Exactly one of them is 1; denote the corresponding index by i_0 . Then

$$\varphi(e_{i_0}) = 1, \quad \varphi(e_j) = 0 \quad \text{for } j \in J \setminus \{i_0\}.$$

Therefore, for every $a \in A$,

$$\varphi(a) = \varphi(e_{i_0}a).$$

Thus φ depends only on the i_0 -th coordinate.

Define

$$\varphi_{i_0} : A_{i_0} \rightarrow B, \quad \varphi_{i_0}(b) := \varphi(a),$$

where $a \in A$ is any element satisfying $a_{i_0} = b$. The preceding paragraph shows that this is well-defined, and then

$$\varphi = \varphi_{i_0} \circ \text{pr}_{i_0}.$$

The continuity of φ_{i_0} follows from the continuity of φ , because

$$\varphi_{i_0}(b) = \varphi(e_{i_0}a)$$

for any a with $a_{i_0} = b$, equivalently φ_{i_0} is the composition of the continuous inclusion of the i_0 -th factor into A with φ . \square

Proposition 1.5.3 *Small limits exist in $\mathcal{T}\text{opAlg}$ and commute with the forgetful functor to the category \mathcal{LCS} .*

Given a directed set I and an inductive system $(A_i)_{i \in I}$ in $\mathcal{T}\text{opAlg}$, let $A = \varinjlim_{i \in I} A_i$ denote the colimit in \mathcal{LCS} . If the multiplication $A \times A \rightarrow A$ is continuous, then A is also the colimit in $\mathcal{T}\text{opAlg}$.

The multiplication is automatically continuous if A is a DFS space, see [Theorem 1.4.4](#).

Proof This is a consequence of [Theorem 1.4.1](#) and the general category theory. \square

Proposition 1.5.4 *Let $(A_i)_{i \in I}$ be a family of topological algebras. Then the natural morphism of locally \mathbb{C} -ringed spaces*

$$\bigsqcup_{i \in I} \text{Sp } A_i \rightarrow \text{Sp } \prod_{i \in I} A_i \tag{1.28}$$

is an isomorphism.

The map (1.28) is just the collection of morphisms $\text{Sp } A_k \rightarrow \text{Sp } \prod_{i \in I} A_i$ defined by applying Sp to the projection map $\prod_{i \in I} A_i \rightarrow A_k$.

Proof By [Lemma 1.5.1](#), the underlying map of sets in (1.28) is a bijection.

Next, we shall prove that (1.28) is a homeomorphism. The map is continuous and bijective, so it suffices to show that it is open. Fix $k \in I$, it suffices to show that

$$\text{Sp } A_k \rightarrow \text{Sp } \prod_{i \in I} A_i$$

is open.

By definition, a basis of open subsets of the source is given as follows: Take finitely many elements $a_1, \dots, a_n \in A_k$, and open subsets $U_1, \dots, U_n \subseteq \mathbb{C}$, the basic open sets are just

$$\{x \in \mathrm{Sp} A_k : \widehat{a}_j(x) \in U_j \text{ for all } j\}.$$

The image of this open set with respect to (1.28) is just

$$\bigcap_{j=1}^n \left\{ x \in \mathrm{Sp} \prod_{i \in I} A_i : x(e_{\{k\}}) \in \mathbb{C} \setminus \{0\}, x(\alpha_j) \in U_j \right\},$$

where $\alpha_j \in \prod_i A_i$ is the element with the k -th component equal to a_j but all other components equal to 1, and $e_{\{k\}}$ is defined by (1.27). In fact, from the proof of **Lemma 1.5.1**, $x(e_{\{k\}}) \in \mathbb{C} \setminus \{0\}$ if and only if x factorizes through A_k .

It remains to identify the stalks. Let $x \in \mathrm{Sp} A_k$, and let y be its image in $\mathrm{Sp} \prod_i A_i$. Then

$$\mathfrak{m}(y) = \prod_{i \neq k} A_i \times \mathfrak{m}(x).$$

The projection

$$\prod_i A_i \rightarrow A_k$$

induces a homomorphism

$$\left(\prod_i A_i \right)_{\mathfrak{m}(y)} \longrightarrow \widehat{(A_k)_{\mathfrak{m}(x)}}.$$

This is an isomorphism. Indeed, if

$$e = e_{I \setminus \{k\}},$$

then $e \in \mathfrak{m}(y)$ and $e^r = e$ for every $r \geq 1$, so e maps to zero in the $\mathfrak{m}(y)$ -adic completion; hence all components away from k vanish in the completion. The remaining component is precisely the $\mathfrak{m}(x)$ -adic completion of A_k .

Using the description

$$\mathcal{O}_{\mathrm{Sp} \prod_i A_i, y} \cong \mathrm{Im} \left(\mathbb{C}\{\mathfrak{m}(y)\} \rightarrow \left(\prod_i A_i \right)_{\mathfrak{m}(y)} \right),$$

we get

$$\mathcal{O}_{\mathrm{Sp} \prod_i A_i, y} \cong \mathcal{O}_{\mathrm{Sp} A_k, x}.$$

Thus the morphism is an isomorphism of locally \mathbb{C} -ringed spaces. \square

Proposition 1.5.5 *Let $(A_i)_{i \in I}$ be an inductive system in the category of $\mathcal{T}\mathrm{op}\mathcal{A}\mathrm{lg}$ indexed by a directed set. Then the natural morphism of locally \mathbb{C} -ringed spaces*

$$\mathrm{Sp} \varinjlim_{i \in I} A_i \rightarrow \varprojlim_{i \in I} \mathrm{Sp} A_i \quad (1.29)$$

is a homeomorphism. Here $\varinjlim_{i \in I} A_i$ is taken in the category of \mathcal{TopAlg} .

The morphism is in general not an isomorphism of locally \mathbb{C} -ringed spaces.

Proof Recall that the topological space underlying $\varinjlim_{i \in I} \mathrm{Sp} A_i$ is just the cofiltered limit of the topological spaces $\mathrm{Sp} A_i$, as shown in [Gil1, Corollary 5, Theorem 4] and their proofs.

Let

$$A = \varinjlim_i A_i.$$

Then, by [Theorem 1.4.3](#),

$$A^* \cong \varprojlim_i A_i^*$$

as topological spaces.

Under this identification, a continuous linear functional $\chi \in A^*$ corresponds to a compatible family

$$(\chi_i)_{i \in I}, \quad \chi_i \in A_i^*.$$

The functional χ is a character of A if and only if each χ_i is a character of A_i . Indeed, every element of A comes from some A_i , and any two elements of A come from a common A_j because I is directed. Therefore the conditions

$$\chi(1) = 1, \quad \chi(ab) = \chi(a)\chi(b)$$

are equivalent to the corresponding conditions on all χ_i .

Hence

$$\mathrm{Sp} A \cong \varprojlim_i \mathrm{Sp} A_i$$

as subsets of

$$A^* \cong \varprojlim_i A_i^*.$$

The topologies agree because all spaces are endowed with the weak topologies. This proves the asserted homeomorphism. \square

1.5.5 Fréchet algebras

Definition 1.5.12 A topological algebra A is a *Fréchet algebra* if as a topological vector space A is a Fréchet space.

A *homomorphism* between Fréchet algebras is a homomorphism of the corresponding topological algebras.

Let \mathcal{FrAlg} be the category of Fréchet algebras. It is a strictly full subcategory of \mathcal{TopAlg} .

Fréchet algebras do not suffice in some cases. For example, if I is an uncountable set, then the product $\prod_{i \in I} \mathbb{C}$ is a topological algebra with the product topology, but it is not a Fréchet algebra. This motivates the following definition.

Definition 1.5.13 A *product-Fréchet space* is a topological vector space isomorphic to a product of Fréchet spaces. A *morphism* between product-Fréchet spaces is a continuous linear map of the underlying topological spaces.

A topological algebra A is a *product-Fréchet algebra* if as a topological vector space, A is a product-Fréchet space. A *morphism* between product-Fréchet algebras is a morphism of the corresponding topological algebras.

Note that a product-Fréchet space is not necessarily Fréchet, as the index set of the product can be uncountable. Let \mathcal{PFRAlg} be the category of product-Fréchet algebras. It is a strictly full subcategory of \mathcal{TopAlg} .

As for modules, we introduce the following:

Definition 1.5.14 Let A be a topological algebra. A *Fréchet A -module* (resp. *product-Fréchet A -module*) is a topological A -module M such that as a topological vector space, M is a Fréchet space (resp. product-Fréchet space). A *homomorphism* between Fréchet or product-Fréchet A -modules is a homomorphism of the corresponding topological A -modules.

Proposition 1.5.6 Let $(X_i)_{i \in I}$ be an inverse system of Fréchet spaces indexed by a countable set I . Then the projective limit $\varprojlim_i X_i$ with the projective limit topology is still a Fréchet space.

Note that we do *not* require that (X_i) be filtered.

Proof Since the inverse limit is a closed subspace of the product $\prod_{i \in I} X_i$, it suffices to show that the product is Fréchet, which is a well-known fact. \square

Chapter 2

Complex spaces

Die Mathematik ist Kunst, Mathematiker sind Künstler!
— Hans Grauert^a

^a Hans Grauert (1930–2011) was one of the leading figures in several complex variables and complex analytic geometry. A student of Heinrich Behnke at Münster, he became professor at Göttingen in 1959. His work, much of it in close dialogue with Reinhold Remmert, helped transform Cartan’s sheaf-theoretic methods and Stein’s function-theoretic ideas into the modern theory of complex spaces. The local analytic algebras studied in this chapter are one of the algebraic languages of that theory.

Throughout the book, a complex space is not assumed to be Hausdorff or paracompact.

Recall that a Hausdorff complex space is paracompact if and only if each of its connected components is σ -compact. Moreover, each connected component is automatically second countable.

2.1 Local analytic algebras

Definition 2.1.1 A *local analytic algebra* is a \mathbb{C} -algebra isomorphic to a proper quotient of $\mathbb{C}\{T_1, \dots, T_n\}$ for some $n \geq 0$.

Here $\mathbb{C}\{T_1, \dots, T_n\} \subseteq \mathbb{C}[[T_1, \dots, T_n]]$ is the \mathbb{C} -subalgebra consisting of convergent power series. Note that a local analytic algebra is noetherian.

A local analytic algebra carries a natural topology, called the *series topology*¹. We first define the topology on $\mathbb{C}\{T_1, \dots, T_n\}$.

Definition 2.1.2 Fix $n \geq 0$. The *series topology* on $\mathbb{C}\{T_1, \dots, T_n\}$ is the final topology with respect to the natural inclusions

$$B_{\mathbf{r}} \hookrightarrow \mathbb{C}\{T_1, \dots, T_n\}, \quad \mathbf{r} \in \mathbb{R}_{>0}^n,$$

where

$$B_{\mathbf{r}} = \left\{ f = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha} \mathbf{T}^{\alpha} : \|f\|_{\mathbf{r}} := \sum_{\alpha \in \mathbb{N}^n} |c_{\alpha}| \mathbf{r}^{\alpha} < \infty \right\}.$$

Equivalently, this is the locally convex inductive-limit topology on

¹ This is translated from German *Folgentopologie*.

$$\mathbb{C}\{T_1, \dots, T_n\} = \varinjlim_{\mathbf{r}} B_{\mathbf{r}},$$

where the inductive limit is taken in the category of topological spaces. But as shown in [GR71, § I.8.6], it is equivalent to taking the topology in the category of locally convex topological vector spaces. In particular, $\mathbb{C}\{T_1, \dots, T_n\}$ is a (locally convex) topological algebra.

Definition 2.1.3 Let A be a local analytic algebra. Its *series topology* is defined as follows: Take $n \geq 0$ so that

$$A \cong \mathbb{C}\{T_1, \dots, T_n\}/I \tag{2.1}$$

for some ideal I . Then I is closed by [GR71, § I.8.6, Satz 3] with respect to the series topology on $\mathbb{C}\{T_1, \dots, T_n\}$ defined in Definition 2.1.2. Then we endow A with the quotient topology induced by the series topology on $\mathbb{C}\{T_1, \dots, T_n\}$.

Note that A is a locally convex topological algebra with respect to the series topology. Furthermore, the series topology on A is independent of the choice of the presentation (2.1), see [GR71, § II.1.2, Satz 4 und Korollar]. Any \mathbb{C} -algebra homomorphism between local analytic algebras is continuous with respect to the series topology. In particular, we find the following:

Proposition 2.1.1 *Let A be a local analytic algebra. Then the map*

$$A \rightarrow A/\mathfrak{m} \cong \mathbb{C}$$

is continuous, where \mathfrak{m} is the maximal ideal in A .

Definition 2.1.4 Let A be a local analytic algebra, and M be a finite A -module. Take $m \geq 0$ and a surjective A -module homomorphism

$$A^m \twoheadrightarrow M. \tag{2.2}$$

Then we define the *series topology* on M as the quotient topology induced from A^m , which is endowed with the product topology induced by the series topology on each A .

Note that the series topology on M is necessarily locally convex. The series topology on M is independent of the choice of the presentation in (2.2). See [GR71, § II.1.3, Satz 8 und Korollar]. Note that M is a topological A -module.

Proposition 2.1.2 *Let A be a local analytic algebra, M, N be finite A -modules. Then the following holds:*

- Any A -module homomorphism from M to N is continuous;
- every submodule of M is closed.

See [GR71, § II.1.3, Satz 8, Satz 10].

Proposition 2.1.3 *Let $\phi: A \rightarrow B$ be a \mathbb{C} -algebra homomorphism of local analytic algebras. Consider a finite A -module M , a finite B -module N , and a ϕ -homomorphism $f: M \rightarrow N$. Then f is continuous.*

Proof First note that ϕ is local. Indeed, a local analytic algebra has only one \mathbb{C} -valued character, namely the residue map. To see this, write $A = \mathbb{C}\{T_1, \dots, T_r\}/I$ and let t_i be the image of T_i . If $\chi: A \rightarrow \mathbb{C}$ is a \mathbb{C} -algebra homomorphism and $a_i = \chi(t_i)$, put $a = (a_1, \dots, a_r)$. Choose a complex linear form ℓ with $\ell(a) \neq 0$ if $a \neq 0$. Then

$$\left(1 - \frac{\ell(T)}{\ell(a)}\right)^{-1} \in \mathbb{C}\{T_1, \dots, T_r\}$$

is a unit in $\mathbb{C}\{T_1, \dots, T_r\}$, but its image under χ would be 0, a contradiction. Hence $a_i = 0$ for all i , and χ is the residue map.

We next show that ϕ is continuous. Choose presentations

$$\mathbb{C}\{T_1, \dots, T_r\} \twoheadrightarrow A, \quad \mathbb{C}\{S_1, \dots, S_s\} \twoheadrightarrow B.$$

Let t_i be the image of T_i in A , and choose lifts $\tilde{b}_i \in \mathbb{C}\{S_1, \dots, S_s\}$ of $\phi(t_i) \in B$. Since ϕ is local, each \tilde{b}_i has zero constant term. For every polyradius $\mathbf{r} \in \mathbb{R}_{>0}^r$, choose a sufficiently small polyradius $\mathbf{s} \in \mathbb{R}_{>0}^s$ such that

$$\|\tilde{b}_i\|_{\mathbf{s}} < r_i, \quad i = 1, \dots, r.$$

Then substitution $T_i \mapsto \tilde{b}_i$ defines a continuous linear map

$$B_{\mathbf{r}}^T \longrightarrow B_{\mathbf{s}}^S$$

between the Banach spaces used in [Definition 2.1.2](#), because $B_{\mathbf{s}}^S$ is a Banach algebra and

$$\left\| \sum_{\alpha} c_{\alpha} \tilde{\mathbf{b}}^{\alpha} \right\|_{\mathbf{s}} \leq \sum_{\alpha} |c_{\alpha}| \mathbf{r}^{\alpha}.$$

Passing to inductive limit topologies and then to quotients gives the continuity of $\phi: A \rightarrow B$.

Now choose a surjection $q: A^m \twoheadrightarrow M$. Let e_1, \dots, e_m be the images of the standard basis vectors, and put $n_i = f(e_i) \in N$. The composition

$$A^m \xrightarrow{q} M \xrightarrow{f} N$$

is

$$(a_1, \dots, a_m) \mapsto \sum_{i=1}^m \phi(a_i) n_i.$$

Each map $a \mapsto \phi(a) n_i$ is continuous: ϕ is continuous, and $b \mapsto b n_i$ is a continuous B -linear map $B \rightarrow N$ by [Proposition 2.1.2](#). Hence $f \circ q$ is continuous. Since q is a quotient map by the definition of the series topology on M , the map f is continuous. \square

Proposition 2.1.4 *Let $A \rightarrow B$ be a finite \mathbb{C} -algebra homomorphism between local analytic algebras, and M be a finite B -module. Then the series topology on M viewed as a B -module is the same as the series topology when M is viewed as an A -module.*

See [GR71, § II.2.6, Satz 7].

Theorem 2.1.1 (Scheja–Storch) *Let A be a local analytic algebra. Then A admits a universally finite \mathbb{C} -derivation. If there is a complex space X and $x \in X$ such that $A \cong \mathcal{O}_{X,x}$, then a universally finite \mathbb{C} -derivation is given by the usual Kähler differential*

$$d_x: \mathcal{O}_{X,x} \rightarrow \Omega_{X,x}$$

See [Definition 1.2.3](#) for the notion of universally finite derivations. See [[SS72](#), Satz 2.6] and its proof.

2.2 The section ring of complex spaces

Let X be a Hausdorff paracompact complex space. We shall construct a topology on $\Gamma(X, \mathcal{O}_X)$, making it a product-Fréchet algebra, as defined in [Definition 1.5.13](#). Similarly, for a coherent sheaf \mathcal{F} on X , we can also construct a topology on $\Gamma(X, \mathcal{F})$, making it a product-Fréchet module. We shall call these topologies the *canonical Fréchet topologies* on $\Gamma(X, \mathcal{O}_X)$ and $\Gamma(X, \mathcal{F})$ respectively.

In fact, let $\{X_i\}_{i \in I}$ be the family of connected components of X . We shall construct a Fréchet-algebra topology on $\Gamma(X_i, \mathcal{O}_{X_i})$ and define the topology on $\Gamma(X, \mathcal{O}_X)$ and on $\Gamma(X, \mathcal{F})$ as the product topology:

$$\Gamma(X, \mathcal{O}_X) = \prod_{i \in I} \Gamma(X_i, \mathcal{O}_{X_i}), \quad \Gamma(X, \mathcal{F}) = \prod_{i \in I} \Gamma(X_i, \mathcal{F}|_{X_i}).$$

In particular, when X has countably many connected components, $\Gamma(X, \mathcal{O}_X)$ is a Fréchet algebra and $\Gamma(X, \mathcal{F})$ is a Fréchet module.

In the sequel, we only handle the case where X is connected. Then X is σ -compact.

2.2.1 The canonical Fréchet topology

Let X be a σ -compact Hausdorff complex space. For the ease of later citations, we shall restate this assumption in each of the theorems below.

Theorem 2.2.1 *Let X be a σ -compact Hausdorff complex space. Let \mathcal{F} be a coherent sheaf on X . There is a unique topology on $\Gamma(X, \mathcal{F})$ such that*

- (1) $\Gamma(X, \mathcal{F})$ is a Fréchet space with respect to this topology;

(2) For each $x \in X$, the germ map

$$\text{ev}_x = \text{ev}_x^{\mathcal{F}}: \Gamma(X, \mathcal{F}) \rightarrow \mathcal{F}_x$$

is continuous, where on \mathcal{F}_x we endow the series topology as in [Definition 2.1.4](#).

In particular, when $\mathcal{F} = \mathcal{O}_X$, we obtain that $\Gamma(X, \mathcal{O}_X)$ is a Fréchet algebra. In general $\Gamma(X, \mathcal{F})$ is a Fréchet $\Gamma(X, \mathcal{O}_X)$ -module.

Proof Step 1. We first prove the uniqueness of the topology.

Let τ_1 and τ_2 be two topologies on $\Gamma(X, \mathcal{F})$ satisfying the assumptions. The graph of the identity map

$$(\Gamma(X, \mathcal{F}), \tau_1) \longrightarrow (\Gamma(X, \mathcal{F}), \tau_2)$$

is closed: indeed, if $s_n \rightarrow s$ for τ_1 and $s_n \rightarrow t$ for τ_2 , then the continuity of the evaluations gives $s_x = t_x$ in \mathcal{F}_x for every $x \in X$, hence $s = t$. By the closed graph theorem the identity map is continuous. Reversing τ_1 and τ_2 gives the continuity of the inverse identity map, so the two topologies coincide.

Step 2. We prove the existence of the topology.

Step 2.1. We first treat the case where X is a σ -compact complex manifold. We begin with the local model. Let $X = \Delta^n$, and suppose that \mathcal{F} is globally finitely generated: There is an epimorphism

$$\mathcal{O}_{\Delta^n}^q \rightarrow \mathcal{F} \rightarrow 0. \quad (2.3)$$

Let \mathcal{K} be the kernel. Since \mathcal{F} is coherent, \mathcal{K} is coherent. By Cartan's Theorem B, the sequence

$$0 \rightarrow \Gamma(\Delta^n, \mathcal{K}) \rightarrow \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}^q) \rightarrow \Gamma(\Delta^n, \mathcal{F}) \rightarrow 0$$

is exact. We endow $\Gamma(\Delta^n, \mathcal{O}_{\Delta^n}^q)$ with the compact-open topology, which is Fréchet since Δ^n is σ -compact. For each $z \in \Delta^n$, the germ map

$$\text{ev}_z^q: \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}^q) \rightarrow \mathcal{O}_{\Delta^n, z}^q$$

is continuous for the series topology on the target. Moreover $\mathcal{K}_z \subseteq \mathcal{O}_{\Delta^n, z}^q$ is closed by [Proposition 2.1.2](#). Hence

$$\Gamma(\Delta^n, \mathcal{K}) = \bigcap_{z \in \Delta^n} (\text{ev}_z^q)^{-1}(\mathcal{K}_z)$$

is closed in $\Gamma(\Delta^n, \mathcal{O}_{\Delta^n}^q)$. We endow $\Gamma(\Delta^n, \mathcal{F})$ with the quotient Fréchet topology.

It remains in this local model to prove continuity of the evaluations. For $z \in \Delta^n$, the diagram

$$\begin{array}{ccc} \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}^q) & \twoheadrightarrow & \Gamma(\Delta^n, \mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\Delta^n, z}^q & \twoheadrightarrow & \mathcal{F}_z \end{array}$$

is commutative. The left vertical map is continuous by the compatibility between compact convergence and the series topology, and the lower horizontal map is continuous by [Proposition 2.1.2](#). Since the upper horizontal map is the quotient map defining the topology on $\Gamma(\Delta^n, \mathcal{F})$, the right vertical map is continuous.

Now let X be an arbitrary σ -compact Hausdorff complex manifold. Since X has a countable basis, we can choose a countable cover $\{U_i\}_{i \geq 1}$ by coordinate polydiscs such that each $\mathcal{F}|_{U_i}$ is generated by finitely many sections. We glue the local topologies as follows. Consider

$$\rho: \Gamma(X, \mathcal{F}) \rightarrow \prod_{i \geq 1} \Gamma(U_i, \mathcal{F}), \quad s \mapsto (s|_{U_i})_{i \geq 1}.$$

The target is Fréchet, and ρ is injective. Its image is closed: if $x \in U_i$, let

$$\rho_{i,x}: \prod_{j \geq 1} \Gamma(U_j, \mathcal{F}) \rightarrow \mathcal{F}_x$$

be projection to $\Gamma(U_i, \mathcal{F})$ followed by the germ map at x . A tuple $(s_i)_i$ comes from a global section if and only if

$$\rho_{i,x}((s_j)_j) = \rho_{k,x}((s_j)_j)$$

for all $x \in U_i \cap U_k$. Since \mathcal{F}_x is Hausdorff, these equalizers are closed, and the image of ρ is their intersection. We transport the induced closed-subspace Fréchet topology to $\Gamma(X, \mathcal{F})$. The germ maps are continuous because locally they factor through one of the projections.

Step 2.2. We next treat the case where X is a closed analytic subspace of a σ -compact Hausdorff complex manifold M , with closed embedding $i: X \hookrightarrow M$. Then $i_*\mathcal{F}$ is coherent on M . Applying Step 2.1 to M and $i_*\mathcal{F}$, and using

$$\Gamma(M, i_*\mathcal{F}) = \Gamma(X, \mathcal{F}),$$

gives a Fréchet topology on $\Gamma(X, \mathcal{F})$. For $x \in X$, the germ map at x is the germ map of $i_*\mathcal{F}$ at $i(x)$; the series topology agrees with the one on \mathcal{F}_x by [Proposition 2.1.4](#).

Step 2.3. We pass to the general case. Since X has a countable basis, we can find a countable open cover $\{U_i\}_{i \geq 1}$ such that each U_i is a closed analytic subspace of a polydisc and $\mathcal{F}|_{U_i}$ satisfies Step 2.2. Applying the same gluing construction from Step 2.1 to this cover gives the desired Fréchet topology on $\Gamma(X, \mathcal{F})$ and makes all germ maps continuous.

Step 3. We finish the proof.

Step 3.1. We first show that $\Gamma(X, \mathcal{O}_X)$ is a Fréchet algebra.

We need to show that the multiplication map

$$\Gamma(X, \mathcal{O}_X) \times \Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(X, \mathcal{O}_X)$$

is continuous. By [Theorem 1.4.4](#), it suffices to show that it is continuous in each variable. By the closed graph theorem, we need to show the following. Suppose $(f_j)_j$

is a sequence in $\Gamma(X, \mathcal{O}_X)$ with limit f , and $g \in \Gamma(X, \mathcal{O}_X)$. Assume that $f_j g \rightarrow h$ for some $h \in \Gamma(X, \mathcal{O}_X)$. Then

$$h = fg. \quad (2.4)$$

Since it suffices to verify (2.4) locally on X , we may assume that X is an analytic subspace of $\Delta^n \subseteq \mathbb{C}^n$.

Since Δ^n is a Stein space, we have the following commutative diagram with surjective vertical maps given by restrictions:

$$\begin{array}{ccc} \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}) \times \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}) & \xrightarrow{\times} & \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}) \\ \downarrow & & \downarrow \\ \Gamma(X, \mathcal{O}_X) \times \Gamma(X, \mathcal{O}_X) & \xrightarrow{\times} & \Gamma(X, \mathcal{O}_X). \end{array}$$

By the open mapping theorem, it suffices to verify the continuity of the upper multiplication, so we are finally reduced to the case $X = \Delta^n$. But in this case (2.4) trivially holds.

Step 3.2. Next we show that $\Gamma(X, \mathcal{F})$ is a Fréchet module over $\Gamma(X, \mathcal{O}_X)$.

We need to show that the multiplication map

$$\Gamma(X, \mathcal{O}_X) \times \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F})$$

is continuous. Similar to Step 3.1, the problem is local, and we can reduce to the case $X = \Delta^n$, and \mathcal{F} is globally of finite type. Taking a surjective homomorphism as in (2.3), we get a commutative diagram:

$$\begin{array}{ccc} \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}) \times \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}^q) & \xrightarrow{\times} & \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}^q) \\ \downarrow & & \downarrow \\ \Gamma(\Delta^n, \mathcal{O}_{\Delta^n}) \times \Gamma(\Delta^n, \mathcal{F}) & \xrightarrow{\times} & \Gamma(\Delta^n, \mathcal{F}). \end{array}$$

We then reduce to the case where $\mathcal{F} = \mathcal{O}_{\Delta^n}^q$. In this case, we can further reduce to $q = 1$, and this case is handled in Step 3.1. \square

Definition 2.2.1 We call the topology in Theorem 2.2.1 the *canonical Fréchet topology* of $\Gamma(X, \mathcal{F})$.

Whenever we mention topological aspects of $\mathcal{O}_X(X)$ or $\Gamma(X, \mathcal{F})$, we always refer to this topology.

The basic properties of the canonical Fréchet topology are summarized as follows:

Proposition 2.2.1 *Let X be a σ -compact Hausdorff complex space. Let \mathcal{F} be a coherent sheaf on X . Then*

(1) *for any open subset $U \subseteq X$, the restriction*

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F})$$

is continuous;

- (2) if $f: Y \rightarrow X$ is a morphism from a σ -compact Hausdorff complex space Y , then the pull-back

$$f^*: \Gamma(X, \mathcal{F}) \rightarrow \Gamma(Y, f^*\mathcal{F})$$

is continuous;

- (3) if $f: Y \rightarrow X$ is a finite morphism from a σ -compact Hausdorff complex space Y , then for any coherent sheaf \mathcal{G} on Y , the natural isomorphism

$$\Gamma(X, f_*\mathcal{G}) \rightarrow \Gamma(Y, \mathcal{G})$$

is an isomorphism of Fréchet spaces;

- (4) if \mathcal{F}' is another coherent sheaf on X and $\mathcal{F} \rightarrow \mathcal{F}'$ is a morphism of coherent sheaves, then the induced map

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}')$$

is continuous;

- (5) if \mathcal{H} is a coherent subsheaf of \mathcal{F} , then $\Gamma(X, \mathcal{H})$ is closed in $\Gamma(X, \mathcal{F})$;
 (6) if \mathcal{F}' is another coherent sheaf on X , then the natural map

$$\Gamma(X, \mathcal{F}) \oplus \Gamma(X, \mathcal{F}') \rightarrow \Gamma(X, \mathcal{F} \oplus \mathcal{F}')$$

is an isomorphism of Fréchet spaces.

Proof We repeatedly use the closed graph theorem.

(1) Since X is second countable and locally compact, every open subset $U \subseteq X$ is again σ -compact. Let $r: \Gamma(X, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F})$ be the restriction map. Suppose that $s_j \rightarrow s$ in $\Gamma(X, \mathcal{F})$ and $r(s_j) \rightarrow t$ in $\Gamma(U, \mathcal{F})$. For every $x \in U$, continuity of the germ maps gives

$$t_x = \lim_j r(s_j)_x = \lim_j (s_j)_x = s_x.$$

Hence $t = s|_U$, so the graph of r is closed. Thus r is continuous.

(2) Let

$$g: \Gamma(X, \mathcal{F}) \rightarrow \Gamma(Y, f^*\mathcal{F})$$

be the pull-back map. Suppose that $s_j \rightarrow s$ in $\Gamma(X, \mathcal{F})$ and $g(s_j) \rightarrow t$ in $\Gamma(Y, f^*\mathcal{F})$. For $y \in Y$, put $x = f(y)$. The natural stalk map

$$\mathcal{F}_x \rightarrow (f^*\mathcal{F})_y$$

is continuous for the series topologies by **Proposition 2.1.3**. Therefore

$$t_y = \lim_j g(s_j)_y = g(s)_y.$$

This holds for every $y \in Y$, hence $t = g(s)$. Thus the graph of g is closed, and g is continuous.

(3) Algebraically, the displayed map is the defining identity $\Gamma(X, f_*\mathcal{G}) = \Gamma(Y, \mathcal{G})$. We compare the two topologies. Transport the canonical Fréchet topology of $\Gamma(Y, \mathcal{G})$ to $\Gamma(X, f_*\mathcal{G})$ through this identity. This transported topology is Fréchet. For $x \in X$, the finite morphism identifies the stalk $(f_*\mathcal{G})_x$ with the finite product $\prod_{y \in f^{-1}(x)} \mathcal{G}_y$, and this identification is a homeomorphism for the series topologies by [Proposition 2.1.4](#). Hence the germ map

$$\Gamma(X, f_*\mathcal{G}) = \Gamma(Y, \mathcal{G}) \rightarrow (f_*\mathcal{G})_x$$

is continuous, since it is given by the finitely many continuous germ maps at the points $y \in f^{-1}(x)$. By the uniqueness part of [Theorem 2.2.1](#), the transported topology is the canonical one. Thus the identity is an isomorphism of Fréchet spaces.

(4) Let $u: \mathcal{F} \rightarrow \mathcal{F}'$ be the given morphism. Suppose that $s_j \rightarrow s$ in $\Gamma(X, \mathcal{F})$ and $u(s_j) \rightarrow t$ in $\Gamma(X, \mathcal{F}')$. For every $x \in X$, the stalk map $u_x: \mathcal{F}_x \rightarrow \mathcal{F}'_x$ is continuous by [Proposition 2.1.2](#). Hence

$$t_x = \lim_j u_x((s_j)_x) = u_x(s_x).$$

Thus $t = u(s)$, so the graph is closed and the induced map on global sections is continuous.

(5) For each $x \in X$, the submodule $\mathcal{H}_x \subseteq \mathcal{F}_x$ is closed by [Proposition 2.1.2](#). Therefore

$$\Gamma(X, \mathcal{H}) = \bigcap_{x \in X} (\text{ev}_x^{\mathcal{F}})^{-1}(\mathcal{H}_x)$$

is closed in $\Gamma(X, \mathcal{F})$.

(6) Let

$$i_1: \mathcal{F} \rightarrow \mathcal{F} \oplus \mathcal{F}', \quad i_2: \mathcal{F}' \rightarrow \mathcal{F} \oplus \mathcal{F}'$$

be the two natural inclusions, and let

$$p_1: \mathcal{F} \oplus \mathcal{F}' \rightarrow \mathcal{F}, \quad p_2: \mathcal{F} \oplus \mathcal{F}' \rightarrow \mathcal{F}'$$

be the two natural projections. By (4), the maps induced by i_1, i_2, p_1 and p_2 on global sections are continuous. The displayed natural map is algebraically the map

$$(s, t) \mapsto i_1(s) + i_2(t),$$

hence it is continuous. Its inverse is

$$u \mapsto (p_1(u), p_2(u)),$$

which is continuous as well. Therefore the natural map is an isomorphism of Fréchet spaces. \square

Corollary 2.2.1 *Let X be a σ -compact Hausdorff complex space, and \mathcal{F} be a coherent \mathcal{O}_X -module. Suppose that $s_1, \dots, s_n \in \Gamma(X, \mathcal{F})$. Suppose that for each $i = 1, \dots, n$, we are given a sequence $(a_i^j)_j$ in $\Gamma(X, \mathcal{O}_X)$ with limit $a_i \in \Gamma(X, \mathcal{O}_X)$.*

Then

$$\sum_{i=1}^n a_i^j s_i \rightarrow \sum_{i=1}^n a_i s_i$$

as $j \rightarrow \infty$.

Proof The sections s_i induce a homomorphism of \mathcal{O}_X -modules:

$$\mathcal{O}_X^n \xrightarrow{\cdot(s_1, \dots, s_n)} \mathcal{F}.$$

Therefore, by [Proposition 2.2.1](#)(4) and (6), the induced map

$$\Gamma(X, \mathcal{O}_X)^n \rightarrow \Gamma(X, \mathcal{F}), \quad (a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_i s_i$$

is continuous. Our assertion follows. \square

Corollary 2.2.2 *Let X be a σ -compact Hausdorff complex space and \mathcal{F} be a coherent \mathcal{O}_X -module. Take $x \in X$. Then in \mathcal{LCS} , the natural map*

$$\varinjlim_{U \ni x} \Gamma(U, \mathcal{F}) \rightarrow \mathcal{F}_x \quad (2.5)$$

is an isomorphism, where $U \subseteq X$ runs over all open neighborhoods of x .

The filtered colimit in (2.5) is taken in the category \mathcal{LCS} .

Proof Note that (2.5) is well-defined, since the series topology on \mathcal{F}_x is locally convex. It is an isomorphism of the underlying $\mathcal{O}_{X,x}$ -modules, and is continuous.

It remains to prove that it is an open map. The problem is local, so we may assume that X admits a closed immersion $i: X \hookrightarrow \Delta^n$ with $i(x) = 0$.

We have a commutative diagram in \mathcal{LCS} :

$$\begin{array}{ccc} \varinjlim_{U \ni x} \Gamma(U, \mathcal{F}) & \longrightarrow & \mathcal{F}_x \\ \uparrow & & \cong \uparrow \\ \varinjlim_{V \ni 0} \Gamma(V, i_* \mathcal{F}) & \longrightarrow & (i_* \mathcal{F})_0 \end{array}$$

By [Proposition 2.1.4](#), the right-vertical map is also a homeomorphism. The left-vertical map is also a homeomorphism by [Proposition 2.2.1](#)(3). Therefore, in order to show that the upper horizontal map is an open map, it suffices to show that the lower horizontal map is. In other words, we have reduced to the case $X = \Delta^n$ and $x = 0$.

Step 1. We first handle the case $\mathcal{F} = \mathcal{O}_X^m$ for some $m \geq 0$. We may easily reduce to the case $m = 1$.

In this case, recall that the series topology is defined as the inductive-limit topology on B_r as r runs over $(0, 1)^n$, see [Definition 2.1.2](#). But the Banach topology on

B_r is clearly finer than the compact-open topology, so our assertion follows in this case.

Step 2. We handle the general case.

After shrinking X around x , we may assume that there is a surjection $\mathcal{O}_X^m \twoheadrightarrow \mathcal{F}$ for some $m \geq 0$. Then for each polydisk $U \subseteq X$ with center 0 we get a commutative diagram in \mathcal{LCS} :

$$\begin{array}{ccc} \varinjlim_{U \ni 0} \Gamma(U, \mathcal{O}_X^m) & \longrightarrow & \mathcal{O}_{X,0}^m \\ \downarrow & & \downarrow \\ \varinjlim_{U \ni 0} \Gamma(U, \mathcal{F}) & \longrightarrow & \mathcal{F}_0. \end{array}$$

The upper horizontal map is a homeomorphism by Step 1, and the right vertical map is open by definition of the series topology. The left vertical map is a surjective open map by [Theorem 1.4.2](#). Therefore, the lower horizontal map is open. \square

Proposition 2.2.2 *Let X be a σ -compact Hausdorff complex space. Let $\{U_i\}_{i \in I}$ be a countable open covering of X . Then the natural map*

$$\Gamma(X, \mathcal{O}_X) \rightarrow \prod_{i \in I} \Gamma(U_i, \mathcal{O}_X), \quad f \mapsto (f|_{U_i})_i \quad (2.6)$$

defines a topological embedding with closed image.

Proof By [Proposition 2.2.1\(1\)](#), the map is continuous. It is injective since $\{U_i\}$ is a covering. The image is closed since it can be defined as an equalizer.

That [\(2.6\)](#) is a topological embedding follows from the open mapping theorem. \square

Corollary 2.2.3 *Let X be a σ -compact Hausdorff complex space. Let $U_1 \subseteq U_2 \subseteq \dots$ be an increasing countable covering of X by open subsets. Then in the category of locally convex topological vector spaces, we have*

$$\Gamma(X, \mathcal{O}_X) \xrightarrow{\sim} \varprojlim_{i > 0} \Gamma(U_i, \mathcal{O}_X).$$

Proof This follows from [Proposition 2.2.2](#). \square

Corollary 2.2.4 *Let $(X_i)_{i \in I}$ be a countable family of σ -compact Hausdorff complex spaces. Set $X = \sqcup_{i \in I} X_i$. Then the natural map*

$$\Gamma(X, \mathcal{O}_X) \rightarrow \prod_{i \in I} \Gamma(X_i, \mathcal{O}_{X_i})$$

is an isomorphism of Fréchet algebras.

Proof This follows from [Proposition 2.2.2](#). \square

Proposition 2.2.3 *Assume that X is a reduced σ -compact Hausdorff complex space. Then the canonical Fréchet topology on $\Gamma(X, \mathcal{O}_X)$ is just the compact-open topology.*

See [GR77, § V.6, Satz 8].

Lemma 2.2.1 *Let X be a σ -compact Hausdorff complex space, and $x \in X$. Then the map*

$$\Gamma(X, \mathcal{O}_X) \rightarrow \mathbb{C}, \quad f \mapsto f(x) \quad (2.7)$$

is continuous.

Proof We write the map (2.7) as a composition

$$\Gamma(X, \mathcal{O}_X) \xrightarrow{\text{ev}_x} \mathcal{O}_{X,x} \rightarrow \mathbb{C},$$

the first map is continuous by definition of the canonical topology, while the second is continuous by Proposition 2.1.1. \square

Proposition 2.2.4 *Let X be a σ -compact Hausdorff complex space. The homomorphism*

$$\Gamma(X, \mathcal{O}_X) \rightarrow C^0(X)$$

sending a holomorphic function f to the associated continuous function is continuous.

Here $C^0(X)$ is endowed with the compact-open topology.

Proof By the closed graph theorem, it suffices to verify the following. Suppose that $(f_j)_j$ is a sequence in $\Gamma(X, \mathcal{O}_X)$ with limit f with respect to the Fréchet topology, and assume that $f_j \rightarrow g$ in $C^0(X)$ for some g . Then g is the continuous function associated with f .

Take $x \in X$. It suffices to show that $f_j(x) \rightarrow f(x)$, but this follows from Lemma 2.2.1. \square

2.2.2 Envelopes

Definition 2.2.2 Let X be a σ -compact Hausdorff complex space. We define the *envelope* \tilde{X} of X as the spectrum $\text{Sp } \Gamma(X, \mathcal{O}_X)$.

Here the spectrum Sp is defined in Section 1.5. We regard \tilde{X} as a locally \mathbb{C} -ringed space.

The envelope is functorial. More precisely, suppose that $F: Y \rightarrow X$ is a morphism between σ -compact Hausdorff complex spaces. Then we define $\tilde{F}: \tilde{Y} \rightarrow \tilde{X}$ as follows: if $\tilde{y} \in \tilde{Y}$, then $\tilde{x} = \tilde{F}(\tilde{y})$ is the map making the following diagram commutative:

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \xrightarrow{F^*} & \Gamma(Y, \mathcal{O}_Y) \\ & \searrow \tilde{x} & \swarrow \tilde{y} \\ & \mathbb{C} & \end{array}$$

Definition 2.2.3 Let X be a σ -compact Hausdorff complex space. The *character map* $\chi = \chi_X: X \rightarrow \tilde{X}$ is defined as follows: Given $x \in X$,

$$\chi(x) = \chi_X(x): \Gamma(X, \mathcal{O}_X) \rightarrow \mathbb{C}, \quad f \mapsto f(x).$$

Note that χ_X is continuous: To see this, it suffices to show that for each $f \in \Gamma(X, \mathcal{O}_X)$, the composition

$$X \xrightarrow{\chi_X} \tilde{X} \xrightarrow{f} \mathbb{C}$$

is continuous, but the composition is nothing but the function associated with f .

Proposition 2.2.5 Let X be a σ -compact Hausdorff complex space. Then the character map can be canonically enhanced to a morphism

$$\chi_X: X \rightarrow \tilde{X} \tag{2.8}$$

of locally \mathbb{C} -ringed spaces.

Proof Put $A = \Gamma(X, \mathcal{O}_X)$.

We construct a homomorphism

$$\chi_X^{-1} \mathcal{O}_{\mathrm{Sp} A} \longrightarrow \mathcal{O}_X.$$

Since $\mathcal{O}_{\mathrm{Sp} A}$ is obtained from $\mathcal{E}(A)$ by quotienting by $\mathcal{I}(A)$, it is enough to construct a homomorphism

$$\chi_X^{-1} \mathcal{E}(A) \longrightarrow \mathcal{O}_X$$

which annihilates $\chi_X^{-1} \mathcal{I}(A)$.

Let $U \subseteq X$ be open, and let $V \subseteq \mathrm{Sp} A$ be open with $\chi_X(U) \subseteq V$. We define a compatible homomorphism

$$\Gamma(V, \mathcal{E}(A)) \longrightarrow \Gamma(U, \mathcal{O}_X).$$

Let $\alpha \in \Gamma(V, \mathcal{E}(A))$ and let $x \in U$. Set $y = \chi_X(x)$. By the definition of $\mathcal{E}(A)$, after shrinking V around y , the germ α_y can be represented in the form

$$P(T_{a_1-y(a_1)}, \dots, T_{a_n-y(a_n)}), \quad P \in \mathbb{C}\{T_1, \dots, T_n\},$$

for some $a_1, \dots, a_n \in A$. We send this germ to

$$P((a_1 - a_1(x))_x, \dots, (a_n - a_n(x))_x) \in \mathcal{O}_{X,x}.$$

This construction is local in x . On a sufficiently small open neighborhood $W \subseteq U$ of x , it is represented by the holomorphic section

$$P(a_1 - a_1(x), \dots, a_n - a_n(x))|_W.$$

The compatibility of different local representations follows from the sheaf property in the definition of $\mathcal{E}(A)$. Hence the above germs glue to a section of \mathcal{O}_X over U .

This gives a homomorphism

$$\chi_X^{-1}\mathcal{E}(A) \longrightarrow \mathcal{O}_X.$$

We now show that this homomorphism annihilates $\chi_X^{-1}\mathcal{I}(A)$. It suffices to check this on stalks. Let $x \in X$ and $y = \chi_X(x)$. An element of $\mathcal{I}(A)_y$ is locally generated by elements

$$\alpha = \sum_{\mathbf{v} \in \mathbb{N}^n} c_{\mathbf{v}} T_{\mathbf{f}}^{\mathbf{v}}, \quad \mathbf{f} = (f_1, \dots, f_n) \in \mathfrak{m}(y)^n,$$

satisfying

$$\sum_{|\mathbf{v}| < r} c_{\mathbf{v}} \mathbf{f}^{\mathbf{v}} \in \mathfrak{m}(y)^r \quad \text{for every } r > 0.$$

After substituting T_{f_i} by $(f_i)_x$, the finite partial sum lies in $\mathfrak{m}_{X,x}^r$, and the tail lies in $\mathfrak{m}_{X,x}^r$ as well. Hence the image of α lies in

$$\bigcap_{r > 0} \mathfrak{m}_{X,x}^r = 0$$

by the Krull intersection theorem for the noetherian local ring $\mathcal{O}_{X,x}$. Thus the homomorphism factors through $\chi_X^{-1}\mathcal{O}_{\text{Sp}A}$.

On stalks the resulting homomorphism is precisely the substitution map

$$\mathbb{C}\{\mathfrak{m}(\chi_X(x))\}/\mathcal{I}(A)_{\chi_X(x)} \longrightarrow \mathcal{O}_{X,x}, \quad T_f \longmapsto f_x.$$

It is local, because T_f maps into $\mathfrak{m}_{X,x}$ whenever $f \in \mathfrak{m}(\chi_X(x))$. Therefore χ_X is a morphism of locally \mathbb{C} -ringed spaces. \square

Proposition 2.2.6 *Let $F: Y \rightarrow X$ be a morphism of σ -compact Hausdorff complex spaces. Then the following diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\chi_Y} & \tilde{Y} \\ F \downarrow & & \downarrow \tilde{F} \\ X & \xrightarrow{\chi_X} & \tilde{X} \end{array}$$

is commutative in the category of locally \mathbb{C} -ringed spaces.

Proof Put

$$A = \Gamma(X, \mathcal{O}_X), \quad B = \Gamma(Y, \mathcal{O}_Y), \quad \alpha = F^*: A \rightarrow B.$$

By [Proposition 2.2.1\(2\)](#), the homomorphism α is continuous, so \tilde{F} is the morphism $\text{Sp } \alpha: \text{Sp } B \rightarrow \text{Sp } A$ from [Theorem 1.5.1](#).

We first check the underlying maps. For $y \in Y$ and $a \in A$, one has

$$((\tilde{F} \circ \chi_Y)(y))(a) = \chi_Y(y)(\alpha(a)) = (F^*a)(y) = a(F(y)) = \chi_X(F(y))(a).$$

Thus $\tilde{F} \circ \chi_Y = \chi_X \circ F$ as maps of topological spaces.

It remains to compare the homomorphisms on the structure sheaves. Since $\mathcal{O}_{\tilde{X}}$ is obtained from $\mathcal{E}(A)$ by quotienting by $\mathcal{I}(A)$, it is enough to compare the two pullbacks on germs represented in $\mathcal{E}(A)$. Fix $y \in Y$, set $x = F(y)$, and let

$$\beta = \sum_{\nu \in \mathbb{N}^n} c_\nu T_{\mathbf{a}-\mathbf{a}(x)}^\nu \in \mathbb{C}\{\mathfrak{m}(\chi_X(x))\}$$

be such a representative, with $\mathbf{a} = (a_1, \dots, a_n) \in A^n$. Pulling β back along $\chi_X \circ F$ gives the germ

$$\sum_{\nu \in \mathbb{N}^n} c_\nu (F^*\mathbf{a} - \mathbf{a}(x))^\nu \in \mathcal{O}_{Y,y}.$$

On the other hand, the morphism $\tilde{F} = \text{Sp } \alpha$ sends the above representative to

$$\sum_{\nu \in \mathbb{N}^n} c_\nu T_{\alpha(\mathbf{a}) - \chi_Y(y)(\alpha(\mathbf{a}))}^\nu \in \mathbb{C}\{\mathfrak{m}(\chi_Y(y))\}.$$

Pulling this further back by χ_Y gives

$$\sum_{\nu \in \mathbb{N}^n} c_\nu (F^*\mathbf{a} - (F^*\mathbf{a})(y))^\nu.$$

This is the same germ, because $(F^*a_i)(y) = a_i(F(y)) = a_i(x)$ for every i . Hence the two morphisms have the same pullback on all stalks of $\mathcal{O}_{\tilde{X}}$, and the diagram commutes in the category of locally \mathbb{C} -ringed spaces. \square

2.3 Auxiliary results about complex spaces

Lemma 2.3.1 *Let X be a complex space, let $K \subseteq X$ be a compact subset, and let $f: Z \rightarrow X$ be a finite holomorphic map. Assume that for every analytic subspace Y of some open neighborhood of K in X , the set $Y \cap K$ has only finitely many connected components. Then $f^{-1}(K)$ has only finitely many connected components.*

Proof Since the assertion only concerns the underlying topological spaces, we may pass to the reductions of X and Z . By local finiteness of irreducible components and compactness of K , after replacing X by the finite union of the irreducible components meeting K , we may assume that X has finite dimension and only finitely many irreducible components. Replacing Z by the union of the irreducible components meeting $f^{-1}(K)$, we may also assume that Z has only finitely many irreducible components.

We argue by induction on $\dim X$. If $\dim X = 0$, then K is finite, and hence $f^{-1}(K)$ is finite because f is finite. Thus the assertion is clear.

Assume now that $\dim X = n \geq 1$, and that the assertion is known for spaces of dimension $< n$.

We first reduce to the case where Z is normal and connected. Let

$$\nu: \tilde{Z} \rightarrow Z$$

be the normalization of Z . Then ν is finite and surjective. If $(f \circ \nu)^{-1}(K)$ has only finitely many connected components, then so does $f^{-1}(K)$, because every connected component of $f^{-1}(K)$ has non-empty inverse image in $(f \circ \nu)^{-1}(K)$. Thus we may replace Z by \tilde{Z} and assume that Z is normal. Since Z has only finitely many irreducible components, its normalization has only finitely many connected components. Hence it is enough to prove the assertion on each connected component separately. We may therefore assume that Z is normal and connected.

Replacing X by the reduced analytic subspace $f(Z) \subseteq X$ and K by $K \cap f(Z)$, we may assume that f is surjective. Since Z is normal and connected, Z is irreducible; hence X is irreducible.

Let

$$\pi: \tilde{X} \rightarrow X$$

be the normalization of X . Since Z is normal, the morphism f factors uniquely through π ; write

$$f = \pi \circ g, \quad g: Z \rightarrow \tilde{X}.$$

The map g is finite. Indeed, this is local on \tilde{X} and X : if $A \rightarrow B$ is the finite homomorphism corresponding to π and $A \rightarrow C$ is the finite homomorphism corresponding to f , then the factorization gives $A \rightarrow B \rightarrow C$, and any finite set of A -generators of C also generates C as a B -module. Moreover, g is surjective: its image is closed, and over X^{Reg} the normalization map π is an isomorphism, so the image of g contains the dense open subset $\pi^{-1}(X^{\text{Reg}})$ of \tilde{X} . Thus g is finite and surjective. By the open mapping theorem for complex spaces, g is open.

We shall use the following elementary topological fact. Let $p: E \rightarrow B$ be a continuous closed open map with finite fibers. If $C \subseteq B$ is compact and has only finitely many connected components, then $p^{-1}(C)$ has only finitely many connected components. Indeed, after replacing B by C and E by $p^{-1}(C)$, it suffices to treat the case where C is connected. Let H be a connected component of E . For every clopen neighborhood W of H in E , the set $p(W)$ is non-empty, open and closed in C , hence $p(W) = C$. Since E is compact Hausdorff, connected components coincide with quasi-components. Hence, for any fixed $c \in C$, the finite set $p^{-1}(c)$ meets every clopen neighborhood of H , and therefore meets H itself. Thus every connected component of E meets the finite fiber $p^{-1}(c)$, so there are only finitely many such components.

Applying this fact to g , it is enough to prove that $\pi^{-1}(K)$ has only finitely many connected components.

Set

$$S := \pi^{-1}(X^{\text{Sing}}).$$

Then π restricts to a finite map

$$\pi': S \rightarrow X^{\text{Sing}}.$$

Since $\dim X^{\text{Sing}} < n$, the inductive hypothesis applied to π' gives that

$$S \cap \pi^{-1}(K) = \pi^{-1}(K \cap X^{\text{Sing}})$$

has only finitely many connected components. Write these components as

$$D_1, \dots, D_m.$$

Let $\{E_i\}_{i \in I}$ be the set of connected components of $\pi^{-1}(K)$. If E_i meets S , then E_i contains one of the components D_j . Hence only finitely many of the E_i meet S .

It remains to consider those E_i disjoint from S . On $\tilde{X} \setminus S$, the map π is a biholomorphism onto $X^{\text{Reg}} = X \setminus X^{\text{Sing}}$. Therefore each such E_i maps homeomorphically onto a connected component of

$$K \cap X^{\text{Reg}} = K \setminus X^{\text{Sing}}.$$

Since E_i is compact, $\pi(E_i)$ is a compact connected component of $K \setminus X^{\text{Sing}}$. A compact connected component of the complement of a closed subset in a compact Hausdorff space is a connected component of the whole compact space. Hence each $\pi(E_i)$ is a connected component of K .

By the hypothesis applied to the analytic subspace $Y = X$, the compact set K has only finitely many connected components. Hence there are only finitely many components E_i disjoint from S . Together with the previous paragraph, this proves that $\pi^{-1}(K)$ has only finitely many connected components. Consequently $g^{-1}(\pi^{-1}(K)) = f^{-1}(K)$ has only finitely many connected components, as required. \square

Theorem 2.3.1 (Gluing of complex spaces) *Let $\{X_i\}_{i \in I}$ be a family of complex spaces. For each pair $(i, j) \in I^2$, suppose that we are given an open subset $U_{ij} \subseteq X_i$, and an isomorphism*

$$\varphi_{ij}: U_{ij} \xrightarrow{\sim} U_{ji}$$

of complex spaces. When $i = j$, we assume that $U_{ii} = X_i$. For each $i, j, k \in I$, assume that

- (1) $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$, and
- (2) *the diagram*

$$\begin{array}{ccc} U_{ij} \cap U_{ik} & \xrightarrow{\varphi_{ik}} & U_{ki} \cap U_{kj} \\ & \searrow \varphi_{ij} \quad \nearrow \varphi_{jk} & \\ & U_{ji} \cap U_{jk} & \end{array}$$

is commutative.

Then there is a complex space X with open subsets $\{U_i\}_{i \in I}$ and isomorphisms $\varphi_i: X_i \rightarrow U_i$ with the following properties:

- (1) $X = \bigcup_{i \in I} U_i$;
- (2) $\varphi_i(U_{ij}) = U_i \cap U_j$;
- (3) $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

Furthermore, if $(X', \{U'_i\}_{i \in I}, \{\varphi'_i\}_{i \in I})$ is another datum satisfying the above properties, then there is a unique isomorphism of complex spaces $X \xrightarrow{\sim} X'$ sending U_i to U'_i and φ_i to φ'_i .

Proof From [Stacks, Tag 01JB], the gluing X exists as a locally ringed space and is unique. Furthermore, the gluing X is clearly a locally \mathbb{C} -ringed space. It remains to show that X is a complex space. But this is clear since X is covered by the open subspaces U_i , which are complex spaces. \square

Theorem 2.3.2 (Gluing of morphisms) *Let X be a complex space, and let $\{U_i\}_{i \in I}$ be an open covering of X . For each $i \in I$, let $f_i: Y_i \rightarrow U_i$ be a complex space over U_i .*

Suppose that for every pair (i, j) we are given an isomorphism of complex spaces over $U_i \cap U_j$:

$$\varphi_{ij}: Y_i \times_{U_i} (U_i \cap U_j) \xrightarrow{\sim} Y_j \times_{U_j} (U_i \cap U_j),$$

such that the following conditions hold:

- (1) $\varphi_{ii} = \text{id}$ for all i ;
- (2) $\varphi_{ij} = \varphi_{ji}^{-1}$ for all i, j ;
- (3) for all i, j, k , on $U_i \cap U_j \cap U_k$ we have

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}.$$

Then there exists a complex space morphism $f: Y \rightarrow X$ together with isomorphisms of complex spaces over U_i :

$$\psi_i: Y \times_X U_i \xrightarrow{\sim} Y_i,$$

such that for all i, j , the induced identifications on $U_i \cap U_j$ satisfy

$$\varphi_{ij} = \psi_j \circ \psi_i^{-1}.$$

Moreover, $f: Y \rightarrow X$ is unique up to a unique isomorphism over X with these properties.

Proof This is a special case of [Theorem 2.3.1](#). \square

We recall a well-known construction.

Definition 2.3.1 Let X be a complex space and let $I \subseteq \Gamma(X, \mathcal{O}_X)$ be an ideal. We define the *coherent ideal IO_X associated with I* as the ideal on X defined stalkwise by declaring $(IO_X)_x$ to be the ideal in $\mathcal{O}_{X,x}$ generated by the germs f_x for all $f \in I$. Note that IO_X is coherent. To see this, by the Noetherian property of coherent ideals, we may assume that I is finitely generated. Then the assertion is clear. There is a natural inclusion

$$I \subseteq \Gamma(X, IO_X). \tag{2.9}$$

We remind the reader that this inclusion is not an equality in general. This can be seen for example from the fact that $\Gamma(X, IO_X)$ is always closed by [Proposition 2.2.1](#).

Proposition 2.3.1 *Let $f: X \rightarrow Y$ be a proper morphism of Hausdorff complex spaces. Let $(\mathcal{F}_i)_{i \in I}$ be a filtered inductive system of Abelian sheaves on X , and set $\mathcal{F} = \varinjlim_i \mathcal{F}_i$. Then for every $p \geq 0$, the canonical morphism*

$$\varinjlim_{i \in I} \mathbf{R}^p f_* \mathcal{F}_i \longrightarrow \mathbf{R}^p f_* \mathcal{F}$$

is an isomorphism.

Proof It suffices to check the assertion on stalks. Let $y \in Y$ and form the cartesian square of complex spaces:

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \{y\} & \longrightarrow & Y. \end{array}$$

Since f is proper, proper base change for topological spaces gives

$$(\mathbf{R}^p f_* \mathcal{F})_y \cong \mathbf{H}^p(X_y, \mathcal{F}|_{X_y}), \quad (\mathbf{R}^p f_* \mathcal{F}_i)_y \cong \mathbf{H}^p(X_y, \mathcal{F}_i|_{X_y});$$

see [Stacks, Tag 09V6].

The fibre X_y is compact Hausdorff. By Lemma 1.1.10,

$$\begin{aligned} \left(\varinjlim_i \mathbf{R}^p f_* \mathcal{F}_i \right)_y &\cong \varinjlim_i (\mathbf{R}^p f_* \mathcal{F}_i)_y \\ &\cong \varinjlim_i \mathbf{H}^p(X_y, \mathcal{F}_i|_{X_y}) \\ &\cong \mathbf{H}^p(X_y, \mathcal{F}|_{X_y}) \\ &\cong (\mathbf{R}^p f_* \mathcal{F})_y. \end{aligned}$$

Thus the desired morphism is an isomorphism on all stalks, hence is an isomorphism. \square

Proposition 2.3.2 *Let Y be a normal complex analytic space, and let $U \subseteq Y$ be an open subset whose complement is analytic and nowhere dense. Let*

$$f_i: X_i \rightarrow Y, \quad i = 1, 2,$$

be finite holomorphic maps such that X_i is normal and $X_{i,U} := f_i^{-1}(U)$ is dense in X_i . Then the restriction map

$$\mathrm{Hom}_{\mathrm{CompSpace}_Y}(X_1, X_2) \longrightarrow \mathrm{Hom}_{\mathrm{CompSpace}_U}(X_{1,U}, X_{2,U}) \quad (2.10)$$

is bijective.

Proof We first prove injectivity. Let

$$\varphi, \psi: X_1 \longrightarrow X_2$$

be two morphisms over Y whose restrictions to $X_{1,U}$ agree. Since $X_2 \rightarrow Y$ is finite, it is separated. Hence the equalizer of φ and ψ is a closed analytic subspace of X_1 . It contains the dense open subset $X_{1,U}$, so it is all of X_1 . Thus $\varphi = \psi$. We now prove surjectivity. Let

$$\varphi_U: X_{1,U} \longrightarrow X_{2,U}$$

be a morphism over U . By the injectivity already proved, the extension, if it exists, is unique. Hence the problem is local on Y . We may therefore assume that f_2 is represented by a finite coherent \mathcal{O}_Y -algebra

$$\mathcal{B} = f_{2,*}\mathcal{O}_{X_2}.$$

A morphism $X_1 \rightarrow X_2$ over Y is the same as an \mathcal{O}_Y -algebra homomorphism

$$\mathcal{B} \longrightarrow f_{1,*}\mathcal{O}_{X_1}.$$

The given morphism φ_U is equivalently an \mathcal{O}_U -algebra homomorphism

$$\varphi_U^\# : \mathcal{B}|_U \longrightarrow f_{1,U,*}\mathcal{O}_{X_{1,U}}.$$

We extend this homomorphism across $Y \setminus U$. Let b be a local section of \mathcal{B} . Since \mathcal{B} is finite over \mathcal{O}_Y , the section b is integral over \mathcal{O}_Y . Thus, locally on Y , it satisfies a monic equation

$$b^N + a_1 b^{N-1} + \cdots + a_N = 0, \quad a_i \in \mathcal{O}_Y.$$

The image

$$c_U := \varphi_U^\#(b|_U)$$

is a holomorphic function on $X_{1,U}$. On $X_{1,U}$ it satisfies

$$c_U^N + f_1^\#(a_1)c_U^{N-1} + \cdots + f_1^\#(a_N) = 0.$$

Hence c_U is locally bounded near $X_1 \setminus X_{1,U}$: locally the coefficients $f_1^\#(a_i)$ are bounded, and the roots of a monic polynomial with bounded coefficients are uniformly bounded. Since X_1 is normal, the Riemann extension theorem for normal complex spaces implies that c_U extends uniquely to a holomorphic function on X_1 . These extensions are compatible with addition, multiplication and the unit, because the corresponding identities hold on the dense open subset $X_{1,U}$ and both sides are holomorphic on X_1 . Thus $\varphi_U^\#$ extends uniquely to an \mathcal{O}_Y -algebra homomorphism

$$\mathcal{B} \longrightarrow f_{1,*}\mathcal{O}_{X_1}.$$

This homomorphism defines a morphism

$$\varphi: X_1 \longrightarrow X_2$$

over Y , whose restriction to U is φ_U . Hence (2.10) is surjective. This proves the proposition. \square

Theorem 2.3.3 (Grauert–Remmert extension theorem) *Let Y be a normal complex analytic space, and let $U \subseteq Y$ be an open subset whose complement $Z := Y \setminus U$ is analytic and nowhere dense. Let $f_U: X_U \rightarrow U$ be a finite étale analytic covering. Then there exists a finite holomorphic map $f: X \rightarrow Y$ with X normal, together with an open immersion $X_U \hookrightarrow X$, such that the diagram*

$$\begin{array}{ccc} X_U & \hookrightarrow & X \\ f_U \downarrow & \square & \downarrow f \\ U & \hookrightarrow & Y \end{array}$$

is Cartesian. Moreover, such an extension is unique up to a unique isomorphism over Y .

We refer to [Hou61a] for the definition of analytic spectrum.

Proof We first prove uniqueness. Let

$$f_i: X_i \rightarrow Y, \quad i = 1, 2,$$

be two finite extensions of f_U , with X_i normal. Then $X_i \times_Y U$ is identified with X_U , and is dense in X_i . Hence the induced isomorphism

$$X_1 \times_Y U \xrightarrow{\sim} X_2 \times_Y U$$

over U extends uniquely to an isomorphism $X_1 \xrightarrow{\sim} X_2$ over Y by Proposition 2.3.2. This proves uniqueness. We now prove existence. By uniqueness, the construction is local on Y . We may therefore shrink Y around an arbitrary point. By embedded resolution of singularities for complex analytic spaces, applied to the pair (Y, Z) , we may choose a proper modification $\pi: Y' \rightarrow Y$ such that Y' is smooth, π is an isomorphism over a dense open subset of Y , and $D := \pi^{-1}(Z)^{\text{red}}$ is a divisor with simple normal crossings. Put

$$U' := Y' \setminus D = \pi^{-1}(U).$$

Base changing f_U gives a finite étale covering

$$f_{U'}: X_{U'} := X_U \times_U U' \longrightarrow U'.$$

We first extend $f_{U'}$ across D . This is local on Y' . Hence we may assume that

$$Y' = \Delta^n, \quad D = \{z_1 \cdots z_r = 0\}, \quad U' = (\Delta^*)^r \times \Delta^{n-r}.$$

After decomposing $X_{U'}$ into connected components, it is enough to treat the case where $X_{U'}$ is connected. Since U' is a complex manifold, the finite étale covering $X_{U'} \rightarrow U'$ is a finite topological covering. The fundamental group of U' is $\pi_1(U') \simeq \mathbb{Z}^r$. The connected covering $X_{U'} \rightarrow U'$ corresponds to a finite-index subgroup $H \subseteq \mathbb{Z}^r$. Choose positive integers m_1, \dots, m_r such that

$$m_1\mathbb{Z} \oplus \cdots \oplus m_r\mathbb{Z} \subseteq H.$$

Then the Kummer covering

$$\rho_{U'}: V_{U'} \rightarrow U', \quad z_i = t_i^{m_i} \quad (1 \leq i \leq r),$$

dominates $X_{U'} \rightarrow U'$. Here

$$V_{U'} = (\Delta^*)^r \times \Delta^{n-r}.$$

The covering $\rho_{U'}$ extends across D to the finite morphism

$$\rho: V = \Delta^n \rightarrow \Delta^n = Y', \quad z_i = t_i^{m_i} \quad (1 \leq i \leq r),$$

with $z_j = t_j$ for $j > r$. Let $G = \prod_{i=1}^r \mu_{m_i}$ be the Galois group of $\rho_{U'}$. It acts on $V_{U'}$ and the action extends holomorphically to V . Since $V_{U'} \rightarrow U'$ dominates $X_{U'} \rightarrow U'$, there is a finite G -set I such that

$$X_{U'} \cong (V_{U'} \times I)/G.$$

Define

$$X' := (V \times I)/G.$$

The analytic quotient exists because G is finite. The induced morphism $f': X' \rightarrow Y'$ is finite. Moreover, X' is normal, since $V \times I$ is normal and finite quotients preserve normality. By construction, $X' \times_{Y'} U' \cong X_{U'}$. Thus $f_{U'}$ extends to a finite morphism from a normal complex space over Y' . We now descend this finite normal cover from Y' to Y . Let

$$g := \pi \circ f': X' \rightarrow Y.$$

This morphism is proper. By Grauert's direct image theorem, $\mathcal{A} := g_*\mathcal{O}_{X'}$ is a coherent \mathcal{O}_Y -algebra. Define

$$X := \underline{\mathrm{Spec}}_Y^{\mathrm{an}}(\mathcal{A}).$$

Then $f: X \rightarrow Y$ is finite. We claim that this finite morphism restricts to the original covering over U . Let $\pi_U: U' \rightarrow U$ be the restriction of π . Since Y is normal and π is a proper modification, one has

$$\pi_{U,*}\mathcal{O}_{U'} = \mathcal{O}_U.$$

Moreover, since f_U is finite étale, the sheaf

$$\mathcal{B} := f_{U,*}\mathcal{O}_{X_U}$$

is locally free of finite rank, and finite flat base change gives

$$f_{U',*}\mathcal{O}_{X_{U'}} \cong \pi_U^*\mathcal{B}.$$

Therefore, by the projection formula,

$$\mathcal{A}|_U = g_*\mathcal{O}_{X'}|_U \cong \pi_{U,*}f_{U',*}\mathcal{O}_{X_{U'}} \cong \pi_{U,*}\pi_U^*\mathcal{B} \cong \mathcal{B} \otimes_{\mathcal{O}_U} \pi_{U,*}\mathcal{O}_{U'} \cong \mathcal{B}.$$

Hence

$$X \times_Y U \cong \underline{\text{Spec}}_U^{\text{an}}(\mathcal{B}) \cong X_U.$$

This gives the required Cartesian diagram. It remains to prove that X is normal. Since this is local on Y , we may assume that Y is Stein. Then X' is a normal complex space proper over the Stein space Y . The finite space $X = \underline{\text{Spec}}_Y^{\text{an}}(g_*\mathcal{O}_{X'})$ is the Stein factor of g . For every Stein open subset $W \subseteq Y$, the ring

$$\Gamma(g^{-1}(W), \mathcal{O}_{X'})$$

is normal, because $g^{-1}(W)$ is a normal complex space. Hence the corresponding finite analytic algebra defining $X|_W$ is normal. Thus X is normal. This proves existence. Together with uniqueness, the theorem follows. \square

2.4 Holomorphically convex hull

Let X be a complex space.

Definition 2.4.1 Let $A \subseteq X$ be a subset. The $\mathcal{O}_X(X)$ -convex hull of A in X is the set

$$\hat{A}_X := \bigcap_{f \in \mathcal{O}_X(X)} \{x \in X : |f(x)| \leq \|f\|_{C^0(A)}\}.$$

When there is no risk of confusion, we also write \hat{A} instead of \hat{A}_X .

We say A is $\mathcal{O}_X(X)$ -convex if $A = \hat{A}_X$.

For simplicity, we sometimes write $\mathcal{O}(X)$ in place of $\mathcal{O}_X(X)$ as well.

The elementary properties of the hull are summarized in the following:

Proposition 2.4.1 Let $A, B \subseteq X$ be subsets. Then

- (1) the set \hat{A} is closed;
- (2) we have $A \subseteq \hat{A}$;
- (3) we have $\hat{\hat{A}} = \hat{A}$;
- (4) if $A \subseteq B$, then $\hat{A} \subseteq \hat{B}$;
- (5) we have $\widehat{A \cap B} \subseteq \hat{A} \cap \hat{B}$;

(6) if $F: Y \rightarrow X$ is a morphism of analytic spaces, then

$$\widehat{F^{-1}(A)}_Y \subseteq F^{-1}(\hat{A}_X).$$

See [GR77, § IV.2].

Definition 2.4.2 A complex space X is *holomorphically convex* if for any compact subset $K \subseteq X$, the set \hat{K}_X is also compact.

Chapter 3

Stein algebras

Die Franzosen haben Panzer, wir nur Pfeile und Bogen.
— Karl Stein, 1953^a

^a Karl Stein (1913–2000) was one of the central figures of the Münster school of several complex variables. A student of Heinrich Behnke, he worked on analytic continuation, Cousin problems, Runge approximation on Riemann surfaces, and the notion of holomorphic completeness. The manifolds now bearing his name were called *variétés de Stein* by Henri Cartan at the 1953 Brussels colloquium. His work with Behnke and Remmert helped prepare the language in which modern complex analytic geometry, and especially the Cartan–Serre theory of coherent analytic sheaves, came to be formulated.

We shall use the following definition of Stein spaces:

Definition 3.0.1 A *Stein space* is a σ -compact Hausdorff complex space S such that

- (1) (Holomorphic separability) For any two distinct points $x, y \in S$, there exists a holomorphic function $f \in \Gamma(S, \mathcal{O}_S)$ such that $f(x) \neq f(y)$;
- (2) (Holomorphic convexity) For any compact subset $K \subseteq S$, the holomorphically convex hull \widehat{K} is compact.

In particular, the empty space is Stein. Note that a Stein space is automatically second-countable. Therefore, any open subset is automatically σ -compact. We shall constantly use this fact implicitly.

Note that uncountably many connected components of a Stein space are not allowed in the definition.

Throughout this section, we assume some familiarity with the basic theory of Stein spaces, such as in the book [GR77].

3.1 Preliminary results about Stein spaces

We shall frequently use the famous *Reduktionssatz* of Grauert:

Theorem 3.1.1 (Reduktionssatz) *Let X be a complex space. Then the following are equivalent:*

- (1) X is Stein;
- (2) X^{red} is Stein.

See [GR77, § V.4, Satz 5].

The following embedding theorem will be crucial.

Theorem 3.1.2 *Let S be a Stein space of finite dimension n and finite embedded dimension m . Then there is a closed immersion $S \hookrightarrow \mathbb{C}^N$ with $N = \max\{n+m, 2n+1\}$.*

Given a point x on a complex space X , the *embedded dimension* of X at x is the \mathbb{C} -dimension of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. This is a Zariski upper semi-continuous quantity of x . The supremum as x runs over all points on X is called the *embedded dimension* of X .

Definition 3.1.1 Let X be a complex space. A compact subset $K \subseteq X$ is *Stein* if there is a fundamental system of open neighborhoods $\{V_i\}_i$ of K in X such that each V_i is Stein.

Lemma 3.1.1 *Let S be a Stein space, and K be an $\mathcal{O}(S)$ -convex compact subset of S . Then K is Stein.*

Proof Let $U \subseteq S$ be a relatively compact open neighborhood of K . We need to construct a Stein open neighborhood $V \subseteq S$ of K contained in U .

For each $x \in \partial U$, we can find $f \in \Gamma(S, \mathcal{O}_S)$ with $|f(x)| > 1$ and $|f| < 1$ on K . Since ∂U is compact, we can find finitely many such functions $f_1, \dots, f_m \in \Gamma(S, \mathcal{O}_S)$ such that for each $x \in \partial U$, there exists i with $|f_i(x)| > 1$ while for all i , $|f_i| < 1$ on K . Set

$$W := \{y \in S : |f_i(y)| < 1 \text{ for all } i = 1, \dots, m\}.$$

Then W is Stein and $W \cap \partial U = \emptyset$.

Then we can set

$$V := W \cap U.$$

Then $V \subseteq S$ is Stein since it is the union of some connected components of W . Note that V satisfies all our requirements. \square

Recall the following well-known consequence of Cartan's Theorem B:

Proposition 3.1.1 *Let S be a Stein space, and let \mathcal{F} be a coherent \mathcal{O}_S -module. If \mathcal{F} is generated by global sections $f_1, \dots, f_n \in \Gamma(S, \mathcal{F})$, then for any $s \in \Gamma(S, \mathcal{F})$, we can find $a_1, \dots, a_n \in \Gamma(S, \mathcal{O}_S)$ such that*

$$s = \sum_{i=1}^n a_i f_i.$$

In particular, suppose that $g_1, \dots, g_n \in \Gamma(S, \mathcal{O}_S)$ do not have a common zero on S . Then there is $a_1, \dots, a_n \in \Gamma(S, \mathcal{O}_S)$ such that

$$\sum_{i=1}^n a_i g_i = 1.$$

Proof We define a homomorphism of \mathcal{O}_S -modules:

$$\mathcal{O}_S^n \xrightarrow{\cdot(f_1, \dots, f_n)} \mathcal{F}.$$

By assumption, this map is surjective. Therefore, it follows from Cartan's Theorem B that the following map is also surjective:

$$\Gamma(S, \mathcal{O}_S)^n \rightarrow \Gamma(S, \mathcal{F}), \quad (a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_i f_i.$$

Our assertion follows. \square

Lemma 3.1.2 *Let S be a Stein space, and $\mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of coherent \mathcal{O}_S -modules. Then the image of $\Gamma(S, \mathcal{F})$ in $\Gamma(S, \mathcal{G})$ is closed.*

Proof Let $\mathcal{H} \subseteq \mathcal{G}$ be the image of \mathcal{F} . By Cartan's Theorem B, the image of $\Gamma(S, \mathcal{F})$ in $\Gamma(S, \mathcal{G})$ is just $\Gamma(S, \mathcal{H})$, which is closed by [Proposition 2.2.1\(5\)](#). \square

Proposition 3.1.2 *Let S be a reduced Stein space. Then*

$$\chi_S: S \rightarrow \mathrm{Sp} \Gamma(S, \mathcal{O}_S)$$

is a bijection.

See [Definition 2.2.3](#) for the general construction of χ_S . We will later show that χ_S is even an isomorphism of locally \mathbb{C} -ringed spaces, even when S is not necessarily reduced, see [Lemma 3.3.3](#).

Proof Let $A = \Gamma(S, \mathcal{O}_S)$. Since S is reduced, the canonical Fréchet topology on A is the compact-open topology, cf. [Proposition 2.2.3](#). Let $\chi \in \mathrm{Sp} A$ be a character. We show that there exists a unique $x \in S$ such that $\chi(f) = f(x)$ for all $f \in A$. The uniqueness is clear since S is holomorphically separable. We only need to show the existence.

Since χ is continuous, we can find a compact subset $K \subseteq S$ and a constant $C > 0$ such that

$$|\chi(f)| \leq C \sup_K |f|$$

for all $f \in A$. Since χ is multiplicative, we can take $C = 1$. Namely,

$$|\chi(f)| \leq \sup_K |f|, \quad \forall f \in A. \quad (3.1)$$

Since S is Stein, we may replace K by \widehat{K}_S and assume that K is $\mathcal{O}(S)$ -convex.

We claim that there exists $x \in K$ such that $\chi(f) = f(x)$ for every $f \in A$.

Suppose not. Then for every $x \in K$, there exists $f_x \in A$ such that

$$f_x(x) \neq 0, \quad \chi(f_x) = 0.$$

By compactness of K , we may choose finitely many $f_1, \dots, f_m \in A$ such that $\chi(f_i) = 0$ for all i . Moreover, for each $x \in K$, there exists i such that $f_i(x) \neq 0$. In other words, the functions f_1, \dots, f_m have no common zero on K .

By [Lemma 3.1.1](#), we can find an open Stein neighborhood V of K such that f_1, \dots, f_m have no common zero on V . By [Proposition 3.1.1](#), there exist

$$a_1, \dots, a_m \in \Gamma(V, \mathcal{O}_V)$$

such that

$$\sum_{i=1}^m a_i f_i = 1$$

on V .

By the Oka–Weil theorem we can find $b_1, \dots, b_m \in A$ approximating a_1, \dots, a_m uniformly on K . Thus we may arrange that

$$\sup_K \left| 1 - \sum_{i=1}^m b_i f_i \right| < 1.$$

On the other hand, since $\chi(f_i) = 0$, we find

$$\chi \left(1 - \sum_{i=1}^m b_i f_i \right) = 1.$$

Using (3.1), we get

$$1 = \left| \chi \left(1 - \sum_{i=1}^m b_i f_i \right) \right| \leq \sup_K \left| 1 - \sum_{i=1}^m b_i f_i \right| < 1,$$

a contradiction. □

3.2 Runge subsets

Definition 3.2.1 Let S be a Stein space. An open subset $U \subseteq S$ is *Runge* if

- (1) U is Stein;
- (2) the restriction map $\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(U, \mathcal{O}_U)$ has dense image.

We also say (S, U) is *Runge* or U is *Runge* in S .

Here on the section space $\Gamma(U, \mathcal{O}_U)$, we used the canonical Fréchet topology, as studied in Section 2.2.

Lemma 3.2.1 Let $U \subseteq S$ be an open subset. Let $\{S_i\}_{i \in I}$ be the connected components of S . Then the following are equivalent:

- (1) U is Runge in S ;
- (2) for each $i \in I$, $U \cap S_i$ is Runge in S_i .

Similarly, for each coherent \mathcal{O}_S -module, the restriction map

$$\Gamma(S, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F})$$

has dense image if and only if for each $i \in I$,

$$\Gamma(S_i, \mathcal{F}) \rightarrow \Gamma(U \cap S_i, \mathcal{F}|_{S_i}).$$

has dense image.

Proof This follows immediately from [Lemma 1.1.3](#). □

Lemma 3.2.2 *Let S be a Stein space, and let $U \subseteq S$ be a Runge open subset. If $V \subseteq U$ is a union of connected components of U , then V is Runge in S .*

Proof Since U is Stein, each connected component of U is Stein. Hence V , being a union of connected components of U , is Stein as well.

It remains to prove the density of the restriction map

$$\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(V, \mathcal{O}_V).$$

By assumption, the image of

$$\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(U, \mathcal{O}_U)$$

is dense. The restriction map

$$\Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(V, \mathcal{O}_V)$$

is continuous and surjective, because a section on V extends to U by putting it equal to 0 on the complementary union of connected components. Therefore the image of $\Gamma(S, \mathcal{O}_S)$ in $\Gamma(V, \mathcal{O}_V)$ is dense. □

A convenient criterion for checking the Runge property is the following:

Lemma 3.2.3 *Let S be a Stein space, let $U \subseteq S$ be an open Stein subspace, and let $K \subseteq U$ be compact. Suppose that $\widehat{K}_S \subseteq U$. Then*

$$\widehat{K}_U = \widehat{K}_S.$$

Proof In general, we always have $\widehat{K}_U \subseteq \widehat{K}_S$, see [Proposition 2.4.1](#).

Conversely, let $x \in \widehat{K}_S$. Note that by our assumption $x \in U$. We show that $x \in \widehat{K}_U$. Suppose not. Then there exists $g \in \Gamma(U, \mathcal{O}_U)$ such that

$$|g(x)| > \sup_K |g|.$$

Since S is Stein, \widehat{K}_S is a compact $\mathcal{O}(S)$ -convex subset of S . By assumption, $\widehat{K}_S \subseteq U$. By the Oka–Weil theorem, we can choose $f \in \Gamma(S, \mathcal{O}_S)$ sufficiently close to g on \widehat{K}_S and at x so that

$$|f(x)| > \sup_K |f|.$$

This contradicts $x \in \widehat{K}_S$. □

Lemma 3.2.4 *Let S be a reduced Stein space, and $U \subseteq S$ be an open subset. Then the following are equivalent:*

- (1) (S, U) is Runge;
- (2) U is Stein, and for any compact subset $K \subseteq U$ we have $\widehat{K}_S = \widehat{K}_U$;
- (3) U is Stein, and for any coherent \mathcal{O}_S -module \mathcal{F} , the restriction

$$\Gamma(S, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F}|_U) \quad (3.2)$$

has dense image.

Proof In all of (1), (2), and (3), the set U is Stein by assumption. Thus we shall always assume this in the proof.

(1) \implies (2). Assume (1). Let $K \subseteq U$ be a compact subset. In view of [Lemma 3.2.3](#), it suffices to show that $\widehat{K}_S \subseteq U$.

Assume by contradiction that there exists $x \in \widehat{K}_S \setminus U$. Then for any $f \in \Gamma(S, \mathcal{O}_S)$, we have

$$|f(x)| \leq \sup_K |f|.$$

Let F denote the image of $\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(U, \mathcal{O}_U)$. We define a map $\chi: F \rightarrow \mathbb{C}$ as follows: Given any $g \in F$, we find $f \in \Gamma(S, \mathcal{O}_S)$, and define $\chi(g) = f(x)$. Note that $\chi(g)$ is independent of the choice of f . Therefore, χ defines a character on F .

We claim that χ is continuous. For this purpose, take a sequence $g_i \rightarrow 0$ in F . Lift each g_i to $f_i \in \Gamma(S, \mathcal{O}_S)$. Then

$$|f_i(x)| \leq \sup_K |f_i| \leq \sup_K |g_i|.$$

Thus $\chi(g_i) \rightarrow 0$. Hence χ is indeed continuous.

By assumption, F is dense in $\Gamma(U, \mathcal{O}_U)$, χ admits a unique continuous extension to a continuous character

$$\chi: \Gamma(U, \mathcal{O}_U) \rightarrow \mathbb{C}.$$

By [Proposition 3.1.2](#), there is therefore $y \in U$ with $\chi(g) = g(y)$ for all $g \in \Gamma(U, \mathcal{O}_U)$. In particular, $f(x) = f(y)$ for all $f \in \Gamma(S, \mathcal{O}_S)$. This contradicts the fact that S is holomorphically separable.

(2) \implies (1). Assume (2).

We prove that the restriction map

$$\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(U, \mathcal{O}_U)$$

has dense image.

Let $f \in \Gamma(U, \mathcal{O}_U)$, let $K \subseteq U$ be compact, and let $\epsilon > 0$. We shall find $F \in \Gamma(S, \mathcal{O}_S)$ such that

$$\sup_K |F - f| < \epsilon. \quad (3.3)$$

Replacing K by \widehat{K}_U , we may assume that it is $\mathcal{O}(U)$ -convex. By our assumption, it is also $\mathcal{O}(S)$ -convex. Thus (3.3) follows from the Oka–Weil theorem on S .

(3) \implies (1). This is obvious.

(1) \implies (3). Assume (1).

Let $s \in \Gamma(U, \mathcal{F}|_U)$, let $K \subseteq U$ be compact. It suffices to show that s can be approximated on K by the image of (3.2). Since U is Stein, after replacing K by its holomorphic hull in U , we may assume that K is $\mathcal{O}(U)$ -convex. It is also $\mathcal{O}(S)$ -convex by the already proved (1) \implies (2).

By Lemma 3.1.1 and Cartan's Theorem A, we can find finitely many $e_1, \dots, e_N \in \Gamma(S, \mathcal{F})$ generating the germs of \mathcal{F} on a Stein open neighborhood $V \subseteq U$ of K . By Proposition 3.1.1, we can find $a_1, \dots, a_N \in \Gamma(V, \mathcal{O}_V)$ such that

$$s|_V = \sum_{i=1}^N a_i e_i|_V.$$

The Oka–Weil theorem gives functions $b_1, \dots, b_N \in \Gamma(S, \mathcal{O}_S)$ such that $\sup_K |b_i - a_i|$ is arbitrarily small. Our assertion now follows from Corollary 2.2.1 and Proposition 2.2.3. \square

Lemma 3.2.5 *Let S be a reduced Stein space. Let $K \subseteq S$ be an $\mathcal{O}(S)$ -convex compact subset. Then K has a fundamental system of open Runge neighborhoods.*

Proof Let $U \subseteq S$ be a relatively compact open neighborhood of K . We need to construct a Runge open neighborhood $V \subseteq S$ of K contained in U .

For each $x \in \partial U$, we can find $f \in \Gamma(S, \mathcal{O}_S)$ with $|f(x)| > 1$ and $|f| < 1$ on K . Since ∂U is compact, we can find finitely many such functions $f_1, \dots, f_m \in \Gamma(S, \mathcal{O}_S)$ such that for each $x \in \partial U$, there exists i with $|f_i(x)| > 1$ while for all i , $|f_i| < 1$ on K . Then we can set

$$W := \{y \in S : |f_i(y)| < 1 \text{ for all } i = 1, \dots, m\}.$$

Note that $W \supseteq K$ and $W \cap \partial U = \emptyset$. We claim that $W \subseteq S$ is Runge.

It is clearly Stein. By Lemma 3.2.3 and Lemma 3.2.4 it remains to show that for each compact subset $L \subseteq W$, we have $\widehat{L}_S \subseteq W$. If not, there exists $x \in \widehat{L}_S$ with $x \notin W$. Then there exists i such that $|f_i(x)| \geq 1$. On the other hand, since $L \subseteq W$, we have $|f_i| < 1$ on L . This contradicts $x \in \widehat{L}_S$.

Now $W \cap \partial U$ is empty. The set $V := W \cap U$ is the union of some connected components of W . Therefore, V is Runge in S by Lemma 3.2.2. \square

Lemma 3.2.6 *Let S be a reduced Stein space. Then there is a sequence of relatively compact Runge open subsets*

$$U_1 \Subset U_2 \Subset \dots$$

exhausting S .

Proof Since S is σ -compact, we can find a sequence $K_1 \subseteq K_2 \subseteq \dots$ of compact subsets of S exhausting it. By an inductive modification of each K_i , we may further assume that

(1) each K_i is $\mathcal{O}(S)$ -convex;

(2) $K_i \subseteq \text{Int } K_{i+1}$.

Applying [Lemma 3.2.5](#), we can find an open Runge neighborhood $U_i \subseteq S$ of K_i compactly contained in $\text{Int } K_{i+1}$. Our assertion follows. \square

The following result is the key for handling Runge subsets in the non-reduced setting.

Theorem 3.2.1 *Let S be a Stein space. Let $U \subseteq S$ be an open subset. Then the following assertions for (S, U) are equivalent, and they are equivalent to the corresponding assertions for $(S^{\text{red}}, U^{\text{red}})$:*

- (1) (S, U) is Runge;
- (2) U is Stein, and for any compact subset $K \subseteq U$ we have $\widehat{K}_S = \widehat{K}_U$;
- (3) U is Stein, and for any coherent \mathcal{O}_S -module \mathcal{F} , the restriction

$$\Gamma(S, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F}|_U) \quad (3.4)$$

has dense image.

Proof We write (*r) for the statement for the reduced pair $(S^{\text{red}}, U^{\text{red}})$. In the proof below, we shall use [Theorem 3.1.1](#) without explicitly mentioning.

By [Lemma 3.2.4](#), (1r), (2r) and (3r) are equivalent.

(3) \implies (1): This is obvious.

(2) \iff (2r): Since U is Stein, the restriction

$$\Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(U^{\text{red}}, \mathcal{O}_{U^{\text{red}}})$$

is surjective, so $\widehat{K}_U = \widehat{K}_{U^{\text{red}}}$. Similarly, $\widehat{K}_S = \widehat{K}_{S^{\text{red}}}$. Therefore, $\widehat{K}_{S^{\text{red}}} = \widehat{K}_{U^{\text{red}}}$.

(1) \implies (1r): We have a natural commutative diagram of Fréchet spaces induced by restriction:

$$\begin{array}{ccc} \Gamma(S, \mathcal{O}_S) & \longrightarrow & \Gamma(S^{\text{red}}, \mathcal{O}_{S^{\text{red}}}) \\ \downarrow & & \downarrow \\ \Gamma(U, \mathcal{O}_U) & \longrightarrow & \Gamma(U^{\text{red}}, \mathcal{O}_{U^{\text{red}}}). \end{array}$$

The two horizontal maps are surjective, since both S and U are Stein, under the assumption of either (1) or (1r). Therefore, if the left-vertical map has dense image, so is the right-vertical map.

(3r) \implies (3): Assume (3r). Let \mathcal{N} be the nilradical of S .

By [Lemma 3.2.6](#), we can find a family of relatively compact open subsets of U :

$$U_1 \Subset U_2 \Subset \dots$$

exhausting U , and $(U^{\text{red}}, U_i^{\text{red}})$ is Runge for each i . We observe that $(S^{\text{red}}, U_i^{\text{red}})$ is then Runge for each i , and hence (3r) holds with U_i in place of U . By [Lemma 1.1.6](#) and [Corollary 2.2.3](#), we may replace U by each U_i and assume that there is $k > 0$ with $\mathcal{N}^k = 0$ on U .

By [Lemma 1.1.9](#), the class of sheaves \mathcal{F} for which [\(3.4\)](#) has dense image is stable under extensions, so we may reduce to the case $\mathcal{N}\mathcal{F} = 0$. In other words, \mathcal{F} is the pushforward of a coherent $\mathcal{O}_{S^{\text{red}}}$ -module. Then [\(3\)](#) follows immediately from [\(3r\)](#). \square

Corollary 3.2.1 *Let S be a Stein space. Let $K \subseteq S$ be an $\mathcal{O}(S)$ -convex compact subset. Then K has a fundamental system of open Runge neighborhoods.*

In particular, K is Stein.

Proof Since S is Stein, the restriction

$$\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(S^{\text{red}}, \mathcal{O}_{S^{\text{red}}})$$

is surjective. Therefore,

$$\widehat{K}_S = \widehat{K}_{S^{\text{red}}}.$$

In particular, K is also $\mathcal{O}(S^{\text{red}})$ -convex. Our assertion now follows immediately from [Lemma 3.2.5](#) and [Theorem 3.2.1](#). \square

Corollary 3.2.2 *Let S be a Stein space. There is a sequence of relatively compact Runge open subsets*

$$U_1 \Subset U_2 \Subset \cdots$$

exhausting S .

Proof Apply [Lemma 3.2.6](#) to S^{red} , and then use [Theorem 3.2.1](#) to lift the Runge property from S^{red} to S . \square

3.3 Stein algebras

Definition 3.3.1 A Fréchet algebra A is a *Stein algebra* if there is a Stein space S with $A \cong \Gamma(S, \mathcal{O}_S)$ as Fréchet algebras.

We define \mathcal{SAlg} as the following category:

- (1) The objects are Stein algebras;
- (2) the morphisms are homomorphisms of \mathbb{C} -algebras;
- (3) the compositions are the usual compositions.

Note that at this stage, we cannot assert that \mathcal{SAlg} is a full subcategory of the category of Fréchet algebras \mathcal{FrAlg} yet, as the morphisms in \mathcal{SAlg} are not assumed continuous. We will see later that they are indeed continuous automatically.

3.3.1 0-dimensional case

The 0-dimensional case is easy to understand.

Proposition 3.3.1 *For a Fréchet algebra A , the following are equivalent:*

- (1) A is the Stein algebra of a 0-dimensional Stein space;
- (2) A is isomorphic to a countable product of finite-dimensional non-zero \mathbb{C} -algebras.

On a finite-dimensional \mathbb{C} -algebra, we use the usual Euclidean topology.

Proof (1) \implies (2). Let S be a 0-dimensional Stein space with

$$A \cong \Gamma(S, \mathcal{O}_S).$$

Since S is 0-dimensional and σ -compact, the underlying topological space of S is a disjoint union of countably many points. Thus we may reduce to the case where $S = \{s\}$ consists of a single point. Then

$$A \cong \mathcal{O}_{S,s}$$

is an Artinian local \mathbb{C} -algebra. In particular, A is finite-dimensional over \mathbb{C} , and therefore its Fréchet topology is the Euclidean topology.

(2) \implies (1). Suppose

$$A = \prod_{i \in I} A_i,$$

where I is countable and each A_i is a finite-dimensional non-zero \mathbb{C} -algebra, endowed with its Euclidean topology. It suffices to treat the case where $A \neq 0$ is finite-dimensional over \mathbb{C} . Then A is Artinian, hence a finite product of Artinian local rings; reducing further, we may assume that A is local.

Let \mathfrak{m} be the maximal ideal of A . Since A/\mathfrak{m} is a finite extension of \mathbb{C} , we have

$$A/\mathfrak{m} \cong \mathbb{C}.$$

In particular,

$$A = \mathbb{C} + \mathfrak{m}.$$

Choose generators x_1, \dots, x_n of \mathfrak{m} . The universal property of convergent power series yields a surjective \mathbb{C} -algebra homomorphism

$$\varphi: \mathbb{C}\{z_1, \dots, z_n\} \rightarrow A, \quad z_j \mapsto x_j.$$

This map is well-defined since $\mathfrak{m}^N = 0$ for some $N > 0$, so any power series evaluates to a finite sum.

Let $I = \ker(\varphi)$. Then

$$A \cong \mathbb{C}\{z_1, \dots, z_n\}/I.$$

We now determine \sqrt{I} . Since A is Artinian local, its nilradical coincides with \mathfrak{m} , hence

$$A^{\text{red}} = A/\mathfrak{m} \cong \mathbb{C}.$$

On the other hand,

$$A^{\text{red}} \cong \mathbb{C}\{z_1, \dots, z_n\}/\sqrt{I}.$$

Therefore,

$$\sqrt{I} = (z_1, \dots, z_n).$$

Choose generators f_1, \dots, f_r of the ideal $I \subseteq \mathbb{C}\{z_1, \dots, z_n\}$. After shrinking a polydisc $U \subseteq \mathbb{C}^n$ around 0, we may regard all f_j as holomorphic functions on U . Let $\mathcal{I} \subseteq \mathcal{O}_U$ be the coherent ideal sheaf generated by these functions. Since $\sqrt{I} = (z_1, \dots, z_n)$, the common zero germ of the f_j 's is just $\{0\}$; after shrinking U once more, the support of $\mathcal{O}_U/\mathcal{I}$ is therefore $\{0\}$.

Define S to be the closed analytic subspace of U defined by \mathcal{I} . Then S is a closed analytic subspace of the Stein space U , hence is Stein, and its underlying topological space is $\{0\}$.

Moreover,

$$\Gamma(S, \mathcal{O}_S) = \Gamma(U, \mathcal{O}_U/\mathcal{I}) = (\mathcal{O}_U/\mathcal{I})_0 = \mathbb{C}\{z_1, \dots, z_n\}/I \cong A.$$

We conclude the proof. \square

Corollary 3.3.1 *The Stein algebra of a reduced 0-dimensional Stein space is just a countable product of \mathbb{C} .*

Proof This follows immediately from [Proposition 3.3.1](#). \square

3.3.2 Satz von H. Cartan

Let S be a Stein space with corresponding Stein algebra $A = \Gamma(S, \mathcal{O}_S)$.

Theorem 3.3.1 (H. Cartan) *Let $I \subseteq A$ be an ideal. Then*

$$\bar{I} = \Gamma(S, I\mathcal{O}_S). \quad (3.5)$$

Recall that the ideal sheaf $I\mathcal{O}_S$ is constructed in [Definition 2.3.1](#) and is coherent.

Proof The inclusion $\bar{I} \subseteq \Gamma(S, I\mathcal{O}_S)$ holds for any σ -compact Hausdorff complex space X , see [\(2.9\)](#) and [Proposition 2.2.1\(5\)](#).

It remains to prove the reverse inclusion.

Step 1. We first assume that $S = \mathbb{C}^n$.

Let $f \in \Gamma(S, I\mathcal{O}_S)$. We show that $f \in \bar{I}$. Since S is reduced, the topology on A is the compact-open topology, see [Proposition 2.2.3](#). Let $K \subseteq S$ be compact and let $\epsilon > 0$. It suffices to find $g \in I$ such that

$$\sup_K |f - g| < \epsilon. \quad (3.6)$$

Without loss of generality, we may replace K by a larger convex subset and assume that K is a closed polydisk centered at $0 \in \mathbb{C}^n$.

For each $x \in K$, we can find holomorphic functions $g_1, \dots, g_m \in I$ generating IO_S in an open neighborhood of x . By the compactness of K , we may then assume that $g_1, \dots, g_m \in I$ generate IO_S on an open polydisk U containing K .

Then we have a surjective homomorphism of coherent \mathcal{O}_U -modules:

$$\mathcal{O}_S|_U^m \xrightarrow{(g_1, \dots, g_m)} (IO_S)|_U.$$

By [Proposition 3.1.1](#), we can find $f_1, \dots, f_m \in \Gamma(U, \mathcal{O}_S)$ so that

$$f|_U = \sum_{i=1}^m f_i g_i|_U.$$

By the Oka–Weil theorem, for each $i = 1, \dots, m$, we can find $f'_i \in A$ with

$$\sup_K |f_i - f'_i| \leq m^{-1} \epsilon \cdot \left(1 + \sup_K |g_i|\right)^{-1}.$$

Then [\(3.6\)](#) follows with

$$g = \sum_{i=1}^m f'_i g_i.$$

Step 2. We assume that S admits a closed immersion $i: S \hookrightarrow \mathbb{C}^n$.

Let A' be the Stein algebra of \mathbb{C}^n . Then we have a natural continuous surjective map $F: A' \rightarrow A$. Note that F is also an open map by the open mapping theorem. By [Lemma 1.1.4](#), we find that

$$\bar{I} = F\left(\overline{F^{-1}(I)}\right).$$

By Step 1, we have

$$\overline{F^{-1}(I)} = \Gamma\left(\mathbb{C}^n, F^{-1}(I)\mathcal{O}_{\mathbb{C}^n}\right).$$

Thus it remains to argue that any $f \in \Gamma(S, IO_S)$ can be written as $F(g)$ for some $g \in \Gamma\left(\mathbb{C}^n, F^{-1}(I)\mathcal{O}_{\mathbb{C}^n}\right)$. For this purpose, observe that we have a natural surjective homomorphism of coherent $\mathcal{O}_{\mathbb{C}^n}$ -modules

$$F^{-1}(I)\mathcal{O}_{\mathbb{C}^n} \rightarrow i_*(IO_S).$$

By Cartan's Theorem B again, our assertion follows.

Step 3. We handle the general case. Let $f \in \Gamma(S, IO_S)$. We prove that $f \in \bar{I}$.

Choose a Runge Stein exhaustion of S :

$$U_1 \Subset U_2 \Subset \dots.$$

The existence is guaranteed by [Corollary 3.2.2](#). Write A_i for the Stein algebra of U_i . Write the restriction map as $F_i: A \rightarrow A_i$.

Note that A is the inverse limit of A_i in the category of locally convex topological vector spaces, by [Corollary 2.2.3](#). Therefore, in view of [Lemma 1.1.5](#), it suffices to

verify that for each i ,

$$f|_{U_i} \in \overline{F_i(I)}. \quad (3.7)$$

Fix i . Note that each U_i admits a closed immersion into some \mathbb{C}^n and hence Step 2 is applicable. Step 2 gives

$$\overline{IA_i} = \Gamma(U_i, (IA_i \mathcal{O}_S)|_{U_i}).$$

Therefore,

$$f|_{U_i} \in \overline{IA_i}.$$

Hence in order to prove (3.7), it suffices to prove

$$\overline{F_i(I)} = \overline{IA_i}. \quad (3.8)$$

The inclusion $\overline{F_i(I)} \subseteq \overline{IA_i}$ is clear, so it suffices to prove that every element of $\overline{IA_i}$ belongs to $\overline{F_i(I)}$.

Let $g \in \overline{IA_i}$. Then

$$g = \sum_{j=1}^N a_j F_i(b_j)$$

for some $a_j \in A_i$ and $b_j \in I$. Since $F_i(A)$ is dense in A_i , we can approximate each coefficient a_j in A_i by elements $a_{j,k} \in A$. Then

$$g_k := \sum_{j=1}^N a_{j,k} b_j \in I,$$

and

$$F_i(g_k) = \sum_{j=1}^N F_i(a_{j,k}) F_i(b_j) \rightarrow \sum_{j=1}^N a_j F_i(b_j) = g$$

in A_i as $k \rightarrow \infty$. Here we applied [Corollary 2.2.1](#). Hence $g \in \overline{F_i(I)}$. This proves (3.8). \square

Corollary 3.3.2 *Let $I \subseteq A$ be an ideal. Then I is closed if and only if $I = \Gamma(S, IO_S)$.*

Proof This follows immediately from [Theorem 3.3.1](#). \square

Corollary 3.3.3 *Let $I \subseteq A$ be a proper closed ideal. Then $\mathbb{V}(IO_S)$ is non-empty.*

Here $\mathbb{V}(I)$ denotes the analytic subspace of S defined by the coherent ideal I .

Proof If $\mathbb{V}(IO_S)$ is empty, then $IO_S = \mathcal{O}_S$, so by [Corollary 3.3.2](#), we have $I = \Gamma(S, IO_S) = A$. \square

There are plenty of closed ideals in A .

Proposition 3.3.2 *Any finitely generated ideal in A is closed.*

The converse fails.

Proof Let $I \subseteq A$ be a finitely generated ideal, say with generators f_1, \dots, f_n . Then we have a homomorphism of \mathcal{O}_S -modules

$$\mathcal{O}_S^n \xrightarrow{(f_1, \dots, f_n)} \mathcal{O}_S.$$

By **Lemma 3.1.2**, we then conclude that I is closed. \square

Lemma 3.3.1 *Let $x \in S$, and let $f_1, \dots, f_n \in A$. Assume that*

- (1) x is the unique common zero of f_1, \dots, f_n ;
- (2) the germs $f_{1,x}, \dots, f_{n,x}$ generate the maximal ideal $\mathfrak{m}_{S,x} \subseteq \mathcal{O}_{S,x}$.

Then f_1, \dots, f_n generate the ideal $\ker \chi_S(x)$.

Recall from **Definition 2.2.3** that $\chi_S: S \rightarrow \text{Sp } A$ is the character map.

Proof Let $g \in \ker \chi_S(x)$. Namely $g \in A$ and $g(x) = 0$. Let

$$\mathcal{J} := (f_1, \dots, f_n)\mathcal{O}_S \subseteq \mathcal{O}_S$$

be the coherent ideal sheaf generated by the f_i 's. Then $g \in \Gamma(S, \mathcal{J})$, as can be checked stalkwise.

Now by **Proposition 3.1.1**, there exist $a_1, \dots, a_n \in A$ such that

$$g = \sum_{i=1}^n a_i f_i.$$

Our assertion follows. \square

Definition 3.3.2 Let S be a Stein space, and let $x \in S$. We shall write

$$\mathfrak{m}(x) := \ker \chi_S(x) = \{f \in A : f(x) = 0\}.$$

Then $\mathfrak{m}(x)$ is a closed maximal ideal of A .

This should not be confused with $\mathfrak{m}_{S,x}$ denoting the maximal ideal of $\mathcal{O}_{S,x}$.

Corollary 3.3.4 *Let \mathfrak{m} be a maximal ideal of A . Then the following are equivalent:*

- (1) \mathfrak{m} is closed;
- (2) $\mathfrak{m} = \mathfrak{m}(x)$ for some $x \in S$;
- (3) \mathfrak{m} is finitely generated.

Note that in (2), the point $x \in S$ is uniquely determined, since S is holomorphically separable.

We remind the reader that there are many non-closed maximal ideals on A , already when $S = \mathbb{C}$.

Proof (1) \implies (2). Assume that \mathfrak{m} is closed. Then there is some point $x \in \mathbb{V}(\mathfrak{m}\mathcal{O}_S)$ by [Corollary 3.3.3](#). Then $\mathfrak{m} \subseteq \mathfrak{m}(x)$ by definition. Since \mathfrak{m} is maximal, we find that equality holds.

(2) \implies (3). Choose $x \in S$. Let S' be the union of all irreducible components of S containing x . Then S' is a finite-dimensional Stein space. We shall construct finitely many $f_1, \dots, f_n \in A$ with $f_i(x) = 0$ such that the restrictions $f_i|_{S'}$ have no common zeros except x .

This is possible, for example, by the following argument. By the embedding theorem [Theorem 3.1.2](#), we can find a closed immersion $S' \hookrightarrow \mathbb{C}^n$ sending x to 0. Let z_1, \dots, z_n be the coordinate functions on \mathbb{C}^n . Then the restrictions $z_i|_{S'}$ have no common zeros except x . Since S is Stein, we can lift each $z_i|_{S'}$ to $f_i \in A$.

Let S_1 be the union of all irreducible components of S not containing x . Then S_1 is a closed analytic subset of S and $x \notin S_1$. By Cartan's Theorem B applied to the closed analytic subspace $S_1 \cup \{x\}$, there exists $f_{n+1} \in A$ such that

$$f_{n+1}(x) = 0, \quad f_{n+1}|_{S_1} = 1.$$

Then f_1, \dots, f_{n+1} have no common zero on S except x .

Take functions $g_1, \dots, g_m \in A$ with $g_{i,x}$ generating $\mathfrak{m}_{S,x}$. Such g_i 's exist by Cartan's Theorem A. In particular, the ideal generated by $f_1, \dots, f_{n+1}, g_1, \dots, g_m$ is $\mathfrak{m}(x)$ by [Lemma 3.3.1](#).

(3) \implies (1). This is a consequence of [Proposition 3.3.2](#). \square

Corollary 3.3.5 *Let $x \in S$. Consider a closed primary ideal $\mathfrak{p} \subseteq A$ contained in $\mathfrak{m}(x)$. Then for any $f \in A$, the following are equivalent:*

- (1) $f \in \mathfrak{p}$;
- (2) $f_x \in (\mathfrak{p}\mathcal{O}_S)_x$.

Note that (2) means precisely that f_x lies in the $\mathcal{O}_{S,x}$ -ideal generated by \mathfrak{p}_x .

Proof (1) \implies (2): This is trivial.

(2) \implies (1): Assume (2). The colon ideal $\mathcal{F} := (\mathfrak{p}\mathcal{O}_S : (f)\mathcal{O}_S)$ is coherent, and the germ at x is precisely $\mathcal{O}_{S,x}$ by assumption. By Cartan's Theorem A, we can find $g \in \Gamma(S, \mathcal{F})$ with $g(x) \neq 0$. Then $gf \in \Gamma(S, \mathfrak{p}\mathcal{O}_S)$, so $gf \in \Gamma(S, \mathfrak{p}\mathcal{O}_S) = \mathfrak{p}$ by [Corollary 3.3.2](#). Suppose by contradiction that (1) fails. Then $g^n \in \mathfrak{p}$ for some $n > 0$. But $g^n(x) \neq 0$, so we get a contradiction. \square

Lemma 3.3.2 *Let $x \in S$ and $n \in \mathbb{N}$. Then*

$$(\mathfrak{m}(x)^n \mathcal{O}_S)_x = \mathfrak{m}_{S,x}^n. \tag{3.9}$$

Proof The \subseteq direction in (3.9) is trivial.

For the reverse direction, let $g \in \mathfrak{m}_{S,x}^n$. We shall show that $g \in (\mathfrak{m}(x)^n \mathcal{O}_S)_x$. For this purpose, we may assume that $g = g_1 \cdots g_n$ with $g_i \in \mathfrak{m}_{S,x}$.

Take $f_1, \dots, f_m \in A$ so that $f_{1,x}, \dots, f_{m,x}$ generate $\mathfrak{m}_{S,x}$ as an $\mathcal{O}_{S,x}$ -module. The existence is guaranteed by Cartan's Theorem A. Then we can express each g_i as

$$g_i = \sum_{j=1}^m g_{ij} f_{j,x}, \quad g_{ij} \in \mathcal{O}_{S,x}.$$

Then observe that an arbitrary product of n -elements (possibly with repetition) among the f_j 's lies in $\mathfrak{m}(x)^n$. \square

Corollary 3.3.6 (Forster) *Let $x \in S$ and $n \in \mathbb{N}$. Then for any $f \in A$, the following are equivalent:*

- (1) $f \in \mathfrak{m}(x)^n$;
- (2) $f_x \in \mathfrak{m}_{S,x}^n$.

Proof There is nothing to prove when $n = 0$. We may assume that $n > 0$. Then $\mathfrak{m}(x)^n$ is a primary ideal in A by [Lemma 1.2.1](#). Thus our assertion follows from [Lemma 3.3.2](#) and [Corollary 3.3.5](#). \square

We also observe the following simple converse to [Theorem 3.3.1](#).

Proposition 3.3.3 *Let \mathcal{I} be a coherent ideal sheaf on S . Then*

$$\mathcal{I} = \Gamma(S, \mathcal{I})\mathcal{O}_S. \quad (3.10)$$

In particular, the map $\mathcal{I} \mapsto \Gamma(S, \mathcal{I})$ is a bijection between coherent ideal sheaves on S and closed ideals of A .

Proof Since both sides of (3.10) are ideal sheaves, it suffices to show that they have the same germ everywhere, which is clear by Cartan's Theorem A.

The bijection follows from (3.10) and [Theorem 3.3.1](#). \square

3.3.3 The equivalence of categories

Lemma 3.3.3 *Let S be a Stein space. Then the canonical morphism*

$$\chi: S \rightarrow \mathrm{Sp}\Gamma(S, \mathcal{O}_S) \quad (3.11)$$

of locally \mathbb{C} -ringed spaces is an isomorphism.

The morphism (3.11) is defined in [Proposition 2.2.5](#).

Proof For simplicity, we write $A = \Gamma(S, \mathcal{O}_S)$.

Step 1. We first show that the underlying topological space map is a homeomorphism.

It is clear that χ_S is injective and it is surjective by [Corollary 3.3.4](#). It only remains to show that χ_S is an open map.

Take $x \in S$ and an open neighborhood $U \subseteq S$ of x . Let S_0 be the union of all irreducible components of S containing x , and S_1 be the union of all other components. Then S_0 is a finite-dimensional Stein space and by the embedding

theorem **Theorem 3.1.2**, we can find holomorphic functions $f_1, \dots, f_N \in \Gamma(S_0, \mathcal{O}_{S_0})$ and $\epsilon \in (0, 1)$ so that

$$\{y \in S_0 : |f_i(y) - f_i(x)| < \epsilon \quad \forall i = 1, \dots, N\} \subseteq U.$$

Since S is Stein, we can lift f_i to $g_i \in A$. Next, since S is Stein, we can find $g_0 \in A$ with $g_0(x) = 0$ and $g_0|_{S_1} = 1$. Then

$$x \in \{y \in S : |g_i(y) - g_i(x)| < \epsilon \quad \forall i = 0, \dots, N\} \subseteq U.$$

Therefore, $\chi(U)$ is open.

Step 2. We prove that (3.11) is also an isomorphism of \mathbb{C} -locally ringed spaces.

Fix $x \in S$. We need to show that the \mathbb{C} -algebra homomorphism

$$\mathcal{O}_{\text{Sp } A, \chi(x)} = \mathbb{C}\{\mathfrak{m}(x)\}/J_x \rightarrow \mathcal{O}_{S,x} \quad (3.12)$$

defined in **Proposition 2.2.5** is an isomorphism. Here J_x is the ideal generated by all

$$\alpha = \sum_{\nu \in \mathbb{N}^n} c_\nu T_{\mathbf{f}}^\nu, \quad \mathbf{f} = (f_1, \dots, f_n) \in \mathfrak{m}(x)^n \quad (3.13)$$

satisfying

$$\sum_{|\nu| < r} c_\nu \mathbf{f}^\nu \in \mathfrak{m}(x)^r, \quad \forall r > 0. \quad (3.14)$$

Recall that this homomorphism is defined by replacing the formal variable T_f with $f \in \mathfrak{m}(x)$ by $f_x \in \mathfrak{m}_{S,x}$.

Step 2.1. We first argue the surjectivity.

Take $g_1, \dots, g_m \in A$ such that $g_i \in \mathfrak{m}(x)$ and the classes of $g_{i,x}$ generate the \mathbb{C} -linear space $\mathfrak{m}_{S,x}/\mathfrak{m}_{S,x}^2$. This is possible by Cartan's Theorem A. By Nakayama's lemma, $g_{1,x}, \dots, g_{m,x}$ generate $\mathfrak{m}_{S,x}$ as an ideal of $\mathcal{O}_{S,x}$. Then the substitution homomorphism

$$\mathbb{C}\{z_1, \dots, z_m\} \rightarrow \mathcal{O}_{S,x}, \quad z_i \mapsto g_{i,x}$$

is surjective. This homomorphism factors as

$$\mathbb{C}\{z_1, \dots, z_m\} \rightarrow \mathbb{C}\{\mathfrak{m}(x)\}/J_x = \mathcal{O}_{\text{Sp } A, \chi(x)} \rightarrow \mathcal{O}_{S,x},$$

where the first map sends z_i to the class of T_{g_i} . Therefore (3.12) is surjective.

Step 2.2. We now argue the injectivity.

Now let α be an element of $\mathbb{C}\{\mathfrak{m}(x)\}$ mapping to 0 in $\mathcal{O}_{S,x}$ and admitting an expansion as in (3.13). We want to show that (3.14) holds. Since

$$\sum_{\nu \in \mathbb{N}^n} c_\nu \mathbf{f}_x^\nu = 0,$$

we have

$$\sum_{|\nu| < r} c_\nu \mathbf{f}_x^\nu = - \sum_{|\nu| \geq r} c_\nu \mathbf{f}_x^\nu \in \mathfrak{m}_{S,x}^r.$$

Here we used the fact that $\mathfrak{m}_{S,x}^r$ is closed. Now (3.14) follows from [Corollary 3.3.6](#). \square

Lemma 3.3.4 *Let A be a Stein algebra. Then the canonical homomorphism of \mathbb{C} -algebras*

$$A \rightarrow \Gamma(\mathrm{Sp} A, \mathcal{O}_{\mathrm{Sp} A})$$

in [Proposition 1.5.2](#) is an isomorphism of Fréchet algebras.

Proof Let S be a Stein space and choose an isomorphism of Fréchet algebras

$$\varphi: A_S := \Gamma(S, \mathcal{O}_S) \xrightarrow{\sim} A.$$

Write η_B for the canonical map in [Proposition 1.5.2](#) attached to a topological algebra B . We first check that

$$\eta_{A_S}: A_S \rightarrow \Gamma(\mathrm{Sp} A_S, \mathcal{O}_{\mathrm{Sp} A_S})$$

is the inverse of the pullback on global sections induced by the isomorphism $\chi_S: S \rightarrow \mathrm{Sp} A_S$ of [Lemma 3.3.3](#). Indeed, for $f \in A_S$, the section $\eta_{A_S}(f)$ is represented at $y \in \mathrm{Sp} A_S$ by

$$y(f) + T_{f-y(f)} \in \mathbb{C}\{\mathfrak{m}(y)\}.$$

At $y = \chi_S(x)$, the structural map of χ_S sends this germ to

$$f(x) + (f - f(x))_x = f_x \in \mathcal{O}_{S,x}.$$

Hence $\chi_S^* \eta_{A_S}(f) = f$ for every $f \in A_S$. Since χ_S is an isomorphism of locally \mathbb{C} -ringed spaces, χ_S^* , and therefore also η_{A_S} , is an isomorphism of Fréchet algebras.

By the functoriality in [Proposition 1.5.2](#) applied to φ , we have a natural commutative diagram

$$\begin{array}{ccc} A_S & \xrightarrow{\varphi} & A \\ \eta_{A_S} \downarrow & & \downarrow \eta_A \\ \Gamma(\mathrm{Sp} A_S, \mathcal{O}_{\mathrm{Sp} A_S}) & \longrightarrow & \Gamma(\mathrm{Sp} A, \mathcal{O}_{\mathrm{Sp} A}) \end{array}$$

in the category of Fréchet algebras. The upper horizontal map is an isomorphism by the choice of φ , and the lower horizontal map is an isomorphism because it is induced by φ . The left vertical map is an isomorphism by the preceding paragraph. It follows that the remaining right vertical map is also an isomorphism. \square

Lemma 3.3.5 (Benoist) *Let S be a Stein space. Then there is a holomorphic map $F: S \rightarrow \mathbb{C}^2$ such that all fibers are finite-dimensional.*

The proof relies on Oka theory, which is beyond the scope of this book. See [[Ben25b](#)].

Lemma 3.3.6 *Let B be a d -dimensional analytic subset of a Stein space S with $\infty > d \geq 1$. Then there is a holomorphic map $f \in \Gamma(S, \mathcal{O}_S)$ such that, if $f: S \rightarrow \mathbb{C}$ denotes the associated holomorphic map, then all fibers of $f|_B$ have dimension at most $d - 1$.*

Proof We may discard the irreducible components of B with dimension less than d , and hence assume that B is equidimensional. Let B_0, B_1, \dots be the irreducible components of B . The index set I could be either a finite set of the form $\{0, 1, \dots, n\}$ or \mathbb{N} .

For each $i \in I$, choose distinct points $p_i, q_i \in B_i$, not lying on any other component. Then the set $\{p_i, q_i : i \in I\}$ is discrete. Since S is Stein, we can find a holomorphic function $f \in \Gamma(S, \mathcal{O}_S)$ assigning different values to each of these points. Now we assert that $f|_B$ does not have d -dimensional fibers. Otherwise, some fiber of f would contain some B_i , which is impossible since $f(p_i) \neq f(q_i)$. \square

Corollary 3.3.7 *Let A be a Stein algebra. Then all characters of A are continuous.*

Proof Let S be a Stein space with Stein algebra A .

Let $\chi: A \rightarrow \mathbb{C}$ be a character. We show that χ is continuous.

Step 1. We reduce to the case where S is finite-dimensional.

Consider $F: S \rightarrow \mathbb{C}^2$ as in [Lemma 3.3.5](#) with components $f_1, f_2 \in A$. Consider the fiber T of F defined by $f_1 = \chi(f_1)$ and $f_2 = \chi(f_2)$. Then χ factorizes through the surjection

$$A \rightarrow \Gamma(T, \mathcal{O}_T).$$

Therefore, we may replace S by T and assume that it is finite-dimensional, say of dimension d .

Step 2. We prove the continuity of χ by induction on $d \geq 0$.

When $d = 0$, by [Proposition 3.3.1](#) and [Proposition 1.2.5](#), we may assume that A is a finite dimensional \mathbb{C} -algebra. The continuity of χ follows from the general fact that any linear map between finite-dimensional linear spaces is continuous.

Assume that $d > 0$ and for all smaller d , the result is known. We take $f \in A$ so that all fibers of f have dimension at most $d - 1$. The existence of f is guaranteed by [Lemma 3.3.6](#). We may also assume that $\chi(f) = 0$ after adding a constant to f . Let $S' = \mathbb{V}(f)$. By [Corollary 3.3.3](#), S' is non-empty. Now χ factorizes through the surjection $A \rightarrow \Gamma(S', \mathcal{O}_{S'})$. Therefore, χ is continuous by the inductive hypothesis. \square

Theorem 3.3.2 *All \mathbb{C} -algebra homomorphisms between Stein algebras are continuous.*

In particular, $\mathcal{S}\mathcal{A}\text{lg}$ is a full subcategory of $\mathcal{F}\text{r}\mathcal{A}\text{lg}$, and Sp is a well-defined contravariant functor from $\mathcal{S}\mathcal{A}\text{lg}$ to SteinSpace .

Proof Let $F: A \rightarrow B$ be a \mathbb{C} -algebra homomorphism between Stein algebras. By [Corollary 3.3.7](#), F is spectral in the sense of [Definition 1.5.4](#). The morphism $\text{Sp } F: \text{Sp } B \rightarrow \text{Sp } A$ of complex spaces is constructed in [Section 1.5.3.2](#).

Now we have the commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \downarrow & & \downarrow \\ \Gamma(\text{Sp } A, \mathcal{O}_{\text{Sp } A}) & \xrightarrow{\Gamma(\text{Sp } F)} & \Gamma(\text{Sp } B, \mathcal{O}_{\text{Sp } B}) \end{array}$$

In the category of \mathbb{C} -algebras. The two vertical maps are isomorphisms of Fréchet algebras by [Lemma 3.3.4](#), and the lower horizontal map is continuous. It follows that F is also continuous.

Theorem 3.3.3 (Forster, Benoist) *The functor*

$$\Gamma: \text{SteinSpace}^{\text{op}} \rightarrow \mathcal{S}\text{Alg}$$

is an equivalence with quasi-inverse Sp .

Proof It follows from [Theorem 3.3.2](#) that $\text{Sp}: \mathcal{S}\text{Alg}^{\text{op}} \rightarrow \text{SteinSpace}$ is well-defined. It is the quasi-inverse of Γ by [Lemma 3.3.3](#) and [Lemma 3.3.4](#). \square

Corollary 3.3.8 *Let A be a Stein algebra. There are canonical bijections between the following sets:*

- (1) $\text{Sp } A$;
- (2) *the set of all characters $A \rightarrow \mathbb{C}$;*
- (3) *the set of finitely generated maximal ideals in A ;*
- (4) *the set of closed maximal ideals in A ;*
- (5) *the set $\text{Spec } A(\mathbb{C})$.*

Proof This is a consequence of [Corollary 3.3.4](#) and [Corollary 3.3.7](#). \square

Corollary 3.3.9 *Given a Stein algebra A , there is a canonical morphism of locally \mathbb{C} -ringed spaces*

$$i_{\text{Spec } A}: \text{Sp } A \rightarrow \text{Spec } A. \quad (3.15)$$

The map is injective on the underlying sets.

Proof The morphism is induced by [[Stacks, Tag 01I1](#)]. To be more precise, the stacks project result only shows that (3.15) is a morphism of locally ringed spaces. It is easy to see that it is indeed a morphism between locally \mathbb{C} -ringed spaces by abstract nonsense, but let us give a more concrete description of this morphism for later use.

Given $x \in \text{Sp } A$, the image in $\text{Spec } A$ is just $\mathfrak{m}(x)$. This defines the underlying map of topological spaces (3.15). Next given $g \in A$, we consider the basic open set $\mathbb{D}(g) \subseteq \text{Spec } A$. The map on the structure sheaves is given by the obvious homomorphism

$$A_g \rightarrow \Gamma(U_g, \mathcal{O}_{\text{Sp } A}),$$

where U_g is the set of $y \in \text{Sp } A$ with $g(y) \neq 0$. In particular, (3.15) is indeed a morphism of locally \mathbb{C} -ringed spaces. \square

Corollary 3.3.10 *Let X be a complex space, and let A be a Stein algebra. Then we have a natural bijection*

$$\text{Hom}_{\mathcal{L}\text{ocRing}/\mathbb{C}}(X, \text{Sp } A) \xrightarrow{\sim} \text{Hom}_{\mathcal{L}\text{ocRing}/\mathbb{C}}(X, \text{Spec } A), \quad f \mapsto i_{\text{Spec } A} \circ f \quad (3.16)$$

Proof The assertion is local on X . Hence we may cover X by Stein open subsets and prove the statement on each member of the cover, with compatibility on overlaps. Thus we reduce to the case where X is Stein. In this case, our assertion follows from [Theorem 3.3.3](#) and [\[Stacks, Tag 01H1\]](#). \square

Proposition 3.3.4 *Let S be a Stein space with associated Stein algebra A . Consider $x \in S$. Then for each $n \geq 0$, the germ map $A \rightarrow \mathcal{O}_{S,x}$ induces isomorphisms*

$$A/\mathfrak{m}(x)^n \xrightarrow{\sim} \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}^n. \quad (3.17)$$

In particular, we get a natural isomorphism

$$\widehat{A_{\mathfrak{m}(x)}} \xrightarrow{\sim} \widehat{\mathcal{O}_{S,x}}. \quad (3.18)$$

We remind the reader that in general A is not noetherian, so the fact that $\widehat{A_{\mathfrak{m}(x)}}$ is noetherian follows from (3.18) and is non-trivial, cf. [\[Stacks, Tag 05JA\]](#).

Proof The case $n = 0$ is trivial. Assume that $n > 0$. Let $\mathcal{I}_x \subseteq \mathcal{O}_S$ be the ideal sheaf of the reduced point $\{x\}$. The coherent sheaf $\mathcal{O}_S/\mathcal{I}_x^n$ is supported at $\{x\}$, and its only non-zero stalk is

$$(\mathcal{O}_S/\mathcal{I}_x^n)_x = \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}^n.$$

Consider the exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{I}_x^n \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S/\mathcal{I}_x^n \rightarrow 0.$$

Since S is Stein, Cartan's Theorem B gives the surjectivity of

$$A = \Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(S, \mathcal{O}_S/\mathcal{I}_x^n) = \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}^n.$$

Under the above identification, this map is precisely the germ map modulo $\mathfrak{m}_{S,x}^n$. Its kernel consists of those $f \in A$ with $f_x \in \mathfrak{m}_{S,x}^n$, which is exactly $\mathfrak{m}(x)^n$ by [Corollary 3.3.6](#). This proves (3.17).

Finally (3.18) follows by taking the inverse limit in n in (3.17). \square

Lemma 3.3.7 *Let S be a Stein space, and let $x \in S$. Then the germ map*

$$\Gamma(S, \mathcal{O}_S) \rightarrow \mathcal{O}_{S,x}, \quad f \mapsto f_x \quad (3.19)$$

is flat.

See [Proposition 4.3.4](#) for a generalization.

Proof We shall use the equational criterion for flatness [\[Stacks, Tag 00HK\]](#).

Let $A := \Gamma(S, \mathcal{O}_S)$. Take $f_1, \dots, f_n \in A$ and suppose we are given a relation

$$\sum_{i=1}^n f_{i,x} b_i = 0$$

with $b_i \in \mathcal{O}_{S,x}$. Consider the morphism of coherent analytic sheaves

$$\varphi: \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{O}_S, \quad (a_1, \dots, a_n) \longmapsto \sum_{i=1}^n a_i f_i.$$

Let $\mathcal{R} := \ker(\varphi)$. Note that \mathcal{R} is coherent, and

$$b := (b_1, \dots, b_n) \in \mathcal{O}_{S,x}^{\oplus n}$$

lies in the stalk \mathcal{R}_x .

By Cartan's Theorem A, there exist global sections

$$r^{(1)}, \dots, r^{(m)} \in \Gamma(S, \mathcal{R}) \subseteq A^{\oplus n}$$

and elements $c_1, \dots, c_m \in \mathcal{O}_{S,x}$ such that

$$b = \sum_{j=1}^m c_j r_x^{(j)}.$$

Write

$$r^{(j)} = (r_{1j}, \dots, r_{nj}) \in A^{\oplus n}.$$

Then for each i we have

$$b_i = \sum_{j=1}^m c_j r_{ij} \quad \text{in } \mathcal{O}_{S,x}.$$

On the other hand, since each $r^{(j)}$ is a global section of \mathcal{R} , we have

$$\sum_{i=1}^n r_{ij} f_i = 0 \quad \text{in } A$$

for every j .

The desired flatness follows. \square

Corollary 3.3.11 *Let A be a Stein algebra and \mathfrak{m} be a closed maximal ideal in A . Then $A_{\mathfrak{m}}$ is noetherian.*

Proof Let $S = \text{Sp } A$ and $x \in S$ be the point such that $\mathfrak{m} = \mathfrak{m}(x)$, cf. [Corollary 3.3.8](#). By [Lemma 3.3.7](#), the map

$$A_{\mathfrak{m}} \rightarrow \mathcal{O}_{S,x}$$

induced by the germ map (3.19) by localization is faithfully flat. Now $A_{\mathfrak{m}}$ is noetherian by [\[Stacks, Tag 033E\]](#). \square

Apart from the trivial situation, a Stein algebra is never noetherian:

Proposition 3.3.5 *A Stein algebra A is noetherian if and only if $\text{Sp } A$ is finite as a set.*

Proof Let $S = \text{Sp } A$.

Suppose that S is finite as a set. Then A is a finite-dimensional \mathbb{C} -algebra by [Proposition 3.3.1](#), and hence noetherian.

Conversely, assume that A is noetherian. We shall prove that S is finite. Suppose, for contradiction, that S is infinite.

We may first reduce to the reduced case. Indeed, there is a natural surjection

$$\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(S^{\text{red}}, \mathcal{O}_{S^{\text{red}}}).$$

Since quotients of noetherian rings are noetherian, it is enough to show that $\Gamma(S^{\text{red}}, \mathcal{O}_{S^{\text{red}}})$ is not noetherian when S^{red} is infinite. Replacing S by S^{red} , we may therefore assume that S is reduced.

Since S is an infinite Stein space, we can find a closed discrete sequence of pairwise distinct points

$$x_1, x_2, \dots \in S.$$

Consider the ideal

$$I := \{f \in A : f(x_i) = 0 \text{ for almost all } i\}.$$

This is a proper ideal of A , since $1 \notin I$.

We claim that I is dense in A . Assuming this, by [Proposition 3.3.2](#), I is not finitely generated, and hence A is not noetherian.

Let us prove the claim. Since S is reduced, the topology on A is nothing but the compact-open topology, see [Proposition 2.2.3](#). Let $K \subseteq S$ be compact and let $\epsilon > 0$. We shall construct $f \in I$ such that

$$\sup_K |f - 1| < \epsilon. \quad (3.20)$$

For this purpose, we may replace K by \widehat{K} and assume that K is $\mathcal{O}(S)$ -convex. Since $\{x_i\}_{i \geq 1}$ is closed and discrete, only finitely many x_i 's lie in K . Hence, for $N \gg 1$, the closed discrete analytic subset

$$D := \{x_i : i \geq N\}$$

is disjoint from K .

By the Oka–Weil theorem ([\[For17, Theorem 2.8.4\]](#)), there exists $f \in A$ such that $f|_D = 0$ and (3.20) holds. Since f vanishes at x_i for all $i \geq N$, we have $f \in I$. Thus $1 \in \overline{I}$. \square

Lemma 3.3.8 *Let U be a relatively compact Stein open subset in a Stein space S , and let \mathcal{F} be a coherent \mathcal{O}_S -module. Then the $\Gamma(U, \mathcal{O}_U)$ -module $\Gamma(U, \mathcal{F})$ is finitely generated and the natural map*

$$\Gamma(S, \mathcal{F}) \otimes_{\Gamma(S, \mathcal{O}_S)} \Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(U, \mathcal{F}) \quad (3.21)$$

is an isomorphism.

Proof Since U is relatively compact, by Cartan's Theorem A, we can find $f_1, \dots, f_n \in \Gamma(S, \mathcal{F})$ generating \mathcal{F} on U . We get a surjective morphism of \mathcal{O}_U -modules

$$\mathcal{O}_U^n \rightarrow \mathcal{F}, \quad (g_1, \dots, g_n) \mapsto \sum_{i=1}^n g_i f_i.$$

By Cartan's Theorem B, we find a surjective homomorphism

$$\Gamma(U, \mathcal{O}_U^n) \twoheadrightarrow \Gamma(U, \mathcal{F}), \quad (g_1, \dots, g_n) \mapsto \sum_{i=1}^n g_i f_i. \quad (3.22)$$

Therefore, $\Gamma(U, \mathcal{F})$ is finitely generated, and the surjectivity of (3.21) follows.

Next, we prove the injectivity. Consider $h_1, \dots, h_r \in \Gamma(S, \mathcal{F})$ and $c_1, \dots, c_r \in \Gamma(U, \mathcal{O}_U)$ with

$$\sum_{i=1}^r c_i h_i|_U = 0.$$

Then we have a morphism of \mathcal{O}_S -modules:

$$\mathcal{O}_S^r \rightarrow \mathcal{F}, \quad (g_1, \dots, g_r) \mapsto \sum_{i=1}^r g_i h_i.$$

Let \mathcal{K} be the kernel. Then $(c_1, \dots, c_r) \in \Gamma(U, \mathcal{K})$.

Let $d_1, \dots, d_s \in \Gamma(S, \mathcal{K})$ be sections generating $\mathcal{K} \subseteq \mathcal{O}_S^r$ on U . We denote the components of d_j by $d_j^1, \dots, d_j^r \in \Gamma(S, \mathcal{O}_S)$. Then by the surjectivity (3.22) applied to \mathcal{K} in place of \mathcal{F} , we can express

$$c_i = \sum_{j=1}^s e_j d_j^i, \quad \forall i = 1, \dots, r,$$

for some $e_1, \dots, e_s \in \Gamma(U, \mathcal{O}_U)$. Therefore,

$$\sum_{i=1}^r h_i \otimes c_i = \sum_{i=1}^r \sum_{j=1}^s h_i \otimes e_j d_j^i = \sum_{j=1}^s \left(\sum_{i=1}^r d_j^i h_i \right) \otimes e_j = 0.$$

We conclude the proof. \square

Corollary 3.3.12 *Let S be a Stein space, \mathcal{F} be a coherent \mathcal{O}_S -module and $x \in S$. Then the natural homomorphism*

$$\Gamma(S, \mathcal{F}) \otimes_{\Gamma(S, \mathcal{O}_S)} \mathcal{O}_{S, x} \rightarrow \mathcal{F}_x$$

is an isomorphism.

Proof This follows from [Stacks, Tag 00DD] and Lemma 3.3.8. \square

Theorem 3.3.4 *Let S be a Stein space, and $U \subseteq S$ be a relatively compact open subspace. Then the restriction*

$$\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(U, \mathcal{O}_U)$$

is flat.

I do not know if the flatness holds if U is not relatively compact.

Proof Let I be a finitely generated ideal in $\Gamma(S, \mathcal{O}_S)$. It suffices to show that

$$I \otimes_{\Gamma(S, \mathcal{O}_S)} \Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(U, \mathcal{O}_U)$$

is injective. Let \mathcal{I} be the coherent ideal on S defined by I . Then the above map can be identified with

$$\Gamma(U, \mathcal{I}) \rightarrow \Gamma(U, \mathcal{O}_U)$$

by [Lemma 3.3.8](#), which is clearly injective. \square

Proposition 3.3.6 *The following are equivalent:*

- (1) S is reduced;
- (2) A is reduced.

Proof (1) \implies (2). Assume (1). Suppose $f \in A$ and $f^n = 0$ for some $n > 1$. Then for any $x \in S$, we have $f_x^n = 0$ and hence $f_x = 0$. Therefore, $f = 0$.

(2) \implies (1). Assume that S is not reduced. Let $\mathcal{N} \subseteq \mathcal{O}_S$ be the nilradical sheaf. Choose $x \in S$ such that $\mathcal{N}_x \neq 0$. By Cartan's Theorem A, there exists $f \in \Gamma(S, \mathcal{N})$ such that $f_x \neq 0$.

Since f_x is nilpotent in $\mathcal{O}_{S,x}$, there exists $m > 1$ such that $f_x^m = 0$. Consider the coherent ideal sheaf

$$\mathcal{I} := \text{Ann}_{\mathcal{O}_S}(f^m) \subseteq \mathcal{O}_S.$$

Since $f_x^m = 0$, we have $\mathcal{I}_x = \mathcal{O}_{S,x}$. By Cartan's Theorem A, there exists $g \in \Gamma(S, \mathcal{I})$ such that g_x is a unit.

Set $h = gf \in A$. Then

$$h_x = g_x f_x \neq 0,$$

so $h \neq 0$. On the other hand,

$$h^m = g^{m-1}(gf^m) = 0.$$

Therefore A is not reduced. \square

3.4 Stein modules

Throughout this section, let S be a Stein space with Stein algebra $A = \Gamma(S, \mathcal{O}_S)$.

Definition 3.4.1 A *Stein A -module* is a topological A -module M such that there is a coherent \mathcal{O}_S -module \mathcal{M} with $M \cong \Gamma(S, \mathcal{M})$ as topological A -modules.

A *morphism* between Stein A -modules is an A -linear map.

Note that a morphism is not assumed continuous. The category of Stein A -modules is denoted by $\text{Mod}_A^{\text{Stein}}$.

The global section functor $\Gamma(S, \bullet)$ gives a covariant functor from the category $\text{Coh}_{\mathcal{O}_S}$ of coherent \mathcal{O}_S -modules to $\text{Mod}_A^{\text{Stein}}$.

Definition 3.4.2 Given an A -module M , we construct an \mathcal{O}_S -module \tilde{M} as follows: if $U \subseteq S$ is an open subset, then $U \mapsto M \otimes_A \Gamma(U, \mathcal{O}_U)$ defines a presheaf of \mathcal{O}_S -modules; the sheaf \tilde{M} is the sheafification of this presheaf.

We call \tilde{M} the *associated \mathcal{O}_S -module* of M .

Equivalently, \tilde{M} is the pull-back of M via the morphism of ringed spaces $S \rightarrow (\text{pt}, A)$.

From this description, it is easy to see that $M \mapsto \tilde{M}$ defines a covariant functor from the category Mod_A of A -modules to the category $\text{Mod}_{\mathcal{O}_S}$ of \mathcal{O}_S -modules.

Observe the following simple fact:

$$\tilde{M}_x \cong M \otimes_A \mathcal{O}_{S,x}, \quad \forall x \in S. \quad (3.23)$$

Example 3.4.1 If I is an ideal in A , then $\tilde{I} = I\mathcal{O}_S$. In particular, \tilde{I} is coherent.

To see this, observe that if $U \subseteq S$ is a relatively compact Stein open subset, then by [Lemma 3.3.8](#) and [Theorem 3.3.1](#), we have a natural isomorphism

$$\tilde{I} \otimes_A \Gamma(U, \mathcal{O}_U) \xrightarrow{\sim} \Gamma(U, I\mathcal{O}_S).$$

Since relatively compact Stein open subsets of S form a basis, $\tilde{I} = I\mathcal{O}_S$ follows.

Stein modules do not have the apparent functoriality.

Example 3.4.2 Let $X = \mathbb{N}$ be the discrete reduced complex space. Then X is Stein and

$$A = \Gamma(X, \mathcal{O}_X) = \mathbb{C}^{\mathbb{N}}.$$

Let \mathcal{M} be the coherent \mathcal{O}_X -module defined by $\mathcal{M}_i = \mathbb{C}^i$ at the point $i \in \mathbb{N}$. Let $Y = \mathbb{N} \times \mathbb{N}$, and let $F: Y \rightarrow X$ be the projection $F(i, j) = i$. Then Y is Stein and

$$B = \Gamma(Y, \mathcal{O}_Y) = \mathbb{C}^{\mathbb{N} \times \mathbb{N}}.$$

We have

$$\Gamma(X, \mathcal{M}) = \prod_{i \geq 1} \mathbb{C}^i, \quad \Gamma(Y, F^* \mathcal{M}) = \prod_{(i,j) \in \mathbb{N} \times \mathbb{N}} \mathbb{C}^i.$$

Consider the natural morphism

$$B \otimes_A \Gamma(X, \mathcal{M}) \longrightarrow \Gamma(Y, F^* \mathcal{M}).$$

An element of the source is represented by a finite sum

$$\sum_{\ell=1}^r b_{\ell} \otimes m_{\ell}.$$

Its image has value at (i, j) equal to

$$\sum_{\ell=1}^r b_{\ell}(i, j) m_{\ell}(i) \in \mathbb{C}^i.$$

Thus, for fixed i , all values as j varies lie in the subspace

$$\text{Span}_{\mathbb{C}}\{m_1(i), \dots, m_r(i)\} \subseteq \mathbb{C}^i,$$

which has dimension at most r .

Now define a section

$$s \in \Gamma(Y, F^* \mathcal{M})$$

by

$$s(i, j) = e_j \in \mathbb{C}^i \quad \text{for } 1 \leq j \leq i,$$

where e_1, \dots, e_i is the standard basis of \mathbb{C}^i , and set $s(i, j) = 0$ for $j > i$. If s were in the image, then for some fixed r , the vectors e_1, \dots, e_i would lie in a subspace of \mathbb{C}^i of dimension at most r for every i . Taking $i > r$ gives a contradiction.

Therefore the natural map

$$B \otimes_A \Gamma(X, \mathcal{M}) \longrightarrow \Gamma(Y, F^* \mathcal{M})$$

is not surjective, hence is not an isomorphism.

Proposition 3.4.1 *The functor*

$$\text{Mod}_A \rightarrow \text{Mod}_{O_S}, \quad M \mapsto \widetilde{M}$$

is exact.

Proof It follows from [Lemma 3.3.7](#) that the morphism $S \rightarrow (\text{pt}, A)$ is flat. Therefore, the pull-back functor $M \mapsto \widetilde{M}$ is exact. \square

Lemma 3.4.1 *Let \mathcal{M} be a coherent O_S -module. Then the natural morphism*

$$\Gamma(\widetilde{S, \mathcal{M}}) \longrightarrow \mathcal{M} \tag{3.24}$$

is an isomorphism.

The morphism [\(3.24\)](#) is induced by the identity map $\Gamma(S, \mathcal{M}) \rightarrow \Gamma(S, \mathcal{M})$ by adjunction.

Proof It suffices to check the isomorphism on stalks. Fix $x \in S$. The germ at x of [\(3.24\)](#) is just

$$\Gamma(S, \mathcal{M}) \otimes_A \mathcal{O}_{S,x} \rightarrow \mathcal{M}_x,$$

which is an isomorphism by [Corollary 3.3.12](#). \square

In general, \widetilde{M} is a mysterious object. However, if M is finitely presented, \widetilde{M} is always coherent and we have:

Proposition 3.4.2 *The functor $M \mapsto \widetilde{M}$ is a quasi-inverse of the global section functor from the category of globally finitely presented \mathcal{O}_S -modules to the category of finitely presented A -modules.*

Here we say that an \mathcal{O}_S -module \mathcal{M} is *globally finitely presented* if there is an exact sequence of \mathcal{O}_S -modules

$$\mathcal{O}_S^m \rightarrow \mathcal{O}_S^n \rightarrow \mathcal{M} \rightarrow 0 \quad (3.25)$$

for some $m, n \in \mathbb{N}$.

Proof We first observe that for any globally finitely presented \mathcal{O}_S -module \mathcal{M} , $\Gamma(S, \mathcal{M})$ is indeed a finitely presented A -module. To see this, it suffices to apply Γ to the exact sequence (3.25) and use Cartan's Theorem B.

Conversely, given a finitely presented A -module M , we can find $m, n \in \mathbb{N}$ and an exact sequence of A -modules:

$$A^m \rightarrow A^n \rightarrow M \rightarrow 0. \quad (3.26)$$

By [Proposition 3.4.1](#), we get an exact sequence of \mathcal{O}_S -modules:

$$\mathcal{O}_S^m \rightarrow \mathcal{O}_S^n \rightarrow \widetilde{M} \rightarrow 0.$$

This shows that \widetilde{M} is globally finitely presented.

Next, take a globally finitely presented \mathcal{O}_S -module \mathcal{M} . The natural map $\widetilde{\Gamma(S, \mathcal{M})} \rightarrow \mathcal{M}$ is an isomorphism as a special case of [Lemma 3.4.1](#).

Conversely, given a finitely presented A -module M , we want to show that the natural map $M \rightarrow \Gamma(S, \widetilde{M})$ is an isomorphism. Taking a presentation of M as in (3.26), we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} A^m & \longrightarrow & A^n & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \Gamma(S, \widetilde{A^m}) & \longrightarrow & \Gamma(S, \widetilde{A^n}) & \longrightarrow & \Gamma(S, \widetilde{M}) & \longrightarrow & 0 \end{array}$$

Therefore, in order to show that $M \rightarrow \Gamma(S, \widetilde{M})$ is an isomorphism, it suffices to show that the left two vertical maps are isomorphisms. But this is clear since $\widetilde{A^m} \cong \mathcal{O}_S^m$ and $\widetilde{A^n} \cong \mathcal{O}_S^n$. \square

On the other hand, if M is not finitely presented, \widetilde{M} could be very wild. But if M is a Stein A -module, \widetilde{M} is still coherent.

Theorem 3.4.1 *The global section functor is an equivalence from $\text{Coh}_{\mathcal{O}_S} \rightarrow \text{Mod}_A^{\text{Stein}}$ with quasi-inverse $M \mapsto \widetilde{M}$.*

Proof First observe that if M is a Stein module, say associated with a coherent \mathcal{O}_S -module \mathcal{M} , then the natural map $\widetilde{M} \rightarrow \mathcal{M}$ is an isomorphism by [Lemma 3.4.1](#). In particular, \widetilde{M} is coherent. Thus the functor $M \mapsto \widetilde{M}$ is well-defined from $\text{Mod}_A^{\text{Stein}}$ to $\text{Coh}_{\mathcal{O}_S}$.

It remains to show that if M is a Stein module, say associated with \mathcal{M} , then the natural map $M \rightarrow \Gamma(S, \widetilde{M})$ is an isomorphism. From the above, we have $\widetilde{M} \cong \mathcal{M}$, so it suffices to show that the natural map $M \rightarrow \Gamma(S, \mathcal{M})$ is an isomorphism. But this follows from the chosen presentation of M as $\Gamma(S, \mathcal{M})$. \square

Corollary 3.4.1 *Any A -module homomorphism between Stein modules is automatically continuous and has closed image.*

As a consequence, on each A -module, there is at most one topology making it a Stein A -module. Now we can say an A -module M is a Stein A -module if there is a topology on M such that M becomes a Stein A -module with respect to this topology.

Proof Let $f: M \rightarrow N$ be a homomorphism between Stein A -modules. By [Theorem 3.4.1](#), we can find a homomorphism between coherent \mathcal{O}_S -modules $F: \widetilde{M} \rightarrow \widetilde{N}$ such that f is the global section map associated with F . Our assertion now follows from [Lemma 3.1.2](#) and [Proposition 2.2.1\(4\)](#). \square

Corollary 3.4.2 *Any finitely presented A -module is a Stein module.*

Proof This is a consequence of [Proposition 3.4.2](#) and [Theorem 3.4.1](#). \square

In the case of ideal sheaves, we have the following more precise result:

Proposition 3.4.3 *Any closed ideal of A is a Stein module. In particular, any finitely generated ideal of A is a Stein module.*

In particular, a Stein module is not necessarily finitely generated.

Proof The first assertion follows from [Theorem 3.3.1](#), and the second assertion is a consequence of the first one and [Proposition 3.3.2](#). \square

3.5 The entire function ring

In this section, we study the simplest non-trivial example of a Stein algebra beyond the 0-dimensional examples in [Section 3.3.1](#). Let A be the ring of entire functions on \mathbb{C} , i.e. $A = \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$.

3.5.1 Divisors of entire functions

Let $\text{Div}^+(\mathbb{C})$ denote the commutative monoid of effective (locally finite) divisors on \mathbb{C} . Thus an element $D \in \text{Div}^+(\mathbb{C})$ is a function $D: \mathbb{C} \rightarrow \mathbb{N}$ whose support

$$|D| = \{z \in \mathbb{C} : D(z) > 0\}$$

is a closed discrete subset of \mathbb{C} . The monoid structure is pointwise addition. We write $D \leq E$ if $D(z) \leq E(z)$ for all $z \in \mathbb{C}$, and we write $D \wedge E$ for the pointwise minimum.

For a nonzero entire function $f \in A$, let

$$\text{div}(f) \in \text{Div}^+(\mathbb{C})$$

be its zero divisor.

We shall use the following standard facts from complex analysis.

Lemma 3.5.1 *The following hold:*

- (1) *The map $\text{div}: A \setminus \{0\} \rightarrow \text{Div}^+(\mathbb{C})$ is surjective.*
- (2) *If $0 \neq f, g \in A$, then f divides g in A if and only if $\text{div}(f) \leq \text{div}(g)$.*
- (3) *The units of A are precisely the nowhere vanishing entire functions.*

We shall also use the following classical theorem of Helmer.

Theorem 3.5.1 (Helmer–Bézout theorem) *Let $f_1, \dots, f_r \in A$ be nonzero entire functions. Then there exist $d, e_1, \dots, e_r \in A$ such that*

$$\text{div}(d) = \text{div}(f_1) \wedge \dots \wedge \text{div}(f_r)$$

and

$$d = e_1 f_1 + \dots + e_r f_r.$$

In particular, every finitely generated ideal of A is principal.

Proof Choose $d \in A$ such that

$$\text{div}(d) = \text{div}(f_1) \wedge \dots \wedge \text{div}(f_r),$$

which is possible by **Lemma 3.5.1**. Then each f_i is divisible by d , say $f_i = dg_i$ for some $g_i \in A$.

By construction,

$$\text{div}(g_1) \wedge \dots \wedge \text{div}(g_r) = 0.$$

Equivalently, the functions g_1, \dots, g_r have no common zero. Therefore, by **Proposition 3.1.1**, there exist $a_1, \dots, a_r \in A$ such that

$$a_1 g_1 + \dots + a_r g_r = 1.$$

Multiplying by d , we obtain

$$d = a_1 f_1 + \cdots + a_r f_r.$$

Thus $d \in (f_1, \dots, f_r)$. \square

3.5.2 Divisor filters

Definition 3.5.1 A *divisor filter* on \mathbb{C} is a filter on the set $\text{Div}^+(\mathbb{C})$.

In other words, a divisor filter is a subset

$$\mathcal{F} \subseteq \text{Div}^+(\mathbb{C})$$

satisfying the following two conditions:

- (1) If $D \in \mathcal{F}$ and $E \geq D$, then $E \in \mathcal{F}$.
- (2) If $D, E \in \mathcal{F}$, then $D \wedge E \in \mathcal{F}$.

The empty subset is allowed and is regarded as a divisor filter.

Given an ideal $I \subseteq A$, define

$$\mathcal{F}_I = \{\text{div}(f) : 0 \neq f \in I\} \subseteq \text{Div}^+(\mathbb{C}). \quad (3.27)$$

Lemma 3.5.2 For every ideal $I \subseteq A$, the subset \mathcal{F}_I is a divisor filter.

Proof First suppose $D \in \mathcal{F}_I$ and $E \geq D$. Choose $0 \neq f \in I$ with $\text{div}(f) = D$. By **Lemma 3.5.1**, choose $g \in A$ with $\text{div}(g) = E$. Since $D \leq E$, **Lemma 3.5.1** implies that f divides g . Thus $g = fh$ for some $h \in A$. Since $f \in I$, we get $g \in I$. Hence $E \in \mathcal{F}_I$.

Now let $D, E \in \mathcal{F}_I$. Choose $0 \neq f, g \in I$ such that

$$\text{div}(f) = D, \quad \text{div}(g) = E.$$

By **Theorem 3.5.1**, there exists $d \in (f, g) \subseteq I$ such that $\text{div}(d) = D \wedge E$. Hence $D \wedge E \in \mathcal{F}_I$. Therefore \mathcal{F}_I is a divisor filter. \square

Conversely, every divisor filter defines an ideal.

Lemma 3.5.3 Let $\mathcal{F} \subseteq \text{Div}^+(\mathbb{C})$ be a divisor filter. Then

$$I_{\mathcal{F}} = \{0\} \cup \{0 \neq f \in A : \text{div}(f) \in \mathcal{F}\} \quad (3.28)$$

is an ideal of A .

Proof It is clear that $0 \in I_{\mathcal{F}}$ and that $I_{\mathcal{F}}$ is stable under multiplication by arbitrary elements of A . Indeed, if $0 \neq f \in I_{\mathcal{F}}$ and $0 \neq h \in A$, then

$$\text{div}(fh) = \text{div}(f) + \text{div}(h) \geq \text{div}(f).$$

Since \mathcal{F} is upward closed, $\operatorname{div}(fh) \in \mathcal{F}$, so $fh \in I_{\mathcal{F}}$.

It remains to check stability under addition. Let $f, g \in I_{\mathcal{F}}$. If one of them is zero, there is nothing to prove. Assume $f, g \neq 0$. For every $a \in \mathbb{C}$, the order of vanishing satisfies

$$\operatorname{ord}_a(f + g) \geq \min\{\operatorname{ord}_a(f), \operatorname{ord}_a(g)\}.$$

Hence

$$\operatorname{div}(f + g) \geq \operatorname{div}(f) \wedge \operatorname{div}(g)$$

if $f + g \neq 0$. Since $\operatorname{div}(f), \operatorname{div}(g) \in \mathcal{F}$ and \mathcal{F} is closed under finite minima, we have

$$\operatorname{div}(f) \wedge \operatorname{div}(g) \in \mathcal{F}.$$

By upward closure, $\operatorname{div}(f + g) \in \mathcal{F}$. Thus $f + g \in I_{\mathcal{F}}$. Therefore $I_{\mathcal{F}}$ is an ideal. \square

Proposition 3.5.1 *There is a canonical bijection between*

- (1) *the set of ideals in A ;*
- (2) *the set of divisor filters on \mathbb{C} .*

The map from (1) to (2) is just $I \mapsto \mathcal{F}_I$ of (3.27), and the map from (2) to (1) is $\mathcal{F} \mapsto I_{\mathcal{F}}$ of (3.28).

Proof Let $I \subseteq A$ be an ideal. We show that

$$I_{\mathcal{F}_I} = I.$$

If $0 \neq f \in I$, then $\operatorname{div}(f) \in \mathcal{F}_I$, so $f \in I_{\mathcal{F}_I}$. Thus $I \subseteq I_{\mathcal{F}_I}$.

Conversely, if $0 \neq f \in I_{\mathcal{F}_I}$, then $\operatorname{div}(f) \in \mathcal{F}_I$. Hence there exists $0 \neq g \in I$ such that

$$\operatorname{div}(g) = \operatorname{div}(f).$$

By **Lemma 3.5.1**, $f = ug$ for some unit $u \in A$. Since $g \in I$, we get $f \in I$. Hence $I_{\mathcal{F}_I} \subseteq I$.

Now let \mathcal{F} be a divisor filter. We show that

$$\mathcal{F}_{I_{\mathcal{F}}} = \mathcal{F}.$$

If $D \in \mathcal{F}$, choose $0 \neq f \in A$ with $\operatorname{div}(f) = D$. Then $f \in I_{\mathcal{F}}$, so $D \in \mathcal{F}_{I_{\mathcal{F}}}$.

Conversely, if $D \in \mathcal{F}_{I_{\mathcal{F}}}$, then $D = \operatorname{div}(f)$ for some $0 \neq f \in I_{\mathcal{F}}$. By definition of $I_{\mathcal{F}}$, this means $\operatorname{div}(f) \in \mathcal{F}$, hence $D \in \mathcal{F}$. Therefore the two constructions are inverse to each other. \square

3.5.3 Prime divisor filters and the spectrum

Definition 3.5.2 A divisor filter $\mathcal{F} \subseteq \operatorname{Div}^+(\mathbb{C})$ is called *prime* if it is proper and for all $D, E \in \operatorname{Div}^+(\mathbb{C})$,

$$D + E \in \mathcal{F} \implies D \in \mathcal{F} \text{ or } E \in \mathcal{F}.$$

Theorem 3.5.2 Under the bijection of [Proposition 3.5.1](#), the following sets correspond to each other:

- (1) $\text{Spec } A$;
- (2) the set of prime divisor filters on \mathbb{C} .

Proof Let \mathcal{F} be a divisor filter. We claim that $I_{\mathcal{F}}$ is prime if and only if \mathcal{F} is prime.

We first observe that $I_{\mathcal{F}}$ is proper if and only if $0_{\text{div}} \notin \mathcal{F}$.

Suppose first that $I_{\mathcal{F}}$ is prime. Let $D, E \in \text{Div}^+(\mathbb{C})$ and assume $D + E \in \mathcal{F}$. Choose nonzero entire functions $f, g \in A$ such that

$$\text{div}(f) = D, \quad \text{div}(g) = E.$$

Then

$$\text{div}(fg) = D + E \in \mathcal{F},$$

so $fg \in I_{\mathcal{F}}$. Since $I_{\mathcal{F}}$ is prime, either $f \in I_{\mathcal{F}}$ or $g \in I_{\mathcal{F}}$. Equivalently, either $D \in \mathcal{F}$ or $E \in \mathcal{F}$. Thus \mathcal{F} is prime.

Conversely, suppose that \mathcal{F} is prime. Let $f, g \in A$ and assume $fg \in I_{\mathcal{F}}$. If $f = 0$ or $g = 0$, there is nothing to prove. Otherwise,

$$\text{div}(f) + \text{div}(g) = \text{div}(fg) \in \mathcal{F}.$$

Since \mathcal{F} is prime, either $\text{div}(f) \in \mathcal{F}$ or $\text{div}(g) \in \mathcal{F}$. Therefore either $f \in I_{\mathcal{F}}$ or $g \in I_{\mathcal{F}}$. Hence $I_{\mathcal{F}}$ is prime. \square

3.5.4 Fixed and free ideals

Definition 3.5.3 Let $I \subseteq A$ be a proper ideal.

- (1) We say that I is *fixed* if

$$\bigcap_{f \in I} Z(f) \neq \emptyset.$$

- (2) We say that I is *free* if

$$\bigcap_{f \in I} Z(f) = \emptyset.$$

Here

$$Z(f) = \{z \in \mathbb{C} : f(z) = 0\}.$$

For $a \in \mathbb{C}$, recall that

$$\mathfrak{m}(a) = \{f \in A : f(a) = 0\}.$$

Proposition 3.5.2 *The fixed non-zero prime ideals of A are precisely the ideals*

$$\mathfrak{m}(a) = \{f \in A : f(a) = 0\}, \quad a \in \mathbb{C}.$$

Proof Clearly, each ideal $\mathfrak{m}(a)$ is fixed.

Conversely, let $\mathfrak{p} \subseteq A$ be a fixed prime ideal. Choose

$$a \in \bigcap_{f \in \mathfrak{p}} Z(f).$$

Then every element of \mathfrak{p} vanishes at a , so $\mathfrak{p} \subseteq \mathfrak{m}(a)$.

Since \mathfrak{p} is nonzero, choose $0 \neq f \in \mathfrak{p}$. Since $f(a) = 0$, we may write

$$f = (z - a)^m u$$

for some integer $m \geq 1$ and some entire function u with $u(a) \neq 0$. More precisely, u is entire and nonzero at a ; in particular, one factor $z - a$ divides f .

Because \mathfrak{p} is prime and $f \in \mathfrak{p}$, repeated application of primeness gives $z - a \in \mathfrak{p}$. But

$$\mathfrak{m}(a) = (z - a).$$

Indeed, if $g(a) = 0$, then $g/(z - a)$ extends holomorphically across a and is entire. Hence $g \in (z - a)$. Therefore,

$$\mathfrak{p} = \mathfrak{m}(a).$$

Consequently, the remaining non-zero prime ideals are precisely the free prime ideals.

3.5.5 Finitely generated ideals

Corollary 3.5.1 *No free ideal of A is finitely generated.*

Proof Let $I \subseteq A$ be a finitely generated proper ideal. By [Theorem 3.5.1](#), there exists $d \in A$ such that $I = (d)$. Since I is proper, d is not a unit. Therefore d has a zero. If $a \in Z(d)$, then every element of (d) vanishes at a . Hence

$$\bigcap_{f \in I} Z(f) \neq \emptyset.$$

Thus I is fixed. Therefore a free ideal cannot be finitely generated. □

Corollary 3.5.2 *No nonzero polynomial belongs to a free ideal of A .*

Proof Let $I \subseteq A$ be a free ideal and suppose that $0 \neq p \in I$ is a polynomial. The zero set $Z(p)$ is finite.

For every $f \in I$, the ideal (p, f) is contained in I , hence is proper. By **Theorem 3.5.1**, the ideal (p, f) is generated by an entire function whose zero divisor is

$$\operatorname{div}(p) \wedge \operatorname{div}(f).$$

Since (p, f) is proper, this divisor is nonzero. Thus

$$Z(p) \cap Z(f) \neq \emptyset$$

for every $f \in I$.

The finite collection of subsets

$$Z(p) \cap Z(f), \quad f \in I,$$

has the finite intersection property. Indeed, for $f_1, \dots, f_r \in I$, the ideal

$$(p, f_1, \dots, f_r)$$

is contained in I , hence is proper; by the Helmer–Bézout theorem, this means that

$$Z(p) \cap Z(f_1) \cap \dots \cap Z(f_r) \neq \emptyset.$$

Since $Z(p)$ is finite, the total intersection is nonempty:

$$Z(p) \cap \bigcap_{f \in I} Z(f) \neq \emptyset.$$

This contradicts the assumption that I is free. Hence no nonzero polynomial belongs to a free ideal. \square

3.5.6 Maximal ideals

The maximal ideals of A are of two kinds:

$$\mathfrak{m}(a), \quad a \in \mathbb{C},$$

and maximal free ideals.

Theorem 3.5.3 *Let $M \subset A$ be a free ideal. Then M is maximal if and only if it satisfies the following condition:*

For every infinite closed discrete subset $D \subseteq \mathbb{C}$, if

$$D \cap Z(f) \neq \emptyset \quad \text{for every } f \in M,$$

then there exists $g \in M$ such that

$$\operatorname{div}(g) = \sum_{a \in D} [a].$$

Proof Assume first that M is maximal. Let $D \subseteq \mathbb{C}$ be an infinite closed discrete subset such that

$$D \cap Z(f) \neq \emptyset$$

for every $f \in M$.

Choose $g \in A$ with $Z(g) = D$, which is possible by the Weierstrass product theorem. We claim that the ideal $M + (g)$ is proper. Indeed, if $1 \in M + (g)$, then there exist $m \in M$ and $h \in A$ such that $1 = m + hg$. By assumption, $D \cap Z(m) \neq \emptyset$. Choose $a \in D \cap Z(m)$. Then $g(a) = 0$ and $m(a) = 0$, so evaluating the equality above at a gives $1 = 0$, a contradiction. Hence $M + (g)$ is proper. Since M is maximal, we must have $M + (g) = M$, so $g \in M$. Thus D is the zero set of an element of M .

Conversely, suppose M satisfies the stated condition. We prove that M is maximal. Let N be a proper ideal with $M \subseteq N$. We show that $N = M$.

Let $g \in N$. If $g = 0$, there is nothing to prove. We prove that $g \in M$. Since N is proper, for every $f \in M$ the ideal (f, g) is proper. By [Theorem 3.5.1](#), this implies

$$Z(f) \cap Z(g) \neq \emptyset.$$

Thus $Z(g)$ meets $Z(f)$ for every $f \in M$.

We claim that $Z(g)$ is infinite. If $Z(g)$ were finite, then the family

$$\{Z(g) \cap Z(f) : f \in M\}$$

would be a family of nonempty subsets of the finite set $Z(g)$ with the finite intersection property. Indeed, for $f_1, \dots, f_r \in M$, the ideal

$$(g, f_1, \dots, f_r)$$

is contained in N , hence is proper; by the Helmer–Bézout theorem,

$$Z(g) \cap Z(f_1) \cap \dots \cap Z(f_r) \neq \emptyset.$$

Since $Z(g)$ is finite, the total intersection is nonempty:

$$Z(g) \cap \bigcap_{f \in M} Z(f) \neq \emptyset.$$

This contradicts the assumption that M is free. Therefore $Z(g)$ is infinite.

Since g is entire and nonzero, $Z(g)$ is a closed discrete subset of \mathbb{C} . By the assumed criterion applied to $D = Z(g)$, there exists $h \in M$ such that $Z(h) = Z(g)$. Equivalently, $\operatorname{div}(h)$ and $\operatorname{div}(g)$ have the same support. Replacing h by a suitable entire function with the same zero set and simple zeros, we may assume that h divides g ; alternatively, using the divisor language, choose $h_0 \in M$ whose divisor is bounded above by $\operatorname{div}(g)$ and has the same support. Then [Lemma 3.5.1](#) gives $g = h_0 u$ for some $u \in A$. Hence $g \in M$.

Thus every $g \in N$ lies in M , so $N = M$. Therefore M is maximal. \square

Corollary 3.5.3 *Maximal free ideals exist.*

Proof It suffices to exhibit a free ideal. Let

$$z_1, z_2, z_3, \dots$$

be a sequence of distinct complex numbers tending to infinity. For each $N \geq 1$, choose an entire function F_N such that

$$Z(F_N) = \{z_N, z_{N+1}, z_{N+2}, \dots\}.$$

Let

$$I = (F_1, F_2, F_3, \dots).$$

Then I is free: given any z_k , the function F_{k+1} does not vanish at z_k , so no point belongs to the zero set of every element of I .

By Zorn's lemma, I is contained in a maximal proper ideal M . Since I is free, the ideal M is also free. Therefore maximal free ideals exist. \square

3.5.7 Residue fields of maximal ideals

For the fixed maximal ideals, one has

$$A/\mathfrak{m}(a) \cong \mathbb{C}$$

via evaluation at a .

The residue fields of maximal free ideals are more subtle.

Proposition 3.5.3 *Let $M \subseteq A$ be a maximal free ideal. Then, as a \mathbb{C} -algebra, A/M contains a subfield isomorphic to $\mathbb{C}(z)$.*

Proof By [Corollary 3.5.2](#), the ideal M contains no nonzero polynomial. Hence the natural map

$$\mathbb{C}[z] \longrightarrow A/M$$

is injective. Since A/M is a field, this injective map extends uniquely to an embedding

$$\mathbb{C}(z) \hookrightarrow A/M.$$

Thus, as a \mathbb{C} -algebra, the residue field of a maximal free ideal is a proper extension of \mathbb{C} .

Nevertheless, as an abstract field, it is isomorphic to \mathbb{C} .

Theorem 3.5.4 *Let $M \subset A$ be a maximal free ideal. Then A/M is isomorphic to \mathbb{C} as an abstract field.*

Proof We first prove that A/M is algebraically closed.

Let

$$P(T) = \bar{f}_0 + \bar{f}_1 T + \cdots + \bar{f}_n T^n \in (A/M)[T]$$

be a nonconstant polynomial, with $n \geq 1$ and

$$f_n \notin M.$$

We must find $\bar{g} \in A/M$ such that

$$P(\bar{g}) = 0.$$

Since $f_n \notin M$ and M is maximal, the ideal $M + (f_n)$ is equal to A . In particular, it is not proper. It follows that there exists $h \in M$ such that

$$Z(h) \cap Z(f_n) = \emptyset.$$

Indeed, if every $h \in M$ met $Z(f_n)$, then the argument used in the proof of [Theorem 3.5.3](#) would show that $M + (f_n)$ is proper, a contradiction.

Since $h \in M$ and M is proper, h is not a unit, so $Z(h) \neq \emptyset$. If $Z(h)$ were finite, then $h = pu$ where p is a nonzero polynomial and u is a nowhere vanishing entire function. Since u is a unit and $h \in M$, we would get $p \in M$, contradicting [Corollary 3.5.2](#). Thus $Z(h)$ is infinite. Write

$$Z(h) = \{a_1, a_2, a_3, \dots\}.$$

For each k , the complex polynomial

$$f_0(a_k) + f_1(a_k)T + \cdots + f_n(a_k)T^n$$

has degree n , because $f_n(a_k) \neq 0$. Since \mathbb{C} is algebraically closed, choose a root $\lambda_k \in \mathbb{C}$.

By the interpolation theorem for entire functions, there exists $g \in A$ such that

$$g(a_k) = \lambda_k \quad \text{for all } k.$$

Then the entire function

$$F = f_0 + f_1 g + \cdots + f_n g^n$$

vanishes at every point of $Z(h)$.

We now use the following consequence of maximality: if $h \in M$ and an entire function F vanishes on the reduced zero set of h , then $F \in M$. To see this, let q be an entire function whose zero set is exactly $Z(h)$, with all zeros simple. The ideal $M + (q)$ is proper, because every element of M has a common zero with q . Since M is maximal, $q \in M$. Since q divides F , we get $F \in M$.

Therefore

$$f_0 + f_1 g + \cdots + f_n g^n \in M,$$

which means

$$P(\bar{g}) = 0$$

in A/M . Hence A/M is algebraically closed.

Next we compute the cardinality of A/M . The set A has cardinality \mathfrak{c} , the cardinality of the continuum. Indeed, an entire function is determined by its values on any countable subset of \mathbb{C} with an accumulation point, for example on $\mathbb{Q} + i\mathbb{Q}$. Hence

$$|A| \leq |\mathbb{C}|^{\aleph_0} = \mathfrak{c},$$

and clearly $|A| \geq \mathfrak{c}$ because A contains the constant functions. Thus

$$|A| = \mathfrak{c}.$$

Therefore

$$|A/M| \leq \mathfrak{c}.$$

On the other hand, the constant functions inject into A/M , because $M \cap \mathbb{C} = 0$. Hence

$$|A/M| \geq \mathfrak{c}.$$

Thus

$$|A/M| = \mathfrak{c}.$$

The field A/M is algebraically closed of characteristic 0 and has cardinality \mathfrak{c} . Hence its transcendence degree over \mathbb{Q} is \mathfrak{c} . The same is true of \mathbb{C} . By the classification of algebraically closed fields by characteristic and transcendence degree, we get an abstract field isomorphism

$$A/M \cong \mathbb{C}.$$

3.5.8 Summary of the spectrum

Putting the preceding results together, we obtain the following description.

Theorem 3.5.5 *Let $A = O(\mathbb{C})$. Then $\text{Spec } A$ is canonically identified with the set of prime divisor filters on $\text{Div}^+(\mathbb{C})$.*

Under this identification:

- (1) *The zero ideal corresponds to the empty filter.*
- (2) *For every $a \in \mathbb{C}$, the evaluation ideal*

$$\mathfrak{m}(a) = \{f \in A : f(a) = 0\}$$

is a maximal ideal. These are exactly the non-zero fixed prime ideals.

- (3) *Every other non-zero prime ideal is free.*
- (4) *The maximal ideals are precisely the ideals $\mathfrak{m}(a)$ together with the maximal free ideals.*

(5) If M is maximal free, then A/M contains $\mathbb{C}(z)$ as a \mathbb{C} -subalgebra, but $A/M \cong \mathbb{C}$ as an abstract field.

Proof The divisor-filter description of $\text{Spec } A$ is [Theorem 3.5.2](#). The classification of fixed prime ideals is [Proposition 3.5.2](#). Since a prime ideal is either fixed or free by definition, all non-fixed prime ideals are free. The description of maximal ideals follows from the fixed case and the definition of maximal free ideals. The assertion about residue fields is [Proposition 3.5.3](#) and [Theorem 3.5.4](#). \square

3.5.9 Zariski topology

Finally, in terms of divisor filters, the Zariski topology is described as follows. For $f \in A \setminus \{0\}$, the basic open subset

$$\mathbb{D}(f) = \{\mathfrak{p} \in \text{Spec } A : f \notin \mathfrak{p}\}$$

corresponds to

$$\{\mathcal{F} : \text{div}(f) \notin \mathcal{F}\}.$$

Equivalently, the basic closed subset

$$\mathbb{V}(f) = \{\mathfrak{p} \in \text{Spec } A : f \in \mathfrak{p}\}$$

corresponds to

$$\{\mathcal{F} : \text{div}(f) \in \mathcal{F}\}.$$

Chapter 4

Stein compacta

Bourbaki is not eternal.
— Henri Cartan^a

^a Henri Cartan (1904–2008) was a founding member of Bourbaki and one of the architects of modern several complex variables. His work on coherent analytic sheaves and Stein manifolds, especially Cartan's Theorems A and B, brought sheaf-theoretic methods into complex analysis and gave Stein spaces their cohomological character. Through the Séminaire Cartan and his students, these ideas became part of the common language of complex analytic geometry.

4.1 The algebra of subsets

4.1.1 The algebra of germs along general subsets

Recall that \mathcal{LCS} is the category of locally convex topological vector spaces, not necessarily Hausdorff.

Definition 4.1.1 Let X be a complex space, and $S \subseteq X$ be a subset. Assume that S admits a σ -compact and Hausdorff open neighborhood in X .¹

We define

$$\Gamma(S, \mathcal{O}_X) := \varinjlim_{U \supseteq S} \Gamma(U, \mathcal{O}_X), \quad (4.1)$$

where U runs over the directed system of σ -compact and Hausdorff open neighborhoods of S in X . The inductive limit is taken in \mathcal{LCS} .

The topology on $\Gamma(S, \mathcal{O}_X)$ is called the *canonical topology*. When we refer to topological aspects of $\Gamma(S, \mathcal{O}_X)$, this topology is always understood.

Note that when S is open, the canonical topology on $\Gamma(S, \mathcal{O}_X)$ is the same as the canonical Fréchet topology as in [Section 2.2](#).

Also note that (4.1) holds in the category of vector spaces by [Proposition 1.5.3](#). In other words, the underlying linear space of $\Gamma(S, \mathcal{O}_X)$ is indeed the usual algebra of germs along S .

In general I do not know if $\Gamma(S, \mathcal{O}_X)$ is a topological algebra.

Example 4.1.1 Let X be a complex space. When $S = \{x\}$ is a singleton, the canonical topology on $\Gamma(S, \mathcal{O}_X) = \mathcal{O}_{X,x}$ is just the series topology by [Corollary 2.2.2](#).

¹ This assumption is satisfied if X is Hausdorff and S is relatively compact.

Proposition 4.1.1 *Let X be a complex space, $S \subseteq X$ be a subset. Assume that S admits a σ -compact and Hausdorff open neighborhood in X . Then there is a morphism in \mathcal{LCS}*

$$\Gamma(S, \mathcal{O}_X) \rightarrow C^0(S),$$

induced by the family of continuous homomorphisms

$$\Gamma(U, \mathcal{O}_X) \rightarrow C^0(S), \quad f \mapsto (s \mapsto f(s)) \quad (4.2)$$

for each σ -compact and Hausdorff open neighborhood $U \subseteq X$ of S .

Note that this map is not injective even when X is smooth.

Proof Since S is Hausdorff, $C^0(S)$ with the compact-open topology is a topological algebra.

It suffices to verify the continuity of (4.2), which follows immediately from Proposition 2.2.4. \square

Proposition 4.1.2 *Let X be a complex space, $T \subseteq X$ be a subset. Assume that T admits a σ -compact and Hausdorff open neighborhood in X . Let $S \subseteq T$ be a subset. Then the restriction*

$$\Gamma(T, \mathcal{O}_X) \rightarrow \Gamma(S, \mathcal{O}_X)$$

is continuous.

Proof Replacing X by an open neighborhood of T , we may assume that X is σ -compact and Hausdorff.

By the definition of the topology on $\Gamma(T, \mathcal{O}_X)$, it suffices to verify the following assertion: If U is an open neighborhood of T in X , then the restriction

$$\Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(S, \mathcal{O}_X)$$

is continuous. But U is also an open neighborhood of S in X , so this follows from the definition of the topology on $\Gamma(S, \mathcal{O}_X)$. \square

Observe the following functoriality:

Proposition 4.1.3 *Let $F: Y \rightarrow X$ be a morphism of complex spaces. Assume that $T \subseteq Y$ and $S \subseteq X$ are two sets with $F(T) \subseteq S$. Assume that T (resp. S) admits a σ -compact Hausdorff open neighborhood in Y (resp. X). Then the pull-back*

$$F^*: \Gamma(S, \mathcal{O}_X) \rightarrow \Gamma(T, \mathcal{O}_Y)$$

is continuous.

Proof Replacing X (resp. Y) by an open neighborhood of S (resp. T), we may assume that X and Y are σ -compact and Hausdorff. By definition of the topology on $\Gamma(S, \mathcal{O}_X)$, we may assume without loss of generality that S is open. Then by Proposition 2.2.1(2) and Proposition 4.1.2, the following two homomorphisms are continuous:

$$\Gamma(S, \mathcal{O}_X) \xrightarrow{F^*} \Gamma(F^{-1}(S), \mathcal{O}_Y) \rightarrow \Gamma(T, \mathcal{O}_Y),$$

so the desired continuity follows. \square

4.2 Germs along subspaces

4.2.1 Generalities

Definition 4.2.1 Consider the category \mathcal{C} as follows:

- (1) An object is a pair (X, S) , where X is a complex space, and $S \subseteq X$ is a subset admitting a σ -compact Hausdorff open neighborhood;
- (2) a morphism $(X, S) \rightarrow (X', S')$ is a morphism $F: X \rightarrow X'$ of complex spaces with $F(S) \subseteq S'$;
- (3) the composition of morphisms is the composition of the underlying morphisms of complex spaces.

Consider the set \mathcal{S} of arrows $F: (X, S) \rightarrow (X', S')$ such that there are open neighborhoods U (resp. U') of S (resp. S') in X (resp. X') for which F restricts to an isomorphism $U \rightarrow U'$ between complex spaces with $F(S) = S'$.

It is easy to see that \mathcal{S} is a right-multiplicative system. We define the category \mathcal{Germ} as the category of right-fractions of \mathcal{C} with respect to \mathcal{S} . An object α in \mathcal{Germ} is called a *germ (of pairs of complex spaces)*.

See [Stacks, Tag 04VB] for the basic notions of localizations of categories.

Definition 4.2.2 A germ α is a *compactum* if for one representative (X, K) of α , the set K is compact and X is Hausdorff.

Note that in this case, we can always take X to be σ -compact as well. Let $\mathcal{Compactum}$ be the category of compacta, regarded as a full subcategory of \mathcal{Germ} .

Given germs α and β represented by (X, S) and (X', S') , a morphism $\alpha \rightarrow \beta$ is represented by a morphism $U \rightarrow X'$ of complex spaces sending S to S' , where U is an open neighborhood of S in X . Two such morphisms are identified if they agree on an open neighborhood of S .

Example 4.2.1 The full subcategory of $\mathcal{CompSpace}$ consisting of σ -compact Hausdorff complex spaces is a full subcategory of \mathcal{Germ} . We identify a σ -compact Hausdorff complex space X with the germ represented by (X, X) .

Example 4.2.2 We write \mathbb{C}^0 for the germ represented by $(\mathbb{C}^0, \mathbb{C}^0)$.

The set S with its topology can also be viewed more intrinsically.

Definition 4.2.3 Let α be a germ. We define $|\alpha|$ as the set of morphisms $\mathbb{C}^0 \rightarrow \alpha$. If we represent α by a pair (X, S) , then $|\alpha|$ as a set is nothing but S . In particular, $|\alpha|$ has a topology induced from X , which is independent of the choice of (X, S) .

Given any subset $T \subseteq |\alpha|$, we write α_T for the germ represented by (X, T) if α is represented by (X, S) . When $x \in |\alpha|$, we write α_x instead of $\alpha_{\{x\}}$.

The object $|\alpha|$ is functorial: Given a morphism $f: \alpha \rightarrow \beta$ of germs, there is an induced continuous map

$$|f|: |\alpha| \rightarrow |\beta|. \quad (4.3)$$

Definition 4.2.4 Let α be a germ, represented by (X, S) . Then we let \mathcal{O}_α be the following sheaf of rings on S :

$$\mathcal{O}_\alpha := i^{-1}\mathcal{O}_X,$$

where $i: S \rightarrow X$ is the inclusion.

It is clear that \mathcal{O}_α is independent of the choice of (X, S) . Moreover, $(|\alpha|, \mathcal{O}_\alpha)$ is a locally \mathbb{C} -ringed space, and the inclusion i becomes a morphism of locally \mathbb{C} -ringed spaces. We have

$$\mathcal{O}_{\alpha,x} = \mathcal{O}_{X,x}, \quad \forall x \in |\alpha|. \quad (4.4)$$

The maximal ideal in $\mathcal{O}_{\alpha,x}$ will be denoted by $\mathfrak{m}_{\alpha,x}$.

Observe that given any morphism of germs $f: \beta \rightarrow \alpha$, we can enhance (4.3) to a functorial morphism between the corresponding locally \mathbb{C} -ringed spaces:

$$f: (|\beta|, \mathcal{O}_\beta) \rightarrow (|\alpha|, \mathcal{O}_\alpha). \quad (4.5)$$

We have chosen to use the same notation f .

Theorem 4.2.1 (H. Cartan) *Let (X, K) be a representative of a compactum. Then the natural functor*

$$\varinjlim_{U \supseteq K} \text{Coh}_{\mathcal{O}_U} \rightarrow \text{Coh}_{\mathcal{O}_\alpha} \quad (4.6)$$

is an equivalence of categories, where the colimit is taken over the filtered category of open neighborhoods U of K in X , with transition functors given by restriction.

The colimit is taken in the 2-category sense.

Proof Let $i: K \hookrightarrow X$ be the inclusion.

Step 1. First we prove full faithfulness.

Let U, V be open neighborhoods of K , and let

$$\mathcal{G} \in \text{Coh}_{\mathcal{O}_U}, \quad \mathcal{H} \in \text{Coh}_{\mathcal{O}_V}.$$

Replacing U and V by $U \cap V$, we may assume $U = V$. Put

$$\mathcal{P} = \text{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H}).$$

Then \mathcal{P} is coherent.

For every open neighborhood $W \subseteq U$ of K , we have

$$\mathrm{Hom}_{\mathcal{O}_W}(\mathcal{G}|_W, \mathcal{H}|_W) = \Gamma(W, \mathcal{P}|_W).$$

Since K is compact, by [Stacks, Tag 09V3], we find

$$\varinjlim_{W \supseteq K} \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{G}|_W, \mathcal{H}|_W) = \varinjlim_{W \supseteq K} \Gamma(W, \mathcal{P}|_W) \xrightarrow{\sim} \Gamma(K, i_U^{-1}\mathcal{P}).$$

Here $i_U : K \hookrightarrow U$ denotes the inclusion.

It remains to identify $i_U^{-1}\mathcal{P}$ with the internal Hom sheaf on K . We claim that the natural morphism

$$i_U^{-1}\mathcal{H}om_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{i_U^{-1}\mathcal{O}_U}(i_U^{-1}\mathcal{G}, i_U^{-1}\mathcal{H}) \quad (4.7)$$

is an isomorphism. This may be checked on stalks. For $x \in K$, the stalk of the left-hand side is $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H})_x$. Since \mathcal{G} is of finite presentation, this is naturally $\mathrm{Hom}_{\mathcal{O}_{U,x}}(\mathcal{G}_x, \mathcal{H}_x)$. On the other hand, the stalk of the right-hand side is

$$\mathrm{Hom}_{(i_U^{-1}\mathcal{O}_U)_x}((i_U^{-1}\mathcal{G})_x, (i_U^{-1}\mathcal{H})_x),$$

which is the same group. This proves that (4.7) is an isomorphism and (4.6) is fully faithful.

Step 2. We next prove the essential surjectivity.

Let

$$\mathcal{F} \in \mathrm{Coh}(i^{-1}\mathcal{O}_X).$$

Since \mathcal{F} is coherent, there exists an open cover of K in the relative topology such that \mathcal{F} admits a finite presentation on each member of the cover. Since K is compact, we may choose finitely many open subsets $\Omega_1, \dots, \Omega_r$ of K such that

$$K = \bigcup_{a=1}^r \Omega_a$$

and such that, on each Ω_a , there is an exact sequence

$$(i^{-1}\mathcal{O}_X|_{\Omega_a})^{\oplus p_a} \xrightarrow{\alpha_a} (i^{-1}\mathcal{O}_X|_{\Omega_a})^{\oplus q_a} \longrightarrow \mathcal{F}|_{\Omega_a} \longrightarrow 0.$$

Choose open subsets $\Omega'_a \subseteq K$ such that

$$K = \bigcup_{a=1}^r \Omega'_a$$

and such that, if $L_a := \overline{\Omega'_a}^K$, then $L_a \subseteq \Omega_a$. Each L_a is compact.

Restricting the presentation above to L_a , we get

$$(i^{-1}\mathcal{O}_X|_{L_a})^{\oplus p_a} \xrightarrow{\alpha_a|_{L_a}} (i^{-1}\mathcal{O}_X|_{L_a})^{\oplus q_a} \longrightarrow \mathcal{F}|_{L_a} \longrightarrow 0.$$

By Step 1, applied to the compact subset L_a , after choosing an open neighborhood V_a of L_a in X , the morphism $\alpha_a|_{L_a}$ is induced by a morphism

$$\tilde{\alpha}_a: \mathcal{O}_{V_a}^{\oplus p_a} \rightarrow \mathcal{O}_{V_a}^{\oplus q_a}.$$

Set $\mathcal{G}_a := \text{coker}(\tilde{\alpha}_a)$. Since \mathcal{O}_{V_a} is coherent and \mathcal{G}_a is finitely presented, \mathcal{G}_a is a coherent \mathcal{O}_{V_a} -module. Moreover, by construction, there is an isomorphism

$$\varphi_a: i_{L_a}^{-1}\mathcal{G}_a \xrightarrow{\sim} \mathcal{F}|_{L_a},$$

where $i_{L_a}: L_a \hookrightarrow V_a$ is the inclusion.

For a, b , put $L_{ab} := L_a \cap L_b$. On L_{ab} , the isomorphisms φ_a and φ_b induce an isomorphism

$$\psi_{ab} := \varphi_b^{-1} \circ \varphi_a: i_{L_{ab}}^{-1}\mathcal{G}_a \xrightarrow{\sim} i_{L_{ab}}^{-1}\mathcal{G}_b.$$

Since L_{ab} is compact, Step 1 implies that, after replacing the common domain by an open neighborhood $V_{ab} \subseteq V_a \cap V_b$ of L_{ab} , the morphism ψ_{ab} is induced by a morphism

$$\tilde{\psi}_{ab}: \mathcal{G}_a|_{V_{ab}} \rightarrow \mathcal{G}_b|_{V_{ab}}.$$

Similarly, ψ_{ab}^{-1} is induced, after perhaps shrinking V_{ab} , by a morphism

$$\tilde{\psi}_{ba}: \mathcal{G}_b|_{V_{ab}} \rightarrow \mathcal{G}_a|_{V_{ab}}.$$

Since the compositions $\psi_{ba} \circ \psi_{ab}$, $\psi_{ab} \circ \psi_{ba}$ are the identity on L_{ab} , another application of Step 1 allows us, after shrinking V_{ab} , to assume that

$$\tilde{\psi}_{ba} \circ \tilde{\psi}_{ab} = \text{id}, \quad \tilde{\psi}_{ab} \circ \tilde{\psi}_{ba} = \text{id}.$$

Thus $\tilde{\psi}_{ab}$ is an isomorphism on V_{ab} .

Now let $L_{abc} := L_a \cap L_b \cap L_c$. On L_{abc} , the transition maps satisfy $\psi_{bc} \circ \psi_{ab} = \psi_{ac}$. By Step 1, after shrinking an open neighborhood of L_{abc} , we may assume that

$$\tilde{\psi}_{bc} \circ \tilde{\psi}_{ab} = \tilde{\psi}_{ac}$$

near L_{abc} .

Since there are only finitely many indices, we may now choose open neighborhoods $W_a \subseteq V_a$ of L_a such that $W_a \cap W_b \subseteq V_{ab}$ for all a, b , and such that the triple intersections $W_a \cap W_b \cap W_c$ are contained in the neighborhoods on which the cocycle identities hold. After replacing \mathcal{G}_a by $\mathcal{G}_a|_{W_a}$ and $\tilde{\psi}_{ab}$ by its restriction to $W_a \cap W_b$, the $\tilde{\psi}_{ab}$ are transition isomorphisms satisfying the cocycle condition.

Therefore the coherent sheaves $\mathcal{G}_a|_{W_a}$ glue to a coherent \mathcal{O}_U -module \mathcal{G} on

$$U := \bigcup_{a=1}^r W_a.$$

Since the open subsets Ω'_a cover K and $\Omega'_a \subseteq L_a \subseteq W_a$, the set U is an open neighborhood of K .

Finally, the isomorphisms

$$\varphi_a: i_{L_a}^{-1} \mathcal{G}_a \xrightarrow{\sim} \mathcal{F}|_{L_a}$$

are compatible with the transition isomorphisms. Since the open sets Ω'_a cover K , they glue to an isomorphism

$$i^{-1} \mathcal{G} \xrightarrow{\sim} \mathcal{F}.$$

This proves essential surjectivity. \square

4.2.2 Properties of germs

We introduce several notions of germs. They generalize the corresponding notions of σ -compact Hausdorff complex spaces under [Example 4.2.1](#).

Definition 4.2.5 We say a germ α is *smooth* (resp. *reduced*) if there is a representative (X, S) of α such that X is smooth (resp. reduced).

Definition 4.2.6 A morphism $\alpha \rightarrow \beta$ of germs is a *closed immersion* if we can find (X, S) representing α and (Y, T) representing β and a morphism $f: X \rightarrow Y$ representing $\alpha \rightarrow \beta$ such that

- (1) f is a closed immersion;
- (2) $S = f^{-1}(T)$.

Definition 4.2.7 An *analytic subspace* of a germ α is a germ β together with a closed immersion $\beta \rightarrow \alpha$. When the closed immersion is obvious, we also say β is an analytic subspace of α .

Two analytic subspaces β and β' are identified if there is an isomorphism $\beta \rightarrow \beta'$ compatible with the morphisms to α .

In this case, we can identify $|\beta|$ with a closed subspace of $|\alpha|$. Given two analytic subspaces β and γ of α , we say $\beta \subseteq \gamma$ if $\beta \rightarrow \alpha$ factorizes through $\gamma \rightarrow \alpha$.

Example 4.2.3 Let α be a germ represented by (X, S) . We write α^{Sing} for the germ represented by $(X^{\text{Sing}}, S \cap X^{\text{Sing}})$. It is independent of the choice of (X, S) , and α^{Sing} is an analytic subspace of α .

Example 4.2.4 Let α be a germ represented by (X, S) . We write α^{red} for the germ represented by $(X^{\text{red}}, S \cap X^{\text{red}})$. It is independent of the choice of (X, S) , and α^{red} is an analytic subspace of α .

Note that α^{red} has the following universal property: if β is a reduced germ and $\beta \rightarrow \alpha$ is a morphism, then β admits a unique factorization through α^{red} .

Indeed, choose a representative (Y, T) of β with Y reduced, so that the morphism $\beta \rightarrow \alpha$ is represented by a morphism $Y \rightarrow X$. The usual universal property of the reduction of a complex space gives a unique factorization

$$Y \rightarrow X^{\text{red}} \rightarrow X.$$

The first arrow sends T into S , so it represents a morphism $\beta \rightarrow \alpha^{\text{red}}$ whose composite with $\alpha^{\text{red}} \rightarrow \alpha$ is the given morphism.

For uniqueness, suppose two morphisms $\beta \rightarrow \alpha^{\text{red}}$ have the same composite with α . After choosing representatives and shrinking the source around T , their composites with $X^{\text{red}} \rightarrow X$ are equal as morphisms to X . By the uniqueness in the universal property of $X^{\text{red}} \rightarrow X$, the two representatives to X^{red} are equal after the same shrinking. Hence the two morphisms are equal in $\mathcal{G}\text{erm}$.

Definition 4.2.8 We say an analytic subspace β of α is *essentially irreducible* at $x \in |\beta|$ if there is a representative (X, S) of α with β represented by an analytic subspace $Y \subseteq X$ such that, for any open subset $U \subseteq X$ containing S , $Y \cap U$ has only one irreducible component containing x .

Proposition 4.2.1 *Let α be a compactum. There is a canonical bijection between the following sets:*

- (1) *The set of analytic subspaces of α ;*
- (2) *the set of coherent ideals in \mathcal{O}_α .*

Given a coherent ideal \mathcal{I} in \mathcal{O}_α , we write the corresponding analytic subspace of α as $\mathbb{V}(\mathcal{I})$.

Proof The map from (1) to (2) is constructed as follows: suppose that $\beta \rightarrow \alpha$ is a closed immersion. Represent it by a closed immersion of analytic spaces $Y \hookrightarrow X$. Then the ideal of Y in X is a coherent ideal on X . It induces a coherent ideal in \mathcal{O}_α . The ideal is independent of the choices by [Theorem 4.2.1](#).

The map from (2) to (1) is constructed as follows. If \mathcal{I} is a coherent ideal in \mathcal{O}_α , then by [Theorem 4.2.1](#) again, we find a representative (X, K) of α such that \mathcal{I} comes from a coherent ideal \mathcal{I}' on X . We define $\mathbb{V}(\mathcal{I})$ as the germ represented by $(\mathbb{V}(\mathcal{I}'), \mathbb{V}(\mathcal{I}') \cap K)$.

The two maps are inverse to each other. □

Corollary 4.2.1 *Let α be a compactum, and β be an analytic subspace of α . Then the following are equivalent:*

- (1) *β is reduced;*
- (2) *the ideal of β in α is radical.*

Proof (1) \implies (2). Take a representative (X, K) of α such that $\beta \hookrightarrow \alpha$ can be represented by a closed immersion $Y \hookrightarrow X$. Since β is reduced, we can find a reduced open neighborhood V of $K \cap Y$ in Y . We can find an open neighborhood U of K in X such that $U \cap Y = V$. We then replace X by U and Y by $Y \cap U$ and reduce to

the case where Y is reduced. In this case, we know that the ideal of Y in X is radical, and hence so is the ideal of β in α .

(2) \implies (1). Assume (2). Denote by $\mathcal{I} \subseteq \mathcal{O}_\alpha$ the ideal of β in α . Represent α by (X, K) . By [Theorem 4.2.1](#), we may assume that \mathcal{I} comes from a coherent ideal sheaf \mathcal{J} on X . Observe that the set of $x \in X$ with \mathcal{J}_x radical is open, so after replacing X by an open neighborhood of K , we may assume that \mathcal{J} is radical. Then it follows that the closed subspace Y it defines is reduced, and hence β is reduced. \square

Corollary 4.2.2 *Let α be a compactum, and β be an analytic subspace of α . Let \mathcal{I} (resp. \mathcal{J}) denote the ideal of β (resp. β^{red}) in α . Then*

$$\mathcal{J} = \sqrt{\mathcal{I}}.$$

In particular, for any coherent ideal sheaf \mathcal{I} on α , $\sqrt{\mathcal{I}}$ is coherent.

Proof Since β^{red} is reduced, [Corollary 4.2.1](#) implies that \mathcal{J} is radical. The closed immersion $\beta^{\text{red}} \hookrightarrow \beta$ gives $\mathcal{I} \subseteq \mathcal{J}$, hence $\sqrt{\mathcal{I}} \subseteq \mathcal{J}$.

Conversely, let γ be the analytic subspace of α defined by $\sqrt{\mathcal{I}}$. By [Corollary 4.2.1](#), γ is reduced. Since $\mathcal{I} \subseteq \sqrt{\mathcal{I}}$, we have a closed immersion $\gamma \hookrightarrow \beta$. The universal property of the reduction in [Example 4.2.4](#) gives a factorization

$$\gamma \longrightarrow \beta^{\text{red}} \longrightarrow \beta.$$

Therefore $\sqrt{\mathcal{I}} \supseteq \mathcal{J}$. This proves the claim. \square

4.3 Stein compactum

4.3.1 Stein compact subsets

Recall that the notion of Stein compact subsets is introduced in [Definition 3.1.1](#).

Proposition 4.3.1 *Let X be a Hausdorff complex space. Let K_1 and K_2 be Stein compact subsets of X . Then so is $K_1 \cap K_2$.*

Proof Let U_1 and U_2 be Stein open neighborhoods of K_1 and K_2 in X . Since X is Hausdorff, the diagonal of X is closed. Hence $U_1 \cap U_2$ identifies with a closed analytic subspace of $U_1 \times U_2$. Since $U_1 \times U_2$ is Stein, the space $U_1 \cap U_2$ is Stein. Thus $K_1 \cap K_2$ has a Stein open neighborhood. \square

Proposition 4.3.2 *Let X be a complex space. Let Y be an analytic subspace of X and K be a compact subset of Y . Then the following are equivalent:*

- (1) K is Stein in Y ;
- (2) K is Stein in X .

Proof We only need to show (1) \implies (2). Assume (1). Let U be an open neighborhood of K in X . We need to find a Stein open neighborhood of K in X contained in U . Note that K is Stein in $Y \cap U$ as well. Thus after replacing X by U and Y by $Y \cap U$, it suffices to find a Stein open neighborhood of K in X . Let V be a Stein open neighborhood of K in Y . Then our assertion follows from Siu's theorem on Stein neighborhoods of V [For17, Theorem 3.1.1]. \square

Proposition 4.3.3 *Let Y be an analytic subspace of X and K be a Stein compact subset of Y . Then the natural map*

$$\Gamma(K, \mathcal{O}_X) \rightarrow \Gamma(K, \mathcal{O}_Y) \quad (4.8)$$

is surjective.

Proof Let $f \in \Gamma(K, \mathcal{O}_Y)$. We show that f lies in the image of (4.8). Take an open neighborhood V of K in Y on which f is defined.

By Proposition 4.3.2, K is Stein in X . Therefore, we can find a Stein open neighborhood U of K in X such that $U \cap Y \subseteq V$. Then $f|_{U \cap Y}$ lifts to a holomorphic function in $\Gamma(U, \mathcal{O}_X)$, whose image in $\Gamma(K, \mathcal{O}_X)$ is a preimage of f with respect to (4.8). \square

Definition 4.3.1 A compactum α is called a *Stein compactum* if, for one (hence any) representative (X, K) of α , the set K is a Stein compact set in X .

If, in addition, X can be chosen smooth, we say that α is a *smooth Stein compactum*.

Definition 4.3.2 Let α be a Stein compactum and let \mathcal{F} be a coherent \mathcal{O}_α -module. Choose a representative (X, K) of α and a coherent \mathcal{O}_X -module \mathcal{G} representing \mathcal{F} near K . We define

$$\Gamma(|\alpha|, \mathcal{F}) := \varinjlim_{U \supseteq K} \Gamma(U, \mathcal{G}|_U),$$

where U runs over the open neighborhoods of K in X for which $\Gamma(U, \mathcal{G}|_U)$ carries its canonical Fréchet topology, and the colimit is taken in \mathcal{LCS} .

This locally convex topology is independent of the representative (X, K) and of \mathcal{G} , by Theorem 4.2.1. In particular,

$$\Gamma(\alpha) := \Gamma(|\alpha|, \mathcal{O}_\alpha)$$

is a topological algebra by [BS76, Proposition 2.10] and Proposition 1.5.3, and $\Gamma(|\alpha|, \mathcal{F})$ is a topological $\Gamma(\alpha)$ -module.

Note that $\Gamma(\alpha)$ is functorial in α , cf. (4.5).

We introduce the following notation. If α is a Stein compactum, β is an analytic subspace of α , and $f \in \Gamma(\alpha)$, then we write $f|_\beta$ for the image of f with respect to $\Gamma(\alpha) \rightarrow \Gamma(\beta)$.

Definition 4.3.3 Given a Stein compactum α , we define its *character map* $\chi = \chi_\alpha: |\alpha| \rightarrow \mathrm{Sp} \Gamma(\alpha)$ as follows: Given $x \in |\alpha|$, $\chi_\alpha(x): \Gamma(\alpha) \rightarrow \mathbb{C}$ sends f to $f(x)$.

The character map is a well-defined continuous map. If α is represented by (X, K) , we also write $\chi_K = \chi: K \rightarrow \mathrm{Sp} \Gamma(K, \mathcal{O}_X)$.

Cartan's Theorem A and Theorem B hold in the following form:

Theorem 4.3.1 *Let α be a Stein compactum. Consider a coherent \mathcal{O}_α -module \mathcal{F} . Then*

- (1) *for any $x \in |\alpha|$, the germ \mathcal{F}_x is generated by $\Gamma(|\alpha|, \mathcal{F})$ as an $\mathcal{O}_{X,x}$ -module;*
- (2) *for any $p \geq 1$, we have $H^p(|\alpha|, \mathcal{F}) = 0$.*

Proof By [Theorem 4.2.1](#), we can take a representative (X, K) of α with X Stein and \mathcal{F} represented by a coherent sheaf on X .

- (1) This follows immediately from the usual Cartan's Theorem A.
- (2) This follows from [\[Stacks, Tag 09V3\]](#) and the usual Cartan's Theorem B. \square

Corollary 4.3.1 *Let α be a Stein compactum and let \mathcal{F} be a coherent \mathcal{O}_α -module. Put $A := \Gamma(\alpha)$. Then:*

- (1) *$\Gamma(|\alpha|, \mathcal{F})$ is a finite A -module;*
- (2) *for every $x \in |\alpha|$, putting $\mathfrak{m} = \mathfrak{m}(x)$, the natural map*

$$\Gamma(|\alpha|, \mathcal{F})_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} \mathcal{O}_{\alpha,x} \longrightarrow \mathcal{F}_x$$

is an isomorphism.

Proof By [Theorem 4.3.1\(1\)](#), for every $x \in |\alpha|$ the stalk \mathcal{F}_x is generated by the germs of global sections of \mathcal{F} . Thus we may choose finitely many sections

$$s_{x,1}, \dots, s_{x,n_x} \in \Gamma(|\alpha|, \mathcal{F})$$

whose germs generate \mathcal{F}_x . The induced morphism

$$\mathcal{O}_\alpha^{\oplus n_x} \longrightarrow \mathcal{F}$$

is surjective at x , hence is surjective on some open neighborhood of x . Since $|\alpha|$ is compact, finitely many such neighborhoods cover $|\alpha|$. Collecting the corresponding sections, we obtain a surjective morphism

$$\mathcal{O}_\alpha^{\oplus N} \twoheadrightarrow \mathcal{F}$$

for some $N \geq 0$.

Let \mathcal{K} be its kernel. Then \mathcal{K} is coherent, and we have a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}_\alpha^{\oplus N} \rightarrow \mathcal{F} \rightarrow 0.$$

Taking global sections and using [Theorem 4.3.1\(2\)](#) for \mathcal{K} , we get a surjection

$$A^{\oplus N} = \Gamma(|\alpha|, \mathcal{O}_\alpha^{\oplus N}) \twoheadrightarrow \Gamma(|\alpha|, \mathcal{F}).$$

Hence $\Gamma(|\alpha|, \mathcal{F})$ is a finite A -module.

We now prove the base change assertion. Applying the preceding finite generation argument to \mathcal{K} , we can choose a surjection

$$\mathcal{O}_\alpha^{\oplus M} \twoheadrightarrow \mathcal{K}.$$

Let \mathcal{L} be its kernel. Taking global sections in

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{O}_\alpha^{\oplus M} \rightarrow \mathcal{K} \rightarrow 0$$

and using [Theorem 4.3.1\(2\)](#) for \mathcal{L} , we see that

$$A^{\oplus M} \rightarrow A^{\oplus N} \rightarrow \Gamma(|\alpha|, \mathcal{F}) \rightarrow 0$$

is exact. For $x \in |\alpha|$, localizing this presentation at $\mathfrak{m} = \mathfrak{m}(x)$ and tensoring with $\mathcal{O}_{\alpha,x}$ gives a presentation whose first two terms are $\mathcal{O}_{\alpha,x}^{\oplus M}$ and $\mathcal{O}_{\alpha,x}^{\oplus N}$. This is precisely the stalk at x of the sheaf presentation

$$\mathcal{O}_\alpha^{\oplus M} \rightarrow \mathcal{O}_\alpha^{\oplus N} \rightarrow \mathcal{F} \rightarrow 0.$$

Taking cokernels gives the desired isomorphism. \square

Proposition 4.3.4 *Let S be a Stein space and let $K \subseteq S$ be a Stein compact subset. Then the natural map*

$$\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(K, \mathcal{O}_S)$$

is flat.

This is a generalization of [Lemma 3.3.7](#).

Proof We shall use the equational criterion for flatness [[Stacks, Tag 00HK](#)].

Let $A := \Gamma(S, \mathcal{O}_S)$. Put $B = \Gamma(K, \mathcal{O}_S)$. Let $f_1, \dots, f_n \in A$ and suppose that

$$\sum_{i=1}^n f_i|_K b_i = 0 \quad \text{in } B.$$

Consider the morphism of coherent analytic sheaves

$$\varphi: \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{O}_S, \quad (a_1, \dots, a_n) \longmapsto \sum_{i=1}^n a_i f_i.$$

Let $\mathcal{R} := \ker(\varphi)$. Note that \mathcal{R} is coherent. Let

$$b := (b_1, \dots, b_n),$$

then $b_x \in \mathcal{R}_x$ for any $x \in K$.

By Cartan's Theorem A and the compactness of K , there exist global sections

$$r^{(1)}, \dots, r^{(m)} \in \Gamma(S, \mathcal{R}) \subseteq A^{\oplus n},$$

which generate \mathcal{R}_x for each $x \in K$. Then we have a surjective morphism of $\mathcal{O}_S|_K$ -modules

$$(\mathcal{O}_S|_K)^m \rightarrow \mathcal{R}|_K, \quad (c_1, \dots, c_m) \mapsto \sum_{j=1}^m c_j r^{(j)}|_K.$$

By Cartan's Theorem B in [Theorem 4.3.1](#), we can therefore find $c_1, \dots, c_m \in \Gamma(K, \mathcal{O}_S)$ such that

$$b_x = \sum_{j=1}^m c_j r_x^{(j)}, \quad \forall x \in K.$$

Write

$$r^{(j)} = (r_{1j}, \dots, r_{nj}) \in A^{\oplus n}.$$

Then for each i we have

$$b_i = \sum_{j=1}^m c_j r_{ij}|_K.$$

On the other hand, since each $r^{(j)}$ is a global section of \mathcal{R} , we have

$$\sum_{i=1}^n r_{ij} f_i = 0 \quad \text{in } A$$

for every j .

The desired flatness follows. \square

Corollary 4.3.2 *Let X be a complex space and $K \subseteq K'$ be Stein compact sets. Then the natural map*

$$\Gamma(K', \mathcal{O}_X) \rightarrow \Gamma(K, \mathcal{O}_X) \tag{4.9}$$

is flat.

Proof By [Proposition 4.3.4](#), for each Stein open neighborhood U of K' , the restriction

$$\Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(K, \mathcal{O}_X)$$

is flat. Therefore, (4.9) is also flat by [\[Stacks, Tag 05UU\]](#). \square

Proposition 4.3.5 *Let X be a Hausdorff complex space, and let $K \subseteq X$ be a Stein compact subset. Then the natural map*

$$\mathrm{Sp} \Gamma(K, \mathcal{O}_X) \rightarrow \varprojlim_{U \supseteq K} \mathrm{Sp} \Gamma(U, \mathcal{O}_X)$$

of locally \mathbb{C} -ringed spaces is a homeomorphism on the underlying topological spaces, where U runs over the σ -compact open neighborhoods of K in X .

Proof This follows from the general fact [Proposition 1.5.5](#). \square

Corollary 4.3.3 *Let X be a Hausdorff complex space. Suppose that $K \subseteq X$ is a Stein compact subset. Then the character map $\chi_K : K \rightarrow \mathrm{Sp}\Gamma(K, \mathcal{O}_X)$ is a homeomorphism.*

Recall that a Stein compact subset is defined in [Definition 3.1.1](#).

Proof For each Stein open neighborhood U of K , the restriction homomorphism

$$\Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(K, \mathcal{O}_X)$$

induces a commutative diagram in the category of topological spaces:

$$\begin{array}{ccc} K & \hookrightarrow & U \\ \downarrow \chi_K & & \downarrow \chi_U \\ \mathrm{Sp}\Gamma(K, \mathcal{O}_X) & \longrightarrow & \mathrm{Sp}\Gamma(U, \mathcal{O}_U). \end{array}$$

Indeed, evaluating a section of $\Gamma(U, \mathcal{O}_U)$ at a point $x \in K$ gives the same value before and after restriction to K . Since U is Stein, the map χ_U is a homeomorphism by [Lemma 3.3.3](#). Passing to the inverse limit over all Stein open neighborhoods U of K , we get a commutative diagram in the category of topological spaces:

$$\begin{array}{ccc} K & \longrightarrow & \varprojlim_U U \\ \downarrow \chi_K & & \downarrow \\ \mathrm{Sp}\Gamma(K, \mathcal{O}_X) & \longrightarrow & \varprojlim_U \mathrm{Sp}\Gamma(U, \mathcal{O}_U), \end{array}$$

where U runs over all Stein open neighborhoods of K . Note that the upper horizontal map is a homeomorphism and the right-vertical map is also a homeomorphism. Since the Stein open neighborhoods of K are cofinal among the open neighborhoods of K , the lower-horizontal map is a homeomorphism by [Proposition 4.3.5](#). Thus χ_K is also a homeomorphism. \square

4.3.2 Morphisms to Stein compacta

Lemma 4.3.1 *Let $K \subseteq \mathbb{C}^n$ be a Stein compact subset, and let β be a compactum. Then any continuous \mathbb{C} -algebra homomorphism*

$$\Phi : \Gamma(K, \mathcal{O}_{\mathbb{C}^n}) \rightarrow \Gamma(\beta)$$

is uniquely determined by its values $\Phi(z_i)$ for $i = 1, \dots, n$.

Here and in the sequel, z_1, \dots, z_n denote the coordinates on \mathbb{C}^n .

Proof Let

$$\Phi' : \Gamma(K, \mathcal{O}_{\mathbb{C}^n}) \rightarrow \Gamma(\beta)$$

be another continuous homomorphism, and assume that

$$\Phi(z_i) = \Phi'(z_i), \quad i = 1, \dots, n.$$

We prove that $\Phi(f) = \Phi'(f)$ for every $f \in \Gamma(K, \mathcal{O}_{\mathbb{C}^n})$.

It suffices to prove equality on stalks at every $y \in |\beta|$. Since $\mathcal{O}_{\beta, y}$ is a Noetherian local ring, Krull's intersection theorem gives

$$\bigcap_{M \geq 1} \mathfrak{m}_{\beta, y}^M = 0.$$

Hence it is enough to prove that, for every $y \in |\beta|$ and every integer $M \geq 1$,

$$\Phi(f)_y - \Phi'(f)_y \in \mathfrak{m}_{\beta, y}^M. \quad (4.10)$$

Fix $y \in |\beta|$. The compositions

$$\chi_{\beta}(y) \circ \Phi, \quad \chi_{\beta}(y) \circ \Phi'$$

are continuous characters of $\Gamma(K, \mathcal{O}_{\mathbb{C}^n})$. Since K is a Stein compact subset of \mathbb{C}^n , the character map

$$K \rightarrow \mathrm{Sp} \Gamma(K, \mathcal{O}_{\mathbb{C}^n})$$

is a homeomorphism by [Corollary 4.3.3](#). Therefore these two characters correspond to points of K . They have the same values on the coordinate functions z_i , hence they correspond to the same point

$$x = (x_1, \dots, x_n) \in K.$$

Equivalently,

$$x_i = \Phi(z_i)(y) = \Phi'(z_i)(y), \quad i = 1, \dots, n.$$

Thus

$$\Phi(z_i)_y - x_i \in \mathfrak{m}_{\beta, y}, \quad i = 1, \dots, n. \quad (4.11)$$

Choose an open neighborhood of K on which f is represented by a holomorphic function, still denoted by f . Since $x \in K$, the germ f_x has a Taylor expansion at x . Subtracting its Taylor polynomial of degree $< M$, we may assume that

$$f_x \in \mathfrak{m}_{\mathbb{C}^n, x}^M.$$

Let \mathcal{J}_M be the coherent ideal sheaf on \mathbb{C}^n generated by the monomials

$$(z - x)^\alpha, \quad |\alpha| = M.$$

Then $(\mathcal{J}_M)_x = \mathfrak{m}_{\mathbb{C}^n, x}^M$, while $(\mathcal{J}_M)_q = \mathcal{O}_{\mathbb{C}^n, q}$ for $q \neq x$. Hence f defines a section of $i^{-1}\mathcal{J}_M$ over K , where $i: K \hookrightarrow \mathbb{C}^n$ is the inclusion.

By [Theorem 4.3.1\(2\)](#), applied to the surjection

$$\mathcal{O}_{\mathbb{C}^n}^{\oplus N} \longrightarrow \mathcal{J}_M$$

induced by the monomial generators, the induced map on sections over K is surjective. Therefore we can write

$$f = \sum_{|\alpha|=M} g_\alpha (z-x)^\alpha$$

with $g_\alpha \in \Gamma(K, \mathcal{O}_{\mathbb{C}^n})$.

Applying Φ and Φ' , and using that $\Phi(z_i) = \Phi'(z_i)$, we get

$$\Phi(f) - \Phi'(f) = \sum_{|\alpha|=M} (\Phi(g_\alpha) - \Phi'(g_\alpha)) (\Phi(z) - x)^\alpha.$$

By (4.11), each monomial $(\Phi(z) - x)^\alpha$ with $|\alpha| = M$ lies in $\mathfrak{m}_{\beta, y}^M$. This proves (4.10), and hence $\Phi(f) = \Phi'(f)$. \square

Lemma 4.3.2 *Let α be a compactum, let β be a Stein compactum, and let γ be an analytic subspace of β . Suppose that $h: \alpha \rightarrow \beta$ is a morphism. Then the following are equivalent:*

- (1) h factors through γ ;
- (2) for every $f \in \Gamma(\beta)$ with $f|_\gamma = 0$, we have $h^*(f) = 0$.

Proof The implication (1) \implies (2) is immediate.

We prove (2) \implies (1). Let

$$\mathcal{I} \subseteq \mathcal{O}_\beta$$

be the coherent ideal sheaf defining γ . We need to show that the composite

$$h^{-1}\mathcal{I} \longrightarrow h^{-1}\mathcal{O}_\beta \longrightarrow \mathcal{O}_\alpha$$

is zero.

Since β is a Stein compactum, applying [Theorem 4.3.1](#) to the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_\beta \longrightarrow \mathcal{O}_\gamma \longrightarrow 0$$

gives

$$\Gamma(|\beta|, \mathcal{I}) = \ker(\Gamma(\beta) \rightarrow \Gamma(\gamma)).$$

Hence assumption (2) says precisely that

$$h^*s = 0, \quad \forall s \in \Gamma(|\beta|, \mathcal{I}).$$

By [Theorem 4.3.1](#)(1), the sheaf \mathcal{I} is generated stalkwise by its global sections. Therefore the image of $h^{-1}\mathcal{I}$ in \mathcal{O}_α is generated stalkwise by the pullbacks of these global sections. These pullbacks are zero by the preceding paragraph. Hence the morphism

$$h^{-1}\mathcal{I} \rightarrow \mathcal{O}_\alpha$$

is zero, and h factors through γ . \square

Theorem 4.3.2 *Let $\alpha, \beta \in \text{Compact}$. Assume that α is a Stein compactum. Then the natural map*

$$\text{Hom}_{\mathcal{G}\text{erm}}(\beta, \alpha) \rightarrow \text{Hom}(\Gamma(\alpha), \Gamma(\beta)), \quad h \mapsto h^* \quad (4.12)$$

is bijective. Here $\text{Hom}(\Gamma(\alpha), \Gamma(\beta))$ denotes the set of continuous \mathbb{C} -algebra homomorphisms $\Gamma(\alpha) \rightarrow \Gamma(\beta)$.

In general, I do not know if $\Gamma(\beta)$ is a topological algebra or not.

Proof Choose a representative (X, K) of α such that X is a Stein space and K is a Stein compact subset of X . By compactness of K , after shrinking X around K , we may assume that X has finite dimension and finite embedded dimension. Hence, by the embedding theorem [Theorem 3.1.2](#), we may further assume that X is a closed analytic subspace of some \mathbb{C}^n .

Let γ be the germ represented by (\mathbb{C}^n, K) . By [Proposition 4.3.2](#), K is a Stein compact subset of \mathbb{C}^n , so γ is a Stein compactum. The inclusion $i: X \hookrightarrow \mathbb{C}^n$ induces a closed immersion of germs $i: \alpha \hookrightarrow \gamma$.

Step 1. We first prove injectivity.

The morphism i induces a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{G}\text{erm}}(\beta, \alpha) & \longrightarrow & \text{Hom}(\Gamma(\alpha), \Gamma(\beta)) \\ i \circ (-) \downarrow & & \downarrow (-) \circ i^* \\ \text{Hom}_{\mathcal{G}\text{erm}}(\beta, \gamma) & \longrightarrow & \text{Hom}(\Gamma(\gamma), \Gamma(\beta)). \end{array}$$

The left vertical arrow is injective because a closed immersion of germs is a monomorphism. Therefore it suffices to prove injectivity when $\alpha = \gamma$, namely when α is represented by (\mathbb{C}^n, K) .

Assume now that α is represented by (\mathbb{C}^n, K) . Let $z_1, \dots, z_n \in \Gamma(\alpha)$ be the coordinate functions. Suppose that

$$h, h': \beta \rightarrow \alpha$$

are two morphisms of germs with $h^* = h'^*$. After choosing a representative (Y, S) of β and shrinking it around S , we may represent h and h' by morphisms $F, F': Y \rightarrow \mathbb{C}^n$. Since $h^* = h'^*$, the germs $F^*(z_i) = F'^*(z_i)$ agree along S for every $i = 1, \dots, n$. After shrinking Y around S , these equalities hold on Y . Hence $F = F'$ near S , and therefore $h = h'$.

Step 2. We reduce surjectivity to the case where α is represented by (\mathbb{C}^n, K) .

Assume surjectivity is known for γ in place of α . Let $\Phi: \Gamma(\alpha) \rightarrow \Gamma(\beta)$ be a continuous homomorphism. By the surjectivity for γ , there is a morphism of germs $h: \beta \rightarrow \gamma$ such that

$$h^* = \Phi \circ i^*: \Gamma(\gamma) \rightarrow \Gamma(\beta).$$

We claim that h factors through $\alpha \hookrightarrow \gamma$. By [Lemma 4.3.2](#), it suffices to show that $h^*(f) = 0$ for every

$$f \in \ker(i^* : \Gamma(\gamma) \rightarrow \Gamma(\alpha)).$$

This is immediate from $h^* = \Phi \circ i^*$.

Thus there exists a morphism $g : \beta \rightarrow \alpha$ such that $h = i \circ g$. Then

$$g^* \circ i^* = h^* = \Phi \circ i^*.$$

The map

$$i^* : \Gamma(\gamma) \rightarrow \Gamma(\alpha)$$

is surjective by [Proposition 4.3.3](#). Hence $g^* = \Phi$.

Step 3. We prove surjectivity when α is represented by (\mathbb{C}^n, K) .

Let $\Phi : \Gamma(\alpha) \rightarrow \Gamma(\beta)$ be a continuous homomorphism. Define

$$g_i := \Phi(z_i) \in \Gamma(\beta), \quad i = 1, \dots, n.$$

Choose a representative (Y, S) of β such that all g_i are represented by holomorphic functions on Y , still denoted by g_i . These functions define a holomorphic map

$$G : Y \rightarrow \mathbb{C}^n.$$

We claim that $G(S) \subseteq K$. To see this, define

$$h_0 := \chi_\alpha^{-1} \circ \text{Sp}(\Phi) \circ \chi_\beta : |\beta| \rightarrow |\alpha| = K.$$

This is well-defined because χ_α is a homeomorphism by [Corollary 4.3.3](#). For $s \in S$ and for each $i = 1, \dots, n$, we have

$$z_i(h_0(s)) = (\chi_\beta(s) \circ \Phi)(z_i) = \chi_\beta(s)(g_i) = g_i(s) = G_i(s).$$

Since the coordinate functions separate points of \mathbb{C}^n , it follows that $G(s) = h_0(s) \in K$. Hence G maps S into K , and therefore defines a morphism of germs $g : \beta \rightarrow \alpha$.

By construction,

$$g^*(z_i) = g_i = \Phi(z_i), \quad i = 1, \dots, n.$$

Therefore [Lemma 4.3.1](#) gives $g^* = \Phi$. This proves surjectivity and hence the theorem. \square

4.4 The section rings of Stein compacta

Let α be a Stein compactum. We wish to understand the ring structure of $\Gamma(\alpha)$.

4.4.1 The ideals

The maximal ideals in $\Gamma(\alpha)$ are easily understood.

Proposition 4.4.1 *Let α be a Stein compactum. There is a natural bijection between the following sets:*

- (1) $|\alpha|$;
- (2) $\text{Sp } \Gamma(\alpha)$;
- (3) *the set of maximal ideals in $\Gamma(\alpha)$.*

We shall write the ideal corresponding to $x \in |\alpha|$ as $\mathfrak{m}(x)$.

Proof The bijection between (1) and (2) is given by [Corollary 4.3.3](#). More specifically, a point $x \in |\alpha|$ is sent to $\chi(x): \Gamma(\alpha) \rightarrow \mathbb{C}$, sending f to $f(x)$.

The map from (1) to (3) sends $x \in |\alpha|$ to $\ker \chi(x)$. The map is clearly injective. Let us show that any maximal ideal \mathfrak{m} of $\Gamma(\alpha)$ is of this form.

Suppose not. Then for any $x \in |\alpha|$, we can find $f^x \in \mathfrak{m} \subseteq \Gamma(\alpha)$ with $f^x(x) \neq 0$. Take an open neighborhood U_x of x in $|\alpha|$ on which $f^x \neq 0$. By the compactness of $|\alpha|$, there are finitely many $f_1, \dots, f_n \in \mathfrak{m}$ without common zeros on $|\alpha|$. The sheaf homomorphism

$$\mathcal{O}_\alpha^n \xrightarrow{\cdot(f_1, \dots, f_n)} \mathcal{O}_\alpha$$

is then surjective. By Cartan's Theorem B in [Theorem 4.3.1](#), we can therefore find $g_1, \dots, g_n \in \Gamma(\alpha)$ such that

$$1 = \sum_{i=1}^n g_i f_i \in \mathfrak{m},$$

which is a contradiction. □

Finitely generated ideals in $\Gamma(\alpha)$ are also easily understood. Given a subset E of $\Gamma(\alpha)$, we write $E\mathcal{O}_\alpha \subseteq \mathcal{O}_\alpha$ for the coherent ideal sheaf generated by E , similar to [Definition 2.3.1](#).

Proposition 4.4.2 *Let α be a Stein compactum. There is a natural bijection between*

- (1) *the set of coherent ideal sheaves in \mathcal{O}_α , and*
- (2) *the set of finitely generated ideals in $\Gamma(\alpha)$.*

In particular, if β is an analytic subspace of α , then the kernel of

$$\Gamma(\alpha) \rightarrow \Gamma(\beta)$$

is finitely generated.

Proof The map from (2) to (1) sends a finitely generated ideal $I \subseteq \Gamma(\alpha)$ to $I\mathcal{O}_\alpha$.

Conversely, the map from (1) to (2) sends a coherent ideal \mathcal{I} to $\Gamma(|\alpha|, \mathcal{I})$. From Cartan's Theorem A and the compactness of $|\alpha|$, we can find $f_1, \dots, f_n \in \Gamma(|\alpha|, \mathcal{I})$ such that the homomorphism

$$\mathcal{O}_\alpha^n \rightarrow \mathcal{I}, \quad (g_1, \dots, g_n) \mapsto \sum_{i=1}^n g_i f_i$$

is surjective. By Cartan's Theorem B, it follows immediately that $\Gamma(|\alpha|, \mathcal{I})$ is finitely generated.

These two operations are inverse to each other. Suppose that I is a finitely generated ideal in $\Gamma(\alpha)$. Then we show that

$$I = \Gamma(|\alpha|, I\mathcal{O}_\alpha). \quad (4.13)$$

The inclusion \subseteq is obvious. It suffices to prove the reverse inclusion. For this purpose, take finitely many generators f_1, \dots, f_n of I . Then we have a surjective morphism of \mathcal{O}_α -modules:

$$\mathcal{O}_\alpha^n \rightarrow I\mathcal{O}_\alpha$$

induced by (f_1, \dots, f_n) . By Cartan's Theorem B, we find that the following map is surjective:

$$\Gamma(\alpha)^n \rightarrow \Gamma(|\alpha|, I\mathcal{O}_\alpha), \quad (g_1, \dots, g_n) \mapsto \sum_{i=1}^n g_i f_i.$$

Therefore, (4.13) follows.

Conversely, if \mathcal{I} is a coherent ideal sheaf in \mathcal{O}_α , we need to show that the natural map

$$\Gamma(|\alpha|, \mathcal{I})\mathcal{O}_\alpha \rightarrow \mathcal{I}$$

is an isomorphism. This follows immediately from Cartan's Theorem A. \square

Corollary 4.4.1 *Let α be a Stein compactum and let I be a finitely generated ideal in $\Gamma(\alpha)$. Then \sqrt{I} is also finitely generated.*

Proof We claim that

$$\sqrt{I} = \Gamma(|\alpha|, \sqrt{I\mathcal{O}_\alpha}). \quad (4.14)$$

Note that $\sqrt{I\mathcal{O}_\alpha}$ is coherent by [Corollary 4.2.2](#). Therefore, \sqrt{I} is finitely generated by [Proposition 4.4.2](#).

It remains to argue (4.14). The \subseteq direction is trivial. Conversely, take $f \in \Gamma(|\alpha|, \sqrt{I\mathcal{O}_\alpha})$. Then for any $x \in |\alpha|$, we find an integer $n > 0$ with $f_x^n \in (I\mathcal{O}_\alpha)_x$. The same holds on an open neighborhood of x as well. Thus by the compactness of $|\alpha|$ we can find $n > 0$ large enough with $f^n \in \Gamma(|\alpha|, I\mathcal{O}_\alpha) = I$, where the equality is just (4.13). \square

Corollary 4.4.2 *Let α be a Stein compactum. All maximal ideals of $\Gamma(\alpha)$ are finitely generated.*

Proof By [Proposition 4.4.1](#), all maximal ideals are of the form $\mathfrak{m}(x)$ for some $x \in |\alpha|$. It therefore suffices to show that any such ideal is finitely generated. We regard $\{x\}$ as a reduced complex space and hence a germ by [Example 4.2.1](#). Then

there is a closed immersion $i: \{x\} \rightarrow \alpha$, and we have a short exact sequence of coherent \mathcal{O}_α -modules:

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_\alpha \rightarrow i_*\mathcal{O}_{\{x\}} \rightarrow 0,$$

where \mathcal{I} is defined by this short exact sequence. By Cartan's Theorem B, we find a short exact sequence

$$0 \rightarrow \Gamma(|\alpha|, \mathcal{I}) \rightarrow \Gamma(\alpha) \xrightarrow{\chi(x)} \mathbb{C} \rightarrow 0.$$

Therefore, $\mathfrak{m}(x) = \Gamma(|\alpha|, \mathcal{I})$ is finitely generated by [Proposition 4.4.2](#). \square

Next we wish to understand the prime ideals in $\Gamma(\alpha)$.

Lemma 4.4.1 *Let α be a Stein compactum, let $\mathfrak{p} \subseteq \Gamma(\alpha)$ be a prime ideal, and let $x \in |\alpha|$ be such that $\mathfrak{p} \subseteq \mathfrak{m}(x)$. Let $I \subseteq \mathfrak{p}$ be a finitely generated ideal. If $f \in \Gamma(\alpha)$ satisfies*

$$f_x \in \left(\sqrt{IO_\alpha} \right)_x,$$

then $f \in \mathfrak{p}$.

Proof By assumption there exists $N > 0$ such that $f_x^N \in (IO_\alpha)_x$. Consider the coherent colon ideal sheaf

$$\mathcal{C} := (IO_\alpha : f^N).$$

Its stalk at x is the whole local ring $\mathcal{O}_{\alpha,x}$. By [Theorem 4.3.1\(1\)](#), there exists $h \in \Gamma(|\alpha|, \mathcal{C})$ such that h_x is invertible. Hence $h(x) \neq 0$, so $h \notin \mathfrak{p}$ because $\mathfrak{p} \subseteq \mathfrak{m}(x)$. On the other hand,

$$hf^N \in \Gamma(|\alpha|, IO_\alpha) = I \subseteq \mathfrak{p}$$

by [Proposition 4.4.2](#). Since \mathfrak{p} is prime, it follows that $f^N \in \mathfrak{p}$, hence $f \in \mathfrak{p}$. \square

Recall that essential irreducibility is defined in [Definition 4.2.8](#).

Proposition 4.4.3 *Let α be a Stein compactum. Take a reduced analytic subspace β of α and a point $x \in |\beta|$ at which β is essentially irreducible. Let*

$$\mathfrak{p}_{\beta,x} := \{f \in \Gamma(\alpha) : (f|_\beta)_x = 0\}. \quad (4.15)$$

Then $\mathfrak{p}_{\beta,x}$ is a prime ideal in $\Gamma(\alpha)$.

Conversely, every prime ideal \mathfrak{p} in $\Gamma(\alpha)$ arises in this way. In fact, x can be taken to be any point such that $\mathfrak{p} \subseteq \mathfrak{m}(x)$.

Proof We first show that $\mathfrak{p}_{\beta,x}$ is a prime ideal. It is clearly an ideal. Let $f, g \in \Gamma(\alpha)$, and assume that

$$fg \in \mathfrak{p}_{\beta,x}, \quad f \notin \mathfrak{p}_{\beta,x}.$$

We prove that $g \in \mathfrak{p}_{\beta,x}$.

Choose a representative (X, K) of α such that β is represented by a reduced analytic subspace $Y \subseteq X$ with $|\beta| = Y \cap K$, and such that the condition in [Definition 4.2.8](#)

is satisfied. After shrinking X around K , we may assume that f and g are represented by holomorphic functions on X . By the essential irreducibility assumption, after this shrinking, there is a unique irreducible component of Y containing x ; denote it by Y_0 .

Since $fg \in \mathfrak{p}_{\beta,x}$, the germ of $(fg)|_Y$ at x is zero. Thus there is an open neighborhood V of x in Y such that $(fg)|_V = 0$. In particular,

$$(fg)|_{V \cap Y_0} = 0.$$

We claim that $f|_{Y_0}$ is not identically zero near x . Indeed, if $f|_{Y_0}$ were identically zero near x , then, since Y_0 is the only irreducible component of Y containing x , the germ $(f|_Y)_x$ would be zero. This would mean $f \in \mathfrak{p}_{\beta,x}$, contradicting the assumption.

Therefore $f|_{Y_0}$ is a nonzero holomorphic function on the reduced irreducible complex space Y_0 near x . By the *Identitätssatz*, its zero locus is a proper analytic subset of Y_0 ; in particular, its complement is dense in a neighborhood of x in Y_0 . On this dense complement we have $g|_{Y_0} = 0$, because $(fg)|_{V \cap Y_0} = 0$.

Since Y_0 is reduced, a holomorphic function on Y_0 which vanishes on a dense open subset vanishes identically. Hence $g|_{Y_0}$ vanishes near x . Again using that Y_0 is the only irreducible component of Y containing x , we conclude that $(g|_Y)_x = 0$. Thus $g \in \mathfrak{p}_{\beta,x}$. Hence $\mathfrak{p}_{\beta,x}$ is prime.

Conversely, let \mathfrak{p} be a prime ideal in $\Gamma(\alpha)$. Choose $x \in |\alpha|$ such that

$$\mathfrak{p} \subseteq \mathfrak{m}(x),$$

which is possible by [Proposition 4.4.1](#). Let $\mathfrak{p}_x \subseteq \mathcal{O}_{\alpha,x}$ be the ideal generated by the germs f_x for $f \in \mathfrak{p}$. Since $\mathcal{O}_{\alpha,x}$ is Noetherian, we may choose $f_1, \dots, f_r \in \mathfrak{p}$ such that their germs generate \mathfrak{p}_x . Let

$$I := (f_1, \dots, f_r) \subseteq \mathfrak{p}, \quad \mathcal{J} := \sqrt{I\mathcal{O}_\alpha}.$$

Let β be the reduced analytic subspace of α defined by \mathcal{J} .

We claim first that

$$\mathfrak{p} = \mathfrak{p}_{\beta,x}.$$

If $f \in \mathfrak{p}$, then $f_x \in \mathfrak{p}_x = I_x \subseteq \mathcal{J}_x$, so $(f|_\beta)_x = 0$. Hence $\mathfrak{p} \subseteq \mathfrak{p}_{\beta,x}$. Conversely, if $f \in \mathfrak{p}_{\beta,x}$, then $f_x \in \mathcal{J}_x$. By [Lemma 4.4.1](#), we get $f \in \mathfrak{p}$. Thus the claim follows.

It remains to show that β is essentially irreducible at x . Suppose not. Choose a Stein representative (X, K) of α such that β is represented by a reduced analytic subspace $Y \subseteq X$. After shrinking X around K , we may assume that Y has at least two irreducible components through x and only finitely many irreducible components meeting K .

Write

$$Y = Y_1 \cup Y_2,$$

where Y_1 is one irreducible component through x , and Y_2 is the union of the remaining irreducible components. Let \mathcal{I}_1 and \mathcal{I}_2 be the defining ideal sheaves of Y_1 and Y_2 in

X. Since neither branch contains the other near x , the coherent sheaves

$$\mathcal{I}_1/(\mathcal{I}_1 \cap \mathcal{I}_2), \quad \mathcal{I}_2/(\mathcal{I}_1 \cap \mathcal{I}_2)$$

have nonzero stalks at x . By Cartan's Theorem A on the Stein space X , we can choose

$$a \in \Gamma(X, \mathcal{I}_1), \quad b \in \Gamma(X, \mathcal{I}_2)$$

whose images in $\mathcal{O}_{Y,x}$ are both nonzero. Since

$$ab \in \mathcal{I}_1 \mathcal{I}_2 \subseteq \mathcal{I}_1 \cap \mathcal{I}_2 = \mathcal{I}_Y,$$

we have

$$(ab|_{\beta})_x = 0.$$

Thus $ab \in \mathfrak{p}_{\beta,x} = \mathfrak{p}$. Since \mathfrak{p} is prime, either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. But $\mathfrak{p} = \mathfrak{p}_{\beta,x}$, while the images of both a and b in $\mathcal{O}_{\beta,x}$ are nonzero. This is a contradiction.

Hence β is essentially irreducible at x , and the proof is complete. \square

Corollary 4.4.3 *Let α be a Stein compactum. Then the nilradical of $\Gamma(\alpha)$ is precisely the kernel of*

$$\Gamma(\alpha) \rightarrow \Gamma(\alpha^{\text{red}}).$$

In particular, the nilradical of $\Gamma(\alpha)$ is finitely generated.

Proof Let N denote the kernel of $\Gamma(\alpha) \rightarrow \Gamma(\alpha^{\text{red}})$. If $f \in N$, then $f|_{\alpha^{\text{red}}} = 0$, so for every reduced analytic subspace β of α and every $x \in |\beta|$, we have

$$(f|_{\beta})_x = 0.$$

By [Proposition 4.4.3](#), f lies in every prime ideal of $\Gamma(\alpha)$. Hence f lies in the nilradical.

Conversely, every nilpotent element maps to zero in $\Gamma(\alpha^{\text{red}})$, because α^{red} is reduced. Thus N is exactly the nilradical.

The final assertion follows from [Proposition 4.4.2](#), since α^{red} is an analytic subspace of α . \square

Proposition 4.4.4 *Let α be a Stein compactum. Consider $x \in |\alpha|$. Then for any $n \geq 1$, the natural map*

$$\Gamma(\alpha)/\mathfrak{m}(x)^n \rightarrow \mathcal{O}_{\alpha,x}/\mathfrak{m}_{\alpha,x}^n \tag{4.16}$$

is an isomorphism of rings. In particular, we have a natural isomorphism

$$\widehat{\Gamma(\alpha)_{\mathfrak{m}(x)}} \xrightarrow{\sim} \widehat{\mathcal{O}_{\alpha,x}}.$$

Proof The natural map (4.16) is induced by the homomorphism $\Gamma(\alpha) \rightarrow \mathcal{O}_{\alpha,x}$ sending f to f_x . It is surjective by [Proposition 3.3.4](#).

It remains to show that it is injective. Let $f \in \Gamma(\alpha)$ with $f_x \in \mathfrak{m}_{\alpha,x}^n$. Then $(\mathfrak{m}(x)^n \mathcal{O}_{\alpha} : (f))$ is coherent, and its germ at x is precisely $\mathcal{O}_{\alpha,x}$, in view of [Lemma 3.3.2](#).

By Cartan's Theorem A, we can find $g \in \Gamma(\alpha)$ with g_x invertible and $gf \in \Gamma(|\alpha|, \mathfrak{m}(x)^n \mathcal{O}_\alpha) = \mathfrak{m}(x)^n$, where the equality follows from the fact that $\mathfrak{m}(x)^n$ is finitely generated by [Corollary 4.4.2](#) and from [Proposition 4.4.2](#).

Suppose that $f \notin \mathfrak{m}(x)^n$. Then $g^m \in \mathfrak{m}(x)^n$ for some $m > 0$ since $\mathfrak{m}(x)^n$ is primary, cf. [Lemma 1.2.1](#). But this contradicts the fact that g_x is invertible. \square

Proposition 4.4.5 *Let α be a Stein compactum. Then the ring $\Gamma(\alpha)$ is stalkwise Noetherian.*

Recall that stalkwise noetherian rings are defined in [Definition 1.2.1](#).

Proof By [Corollary 4.4.3](#), the nilradical N of $\Gamma(\alpha)$ is finitely generated. Since every element of N is nilpotent, the ideal N is nilpotent. Hence, by [Proposition 1.2.2](#), it suffices to prove the assertion for α^{red} . We may therefore assume that α is reduced.

By [Proposition 4.4.1](#), the maximal ideals of $\Gamma(\alpha)$ are the ideals $\mathfrak{m}(x)$ for $x \in |\alpha|$. Fix $x \in |\alpha|$. It suffices to show that $\Gamma(\alpha)_{\mathfrak{m}(x)}$ is Noetherian.

Put

$$A := \Gamma(\alpha), \quad \mathfrak{m} := \mathfrak{m}(x).$$

Consider the natural homomorphism

$$\gamma: A \longrightarrow \mathcal{O}_{\alpha,x}, \quad f \longmapsto f_x.$$

Since every element of $A \setminus \mathfrak{m}$ maps to a unit in $\mathcal{O}_{\alpha,x}$, this homomorphism induces a homomorphism

$$\gamma': A_{\mathfrak{m}} \longrightarrow \mathcal{O}_{\alpha,x}.$$

We first show that γ' is injective. Let $f/s \in A_{\mathfrak{m}}$ be mapped to zero by γ' , where $s \notin \mathfrak{m}$. Since s_x is a unit in $\mathcal{O}_{\alpha,x}$, we have $f_x = 0$. Choose a reduced Stein representative (X, K) of α with $x \in K$. After shrinking X around K , we may assume that X has only finitely many irreducible components and that f and s are represented by holomorphic functions on X .

Since $f_x = 0$, the function f vanishes on a neighborhood of x in X . Hence, by the *Identitätssatz*, f vanishes identically on every irreducible component of X containing x . Let E be the union of all irreducible components of X not containing x . This is a closed analytic subspace of X . If $E = \emptyset$, put $h = 1$. Otherwise, by Cartan's Theorem A applied to the ideal sheaf of E , we can find $h \in \Gamma(X, \mathcal{O}_X)$ such that

$$h|_E = 0, \quad h(x) = 1.$$

Then $hf = 0$ on every irreducible component of X . Since X is reduced, this gives $hf = 0$ in A . Moreover $h \notin \mathfrak{m}$. Therefore $\frac{f}{s} = 0$ in $A_{\mathfrak{m}}$. This proves that γ' is injective.

Let

$$R := \text{Im}(\gamma') \subseteq \mathcal{O}_{\alpha,x}.$$

Since γ' is injective, it remains to prove that R is Noetherian.

By Cohen's theorem [Theorem 1.2.1](#), it suffices to show that every prime ideal of R is finitely generated. Let $P \subseteq R$ be a prime ideal. We will prove that P is finitely generated.

Let

$$\mathfrak{p} := \gamma^{-1}(P) \subseteq A.$$

Then \mathfrak{p} is a prime ideal of A . Moreover, $\mathfrak{p} \subseteq \mathfrak{m}$. Indeed, if $g \in A$ and $g(x) \neq 0$, then $g \notin \mathfrak{m}$, so $\gamma(g)$ is a unit in R , because its inverse is the image of $1/g \in A_{\mathfrak{m}}$. Hence $\gamma(g) \notin P$, and therefore $g \notin \mathfrak{p}$.

By [Proposition 4.4.3](#), applied to the point x with $\mathfrak{p} \subseteq \mathfrak{m}(x)$, there exists a reduced analytic subspace β of α , essentially irreducible at x , such that

$$\mathfrak{p} = \{f \in A : (f|_{\beta})_x = 0\}.$$

Let

$$J := \ker(A \rightarrow \Gamma(\beta)).$$

By [Proposition 4.4.2](#), the ideal J is finitely generated. Choose generators $f_1, \dots, f_r \in J$.

We claim that

$$\gamma(f_1), \dots, \gamma(f_r)$$

generate P as an ideal of R . First observe that P is generated by $\gamma(\mathfrak{p})$. Indeed, every element of R is of the form $\gamma'(a/s)$ with $a \in A$ and $s \notin \mathfrak{m}$. If $\gamma'(a/s) \in P$, then multiplying by the unit $\gamma'(s)$ gives

$$\gamma(a) \in P,$$

hence $a \in \mathfrak{p}$. Therefore it suffices to show that for every $g \in \mathfrak{p}$, the element $\gamma(g)$ lies in the ideal of R generated by the $\gamma(f_i)$.

Fix $g \in \mathfrak{p}$. Choose a reduced Stein representative (X, K) of α such that β is represented by a reduced analytic subspace $Y \subseteq X$, and such that g, f_1, \dots, f_r are represented by holomorphic functions on X . After shrinking X around K , we may assume that Y has only finitely many irreducible components meeting K , and that $Y = Y_0 \cup Y_1$, where Y_0 is the unique irreducible component of Y containing x , and Y_1 is the union of the remaining irreducible components.

Since $g \in \mathfrak{p}$, we have $(g|_{\beta})_x = 0$. Thus g vanishes on a neighborhood of x in Y . In particular, g vanishes on a nonempty open subset of Y_0 . Since Y_0 is reduced and irreducible, the *Identitätssatz* implies that $g|_{Y_0} = 0$ as a germ along $K \cap Y_0$.

We now choose a holomorphic function separating Y_0 from the other components. If $Y_1 = \emptyset$, put $h = 1$. Otherwise, by Cartan's Theorem A applied to the ideal sheaf of Y_1 in X , we can find $h \in \Gamma(X, \mathcal{O}_X)$ such that

$$h|_{Y_1} = 0, \quad h(x) = 1.$$

We denote the image of h in A by the same symbol.

Since $g|_{Y_0} = 0$ and $h|_{Y_1} = 0$, the product gh vanishes on Y as a germ along $K \cap Y$. Hence $gh \in J$. Therefore there exist $a_1, \dots, a_r \in A$ such that

$$gh = \sum_{i=1}^r a_i f_i.$$

Applying γ , we get

$$\gamma(g)\gamma(h) = \sum_{i=1}^r \gamma(a_i)\gamma(f_i).$$

Since $h(x) = 1$, the element $\gamma(h)$ is a unit in R . Hence $\gamma(g)$ lies in the ideal of R generated by $\gamma(f_1), \dots, \gamma(f_r)$.

This proves that P is finitely generated. By Cohen's theorem, R is Noetherian. Since $A_{\mathfrak{m}} \cong R$, the local ring $A_{\mathfrak{m}}$ is Noetherian. This completes the proof. \square

4.4.2 Excellence

Definition 4.4.1 A Stein compactum α is *excellent* if $\Gamma(\alpha)$ is noetherian.

Lemma 4.4.2 A Stein compactum α is excellent if and only if α^{red} is excellent.

Proof Let N be the kernel of the surjection $\Gamma(\alpha) \rightarrow \Gamma(\alpha^{\text{red}})$. By **Corollary 4.4.3**, the ideal N is finitely generated and consists of nilpotent elements. Hence N is nilpotent. Therefore $\Gamma(\alpha)$ is noetherian if and only if $\Gamma(\alpha^{\text{red}})$ is noetherian. \square

Lemma 4.4.3 Let X be a reduced complex space, and let $K \subseteq X$ be a Stein compact subset. Assume that for every analytic subspace Y of some open neighborhood of K in X , the set $Y \cap K$ has only finitely many connected components. Then for every $x \in K$, the ideal

$$\mathfrak{p}_x := \{f \in \Gamma(K, \mathcal{O}_X) : f_x = 0\}$$

is finitely generated.

Proof Since all assertions are assertions about germs along K , we may replace X by an open neighborhood of K whenever needed. In particular, since K is a Stein compact subset, we may first replace X by a Stein open neighborhood of K . Thus we may assume that X is Hausdorff. Let $\pi: \tilde{X} \rightarrow X$ be the normalization of X . The map π is finite. By **Lemma 2.3.1**, the compact set $\pi^{-1}(K)$ has only finitely many connected components.

Step 1. We show that, after replacing X by an open neighborhood of K , the natural map

$$\pi_0(\pi^{-1}(K)) \rightarrow \pi_0(\tilde{X}) \tag{4.17}$$

is injective. Write

$$\pi^{-1}(K) = C_1 \sqcup \dots \sqcup C_r$$

for the decomposition into connected components. The sets C_j are compact and pairwise disjoint. Since \tilde{X} is Hausdorff, we can choose pairwise disjoint open neighborhoods $U_j \subseteq \tilde{X}$ of C_j . Set

$$W := \bigcup_{j=1}^r U_j.$$

Then W is an open neighborhood of $\pi^{-1}(K)$. Since π is finite, it is closed. Hence

$$\Omega := X \setminus \pi(\tilde{X} \setminus W)$$

is an open neighborhood of K in X . Moreover,

$$\pi^{-1}(\Omega) \subseteq W.$$

Replacing X by Ω and \tilde{X} by $\pi^{-1}(\Omega)$, which is the normalization of Ω , we may assume that every connected component of \tilde{X} is contained in one of the U_j . Therefore every connected component of \tilde{X} meets at most one connected component of $\pi^{-1}(K)$. This proves that (4.17) is injective.

Step 2. We now assume that (4.17) holds. Since π is finite, the fiber $\pi^{-1}(x)$ is finite. Write

$$\pi^{-1}(x) = \{y_1, \dots, y_n\}.$$

Let H_i be the connected component of \tilde{X} containing y_i . Since \tilde{X} is normal, its connected components are irreducible. Put

$$H := \bigcup_{i=1}^n H_i.$$

This is a union of connected components of \tilde{X} , hence a closed analytic subspace of \tilde{X} . Since π is finite, Remmert's proper mapping theorem shows that $V := \pi(H)$ is a closed analytic subspace of X . We endow V with its reduced induced complex space structure. The restricted map $\pi_H : H \rightarrow V$ is finite and surjective. We claim that

$$\mathfrak{p}_x = \{f \in \Gamma(K, \mathcal{O}_X) : f|_V = 0 \text{ as a germ along } K \cap V\}. \quad (4.18)$$

First let $f \in \mathfrak{p}_x$. After shrinking X around K , we may assume that f is represented by a holomorphic function on X . The condition $f_x = 0$ means that f vanishes on an open neighborhood of x in X . Therefore $\pi^* f$ vanishes on an open neighborhood of each y_i . Since H_i is irreducible, the *Identitätssatz* implies that $(\pi^* f)|_{H_i} = 0$ for every i . Hence $(\pi^* f)|_H = 0$. Equivalently, $\pi_H^*(f|_V) = 0$. Since V is reduced and $\pi_H : H \rightarrow V$ is finite and surjective, the natural morphism $\mathcal{O}_V \rightarrow (\pi_H)_* \mathcal{O}_H$ is injective. Thus $f|_V = 0$ as a germ along $K \cap V$. This proves the inclusion \subseteq in (4.18). Conversely, suppose that $f|_V = 0$ as a germ along $K \cap V$. After shrinking X around K , we may assume that f is represented by a holomorphic function on X and that $f|_V = 0$. Then $(\pi^* f)|_H = 0$. In particular, for every $i = 1, \dots, n$, $(\pi^* f)_{y_i} = 0$. Since X is reduced and $\pi : \tilde{X} \rightarrow X$ is the normalization, the natural morphism

$\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\tilde{X}}$ is injective. Taking stalks at x , we get an injection

$$\mathcal{O}_{X,x} \hookrightarrow \prod_{y \in \pi^{-1}(x)} \mathcal{O}_{\tilde{X},y}.$$

Since the image of f_x in each factor is zero, this injection gives $f_x = 0$. Thus $f \in \mathfrak{p}_x$. This proves the reverse inclusion and hence (4.18). Finally, let α be the Stein compactum represented by (X, K) , and let β be the analytic subspace of α represented by $(V, K \cap V)$. Then the right-hand side of (4.18) is exactly the kernel of the restriction homomorphism

$$\Gamma(\alpha) = \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(\beta) = \Gamma(K \cap V, \mathcal{O}_V).$$

By Proposition 4.4.2, this kernel is finitely generated. Therefore \mathfrak{p}_x is finitely generated. \square

Theorem 4.4.1 (Siu) *Let α be a Stein compactum. Then the following are equivalent:*

- (1) α is excellent;
- (2) for any analytic subspace β of α , $|\beta|$ has only finitely many connected components.

Proof By Lemma 4.4.2, we may assume that α is reduced.

(1) \implies (2). Suppose that $|\beta|$ has infinitely many connected components. Since $|\beta|$ is compact Hausdorff, at least one connected component L is not open. Consider the ideal

$$I = \{f \in \Gamma(\beta) : f \text{ vanishes on an open neighborhood of } L\}.$$

We claim that I is not finitely generated. Suppose otherwise, and choose generators f_1, \dots, f_r of I . Then there is an open neighborhood U of L in $|\beta|$ on which all f_i vanish. Hence every element of I vanishes on U .

Since L is not open, we can choose $x \in U \setminus L$. In a compact Hausdorff space, connected components coincide with quasi-components; hence there is a clopen subset $W \subseteq |\beta|$ such that $L \subseteq W$ and $x \notin W$. The characteristic function $\mathbb{1}_{|\beta| \setminus W}$ is a section of \mathcal{O}_β : it is obtained by gluing the constant holomorphic functions 0 and 1 on the clopen decomposition $|\beta| = W \sqcup (|\beta| \setminus W)$. This section belongs to I , but it takes the value 1 at x , contradicting the previous paragraph. Thus I is not finitely generated.

Therefore $\Gamma(\beta)$ is not Noetherian. Since $\Gamma(\alpha) \rightarrow \Gamma(\beta)$ is surjective, $\Gamma(\alpha)$ is not Noetherian.

(2) \implies (1). Suppose that (2) holds. We prove that $\Gamma(\alpha)$ is Noetherian. By Cohen's theorem Theorem 1.2.1, it suffices to show that every prime ideal of $\Gamma(\alpha)$ is finitely generated.

By Proposition 4.4.3, it suffices to consider a prime ideal of the form $\mathfrak{p}_{\beta,x}$, where β is a reduced analytic subspace of α and $x \in |\beta|$ is a point at which β is essentially irreducible. The kernel of the surjection $\Gamma(\alpha) \rightarrow \Gamma(\beta)$ is finitely generated by Proposition 4.4.2. Thus it is enough to prove that the ideal

$$\{g \in \Gamma(\beta) : g_x = 0\}$$

is finitely generated. This follows from [Lemma 4.4.3](#), applied to β . \square

Corollary 4.4.4 (Frisch) *Any semianalytic Stein compactum is excellent.*

Proof Let (X, K) be a representative of the Stein compactum such that K is semianalytic. Let Y be an analytic subset of X . Then by [Theorem 4.4.1](#) it suffices to show that $Y \cap K$ has finitely many connected components. But this is clear since $Y \cap K$ is semianalytic and compact. \square

Lemma 4.4.4 *Let X be a Stein space, and K be a compact subset. Then K has a semianalytic $\mathcal{O}_X(X)$ -convex compact neighborhood in X .*

Proof We may assume that X is reduced. Then by the solution to Levi's problem, there is a real-analytic strongly psh exhaustion function ρ of X . The desired neighborhood is just given by

$$\{x \in X : \rho(x) \leq 1 + \sup_K \rho\}.$$

Proposition 4.4.6 *Let X be a complex space and K be a Stein compact subset. Then K has a fundamental system of semianalytic Stein compact neighborhoods.*

In particular, K has a fundamental system of excellent Stein compact neighborhoods.

Proof Let U be an open neighborhood of K in X . Since K is a Stein compact subset, there is a Stein open neighborhood V of K with

$$K \subseteq V \subseteq U.$$

Applying [Lemma 4.4.4](#) to the Stein space V , we obtain a semianalytic $\mathcal{O}(V)$ -convex compact neighborhood L of K in V . By [Lemma 3.1.1](#), L is a Stein compact subset of V , hence also of X . By [Corollary 4.4.4](#), it is excellent. This gives a fundamental system of semianalytic excellent Stein compact neighborhoods of K . \square

Theorem 4.4.2 *Assume that X is a complex space and $K \subseteq X$ is an excellent Stein compact subset. Then $\Gamma(K, \mathcal{O}_X)$ is excellent.*

Proof Write $A = \Gamma(K, \mathcal{O}_X)$. By definition of excellence for Stein compacta, A is noetherian.

By [Theorem 1.2.3](#), it suffices to construct a universally finite \mathbb{C} -derivation of A .

Let Ω_X denote the sheaf of analytic Kähler differentials on X , and set $M := \Gamma(K, \Omega_X)$. By [Corollary 4.3.1](#), M is a finite A -module. The analytic differential

$$d: \mathcal{O}_X \rightarrow \Omega_X$$

induces a finite \mathbb{C} -derivation

$$d: A \rightarrow M.$$

We claim that d is universally finite. By [Theorem 1.2.2](#), it suffices to check this after localizing at maximal ideals. By [Proposition 4.4.1](#), every maximal ideal of A is of the form $\mathfrak{m}(x)$ for a unique $x \in K$. Fix such an x , and put $\mathfrak{m} = \mathfrak{m}(x)$. We need to prove that

$$d_{\mathfrak{m}}: A_{\mathfrak{m}} \rightarrow M_{\mathfrak{m}}$$

is universally finite.

The natural local homomorphism $A_{\mathfrak{m}} \rightarrow O_{X,x}$ is faithfully flat by [Corollary 4.3.2](#). Moreover, by [Proposition 4.4.4](#), the induced homomorphism on completions is an isomorphism

$$\widehat{A_{\mathfrak{m}}} \xrightarrow{\sim} \widehat{O_{X,x}}.$$

By [Corollary 4.3.1](#), we have a natural isomorphism

$$M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} O_{X,x} \xrightarrow{\sim} \Omega_{X,x}.$$

Under this identification, the base change of $d_{\mathfrak{m}}$ to $O_{X,x}$ is the analytic differential

$$d_x: O_{X,x} \rightarrow \Omega_{X,x}.$$

Since $M_{\mathfrak{m}}$ and $\Omega_{X,x}$ are finite modules, completion commutes with the preceding base change. Hence

$$\widehat{M_{\mathfrak{m}}} \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} \widehat{A_{\mathfrak{m}}} \cong (M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} O_{X,x}) \otimes_{O_{X,x}} \widehat{O_{X,x}} \cong \widehat{\Omega_{X,x}}.$$

Therefore, under the identifications

$$\widehat{A_{\mathfrak{m}}} \simeq \widehat{O_{X,x}}, \quad \widehat{M_{\mathfrak{m}}} \simeq \widehat{\Omega_{X,x}},$$

the completed derivation

$$\widehat{d_{\mathfrak{m}}}: \widehat{A_{\mathfrak{m}}} \rightarrow \widehat{M_{\mathfrak{m}}}$$

is identified with the completion of

$$d_x: O_{X,x} \rightarrow \Omega_{X,x}.$$

Since $O_{X,x}$ is a local analytic \mathbb{C} -algebra, [Theorem 2.1.1](#) shows that it admits a universally finite derivation, and this derivation is identified with the analytic differential

$$d_x: O_{X,x} \rightarrow \Omega_{X,x}.$$

By [Proposition 1.2.7](#), its completion is universally finite. Hence $\widehat{d_{\mathfrak{m}}}$ is universally finite. Applying [Proposition 1.2.7](#) once more, now to the noetherian local \mathbb{C} -algebra $A_{\mathfrak{m}}$, we conclude that

$$d_{\mathfrak{m}}: A_{\mathfrak{m}} \rightarrow M_{\mathfrak{m}}$$

is universally finite.

Thus d is universally finite by [Theorem 1.2.2](#). Therefore [Theorem 1.2.3](#) shows that A is excellent. \square

Proposition 4.4.7 *Let α be a Stein compactum, and let \mathfrak{p} be a prime ideal in $\Gamma(\alpha)$ defined by a reduced analytic subspace β and a point $x \in |\beta|$ at which β is essentially irreducible. Then the following are equivalent:*

- (1) *The local ring $\Gamma(\alpha)_{\mathfrak{p}}$ is regular;*
- (2) *the germ β_x is not contained in α_x^{Sing} .*

Proof Choose a Stein representative (X, K) of α such that β is represented by a reduced analytic subspace $Y \subseteq X$.

(1) \implies (2). Assume that $\Gamma(\alpha)_{\mathfrak{p}}$ is regular. By [Proposition 4.4.6](#), we can choose an excellent Stein compact neighborhood L of K in X . Put

$$B := \Gamma(L, \mathcal{O}_X), \quad A := \Gamma(K, \mathcal{O}_X).$$

Let $\mathfrak{q} \subseteq B$ be the prime ideal defined by the same analytic subspace Y and the same point x . Then \mathfrak{q} is the inverse image of \mathfrak{p} under $B \rightarrow A$. The local homomorphism

$$B_{\mathfrak{q}} \rightarrow A_{\mathfrak{p}}$$

is faithfully flat by [Corollary 4.3.2](#). Since $A_{\mathfrak{p}}$ is regular, $B_{\mathfrak{q}}$ is regular by faithfully flat descent of regularity.

By [Theorem 4.4.2](#), the ring B is excellent. Hence the singular locus of $\text{Spec } B$ is closed. Let $I \subseteq B$ be an ideal defining this singular locus. Since $B_{\mathfrak{q}}$ is regular, we have $I \not\subseteq \mathfrak{q}$; choose

$$f \in I \setminus \mathfrak{q}.$$

The condition $f \notin \mathfrak{q}$ means that $f|_Y$ is not zero as a germ at x . Thus every neighborhood of x in Y contains a point $t \in Y \cap L$ such that $f(t) \neq 0$. For such t , the maximal ideal $\mathfrak{m}(t) \subseteq B$ does not contain I , so $B_{\mathfrak{m}(t)}$ is regular. By [Proposition 4.4.4](#), the completion of $B_{\mathfrak{m}(t)}$ is isomorphic to $\widehat{\mathcal{O}}_{X,t}$; hence $\mathcal{O}_{X,t}$ is regular. Therefore the germ Y_x cannot be contained in X_x^{Sing} . This proves (2).

(2) \implies (1). Assume that β_x is not contained in α_x^{Sing} . Let Y' be the irreducible component of the germ Y_x corresponding to β_x , and let

$$\mathfrak{p}' \subseteq \mathcal{O}_{X,x}$$

be the prime ideal of functions vanishing on Y' . By [[Hou61b](#), Théorème 3], the local ring

$$(\mathcal{O}_{X,x})_{\mathfrak{p}'}$$

is regular.

The homomorphism

$$\Gamma(\alpha)_{\mathfrak{p}} \longrightarrow (\mathcal{O}_{X,x})_{\mathfrak{p}'}$$

is faithfully flat: it is obtained by localizing the flat local map

$$\Gamma(\alpha)_{\mathfrak{m}(x)} \rightarrow \mathcal{O}_{X,x}$$

from [Corollary 4.3.2](#). Since $\Gamma(\alpha)_{\mathfrak{p}}$ is Noetherian by [Proposition 4.4.5](#), regularity descends from the faithfully flat extension above. Hence $\Gamma(\alpha)_{\mathfrak{p}}$ is regular. \square

Corollary 4.4.5 *Let α be a smooth Stein compactum. Then $\Gamma(\alpha)$ is stalkwise regular.*

See [Definition 1.2.2](#) for the definition of stalkwise regular rings.

Proof Let $\mathfrak{p} \subseteq \Gamma(\alpha)$ be a prime ideal. By [Proposition 4.4.3](#), \mathfrak{p} is defined by a reduced analytic subspace β of α and a point $x \in |\beta|$ at which β is essentially irreducible. Since α is smooth, we have $\alpha^{\text{Sing}} = \emptyset$. Thus

$$\beta_x \not\subseteq \alpha_x^{\text{Sing}}.$$

By [Proposition 4.4.7](#), $\Gamma(\alpha)_{\mathfrak{p}}$ is regular. This proves the assertion. \square

Corollary 4.4.6 *Let α be a Stein compactum. Then $\text{Spec } \Gamma(\alpha)$ admits a closed immersion into a regular affine scheme.*

Proof Choose a Stein representative (X, K) of α . After shrinking X around K , we may assume that X has finite dimension and finite embedded dimension. By [Theorem 3.1.2](#), there is a closed immersion

$$X \hookrightarrow \mathbb{C}^n$$

for some n . By [Proposition 4.3.3](#), the induced homomorphism

$$\Gamma(K, \mathcal{O}_{\mathbb{C}^n}) \longrightarrow \Gamma(K, \mathcal{O}_X) = \Gamma(\alpha)$$

is surjective. Therefore

$$\text{Spec } \Gamma(\alpha) \hookrightarrow \text{Spec } \Gamma(K, \mathcal{O}_{\mathbb{C}^n})$$

is a closed immersion. The compactum represented by (\mathbb{C}^n, K) is smooth and Stein, so $\Gamma(K, \mathcal{O}_{\mathbb{C}^n})$ is regular by [Corollary 4.4.5](#). Hence the target is regular. \square

Corollary 4.4.7 *Let X be a complex space, and let $K \subseteq L \subseteq X$ be two excellent Stein compact subsets. Then the ring homomorphism*

$$\Gamma(L, \mathcal{O}_X) \rightarrow \Gamma(K, \mathcal{O}_X)$$

is regular.

Proof We may freely shrink X around L . Put

$$A_L := \Gamma(L, \mathcal{O}_X), \quad A_K := \Gamma(K, \mathcal{O}_X).$$

The map $A_L \rightarrow A_K$ is flat by [Corollary 4.3.2](#). It remains to show that its fibers are geometrically regular.

Let $\mathfrak{q} \in \text{Spec } A_K$, and let

$$\mathfrak{p} = \mathfrak{q} \cap A_L.$$

Since the residue field $\kappa(\mathfrak{p})$ has characteristic 0, it is enough to show that the local fiber

$$(A_K)_{\mathfrak{q}}/\mathfrak{p}(A_K)_{\mathfrak{q}}$$

is regular.

Since A_L is Noetherian, the prime ideal \mathfrak{p} is finitely generated. By [Proposition 4.4.2](#), after shrinking X around L , the ideal \mathfrak{p} is represented by a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. Let

$$Z := \mathbb{V}(\mathcal{I})$$

be the corresponding analytic subspace. Replacing X by Z and replacing K, L by $K \cap Z, L \cap Z$, we reduce to the case

$$\mathfrak{p} = 0.$$

Thus A_L is an integral domain, and we need to prove that $(A_K)_{\mathfrak{q}}$ is regular.

By [Proposition 4.4.3](#), the prime ideal \mathfrak{q} is defined by a reduced analytic subspace β of the compactum represented by (X, K) and a point $x \in |\beta|$ at which β is essentially irreducible. By [Proposition 4.4.7](#), it suffices to prove that

$$\beta_x \not\subseteq X_x^{\text{Sing}}. \quad (4.19)$$

Suppose, to the contrary, that $\beta_x \subseteq X_x^{\text{Sing}}$. Since A_L is an excellent domain, the singular locus of $\text{Spec } A_L$ is a proper closed subset. Hence there exists a nonzero element $f \in A_L$ which vanishes on the singular locus of $\text{Spec } A_L$.

For a point $t \in L$, the local ring $(A_L)_{\mathfrak{m}(t)}$ is regular if and only if $\mathcal{O}_{X,t}$ is regular, by [Proposition 4.4.4](#). Therefore, after shrinking X around L , the section f vanishes on the analytic singular locus X^{Sing} near L . Hence we can choose a nonzero section $f \in A_L$ which vanishes on X^{Sing} near L . Since $\beta_x \subseteq X_x^{\text{Sing}}$, the image of f in A_K belongs to \mathfrak{q} . But $\mathfrak{q} \cap A_L = 0$, because we have reduced to $\mathfrak{p} = 0$. This contradicts $f \neq 0$.

Therefore [\(4.19\)](#) follows. We conclude the proof. \square

Chapter 5

Analytifications

Or, nous, devant le beau système de problèmes à F. Hartogs et aux successeurs, voulons léguer des nouveaux problèmes à ceux qui nous suivront.
— Kiyoshi Oka^a, at the end of his 1943 paper

^a Kiyoshi Oka (岡潔, 1901–1978) was a Japanese mathematician whose work transformed the theory of functions of several complex variables. After studying at Kyoto Imperial University and spending several years in Paris, he developed the ideas that led to the solution of the Cousin problems, the Levi problem, and the theory of domains of holomorphy. His coherence theorem and the principles now bearing his name became part of the foundation from which the Cartan–Serre theory of coherent analytic sheaves and modern complex analytic geometry emerged.

5.1 List of properties

5.1.1 Fpqc local properties

Consider the following properties of morphisms of schemes:

- open immersion/closed immersion/quasi-compact immersion;
- quasi-compact;
- universally closed/universally open;
- quasi-separated/separated/proper;
- monomorphism/surjective/universally injective;
- affine/quasi-affine;
- of finite type/locally of finite type/of finite presentation/locally of finite presentation/locally of finite type of relative dimension d ;
- quasi-finite/locally quasi-finite/finite/integral/finite locally free;
- flat/syntomic/smooth/unramified/G-unramified/étale;
- universally submersive/universal homeomorphism.

See [[Stacks](#), [Tag 02WE](#)].

All these properties of morphisms of schemes are

- (1) stable under base change;
- (2) fpqc local on the base:

5.1.2 Properties of modules

Fix an integer $n \geq 0$. Consider the following properties of a finite module M over a noetherian local ring R :

- (1) $M = 0$;
- (2) M is free;
- (3) M can be generated by n elements;
- (4) $\text{pd}_R(M) \leq n$;
- (5) M is torsion-free;
- (6) M is reflexive;
- (7) M satisfies (S_n) ;
- (8) M is Cohen–Macaulay;
- (9) $\text{codep}_R(M) \leq n$.

For (9), we use the convention

$$\text{codep}_R(M) = \dim_R(M) - \text{dep}_R(M)$$

for $M \neq 0$, and $\text{codep}_R(0) = -\infty$.

Proposition 5.1.1 *Let $f: (R, \mathfrak{m}) \rightarrow (R', \mathfrak{m}')$ be a flat local homomorphism of noetherian local rings, and let M be a finite R -module. Put $M' = M \otimes_R R'$. Then M satisfies any one of the properties (1)–(6) over R if and only if M' satisfies the corresponding property over R' . If, in addition, f is regular, then the same assertion holds for (7)–(9).*

Proof Since a flat local homomorphism of local rings is faithfully flat, R' is faithfully flat over R .

(1) The assertion is immediate from faithful flatness.

(2) The implication from M to M' is clear. Conversely, if M' is free, then M' is projective. Hence M is projective by faithfully flat descent of projectivity, [Stacks, Tag 05A9]. Since R is local and M is finite, M is free.

(3) By Nakayama’s lemma, M can be generated by n elements if and only if $\dim_{\kappa(\mathfrak{m})} M/\mathfrak{m}M \leq n$. Since

$$M'/\mathfrak{m}'M' \cong (M/\mathfrak{m}M) \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m}'),$$

the assertion follows from the same criterion over R' .

(4) The implication from M to M' follows from [Stacks, Tag 066M]. Conversely, assume that $\text{pd}_{R'}(M') \leq n$. For every R -module N and every $i > n$, flat base change gives

$$\text{Tor}_i^R(M, N) \otimes_R R' \cong \text{Tor}_i^{R'}(M', N \otimes_R R') = 0.$$

Faithful flatness gives $\text{Tor}_i^R(M, N) = 0$ for all $i > n$, hence $\text{pd}_R(M) \leq n$.

(5) For a finite module over a noetherian ring, torsion-freeness is equivalent to $\text{Ass}_R(M) \subseteq \text{Ass}_R(R)$, by [Stacks, Tag 00LD]. By flat base change for associated primes, [Stacks, Tag 05C2], we have

$$\text{Ass}_{R'}(M') = \bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} \text{Ass}_{R' \otimes_R \kappa(\mathfrak{p})}(R' \otimes_R \kappa(\mathfrak{p})),$$

as subsets of $\text{Spec } R'$, and the same formula holds with $M = R$. If $\text{Ass}_R(M) \subseteq \text{Ass}_R(R)$, the displayed formula gives $\text{Ass}_{R'}(M') \subseteq \text{Ass}_{R'}(R')$. Conversely, assume that $\text{Ass}_{R'}(M') \subseteq \text{Ass}_{R'}(R')$, and let $\mathfrak{p} \in \text{Ass}_R(M)$. Since R' is faithfully flat over R , the ring $R' \otimes_R \kappa(\mathfrak{p})$ is nonzero. It is noetherian, hence it has an associated prime \mathfrak{q} . The displayed formula gives $\mathfrak{q} \in \text{Ass}_{R'}(M')$, hence $\mathfrak{q} \in \text{Ass}_{R'}(R')$. Applying the displayed formula to R shows that \mathfrak{q} lies over some $\mathfrak{p}_0 \in \text{Ass}_R(R)$. Since \mathfrak{q} also lies over \mathfrak{p} , we get $\mathfrak{p} = \mathfrak{p}_0 \in \text{Ass}_R(R)$. Hence torsion-freeness is equivalent before and after base change.

(6) Since R is noetherian and M is finite, M is finitely presented. For every finite R -module N , choosing a finite presentation of M and using flatness of R' gives

$$\text{Hom}_R(M, N) \otimes_R R' \cong \text{Hom}_{R'}(M', N \otimes_R R').$$

Taking $N = R$ and then $N = M^\vee$ shows that the base change of the canonical map $M \rightarrow M^{\vee\vee}$ is the canonical map $M' \rightarrow (M')^{\vee\vee}$. Faithful flatness detects isomorphisms, so M is reflexive if and only if M' is reflexive.

(7)–(9) Assume now that f is regular. Let $\mathfrak{q} \in \text{Supp}_{R'}(M')$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. Put

$$c(\mathfrak{q}) = \dim(R'_\mathfrak{q}/\mathfrak{p}R'_\mathfrak{q}).$$

By the depth formula for flat local homomorphisms, [Stacks, Tag 0336], the dimension formula, [Stacks, Tag 000N], and the compatibility of supports with flat base change, [Stacks, Tag 056J], we have

$$\text{dep}_{R'_\mathfrak{q}}(M'_\mathfrak{q}) = \text{dep}_{R_\mathfrak{p}}(M_\mathfrak{p}) + c(\mathfrak{q}), \quad \dim_{R'_\mathfrak{q}}(M'_\mathfrak{q}) = \dim_{R_\mathfrak{p}}(M_\mathfrak{p}) + c(\mathfrak{q}),$$

because the local fibre $R'_\mathfrak{q}/\mathfrak{p}R'_\mathfrak{q}$ is regular, hence Cohen–Macaulay.

(7) Suppose first that M satisfies (S_n) . For every $\mathfrak{q} \in \text{Supp}_{R'}(M')$, the preceding formulas give

$$\text{dep}_{R'_\mathfrak{q}}(M'_\mathfrak{q}) \geq \min\{n, \dim_{R'_\mathfrak{q}}(M'_\mathfrak{q})\},$$

so M' satisfies (S_n) . Conversely, assume that M' satisfies (S_n) and let $\mathfrak{p} \in \text{Supp}_R(M)$. Since R' is faithfully flat over R , the fibre over \mathfrak{p} is nonempty. Choose $\mathfrak{q} \in \text{Spec } R'$ lying over \mathfrak{p} and minimal in that fibre. Then $\mathfrak{q} \in \text{Supp}_{R'}(M')$ and $c(\mathfrak{q}) = 0$. The (S_n) inequality for $M'_\mathfrak{q}$ is therefore exactly the (S_n) inequality for $M_\mathfrak{p}$. Hence M satisfies (S_n) .

(9) Taking $\mathfrak{q} = \mathfrak{m}'$ and $\mathfrak{p} = \mathfrak{m}$ in the displayed formulas gives

$$\text{codep}_{R'}(M') = \text{codep}_R(M),$$

with our convention for the zero module. Hence $\text{codepth} \leq n$ is equivalent before and after base change.

(8) If $M = 0$, then $M' = 0$ and there is nothing to prove. Otherwise M is Cohen–Macaulay if and only if $\text{codep}_R(M) = 0$, and similarly for M' . Thus (8) follows from (9). \square

Proposition 5.1.2 *Let X be a locally noetherian scheme and let \mathcal{F} be a coherent \mathcal{O}_X -module. For each of the properties (1)–(6), the subset of points $x \in X$ such that \mathcal{F}_x satisfies the corresponding property over $\mathcal{O}_{X,x}$ is Zariski open. If X is quasi-excellent, then the same assertion holds for (7) and (8). If X locally admits an immersion into a regular scheme, then the same assertion holds for (9).*

Proof (1) The locus is $X \setminus \text{Supp}(\mathcal{F})$, which is open by [Stacks, Tag 056J].

(2) Since X is locally noetherian and \mathcal{F} is coherent, \mathcal{F} is finitely presented by [Stacks, Tag 01XY]; the assertion follows from the openness of the free locus, [Stacks, Tag 0GWM].

(3) The locus where \mathcal{F}_x can be generated by n elements is the locus where $\text{Fitt}_n(\mathcal{F})_x = \mathcal{O}_{X,x}$, hence is open by the construction and local criterion for Fitting ideals, [Stacks, Tag 0C3C].

(4) If $\text{pd}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \leq n$, then by [Stacks, Tag 0CXF] there is a finite free resolution of \mathcal{F}_x of length n . Since coherent sheaves on locally noetherian schemes are finitely presented and exactness of coherent complexes spreads to an open neighborhood, by [Stacks, Tag 01XY], after shrinking around x this resolution spreads to a locally free resolution of \mathcal{F} of length n . Thus the locus is open.

(5) This is Lemma 1.2.2.

(6) Let $\delta : \mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$ be the canonical morphism. By [Stacks, Tag 01XY], \mathcal{F}^{\vee} and $\mathcal{F}^{\vee\vee}$ are coherent. The reflexive locus is the isomorphism locus of δ , hence is open by [Stacks, Tag 05GF].

(7) Under the quasi-excellence hypothesis this is Proposition 1.2.3.

(8) Under the quasi-excellence hypothesis this is [EGA IV₂, Corollaire 6.11.5].

(9) The assertion is local on X and follows from [EGA IV₂, Proposition 6.11.2]. \square

5.1.3 Properties of ideals

Consider the following properties of a pair (R, I) consisting of a noetherian local ring R and an ideal I :

- (1) R satisfies (R_n) ;
- (2) R is regular;
- (3) R is reduced;
- (4) R is normal;
- (5) I is a complete intersection;
- (6) R is Gorenstein;
- (7) the height of I is at least n .

Recall that the height of an ideal I is the minimal height of prime ideals containing I . We say I is a complete intersection if I can be generated by a regular sequence in R .

Proposition 5.1.3 *Let (Q) be one of the properties listed above. Let $f: (R, \mathfrak{m}) \rightarrow (R', \mathfrak{m}')$ be a regular local homomorphism between excellent noetherian local rings, and let I be an ideal of R . Then (Q) holds for (R, I) if and only if it holds for (R', IR') .*

Proof Since f is regular, it is flat. Since it is also local, it is faithfully flat by [Stacks, Tag 00HR]. Moreover all fibres of f are geometrically regular, hence regular.

(1) The ascent follows from [Stacks, Tag 033A], because the fibres of f are regular and hence satisfy (R_n) . The descent follows from faithfully flat descent for (R_n) , [Stacks, Tag 0353].

(2) The ascent follows from [Stacks, Tag 0H7S], and the descent follows from faithfully flat descent for regularity, [Stacks, Tag 07NG].

(3) The ascent follows from [Stacks, Tag 07QK], and the descent follows from faithfully flat descent for reducedness, [Stacks, Tag 033F].

(4) The ascent follows from [Stacks, Tag 0BFF], and the descent follows from faithfully flat descent for normality, [Stacks, Tag 033G].

(5) If $I = R$, then $IR' = R'$, and conversely $IR' = R'$ implies $(R/I) \otimes_R R' = 0$, hence $R/I = 0$ by faithful flatness. Thus we may assume that I and IR' are proper. Let a_1, \dots, a_r be a minimal system of generators of I , and put $a'_i = f(a_i)$. By Nakayama's lemma and the isomorphism

$$IR'/\mathfrak{m}'IR' \cong (I/\mathfrak{m}I) \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m}'),$$

the elements a'_1, \dots, a'_r form a minimal system of generators of IR' . If I is a complete intersection, then we may choose a_1, \dots, a_r to be a regular sequence, and a'_1, \dots, a'_r is a regular sequence by [Stacks, Tag 00LM]. Hence IR' is a complete intersection. Conversely, assume that IR' is a complete intersection. Choose a regular sequence b'_1, \dots, b'_s generating IR' . Since IR' is proper, all b'_j lie in \mathfrak{m}' . A regular sequence is quasi-regular by [Stacks, Tag 00LN]; in particular, the classes of b'_1, \dots, b'_s form a basis of $IR'/\mathfrak{m}'IR'$ over $\kappa(\mathfrak{m}')$. Hence $s = r$. Since a'_1, \dots, a'_r and b'_1, \dots, b'_r are two minimal systems of generators of IR' , they differ by an invertible matrix over the local ring R' . By [Stacks, Tag 0625], their Koszul complexes are isomorphic. Since b'_1, \dots, b'_r is a regular sequence, it is Koszul-regular by [Stacks, Tag 062F]. Thus a'_1, \dots, a'_r is Koszul-regular. By [Stacks, Tag 09CC], a'_1, \dots, a'_r is a regular sequence. Applying [Stacks, Tag 00LM] again, a_1, \dots, a_r is a regular sequence. Hence I is a complete intersection.

(6) The closed fibre $R'/\mathfrak{m}R'$ is regular, hence Gorenstein by [Stacks, Tag 0AWX]. The equivalence follows from the flat local criterion for Gorenstein rings, [Stacks, Tag 0BJL].

(7) Since f is flat local, the assertion follows from Proposition 1.2.4. \square

Proposition 5.1.4 *Let X be a quasi-excellent scheme and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf. For each property, the set of points $x \in X$ such that the pair $(\mathcal{O}_{X,x}, \mathcal{I}_x)$ satisfies that property is Zariski open.*

Proof The assertion is local on X , so we may work affine-locally whenever needed.

(1) This is the openness of the (R_n) -locus on a quasi-excellent scheme, [EGA IV₂, Proposition 6.12.9].

(2) Let $U = \text{Spec } A$ be an affine open subset of X . Since X is quasi-excellent, A is quasi-excellent, hence A is J-2 by [Stacks, Tag 07QT]. By the definition of J-2, [Stacks, Tag 07P7], the regular locus of $\text{Spec } A$ is open. Thus the regular locus of X is open.

(3) By [Stacks, Tag 031R], a noetherian ring is reduced if and only if it satisfies (R_0) and (S_1) . The (R_0) -locus is open by (1), and the (S_1) -locus for \mathcal{O}_X is open by Proposition 5.1.2. Hence the reduced locus is open.

(4) By Serre's criterion for normality, [Stacks, Tag 031S], a noetherian ring is normal if and only if it satisfies (R_1) and (S_2) . The (R_1) -locus is open by (1), and the (S_2) -locus for \mathcal{O}_X is open by Proposition 5.1.2. Hence the normal locus is open.

(5) Let $x \in X$ and suppose that \mathcal{I}_x is a complete intersection ideal of $\mathcal{O}_{X,x}$. After replacing X by an affine open neighborhood of x , choose sections $a_1, \dots, a_r \in \Gamma(X, \mathcal{I})$ whose images in $\mathcal{O}_{X,x}$ generate \mathcal{I}_x and form a regular sequence. The morphism

$$\mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{I}, \quad (b_1, \dots, b_r) \longmapsto \sum_i b_i a_i$$

is surjective at x . Its cokernel is coherent, hence its support is closed by [Stacks, Tag 056J]. After shrinking around x , the sections a_1, \dots, a_r generate \mathcal{I} . By [Stacks, Tag 061L], after shrinking further, the sequence a_1, \dots, a_r remains regular in every stalk. Therefore \mathcal{I}_y is a complete intersection ideal of $\mathcal{O}_{X,y}$ for all y in some open neighborhood of x .

(6) Under the present quasi-excellence hypothesis this is the openness theorem for the Gorenstein locus, [EGA IV₂, IV₂, §6.11]. If one assumes instead that X locally admits a dualizing complex, this reference can be replaced by [Stacks, Tag 0BFQ].

(7) This is [EGA IV₁, Proposition 0.14.2.6]. \square

5.1.4 Properties of morphisms of schemes and complex spaces

Consider the following properties of morphisms between schemes and complex spaces.

- (1) smooth;
- (2) unramified;
- (3) étale;
- (4) flat;
- (5) surjective;
- (6) universally injective;
- (7) open immersion;
- (8) closed immersion;
- (9) immersion;
- (10) isomorphism;

- (11) monomorphism;
- (12) separated;
- (13) proper;
- (14) finite;
- (15) dominant;
- (16) quasi-finite.

In the case of complex spaces, universally injective is precisely the same as injective by [Corollary 1.3.1](#).

Proposition 5.1.5 *Let (Q) be one of the properties (1)–(4). Let $A \rightarrow A'$ be a flat homomorphism between noetherian rings, and let $f: X \rightarrow Y$ be an A -morphism between A -schemes of finite presentation. Let*

$$f': X' = X \times_{\text{Spec } A} \text{Spec } A' \longrightarrow Y' = Y \times_{\text{Spec } A} \text{Spec } A'$$

be the base change, and let $p: X' \rightarrow X$ be the projection. Let \mathcal{T} be the set of points $x \in X$ such that f does not satisfy (Q) on any open neighborhood of x , and define $\mathcal{T}' \subseteq X'$ similarly for f' . Then \mathcal{T} and \mathcal{T}' are Zariski closed, and

$$\mathcal{T}' = p^{-1}(\mathcal{T}).$$

Proof For a morphism $h: Z \rightarrow W$, write $W_{(Q)}(h) \subseteq Z$ for the locus of points $z \in Z$ such that h satisfies (Q) on some open neighborhood of z . Since each of the properties (1)–(4) is local on the source, we have

$$\mathcal{T} = X \setminus W_{(Q)}(f), \quad \mathcal{T}' = X' \setminus W_{(Q)}(f').$$

It is therefore enough to prove that $W_{(Q)}(f)$ and $W_{(Q)}(f')$ are open and that

$$W_{(Q)}(f') = p^{-1}(W_{(Q)}(f)).$$

- (1) This is exactly [[Stacks, Tag 02V4](#)], applied to the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ \downarrow f' & & \downarrow f \\ Y' & \longrightarrow & Y, \end{array}$$

because f is locally of finite presentation and $Y' \rightarrow Y$ is flat.

(2) This is [[Stacks, Tag 0475](#)]. Indeed, f is locally of finite presentation, hence locally of finite type, and the open locus where f is unramified commutes with arbitrary base change.

(3) This is [[Stacks, Tag 0476](#)], again using that f is locally of finite presentation and $Y' \rightarrow Y$ is flat.

(4) The loci $W(f)$ and $W(f')$ are open by the openness of the flat locus, [[Stacks, Tag 0399](#)]. Let $x' \in X'$ and put $x = p(x')$, $y = f(x)$, and $y' = f'(x')$. If f is flat at

x , then f' is flat at x' by flat base change, [Stacks, Tag 01U8]. Conversely, assume that f' is flat at x' . Since $\mathcal{Y}' \rightarrow \mathcal{Y}$ is flat, the projection $p: \mathcal{X}' \rightarrow \mathcal{X}$ is flat at x' by [Stacks, Tag 01U8]. Moreover, the composite $\mathcal{X}' \rightarrow \mathcal{Y}' \rightarrow \mathcal{Y}$ is flat at x' by [Stacks, Tag 01U6]. Applying [Stacks, Tag 02JZ] to the morphism $p: \mathcal{X}' \rightarrow \mathcal{X}$ over \mathcal{Y} and to the sheaf $\mathcal{O}_{\mathcal{X}}$, we conclude that \mathcal{X} is flat over \mathcal{Y} at x . Thus f is flat at x . Hence $W(f') = p^{-1}(W(f))$. Taking complements gives $\mathcal{T}' = p^{-1}(\mathcal{T})$ in all four cases, and the closedness of \mathcal{T} and \mathcal{T}' follows from the openness of the corresponding good loci. \square

5.2 Interior properties

Let S be a Stein space with associated Stein algebra $A = \Gamma(S, \mathcal{O}_S)$.

For simplicity, we first introduce the following shorthand notation. Suppose that $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of A -schemes, and let $T \subseteq S$ be an arbitrary subset. We define $f_T: \mathcal{X}_T \rightarrow \mathcal{Y}_T$ as the morphism defined by the following Cartesian diagram in the category of schemes:

$$\begin{array}{ccc} \mathcal{X}_T & \longrightarrow & \mathcal{X} \\ \downarrow f_T & & \downarrow f \\ \mathcal{Y}_T & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \text{Spec } \Gamma(T, \mathcal{O}_S) & \longrightarrow & \text{Spec } A. \end{array}$$

More generally, if $S' \rightarrow S$ is a morphism of Stein spaces and $T' \subseteq S'$ is a subset, we write $f_{T'}: \mathcal{X}_{T'} \rightarrow \mathcal{Y}_{T'}$ for the morphism defined by the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_{T'} & \longrightarrow & \mathcal{X} \\ \downarrow f_{T'} & & \downarrow f \\ \mathcal{Y}_{T'} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \text{Spec } \Gamma(T', \mathcal{O}_{S'}) & \longrightarrow & \text{Spec } A. \end{array}$$

For any $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{F} , we write $\mathcal{F}_{T'}$ for the inverse image with respect to $\mathcal{X}_{T'} \rightarrow \mathcal{X}$. We regard $\mathcal{F}_{T'}$ as an $\mathcal{O}_{\mathcal{X}_{T'}}$ -module.

Let (P) be a property of schemes which is

- (1) stable under base change;
- (2) fpqc local on the base.

For a list of examples, we refer to [Section 5.1.1](#).

Definition 5.2.1 We say a morphism of A -schemes $f: X \rightarrow Y$ *interiorly satisfies* (P) if there is an open covering $\{U_i\}_{i \in I}$ of S such that the base changes f_{U_i} satisfy (P) for each $i \in I$.

We say an A -scheme X *interiorly satisfies* (P) if the structure morphism $X \rightarrow \text{Spec } A$ interiorly satisfies (P).

In particular, if f satisfies (P), then it interiorly satisfies (P) as well. In this definition, we may equivalently assume that the U_i 's are Stein.

Lemma 5.2.1 *Let $f: X \rightarrow Y$ be a morphism of A -schemes interiorly satisfying (P). Then*

- (1) *for any compact subset $K \subseteq S$, the induced morphism $f_K: X_K \rightarrow Y_K$ satisfies (P);*
- (2) *for any open and relatively compact subset $U \subseteq S$, the induced morphism $f_U: X_U \rightarrow Y_U$ satisfies (P).*

Proof (1) Since (P) is stable under base change, when proving (1), we could always replace K by a larger subset. Therefore, without loss of generality, we may assume that K is $\mathcal{O}(S)$ -convex, and hence Stein.

Next we construct a finite family K_1, \dots, K_n of Stein compact subsets of S contained in K such that f_{K_i} satisfies (P) for each i .

By assumption, each $s \in K$ has a Stein compact neighborhood K_s such that f_{K_s} satisfies (P). By the compactness of K , we can then find finitely many $s_1, \dots, s_n \in K$ such that the corresponding Stein compact neighborhoods K_1, \dots, K_n cover K . Replacing K_i by $K_i \cap K$, we get the desired family.

Now we claim that the natural map

$$\Gamma(K, \mathcal{O}_S) \rightarrow \prod_{i=1}^n \Gamma(K_i, \mathcal{O}_S)$$

is faithfully flat. The flatness is already proved in [Corollary 4.3.2](#), so it remains to show that every maximal ideal of $\Gamma(K, \mathcal{O}_S)$ is the inverse image of a prime ideal in some $\Gamma(K_i, \mathcal{O}_S)$. This follows immediately from [Proposition 4.4.1](#).

Since (P) is fpqc local, we conclude that f_K satisfies (P).

(2) This is a simple consequence of (1) applied to the compact set \bar{U} . □

Corollary 5.2.1 *Let X be an A -scheme interiorly locally of finite type (resp. interiorly of finite type). Then X is interiorly locally of finite presentation (resp. interiorly of finite presentation).*

Proof Let $s \in S$. It suffices to construct a compact neighborhood K of s in S such that X_K is locally of finite presentation (resp. of finite presentation).

By [Proposition 4.4.6](#), s admits an excellent Stein compact neighborhood K . By [Lemma 5.2.1](#), X_K is locally of finite type (resp. of finite type) over $\Gamma(K, \mathcal{O}_S)$. But the latter ring is noetherian, and hence X_K is locally of finite presentation (resp. of finite presentation). □

Proposition 5.2.1 *Let $\mathcal{X} \rightarrow \mathcal{Z}, \mathcal{Y} \rightarrow \mathcal{Z}$ be morphisms between interiorly locally of finite type A -schemes. Then the fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is interiorly locally of finite type as A -scheme.*

Proof Take an open covering $\{U_i\}_{i \in I}$ of S by relatively compact subsets. Then for each $i \in I$, the schemes $\mathcal{X}_{U_i}, \mathcal{Y}_{U_i}$, and \mathcal{Z}_{U_i} are all locally of finite type by [Lemma 5.2.1](#). Thus

$$\mathcal{X}_{U_i} \times_{\mathcal{Z}_{U_i}} \mathcal{X}_{U_i} \cong (\mathcal{X} \times_{\mathcal{Z}} \mathcal{X}) \times_{\text{Spec } A} \text{Spec } \Gamma(U_i, S)$$

is locally of finite type for each i . Our assertion follows. \square

Proposition 5.2.2 *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of A -schemes that interiorly satisfies (P). Let $S' \rightarrow S$ be a morphism of Stein spaces corresponding to a morphism of Stein algebras $A \rightarrow A'$. Then the base change $f_{S'}: \mathcal{X}_{S'} \rightarrow \mathcal{Y}_{S'}$, as a morphism of A' -schemes, also interiorly satisfies (P).*

Proof Let $\{U_i\}_{i \in I}$ be an open covering of S such that f_{U_i} satisfies (P) for each $i \in I$. Let U'_i be the inverse image of U_i in S' . Then $f_{U'_i}$ satisfies (P). \square

5.3 Analytification functor

Let S be a Stein space with associated Stein algebra $A = \Gamma(S, \mathcal{O}_S)$.

Recall that $\mathcal{L}oc\mathcal{R}ing_A$ is the category of locally A -ringed spaces, as defined in [Definition 1.3.1](#). Note that given a locally ringed space Z with a morphism $Z \rightarrow S$, we can view Z as an element in $\mathcal{L}oc\mathcal{R}ing_A$ by considering the composition

$$Z \rightarrow S \xrightarrow{i_{\text{Spec } A}} \text{Spec } A,$$

where $i: S \rightarrow \text{Spec } A$ is defined in [Corollary 3.3.9](#).

Definition 5.3.1 Let \mathcal{X} be an A -scheme. We define a functor:

$$F_{\mathcal{X}}: \text{CompSpace}_{/S}^{\text{op}} \rightarrow \text{Set}$$

as follows:

(1) Given a complex space Z over S , we define

$$F_{\mathcal{X}}(Z) := \text{Hom}_{\mathcal{L}oc\mathcal{R}ing_A}(Z, \mathcal{X});$$

(2) given a morphism $h: Z \rightarrow W$ of complex spaces over S we define the corresponding map

$$F_{\mathcal{X}}(h): \text{Hom}_{\mathcal{L}oc\mathcal{R}ing_A}(W, \mathcal{X}) \rightarrow \text{Hom}_{\mathcal{L}oc\mathcal{R}ing_A}(Z, \mathcal{X}), \quad f \mapsto f \circ h.$$

Note that $F_{\mathcal{X}}$ is functorial in the following sense. Suppose that we are given

- (1) a Stein space morphism $S' \rightarrow S$, where S' has the Stein algebra A' ;
- (2) an A' -scheme \mathcal{X}' ;
- (3) a morphism of A -schemes $\mathcal{X}' \rightarrow \mathcal{X}$.

Then there is a natural transformation

$$F_{\mathcal{X}'} \implies F_{\mathcal{X}} \circ (\text{CompSpace}_{/S'} \rightarrow \text{CompSpace}_{/S}). \quad (5.1)$$

Here $(\text{CompSpace}_{/S'} \rightarrow \text{CompSpace}_{/S})$ denotes the composition with $S' \rightarrow S$. The natural transformation is defined as follows. Let Z be a complex space over S' . We define the map between sets

$$\text{Hom}_{\text{LocRing}_{/A'}}(Z, \mathcal{X}') \rightarrow \text{Hom}_{\text{LocRing}_{/A}}(Z, \mathcal{X}) \quad (5.2)$$

by composition with the morphism $\mathcal{X}' \rightarrow \mathcal{X}$ in (3).

Lemma 5.3.1 *Let \mathcal{X} be an A -scheme. Suppose that $S' \rightarrow S$ is a morphism of Stein spaces with corresponding morphism of Stein algebras $A \rightarrow A'$. Let \mathcal{X}' be the A' -scheme defined by the following Cartesian diagram in the category of schemes:*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \\ \text{Spec } A' & \longrightarrow & \text{Spec } A. \end{array} \quad (5.3)$$

Then the natural transformation (5.1) is a natural isomorphism.

Proof Observe that (5.3) is also a Cartesian diagram in the category of locally A -ringed spaces. This follows for example from [Stacks, Tag 01JN] and Proposition 1.3.1.

Then it follows that (5.2) is bijective and (5.1) is a natural isomorphism. \square

Example 5.3.1 Let $B = A[t_1, \dots, t_n]$ for some $n \geq 0$ and let $\mathcal{X} = \text{Spec } B$. We claim that the functor $F_{\mathcal{X}}$ is represented by $S \times \mathbb{C}^n$ over S .

Let $p: S \times \mathbb{C}^n \rightarrow S$ be the projection, and let z_1, \dots, z_n denote the standard coordinates on \mathbb{C}^n . The morphism

$$i_{\mathcal{X}}: S \times \mathbb{C}^n \longrightarrow \text{Spec } B$$

is the morphism of locally A -ringed spaces induced by the homomorphism of A -algebras

$$B = A[t_1, \dots, t_n] \longrightarrow \Gamma(S \times \mathbb{C}^n, \mathcal{O}_{S \times \mathbb{C}^n}), \quad a \mapsto p^* a, \quad t_i \mapsto z_i.$$

Let $q: Z \rightarrow S$ be a complex space over S . A morphism $Z \rightarrow S \times \mathbb{C}^n$ over S is the same as a morphism $Z \rightarrow \mathbb{C}^n$, and hence is the same as an n -tuple

$$(g_1, \dots, g_n) \in \Gamma(Z, \mathcal{O}_Z)^n.$$

On the other hand, by the universal property of affine schemes, a morphism $Z \rightarrow \text{Spec } B$ in $\mathcal{L}oc\mathcal{R}ing_A$ is the same as an A -algebra homomorphism

$$B = A[t_1, \dots, t_n] \longrightarrow \Gamma(Z, \mathcal{O}_Z).$$

Such a homomorphism is uniquely determined by the images of t_1, \dots, t_n , and these images are arbitrary elements $g_1, \dots, g_n \in \Gamma(Z, \mathcal{O}_Z)$. Therefore composition with i_X induces a natural bijection

$$\text{Hom}_{\text{CompSpace}/S}(Z, S \times \mathbb{C}^n) \xrightarrow{\sim} \text{Hom}_{\mathcal{L}oc\mathcal{R}ing_A}(Z, \mathcal{X}).$$

Thus $S \times \mathbb{C}^n$ represents F_X .

We also record the corresponding local computation. Let $x = (s, \mathbf{a}) \in S \times \mathbb{C}^n$, where $\mathbf{a} = (a_1, \dots, a_n)$. Let $\mathfrak{m}(s)$ be the maximal ideal of A corresponding to s . Then

$$i_X(x) = \mathfrak{m}(s)B + (t_1 - a_1, \dots, t_n - a_n).$$

Put $u_i = t_i - a_i$. The local homomorphism

$$i_{X,x}^\# : \mathcal{O}_{X,i_X(x)} \longrightarrow \mathcal{O}_{S \times \mathbb{C}^n, x}$$

identifies with

$$(A_{\mathfrak{m}(s)}[u_1, \dots, u_n])_{\mathfrak{m}(s)A_{\mathfrak{m}(s)}[u_1, \dots, u_n] + (u_1, \dots, u_n)} \longrightarrow \mathcal{O}_{S,s}\{u_1, \dots, u_n\}.$$

The source is noetherian, since $A_{\mathfrak{m}(s)}$ is noetherian by [Corollary 3.3.11](#).

Taking completions at the maximal ideals, we obtain the homomorphism

$$\widehat{A_{\mathfrak{m}(s)}}[[u_1, \dots, u_n]] \longrightarrow \widehat{\mathcal{O}_{S,s}}[[u_1, \dots, u_n]].$$

This is the formal power series extension of the isomorphism

$$\widehat{A_{\mathfrak{m}(s)}} \xrightarrow{\sim} \widehat{\mathcal{O}_{S,s}}$$

from [Proposition 3.3.4](#). Hence $i_{X,x}^\#$ induces an isomorphism on completions. In particular, $i_{X,x}^\#$ is flat.

Theorem 5.3.1 (Bingener) *Let X be an A -scheme interiorly locally of finite type. Then the functor F_X is representable.*

In other words, there is a complex space X^{an} over S and a morphism of locally A -ringed spaces $i_X : X^{\text{an}} \rightarrow X$ such that for any complex space Z over S , composition with i_X induces a bijection

$$\text{Hom}_{\text{CompSpace}/S}(Z, X^{\text{an}}) \xrightarrow{\sim} \text{Hom}_{\mathcal{L}oc\mathcal{R}ing_A}(Z, \mathcal{X}). \quad (5.4)$$

Furthermore, for each $x \in \mathcal{X}^{\text{an}}$, we have

- (1) $\mathcal{O}_{\mathcal{X}, i_{\mathcal{X}}(x)}$ is noetherian;
- (2) the induced homomorphism

$$i_{\mathcal{X}, x}^{\#} : \mathcal{O}_{\mathcal{X}, i_{\mathcal{X}}(x)} \rightarrow \mathcal{O}_{\mathcal{X}^{\text{an}}, x} \quad (5.5)$$

induces an isomorphism on the completion. In particular, $i_{\mathcal{X}}$ is flat.

In particular, (5.5) is regular. Furthermore, for any $x \in \mathcal{X}^{\text{an}}$, the image $i_{\mathcal{X}}(x)$ is necessarily a closed point with residue \mathbb{C} .

By [Corollary 3.3.10](#), our previous notation $i_{\text{Spec } A} : S \rightarrow \text{Spec } A$ is compatible with the notation here, and S is canonically isomorphic to $(\text{Spec } A)^{\text{an}}$.

Proof We first record two reductions.

Step 1: base change to a Stein open subset. Let $S' \subseteq S$ be a Stein open subset, and put $A' = \Gamma(S', \mathcal{O}_{S'})$. Let $\mathcal{X}' = \mathcal{X} \times_{\text{Spec } A} \text{Spec } A'$. Suppose that $F_{\mathcal{X}}$ is represented by $i_{\mathcal{X}} : \mathcal{X}^{\text{an}} \rightarrow \mathcal{X}$. Then the open subspace $\mathcal{X}'^{\text{an}} := \mathcal{X}^{\text{an}} \times_S S'$ represents $F_{\mathcal{X}'}$. Indeed, for every complex space $Z \rightarrow S'$, we have natural bijections

$$\begin{aligned} \text{Hom}_{\text{CompSpace}_{S'}}(Z, \mathcal{X}'^{\text{an}}) &= \text{Hom}_{\text{CompSpace}_S}(Z, \mathcal{X}^{\text{an}}) \\ &\cong \text{Hom}_{\text{LocRing}_A}(Z, \mathcal{X}) \\ &\cong \text{Hom}_{\text{LocRing}_{A'}}(Z, \mathcal{X}'), \end{aligned}$$

where the last bijection is [Lemma 5.3.1](#). This proves the asserted compatibility of the representing object with base change to Stein open subsets.

Step 2: open and closed subschemes. Assume that the theorem has been proved for an A -scheme \mathcal{X} interiorly locally of finite type.

First let $\mathcal{Y} \subseteq \mathcal{X}$ be an open subscheme. We define $\mathcal{Y}^{\text{an}} := i_{\mathcal{X}}^{-1}(\mathcal{Y})$ with the induced morphism $i_{\mathcal{Y}} : \mathcal{Y}^{\text{an}} \rightarrow \mathcal{Y}$. Then \mathcal{Y}^{an} represents $F_{\mathcal{Y}}$, since a morphism to an open subspace is the same as a morphism whose underlying map has image in that open subspace. The assertions on local rings are immediate.

Now let $\mathcal{Y} \subseteq \mathcal{X}$ be a closed subscheme whose defining ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ is of finite type. Then

$$i_{\mathcal{X}}^{-1} \mathcal{I} \cdot \mathcal{O}_{\mathcal{X}^{\text{an}}}$$

is a coherent ideal sheaf of $\mathcal{O}_{\mathcal{X}^{\text{an}}}$. We define \mathcal{Y}^{an} to be the analytic subspace of \mathcal{X}^{an} defined by this coherent ideal sheaf. Equivalently, it fits into the Cartesian diagram of locally ringed spaces

$$\begin{array}{ccc} \mathcal{Y}^{\text{an}} & \longrightarrow & \mathcal{X}^{\text{an}} \\ \downarrow i_{\mathcal{Y}} & & \downarrow i_{\mathcal{X}} \\ \mathcal{Y} & \hookrightarrow & \mathcal{X}. \end{array}$$

We now verify the universal property. By the universal property of \mathcal{X}^{an} , a morphism $Z \rightarrow \mathcal{X}^{\text{an}}$ over S is the same as a morphism $Z \rightarrow \mathcal{X}$ in LocRing_A . Under this bijection, a morphism $g : Z \rightarrow \mathcal{X}^{\text{an}}$ corresponds to $h = i_{\mathcal{X}} \circ g$. Since \mathcal{Y}^{an} is defined by the ideal sheaf $i_{\mathcal{X}}^{-1} \mathcal{I} \cdot \mathcal{O}_{\mathcal{X}^{\text{an}}}$, the condition that g factors through \mathcal{Y}^{an} is equivalent

to the condition that $h^{-1}\mathcal{I} \rightarrow \mathcal{O}_Z$ is zero. This is precisely the condition that h factors through \mathcal{Y} . Hence the representing bijection for \mathcal{X} restricts to the desired bijection

$$\mathrm{Hom}_{\mathrm{CompSpace}_S}(Z, \mathcal{Y}^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LocRing}_A}(Z, \mathcal{Y}).$$

It remains to check the local ring assertions in the closed case. Let $y \in \mathcal{Y}^{\mathrm{an}}$, let x be its image in $\mathcal{X}^{\mathrm{an}}$, and put $\eta = i_{\mathcal{Y}}(y)$ and $\xi = i_{\mathcal{X}}(x)$. Set $R = \mathcal{O}_{\mathcal{X}, \xi}$ and $R' = \mathcal{O}_{\mathcal{X}^{\mathrm{an}}, x}$. Let $I \subseteq R$ be the stalk of \mathcal{I} at ξ . Then

$$\mathcal{O}_{\mathcal{Y}, \eta} = R/I, \quad \mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y} = R'/IR'.$$

Since R is noetherian and I is finitely generated, R/I is noetherian. Moreover, using the completion isomorphism $\widehat{R} \xrightarrow{\sim} \widehat{R}'$, we get

$$\widehat{R/I} \cong \widehat{R}/I\widehat{R} \cong \widehat{R}'/I\widehat{R}' \cong \widehat{R'/IR'}.$$

Hence $\mathcal{O}_{\mathcal{Y}, \eta} \rightarrow \mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y}$ induces an isomorphism on completions. The local homomorphism is therefore flat.

Combining the open and closed cases, the theorem holds for every locally closed subscheme of \mathcal{X} whose closed embedding into an open subscheme is defined by an ideal sheaf of finite type.

Step 3: the affine finite presentation case. Assume that $\mathcal{X} = \mathrm{Spec} B$, where B is a finitely presented A -algebra. Choose a presentation $B = A[t_1, \dots, t_n]/I$ with I finitely generated. Then \mathcal{X} is a closed subscheme of $\mathbb{A}_A^n = \mathrm{Spec} A[t_1, \dots, t_n]$. The theorem for \mathbb{A}_A^n is exactly [Example 5.3.1](#). Therefore the theorem for \mathcal{X} follows from Step 2.

Step 4: the locally finite presentation case. Now assume that $\mathcal{X} \rightarrow \mathrm{Spec} A$ is locally of finite presentation. Choose an affine open covering $\mathcal{X} = \bigcup_{i \in I} \mathcal{U}_i$ such that each $\mathcal{U}_i = \mathrm{Spec} B_i$ is of finite presentation over A . By Step 3, each $F_{\mathcal{U}_i}$ is represented by some $\mathcal{U}_i^{\mathrm{an}}$.

For $i, j \in I$, $F_{\mathcal{U}_i \cap \mathcal{U}_j}$ is represented by

$$i_{\mathcal{U}_i}^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \subseteq \mathcal{U}_i^{\mathrm{an}}$$

by Step 2. The same open subscheme is also an open subscheme of \mathcal{U}_j , and the corresponding functor is represented by

$$i_{\mathcal{U}_j}^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \subseteq \mathcal{U}_j^{\mathrm{an}}.$$

By the universal property, these two complex spaces are canonically isomorphic. The resulting transition isomorphisms satisfy the cocycle condition. Hence, by [Theorem 2.3.2](#), the $\mathcal{U}_i^{\mathrm{an}}$'s glue to a complex space $\mathcal{X}^{\mathrm{an}}$ over S .

The morphisms $i_{\mathcal{U}_i} : \mathcal{U}_i^{\mathrm{an}} \rightarrow \mathcal{U}_i$ glue to a morphism of locally A -ringed spaces

$$i_{\mathcal{X}} : \mathcal{X}^{\mathrm{an}} \longrightarrow \mathcal{X}.$$

The universal property follows from the usual open-cover description of morphisms into schemes and into complex spaces: both sides of

$$\mathrm{Hom}_{\mathrm{CompSpace}/S}(Z, \mathcal{X}^{\mathrm{an}}) \longrightarrow \mathrm{Hom}_{\mathrm{LocRing}_A}(Z, \mathcal{X})$$

are obtained by gluing compatible morphisms to the open pieces $\mathcal{U}_i^{\mathrm{an}}$ and \mathcal{U}_i , respectively. The local ring assertions are local on $\mathcal{X}^{\mathrm{an}}$, hence follow from Step 3.

Step 5: the interiorly locally finite type case. Finally assume only that \mathcal{X} is interiorly locally of finite type. By [Corollary 5.2.1](#), there exists a Stein open covering $\{S_\alpha\}_{\alpha \in I}$ of S such that, if $A_\alpha = \Gamma(S_\alpha, \mathcal{O}_{S_\alpha})$ and $\mathcal{X}_\alpha = \mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} A_\alpha$, then \mathcal{X}_α is locally of finite presentation over A_α . By Step 4, each $F_{\mathcal{X}_\alpha}$ can be represented by $\mathcal{X}_\alpha^{\mathrm{an}}$ over S_α .

On $S_{\alpha\beta} = S_\alpha \cap S_\beta$, the restrictions of $\mathcal{X}_\alpha^{\mathrm{an}}$ and $\mathcal{X}_\beta^{\mathrm{an}}$ represent the same functor. Indeed, for every complex space $Z \rightarrow S_{\alpha\beta}$, we have natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CompSpace}/S_\alpha}(Z, \mathcal{X}_\alpha^{\mathrm{an}}) &\cong \mathrm{Hom}_{\mathrm{LocRing}_{A_\alpha}}(Z, \mathcal{X}_\alpha) \\ &\cong \mathrm{Hom}_{\mathrm{LocRing}_A}(Z, \mathcal{X}), \end{aligned}$$

and similarly for β . Thus the two restrictions are canonically isomorphic over $S_{\alpha\beta}$, and these canonical isomorphisms satisfy the cocycle condition. We glue the $\mathcal{X}_\alpha^{\mathrm{an}}$'s to obtain a complex space $\mathcal{X}^{\mathrm{an}}$ over S . The morphisms $i_{\mathcal{X}_\alpha}$ glue to a morphism

$$i_{\mathcal{X}}: \mathcal{X}^{\mathrm{an}} \longrightarrow \mathcal{X}.$$

The universal property is checked on the open covering $\{S_\alpha\}$, hence follows from the universal properties of the $\mathcal{X}_\alpha^{\mathrm{an}}$'s.

It remains to verify the local ring assertions for the original scheme \mathcal{X} . Let $x \in \mathcal{X}^{\mathrm{an}}$, and let $s \in S$ be its image. Choose α such that $s \in S_\alpha$. Put $\mathfrak{m}(s) \subseteq A$ and $\mathfrak{m}_\alpha(s) \subseteq A_\alpha$ for the maximal ideals corresponding to s . By [Proposition 3.3.4](#), applied to S and to S_α , the local homomorphism

$$A_{\mathfrak{m}(s)} \longrightarrow (A_\alpha)_{\mathfrak{m}_\alpha(s)}$$

induces an isomorphism on completions. Since both local rings are noetherian by [Corollary 3.3.11](#), this homomorphism is faithfully flat.

The base change \mathcal{X}_α is locally of finite presentation at the point corresponding to x . Since locally finite presentation is fpqc local on the base, it follows that \mathcal{X} is locally of finite presentation over A at the point $i_{\mathcal{X}}(x)$. Hence we may choose an open neighborhood $\mathcal{V} \subseteq \mathcal{X}$ of $i_{\mathcal{X}}(x)$ such that $\mathcal{V} \rightarrow \mathrm{Spec} A$ is locally of finite presentation. The open subspace $i_{\mathcal{X}}^{-1}(\mathcal{V}) \subseteq \mathcal{X}^{\mathrm{an}}$ represents $F_{\mathcal{V}}$, hence is canonically identified with $\mathcal{V}^{\mathrm{an}}$ constructed in Step 4. Therefore the local homomorphism

$$\mathcal{O}_{\mathcal{X}, i_{\mathcal{X}}(x)} \longrightarrow \mathcal{O}_{\mathcal{X}^{\mathrm{an}}, x}$$

has noetherian source and induces an isomorphism on completions by Step 4. It is therefore flat. \square

Definition 5.3.2 Given a scheme \mathcal{X} interiorly locally of finite type over A , we define its analytification \mathcal{X}^{an} as the complex space over S as defined in [Theorem 5.3.1](#).

This construction is functorial: Given any morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of interiorly locally of finite type A -schemes, we write

$$f^{\text{an}}: \mathcal{X}^{\text{an}} \rightarrow \mathcal{Y}^{\text{an}}$$

for the analytification of f .

Example 5.3.2 Let $\mathcal{X} = \mathbb{P}_A^n$. Then \mathcal{X}^{an} is canonically isomorphic to the relative complex projective space

$$\mathbb{P}_S^n := S \times \mathbb{P}_{\mathbb{C}}^n,$$

and the structure morphism $\mathcal{X}^{\text{an}} \rightarrow S$ is the projection $\mathbb{P}_S^n \rightarrow S$.

Let T_0, \dots, T_n be the homogeneous coordinates on \mathbb{P}_A^n . For $i = 0, \dots, n$, put

$$\mathcal{U}_i = D_+(T_i) \subseteq \mathbb{P}_A^n.$$

Then

$$\mathcal{U}_i \cong \text{Spec } A[u_{i,0}, \dots, \widehat{u_{i,i}}, \dots, u_{i,n}], \quad u_{i,j} = \frac{T_j}{T_i} \quad (j \neq i).$$

By [Example 5.3.1](#), the analytification of \mathcal{U}_i is

$$\mathcal{U}_i^{\text{an}} \cong S \times \mathbb{C}^n,$$

with coordinates $z_{i,j}$ corresponding to $u_{i,j}$ for $j \neq i$.

On the overlap $\mathcal{U}_i \cap \mathcal{U}_j$, where $i \neq j$, we have $u_{i,j} \neq 0$. Under the above affine coordinates, the transition functions are

$$u_{j,i} = \frac{1}{u_{i,j}}, \quad u_{j,k} = \frac{u_{i,k}}{u_{i,j}} \quad (k \neq i, j).$$

Therefore the induced analytic transition functions on

$$\mathcal{U}_i^{\text{an}} \cap \mathcal{U}_j^{\text{an}} = \{z_{i,j} \neq 0\} \subseteq S \times \mathbb{C}^n$$

are

$$z_{j,i} = \frac{1}{z_{i,j}}, \quad z_{j,k} = \frac{z_{i,k}}{z_{i,j}} \quad (k \neq i, j).$$

These are exactly the usual transition functions defining the relative complex projective space \mathbb{P}_S^n . Hence the complex spaces obtained by gluing the $\mathcal{U}_i^{\text{an}}$'s are canonically identified with \mathbb{P}_S^n .

Let us also describe the morphism

$$i_{\mathcal{X}}: \mathbb{P}_S^n \longrightarrow \mathbb{P}_A^n.$$

On the standard chart $S \times \mathbb{C}^n \cong \mathcal{U}_i^{\text{an}}$, it is the morphism of locally A -ringed spaces induced by

$$A[u_{i,0}, \dots, \widehat{u_{i,i}}, \dots, u_{i,n}] \longrightarrow \Gamma(S \times \mathbb{C}^n, \mathcal{O}_{S \times \mathbb{C}^n}), \quad a \mapsto p^* a, \quad u_{i,j} \mapsto z_{i,j},$$

where $p : S \times \mathbb{C}^n \rightarrow S$ is the projection. These local morphisms are compatible with the transition functions above, and hence glue to a morphism $i_X : \mathbb{P}_S^n \rightarrow \mathbb{P}_A^n$.

Finally, this glued object has the desired universal property. Indeed, let $Z \rightarrow S$ be a complex space. A morphism $Z \rightarrow \mathbb{P}_S^n$ over S is equivalent to giving an open covering $\{Z_i\}_{i=0}^n$ of Z together with morphisms

$$Z_i \longrightarrow S \times \mathbb{C}^n \cong \mathcal{U}_i^{\text{an}}$$

over S , satisfying the usual compatibility conditions on the overlaps. By [Example 5.3.1](#), this is equivalent to giving compatible morphisms

$$Z_i \longrightarrow \mathcal{U}_i$$

in $\mathcal{L}oc\mathcal{R}ing_A$. Since the \mathcal{U}_i 's form an open covering of \mathbb{P}_A^n , such compatible local morphisms glue uniquely to a morphism

$$Z \longrightarrow \mathbb{P}_A^n$$

in $\mathcal{L}oc\mathcal{R}ing_A$. Thus composition with i_X gives a natural bijection

$$\text{Hom}_{\text{CompSpace}/S}(Z, \mathbb{P}_S^n) \xrightarrow{\sim} \text{Hom}_{\mathcal{L}oc\mathcal{R}ing_A}(Z, \mathbb{P}_A^n).$$

Hence \mathbb{P}_S^n represents F_X , as claimed.

The functoriality (5.1) and [Lemma 5.3.1](#) now give the following:

Proposition 5.3.1 *Let $S' \rightarrow S$ be a morphism of Stein spaces corresponding to a morphism of Stein algebras $A \rightarrow A'$. Suppose that we are given:*

- (1) *an A -scheme X (resp. A' -scheme X') interiorly locally of finite type;*
- (2) *an A -morphism $f : X' \rightarrow X$.*

Then there is a functorial morphism of complex spaces $f^{\text{an}} : X'^{\text{an}} \rightarrow X^{\text{an}}$ making the following diagram commutative:

$$\begin{array}{ccc} X'^{\text{an}} & \xrightarrow{f^{\text{an}}} & X^{\text{an}} \\ \downarrow i_{X'} & & \downarrow i_X \\ X' & \xrightarrow{f} & X. \end{array} \quad (5.6)$$

Furthermore, if X' is the Cartesian product $X \times_{\text{Spec } A} \text{Spec } A'$, then the diagram is Cartesian in the category of locally \mathbb{C} -ringed spaces.

Corollary 5.3.1 *Let X be an A -scheme interiorly locally of finite type. Then the following diagram is Cartesian:*

$$\begin{array}{ccc}
\mathcal{X}^{\text{an}} & \xrightarrow{i_X} & \mathcal{X} \\
\downarrow & \square & \downarrow \\
S & \xrightarrow{i_{\text{Spec } A}} & \text{Spec } A
\end{array}$$

in the category of locally \mathbb{C} -ringed spaces.

Proof This follows immediately from [Proposition 5.3.1](#). \square

Corollary 5.3.2 *Let \mathcal{X} be an A -scheme interiorly locally of finite type. Then for any open subscheme $i: \mathcal{U} \rightarrow \mathcal{X}$, the map $i^{\text{an}}: \mathcal{U}^{\text{an}} \rightarrow \mathcal{X}^{\text{an}}$ is an open immersion, and the following diagram is Cartesian:*

$$\begin{array}{ccc}
\mathcal{U}^{\text{an}} & \xrightarrow{i^{\text{an}}} & \mathcal{X}^{\text{an}} \\
\downarrow i_U & \square & \downarrow i_X \\
\mathcal{U} & \xrightarrow{i} & \mathcal{X}
\end{array}$$

Similarly, if $U \subseteq S$ is an open Stein subset with corresponding morphism of Stein algebras $A \rightarrow A'$. Then the following diagram is Cartesian:

$$\begin{array}{ccc}
\mathcal{X}_U^{\text{an}} & \longrightarrow & \mathcal{X}^{\text{an}} \\
i_{X_U} \downarrow & \square & \downarrow i_X \\
\mathcal{X}_U & \longrightarrow & \mathcal{X} \\
\downarrow & \square & \downarrow \\
U & \longrightarrow & S
\end{array}$$

and $\mathcal{X}_U^{\text{an}} \rightarrow \mathcal{X}^{\text{an}}$ is an open immersion.

We shall identify \mathcal{U}^{an} and $\mathcal{X}_U^{\text{an}}$ with their images in \mathcal{X}^{an} in the sequel.

Proof We first observe that \mathcal{U} is an A -scheme interiorly locally of finite type. The Cartesian diagrams follow from [Proposition 5.3.1](#). \square

Corollary 5.3.3 *Let \mathcal{X} be an A -scheme interiorly locally of finite type. Let $\{\mathcal{U}_i\}_{i \in I}$ be a family of open subschemes of \mathcal{X} . If $\{\mathcal{U}_i\}_{i \in I}$ covers \mathcal{X} , then $\{\mathcal{U}_i^{\text{an}}\}_{i \in I}$ covers \mathcal{X}^{an} .*

Proof Let $x \in \mathcal{X}^{\text{an}}$. Since the open subschemes $\{\mathcal{U}_i\}_{i \in I}$ cover \mathcal{X} , there exists $i \in I$ such that

$$i_X(x) \in \mathcal{U}_i.$$

By [Corollary 5.3.2](#), the analytification of the open immersion $\mathcal{U}_i \hookrightarrow \mathcal{X}$ identifies $\mathcal{U}_i^{\text{an}}$ with the open subspace

$$i_X^{-1}(\mathcal{U}_i) \subseteq \mathcal{X}^{\text{an}}.$$

Hence $x \in i_X^{-1}(\mathcal{U}_i) = \mathcal{U}_i^{\text{an}}$. Since x was arbitrary, the family $\{\mathcal{U}_i^{\text{an}}\}_{i \in I}$ covers \mathcal{X}^{an} . \square

Corollary 5.3.4 *Let X be an A -scheme interiorly locally of finite type. Let $\{U_i\}_{i \in I}$ be a family of Stein open subsets of S . If $\{U_i\}_{i \in I}$ covers S , then $\{\mathcal{X}_{U_i}^{\text{an}}\}_{i \in I}$ covers \mathcal{X}^{an} .*

Proof Let $p: \mathcal{X}^{\text{an}} \rightarrow S$ be the structure morphism. By [Corollary 5.3.2](#), for every $i \in I$, $\mathcal{X}_{U_i}^{\text{an}}$ is naturally identified with the open subspace $p^{-1}(U_i) = \mathcal{X}^{\text{an}} \times_S U_i$ of \mathcal{X}^{an} .

Since $\{U_i\}_{i \in I}$ covers S , we get

$$\mathcal{X}^{\text{an}} = p^{-1}(S) = p^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} p^{-1}(U_i) = \bigcup_{i \in I} \mathcal{X}_{U_i}^{\text{an}}.$$

Therefore $\{\mathcal{X}_{U_i}^{\text{an}}\}_{i \in I}$ covers \mathcal{X}^{an} . \square

Proposition 5.3.2 *Let $X \rightarrow Z$, $Y \rightarrow Z$ be morphisms between interiorly locally of finite type A -schemes. Then the fiber product $\mathcal{X}^{\text{an}} \times_{Z^{\text{an}}} \mathcal{Y}^{\text{an}}$ exists and there is a natural isomorphism*

$$\mathcal{X}^{\text{an}} \times_{Z^{\text{an}}} \mathcal{Y}^{\text{an}} \xrightarrow{\sim} (\mathcal{X} \times_Z \mathcal{Y})^{\text{an}}.$$

Proof First observe that $\mathcal{X} \times_Z \mathcal{Y}$ is again an A -scheme interiorly locally of finite type by [Proposition 5.2.1](#).

The fiber product $\mathcal{X}^{\text{an}} \times_{Z^{\text{an}}} \mathcal{Y}^{\text{an}}$ exists in CompSpace by [Proposition 1.3.2](#). We show that it represents $F_{\mathcal{X} \times_Z \mathcal{Y}}$. Let $T \rightarrow S$ be a complex space over S . By the universal property of the fiber product in CompSpace , we have a natural bijection

$$\begin{aligned} & \text{Hom}_{\text{CompSpace}/S}(T, \mathcal{X}^{\text{an}} \times_{Z^{\text{an}}} \mathcal{Y}^{\text{an}}) \\ & \cong \text{Hom}_{\text{CompSpace}/S}(T, \mathcal{X}^{\text{an}}) \times_{\text{Hom}_{\text{CompSpace}/S}(T, Z^{\text{an}})} \text{Hom}_{\text{CompSpace}/S}(T, \mathcal{Y}^{\text{an}}). \end{aligned}$$

By [Theorem 5.3.1](#), the right-hand side is naturally identified with

$$\text{Hom}_{\mathcal{L}\text{ocRing}_A}(T, \mathcal{X}) \times_{\text{Hom}_{\mathcal{L}\text{ocRing}_A}(T, Z)} \text{Hom}_{\mathcal{L}\text{ocRing}_A}(T, \mathcal{Y}).$$

Since the Cartesian square

$$\begin{array}{ccc} \mathcal{X} \times_Z \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \\ \mathcal{Y} & \longrightarrow & Z \end{array}$$

is also Cartesian in the category of locally A -ringed spaces, this last set is naturally identified with

$$\text{Hom}_{\mathcal{L}\text{ocRing}_A}(T, \mathcal{X} \times_Z \mathcal{Y}) = F_{\mathcal{X} \times_Z \mathcal{Y}}(T).$$

Thus $\mathcal{X}^{\text{an}} \times_{Z^{\text{an}}} \mathcal{Y}^{\text{an}}$ represents $F_{\mathcal{X} \times_Z \mathcal{Y}}$.

On the other hand, by [Theorem 5.3.1](#), $(\mathcal{X} \times_Z \mathcal{Y})^{\text{an}}$ also represents $F_{\mathcal{X} \times_Z \mathcal{Y}}$. By uniqueness of representing objects, there is a unique isomorphism

$$\mathcal{X}^{\text{an}} \times_{Z^{\text{an}}} \mathcal{Y}^{\text{an}} \xrightarrow{\sim} (\mathcal{X} \times_Z \mathcal{Y})^{\text{an}}$$

compatible with the two projections to \mathcal{X}^{an} and \mathcal{Y}^{an} . This is the desired natural isomorphism. \square

Corollary 5.3.5 *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism between interiorly locally of finite type A -schemes. Then, under the natural isomorphism*

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})^{\text{an}} \simeq \mathcal{X}^{\text{an}} \times_{\mathcal{Y}^{\text{an}}} \mathcal{X}^{\text{an}}$$

of [Proposition 5.3.2](#), the analytification of the diagonal morphism

$$\Delta_f: \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

is identified with the diagonal morphism

$$\Delta_{f^{\text{an}}}: \mathcal{X}^{\text{an}} \rightarrow \mathcal{X}^{\text{an}} \times_{\mathcal{Y}^{\text{an}}} \mathcal{X}^{\text{an}}.$$

Proof Let

$$p_1, p_2: \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \rightarrow \mathcal{X}$$

be the two projections, and let

$$q_1, q_2: \mathcal{X}^{\text{an}} \times_{\mathcal{Y}^{\text{an}}} \mathcal{X}^{\text{an}} \rightarrow \mathcal{X}^{\text{an}}$$

be the two analytic projections. Denote by

$$\alpha: (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})^{\text{an}} \xrightarrow{\sim} \mathcal{X}^{\text{an}} \times_{\mathcal{Y}^{\text{an}}} \mathcal{X}^{\text{an}}$$

the isomorphism of [Proposition 5.3.2](#). By construction of α , one has

$$q_i \circ \alpha = (p_i)^{\text{an}}$$

for $i = 1, 2$. Therefore

$$q_i \circ \alpha \circ (\Delta_f)^{\text{an}} = (p_i)^{\text{an}} \circ (\Delta_f)^{\text{an}} = (p_i \circ \Delta_f)^{\text{an}} = \text{id}_{\mathcal{X}^{\text{an}}}$$

for $i = 1, 2$. Hence the morphism

$$\alpha \circ (\Delta_f)^{\text{an}}: \mathcal{X}^{\text{an}} \rightarrow \mathcal{X}^{\text{an}} \times_{\mathcal{Y}^{\text{an}}} \mathcal{X}^{\text{an}}$$

has both projections equal to $\text{id}_{\mathcal{X}^{\text{an}}}$. By the universal property of the fibre product, this morphism is exactly the diagonal morphism associated with

$$f^{\text{an}}: \mathcal{X}^{\text{an}} \rightarrow \mathcal{Y}^{\text{an}}.$$

This proves the claim. \square

Definition 5.3.3 Let \mathcal{X} be an A -scheme interiorly locally of finite type. Given an $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{F} , we define its analytification \mathcal{F}^{an} as the $\mathcal{O}_{\mathcal{X}^{\text{an}}}$ -module given by $i_{\mathcal{X}}^* \mathcal{F}$.

By the flatness part in [Theorem 5.3.1](#), the functor $\mathcal{F} \mapsto \mathcal{F}^{\text{an}}$ is exact.

Proposition 5.3.3 *Let \mathcal{X} be an A -scheme interiorly locally of finite type, and let \mathcal{F} be an $\mathcal{O}_{\mathcal{X}}$ -module of finite presentation. Then \mathcal{F}^{an} is coherent.*

If \mathcal{I} is a quasi-coherent ideal sheaf on \mathcal{X} , then \mathcal{I}^{an} is a coherent ideal sheaf on \mathcal{X}^{an} .

Proof Both assertions are local on \mathcal{X}^{an} . Let $x \in \mathcal{X}^{\text{an}}$, and let $s \in S$ be its image. By the construction in [Theorem 5.3.1](#), after replacing S by a Stein open neighborhood of s and \mathcal{X} by an affine open neighborhood of $i_{\mathcal{X}}(x)$, we may assume that $\mathcal{X} = \text{Spec } B$, where B is a finitely presented A -algebra. In this case \mathcal{X}^{an} is a closed analytic subspace of some $S \times \mathbb{C}^n$, hence is Stein.

We first prove the assertion for \mathcal{F} . Since \mathcal{F} is of finite presentation, after further shrinking \mathcal{X} if necessary, there is an exact sequence

$$\mathcal{O}_{\mathcal{X}}^m \rightarrow \mathcal{O}_{\mathcal{X}}^n \rightarrow \mathcal{F} \rightarrow 0.$$

Pulling back to \mathcal{X}^{an} gives an exact sequence

$$\mathcal{O}_{\mathcal{X}^{\text{an}}}^m \rightarrow \mathcal{O}_{\mathcal{X}^{\text{an}}}^n \rightarrow \mathcal{F}^{\text{an}} \rightarrow 0.$$

Since \mathcal{X}^{an} is a complex space, $\mathcal{O}_{\mathcal{X}^{\text{an}}}$ is coherent, and cokernels of morphisms between coherent analytic sheaves are coherent. Hence \mathcal{F}^{an} is coherent.

We now prove the assertion for \mathcal{I} . Since $\mathcal{X} = \text{Spec } B$, the ideal sheaf \mathcal{I} corresponds to an ideal $I \subseteq B$. Let

$$C = \Gamma(\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}})$$

and let $J \subseteq C$ be the ideal generated by the image of I under the natural homomorphism $B \rightarrow C$. By the definition of analytification of modules,

$$\mathcal{I}^{\text{an}} = J\mathcal{O}_{\mathcal{X}^{\text{an}}}.$$

Since \mathcal{X}^{an} is Stein, [Example 3.4.1](#) implies that $J\mathcal{O}_{\mathcal{X}^{\text{an}}}$ is a coherent ideal sheaf. Thus \mathcal{I}^{an} is coherent. \square

Corollary 5.3.6 *Let \mathcal{X} be an A -scheme interiorly locally of finite type, and $i: \mathcal{Y} \rightarrow \mathcal{X}$ be a closed immersion with ideal \mathcal{I} . Then i^{an} is a closed immersion of complex spaces with ideal \mathcal{I}^{an} .*

Proof We first observe that \mathcal{Y} is an A -scheme interiorly locally of finite type.

By [Proposition 5.3.1](#), we have a Cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}^{\text{an}} & \xrightarrow{i^{\text{an}}} & \mathcal{X}^{\text{an}} \\ \downarrow i_{\mathcal{Y}} & \square & \downarrow i_{\mathcal{X}} \\ \mathcal{Y} & \xrightarrow{i} & \mathcal{X} \end{array}$$

In particular, i^{an} is a closed immersion of locally ringed spaces of ideal $i_{\mathcal{X}}^{-1}\mathcal{I} \cdot \mathcal{O}_{\mathcal{X}^{\text{an}}}$. By the flatness of $i_{\mathcal{X}}$, the latter is just \mathcal{I}^{an} . It is coherent by [Proposition 5.3.3](#). Therefore, i^{an} is a closed immersion of complex spaces with ideal \mathcal{I}^{an} . \square

Corollary 5.3.7 *Let X be an affine A -scheme of finite type. Then X^{an} is a Stein space. Further, the following diagram is commutative:*

$$\begin{array}{ccc} X^{\text{an}} & \xrightarrow{i_X} & X \\ & \searrow^{i_{\text{Spec } \Gamma(\mathcal{O}_{X^{\text{an}}})}} & \nearrow \\ & \text{Spec } \Gamma(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}) & \end{array}$$

Proof Write $X = \text{Spec } B$, where B is a finite type A -algebra. Choose a surjection

$$A[t_1, \dots, t_n] \twoheadrightarrow B.$$

It defines a closed immersion

$$X \hookrightarrow \mathbb{A}_A^n = \text{Spec } A[t_1, \dots, t_n].$$

By [Example 5.3.1](#), we have

$$(\mathbb{A}_A^n)^{\text{an}} \simeq S \times \mathbb{C}^n.$$

By [Corollary 5.3.6](#), the induced morphism

$$X^{\text{an}} \hookrightarrow S \times \mathbb{C}^n$$

is a closed immersion of complex spaces. Since $S \times \mathbb{C}^n$ is Stein and closed analytic subspaces of Stein spaces are Stein, X^{an} is Stein.

It remains to verify the commutativity of the diagram. Put

$$C = \Gamma(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}).$$

Since X^{an} is Stein, we have the canonical morphism

$$i_{\text{Spec } C}: X^{\text{an}} \longrightarrow \text{Spec } C.$$

The morphism $i_X: X^{\text{an}} \rightarrow X = \text{Spec } B$ induces an A -algebra homomorphism

$$\varphi: B = \Gamma(X, \mathcal{O}_X) \longrightarrow C = \Gamma(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}).$$

Let

$$\rho: \text{Spec } C \longrightarrow \text{Spec } B = X$$

be the morphism induced by φ . Then the composite

$$\rho \circ i_{\text{Spec } C}: X^{\text{an}} \longrightarrow X$$

is the morphism corresponding to the homomorphism $\varphi: B \rightarrow C$. But i_X also corresponds to the same homomorphism φ . Hence

$$i_X = \rho \circ i_{\text{Spec } C},$$

which is exactly the commutativity of the displayed diagram. \square

Lemma 5.3.2 *Let \mathcal{X} be an A -scheme interiorly locally of finite type. For every $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{F} , one has*

$$\mathrm{Supp}(\mathcal{F}^{\mathrm{an}}) = i_{\mathcal{X}}^{-1}(\mathrm{Supp} \mathcal{F}).$$

Proof Let $x \in \mathcal{X}^{\mathrm{an}}$. By definition,

$$(\mathcal{F}^{\mathrm{an}})_x = \mathcal{F}_{i_{\mathcal{X}}(x)} \otimes_{\mathcal{O}_{\mathcal{X}, i_{\mathcal{X}}(x)}} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}, x}.$$

But $\mathcal{O}_{\mathcal{X}, i_{\mathcal{X}}(x)} \rightarrow \mathcal{O}_{\mathcal{X}^{\mathrm{an}}, x}$ is faithfully flat by [Theorem 5.3.1](#). Thus $\mathcal{F}_x^{\mathrm{an}} = 0$ if and only if $\mathcal{F}_{i_{\mathcal{X}}(x)} = 0$. \square

Lemma 5.3.3 *Let $K \subseteq S$ be a Stein compact subset. Assume that \mathcal{X} is an A -scheme interiorly of finite type. Consider a constructible subset $\mathcal{T} \subseteq |\mathcal{X}|$. Then the following are equivalent:*

- (1) *there is a Stein open neighborhood $U \subseteq S$ of K such that the inverse image of \mathcal{T} with respect to $\mathcal{X}_U^{\mathrm{an}} \rightarrow \mathcal{X}$ is empty;*
- (2) *the inverse image of \mathcal{T} with respect to $\mathcal{X}_K \rightarrow \mathcal{X}$ is empty.*

Proof We may shrink S around K . Indeed, replacing S by a Stein open neighborhood S_0 of K , and replacing \mathcal{X} by \mathcal{X}_{S_0} and \mathcal{T} by its inverse image in \mathcal{X}_{S_0} , does not change condition (2), and condition (1) is unchanged because it only asks for the existence of a smaller open neighborhood of K .

Thus, by [Lemma 5.2.1](#) and [Corollary 5.2.1](#), we may assume that the structure morphism

$$f: \mathcal{X} \longrightarrow \mathrm{Spec} A$$

is of finite presentation. By Chevalley's theorem [[Stacks, Tag 054J](#)], the image $f(\mathcal{T}) \subseteq \mathrm{Spec} A$ is constructible.

We first reduce the assertion to the case $\mathcal{X} = \mathrm{Spec} A$. We claim that the two conditions for $\mathcal{T} \subseteq |\mathcal{X}|$ are equivalent to the corresponding two conditions for the constructible subset $f(\mathcal{T}) \subseteq \mathrm{Spec} A$.

For condition (2), this follows from the usual set-theoretic behaviour of scheme-theoretic fiber products. Indeed, in the Cartesian square

$$\begin{array}{ccc} \mathcal{X}_K & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow f \\ \mathrm{Spec} \Gamma(K, \mathcal{O}_S) & \longrightarrow & \mathrm{Spec} A, \end{array}$$

the inverse image of \mathcal{T} in \mathcal{X}_K is empty if and only if the inverse image of $f(\mathcal{T})$ in $\mathrm{Spec} \Gamma(K, \mathcal{O}_S)$ is empty.

For condition (1), let $U \subseteq S$ be a Stein open neighborhood of K . We claim that the inverse image of \mathcal{T} in $\mathcal{X}_U^{\mathrm{an}}$ is empty if and only if the inverse image of $f(\mathcal{T})$ in U is empty. One implication is immediate. Conversely, suppose that there exists $s \in U$ whose image in $\mathrm{Spec} A$ lies in $f(\mathcal{T})$. Then the constructible subset

$$\mathcal{T}_S := \mathcal{T} \times_{\text{Spec } A} \text{Spec } \mathbb{C}_S \subseteq |\mathcal{X}_S|$$

is non-empty. Since \mathcal{X}_S is a finite type scheme over \mathbb{C} , it is Jacobson; hence every non-empty constructible subset of $|\mathcal{X}_S|$ contains a closed point. Thus \mathcal{T}_S contains a closed point x . As \mathbb{C} is algebraically closed, x gives a point of the analytic space $\mathcal{X}_S^{\text{an}} \subseteq \mathcal{X}_U^{\text{an}}$, and this point maps to \mathcal{T} . Hence the inverse image of \mathcal{T} in $\mathcal{X}_U^{\text{an}}$ is non-empty. This proves the claim.

Therefore it suffices to prove the assertion in the special case $\mathcal{X} = \text{Spec } A$.

Assume now that $\mathcal{X} = \text{Spec } A$, and let $\mathcal{T} \subseteq \text{Spec } A$ be constructible. We need to prove the equivalence of:

- (1) there exists a Stein open neighborhood $U \subseteq S$ of K such that the inverse image of \mathcal{T} under $U \rightarrow \text{Spec } A$ is empty;
- (2) the inverse image of \mathcal{T} under $\text{Spec } \Gamma(K, \mathcal{O}_S) \rightarrow \text{Spec } A$ is empty.

Since \mathcal{T} is constructible, it is a finite union of subsets of the form

$$V(\mathfrak{a}_i) \cap \mathbb{D}(g_i),$$

where each $\mathfrak{a}_i \subseteq A$ is finitely generated and $g_i \in A$. Since we are testing emptiness of a finite union, it is enough to treat one summand: after treating each summand, we take the intersection of the finitely many neighborhoods obtained. Thus we may assume that

$$\mathcal{T} = V(\mathfrak{a}) \cap \mathbb{D}(g),$$

where $\mathfrak{a} = (h_1, \dots, h_m)$ is a finitely generated ideal of A and $g \in A$.

Put

$$A_K := \Gamma(K, \mathcal{O}_S).$$

The inverse image of \mathcal{T} in $\text{Spec } A_K$ is

$$V(\mathfrak{a}A_K) \cap \mathbb{D}(g) \subseteq \text{Spec } A_K.$$

Hence condition (2) is equivalent to

$$g \in \sqrt{\mathfrak{a}A_K}.$$

On the other hand, for a Stein open neighborhood $U \subseteq S$ of K , the inverse image of \mathcal{T} in U is

$$\{s \in U : h_1(s) = \dots = h_m(s) = 0, g(s) \neq 0\}.$$

By Rückert's Nullstellensatz, this set is empty if and only if

$$g|_U \in \sqrt{\mathfrak{a}\mathcal{O}_U}.$$

Therefore condition (1) is equivalent to the existence of a Stein open neighborhood $U \subseteq S$ of K such that

$$g|_U \in \sqrt{\mathfrak{a}\mathcal{O}_U}.$$

It remains to compare this condition with $g \in \sqrt{\mathfrak{a}A_K}$. If $g|_U \in \sqrt{\mathfrak{a}\mathcal{O}_U}$ for some such U , then after restricting to K we immediately get $g \in \sqrt{\mathfrak{a}A_K}$.

Conversely, assume that $g \in \sqrt{\mathfrak{a}A_K}$. Then there exists $N \geq 1$ such that

$$g^N \in \mathfrak{a}A_K.$$

Since $A_K = \Gamma(K, \mathcal{O}_S)$ is the ring of germs of holomorphic functions along K , this means that, after replacing U by a sufficiently small Stein open neighborhood of K , there exist

$$\varphi_1, \dots, \varphi_m \in \Gamma(U, \mathcal{O}_U)$$

such that

$$g^N = \sum_{j=1}^m h_j \varphi_j \quad \text{in } \Gamma(U, \mathcal{O}_U).$$

Hence $g|_U \in \sqrt{\mathfrak{a}\mathcal{O}_U}$. This proves condition (1), and the lemma follows. \square

5.4 Properties of the analytification functor

Let S be a Stein space with associated Stein algebra A .

5.4.1 Analytification of modules

Theorem 5.4.1 *Let X be an A -scheme interiorly locally of finite type, and let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let (Q) be one of the properties of finite modules over noetherian local rings listed in [Section 5.1.2](#). If \mathcal{F} is an ideal sheaf, then (Q) may also be one of the properties of pairs (R, I) listed in [Section 5.1.3](#). Then, for every excellent Stein compact subset $K \subseteq S$, after replacing S by a Stein open neighborhood of K and replacing X and \mathcal{F} by the corresponding base changes, there exists a closed locally constructible subset $\mathcal{T} \subseteq |X|$ such that:*

- (1) *for every excellent Stein compact subset $K' \subseteq S$ with $K \subseteq K'$, the inverse image of \mathcal{T} in $X_{K'}$ is precisely the set of points where $\mathcal{F}_{K'}$ fails to satisfy (Q) ;*
- (2) *the inverse image of \mathcal{T} in X^{an} is precisely the set of points where \mathcal{F}^{an} fails to satisfy (Q) .*

If X is interiorly quasi-compact, then \mathcal{T} may be chosen constructible.

Proof Fix the property (Q) . For every excellent Stein compact subset $E \subseteq S$, define

$$\mathcal{T}_E := \{x \in |X_E| : \mathcal{F}_{E,x} \text{ fails to satisfy } (Q)\}.$$

If \mathcal{F} is an ideal sheaf and (Q) is one of the properties of pairs, the phrase means that the pair $(\mathcal{O}_{X_{E,x}}, \mathcal{F}_{E,x})$ fails to satisfy (Q) . The ring $\Gamma(E, \mathcal{O}_S)$ is excellent by

Theorem 4.4.2. By [Lemma 5.2.1](#), the scheme \mathcal{X}_E is locally of finite type over $\Gamma(E, \mathcal{O}_S)$, hence is locally noetherian, and \mathcal{F}_E is coherent. Thus \mathcal{T}_E is closed and locally constructible by [Proposition 5.1.2](#) in the module case, and by [Proposition 5.1.4](#) in the ideal-pair case. We next record the compatibility of the sets \mathcal{T}_E under restriction. Let $E' \subseteq E \subseteq S$ be excellent Stein compact subsets. By [Corollary 4.4.7](#), the homomorphism

$$\Gamma(E, \mathcal{O}_S) \longrightarrow \Gamma(E', \mathcal{O}_S)$$

is regular. Hence the morphism $\mathcal{X}_{E'} \rightarrow \mathcal{X}_E$ is regular, and all induced local homomorphisms are regular local homomorphisms. Therefore, by [Proposition 5.1.1](#) in the module case, and by [Proposition 5.1.3](#) in the ideal-pair case, a point of $\mathcal{X}_{E'}$ lies in $\mathcal{T}_{E'}$ if and only if its image in \mathcal{X}_E lies in \mathcal{T}_E . Hence

$$\mathcal{T}_{E'} = \mathcal{X}_{E'} \times_{\mathcal{X}_E} \mathcal{T}_E$$

as subsets of $|\mathcal{X}_{E'}|$. Choose, by [Proposition 4.4.6](#), an excellent Stein compact neighborhood L of K in S , and choose a Stein open neighborhood U of K such that $U \subseteq L$. Let \mathcal{T} be the inverse image of \mathcal{T}_L under the morphism $\mathcal{X}_U \rightarrow \mathcal{X}_L$. Then \mathcal{T} is closed. It is locally constructible by [[Stacks, Tag 054I](#)]. Replacing S by U , and replacing \mathcal{X} and \mathcal{F} by \mathcal{X}_U and \mathcal{F}_U , we regard \mathcal{T} as a subset of $|\mathcal{X}|$. Let $K' \subseteq S$ be an excellent Stein compact subset with $K \subseteq K'$. Since, after the replacement, $S = U \subseteq L$, we have $K' \subseteq L$. The compatibility just proved gives

$$\mathcal{T}_{K'} = \mathcal{X}_{K'} \times_{\mathcal{X}_L} \mathcal{T}_L.$$

But the inverse image of $\mathcal{T} \subseteq |\mathcal{X}| = |\mathcal{X}_U|$ in $\mathcal{X}_{K'}$ is the same fibre product. This proves (1). It remains to prove (2). Let $x \in \mathcal{X}^{\text{an}}$, and let $z \in \mathcal{X}_L$ be its image under the composition

$$\mathcal{X}^{\text{an}} \longrightarrow \mathcal{X} = \mathcal{X}_U \longrightarrow \mathcal{X}_L.$$

The local homomorphism

$$\mathcal{O}_{\mathcal{X}_L, z} \longrightarrow \mathcal{O}_{\mathcal{X}^{\text{an}}, x}$$

is regular. Indeed, this is local on \mathcal{X}_L ; after embedding an affine neighborhood of z into an affine space over $\Gamma(L, \mathcal{O}_S)$, it follows from [Proposition 4.4.4](#) by the same local computation as in the proof of [Theorem 5.3.1](#). Hence, by [Proposition 5.1.1](#) in the module case, and by [Proposition 5.1.3](#) in the ideal-pair case, $\mathcal{F}_x^{\text{an}}$ fails to satisfy (Q) if and only if $\mathcal{F}_{L, z}$ fails to satisfy (Q). Equivalently,

$$x \in \mathcal{X}^{\text{an}} \times_{\mathcal{X}_L} \mathcal{T}_L.$$

Since \mathcal{T} is the inverse image of \mathcal{T}_L in $\mathcal{X} = \mathcal{X}_U$, this is precisely the inverse image of \mathcal{T} in \mathcal{X}^{an} . This proves (2). Finally assume that \mathcal{X} is interiorly quasi-compact. Then after shrinking S , we may assume that it is quasi-compact. Then \mathcal{X}_L is a noetherian scheme, so the closed subset \mathcal{T}_L is constructible. Therefore its inverse image \mathcal{T} in \mathcal{X}_U is constructible by [[Stacks, Tag 054I](#)]. \square

Corollary 5.4.1 *Let X be an A -scheme interiorly of finite type, let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type, and let $K \subseteq S$ be an excellent Stein compact subset. Let (Q) be one of the properties of finite modules over noetherian local rings listed in [Section 5.1.2](#). If \mathcal{F} is an ideal sheaf, then (Q) may also be one of the properties of pairs (R, I) listed in [Section 5.1.3](#). Then the following are equivalent:*

- (1) *the base change \mathcal{F}_K on X_K satisfies (Q) everywhere;*
- (2) *there exists a Stein open neighborhood $U \subseteq S$ of K such that the analytification $\mathcal{F}_U^{\text{an}}$ on X_U^{an} satisfies (Q) everywhere.*

Proof Applying [Theorem 5.4.1](#) to X , \mathcal{F} , K , and (Q) , we get a constructible subset $\mathcal{T} \subseteq |X|$, since X is interiorly of finite type. After the harmless shrinking allowed in [Theorem 5.4.1](#), the inverse image of \mathcal{T} in X_K is precisely the failure locus of (Q) for \mathcal{F}_K , and the inverse image of \mathcal{T} in X^{an} is precisely the failure locus of (Q) for \mathcal{F}^{an} . Thus (1) says that the inverse image of \mathcal{T} in X_K is empty, while (2) says that, after possibly shrinking S to a Stein open neighborhood of K , the inverse image of \mathcal{T} in X^{an} is empty. These two conditions are equivalent by [Lemma 5.3.3](#). \square

Lemma 5.4.1 *Let X be an interiorly locally of finite type A -scheme. Let $\mathcal{T} \subseteq |X|$ be a locally constructible subset, and let T be its inverse image in X^{an} . Then*

$$\overline{T} = i_X^{-1}(\overline{\mathcal{T}}),$$

where the closure on the left is taken in X^{an} and the closure on the right is taken in $|X|$.

Proof The inclusion

$$\overline{T} \subseteq i_X^{-1}(\overline{\mathcal{T}})$$

follows immediately from the continuity of i_X . We prove the reverse inclusion locally on S and on X . We shall use the following standard fact: if $g: Y \rightarrow X$ is flat and $C \subseteq |X|$ is locally constructible, then

$$g^{-1}(\overline{C}) = \overline{g^{-1}(C)}.$$

This is [[EGA IV₂](#), Théorème 2.3.10]. Since $A \rightarrow \Gamma(U, \mathcal{O}_S)$ is flat for every Stein open subset $U \subseteq S$ by [Theorem 3.3.4](#), this fact shows that the desired equality may be checked after replacing S by Stein open subsets. Similarly, since closure commutes with restriction to open subsets, the assertion is local on X . Thus we may work locally on X . Fix an excellent Stein compact subset $K \subseteq S$. By [Proposition 4.4.6](#), after replacing S by a Stein open neighborhood of K , we may choose an excellent Stein compact subset $L \subseteq S$ with $K \subseteq \text{Int}(L)$. Put

$$A_L = \Gamma(L, \mathcal{O}_S).$$

Let $X_L = X \times_{\text{Spec } A} \text{Spec } A_L$, and let \mathcal{T}_L be the inverse image of \mathcal{T} in X_L . By the flatness of $A \rightarrow A_L$ and the preceding paragraph,

$$\overline{\mathcal{T}_L} = X_L \times_X \overline{\mathcal{T}}$$

as subsets of $|\mathcal{X}_L|$. Hence it is enough to prove the corresponding assertion for $\mathcal{T}_L \subseteq |\mathcal{X}_L|$ after shrinking around K . By [Lemma 5.2.1](#), the scheme \mathcal{X}_L is locally of finite type over the noetherian ring A_L , hence is locally noetherian. Since the assertion is local on \mathcal{X}_L , we may assume that \mathcal{X}_L is affine and that \mathcal{T}_L is constructible. Write

$$\mathcal{T}_L = \bigcup_{j=1}^r \mathcal{T}_j$$

as a finite union of locally closed subsets such that each \mathcal{T}_j is open and dense in its closure

$$\mathcal{Z}_j := \overline{\mathcal{T}_j}.$$

Endow \mathcal{Z}_j with the reduced induced scheme structure. Since inverse images and closures commute with finite unions, it suffices to prove the assertion for each pair $(\mathcal{Z}_j, \mathcal{T}_j)$. Replacing \mathcal{X}_L by \mathcal{Z}_j , we are reduced to the case where \mathcal{T}_L is open and dense in \mathcal{X}_L . Put

$$\mathcal{H}_L := \mathcal{X}_L \setminus \mathcal{T}_L.$$

Then \mathcal{H}_L is a closed subset of \mathcal{X}_L containing no irreducible component. Let $\mathcal{I}_L \subseteq \mathcal{O}_{\mathcal{X}_L}$ be the coherent ideal sheaf defining the reduced closed subscheme with underlying set \mathcal{H}_L . Since \mathcal{H}_L contains no irreducible component of the noetherian scheme \mathcal{X}_L , the ideal $(\mathcal{I}_L)_z$ has height at least 1 for every $z \in \mathcal{X}_L$, with the usual convention that the unit ideal has infinite height. By [Corollary 4.4.7](#), the homomorphism

$$A_L \longrightarrow \Gamma(K, \mathcal{O}_S)$$

is regular. Therefore $\mathcal{X}_K \rightarrow \mathcal{X}_L$ is regular, and all induced local homomorphisms are regular local homomorphisms between excellent noetherian local rings. By [Proposition 5.1.3](#), applied to property (7) in [Section 5.1.3](#) with $n = 1$, the ideal $(\mathcal{I}_L)_K$ has height at least 1 at every point of \mathcal{X}_K . Applying [Corollary 5.4.1](#) to the ideal sheaf $(\mathcal{I}_L)_{\text{Int}(L)}$ and to property (7) of [Section 5.1.3](#) with $n = 1$, we may shrink S to a Stein open neighborhood U of K contained in $\text{Int}(L)$ such that the analytified ideal $\mathcal{I}_U^{\text{an}}$ has height at least 1 at every point of $\mathcal{X}_U^{\text{an}}$. Hence the closed analytic subset

$$H_U := \mathcal{H}_L \times_{\mathcal{X}_L} \mathcal{X}_U^{\text{an}}$$

contains no irreducible component of $\mathcal{X}_U^{\text{an}}$, and therefore has empty interior. Equivalently, the inverse image of \mathcal{T}_L in $\mathcal{X}_U^{\text{an}}$ is dense. Thus, after shrinking around every excellent Stein compact subset $K \subseteq S$, the inverse image of \mathcal{T}_L is dense in the inverse image of $\overline{\mathcal{T}_L}$. Since such neighborhoods cover S , this proves

$$i_{\mathcal{X}}^{-1}(\overline{\mathcal{T}}) \subseteq \overline{\mathcal{T}}.$$

Together with the first inclusion, the lemma follows. \square

Corollary 5.4.2 *Let \mathcal{X} be an A -scheme interiorly of finite type, let $\mathcal{T} \subseteq |\mathcal{X}|$ be a constructible subset, and let $K \subseteq S$ be a Stein compact subset. Then the following are equivalent:*

- (1) the inverse image of \mathcal{T} in \mathcal{X}_K is dense, respectively closed, respectively open;
 (2) there exists a Stein open neighborhood $U \subseteq S$ of K such that the inverse image of \mathcal{T} in $\mathcal{X}_U^{\text{an}}$ is dense, respectively closed, respectively open.

Proof In view of [Stacks, Tag 054I], after shrinking S , we may assume that \mathcal{X} is an A -scheme of finite presentation, see Corollary 5.2.1.

Put

$$\mathcal{T}_K := \mathcal{X}_K \times_{\mathcal{X}} \mathcal{T}, \quad T_U := (\mathcal{X}_U^{\text{an}}) \times_{\mathcal{X}} \mathcal{T}.$$

We first treat density. Since \mathcal{T} is constructible and \mathcal{X} is quasi-compact, the subset

$$C := |\mathcal{X}| \setminus \overline{\mathcal{T}}$$

is constructible. The morphism $\mathcal{X}_K \rightarrow \mathcal{X}$ is flat by Proposition 4.3.4. Hence, by the compatibility of flat pullback with closures of constructible subsets, [EGA IV₂, Théorème 2.3.10], we have

$$\overline{\mathcal{T}_K} = \mathcal{X}_K \times_{\mathcal{X}} \overline{\mathcal{T}}.$$

Therefore \mathcal{T}_K is dense in \mathcal{X}_K if and only if the inverse image of C in \mathcal{X}_K is empty. On the other hand, by Lemma 5.4.1, for every Stein open neighborhood U of K we have

$$\overline{T_U} = (\mathcal{X}_U^{\text{an}}) \times_{\mathcal{X}} \overline{\mathcal{T}}.$$

Thus T_U is dense in $\mathcal{X}_U^{\text{an}}$ if and only if the inverse image of C in $\mathcal{X}_U^{\text{an}}$ is empty. The equivalence for density follows from Lemma 5.3.3 applied to C . We next treat closedness. Since \mathcal{T} is constructible and \mathcal{X} is quasi-compact, the subset

$$\mathcal{D} := \overline{\mathcal{T}} \setminus \mathcal{T}$$

is constructible. By the same flat-closure compatibility as above, \mathcal{T}_K is closed in \mathcal{X}_K if and only if the inverse image of \mathcal{D} in \mathcal{X}_K is empty. By Lemma 5.4.1, T_U is closed in $\mathcal{X}_U^{\text{an}}$ if and only if the inverse image of \mathcal{D} in $\mathcal{X}_U^{\text{an}}$ is empty. The equivalence for closedness follows again from Lemma 5.3.3. Finally, openness follows by applying the closedness assertion to the constructible subset

$$|\mathcal{X}| \setminus \mathcal{T}.$$

Indeed, the inverse image of \mathcal{T} is open if and only if the inverse image of its complement is closed, both on \mathcal{X}_K and on $\mathcal{X}_U^{\text{an}}$. \square

5.4.2 Analytification of morphisms

Let S be a Stein space and let A be the corresponding Stein algebra.

Theorem 5.4.2 *Let $K \subseteq S$ be a Stein compact subset. Consider A -schemes \mathcal{X} and \mathcal{Y} interiorly of finite type, and a morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of A -schemes. Let (Q) be one*

of the properties of morphisms of schemes and morphisms of complex spaces listed in [Section 5.1.4](#). Then the following are equivalent:

- (1) $f_K : \mathcal{X}_K \rightarrow \mathcal{Y}_K$ satisfies (Q);
- (2) there is an open neighborhood $U \subseteq S$ of K such that $f_U^{\text{an}} : \mathcal{X}_U^{\text{an}} \rightarrow \mathcal{Y}_U^{\text{an}}$ satisfies (Q).

Proof Fix the property (Q). We first make a reduction. By [Proposition 4.4.6](#), after replacing S by a Stein open neighborhood S' of K , we may choose an excellent Stein compact subset L in the original base such that

$$K \subseteq S \subseteq \text{Int}(L).$$

Put

$$A_L = \Gamma(L, \mathcal{O}_S).$$

By [Lemma 5.2.1](#) and [Corollary 5.2.1](#), the schemes \mathcal{X}_L and \mathcal{Y}_L are of finite presentation over A_L . Since A_L is noetherian and $\mathcal{X}_L, \mathcal{Y}_L$ are of finite presentation over A_L , the morphism

$$f_L : \mathcal{X}_L \longrightarrow \mathcal{Y}_L$$

is of finite presentation. We shall compare both f_K and f^{an} with this finite-presentation model f_L . We shall use the following consequence of [Lemma 5.3.3](#) and [Corollary 5.4.2](#). Let \mathcal{Z}_L be a quasi-compact A_L -scheme of finite presentation, and let $C \subseteq |\mathcal{Z}_L|$ be constructible. Then the inverse image of C in \mathcal{Z}_K is empty if and only if, after possibly shrinking S around K , the inverse image of C in \mathcal{Z}^{an} is empty. Similarly, the inverse image of C in \mathcal{Z}_K is dense, closed, or open if and only if, after possibly shrinking S around K , the inverse image of C in \mathcal{Z}^{an} is respectively dense, closed, or open. We also use repeatedly that analytification commutes with fibre products by [Proposition 5.3.2](#).

(1)–(4) Let $\mathcal{B}_L \subseteq |\mathcal{X}_L|$ be the locus of points at which f_L fails to satisfy (Q) in every open neighborhood. By [Proposition 5.1.5](#), \mathcal{B}_L is Zariski closed and its inverse image in \mathcal{X}_K is exactly the corresponding failure locus for f_K . On the analytic side, the inverse image of \mathcal{B}_L in \mathcal{X}^{an} is exactly the corresponding failure locus for f^{an} . Indeed, this follows from the local criteria for smoothness, unramifiedness, étaleness, and flatness, together with the regularity of the local homomorphisms

$$\mathcal{O}_{\mathcal{X}_L, i_{\mathcal{X}}(x)} \longrightarrow \mathcal{O}_{\mathcal{X}^{\text{an}}, x}$$

from [Theorem 5.3.1](#). Hence f_K satisfies (Q) everywhere if and only if the inverse image of \mathcal{B}_L in \mathcal{X}_K is empty, and f^{an} satisfies (Q) everywhere if and only if the inverse image of \mathcal{B}_L in \mathcal{X}^{an} is empty. The assertion follows from [Lemma 5.3.3](#).

(5) Let

$$C_L = |\mathcal{Y}_L| \setminus f_L(|\mathcal{X}_L|).$$

Since f_L is of finite presentation, Chevalley's theorem gives that $f_L(|\mathcal{X}_L|)$ is constructible, hence C_L is constructible. In a Cartesian square, the complement of the image pulls back to the complement of the image. Therefore the inverse image of C_L

in \mathcal{Y}_K is precisely

$$|\mathcal{Y}_K| \setminus f_K(|\mathcal{X}_K|).$$

Similarly, using [Proposition 5.3.2](#), the inverse image of C_L in \mathcal{Y}^{an} is precisely

$$|\mathcal{Y}^{\text{an}}| \setminus f^{\text{an}}(|\mathcal{X}^{\text{an}}|).$$

Thus f_K is surjective if and only if the inverse image of C_L in \mathcal{Y}_K is empty, and f^{an} is surjective if and only if the inverse image of C_L in \mathcal{Y}^{an} is empty. The assertion follows from [Lemma 5.3.3](#).

(6) For schemes, a morphism is universally injective if and only if its diagonal is surjective, by [\[Stacks, Tag 01S4\]](#). For complex spaces, universally injective is the same as injective by [Corollary 1.3.1](#), and injectivity is likewise equivalent to surjectivity of the diagonal. Since analytification commutes with fibre products, the diagonal of f^{an} is the analytification of the diagonal of f . Hence the assertion follows from (5) applied to

$$\Delta_f : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.$$

(7) A morphism of schemes is an open immersion if and only if it is étale and universally injective, by [\[Stacks, Tag 02LC\]](#). The corresponding analytic statement says that an injective local biholomorphism identifies the source with an open analytic subspace of the target. Thus the assertion follows from (3) and (6).

(10) A morphism is an isomorphism if and only if it is a surjective open immersion, both for schemes and for complex spaces. Hence the assertion follows from (5) and (7).

(11) A morphism is a monomorphism if and only if its diagonal is an isomorphism, both for schemes and for complex spaces; in the algebraic case this is [\[Stacks, Tag 01L3\]](#). In view of [Corollary 5.3.5](#), the assertion follows from (10) applied to

$$\Delta_f : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.$$

(12) Put

$$\mathcal{Z}_L = \mathcal{X}_L \times_{\mathcal{Y}_L} \mathcal{X}_L,$$

and let

$$\mathcal{D}_L = \Delta_{f_L}(|\mathcal{X}_L|) \subseteq |\mathcal{Z}_L|$$

be the image of the diagonal. Since Δ_{f_L} is of finite presentation, \mathcal{D}_L is constructible by Chevalley's theorem. A morphism of schemes locally of finite presentation is separated if and only if the image of its diagonal is closed. The same criterion holds for morphisms of complex spaces. By [Proposition 5.3.2](#), analytification commutes with the formation of \mathcal{Z}_L and of the diagonal. Moreover, by the same fibre-product argument as in (5), the inverse image of \mathcal{D}_L in \mathcal{Z}_K is the image of Δ_{f_K} , and its inverse image in \mathcal{Z}^{an} is the image of $\Delta_{f^{\text{an}}}$. Hence f_K is separated if and only if the inverse image of \mathcal{D}_L in \mathcal{Z}_K is closed, and f^{an} is separated if and only if the inverse image of \mathcal{D}_L in \mathcal{Z}^{an} is closed. The assertion follows from [Corollary 5.4.2](#).

(15) Let

$$\mathcal{I}_L = f_L(|\mathcal{X}_L|) \subseteq |\mathcal{Y}_L|.$$

This subset is constructible by Chevalley's theorem. Its inverse image in \mathcal{Y}_K is the image of f_K , and its inverse image in \mathcal{Y}^{an} is the image of f^{an} , again by the fibre-product argument used in (5). Therefore f_K is dominant if and only if the inverse image of \mathcal{I}_L in \mathcal{Y}_K is dense, and f^{an} is dominant if and only if the inverse image of \mathcal{I}_L in \mathcal{Y}^{an} is dense. The assertion follows from [Corollary 5.4.2](#).

(16) Let $\mathcal{B}_L \subseteq |\mathcal{X}_L|$ be the complement of the quasi-finite locus of f_L . The quasi-finite locus is open by [\[Stacks, Tag 01TI\]](#), hence \mathcal{B}_L is closed. Since f_L is locally of finite presentation, quasi-finiteness at a point is equivalent to that point being isolated in its fibre. This condition is invariant under extension of the residue field. Therefore the inverse image of \mathcal{B}_L in \mathcal{X}_K is the failure locus for quasi-finiteness of f_K , and the inverse image of \mathcal{B}_L in \mathcal{X}^{an} is the failure locus for quasi-finiteness of f^{an} . The assertion follows from [Lemma 5.3.3](#).

(13) Assume first that f_K is proper. Since f_L is of finite presentation, properness spreads out from K : by [\[EGA IV₃, IV₃, Théorème 8.10.5\]](#), after shrinking S around K , the morphism f is proper. The analytification of a proper morphism of finite presentation is proper. Indeed, by Chow's lemma, one reduces to the projective case, and the projective case follows from the properness of complex projective space over the Stein base. Hence, after shrinking S around K , the morphism f^{an} is proper. Conversely, assume that, after shrinking S around K , the morphism

$$f^{\text{an}}: \mathcal{X}^{\text{an}} \longrightarrow \mathcal{Y}^{\text{an}}$$

is proper. By (12), after possibly shrinking S further, the morphism f_K is separated. Since f_K is of finite presentation, it remains to prove that f_K is universally closed. By [\[EGA IV₃, IV₃, Théorème 8.3.11\]](#), it is enough to verify the following condition: for every $n \geq 0$ and every constructible subset

$$C_L \subseteq |\mathcal{X}_{L,n}|, \quad \mathcal{X}_{L,n} := \mathcal{X}_L \times_{\text{Spec } \mathbb{C}} \mathbb{A}_{\mathbb{C}}^n,$$

whose inverse image $C_K \subseteq |\mathcal{X}_{K,n}|$ is closed, the image of C_K under

$$f_{K,n}: \mathcal{X}_{K,n} \longrightarrow \mathcal{Y}_{K,n}, \quad \mathcal{Y}_{K,n} := \mathcal{Y}_K \times_{\text{Spec } \mathbb{C}} \mathbb{A}_{\mathbb{C}}^n,$$

is closed. Fix such an n and such a constructible subset C_L . By [Corollary 5.4.2](#), after shrinking S around K , the inverse image

$$C^{\text{an}} \subseteq |\mathcal{X}^{\text{an}} \times_{\mathbb{C}} \mathbb{C}^n|$$

of C_L is closed. Since f^{an} is proper, its base change

$$f_n^{\text{an}}: \mathcal{X}^{\text{an}} \times_{\mathbb{C}} \mathbb{C}^n \longrightarrow \mathcal{Y}^{\text{an}} \times_{\mathbb{C}} \mathbb{C}^n$$

is proper, hence closed. Therefore $f_n^{\text{an}}(C^{\text{an}})$ is closed. Let

$$\mathcal{W}_L = f_{L,n}(C_L) \subseteq |\mathcal{Y}_{L,n}|, \quad \mathcal{Y}_{L,n} := \mathcal{Y}_L \times_{\text{Spec } \mathbb{C}} \mathbb{A}_{\mathbb{C}}^n.$$

By Chevalley's theorem, \mathcal{W}_L is constructible. The inverse image of \mathcal{W}_L in $\mathcal{Y}_{K,n}$ is precisely $f_{K,n}(C_K)$, and the inverse image of \mathcal{W}_L in $\mathcal{Y}^{\text{an}} \times_{\mathbb{C}} \mathbb{C}^n$ is precisely $f_n^{\text{an}}(C^{\text{an}})$. Since the latter is closed, [Corollary 5.4.2](#) implies that $f_{K,n}(C_K)$ is closed in $\mathcal{Y}_{K,n}$. Thus the criterion above applies, so f_K is universally closed. Since f_K is separated and of finite presentation, it is proper.

(14) A morphism of schemes is finite if and only if it is proper and locally quasi-finite, by [\[Stacks, Tag 02LS\]](#). The same criterion holds for morphisms of complex spaces. Hence the assertion follows from (13) and (16).

(8) A morphism of schemes is a closed immersion if and only if it is a proper monomorphism; the non-trivial direction is [\[Stacks, Tag 04XV\]](#). The same criterion holds for complex spaces. Hence the assertion follows from (13) and (11).

(9) A morphism locally of finite presentation is an immersion if and only if it is a monomorphism and its image is locally closed. The same criterion holds for morphisms of complex spaces. By (11), the monomorphism condition is equivalent on the algebraic and analytic sides. By Chevalley's theorem, the image

$$\mathcal{I}_L = f_L(|\mathcal{X}_L|) \subseteq |\mathcal{Y}_L|$$

is constructible. Its inverse image in \mathcal{Y}_K is $f_K(|\mathcal{X}_K|)$, and its inverse image in \mathcal{Y}^{an} is $f^{\text{an}}(|\mathcal{X}^{\text{an}}|)$. Since local closedness of a constructible subset is equivalent to being open in its closure, [Corollary 5.4.2](#) gives the equivalence of local closedness for these two inverse images. Thus the assertion follows.

This proves the theorem for all properties listed in [Section 5.1.4](#). \square

Lemma 5.4.2 *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of A -schemes interiorly locally of finite type. Then, for every $p \geq 0$, there is a canonical isomorphism*

$$\left(\Omega_{\mathcal{X}/\mathcal{Y}}^p\right)^{\text{an}} \xrightarrow{\sim} \Omega_{\mathcal{X}^{\text{an}}/\mathcal{Y}^{\text{an}}}^p.$$

Proof We first construct the canonical morphism. Let

$$i_{\mathcal{X}}: \mathcal{X}^{\text{an}} \rightarrow \mathcal{X}, \quad i_{\mathcal{Y}}: \mathcal{Y}^{\text{an}} \rightarrow \mathcal{Y}$$

be the analytification morphisms. The universal derivation

$$d: \mathcal{O}_{\mathcal{X}^{\text{an}}} \rightarrow \Omega_{\mathcal{X}^{\text{an}}/\mathcal{Y}^{\text{an}}}^1$$

induces a derivation

$$i_{\mathcal{X}}^{-1} d: \mathcal{O}_{\mathcal{X}} \rightarrow \Omega_{\mathcal{X}^{\text{an}}/\mathcal{Y}^{\text{an}}}^1.$$

This derivation vanishes on the image of $i_{\mathcal{X}}^{-1} f^{-1} d_{\mathcal{Y}}$, because this image factors through $(f^{\text{an}})^{-1} d_{\mathcal{Y}^{\text{an}}}$. Hence, by the universal property of relative Kähler differentials, it induces a morphism

$$i_{\mathcal{X}}^{-1} \Omega_{\mathcal{X}/\mathcal{Y}}^1 \rightarrow \Omega_{\mathcal{X}^{\text{an}}/\mathcal{Y}^{\text{an}}}^1.$$

After tensoring with $\mathcal{O}_{\mathcal{X}^{\text{an}}}$, we get a canonical morphism

$$\theta_f^1 : \left(\Omega_{\mathcal{X}/\mathcal{Y}}^1 \right)^{\text{an}} = i_{\mathcal{X}}^* \Omega_{\mathcal{X}/\mathcal{Y}}^1 \longrightarrow \Omega_{\mathcal{X}^{\text{an}}/\mathcal{Y}^{\text{an}}}^1.$$

Taking exterior powers gives canonical morphisms

$$\theta_f^p : \left(\Omega_{\mathcal{X}/\mathcal{Y}}^p \right)^{\text{an}} \longrightarrow \Omega_{\mathcal{X}^{\text{an}}/\mathcal{Y}^{\text{an}}}^p$$

for all $p \geq 0$. The case $p = 0$ is tautological. It remains to prove that θ_f^1 is an isomorphism, because analytification is a pull-back functor and therefore commutes with finite tensor products, quotients, and exterior powers. The assertion is local on \mathcal{X}^{an} and on \mathcal{Y}^{an} . By [Corollary 5.3.4](#), it is also local on the Stein base S . Using [Corollary 5.2.1](#), we may therefore replace S by a Stein open subset and assume that \mathcal{X} and \mathcal{Y} are locally of finite presentation over A . Then f is locally of finite presentation. After shrinking \mathcal{X} and \mathcal{Y} Zariski-locally, we may assume that

$$\mathcal{Y} = \text{Spec } B, \quad \mathcal{X} = \text{Spec } C, \quad C = B[T_1, \dots, T_n]/J$$

with J finitely generated. Put

$$\mathcal{P} = \mathbb{A}_{\mathcal{Y}}^n = \text{Spec } B[T_1, \dots, T_n].$$

Then $\mathcal{X} \hookrightarrow \mathcal{P}$ is a closed immersion. Let $\mathcal{J} \subseteq \mathcal{O}_{\mathcal{P}}$ be its defining ideal. The conormal exact sequence gives

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{P}/\mathcal{Y}}^1 \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_{\mathcal{X}} \longrightarrow \Omega_{\mathcal{X}/\mathcal{Y}}^1 \longrightarrow 0.$$

Since

$$\Omega_{\mathcal{P}/\mathcal{Y}}^1 \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathcal{P}} dT_i,$$

analytifying this exact sequence gives an exact sequence

$$\left(\mathcal{J}/\mathcal{J}^2 \right)^{\text{an}} \longrightarrow \left(\Omega_{\mathcal{P}/\mathcal{Y}}^1 \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_{\mathcal{X}} \right)^{\text{an}} \longrightarrow \left(\Omega_{\mathcal{X}/\mathcal{Y}}^1 \right)^{\text{an}} \longrightarrow 0.$$

Here exactness follows from the flatness of $i_{\mathcal{X}}$ in [Theorem 5.3.1](#). By [Proposition 5.3.2](#), we have

$$\mathcal{P}^{\text{an}} \cong \mathbb{A}_{\mathcal{Y}^{\text{an}}}^n.$$

Moreover, by [Corollary 5.3.6](#), the closed immersion

$$\mathcal{X}^{\text{an}} \hookrightarrow \mathcal{P}^{\text{an}}$$

is defined by the ideal \mathcal{J}^{an} . Hence

$$\left(\mathcal{J}/\mathcal{J}^2 \right)^{\text{an}} \cong \mathcal{J}^{\text{an}}/(\mathcal{J}^{\text{an}})^2.$$

Also,

$$\left(\Omega_{\mathcal{P}/\mathcal{Y}}^1 \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_{\mathcal{X}}\right)^{\text{an}} \cong \Omega_{\mathcal{P}^{\text{an}}/\mathcal{Y}^{\text{an}}}^1 \otimes_{\mathcal{O}_{\mathcal{P}^{\text{an}}}} \mathcal{O}_{\mathcal{X}^{\text{an}}},$$

because both sides are canonically isomorphic to

$$\bigoplus_{i=1}^n \mathcal{O}_{\mathcal{X}^{\text{an}}} dT_i.$$

On the analytic side, the conormal exact sequence for

$$\mathcal{X}^{\text{an}} \hookrightarrow \mathcal{P}^{\text{an}}$$

is

$$\mathcal{J}^{\text{an}}/(\mathcal{J}^{\text{an}})^2 \longrightarrow \Omega_{\mathcal{P}^{\text{an}}/\mathcal{Y}^{\text{an}}}^1 \otimes_{\mathcal{O}_{\mathcal{P}^{\text{an}}}} \mathcal{O}_{\mathcal{X}^{\text{an}}} \longrightarrow \Omega_{\mathcal{X}^{\text{an}}/\mathcal{Y}^{\text{an}}}^1 \longrightarrow 0.$$

Under the identifications above, the first arrows in the analytified algebraic conormal sequence and in the analytic conormal sequence agree: both send the class of a local section $h \in \mathcal{J}$ to dh . Therefore their cokernels are canonically identified. This says exactly that

$$\theta_f^1 : \left(\Omega_{\mathcal{X}/\mathcal{Y}}^1\right)^{\text{an}} \xrightarrow{\sim} \Omega_{\mathcal{X}^{\text{an}}/\mathcal{Y}^{\text{an}}}^1$$

is an isomorphism. Taking exterior powers gives the assertion for every $p \geq 0$. \square

5.4.3 Cohomology comparison

Let S be a Stein space and A be the corresponding Stein algebra.

Theorem 5.4.3 *Let*

$$f: \mathcal{X} \longrightarrow \mathcal{Y}$$

be an interiorly proper morphism between A -schemes interiorly locally of finite type, and let \mathcal{F} be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -module. Then, for every $p \geq 0$, the canonical morphism

$$(\mathbf{R}^p f_* \mathcal{F})^{\text{an}} \longrightarrow \mathbf{R}^p f_*^{\text{an}} \mathcal{F}^{\text{an}} \tag{5.7}$$

is an isomorphism.

Proof Step 1. Construction of the comparison morphism. Consider the commutative square

$$\begin{array}{ccc} \mathcal{X}^{\text{an}} & \xrightarrow{f^{\text{an}}} & \mathcal{Y}^{\text{an}} \\ i_{\mathcal{X}} \downarrow & & \downarrow i_{\mathcal{Y}} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

For every quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{F} , adjunction gives a natural morphism

$$i_{\mathcal{Y}}^{-1} f_* \mathcal{F} \longrightarrow f_*^{\text{an}} i_{\mathcal{X}}^{-1} \mathcal{F}.$$

Tensoring with $\mathcal{O}_{\mathcal{Y}^{\text{an}}}$ over $i_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{Y}}$ gives a functorial morphism

$$(f_*\mathcal{F})^{\text{an}} = i_{\mathcal{Y}}^* f_*\mathcal{F} \longrightarrow f_*^{\text{an}} i_{\mathcal{X}}^* \mathcal{F} = f_*^{\text{an}} \mathcal{F}^{\text{an}}.$$

Since $i_{\mathcal{Y}}$ is flat by [Theorem 5.3.1](#), the functor $i_{\mathcal{Y}}^*$ is exact. Passing to right derived functors gives the morphism [\(5.7\)](#). This is the usual base-change morphism associated with the displayed square.

Step 2. Local reductions and reduction to finitely presented sheaves. The assertion is local on \mathcal{Y}^{an} and local on the Stein base S . Thus, by the definition of interior properness and by [Corollary 5.2.1](#), we may replace S by Stein open subsets and assume that f is proper and that \mathcal{X} and \mathcal{Y} are locally of finite presentation over A . The morphism f^{an} is then proper by [Theorem 5.4.2](#). The assertion is also local on \mathcal{Y} , so we may assume that \mathcal{Y} is affine. Then \mathcal{X} is quasi-compact and quasi-separated. Write

$$\mathcal{F} = \varinjlim_{i \in I} \mathcal{F}_i$$

as a filtered colimit of finitely presented $\mathcal{O}_{\mathcal{X}}$ -modules, using [\[Stacks, Tag 01PJ\]](#). Since f is quasi-compact and quasi-separated, cohomology of quasi-coherent sheaves commutes with filtered colimits, so

$$\mathbf{R}^p f_* \mathcal{F} \cong \varinjlim_i \mathbf{R}^p f_* \mathcal{F}_i$$

by [\[Stacks, Tag 07TB\]](#). Flatness of analytification gives

$$(\mathbf{R}^p f_* \mathcal{F})^{\text{an}} \cong \varinjlim_i (\mathbf{R}^p f_* \mathcal{F}_i)^{\text{an}}.$$

Moreover,

$$\mathcal{F}^{\text{an}} = \varinjlim_i \mathcal{F}_i^{\text{an}}.$$

Since f^{an} is proper and hence quasi-compact with finite cohomological dimension locally on the target, [Proposition 2.3.1](#) gives

$$\mathbf{R}^p f_*^{\text{an}} \mathcal{F}^{\text{an}} \cong \varinjlim_i \mathbf{R}^p f_*^{\text{an}} \mathcal{F}_i^{\text{an}}.$$

The comparison morphism is compatible with these colimit identifications. Hence it suffices to prove the theorem when \mathcal{F} is finitely presented. In this case \mathcal{F}^{an} is coherent by [Proposition 5.3.3](#).

Step 3. Reduction to the case $\mathcal{Y} = \text{Spec } A$. Assume first that the theorem has been proved when the target is the Stein base. We prove the theorem for affine \mathcal{Y} . Write

$$\mathcal{Y} = \text{Spec } B.$$

Then \mathcal{Y}^{an} is Stein by [Corollary 5.3.7](#). Put

$$C = \Gamma(\mathcal{Y}^{\text{an}}, \mathcal{O}_{\mathcal{Y}^{\text{an}}}).$$

Let

$$\mathcal{X}_C = \mathcal{X} \times_{\text{Spec } B} \text{Spec } C, \quad f_C: \mathcal{X}_C \longrightarrow \text{Spec } C$$

be the base change, and let \mathcal{F}_C be the pull-back of \mathcal{F} . By [Lemma 5.3.1](#), the analytification of \mathcal{X}_C as a C -scheme is naturally identified with \mathcal{X}^{an} over \mathcal{Y}^{an} . Thus the theorem for the morphism f_C gives

$$(\mathbf{R}^p f_{C*} \mathcal{F}_C)^{\text{an}} \xrightarrow{\sim} \mathbf{R}^p f_*^{\text{an}} \mathcal{F}^{\text{an}}.$$

It remains to compare $\mathbf{R}^p f_{C*} \mathcal{F}_C$ with the pull-back of $\mathbf{R}^p f_* \mathcal{F}$. Let $K \subseteq \mathcal{Y}^{\text{an}}$ be an excellent Stein compact subset and put

$$C_K = \Gamma(K, \mathcal{O}_{\mathcal{Y}^{\text{an}}}).$$

The homomorphisms $B \rightarrow C_K$ and $C \rightarrow C_K$ are flat: for $C \rightarrow C_K$ this is [Proposition 4.3.4](#); for $B \rightarrow C_K$ it follows from the flatness of the local analytification maps in [Theorem 5.3.1](#) and the completion criterion, using [Proposition 3.3.4](#). Proper flat base change gives canonical isomorphisms

$$(\mathbf{R}^p f_* \mathcal{F}) \otimes_B C_K \cong \mathbf{R}^p f_{K*} \mathcal{F}_K$$

and

$$(\mathbf{R}^p f_{C*} \mathcal{F}_C) \otimes_C C_K \cong \mathbf{R}^p f_{K*} \mathcal{F}_K.$$

Hence the canonical morphism

$$C \otimes_B \mathbf{R}^p f_* \mathcal{F} \longrightarrow \mathbf{R}^p f_{C*} \mathcal{F}_C$$

becomes an isomorphism after tensoring with C_K for every excellent Stein compact $K \subseteq \mathcal{Y}^{\text{an}}$. Its kernel and cokernel are finite modules. Applying [Corollary 5.4.1](#) to the property of being zero, after shrinking around each such K we find that the analytification of this morphism is an isomorphism. Since the excellent Stein compact subsets form a local basis on \mathcal{Y}^{an} , the analytification is an isomorphism globally. Therefore the theorem for target $\text{Spec } C$ implies the theorem for target $\mathcal{Y} = \text{Spec } B$. Thus it remains to prove the theorem when

$$f: \mathcal{X} \longrightarrow \text{Spec } A$$

is proper of finite presentation and \mathcal{F} is finitely presented.

Step 4. The projective case. We first prove the comparison theorem for projective morphisms. More precisely, let

$$g: \mathcal{Z} \longrightarrow \mathcal{W}$$

be a projective morphism of finite presentation between A -schemes interiorly locally of finite type, and let \mathcal{G} be a finitely presented $\mathcal{O}_{\mathcal{Z}}$ -module. Then the comparison

morphisms

$$(\mathbf{R}^p g_* \mathcal{G})^{\text{an}} \longrightarrow \mathbf{R}^p g_*^{\text{an}} \mathcal{G}^{\text{an}}$$

are isomorphisms for all $p \geq 0$. The assertion is local on \mathcal{W} . We may therefore assume that \mathcal{W} is affine and that g factors as

$$\mathcal{Z} \xrightarrow{j} \mathbb{P}_{\mathcal{W}}^n \xrightarrow{\pi} \mathcal{W},$$

where j is a closed immersion. Since j_* is exact and analytification carries j to the closed immersion j^{an} defined by the analytified ideal, we have

$$(j_* \mathcal{G})^{\text{an}} \cong j_*^{\text{an}} \mathcal{G}^{\text{an}}.$$

Thus Leray reduces the projective case to the case

$$\pi: \mathbb{P}_{\mathcal{W}}^n \longrightarrow \mathcal{W}.$$

Since the assertion is local on \mathcal{W} , we reduce further to $\mathcal{W} = \text{Spec } B$. We prove the result for $\pi: \mathbb{P}_B^n \rightarrow \text{Spec } B$. Its analytification is

$$\pi^{\text{an}}: \mathcal{W}^{\text{an}} \times \mathbb{P}^n \longrightarrow \mathcal{W}^{\text{an}}.$$

First take $\mathcal{G} = \mathcal{O}_{\mathbb{P}_B^n}$. Algebraically,

$$\pi_* \mathcal{O}_{\mathbb{P}_B^n} \cong B, \quad \mathbf{R}^p \pi_* \mathcal{O}_{\mathbb{P}_B^n} = 0 \quad (p > 0).$$

Analytically, let $V \subseteq \mathcal{W}^{\text{an}}$ be Stein. The product $V \times \mathbb{P}^n$ is covered by the $n+1$ Stein open subsets

$$V \times D_+(T_i), \quad i = 0, \dots, n,$$

and all finite intersections are Stein. The Čech complex for this covering is obtained from the standard Čech complex computing

$$\mathbf{H}^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$$

by completed tensor product with $\Gamma(V, \mathcal{O}_V)$. Since

$$\mathbf{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \mathbb{C}, \quad \mathbf{H}^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 \quad (p > 0),$$

we get

$$\pi_*^{\text{an}} \mathcal{O}_{\mathcal{W}^{\text{an}} \times \mathbb{P}^n} \cong \mathcal{O}_{\mathcal{W}^{\text{an}}}, \quad \mathbf{R}^p \pi_*^{\text{an}} \mathcal{O}_{\mathcal{W}^{\text{an}} \times \mathbb{P}^n} = 0 \quad (p > 0).$$

Hence the comparison theorem holds for $\mathcal{O}_{\mathbb{P}_B^n}$. Next we prove it for all twists $\mathcal{O}_{\mathbb{P}_B^n}(m)$, $m \in \mathbb{Z}$, by induction on n . The case $n = 0$ is immediate. Assume $n > 0$ and let

$$H \hookrightarrow \mathbb{P}_B^n$$

be the hyperplane $T_n = 0$, so that $H \cong \mathbb{P}_B^{n-1}$. For every $m \in \mathbb{Z}$ there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_B^n}(m-1) \longrightarrow \mathcal{O}_{\mathbb{P}_B^n}(m) \longrightarrow \mathcal{O}_H(m) \longrightarrow 0.$$

Analytification gives the corresponding analytic exact sequence. By the induction hypothesis the comparison theorem holds for $\mathcal{O}_H(m)$. Comparing the long exact cohomology sequences and applying the five lemma, we see that the theorem holds for $\mathcal{O}_{\mathbb{P}_B^n}(m)$ if and only if it holds for $\mathcal{O}_{\mathbb{P}_B^n}(m-1)$. Since it holds for $m=0$, it holds for every $m \in \mathbb{Z}$. Finally let \mathcal{G} be an arbitrary finitely presented $\mathcal{O}_{\mathbb{P}_B^n}$ -module. The assertion is local on \mathcal{W}^{an} . Fix a point $w \in \mathcal{W}^{\text{an}}$ and put

$$P_w = \mathbb{P}_{\mathcal{O}_{\mathcal{W}^{\text{an}},w}}^n.$$

It suffices to show that, for every coherent \mathcal{O}_{P_w} -module \mathcal{M}_w , the natural comparison map

$$H^p(P_w, \mathcal{M}_w) \longrightarrow (\mathbf{R}^p \pi_*^{\text{an}} \mathcal{M}^{\text{an}})_w$$

is an isomorphism, where \mathcal{M} is any finitely presented model of \mathcal{M}_w after shrinking \mathcal{W} around w . The existence and independence of such a model follow from the limit theorem for finitely presented modules, [Stacks, Tag 01ZR]. We prove this by descending induction on p . Both sides vanish for $p > n$: algebraically this is standard for projective space, and analytically it follows from the Stein covering above and Cartan's Theorem B. Assume the comparison is known in degree $p+1$ for all coherent modules on P_w . Choose a surjection

$$\mathcal{L}_w \longrightarrow \mathcal{M}_w \longrightarrow 0,$$

where

$$\mathcal{L}_w = \bigoplus_{j=1}^r \mathcal{O}_{P_w}(m_j)$$

is a finite direct sum of twists. Let \mathcal{N}_w be the kernel. After shrinking, this descends to an exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow 0$$

on \mathbb{P}_B^n , with \mathcal{L} a finite direct sum of twists. Comparing the long exact algebraic and analytic cohomology sequences gives a commutative diagram

$$\begin{array}{ccccccccc} H^p(P_w, \mathcal{N}_w) & \longrightarrow & H^p(P_w, \mathcal{L}_w) & \longrightarrow & H^p(P_w, \mathcal{M}_w) & \longrightarrow & H^{p+1}(P_w, \mathcal{N}_w) & \longrightarrow & H^{p+1}(P_w, \mathcal{L}_w) \\ \downarrow & & \downarrow \sim & & \downarrow & & \downarrow \sim & & \downarrow \sim \\ (\mathbf{R}^p \pi_*^{\text{an}} \mathcal{N}^{\text{an}})_w & \longrightarrow & (\mathbf{R}^p \pi_*^{\text{an}} \mathcal{L}^{\text{an}})_w & \longrightarrow & (\mathbf{R}^p \pi_*^{\text{an}} \mathcal{M}^{\text{an}})_w & \longrightarrow & (\mathbf{R}^{p+1} \pi_*^{\text{an}} \mathcal{N}^{\text{an}})_w & \longrightarrow & (\mathbf{R}^{p+1} \pi_*^{\text{an}} \mathcal{L}^{\text{an}})_w. \end{array}$$

The comparison is an isomorphism for \mathcal{L} because \mathcal{L} is a finite direct sum of twists, and it is an isomorphism in degree $p+1$ by the induction hypothesis. The four lemma gives surjectivity of the middle vertical arrow. Applying this to \mathcal{N}_w as well, the first vertical arrow is also surjective. Returning to the diagram, the five lemma gives injectivity of the middle vertical arrow. Hence it is an isomorphism. This proves the projective case.

Step 5. The proper case over Spec A . We now assume that

$$f: \mathcal{X} \longrightarrow \operatorname{Spec} A$$

is proper of finite presentation and that \mathcal{F} is finitely presented. We prove the comparison theorem stalkwise on S . Fix $s \in S$ and put

$$R_s = \mathcal{O}_{S,s}, \quad \mathcal{X}_s = \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec} R_s.$$

For a finitely presented $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{M} , let \mathcal{M}_s denote its pull-back to \mathcal{X}_s . Since $A \rightarrow R_s$ is flat by [Lemma 3.3.7](#), flat base change gives

$$(\mathbf{R}^p f_* \mathcal{M})_s^{\text{an}} \cong \mathbf{H}^p(\mathcal{X}_s, \mathcal{M}_s).$$

Thus the comparison morphism at s may be written as

$$\mathbf{H}^p(\mathcal{X}_s, \mathcal{M}_s) \longrightarrow (\mathbf{R}^p f_*^{\text{an}} \mathcal{M}^{\text{an}})_s. \quad (5.8)$$

By the limit theorem for finitely presented modules,

$$\varinjlim_{U \ni s} \operatorname{Mod}^{\text{fp}}(\mathcal{X}_U) \longrightarrow \operatorname{Mod}^{\text{fp}}(\mathcal{X}_s)$$

is an equivalence, where U runs through Stein open neighborhoods of s . Hence the map (5.8) depends only on the coherent $\mathcal{O}_{\mathcal{X}_s}$ -module \mathcal{M}_s . Let \mathcal{E}_s be the full subcategory of $\operatorname{Coh}(\mathcal{X}_s)$ consisting of those coherent modules \mathcal{M}_s for which (5.8) is an isomorphism for every $p \geq 0$. Both sides have finite cohomological dimension locally, and comparison morphisms are compatible with long exact sequences. It follows from the five lemma, by descending induction on p , that \mathcal{E}_s is a Serre subcategory of $\operatorname{Coh}(\mathcal{X}_s)$. We shall show that $\mathcal{E}_s = \operatorname{Coh}(\mathcal{X}_s)$ by dévissage. By [\[Stacks, Tag 01YI\]](#), it suffices to prove the following: for every irreducible closed subset $Z \subseteq |\mathcal{X}_s|$ with generic point ξ , there exists $\mathcal{G}_s \in \mathcal{E}_s$ such that

$$\operatorname{Supp}(\mathcal{G}_s) = Z, \quad (\mathcal{G}_s)_{\xi} \cong \kappa(\xi).$$

Fix such Z . After replacing S by a Stein open neighborhood of s , the reduced closed subscheme $Z \hookrightarrow \mathcal{X}_s$ spreads out to a closed immersion

$$j: \mathcal{Z} \hookrightarrow \mathcal{X}$$

of finite presentation. Since \mathcal{X} is proper over $\operatorname{Spec} A$, so is \mathcal{Z} . By Chow's lemma, after shrinking S around s , there exists a projective A -scheme \mathcal{Z}' of finite presentation and a projective surjective morphism

$$g: \mathcal{Z}' \longrightarrow \mathcal{Z}$$

which is birational over the generic point of $Z_s = Z$. Put

$$h = j \circ g: \mathcal{Z}' \longrightarrow \mathcal{X}.$$

Then h is projective, and $f \circ h$ is projective over $\text{Spec } A$. Choose a g -ample invertible sheaf $\mathcal{O}_{\mathcal{Z}'}(1)$. For $n \gg 0$, relative Serre vanishing gives

$$\mathbf{R}^q g_* \mathcal{O}_{\mathcal{Z}'}(n) = 0 \quad (q > 0).$$

Set

$$\mathcal{G} = h_* \mathcal{O}_{\mathcal{Z}'}(n) = j_* g_* \mathcal{O}_{\mathcal{Z}'}(n),$$

and let \mathcal{G}_s be its pull-back to X_s . Since j_* is exact and $A \rightarrow R_s$ is flat, flat base change gives

$$\mathcal{G}_s \cong h_{s,*} \mathcal{O}_{\mathcal{Z}'_s}(n), \quad \mathbf{R}^q h_{s,*} \mathcal{O}_{\mathcal{Z}'_s}(n) = 0 \quad (q > 0).$$

The morphism $g_s: \mathcal{Z}'_s \rightarrow Z$ is surjective and birational at the generic point ξ . Hence $\text{Supp}(\mathcal{G}_s) = Z$ and

$$(\mathcal{G}_s)_\xi \cong H^0(\text{Spec } \kappa(\xi), \mathcal{O}_{\mathcal{Z}'_s}(n)|_{\text{Spec } \kappa(\xi)}) \cong \kappa(\xi).$$

It remains to prove that $\mathcal{G}_s \in \mathcal{C}_s$. Algebraic Leray gives

$$H^p(X_s, \mathcal{G}_s) \cong H^p(\mathcal{Z}'_s, \mathcal{O}_{\mathcal{Z}'_s}(n)).$$

On the analytic side, the projective case applied to h gives

$$(\mathbf{R}^q h_* \mathcal{O}_{\mathcal{Z}'}(n))^{\text{an}} \cong \mathbf{R}^q h_*^{\text{an}} \mathcal{O}_{\mathcal{Z}^{\text{an}}}(n).$$

Since the algebraic higher direct images vanish for $q > 0$, we get

$$\mathbf{R}^q h_*^{\text{an}} \mathcal{O}_{\mathcal{Z}^{\text{an}}}(n) = 0 \quad (q > 0),$$

and

$$h_*^{\text{an}} \mathcal{O}_{\mathcal{Z}^{\text{an}}}(n) \cong \mathcal{G}^{\text{an}}.$$

Therefore analytic Leray gives

$$(\mathbf{R}^p f_*^{\text{an}} \mathcal{G}^{\text{an}})_s \cong (\mathbf{R}^p (f \circ h)_*^{\text{an}} \mathcal{O}_{\mathcal{Z}^{\text{an}}}(n))_s.$$

Since $f \circ h$ is projective, the projective case applied to $f \circ h$ gives

$$(\mathbf{R}^p (f \circ h)_* \mathcal{O}_{\mathcal{Z}'}(n))_s^{\text{an}} \cong (\mathbf{R}^p (f \circ h)_*^{\text{an}} \mathcal{O}_{\mathcal{Z}^{\text{an}}}(n))_s.$$

By flat base change, the left-hand side identifies with

$$H^p(\mathcal{Z}'_s, \mathcal{O}_{\mathcal{Z}'_s}(n)).$$

Combining these identifications, we obtain

$$H^p(\mathcal{X}_s, \mathcal{G}_s) \cong (\mathbf{R}^p f_*^{\text{an}} \mathcal{G}^{\text{an}})_s$$

for every $p \geq 0$. Hence $\mathcal{G}_s \in \mathcal{E}_s$. The dévissage lemma now gives

$$\mathcal{E}_s = \text{Coh}(\mathcal{X}_s).$$

In particular, the comparison map (5.8) is an isomorphism for \mathcal{F}_s . Since $s \in S$ was arbitrary, the comparison morphism is an isomorphism for proper morphisms over $\text{Spec } A$.

Step 6. Conclusion. By Step 3, the result for proper morphisms over $\text{Spec } A$ implies the result for proper morphisms with affine target. Since the assertion is local on the target, it holds for every proper morphism of finite presentation between A -schemes locally of finite presentation. By Step 2, this proves the theorem for arbitrary quasi-coherent \mathcal{F} . Finally, since the original morphism was only assumed interiorly proper and the schemes were only assumed interiorly locally of finite type, the same conclusion follows by covering S by Stein open subsets on which the preceding finite-presentation proper case applies, and then gluing the resulting local isomorphisms. \square

Corollary 5.4.3 *Let X be an A -scheme interiorly of finite type, and let \mathcal{F} be a finitely presented \mathcal{O}_X -module. Assume that $\text{Supp } \mathcal{F}$ is proper over $\text{Spec } A$, in the sense that the closed subscheme defined by $\text{Fit}_0(\mathcal{F})$ is proper over $\text{Spec } A$. Let $K \subseteq S$ be an excellent Stein compact subset. Then, for every $p \geq 0$, there is a canonical isomorphism*

$$H^p(\mathcal{X}_K, \mathcal{F}_K) \xrightarrow{\sim} \varinjlim_{U \supseteq K} H^p(\mathcal{X}_U^{\text{an}}, \mathcal{F}_U^{\text{an}}), \quad (5.9)$$

where $U \subseteq S$ runs through the Stein open neighborhoods of K .

Proof Let

$$\mathfrak{a} = \text{Fit}_0(\mathcal{F}),$$

and let

$$i: \mathcal{Z} \hookrightarrow \mathcal{X}$$

be the closed subscheme defined by \mathfrak{a} . Since \mathfrak{a} annihilates \mathcal{F} , the sheaf \mathcal{F} descends to a finitely presented $\mathcal{O}_{\mathcal{Z}}$ -module $\mathcal{F}_{\mathcal{Z}}$ such that

$$i_* \mathcal{F}_{\mathcal{Z}} \cong \mathcal{F}.$$

The underlying closed subset of \mathcal{Z} is $\text{Supp } \mathcal{F}$, and \mathcal{Z} is proper over $\text{Spec } A$ by assumption. Since closed immersions are exact on sheaves, and since analytification is compatible with closed immersions by Corollary 5.3.6, for every $p \geq 0$ we have canonical isomorphisms

$$H^p(\mathcal{X}_K, \mathcal{F}_K) \cong H^p(\mathcal{Z}_K, (\mathcal{F}_{\mathcal{Z}})_K)$$

and

$$H^p(\mathcal{X}_U^{\text{an}}, \mathcal{F}_U^{\text{an}}) \cong H^p(\mathcal{Z}_U^{\text{an}}, (\mathcal{F}_{\mathcal{Z}})_U^{\text{an}})$$

for every Stein open neighborhood U of K . Thus we may replace X by Z and \mathcal{F} by \mathcal{F}_Z . Hence we may assume that

$$f: X \longrightarrow \text{Spec } A$$

is proper. By [Theorem 5.4.3](#), the canonical morphism

$$(\mathbf{R}^p f_* \mathcal{F})^{\text{an}} \xrightarrow{\sim} \mathbf{R}^p f_*^{\text{an}} \mathcal{F}^{\text{an}}$$

is an isomorphism. Taking sections over the Stein compactum K gives

$$\Gamma(K, (\mathbf{R}^p f_* \mathcal{F})^{\text{an}}) \xrightarrow{\sim} \Gamma(K, \mathbf{R}^p f_*^{\text{an}} \mathcal{F}^{\text{an}}). \quad (5.10)$$

Put

$$A_K = \Gamma(K, \mathcal{O}_S).$$

Since $A \rightarrow A_K$ is flat by [Proposition 4.3.4](#), flat base change gives

$$(\mathbf{R}^p f_* \mathcal{F}) \otimes_A A_K \cong \mathbf{R}^p (f_K)_* \mathcal{F}_K.$$

Therefore

$$\Gamma(K, (\mathbf{R}^p f_* \mathcal{F})^{\text{an}}) \cong \Gamma(\text{Spec } A_K, \mathbf{R}^p (f_K)_* \mathcal{F}_K) = H^p(X_K, \mathcal{F}_K).$$

On the analytic side, by the definition of sections over a Stein compactum,

$$\Gamma(K, \mathbf{R}^p f_*^{\text{an}} \mathcal{F}^{\text{an}}) = \varinjlim_{U \supseteq K} \Gamma(U, \mathbf{R}^p f_U^{\text{an}} \mathcal{F}_U^{\text{an}}).$$

For every Stein open neighborhood U of K , the morphism f_U^{an} is proper by [Theorem 5.4.2](#), and $\mathcal{F}_U^{\text{an}}$ is coherent by [Proposition 5.3.3](#). Hence $\mathbf{R}^q f_U^{\text{an}} \mathcal{F}_U^{\text{an}}$ is coherent by Grauert's direct image theorem. Since U is Stein, Cartan's Theorem B gives

$$H^r(U, \mathbf{R}^q f_U^{\text{an}} \mathcal{F}_U^{\text{an}}) = 0 \quad (r > 0).$$

Thus the Leray spectral sequence for f_U^{an} degenerates and gives canonical isomorphisms

$$\Gamma(U, \mathbf{R}^p f_U^{\text{an}} \mathcal{F}_U^{\text{an}}) \cong H^p(X_U^{\text{an}}, \mathcal{F}_U^{\text{an}}).$$

Combining these identifications with [\(5.10\)](#), we obtain the canonical isomorphism [\(5.9\)](#). \square

Corollary 5.4.4 *Let X be an A -scheme interiorly of finite type, and let \mathcal{F}, \mathcal{G} be finitely presented \mathcal{O}_X -modules. Assume that*

$$\text{Supp } \mathcal{F} \cap \text{Supp } \mathcal{G}$$

is proper over $\text{Spec } A$. Let $K \subseteq S$ be an excellent Stein compact subset. Then, for every $n \geq 0$, there is a canonical isomorphism

$$\mathrm{Ext}_{\mathcal{O}_{X_K}}^n(\mathcal{F}_K, \mathcal{G}_K) \xrightarrow{\sim} \varinjlim_{U \supseteq K} \mathrm{Ext}_{\mathcal{O}_{X_U^{\mathrm{an}}}}^n(\mathcal{F}_U^{\mathrm{an}}, \mathcal{G}_U^{\mathrm{an}}), \quad (5.11)$$

where $U \subseteq S$ runs through the Stein open neighborhoods of K . In particular, for $n = 0$, there is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{X_K}}(\mathcal{F}_K, \mathcal{G}_K) \xrightarrow{\sim} \varinjlim_{U \supseteq K} \mathrm{Hom}_{\mathcal{O}_{X_U^{\mathrm{an}}}}(\mathcal{F}_U^{\mathrm{an}}, \mathcal{G}_U^{\mathrm{an}}).$$

Proof For every $q \geq 0$, put

$$\mathcal{E}^q = \mathcal{E}xt_{\mathcal{O}_X}^q(\mathcal{F}, \mathcal{G}).$$

Since \mathcal{F} and \mathcal{G} are finitely presented, the sheaves \mathcal{E}^q are finitely presented. Moreover,

$$\mathrm{Supp} \mathcal{E}^q \subseteq \mathrm{Supp} \mathcal{F} \cap \mathrm{Supp} \mathcal{G},$$

so $\mathrm{Supp} \mathcal{E}^q$ is proper over $\mathrm{Spec} A$. We shall use the following base-change identifications. Since \mathcal{F} is finitely presented, it is pseudo-coherent. The morphisms

$$\mathcal{X}_K \longrightarrow \mathcal{X} \quad \text{and} \quad \mathcal{X}_U^{\mathrm{an}} \longrightarrow \mathcal{X}$$

are flat; the first by [Proposition 4.3.4](#), and the second by [Theorem 3.3.4](#) together with the flatness of analytification in [Theorem 5.3.1](#). Hence flat base change for derived Hom , [[Stacks, Tag 0A6A](#)], gives canonical isomorphisms

$$\mathcal{E}_K^q \xrightarrow{\sim} \mathcal{E}xt_{\mathcal{O}_{X_K}}^q(\mathcal{F}_K, \mathcal{G}_K)$$

and

$$(\mathcal{E}_U^q)^{\mathrm{an}} \xrightarrow{\sim} \mathcal{E}xt_{\mathcal{O}_{X_U^{\mathrm{an}}}}^q(\mathcal{F}_U^{\mathrm{an}}, \mathcal{G}_U^{\mathrm{an}})$$

for every Stein open neighborhood U of K . Applying [Corollary 5.4.3](#) to \mathcal{E}^q , we get canonical isomorphisms

$$\mathrm{H}^p(\mathcal{X}_K, \mathcal{E}xt_{\mathcal{O}_{X_K}}^q(\mathcal{F}_K, \mathcal{G}_K)) \xrightarrow{\sim} \varinjlim_{U \supseteq K} \mathrm{H}^p(\mathcal{X}_U^{\mathrm{an}}, \mathcal{E}xt_{\mathcal{O}_{X_U^{\mathrm{an}}}}^q(\mathcal{F}_U^{\mathrm{an}}, \mathcal{G}_U^{\mathrm{an}})). \quad (5.12)$$

Consider the local-to-global Ext spectral sequence on \mathcal{X}_K :

$$E_2^{p,q} = \mathrm{H}^p(\mathcal{X}_K, \mathcal{E}xt_{\mathcal{O}_{X_K}}^q(\mathcal{F}_K, \mathcal{G}_K)) \implies \mathrm{Ext}_{\mathcal{O}_{X_K}}^{p+q}(\mathcal{F}_K, \mathcal{G}_K).$$

For every Stein open neighborhood U of K , we also have the analytic local-to-global Ext spectral sequence

$${}^{\mathrm{an}}E_2^{p,q}(U) = \mathrm{H}^p(\mathcal{X}_U^{\mathrm{an}}, \mathcal{E}xt_{\mathcal{O}_{X_U^{\mathrm{an}}}}^q(\mathcal{F}_U^{\mathrm{an}}, \mathcal{G}_U^{\mathrm{an}})) \implies \mathrm{Ext}_{\mathcal{O}_{X_U^{\mathrm{an}}}}^{p+q}(\mathcal{F}_U^{\mathrm{an}}, \mathcal{G}_U^{\mathrm{an}}).$$

Passing to the filtered colimit over $U \supseteq K$ gives a spectral sequence

$$\varinjlim_{U \supseteq K} {}^{\text{an}} E_2^{p,q}(U) \implies \varinjlim_{U \supseteq K} \text{Ext}_{\mathcal{O}_{X^{\text{an}}}}^{p+q}(\mathcal{F}_U^{\text{an}}, \mathcal{G}_U^{\text{an}}).$$

Indeed, filtered colimits are exact, and these are first-quadrant spectral sequences, so for every fixed total degree only finitely many terms are involved. The canonical base-change morphisms for sheaf Ext and cohomology define a morphism from the algebraic spectral sequence to the filtered colimit of the analytic spectral sequences. By (5.12), this morphism is an isomorphism on the E_2 -page. Therefore it is an isomorphism on the abutments. This gives (5.11). The assertion for Hom is the special case $n = 0$. \square

Corollary 5.4.5 *Let $f: X \rightarrow Y$ be an interiorly proper morphism of A -schemes interiorly locally of finite type. Then, for every $n \geq 0$, the canonical morphism*

$$\left(\mathbf{R}^n f_* \Omega_{X/Y}^\bullet \right)^{\text{an}} \longrightarrow \mathbf{R}^n f_*^{\text{an}} \Omega_{X^{\text{an}}/Y^{\text{an}}}^\bullet \quad (5.13)$$

is an isomorphism.

Proof By Lemma 5.4.2, for each $p \geq 0$, we have a canonical isomorphism

$$\left(\Omega_{X/Y}^p \right)^{\text{an}} \xrightarrow{\sim} \Omega_{X^{\text{an}}/Y^{\text{an}}}^p. \quad (5.14)$$

Consider the algebraic hypercohomology spectral sequence

$$E_1^{p,q} = \mathbf{R}^q f_* \Omega_{X/Y}^p \implies \mathbf{R}^{p+q} f_* \Omega_{X/Y}^\bullet.$$

After analytification, this becomes

$$(E_1^{p,q})^{\text{an}} = \left(\mathbf{R}^q f_* \Omega_{X/Y}^p \right)^{\text{an}}.$$

On the analytic side, we have the corresponding hypercohomology spectral sequence

$${}^{\text{an}} E_1^{p,q} = \mathbf{R}^q f_*^{\text{an}} \Omega_{X^{\text{an}}/Y^{\text{an}}}^p \implies \mathbf{R}^{p+q} f_*^{\text{an}} \Omega_{X^{\text{an}}/Y^{\text{an}}}^\bullet.$$

By the comparison theorem for proper morphisms, applied to the coherent sheaf $\Omega_{X/Y}^p$, the natural morphism

$$\left(\mathbf{R}^q f_* \Omega_{X/Y}^p \right)^{\text{an}} \longrightarrow \mathbf{R}^q f_*^{\text{an}} \left(\Omega_{X/Y}^p \right)^{\text{an}}$$

is an isomorphism. Using (5.14), we obtain

$$\left(\mathbf{R}^q f_* \Omega_{X/Y}^p \right)^{\text{an}} \xrightarrow{\sim} \mathbf{R}^q f_*^{\text{an}} \Omega_{X^{\text{an}}/Y^{\text{an}}}^p$$

for all $p, q \geq 0$.

Thus the morphism of hypercohomology spectral sequences induced by analytification is an isomorphism on the E_1 -page. Consequently it is an isomorphism on the abutments. Therefore, (5.13) is an isomorphism. \square

5.4.3.1 Existence theorem

Theorem 5.4.4 *Let $K \subseteq S$ be an excellent Stein compact subset, and let X be an interiorly separated A -scheme interiorly locally of finite type. Then analytification induces an equivalence of categories*

$$\mathrm{Coh}_{\mathcal{O}_{X_K}}^{\mathrm{ps}/A_K} \xrightarrow{\sim} \varinjlim_{U \supseteq K} \mathrm{Coh}_{\mathcal{O}_{X_U}^{\mathrm{an}}}^{\mathrm{ps}/U}. \quad (5.15)$$

Here $A_K = \Gamma(K, \mathcal{O}_S)$, the category $\mathrm{Coh}_{\mathcal{O}_{X_K}}^{\mathrm{ps}/A_K}$ is the full subcategory of $\mathrm{Coh}_{\mathcal{O}_{X_K}}$ consisting of coherent modules with proper support over $\mathrm{Spec} A_K$, and $\mathrm{Coh}_{\mathcal{O}_{X_U}^{\mathrm{an}}}^{\mathrm{ps}/U}$ is defined similarly.

Proof We first construct the functor. Since every object under consideration has proper support, all questions are local on a quasi-compact open neighborhood of the support. Thus, for the construction and for full faithfulness, we may work after replacing X by a quasi-compact open subscheme. After shrinking S around K , we may then assume that X is a separated A -scheme of finite presentation. By the limit theorem for finitely presented modules, [Stacks, Tag 01ZR], the natural functor

$$\varinjlim_{U \supseteq K} \mathrm{Mod}_{\mathcal{O}_{X_U}}^{\mathrm{fp}} \longrightarrow \mathrm{Coh}_{\mathcal{O}_{X_K}} \quad (5.16)$$

is an equivalence, where U runs through the Stein open neighborhoods of K . Moreover, properness of the support spreads out after shrinking U around K , by the finite-presentation limit theorem for proper morphisms, [Stacks, Tag 081G]. Hence (5.16) restricts to an equivalence

$$\varinjlim_{U \supseteq K} \mathrm{Mod}_{\mathcal{O}_{X_U}}^{\mathrm{fp}, \mathrm{ps}/A_U} \xrightarrow{\sim} \mathrm{Coh}_{\mathcal{O}_{X_K}}^{\mathrm{ps}/A_K}. \quad (5.17)$$

For such a finitely presented module \mathcal{F}_U on X_U , its analytification $\mathcal{F}_U^{\mathrm{an}}$ is coherent by Proposition 5.3.3, and its support is proper over U by Theorem 5.4.2. Thus analytification defines the functor (5.15). We next prove full faithfulness. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}_{\mathcal{O}_{X_K}}^{\mathrm{ps}/A_K}$. After shrinking S around K , choose finitely presented lifts, still denoted \mathcal{F} and \mathcal{G} , on X , with proper support over $\mathrm{Spec} A$. Then $\mathrm{Supp} \mathcal{F} \cap \mathrm{Supp} \mathcal{G}$ is proper over $\mathrm{Spec} A$. Applying Corollary 5.4.4 with $n = 0$ gives

$$\mathrm{Hom}_{\mathcal{O}_{X_K}}(\mathcal{F}_K, \mathcal{G}_K) \xrightarrow{\sim} \varinjlim_{U \supseteq K} \mathrm{Hom}_{\mathcal{O}_{X_U}^{\mathrm{an}}}(\mathcal{F}_U^{\mathrm{an}}, \mathcal{G}_U^{\mathrm{an}}).$$

This is exactly full faithfulness. It remains to prove essential surjectivity. Let \mathcal{G} be an object of the right-hand side of (5.15). After replacing S by a Stein open neighborhood of K , we may assume that \mathcal{G} is a coherent $\mathcal{O}_{X^{\text{an}}}$ -module with proper support over S . Shrinking S again and replacing X by a quasi-compact open neighborhood of the support, we may assume that X is separated and of finite presentation over A . We say that such an analytic coherent module is *liftable* if, after possibly shrinking S around K , it lies in the essential image of (5.15). We record two permanence properties of liftable modules. First, liftable modules are stable under kernels and cokernels of morphisms between liftable modules. Indeed, if $\alpha: \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a morphism between liftable modules, choose algebraic lifts $\mathcal{F}_1, \mathcal{F}_2$ on X_K . By full faithfulness, after shrinking around K , the morphism α is induced by a unique morphism $\beta: \mathcal{F}_1 \rightarrow \mathcal{F}_2$. Since analytification is exact, $\ker \alpha$, $\text{coker } \alpha$, and $\text{im } \alpha$ are respectively the analytifications of $\ker \beta$, $\text{coker } \beta$, and $\text{im } \beta$. These sheaves still have proper support. Hence they are liftable. Second, liftable modules are stable under extensions. Let

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$$

be an exact sequence with \mathcal{G}' and \mathcal{G}'' liftable. Choose algebraic lifts \mathcal{F}' and \mathcal{F}'' on X_K . The extension class of the displayed sequence lies in

$$\varinjlim_{U \supseteq K} \text{Ext}_{\mathcal{O}_{X^{\text{an}}}}^1(\mathcal{F}_U''^{\text{an}}, \mathcal{F}_U'^{\text{an}}).$$

By [Corollary 5.4.4](#), it comes from a unique class in

$$\text{Ext}_{\mathcal{O}_{X_K}}^1(\mathcal{F}'', \mathcal{F}').$$

Let

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be the corresponding algebraic extension. Then \mathcal{F} has proper support and $\mathcal{F}^{\text{an}} \cong \mathcal{G}$ as germs around K . Hence \mathcal{G} is liftable. We shall also use the following closed immersion reduction. Let

$$i: \mathcal{Z} \hookrightarrow X$$

be a closed immersion of separated A -schemes of finite presentation. Let \mathcal{H} be a coherent $\mathcal{O}_{Z^{\text{an}}}$ -module with proper support. If $i_*^{\text{an}} \mathcal{H}$ is liftable on X^{an} , then \mathcal{H} is liftable on Z^{an} . To see this, choose a lift \mathcal{F} of $i_*^{\text{an}} \mathcal{H}$ on X_K , after spreading it out to a neighborhood of K . Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the ideal defining \mathcal{Z} . Since $\mathcal{I}^{\text{an}} i_*^{\text{an}} \mathcal{H} = 0$, we have $(\mathcal{I}\mathcal{F})^{\text{an}} = 0$ as a germ near K . By [Corollary 5.4.1](#), after shrinking around K , $\mathcal{I}_K \mathcal{F} = 0$. Hence \mathcal{F} descends to a coherent module \mathcal{F}_Z on Z_K . Since analytification is compatible with closed immersions, we get

$$i_*^{\text{an}} \mathcal{F}_Z^{\text{an}} \cong (i_{K,*} \mathcal{F}_Z)^{\text{an}} \cong \mathcal{F}^{\text{an}} \cong i_*^{\text{an}} \mathcal{H}.$$

As i_*^{an} is fully faithful for a closed immersion, $\mathcal{F}_Z^{\text{an}} \cong \mathcal{H}$. Thus \mathcal{H} is liftable. We now prove the projective case. Suppose first that

$$\mathcal{X} = \mathbb{P}_A^N.$$

Let $\pi^{\text{an}}: \mathbb{P}_S^N \rightarrow S$ be the analytic projection. By the analytic Serre theorem for projective space over a Stein base, after shrinking S around K , there exist integers a, b, m, ℓ and an exact sequence

$$\mathcal{O}_{\mathbb{P}_S^N}(-m)^{\oplus a} \rightarrow \mathcal{O}_{\mathbb{P}_S^N}(-\ell)^{\oplus b} \rightarrow \mathcal{G} \rightarrow 0.$$

Indeed, for $r \gg 0$, the sheaf $\mathcal{G}(r)$ is generated by $\pi_*^{\text{an}} \mathcal{G}(r)$ near K , and $\pi_*^{\text{an}} \mathcal{G}(r)$ is generated by finitely many sections after shrinking the Stein base; applying the same argument to the kernel gives the displayed two-term presentation. By full faithfulness already proved, the first arrow in this presentation is induced, as a germ around K , by a morphism

$$\mathcal{O}_{\mathbb{P}_{AK}^N}(-m)^{\oplus a} \rightarrow \mathcal{O}_{\mathbb{P}_{AK}^N}(-\ell)^{\oplus b}.$$

The cokernel of this algebraic morphism is a coherent module on \mathbb{P}_{AK}^N , and its analytification is \mathcal{G} . Thus \mathcal{G} is liftable on \mathbb{P}_A^N . If \mathcal{X} is projective over A , choose a closed immersion $\mathcal{X} \hookrightarrow \mathbb{P}_A^N$ and apply the closed immersion reduction. Hence the theorem holds for projective \mathcal{X} . Next assume that \mathcal{X} is quasi-projective over A . Choose an open immersion

$$j: \mathcal{X} \hookrightarrow \overline{\mathcal{X}}$$

with $\overline{\mathcal{X}}$ projective over A . Since $\text{Supp } \mathcal{G}$ is proper over S and $\overline{\mathcal{X}}^{\text{an}}$ is separated over S , the induced map

$$\text{Supp } \mathcal{G} \rightarrow \overline{\mathcal{X}}^{\text{an}}$$

is proper; hence its image is closed and contained in the open subset \mathcal{X}^{an} . Therefore $j_*^{\text{an}} \mathcal{G}$ is a coherent $\mathcal{O}_{\overline{\mathcal{X}}^{\text{an}}}$ -module with proper support. By the projective case, $j_*^{\text{an}} \mathcal{G}$ is liftable. Let $\overline{\mathcal{F}}$ be a lift on $\overline{\mathcal{X}}_K$. The analytification of $\overline{\mathcal{F}}$ vanishes on $\overline{\mathcal{X}}^{\text{an}} \setminus \mathcal{X}^{\text{an}}$ as a germ near K . By [Corollary 5.4.1](#), after shrinking around K , the restriction of $\overline{\mathcal{F}}$ to $\overline{\mathcal{X}}_K \setminus \mathcal{X}_K$ is zero. Hence $\overline{\mathcal{F}}$ is the extension by zero of a coherent module \mathcal{F} on \mathcal{X}_K , and $\mathcal{F}^{\text{an}} \cong \mathcal{G}$. Thus the theorem holds for quasi-projective \mathcal{X} . We finally prove the general case by noetherian induction on closed subsets of $|\mathcal{X}_K|$. For a closed subset $Y \subseteq |\mathcal{X}_K|$, let $P(Y)$ be the assertion that every coherent $\mathcal{O}_{\mathcal{X}^{\text{an}}}$ -module with proper support, whose support is contained near K in the analytification of some closed constructible model of Y , is liftable. The assertion is trivial for $Y = \emptyset$. Assume that $P(Y')$ is known for every proper closed subset $Y' \subsetneq Y$, and let \mathcal{G} be supported over Y . Choose an open subscheme $\mathcal{V} \subseteq \mathcal{X}$, after shrinking around K , such that \mathcal{V}_K meets every irreducible component of Y and is quasi-projective over A . By Chow's lemma in its \mathcal{V} -admissible form, [\[Stacks, Tag 02O2\]](#), after shrinking around K there exists a projective surjective morphism

$$h: \mathcal{X}' \rightarrow \mathcal{X}$$

such that \mathcal{X}' is quasi-projective over A and h is an isomorphism over \mathcal{V} . Since h^{an} is proper, the support of $h^{\text{an}*} \mathcal{G}$ is proper over S . By the quasi-projective case, $h^{\text{an}*} \mathcal{G}$

is liftable on \mathcal{X}^{an} . Therefore $h_*^{\text{an}} h^{\text{an}*} \mathcal{G}$ is liftable on \mathcal{X}^{an} by [Theorem 5.4.3](#), applied to the proper morphism h in degree 0. Consider the adjunction morphism

$$\mathcal{G} \longrightarrow h_*^{\text{an}} h^{\text{an}*} \mathcal{G}.$$

It is an isomorphism over \mathcal{V}^{an} , because h is an isomorphism over \mathcal{V} . Hence its kernel and cokernel are supported over

$$Y \setminus (Y \cap \mathcal{V}_K),$$

which is a proper closed subset of Y . By the induction hypothesis, this kernel and cokernel are liftable. Since liftable modules are stable under kernels, cokernels, and extensions, \mathcal{G} is liftable. Thus $P(Y)$ holds. By noetherian induction, $P(Y)$ holds for every closed subset $Y \subseteq |\mathcal{X}_K|$. Taking $Y = |\mathcal{X}_K|$, we find that every object on the right-hand side of [\(5.15\)](#) is liftable. This proves essential surjectivity, and hence the theorem. \square

Corollary 5.4.6 *Let $K \subseteq S$ be an excellent Stein compact subset, and let \mathcal{X} be an interiorly proper A -scheme interiorly locally of finite type. Then analytification induces an equivalence of categories*

$$\text{Coh}_{\mathcal{O}_{\mathcal{X}_K}} \xrightarrow{\sim} \varinjlim_{U \supseteq K} \text{Coh}_{\mathcal{O}_{\mathcal{X}_U^{\text{an}}}}, \quad (5.18)$$

where $U \subseteq S$ runs through the Stein open neighborhoods of K .

Proof By [Theorem 5.4.4](#), it suffices to observe that every coherent module on \mathcal{X}_K has proper support over $\text{Spec } \Gamma(K, \mathcal{O}_S)$, and, after shrinking S around K , every coherent module on $\mathcal{X}_U^{\text{an}}$ has proper support over U . This follows from the properness of $\mathcal{X}_K \rightarrow \text{Spec } \Gamma(K, \mathcal{O}_S)$ and of $\mathcal{X}_U^{\text{an}} \rightarrow U$, the latter being a consequence of [Theorem 5.4.2](#). \square

Corollary 5.4.7 *Let $K \subseteq S$ be an excellent Stein compact subset, and let \mathcal{X} be an interiorly separated A -scheme interiorly locally of finite type. Let*

$$f: \mathcal{X} \longrightarrow \text{Spec } A$$

be the structure morphism. Then analytification induces a bijection between the set of closed subschemes of \mathcal{X}_K with proper support over $\text{Spec } \Gamma(K, \mathcal{O}_S)$ and the set of germs along $f^{\text{an}, -1}(K)$ of closed analytic subspaces of \mathcal{X}^{an} with proper support over S .

Proof A closed subscheme of \mathcal{X}_K with proper support is equivalently a quotient

$$\mathcal{O}_{\mathcal{X}_K} \longrightarrow \mathcal{B}$$

where \mathcal{B} is a coherent $\mathcal{O}_{\mathcal{X}_K}$ -algebra and $\text{Supp } \mathcal{B}$ is proper over $\text{Spec } \Gamma(K, \mathcal{O}_S)$. Similarly, a closed analytic subspace of a germ of \mathcal{X}^{an} with proper support is equivalently, after shrinking S around K , a quotient

$$\mathcal{O}_{\mathcal{X}_U^{\text{an}}} \longrightarrow \mathcal{B}_U$$

where \mathcal{B}_U is a coherent $\mathcal{O}_{\mathcal{X}_U^{\text{an}}}$ -algebra with proper support over U . By [Theorem 5.4.4](#), the underlying coherent module of \mathcal{B}_U has, as a germ around K , an algebraic lift \mathcal{B} on \mathcal{X}_K with proper support. The unit morphism

$$\mathcal{O}_{\mathcal{X}_U^{\text{an}}} \longrightarrow \mathcal{B}_U$$

is lifted, after shrinking around K , by the case $n = 0$ of [Corollary 5.4.4](#), applied to $\mathcal{O}_{\mathcal{X}}$ and \mathcal{B} . The multiplication

$$\mathcal{B}_U \otimes_{\mathcal{O}_{\mathcal{X}_U^{\text{an}}}} \mathcal{B}_U \longrightarrow \mathcal{B}_U$$

is likewise lifted by the same full-faithfulness statement, applied to $\mathcal{B} \otimes_{\mathcal{O}_{\mathcal{X}_K}} \mathcal{B}$ and \mathcal{B} . The associativity, commutativity, and unit identities hold after analytification, hence hold algebraically by full faithfulness. Thus \mathcal{B} is a coherent $\mathcal{O}_{\mathcal{X}_K}$ -algebra whose analytification is \mathcal{B}_U as an algebra germ. It remains to see that the lifted unit map is surjective. Let C be its cokernel. Then C has proper support, and its analytification is zero as a germ around K , because the analytic unit map is the quotient defining the closed analytic subspace. By the equivalence in [Theorem 5.4.4](#), or equivalently by [Corollary 5.4.1](#) applied to the property of being zero, we get $C = 0$. Therefore

$$\mathcal{B} \simeq \mathcal{O}_{\mathcal{X}_K} / \mathcal{I}$$

for a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}_K}$, and this defines the required closed subscheme. The construction is inverse to analytification. Indeed, if two algebraic closed subschemes have the same analytic germ, then the corresponding quotient algebras are isomorphic as analytic quotient algebras. By full faithfulness, this isomorphism comes from a unique algebraic isomorphism compatible with the quotient maps from $\mathcal{O}_{\mathcal{X}_K}$. Hence the kernels are equal. This proves injectivity, and the lifting argument above proves surjectivity. \square

Corollary 5.4.8 *Let \mathcal{X} be an interiorly proper A -scheme, and let \mathcal{Y} be an interiorly separated A -scheme interiorly locally of finite type. Let $K \subseteq S$ be an excellent Stein compact subset. Then the natural map*

$$\text{Hom}_{\text{Sch}/\text{Spec } \Gamma(K, \mathcal{O}_S)}(\mathcal{X}_K, \mathcal{Y}_K) \longrightarrow \varinjlim_{U \supseteq K} \text{Hom}_{\text{CompSpace}/U}(\mathcal{X}_U^{\text{an}}, \mathcal{Y}_U^{\text{an}})$$

is bijective, where $U \subseteq S$ runs through the Stein open neighborhoods of K .

Proof Put

$$A_K = \Gamma(K, \mathcal{O}_S), \quad \mathcal{Z} = \mathcal{X} \times_{\text{Spec } A} \mathcal{Y}.$$

Since \mathcal{X} is interiorly proper and \mathcal{Y} is interiorly separated and interiorly locally of finite type, the product \mathcal{Z} is interiorly separated and interiorly locally of finite type. By [Proposition 5.3.2](#), one has

$$\mathcal{Z}_U^{\text{an}} \simeq \mathcal{X}_U^{\text{an}} \times_U \mathcal{Y}_U^{\text{an}}$$

for every Stein open neighborhood U of K . A morphism

$$g: \mathcal{X}_K \longrightarrow \mathcal{Y}_K$$

over $\text{Spec } A_K$ is equivalent to its graph

$$\Gamma_g \subseteq \mathcal{X}_K \times_{\text{Spec } A_K} \mathcal{Y}_K = \mathcal{Z}_K.$$

This graph is a closed subscheme because \mathcal{Y}_K is separated over $\text{Spec } A_K$, and its support is proper over $\text{Spec } A_K$ because the first projection

$$\Gamma_g \longrightarrow \mathcal{X}_K$$

is an isomorphism and \mathcal{X}_K is proper over $\text{Spec } A_K$. Conversely, a closed subscheme $\Gamma \subseteq \mathcal{Z}_K$ with proper support is the graph of a morphism $\mathcal{X}_K \rightarrow \mathcal{Y}_K$ if and only if the first projection

$$\Gamma \longrightarrow \mathcal{X}_K$$

is an isomorphism. The same description holds analytically for morphisms

$$\mathcal{X}_U^{\text{an}} \longrightarrow \mathcal{Y}_U^{\text{an}}$$

over U , using graphs inside $\mathcal{X}_U^{\text{an}} \times_U \mathcal{Y}_U^{\text{an}}$. By [Corollary 5.4.7](#), analytification gives a bijection between closed subschemes of \mathcal{Z}_K with proper support and germs around K of closed analytic subspaces of \mathcal{Z}^{an} with proper support over S . Under this bijection, the condition that the first projection be an isomorphism is preserved. One direction is immediate from analytification. Conversely, suppose that the analytic first projection is an isomorphism as a germ around K . After shrinking S around K , spread the corresponding algebraic closed subscheme to a closed subscheme

$$\Gamma_U \subseteq \mathcal{Z}_U.$$

Then the analytification of

$$\Gamma_U \longrightarrow \mathcal{X}_U$$

is an isomorphism as a germ around K . Applying [Theorem 5.4.2](#) to the property of being an isomorphism, we conclude that

$$\Gamma_K \longrightarrow \mathcal{X}_K$$

is an isomorphism. Hence Γ_K is the graph of a unique morphism $\mathcal{X}_K \rightarrow \mathcal{Y}_K$. Thus the graph correspondence identifies the algebraic Hom set with the colimit of the analytic Hom sets, which proves the claim. \square

5.4.3.2 Finite étale covers

Theorem 5.4.5 (Grauert–Frisch injectivity theorem) *Let S be a complex space, let $K \subseteq S$ be a Stein compact subset, and let \mathcal{F} be a coherent \mathcal{O}_S -module. Then, for every open neighborhood W of K in S , there exists a Stein open neighborhood U of K with $U \subseteq W$ such that the natural restriction map*

$$\Gamma(U, \mathcal{F}) \longrightarrow \Gamma(K, \mathcal{F})$$

is injective. In particular, there exists a fundamental system $\{U_\lambda\}_\lambda$ of Stein open neighborhoods of K such that

$$\Gamma(U_\lambda, \mathcal{F}) \longrightarrow \Gamma(K, \mathcal{F})$$

is injective for every λ .

Proof We first recall two standard facts about associated components of coherent analytic sheaves. If \mathcal{G} is a coherent sheaf on a complex space V , then the associated components of \mathcal{G} form a locally finite family of irreducible closed analytic subsets of V . Moreover, if $U \subseteq V$ is open, then the associated components of $\mathcal{G}|_U$ are exactly the non-empty intersections $Z \cap U$, where Z runs through the associated components of \mathcal{G} on V . Finally, if $\mathcal{H} \subseteq \mathcal{G}$ is a non-zero coherent subsheaf, then every associated component of \mathcal{H} is an associated component of \mathcal{G} . Fix an open neighborhood W of K . Since K is a Stein compact subset, there exists a Stein open neighborhood W_0 of K such that

$$K \subseteq W_0 \subseteq W.$$

Let $\{Z_\alpha\}_{\alpha \in I}$ be the locally finite family of associated components of $\mathcal{F}|_{W_0}$. Let

$$B = \bigcup_{Z_\alpha \cap K = \emptyset} Z_\alpha.$$

Since the family $\{Z_\alpha\}_{\alpha \in I}$ is locally finite, the subset B is closed in W_0 . It is disjoint from K . Therefore

$$W_1 := W_0 \setminus B$$

is an open neighborhood of K . Again using that K is Stein compact, choose a Stein open neighborhood U of K with

$$K \subseteq U \subseteq W_1.$$

We claim that every associated component of $\mathcal{F}|_U$ meets K . Indeed, let C be such an associated component. By the restriction property for associated components, there is an associated component Z_α of $\mathcal{F}|_{W_0}$ such that

$$C = Z_\alpha \cap U.$$

If $Z_\alpha \cap K = \emptyset$, then $Z_\alpha \subseteq B$, hence $Z_\alpha \cap U = \emptyset$, a contradiction. Thus $Z_\alpha \cap K \neq \emptyset$. Since $K \subseteq U$, we get

$$C \cap K = (Z_\alpha \cap U) \cap K = Z_\alpha \cap K \neq \emptyset.$$

Now let

$$\sigma \in \Gamma(U, \mathcal{F})$$

be a section whose image in $\Gamma(K, \mathcal{F})$ is zero. By the definition

$$\Gamma(K, \mathcal{F}) = \varinjlim_{V \supseteq K} \Gamma(V, \mathcal{F}),$$

there exists an open neighborhood V of K in U such that

$$\sigma|_V = 0.$$

We prove that $\sigma = 0$. Suppose otherwise. Let $\mathcal{H} \subseteq \mathcal{F}|_U$ be the coherent subsheaf generated by σ , namely the image of

$$\mathcal{O}_U \longrightarrow \mathcal{F}|_U, \quad 1 \longmapsto \sigma.$$

Then \mathcal{H} is non-zero. Hence \mathcal{H} has an associated component C . Since $\mathcal{H} \subseteq \mathcal{F}|_U$, the component C is also an associated component of $\mathcal{F}|_U$. By the construction of U , we have

$$C \cap K \neq \emptyset.$$

On the other hand, $\sigma|_V = 0$, so $\mathcal{H}|_V = 0$. Therefore

$$\text{Supp}(\mathcal{H}) \cap V = \emptyset.$$

Since $C \subseteq \text{Supp}(\mathcal{H})$ and $K \subseteq V$, this implies

$$C \cap K = \emptyset,$$

a contradiction. Hence $\sigma = 0$, and the map

$$\Gamma(U, \mathcal{F}) \longrightarrow \Gamma(K, \mathcal{F})$$

is injective. Since the construction works inside every open neighborhood W of K , the resulting Stein open neighborhoods form a fundamental system. This proves the theorem. \square

Lemma 5.4.3 *Let $\{\mathcal{X}_i\}_{1 \leq i \leq n}$ be a finite family of A -schemes interiorly of finite type, and let $K \subseteq S$ be an excellent Stein compact subset. Then the following conditions are equivalent:*

- (1) *there exists a fundamental system of Stein open neighborhoods $\{U_\lambda\}_\lambda$ of K such that, for every i and every λ , the complex space $\mathcal{X}_{i,U_\lambda}^{\text{an}}$ is irreducible, respectively connected;*

(2) for every i , the scheme $\mathcal{X}_{i,K}$ is irreducible, respectively connected.

Proof Replacing S by a Stein open neighborhood of K is harmless. Thus, by [Corollary 5.2.1](#), we may assume that all \mathcal{X}_i are of finite presentation over A . Since the family is finite, all subsequent shrinkings of S around K may be made simultaneously for all i .

(1) \Rightarrow (2), irreducible case. Assume that $\mathcal{X}_{i,K}$ is reducible for some i . Since $\mathcal{X}_{i,K}$ is noetherian, there exist proper closed subsets

$$Z_K, Z'_K \subsetneq |\mathcal{X}_{i,K}|$$

such that

$$|\mathcal{X}_{i,K}| = Z_K \cup Z'_K.$$

After shrinking S around K , the two closed subsets spread out to closed subschemes

$$\mathcal{Z}, \mathcal{Z}' \subseteq \mathcal{X}_i$$

of finite presentation whose base changes to K have underlying sets Z_K and Z'_K , by [\[Stacks, Tag 01ZM\]](#) and [\[Stacks, Tag 01ZP\]](#). Since $Z_K \cup Z'_K = |\mathcal{X}_{i,K}|$, [Lemma 5.3.3](#) implies, after shrinking S around K , that

$$|\mathcal{X}_i^{\text{an}}| = |\mathcal{Z}^{\text{an}}| \cup |\mathcal{Z}'^{\text{an}}|.$$

Moreover, since Z_K and Z'_K are proper in $|\mathcal{X}_{i,K}|$, the complements

$$|\mathcal{X}_i| \setminus |\mathcal{Z}|, \quad |\mathcal{X}_i| \setminus |\mathcal{Z}'|$$

have non-empty inverse images in $\mathcal{X}_{i,K}$. Again by [Lemma 5.3.3](#), their analytic inverse images cannot become empty on any sufficiently small Stein neighborhood of K . Hence, for every sufficiently small member U_λ of the fundamental system in (1), the space $\mathcal{X}_{i,U_\lambda}^{\text{an}}$ is the union of two proper closed analytic subsets. This contradicts irreducibility.

(1) \Rightarrow (2), connected case. Assume that $\mathcal{X}_{i,K}$ is disconnected for some i . Then it has a non-trivial clopen subset Z_K . By the limit theorem for finitely presented closed and open subschemes, after shrinking S around K , Z_K spreads out to a clopen subscheme

$$\mathcal{Z} \subseteq \mathcal{X}_i$$

whose base change to K is Z_K , see [\[Stacks, Tag 01ZM\]](#), [\[Stacks, Tag 01ZP\]](#), and [\[Stacks, Tag 0EUV\]](#). Since Z_K and its complement are both non-empty, [Lemma 5.3.3](#) implies that, on every sufficiently small Stein neighborhood of K , both \mathcal{Z}^{an} and its complement in $\mathcal{X}_i^{\text{an}}$ are non-empty. Thus $\mathcal{X}_{i,U_\lambda}^{\text{an}}$ is disconnected for sufficiently small U_λ , contradicting (1).

(2) \Rightarrow (1), irreducible case: reduction to the affine case. Assume that every $\mathcal{X}_{i,K}$ is irreducible. After shrinking S around K , choose finite affine open coverings

$$\mathcal{X}_i = \bigcup_{\alpha=1}^{r_i} \mathcal{V}_{i\alpha}$$

such that $\mathcal{V}_{i\alpha,K} \neq \emptyset$ for every α . This can be achieved by removing those affine opens whose base change to K is empty, using [Lemma 5.3.3](#). Since $\mathcal{X}_{i,K}$ is irreducible, each non-empty open subscheme $\mathcal{V}_{i\alpha,K}$ is irreducible, and, for fixed i , the intersections

$$\mathcal{V}_{i\alpha,K} \cap \mathcal{V}_{i\beta,K}$$

are non-empty for all α, β . By [Lemma 5.3.3](#), after shrinking S around K , the analytic intersections

$$\mathcal{V}_{i\alpha,U}^{\text{an}} \cap \mathcal{V}_{i\beta,U}^{\text{an}}$$

are non-empty for every sufficiently small Stein open neighborhood U of K . Therefore, if the assertion is known for the finite family of affine schemes $\{\mathcal{V}_{i\alpha}\}_{i,\alpha}$, then the spaces $\mathcal{X}_{i,U}^{\text{an}}$ are finite unions of irreducible open subspaces with pairwise non-empty intersections, hence are irreducible. Thus we may assume that all \mathcal{X}_i are affine.

(2) \Rightarrow (1), irreducible case: reduction to the normal affine case. Assume that all \mathcal{X}_i are affine and that all $\mathcal{X}_{i,K}$ are irreducible. Put

$$A_K = \Gamma(K, \mathcal{O}_S).$$

By [Theorem 4.4.2](#), A_K is excellent. Hence each reduced scheme $(\mathcal{X}_{i,K})_{\text{red}}$ is excellent, and its normalization

$$v_{i,K} : \tilde{\mathcal{X}}_{i,K} \longrightarrow \mathcal{X}_{i,K}$$

is finite. Since $\mathcal{X}_{i,K}$ is irreducible, the scheme $\tilde{\mathcal{X}}_{i,K}$ is normal and irreducible. By the limit theorem for finite morphisms, after shrinking S around K , there exist finite morphisms

$$v_i : \tilde{\mathcal{X}}_i \longrightarrow \mathcal{X}_i$$

whose base changes to K are the morphisms $v_{i,K}$; see [\[Stacks, Tag 01ZM\]](#), [\[Stacks, Tag 01ZO\]](#), and [\[Stacks, Tag 07RR\]](#). By [Theorem 5.4.2](#), after shrinking S around K , the analytified morphisms

$$v_i^{\text{an}} : \tilde{\mathcal{X}}_i^{\text{an}} \longrightarrow \mathcal{X}_i^{\text{an}}$$

are finite and surjective. Thus, if the desired irreducibility assertion is known for the finite family $\{\tilde{\mathcal{X}}_i\}_i$, then $\tilde{\mathcal{X}}_{i,U}^{\text{an}}$ is irreducible for every sufficiently small U , and its image $\mathcal{X}_{i,U}^{\text{an}}$ is irreducible. Hence we may assume that all \mathcal{X}_i are affine and that all $\mathcal{X}_{i,K}$ are normal and irreducible.

(2) \Rightarrow (1), irreducible case: reduction to the projective normal case. Assume that all \mathcal{X}_i are affine and that all $\mathcal{X}_{i,K}$ are normal and irreducible. We reduce to the case where the \mathcal{X}_i are projective over $\text{Spec } A$ and the $\mathcal{X}_{i,K}$ are normal and irreducible. Fix i . Since $\mathcal{X}_{i,K}$ is affine and of finite presentation over A_K , choose a quasi-compact immersion

$$\mathcal{X}_{i,K} \hookrightarrow \mathbb{P}_{A_K}^N.$$

By the factorization theorem for quasi-compact immersions, [Stacks, Tag 01QV], this factors as an open immersion followed by a closed immersion

$$\mathcal{X}_{i,K} \hookrightarrow \overline{\mathcal{P}}_{i,K} \hookrightarrow \mathbb{P}_{A_K}^N.$$

Replacing $\overline{\mathcal{P}}_{i,K}$ by the reduced scheme-theoretic closure of $\mathcal{X}_{i,K}$, we may assume that $\overline{\mathcal{P}}_{i,K}$ is reduced and irreducible and that $\mathcal{X}_{i,K}$ is a dense open subscheme. Since A_K is excellent, the normalization

$$\rho_{i,K}: \mathcal{P}_{i,K} \longrightarrow \overline{\mathcal{P}}_{i,K}$$

is finite. Since $\mathcal{X}_{i,K}$ is normal, $\rho_{i,K}$ is an isomorphism over $\mathcal{X}_{i,K}$. Hence $\mathcal{P}_{i,K}$ is projective, normal and irreducible over A_K , and $\mathcal{X}_{i,K}$ is identified with an open subscheme of $\mathcal{P}_{i,K}$. By the limit theorem for schemes, morphisms, and open immersions of finite presentation, after shrinking S around K , the open immersion

$$\mathcal{X}_{i,K} \hookrightarrow \mathcal{P}_{i,K}$$

spreads out to an open immersion

$$j_i: \mathcal{X}_i \hookrightarrow \mathcal{P}_i$$

where \mathcal{P}_i is projective over $\text{Spec } A$, and the base change of \mathcal{P}_i to K is $\mathcal{P}_{i,K}$; see [Stacks, Tag 01ZM], [Stacks, Tag 01ZP], and [Stacks, Tag 0EUU]. If the desired assertion is known for the finite family $\{\mathcal{P}_i\}_i$, then, for every sufficiently small U , the spaces $\mathcal{P}_{i,U}^{\text{an}}$ are irreducible. By [Theorem 5.4.2](#), the morphisms

$$j_{i,U}^{\text{an}}: \mathcal{X}_{i,U}^{\text{an}} \longrightarrow \mathcal{P}_{i,U}^{\text{an}}$$

are open immersions after shrinking S around K . Their images are non-empty by [Lemma 5.3.3](#). Hence $\mathcal{X}_{i,U}^{\text{an}}$ is a non-empty open analytic subspace of an irreducible complex space, and is therefore irreducible. Thus it remains to prove the irreducible assertion in the projective case.

(2) \Rightarrow (1), irreducible case: projective normal case. Assume now that each \mathcal{X}_i is projective over $\text{Spec } A$ and that each $\mathcal{X}_{i,K}$ is normal and irreducible. Let

$$f_i: \mathcal{X}_i \longrightarrow \text{Spec } A$$

be the structure morphism, and put

$$\mathcal{G}_i = (f_i^{\text{an}})_* \mathcal{O}_{\mathcal{X}_i^{\text{an}}}.$$

After shrinking S around K , the morphisms f_i^{an} are proper by [Theorem 5.4.2](#), and hence the sheaves \mathcal{G}_i are coherent by Grauert's direct image theorem. Applying [Theorem 5.4.5](#) to the coherent sheaf

$$\mathcal{G} = \bigoplus_{i=1}^n \mathcal{G}_i$$

gives a fundamental system of Stein open neighborhoods U_λ of K such that, for every i and every λ , the restriction map

$$\Gamma(U_\lambda, \mathcal{G}_i) \longrightarrow \Gamma(K, \mathcal{G}_i)$$

is injective. Equivalently,

$$\Gamma(\mathcal{X}_{i,U_\lambda}^{\text{an}}, \mathcal{O}_{\mathcal{X}_{i,U_\lambda}^{\text{an}}}) \longrightarrow \Gamma(K, \mathcal{G}_i)$$

is injective. By [Corollary 5.4.3](#), applied to $\mathcal{O}_{\mathcal{X}_i}$ and $p = 0$, we have a canonical identification

$$\Gamma(K, \mathcal{G}_i) \cong \Gamma(\mathcal{X}_{i,K}, \mathcal{O}_{\mathcal{X}_{i,K}}).$$

Since $\mathcal{X}_{i,K}$ is normal and irreducible, the ring

$$\Gamma(\mathcal{X}_{i,K}, \mathcal{O}_{\mathcal{X}_{i,K}})$$

is an integral domain. Hence

$$\Gamma(\mathcal{X}_{i,U_\lambda}^{\text{an}}, \mathcal{O}_{\mathcal{X}_{i,U_\lambda}^{\text{an}}})$$

is an integral domain for every i and λ . In particular, $\mathcal{X}_{i,U_\lambda}^{\text{an}}$ is connected. On the other hand, by [Corollary 5.4.1](#), applied to the property of being normal, after shrinking S around K we may assume that all $\mathcal{X}_{i,U_\lambda}^{\text{an}}$ are normal. A normal complex space has open and closed irreducible components. Therefore a connected normal complex space is irreducible. Thus all $\mathcal{X}_{i,U_\lambda}^{\text{an}}$ are irreducible. This proves the irreducible case.

(2) \Rightarrow (1), connected case. Assume that every $\mathcal{X}_{i,K}$ is connected. Since the schemes $\mathcal{X}_{i,K}$ are noetherian, each has finitely many irreducible components. After shrinking S around K , choose closed subschemes

$$\mathcal{Y}_{ij} \subseteq \mathcal{X}_i, \quad 1 \leq j \leq m_i,$$

of finite presentation such that the closed subsets $|\mathcal{Y}_{ij,K}|$ are precisely the irreducible components of $|\mathcal{X}_{i,K}|$, with their reduced induced structures. This is possible by [\[Stacks, Tag 01ZM\]](#) and [\[Stacks, Tag 01ZP\]](#). By the irreducible case already proved, after shrinking S around K and choosing a fundamental system of Stein open neighborhoods $\{U_\lambda\}_\lambda$ of K , every $\mathcal{Y}_{ij,U_\lambda}^{\text{an}}$ is irreducible, hence connected. Since the irreducible components of $\mathcal{X}_{i,K}$ cover $\mathcal{X}_{i,K}$, [Lemma 5.3.3](#) implies that

$$\mathcal{X}_{i,U_\lambda}^{\text{an}} = \bigcup_{j=1}^{m_i} \mathcal{Y}_{ij,U_\lambda}^{\text{an}}$$

after shrinking S around K . It remains to check that this finite union is connected. Since $\mathcal{X}_{i,K}$ is connected, the incidence graph of its irreducible components is connected. Thus, for any two indices j, j' , there is a chain

$$j = j_0, j_1, \dots, j_r = j'$$

such that

$$\mathcal{Y}_{ij_\nu, K} \cap \mathcal{Y}_{ij_{\nu+1}, K} \neq \emptyset \quad 0 \leq \nu < r.$$

By [Lemma 5.3.3](#), after shrinking S around K , the analytic intersections

$$\mathcal{Y}_{ij_\nu, U_\lambda}^{\text{an}} \cap \mathcal{Y}_{ij_{\nu+1}, U_\lambda}^{\text{an}}$$

are non-empty for all ν and all sufficiently small U_λ . Hence the union above is connected. Therefore every $\mathcal{X}_{i, U_\lambda}^{\text{an}}$ is connected. This proves the connected case, and hence the lemma. \square

Corollary 5.4.9 *Let $\{\mathcal{X}_i\}_{1 \leq i \leq n}$ be a finite family of A -schemes interiorly of finite type, and let $K \subseteq S$ be an excellent Stein compact subset. Then, after replacing S by a Stein open neighborhood of K , the following assertions hold.*

(1) *There exist closed subschemes of finite presentation*

$$\mathcal{Y}_{ij} \subseteq \mathcal{X}_i, \quad 1 \leq j \leq k_i,$$

and a fundamental system $\{U_\lambda\}_\lambda$ of Stein open neighborhoods of K such that, for every i , the closed subschemes $\mathcal{Y}_{ij, K}$ are the reduced irreducible components of $\mathcal{X}_{i, K}$, and, for every λ , the underlying closed analytic subsets

$$|\mathcal{Y}_{ij, U_\lambda}^{\text{an}}|, \quad 1 \leq j \leq k_i,$$

are precisely the irreducible components of $\mathcal{X}_{i, U_\lambda}^{\text{an}}$.

(2) *There exist open and closed subschemes*

$$\mathcal{Z}_{ij} \subseteq \mathcal{X}_i, \quad 1 \leq j \leq \ell_i,$$

of finite presentation and a fundamental system $\{V_\lambda\}_\lambda$ of Stein open neighborhoods of K such that, for every i , the subschemes $\mathcal{Z}_{ij, K}$ are the connected components of $\mathcal{X}_{i, K}$, and, for every λ , the analytic subspaces

$$\mathcal{Z}_{ij, V_\lambda}^{\text{an}}, \quad 1 \leq j \leq \ell_i,$$

are precisely the connected components of $\mathcal{X}_{i, V_\lambda}^{\text{an}}$.

Proof After shrinking S around K , we may assume by [Corollary 5.2.1](#) that all \mathcal{X}_i are of finite presentation over A . Put

$$A_K = \Gamma(K, \mathcal{O}_S).$$

By [Theorem 4.4.2](#), A_K is noetherian. Hence each $\mathcal{X}_{i, K}$ is noetherian.

We first prove the assertion on irreducible components. For fixed i , let

$$Z_{i1}, \dots, Z_{ik_i}$$

be the irreducible components of $\mathcal{X}_{i,K}$, endowed with their reduced induced closed subscheme structures. By the limit theorem for finitely presented closed subschemes, [Stacks, Tag 01ZM], [Stacks, Tag 01ZP], after shrinking S around K we may choose closed subschemes of finite presentation

$$\mathcal{Y}_{ij} \subseteq \mathcal{X}_i, \quad 1 \leq j \leq k_i,$$

such that

$$\mathcal{Y}_{ij,K} = Z_{ij}$$

for all i, j .

We claim that, after shrinking S around K , the underlying analytic subsets $|\mathcal{Y}_{ij,U}^{\text{an}}|$ are the irreducible components of $\mathcal{X}_{i,U}^{\text{an}}$ for all sufficiently small Stein open neighborhoods U in a fundamental system. First, since the Z_{ij} cover $\mathcal{X}_{i,K}$, the constructible subset

$$|\mathcal{X}_i| \setminus \bigcup_{j=1}^{k_i} |\mathcal{Y}_{ij}|$$

has empty inverse image in $\mathcal{X}_{i,K}$. By Lemma 5.3.3, after shrinking S around K , its inverse image in $\mathcal{X}_i^{\text{an}}$ is empty. Thus

$$|\mathcal{X}_U^{\text{an}}| = \bigcup_{j=1}^{k_i} |\mathcal{Y}_{ij,U}^{\text{an}}|$$

for every sufficiently small Stein open neighborhood U of K .

Next, for $j \neq j'$, the constructible subset

$$|\mathcal{Y}_{ij}| \setminus \bigcup_{j' \neq j} |\mathcal{Y}_{ij'}|$$

has non-empty inverse image in $\mathcal{X}_{i,K}$, because Z_{ij} is an irreducible component of $\mathcal{X}_{i,K}$. Hence, again by Lemma 5.3.3, its inverse image in $\mathcal{X}_i^{\text{an}}$ is non-empty on every Stein open neighborhood of K . In particular, no $|\mathcal{Y}_{ij,U}^{\text{an}}|$ is contained in the union of the others.

Finally, each $\mathcal{Y}_{ij,K}$ is irreducible. Applying Lemma 5.4.3 to the finite family $\{\mathcal{Y}_{ij}\}_{i,j}$, we obtain a fundamental system $\{U_\lambda\}_\lambda$ of Stein open neighborhoods of K such that every $\mathcal{Y}_{ij,U_\lambda}^{\text{an}}$ is irreducible. For such U_λ , the space $\mathcal{X}_{i,U_\lambda}^{\text{an}}$ is a finite union of the irreducible closed analytic subsets $|\mathcal{Y}_{ij,U_\lambda}^{\text{an}}|$, none of which is contained in the union of the others. Therefore these subsets are exactly the irreducible components of $\mathcal{X}_{i,U_\lambda}^{\text{an}}$. This proves (1).

We now prove the assertion on connected components. For fixed i , let

$$C_{i1}, \dots, C_{i\ell_i}$$

be the connected components of $\mathcal{X}_{i,K}$. Since $\mathcal{X}_{i,K}$ is noetherian, these are finitely many open and closed subschemes of $\mathcal{X}_{i,K}$. By the limit theorem for open and closed subschemes, [Stacks, Tag 01ZM], [Stacks, Tag 01ZP], [Stacks, Tag 0EUU], after shrinking S around K we may choose open and closed subschemes

$$\mathcal{Z}_{ij} \subseteq \mathcal{X}_i, \quad 1 \leq j \leq \ell_i,$$

of finite presentation such that

$$\mathcal{Z}_{ij,K} = C_{ij}$$

for all i, j , and such that, for each i ,

$$\mathcal{X}_i = \bigsqcup_{j=1}^{\ell_i} \mathcal{Z}_{ij}.$$

After analytification, this gives

$$\mathcal{X}_U^{\text{an}} = \bigsqcup_{j=1}^{\ell_i} \mathcal{Z}_{ij,U}^{\text{an}}$$

for every Stein open neighborhood U of K .

Each $\mathcal{Z}_{ij,K}$ is connected. Applying [Lemma 5.4.3](#) to the finite family $\{\mathcal{Z}_{ij}\}_{i,j}$, we obtain a fundamental system $\{V_\lambda\}_\lambda$ of Stein open neighborhoods of K such that every $\mathcal{Z}_{ij,V_\lambda}^{\text{an}}$ is connected. Since, for each i , the analytic spaces $\mathcal{Z}_{ij,V_\lambda}^{\text{an}}$ form a finite open and closed disjoint decomposition of $\mathcal{X}_{i,V_\lambda}^{\text{an}}$, they are precisely the connected components of $\mathcal{X}_{i,V_\lambda}^{\text{an}}$. This proves (2). \square

Definition 5.4.1 Let \mathcal{X} be a scheme and let X be a complex space. We denote by $\mathcal{F}\text{Et}(\mathcal{X})$ the full subcategory of Sch/\mathcal{X} consisting of finite étale morphisms

$$\mathcal{Y} \longrightarrow \mathcal{X},$$

and by $\mathcal{F}\text{Et}(X)$ the full subcategory of $\text{CompSpace}/X$ consisting of finite étale morphisms

$$Y \longrightarrow X.$$

Theorem 5.4.6 *Let \mathcal{X} be an A -scheme interiorly of finite type, and let $K \subseteq S$ be an excellent Stein compact subset. Then analytification induces an equivalence of categories*

$$\mathcal{F}\text{Et}(\mathcal{X}_K) \xrightarrow{\sim} \lim_{\substack{\longrightarrow \\ U \supseteq K}} \mathcal{F}\text{Et}(\mathcal{X}_U^{\text{an}}), \quad (5.19)$$

where $U \subseteq S$ runs through the Stein open neighborhoods of K .

Proof By [Corollary 5.2.1](#), after shrinking S around K , we may assume that \mathcal{X} is of finite presentation over A .

We first construct the functor. Let

$$p_K: \mathcal{Y}_K \longrightarrow \mathcal{X}_K$$

be a finite étale morphism. By the limit theorem for schemes and morphisms of finite presentation, together with the spreading-out of the property of being finite étale, after shrinking S around K there exists a finite étale morphism

$$p: \mathcal{Y} \longrightarrow \mathcal{X}$$

whose base change to K is p_K . Then p^{an} is finite étale after shrinking S around K , by [Theorem 5.4.2](#). This defines the functor (5.19). The construction is independent of the chosen model, after passing to the filtered colimit over Stein neighborhoods of K , again by the limit theorem.

We prove full faithfulness. Let

$$p'_K: \mathcal{X}'_K \longrightarrow \mathcal{X}_K, \quad p''_K: \mathcal{X}''_K \longrightarrow \mathcal{X}_K$$

be two finite étale covers. After shrinking S around K , choose finite étale models

$$p': \mathcal{X}' \longrightarrow \mathcal{X}, \quad p'': \mathcal{X}'' \longrightarrow \mathcal{X}.$$

Put

$$\mathcal{Z} = \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'',$$

with projections

$$\pi_1: \mathcal{Z} \longrightarrow \mathcal{X}', \quad \pi_2: \mathcal{Z} \longrightarrow \mathcal{X}''.$$

The morphism π_1 is finite étale.

A morphism

$$u_K: \mathcal{X}'_K \longrightarrow \mathcal{X}''_K$$

over \mathcal{X}_K is the same thing as a section of

$$\pi_{1,K}: \mathcal{Z}_K \longrightarrow \mathcal{X}'_K.$$

Let

$$\mathcal{X}'_K = \coprod_a C_{a,K}$$

be the decomposition of \mathcal{X}'_K into connected components. Since $\pi_{1,K}$ is finite étale, the inverse image $\pi_{1,K}^{-1}(C_{a,K})$ is a finite disjoint union of connected components of \mathcal{Z}_K . For a fixed a , a section over $C_{a,K}$ is equivalent to the choice of a connected component

$$D_{b,K} \subseteq \mathcal{Z}_K$$

contained in $\pi_{1,K}^{-1}(C_{a,K})$ such that

$$\pi_{1,K}|_{D_{b,K}}: D_{b,K} \longrightarrow C_{a,K}$$

is an isomorphism. The corresponding morphism on $C_{a,K}$ is

$$C_{a,K} \xrightarrow{(\pi_{1,K}|_{D_{b,K}})^{-1}} D_{b,K} \hookrightarrow \mathcal{Z}_K \xrightarrow{\pi_{2,K}} \mathcal{X}_K''.$$

Thus a morphism $\mathcal{X}_K' \rightarrow \mathcal{X}_K''$ over \mathcal{X}_K is equivalent to a compatible choice of such components $D_{b,K}$ for all connected components $C_{a,K}$ of \mathcal{X}_K' .

By [Corollary 5.4.9](#), after shrinking S around K , the connected components of \mathcal{X}_K' and \mathcal{Z}_K are represented by open and closed subschemes of finite presentation whose analytifications are exactly the connected components of \mathcal{X}^{an} and \mathcal{Z}^{an} on a fundamental system of Stein neighborhoods of K . Moreover, by [Theorem 5.4.2](#), the condition that

$$\pi_1|_{D_b} : D_b \longrightarrow C_a$$

be an isomorphism is equivalent to the corresponding analytic condition after shrinking around K . Hence the preceding component description gives a bijection

$$\text{Hom}_{\text{Sch}/\mathcal{X}_K}(\mathcal{X}_K', \mathcal{X}_K'') \xrightarrow{\sim} \varinjlim_{U \supseteq K} \text{Hom}_{\text{CompSpace}/\mathcal{X}_U^{\text{an}}}(\mathcal{X}_U'^{\text{an}}, \mathcal{X}_U''^{\text{an}}).$$

This proves full faithfulness.

It remains to prove essential surjectivity. Let

$$q : Y \longrightarrow \mathcal{X}_U^{\text{an}}$$

be a finite étale morphism for some Stein open neighborhood U of K . Replacing S by U , we may assume that $U = S$. Since finite étale covers form a stack for the analytic topology, and since full faithfulness has already been proved, essential surjectivity is local on \mathcal{X} . After shrinking around K , we may therefore assume that \mathcal{X} is affine and of finite presentation over A .

We first reduce to the case where \mathcal{X}_K is reduced. Let

$$\mathcal{X}_{0,K} \hookrightarrow \mathcal{X}_K$$

be the reduction. By the limit theorem for closed subschemes of finite presentation, after shrinking S around K , this closed immersion spreads out to a closed immersion

$$\mathcal{X}_0 \hookrightarrow \mathcal{X}.$$

After shrinking further, the induced closed immersion

$$\mathcal{X}_0^{\text{an}} \hookrightarrow \mathcal{X}^{\text{an}}$$

is a nilpotent thickening as a germ along K . Finite étale covers are invariant under nilpotent thickenings, both algebraically and analytically; algebraically this is [[Stacks, Tag 0BQB](#)]. Hence the assertion for \mathcal{X} is equivalent to the assertion for \mathcal{X}_0 . We may therefore assume that \mathcal{X}_K is reduced.

We next reduce to the case where \mathcal{X}_K is normal. Since K is excellent, the scheme \mathcal{X}_K is excellent. Let

$$\nu_K : \tilde{\mathcal{X}}_K \longrightarrow \mathcal{X}_K$$

be the normalization. It is finite and surjective. By the limit theorem for finite morphisms, after shrinking S around K , there exists a finite morphism

$$\nu : \tilde{\mathcal{X}} \longrightarrow \mathcal{X}$$

whose base change to K is ν_K .

Assume essential surjectivity has been proved for $\tilde{\mathcal{X}}$. Pulling q back along

$$\nu^{\text{an}} : \tilde{\mathcal{X}}^{\text{an}} \longrightarrow \mathcal{X}^{\text{an}},$$

we obtain a finite étale cover of $\tilde{\mathcal{X}}^{\text{an}}$. By the normal case, after shrinking S around K , this cover is the analytification of a finite étale cover

$$\tilde{\mathcal{Y}}_K \longrightarrow \tilde{\mathcal{X}}_K.$$

The analytic cover carries its canonical descent datum relative to

$$\tilde{\mathcal{X}}^{\text{an}} \times_{\mathcal{X}^{\text{an}}} \tilde{\mathcal{X}}^{\text{an}} \rightrightarrows \tilde{\mathcal{X}}^{\text{an}},$$

because it is pulled back from q . By full faithfulness, this analytic descent datum is induced by a unique algebraic descent datum on $\tilde{\mathcal{Y}}_K$ relative to

$$\tilde{\mathcal{X}}_K \times_{\mathcal{X}_K} \tilde{\mathcal{X}}_K \rightrightarrows \tilde{\mathcal{X}}_K.$$

The cocycle condition holds algebraically because it holds after analytification and full faithfulness detects equality of morphisms. Since ν_K is finite surjective, hence integral and surjective, effective descent for étale morphisms applies, [Stacks, Tag 0BTH]. Thus the descent datum is effective and gives an étale morphism

$$\mathcal{Y}_K \longrightarrow \mathcal{X}_K$$

whose analytification is q as a germ along K . Finally, since q is finite, [Theorem 5.4.2](#) applied to the property of being finite shows that, after shrinking around K , the morphism $\mathcal{Y}_K \rightarrow \mathcal{X}_K$ is finite. Hence it is finite étale. This proves the reduction to the normal case.

We now assume that \mathcal{X} is affine and that \mathcal{X}_K is normal. Choose, after shrinking S around K , a projective compactification

$$j : \mathcal{X} \hookrightarrow \mathcal{P}$$

such that \mathcal{P} is projective over $\text{Spec } A$, and such that \mathcal{P}_K is normal and $\mathcal{X}_K \subseteq \mathcal{P}_K$ is a dense open subscheme. This compactification is constructed as in the proof of [Lemma 5.4.3](#): take a projective closure of \mathcal{X}_K , normalize it, and spread it out. By [Theorem 5.4.2](#) and [Corollary 5.4.1](#), after shrinking around K , the analytification

$$j^{\text{an}}: \mathcal{X}^{\text{an}} \hookrightarrow \mathcal{P}^{\text{an}}$$

is a dense open immersion and \mathcal{P}^{an} is normal. Put

$$H = \mathcal{P}^{\text{an}} \setminus \mathcal{X}^{\text{an}}.$$

Then H is a nowhere dense closed analytic subset of \mathcal{P}^{an} .

By the Grauert–Remmert extension theorem, [Theorem 2.3.3](#), the finite étale cover

$$q: Y \longrightarrow \mathcal{X}^{\text{an}}$$

extends uniquely to a finite morphism

$$\bar{q}: \bar{Y} \longrightarrow \mathcal{P}^{\text{an}}$$

with \bar{Y} normal, and whose restriction to \mathcal{X}^{an} is q .

Since \mathcal{P} is interiorly proper and interiorly locally of finite type, [Corollary 5.4.6](#) applies to \mathcal{P} . The coherent analytic algebra

$$\bar{q}_* \mathcal{O}_{\bar{Y}}$$

therefore has an algebraic lift, as a germ along K , to a coherent $\mathcal{O}_{\mathcal{P}_K}$ -module \mathcal{B}_K . By the full faithfulness part of [Corollary 5.4.4](#) with $n = 0$, the multiplication and unit maps

$$\bar{q}_* \mathcal{O}_{\bar{Y}} \otimes_{\mathcal{O}_{\mathcal{P}^{\text{an}}}} \bar{q}_* \mathcal{O}_{\bar{Y}} \longrightarrow \bar{q}_* \mathcal{O}_{\bar{Y}}, \quad \mathcal{O}_{\mathcal{P}^{\text{an}}} \longrightarrow \bar{q}_* \mathcal{O}_{\bar{Y}}$$

come from unique algebraic maps

$$\mathcal{B}_K \otimes_{\mathcal{O}_{\mathcal{P}_K}} \mathcal{B}_K \longrightarrow \mathcal{B}_K, \quad \mathcal{O}_{\mathcal{P}_K} \longrightarrow \mathcal{B}_K.$$

The associativity, commutativity, and unit identities hold analytically; by full faithfulness, they hold algebraically. Hence \mathcal{B}_K is a coherent $\mathcal{O}_{\mathcal{P}_K}$ -algebra.

Define

$$\bar{\mathcal{Y}}_K = \underline{\text{Spec}}_{\mathcal{P}_K}(\mathcal{B}_K).$$

Then

$$\bar{\mathcal{Y}}_K \longrightarrow \mathcal{P}_K$$

is finite, and its analytification is identified, as a germ along K , with

$$\bar{q}: \bar{Y} \longrightarrow \mathcal{P}^{\text{an}}.$$

Now set

$$\mathcal{Y}_K = \bar{\mathcal{Y}}_K \times_{\mathcal{P}_K} \mathcal{X}_K.$$

Then the analytification of

$$\mathcal{Y}_K \longrightarrow \mathcal{X}_K$$

is identified with

$$q: Y \longrightarrow \mathcal{X}^{\text{an}}$$

as a germ along K . Since q is finite étale, [Theorem 5.4.2](#) applied to the properties of being finite and étale implies that $\mathcal{Y}_K \rightarrow \mathcal{X}_K$ is finite étale. Thus q lies in the essential image of [\(5.19\)](#).

This proves essential surjectivity, and hence the theorem. \square

Comments

A brief history

Chapter 1.

The results about universally finite derivations are mostly due to Scheja–Storch [SS72]. Proposition 1.2.7 seems new, even though it has already been used by Bingener [Bin76a].

The construction in Section 1.5.3 is due to Forster [For67]. I reformulated Forster’s constructions in modern language.

Chapter 2.

The basic reference to the local analytic algebras is Grauert–Remmert’s book [GR71].

Theorem 2.3.3 is due to [GR58]. Our proof follows [SGA I, Exposé XII.5.4].

Chapter 3.

Theorem 3.3.1 is usually attributed to H. Cartan [Car50], even though Cartan’s proof only works in the reduced setting. As far as I know, our proof is the first one in the literature.

Theorem 3.3.3 is due to Benoist [Ben25a] and Forster [For67].

I learned the various flatness theorems Lemmas 3.3.7 and 3.3.8, Theorem 3.3.4, and Proposition 4.3.4 from O. Benoist, who attributed the idea of these arguments to Kucharz [Kuc05, Lemma 2.2]. These results are necessary for Bingener’s results [Bin76b]. I do not know any place where the proofs are written explicitly.

Section 3.4 is essentially due to [For67].

Results in Section 3.5 are due to Henriksen [Hen52].

Chapter 4.

Theorem 4.2.1 is due to H. Cartan [Car57].

Theorem 4.3.2 is due to Zame [Zam72a, Zam72b, Zam73]. Proposition 4.4.3 is due to Zame [Zam76]. Theorem 4.4.1 is due to Siu [Siu69]; our proof follows [Zam76] closely. In the cited papers of Zame, there are certain details which I do not fully understand, so the proofs are essentially rewritten.

Theorem 4.4.2 is due to [Bin76a]. However, I was unable to follow Bingener's argument. **Corollary 4.4.7** is due to [Ben25b]. **Corollary 4.4.6** seems new, but it was already used by Bingener in [Bin76b].

Chapter 5.

Most non-trivial results in this chapter are due to Bingener [Bin76b]. I made some marginal improvements and corrections at various places.

Index

A

algebra of germs **101**
algebra of subsets **101**
analytic property **142**
analytic subset **51**
analytic subspace germ **103**
analytification **135**
 cohomology comparison **169**
 of modules **159**
 of morphisms **163**
analytification functor **144**

B

Bingener analytification **144**

C

canonical Fréchet topology **40**
canonical topology **101**
Cartan's theorem A **40, 71**
Cartan's theorem B **71**
category
 coherent modules, Coh **86**
 compacta, Compactum **103**
 complex spaces, CompSpace **21**
 Fréchet algebras, FrAlg **34**
 germs of pairs of complex spaces, Germ **103**
 locally A -ringed spaces, $\text{LocRing}/_A$ **20**
 locally \mathbb{C} -ringed spaces, $\text{LocRing}/_{\mathbb{C}}$ **21**
 locally convex topological vector spaces, LCS **22**
 locally ringed spaces, LocRing **20**
 modules, Mod **86**
 product-Fréchet algebras, PFrAlg **35**

schemes, Sch **184**
sets, Set **144**
Stein A -modules, $\text{Mod}_A^{\text{Stein}}$ **86**
Stein algebras, SAlg **69**
Stein spaces, SteinSpace **79**
topological \mathbb{C} -algebras, TopAlg **24**
topological \mathbb{C} -algebras with spectral
 homomorphisms, TopAlg^s **25**

character map **48**
closed ideal **119**
closed immersion
 of germs **107**
coherent sheaf **40**
 Stein module **85**
cohomology comparison **169**
commutative algebra **9**
compact space **1**
compactum **103**
complex space **20, 37**
 local analytic algebra **37**
 morphisms **140**
continuous character **24**
convergent power series **9**

D

divisor
 of an entire function **90**
divisor filter **91**

E

embedding dimension **37**
entire function
 divisor **90**
entire function ring **89**
 maximal ideals **95**

- envelope **48**
- essentially irreducible germ **107**
- excellence **126**
- F**
- faithfully flat descent **135**
- filter
 - of divisors **91**
- finite presentation **136**
- finitely generated ideal **94**
- fixed ideal **93**
- flat module **136**
- fpqc local property **135**
- Fréchet algebra **34**
- Fréchet module **34**
- Fréchet topology **40**
- free ideal **93**
- G**
- Gelfand spectrum **24**
 - structure sheaf **27**
- Gelfand transform **48**
- germ
 - along a subset **101**
 - along a subspace **103**
- H**
- holomorphic convexity **59**
- holomorphically convex hull **59**
- I**
- ideal
 - finitely generated **94**
 - fixed and free **93**
 - in the section ring of a Stein compactum **119**
- interior property **142**
- K**
- Kähler differentials **9**
- L**
- local analytic algebra **37**
- locally \mathbb{C} -ringed space **27**
- locally compact space **1**
- locally convex topological vector space **21**
- locally ringed space **20**
- M**
- maximal ideal **95**
 - residue field **97**
- module
 - analytification **159**
- morphism
 - analytification **163**
 - to a Stein compactum **114**
- P**
- primary decomposition **71**
- prime divisor filter **92**
- prime ideal **99, 138**
- product-Fréchet algebra **34**
- Q**
- quasi-Noetherian ring **9**
- R**
- radical ideal **138**
- reduced complex space **51**
- reduced germ **107**
- regular local ring **9**
- residue field **97**
- ring of entire functions **89**
- Runge property **64**
- Runge subset **64**
- S**
- scheme **140**
- section ring
 - of a complex space **40**
 - of a Stein compactum **118**
 - of a Stein space **69**
- spectral homomorphism **24**
- spectrum
 - of the entire function ring **92**
 - Zariski topology **100**
- Stein algebra **61, 69**
 - equivalence of categories **76**
- Stein compact subset **61, 109**
- Stein compactum **101, 109**
 - compact subsets **109**
 - excellence **126**
 - morphism **114**
 - section ring **118**
- Stein module **85**
- Stein space **61**
 - equivalence of categories **76**
- structure sheaf **27**
- T**
- topological algebra **23**
- Z**
- Zariski topology **100**
- 0-dimensional Stein algebra **69**

References

- Ben25a. Olivier Benoist. Étale cohomology of algebraic varieties over Stein compacta. *Invent. Math.*, 240(2):497–536, 2025.
- Ben25b. Olivier Benoist. Stein spaces and Stein algebras. *Bull. Pol. Acad. Sci. Math.*, 73(2):183–190, 2025.
- Bin76a. Jürgen Bingener. Holomorph-prävollständige Resträume zu analytischen Mengen in Steinschen Räumen. *J. Reine Angew. Math.*, 285:149–171, 1976.
- Bin76b. Jürgen Bingener. Schemata über Steinschen Algebren. *Schr. Math. Inst. Univ. Münster* (2), page 52, 1976.
- BS76. Constantin Bănică and Octavian Stănășilă. *Algebraic methods in the global theory of complex spaces*. Editura Academiei, Bucharest; John Wiley & Sons, London-New York-Sydney, 1976. Translated from the Romanian.
- Car50. Henri Cartan. Idéaux et modules de fonctions analytiques de variables complexes. *Bull. Soc. Math. France*, 78:29–64, 1950.
- Car57. Henri Cartan. Variétés analytiques réelles et variétés analytiques complexes. *Bull. Soc. Math. France*, 85:77–99, 1957.
- CAS. Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984.
- Čes21. Kestutis Česnavičius. Macaulayfication of Noetherian schemes. *Duke Math. J.*, 170(7):1419–1455, 2021.
- EGA IV₁. Jean Dieudonné and Alexandre Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Première partie*, volume 20. Institut des hautes études scientifiques, 1964.
- EGA IV₂. Jean Dieudonné and Alexandre Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie*, volume 24. Institut des hautes études scientifiques, 1965.
- EGA IV₃. Jean Dieudonné and Alexandre Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie*, volume 28. Institut des hautes études scientifiques, 1966.
- Fis76. Gerd Fischer. *Complex analytic geometry*, volume Vol. 538 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1976.
- For67. Otto Forster. Zur Theorie der Steinschen Algebren und Moduln. *Math. Z.*, 97:376–405, 1967.
- For17. Franc Forstnerič. *Stein manifolds and holomorphic mappings*, volume 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, second edition, 2017. The homotopy principle in complex analysis.

- Gil11. W. D. Gillam. Localization of ringed spaces, 2011.
- GR58. Hans Grauert and Reinhold Remmert. Komplexe Räume. *Math. Ann.*, 136:245–318, 1958.
- GR71. H. Grauert and R. Remmert. *Analytische Stellenalgebren*, volume Band 176 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1971. Unter Mitarbeit von O. Riemenschneider.
- GR77. H. Grauert and R. Remmert. *Theorie der Steinschen Räume*, volume 227 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1977.
- Hen52. Melvin Henriksen. On the ideal structure of the ring of entire functions. *Pacific J. Math.*, 2:179–184, 1952.
- Hou61a. Christian Houzel. Géométrie analytique locale, II. Théorie des morphismes finis. *Séminaire Henri Cartan*, 13(2):1–22, 1960–1961. Exposé no. 19.
- Hou61b. Christian Houzel. Géométrie analytique locale, III. *Séminaire Henri Cartan*, 13(2):1–25, 1960–1961. Exposé no. 20.
- Jar81. Hans Jarchow. *Locally Convex Spaces*. Mathematische Leitfäden. B. G. Teubner, Stuttgart, 1981.
- Kuc05. W. Kucharz. Meromorphic functions and factoriality. *Proc. Amer. Math. Soc.*, 133(7):2013–2021, 2005.
- SGA I. Alexandre Grothendieck and Michele Raynaud. *Revêtements Étales et Groupe Fondamental: Séminaire de Géométrie Algébrique du Bois Marie 1960/61*, volume 224. Springer, 2006.
- Siu69. Yum-tong Siu. Noetherianness of rings of holomorphic functions on Stein compact series. *Proc. Amer. Math. Soc.*, 21:483–489, 1969.
- SS72. Günter Scheja and Uwe Storch. Differentielle Eigenschaften der Lokalisierungen analytischer Algebren. *Math. Ann.*, 197:137–170, 1972.
- Stacks. The Stacks Project Authors. Stacks project. <http://stacks.math.columbia.edu>, 2020.
- SW99. H. H. Schaefer and M. P. Wolff. *Topological vector spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1999.
- Zam72a. William R. Zame. Algebras of analytic germs. *Trans. Amer. Math. Soc.*, 174:275–288, 1972.
- Zam72b. William R. Zame. Homomorphisms of rings of germs of analytic functions. *Proc. Amer. Math. Soc.*, 33:410–414, 1972.
- Zam73. William Zame. Induced homomorphisms of algebras of analytic germs. *Rice Univ. Stud.*, 59(2):157–163, 1973.
- Zam76. William R. Zame. Holomorphic convexity of compact sets in analytic spaces and the structure of algebras of holomorphic germs. *Trans. Amer. Math. Soc.*, 222:107–127, 1976.