ANALYTIC BERTINI THEOREM II — THE LOCAL CASE

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ABSTRACT. We prove the local analytic Bertini theorem, confirming a conjecture of Boucksom in full generality.

1. Introduction

For a non-negative integer k, write Δ^k for the unit polydisc of dimension k. The main result of the paper is the following:

Theorem 1.1 (Local analytic Bertini theorem)**.** *Let m, n be non-negative integers and*

$$X = \Delta_z^m \times \Delta_\eta^n, \qquad \Phi \in \mathrm{PSH}(X). \tag{1.1}$$

For $\eta \in \Delta_\eta^n$, put $X_\eta \coloneqq \Delta_z^m \times \{\eta\}$. There is a pluripolar set $P \subseteq \Delta_\eta^n$ such that, for every $\eta \notin P$, the multiplier ideal sheaves satisfy

$$\mathcal{I}_X(\Phi) \cdot \mathcal{O}_{X_\eta} = \mathcal{I}_{X_\eta}\left(\Phi|_{X_\eta}\right) \tag{1.2}$$

as ideal sheaves on X_η.

Here and throughout, $\mathcal{I}_X(\Phi) \cdot \mathcal{O}_{X_\eta}$ denotes the image ideal in $\mathcal{O}_{X_\eta}$.

The inclusion $\supseteq$ in (1.2) is a simple consequence of the Ohsawa–Takegoshi extension theorem. From the previous works of Fujino–Matsumura and Meng–Zhou [FM21; MZ23], we know that the exceptional set of η is Lebesgue null, and Boucksom asked whether the exceptional set can be taken as pluripolar or not. Previously, for a projective fibration, the author confirmed this conjecture in [Xia22]. The proof relies on the technique of positivity of direct images of Berndtsson [Ber09; HPS18] and does not generalize immediately to the local setting.

In the local setting, the main difficulty is that, due to the non-compactness of the fibers, the direct image of $\mathcal{I}(\Phi)$ on Δ_η^n does not admit any natural geometric structure[1] We shall replace the direct image argument used in the global setting by a jet bundle argument, and Berndtsson's positivity theorem by the local psh variation theorem of Bergman kernels, again due to Berndtsson and generalized by Bao–Guan–Yuan [Ber06; BGY22]. The idea of the proof is very close to the global setting, though the technical details are much more difficult.

As an intermediate step, we prove a general criterion, Theorem 2.3, that upgrades a Lebesgue null set to a pluripolar set. The proof of this criterion relies crucially on the fine pluripotential theory developed by many authors, including El Kadiri, Fuglede, Wiegerinck, and El Marzguioui.

Theorem 1.1 also extends to a general fibration, as shown in Section 5.

Corollary 1.2. *Let $f\colon Y \to Z$ be a morphism of (σ-compact) complex manifolds and let $\Phi \in \mathrm{PSH}(Y)$. There is a pluripolar set $P \subseteq Z$ such that, for every $z \in Z \setminus P$, the fiber $Y_z \coloneqq f^{-1}(z)$ is a (possibly empty) complex manifold and*

$$\mathcal{I}_Y(\Phi) \cdot \mathcal{O}_{Y_z} = \mathcal{I}_{Y_z}(\Phi|_{Y_z}). \tag{1.3}$$

Theorem 1.1 and Corollary 1.2 are the first step towards understanding the deformations of psh singularities, a project which the author wishes to work on in the near future.

[1] It is a quasi-coherent sheaf once the language is properly set up, but this does not seem very useful for the present problem.

Acknowledgments. AI plays a key role in this project. The idea of the proof was due to the author before the AI era. The details were, however, first carried out by the Rethlas system using the ChatGPT 5.6 Sol model. The AI-generated proofs were later simplified and largely rewritten by the author.

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2. Preliminaries

2.1. Multiplier ideals and the almost-everywhere case. Given a psh function φ on a complex manifold Y, the multiplier ideal sheaf $\mathcal{I}(\varphi)=\mathcal{I}_Y(\varphi)$ consists of the holomorphic germs f for which $|f|^2\mathrm{e}^{-\varphi}$ is locally integrable. By Nadel's coherence theorem, $\mathcal{I}_Y(\varphi)$ is coherent. We use the convention $\mathcal{I}_Y(-\infty)=0$.

Given a function $f(z,\eta)$ depending on two variables, we shall write f_η for the function sending z to $f(z,\eta)$.

Fix m,n, X and Φ as in (1.1). For each $\eta\in\Delta^n_\eta$, write

$$J\coloneqq \mathcal{I}_X(\Phi),\qquad J_\eta\coloneqq J\cdot\mathcal{O}_{X_\eta},\qquad I_\eta\coloneqq\mathcal{I}_{X_\eta}(\Phi_\eta). \tag{2.1}$$

The Ohsawa–Takegoshi extension theorem gives

$$I_\eta\subseteq J_\eta. \tag{2.2}$$

The reverse inclusion holds on almost every fiber by Fubini's theorem. Indeed, on product charts $U'=W'\times G'\Subset U=W\times G\Subset X$, choose finitely many generators of $J|_U$. Their weighted squares are integrable on U'. Fubini's theorem therefore gives a null set in G' outside which all restricted generators belong to I_η on W'. Thus $J_\eta\subseteq I_\eta$ there. A countable cover by relatively compact product charts and (2.2) give

$$G_{\mathrm{ae}}\coloneqq\{\eta\in\Delta^n_\eta : I_\eta=J_\eta\},\qquad |\Delta^n_\eta\setminus G_{\mathrm{ae}}|=0, \tag{2.3}$$

where $|\cdot|$ denotes Lebesgue measure. This is also established in [MZ23].

2.2. Weighted Bergman kernels. For $k\in\mathbb{N}$ and $\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^k$, write

$$|\alpha|\coloneqq\sum_{j=1}^k\alpha_j,\qquad \alpha!\coloneqq\prod_{j=1}^k\alpha_j!,\qquad \partial_u^\alpha\coloneqq\prod_{j=1}^k\frac{\partial^{\alpha_j}}{\partial u_j^{\alpha_j}},\qquad D_u^\alpha\coloneqq\frac{1}{\alpha!}\partial_u^\alpha.$$

If $U\subseteq\mathbb{C}^k$ is a domain and $\varphi\in\mathrm{PSH}(U)$, define

$$A^2(U,\mathrm{e}^{-\varphi})\coloneqq\left\{f\in\mathcal{O}(U):\|f\|^2_{U,\varphi}\coloneqq\int_U|f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V<\infty\right\}.$$

Here and in the sequel, $\mathrm{d}V$ denotes standard Lebesgue measure.

Let $q,s\in\mathbb{N}$. Given finitely supported coefficients $(c_\alpha)_{\alpha\in\mathbb{N}^q}$ in $\mathbb{C}$, put

$$\mathcal{D}\coloneqq\sum_{\alpha\in\mathbb{N}^q}c_\alpha D_u^\alpha.$$

Let $V\subseteq\mathbb{C}^q_u$ be a bounded pseudoconvex domain, let $G\subseteq\mathbb{C}^s_\eta$ be a domain, and let $\psi\in\mathrm{PSH}(V\times G)$. For $\eta\in G$, consider the weighted Bergman kernels:

$$K^{\psi_\eta}_{\mathcal{D},V}(u)\coloneqq\sup_{0\neq f\in A^2(V,\mathrm{e}^{-\psi_\eta})}\frac{|(\mathcal{D}f)(u)|^2}{\int_V|f|^2\mathrm{e}^{-\psi_\eta}\,\mathrm{d}V},\qquad u\in V.$$

If the space $A^2(V,\mathrm{e}^{-\psi_\eta})$ is trivial, we understand that $K^{\psi_\eta}_{\mathcal{D},V}\equiv0$.

We use the following form of Berndtsson's psh variation theorem [Ber06].

Theorem 2.1. *With the notations above,*

$$(u,\eta)\longmapsto\log K^{\psi_\eta}_{\mathcal{D},V}(u)$$

is plurisubharmonic or identically $-\infty$ on $V\times G$.

This is a special case of a much more general result of Bao–Guan–Yuan [BGY22, Theorem 1.1]; see also [BG23].

2.3. **A pluripolar zero-set criterion.** Let $\mathcal{F}$ denote the plurifine topology, the coarsest topology making every psh function continuous. Topological notions relative to the plurifine topology will be indicated by the prefix $\mathcal{F}$-. In this section, we assume some familiarity with fine pluripotential theory. The results that we need in this section are proved in several articles; see, for example, [EMW06; EKFW11; EM09]. A good reference is the book [EKF25].

Recall that a function $v\colon V \to [-\infty,\infty)$ on an $\mathcal{F}$-open set $V \subseteq \mathbb{C}^n$ is *(weakly) $\mathcal{F}$-psh* if it is $\mathcal{F}$-upper semicontinuous and, for every complex affine line Λ, its restriction to each fine component of $V \cap \Lambda$ is finely subharmonic or identically $-\infty$. The $\mathcal{F}$-psh functions satisfy the usual sheaf property, and if V is a connected open set, they are the same as the usual notion of psh functions. See [EKF25, Theorem 5.4.2(i), Proposition 5.4.6].

Lemma 2.2. *Every nonempty $\mathcal{F}$-open set has positive Lebesgue measure.*

For completeness, we provide a proof.

Proof. By the Bedford–Taylor base theorem [EKF25, Theorem 5.2.8], it suffices to show that a nonempty set of the form
$$\{x \in B : h(x) > 0\}$$
has positive measure, where $B \subseteq \mathbb{C}^n$ is the unit ball and $h \in \mathrm{PSH}(B)$. If this set were null, then $h \leq 0$ almost everywhere and hence everywhere by [Xia, Proposition 1.2.6]. This is a contradiction. □

For a polydisc B, let $2B$ denote the concentric polydisc with twice its polyradii. The following pluripolarity criterion is the main new ingredient in this paper:

Theorem 2.3. *Let $B \subseteq \mathbb{C}^m$ be a polydisc and let $W \subseteq \mathbb{C}^m$ be open with $\overline{2B} \Subset W$. Let $G \subseteq \mathbb{C}^n$ be open and let $V \subseteq G$ be $\mathcal{F}$-open. For every $\eta \in G$, let $\mathcal{K}_\eta$ be a complex Hilbert space and $F_\eta\colon W \to \mathcal{K}_\eta$ a holomorphic map. Assume that*
$$U(w,\eta) := \log \|F_\eta(w)\|^2_{\mathcal{K}_\eta} \in \mathrm{PSH}(W \times G),$$
and, for some $M > 0$, that
$$U > -M \quad \text{on } \partial B \times V, \qquad \sup_{\overline{2B}\times V} U < \infty. \tag{2.4}$$
If a conull set $G_0 \subseteq G$ (i.e., the complement $G \setminus G_0$ is Lebesgue null) satisfies
$$F_\eta(w) \neq 0 \qquad (\eta \in G_0,\ w \in \overline{2B}),$$
then
$$Z := \{\eta \in V : F_\eta^{-1}(0) \cap B \neq \varnothing\}$$
is pluripolar.

For the elementary facts about Hilbert-space-valued holomorphic functions, we refer to [Rud91, Chapter 3]. The idea of the proof is to use the boundary condition (2.4) to extend U in the w-variable to $\mathbb{P}^m$, and apply Berndtsson's positivity of direct images there in the same way as in [Xia22].

Proof. We may assume that $V \neq \varnothing$. Since, by [EKF25, Corollary 5.2.21], V has at most countably many $\mathcal{F}$-connected components, all of which are $\mathcal{F}$-open, we may replace V by each one of these components and reduce to the case where V is $\mathcal{F}$-connected.

Step 1. Put $C := \sup_{\overline{2B}\times V} U$. Embed $\overline{2B}$ in the standard affine chart of $\mathbb{P}^m$, and let ω_{FS} be the Fubini–Study form. Take $h \in C^\infty(\mathbb{P}^m)$ such that
$$h < -M-3 \quad \text{near } \partial B, \qquad h > C+3 \quad \text{near } \partial(2B).$$
Choose an integer d_0 sufficiently large such that h is ω-psh for $\omega := d_0\omega_{\mathrm{FS}}$ and $K_{\mathbb{P}^m} \otimes \mathcal{O}_{\mathbb{P}^m}(md_0)$ is globally generated. Let $L := \mathcal{O}_{\mathbb{P}^m}(d_0)$ with its Fubini–Study metric h_L of curvature ω, put
$$\mathcal{S} := \mathrm{H}^0(\mathbb{P}^m, K_{\mathbb{P}^m} \otimes L^{\otimes m}),$$

and fix a basis $s_1,\dots,s_\mu$ of $\mathcal{S}$.

For $\eta\in V$, set

$$\Psi_\eta := \begin{cases} U_\eta, & \text{on } B,\\ \max_\varepsilon(U_\eta,h), & \text{on } 2B\setminus B,\\ h, & \text{on } \mathbb{P}^m\setminus 2B.\end{cases}$$

Here $\max_\varepsilon$ denotes the usual regularized maximum.[2] For all sufficiently small $\varepsilon>0$, independently of η, we have $\Psi_\eta\in \mathrm{PSH}(\mathbb{P}^m,\omega)$. Fix such an ε for the rest of the proof.

Step 2. Fix $\eta_0\in V$. By Brelot's Theorem [EKF25, Corollary 5.2.10], there is a compact set K with

$$\eta_0\in \mathrm{int}_{\mathcal{F}} K,\qquad K\subseteq V.$$

Here $\mathrm{int}_{\mathcal{F}} K$ denotes the plurifine interior of K.

Choose standard radial convolutions $U_k\downarrow U$, smooth and psh on a fixed neighborhood of $\overline{2B}\times K$. Then $U_k>-M>h+2$ on $\partial B\times K$. Here we have omitted the obvious pull-back notation. The compact sets

$$\{U_k\geq C+1\}\cap(\partial(2B)\times K)$$

are nested with empty intersection because $U_k\downarrow U\leq C$; hence one of them is empty. After discarding finitely many terms, $U_k<C+1$ on $\partial(2B)\times K$. Therefore,

$$U_k>h+2\quad \text{on } \partial B\times K,\qquad U_k<h-2\quad \text{on } \partial(2B)\times K.$$

These estimates still hold if we replace K by a sufficiently small open neighborhood $O_k\Subset G$. Let $p_1\colon \mathbb{P}^m\times O_k\to\mathbb{P}^m$ be the projection. The function

$$\Psi_k := \begin{cases} U_k, & \text{on } B\times O_k,\\ \max_\varepsilon(U_k,h), & \text{on } (2B\setminus B)\times O_k,\\ h, & \text{on } (\mathbb{P}^m\setminus 2B)\times O_k\end{cases}$$

is smooth and $p_1^*\omega$-psh on $\mathbb{P}^m\times O_k$. Moreover,

$$\Psi_k(p,\eta)\downarrow\Psi_\eta(p)\qquad ((p,\eta)\in\mathbb{P}^m\times K).$$

For any ω-psh function Θ on $\mathbb{P}^m$, set

$$q_\Theta(s) := \mathrm{i}^{m^2}\int_{\mathbb{P}^m}|s|^2_{h_L^m}\mathrm{e}^{-m\Theta}\in[0,\infty],\qquad s\in\mathcal{S}.$$

Here $\mathrm{i}^{m^2}|s|^2_{h_L^m}$ is regarded as a positive Radon measure on $\mathbb{P}^m$.

When q_Θ is finite on $\mathcal{S}$, let $\det q_\Theta$ denote the determinant of its Gram matrix in the fixed basis. For $\eta\in O_k$, put

$$q_{k,\eta} := q_{\Psi_k(\,\cdot\,,\eta)},\qquad \chi_k(\eta) := -\log\det q_{k,\eta}.$$

The form $q_{k,\eta}$ is finite and positive definite. By [Ber09, Theorem 1.2], the function χ_k is psh on O_k.

For $\eta\in V$, put $q_\eta := q_{\Psi_\eta}$ and

$$\chi(\eta) := \begin{cases} -\log\det q_\eta, & q_\eta \text{ is finite on } \mathcal{S},\\ -\infty, & \text{otherwise.}\end{cases}$$

For $\eta\in\mathrm{int}_{\mathcal{F}} K$, monotone convergence gives $q_{k,\eta}(s)\uparrow q_\eta(s)$ for every $s\in\mathcal{S}$. If q_η is finite, the Gram matrices of $q_{k,\eta}$ converge to that of q_η. Otherwise one eigenvalue of the Gram matrix of $q_{k,\eta}$ tends to infinity, while $q_{k,\eta}\geq q_{1,\eta}>0$, so $\det q_{k,\eta}\to\infty$. Thus in both cases

$$\chi_k\downarrow\chi\qquad \text{on } \mathrm{int}_{\mathcal{F}} K.$$

By [EKF25, Theorem 5.4.2(i) and (iv)], χ is $\mathcal{F}$-psh on $\mathrm{int}_{\mathcal{F}} K$ or identically $-\infty$. Since η_0 is arbitrary, χ is $\mathcal{F}$-psh or identically $-\infty$.

[2] One can also use the usual maximum here, since Berndtsson's theorem has been extended to this setting as well, see [HPS18] for example.

By Lemma 2.2, V has positive Lebesgue measure. In particular, $V \setminus Z$, which contains $V \cap G_0$, is nonempty. If $\eta \in V \setminus Z$, the inequality $U_\eta > -M$ on ∂B and the definition of Z show that F_η is nowhere zero on $\overline{B}$. Thus U_η is bounded on $\overline{B}$, Ψ_η is bounded on $\mathbb{P}^m$, and q_η is finite and positive definite. Therefore χ is finite on $V \setminus Z$ and is not identically $-\infty$. We conclude that χ is $\mathcal{F}$-psh.

Step 3. By [EKF25, Theorem 5.4.3(d)], the set $\{x \in V : \chi(x) = -\infty\}$ is pluripolar.

Take $\eta \in Z$ and $w_* \in B$ with $F_\eta(w_*) = 0$. A local bound for ∇F_η, followed by integration along line segments, gives

$$\|F_\eta(w)\|_{\mathcal{K}_\eta} \le C_*|w - w_*|, \qquad U_\eta(w) \le 2\log|w - w_*| + O(1) \tag{2.5}$$

for any $w \in 2B$. Since $K_{\mathbb{P}^m} \otimes L^{\otimes m}$ is globally generated, choose $s \in \mathcal{S}$ with $s(w_*) \neq 0$. As $\Psi_\eta = U_\eta$ on B, there are $r > 0$ and $c_* > 0$ such that (2.5) and a local trivialization give

$$q_\eta(s) = \mathrm{i}^{m^2} \int_{\mathbb{P}^m} |s|^2_{h_L^m} \mathrm{e}^{-m\Psi_\eta} \ge c_* \int_{|w-w_*|<r} |w - w_*|^{-2m}\, \mathrm{d}V = \infty.$$

Thus $Z \subseteq \{x \in V : \chi(x) = -\infty\}$ is pluripolar. □

3. Finite-jet Bergman rank detectors

Fix m, n, X and Φ as in (1.1). The notations J, J_η and I_η defined in (2.1) will be used. Since Theorem 1.1 is trivial when either $m = 0$ or $n = 0$, let us assume that $m, n > 0$.

3.1. **Jet bundles and notation.** All constructions in this section are local. Fix a base polydisc $G \Subset \Delta^n_\eta$ and vertical polydiscs

$$D_0 \Subset D_1 \Subset D_2 \Subset \Delta^m_z.$$

Let $N \in \mathbb{N}$. For $x \in D_1$, put

$$R_x := \mathcal{O}_{D_1,x}, \qquad \mathfrak{m}_x := \{f \in R_x : f(x) = 0\}.$$

Let $\mathrm{Jet}^N \to D_1$ be the holomorphic vector bundle of jets of order at most N, characterized by

$$\mathrm{Jet}^N_x := R_x/\mathfrak{m}_x^{N+1}, \qquad \nu_N := \operatorname{rk} \mathrm{Jet}^N = \binom{m+N}{m}.$$

If f is holomorphic near x, then

$$j^N_x f := f_x \bmod \mathfrak{m}_x^{N+1} \in \mathrm{Jet}^N_x$$

denotes its order-N jet. For each multi-index $\alpha \in \mathbb{N}^m$, set

$$e_{\alpha,x} := j^N_x\big((z - x)^\alpha\big) \in \mathrm{Jet}^N_x, \qquad D^\alpha_x(j^N_x f) := \frac{1}{\alpha!}\partial^\alpha_z f(x).$$

The vectors $(e_{\alpha,x})_{|\alpha|\le N}$ and covectors $(D^\alpha_x)_{|\alpha|\le N}$ are dual holomorphic frames of Jet^N and $(\mathrm{Jet}^N)^*$, respectively, and

$$j^N_x f = \sum_{|\alpha|\le N} D^\alpha_x f\, e_{\alpha,x},$$

where we abbreviate $D^\alpha_x(j^N_x f)$ to $D^\alpha_x f$.

Over $D_0 \times G$, the symbols Jet^N and $(\mathrm{Jet}^N)^*$ mean the pullbacks of $\mathrm{Jet}^N|_{D_0}$ and $(\mathrm{Jet}^N)^*|_{D_0}$ by the vertical projection $\mathrm{pr}_z \colon D_0 \times G \to D_0$.

We identify X_η with Δ^m_z by the vertical coordinate and hence R_x with $\mathcal{O}_{X_\eta,(x,\eta)}$. In particular, for the stalk ideals $I_{\eta,x} := (I_\eta)_{(x,\eta)}$ and $J_{\eta,x} := (J_\eta)_{(x,\eta)}$ in R_x, put

$$E^I_N(x,\eta) := \frac{I_{\eta,x} + \mathfrak{m}_x^{N+1}}{\mathfrak{m}_x^{N+1}}, \qquad E^J_N(x,\eta) := \frac{J_{\eta,x} + \mathfrak{m}_x^{N+1}}{\mathfrak{m}_x^{N+1}}.$$

By (2.2),

$$E^I_N(x,\eta) \subseteq E^J_N(x,\eta).$$

We do not need to endow E^I_N and E^J_N with any additional structures.

For $\eta \in G$, set

$$H_\eta := A^2(D_1, \mathrm{e}^{-\Phi_\eta}) = \left\{ f \in \mathcal{O}(D_1) : \|f\|^2_{H_\eta} < \infty \right\}, \qquad \|f\|^2_{H_\eta} = \int_{D_1} |f|^2 \mathrm{e}^{-\Phi_\eta}\, \mathrm{d}V.$$

We understand that $H_\eta = \{0\}$ when $\Phi_\eta \equiv -\infty$.

If $\Phi_\eta \not\equiv -\infty$, for any $x \in D_1$, choose a polydisc $B_x \Subset D_1$ containing x. Then

$$\|f\|^2_{L^2(B_x)} \leq \mathrm{e}^{\sup_{B_x} \Phi_\eta} \|f\|^2_{H_\eta}, \tag{3.1}$$

so Cauchy inequalities make every order-N jet functional continuous on H_η. If $\Phi_\eta \equiv -\infty$, then $H_\eta = \{0\}$. Thus, in every case, for any $x \in D_1$, the coordinate functionals

$$\ell_{\alpha,\eta}(x) \in H^*_\eta, \qquad \ell_{\alpha,\eta}(x)(f) := D^\alpha_x f, \qquad |\alpha| \leq N,$$

are well defined.

For a Hilbert space H and $d \geq 1$, let $H^{\widehat{\otimes} d}$ be its completed d-fold tensor product and let $\widehat{\bigwedge}^d H$ be the closed alternating subspace. Let $\mathfrak{S}_d$ denote the permutation group of $\{1, \dots, d\}$. For $\ell_1, \dots, \ell_d \in H$, our normalized wedge convention is

$$\ell_1 \wedge \cdots \wedge \ell_d := \frac{1}{\sqrt{d!}} \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sgn}(\sigma) \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(d)}.$$

Thus

$$\|\ell_1 \wedge \cdots \wedge \ell_d\|^2 = \det(\langle \ell_i, \ell_j \rangle)^d_{i,j=1}.$$

3.2. **Jet images of the fiber and restricted ambient ideals.**

Lemma 3.1. *For every $\eta \in G$ and $x \in D_0$,*

$$\mathrm{Im} \left(j^N_x : H_\eta \to \mathrm{Jet}^N_x \right) = E^I_N(x, \eta). \tag{3.2}$$

Proof. Only surjectivity in (3.2) needs proof. Let $\mathcal{M}_{x,N+1} \subseteq \mathcal{O}_{D_2}$ be the order-N fat-point ideal. The quotient morphism

$$I_\eta|_{D_2} \longrightarrow \frac{I_\eta|_{D_2} + \mathcal{M}_{x,N+1}}{\mathcal{M}_{x,N+1}}$$

has coherent kernel and a finite-length target supported at x. Since D_2 is Stein, Cartan's Theorem B implies that the induced map on global sections is surjective; its target identifies with $E^I_N(x, \eta)$. Thus every prescribed jet lifts to a section of $I_\eta|_{D_2}$. The lift restricts to H_η because sections of I_η are locally square integrable with weight $\mathrm{e}^{-\Phi_\eta}$ and $\overline{D_1}$ is compact in D_2. This proves (3.2). □

By Cartan's Theorem A, choose an integer $r \geq 1$ and sections $g_1, \dots, g_r$ of J on a product neighborhood of $\overline{D_1} \times \overline{G}$ that generate J on a neighborhood of $\overline{D_0} \times \overline{G}$. For $1 \leq i \leq r$ and $|\alpha| \leq N$, define the moving-center columns

$$s_{i,\alpha}(x, \eta) := j^N_x \left((z - x)^\alpha g_i(z, \eta) \right) \in \mathrm{Jet}^N_x \,.$$

They give a linear bundle morphism

$$\mathcal{A}_N : \mathcal{O}^{r\nu_N}_{D_0 \times G} \longrightarrow \mathrm{Jet}^N, \qquad (c_{i,\alpha}) \longmapsto \sum_{i,\alpha} c_{i,\alpha} s_{i,\alpha},$$

whose pointwise image is

$$\mathrm{Im}\, \mathcal{A}_N(x, \eta) = E^J_N(x, \eta) = \frac{J_{\eta,x} + \mathfrak{m}^{N+1}_x}{\mathfrak{m}^{N+1}_x}. \tag{3.3}$$

Indeed, if $f = \sum_i b_i g_i$, then modulo $\mathfrak{m}^{N+1}_x$ each holomorphic coefficient b_i may be replaced by its degree-N Taylor polynomial. In normalized jet coordinates, the matrix entries are

$$D^\beta_x \big(s_{i,\alpha}(x, \eta) \big) = \begin{cases} (D^{\beta-\alpha}_z g_i)(x, \eta), & \beta \geq_{\mathrm{comp}} \alpha, \\ 0, & \text{otherwise}. \end{cases}$$

where $\beta \geq_{\mathrm{comp}} \alpha$ means componentwise inequality. Thus $\mathcal{A}_N$ is holomorphic.

3.3. **The rank detector and its Bergman realization.** For $1 \le \rho \le \nu_N$, put
$$\Omega_{N,\rho} := \{(x,\eta) \in D_0 \times G : \dim_{\mathbb{C}} E_N^J(x,\eta) \ge \rho\}.$$
In view of (3.3), its complement is the common zero set of the $\rho \times \rho$ minors of $\mathcal{A}_N$, so $\Omega_{N,\rho}$ is open.

Order multi-indices lexicographically. For $1 \le \rho \le \nu_N$, let
$$\mathfrak{T}_{N,\rho} := \{(\alpha_1,\dots,\alpha_\rho) \in (\mathbb{N}^m)^\rho : |\alpha_i| \le N \ (1 \le i \le \rho),\ \alpha_1 < \cdots < \alpha_\rho\}.$$
For $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_\rho) \in \mathfrak{T}_{N,\rho}$ and $x \in D_1$, put
$$F_{\boldsymbol{\alpha},\eta}(x) := \ell_{\alpha_1,\eta}(x) \wedge \cdots \wedge \ell_{\alpha_\rho,\eta}(x) \in \widehat{\bigwedge}^\rho H_\eta^*.$$
Using the finite Hilbert direct sum, set
$$\mathcal{H}_{N,\rho,\eta} := \bigoplus_{\boldsymbol{\alpha} \in \mathfrak{T}_{N,\rho}} \widehat{\bigwedge}^\rho H_\eta^*, \qquad F_{N,\rho,\eta}(x) := (F_{\boldsymbol{\alpha},\eta}(x))_{\boldsymbol{\alpha} \in \mathfrak{T}_{N,\rho}} \in \mathcal{H}_{N,\rho,\eta}.$$
Set
$$U_{N,\rho}(x,\eta) := \log \|F_{N,\rho,\eta}(x)\|^2_{\mathcal{H}_{N,\rho,\eta}}. \tag{3.4}$$

Proposition 3.2. *For every $1 \le \rho \le \nu_N$, the function $U_{N,\rho}$ is psh or identically $-\infty$ on $D_0 \times G$, and*
$$U_{N,\rho}(x,\eta) = -\infty \iff \dim_{\mathbb{C}} E_N^I(x,\eta) < \rho. \tag{3.5}$$
Consequently,
$$\Omega_{N,\rho} \cap \{U_{N,\rho} = -\infty\} = \{(x,\eta) : \dim_{\mathbb{C}} E_N^I(x,\eta) < \rho \le \dim_{\mathbb{C}} E_N^J(x,\eta)\}. \tag{3.6}$$
For fixed $\eta \in G$, the pole set of $U_{N,\rho}(\cdot,\eta)$ is analytic in D_0, and $F_{N,\rho,\eta}$ is holomorphic on D_0.

Proof. Fix $\rho \ge 1$ and $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_\rho) \in \mathfrak{T}_{N,\rho}$. On D_1^ρ, consider the operator
$$\mathcal{D}_{\boldsymbol{\alpha}} := \sum_{\sigma \in \mathfrak{S}_\rho} \operatorname{sgn}(\sigma) D_{z_1}^{\alpha_{\sigma(1)}} \cdots D_{z_\rho}^{\alpha_{\sigma(\rho)}}.$$
Put
$$\psi(z_1,\dots,z_\rho,\eta) := \sum_{j=1}^\rho \Phi(z_j,\eta).$$
Theorem 2.1 applies with $q = m\rho$, $s = n$, $V = D_1^\rho$, the fixed base polydisc G, and $\mathcal{D} = \mathcal{D}_{\boldsymbol{\alpha}}$. Multiplication of elementary tensors gives canonical unitary identifications
$$H_\eta^{\widehat{\otimes}\rho} \simeq A^2\Big(D_1^\rho, \mathrm{e}^{-\sum_j \Phi_\eta(z_j)}\Big), \qquad (H_\eta^*)^{\widehat{\otimes}\rho} \simeq (H_\eta^{\widehat{\otimes}\rho})^*,$$
see [Dem12, proof of Theorem 15.2(a)] for example. Let T_η denote the first unitary identification, and for $x \in D_1$ let
$$L_{x,\eta}(g) := (\mathcal{D}_{\boldsymbol{\alpha}} g)(x,\dots,x).$$
The analogue of (3.1) on a product polydisc, followed by the Cauchy inequalities, shows that $L_{x,\eta}$ is a continuous functional. Under T_η, an elementary tensor corresponds to
$$T_\eta(f_1 \otimes \cdots \otimes f_\rho)(z_1,\dots,z_\rho) = \prod_{j=1}^\rho f_j(z_j).$$
Consequently, for $f_1,\dots,f_\rho \in H_\eta$,
$$\begin{aligned} &(L_{x,\eta} \circ T_\eta)(f_1 \otimes \cdots \otimes f_\rho) \\ &= \sum_{\sigma \in \mathfrak{S}_\rho} \operatorname{sgn}(\sigma) \prod_{j=1}^\rho D_x^{\alpha_{\sigma(j)}} f_j \\ &= \det(\ell_{\alpha_i,\eta}(x)(f_j))_{i,j=1}^\rho \\ &= \sqrt{\rho!}\, F_{\boldsymbol{\alpha},\eta}(x)(f_1 \otimes \cdots \otimes f_\rho). \end{aligned}$$

There are no additional factorials in the second line, since each normalized derivative acts in a different variable block. In the last line, $F_{\boldsymbol{\alpha},\eta}(x)$ is viewed as a functional on $H_\eta^{\widehat{\otimes}\rho}$ via the second unitary identification, and the factor $\sqrt{\rho!}$ comes from the normalized wedge convention. Since the algebraic tensor product is dense and both functionals are continuous, the equality extends to the completed tensor product:

$$L_{x,\eta} \circ T_\eta = \sqrt{\rho!}\, F_{\boldsymbol{\alpha},\eta}(x).$$

Since T_η is unitary and the defining supremum of the generalized Bergman kernel is the squared dual norm of $L_{x,\eta}$, it follows that

$$K^{\psi_\eta}_{\mathcal{D}_{\boldsymbol{\alpha}},D_1^\rho}(x,\dots,x) = \|L_{x,\eta}\|^2 = \|L_{x,\eta}\circ T_\eta\|^2 = \rho!\,\|F_{\boldsymbol{\alpha},\eta}(x)\|^2.$$

This also covers the case of a trivial Bergman space, when both sides vanish. Thus, by Theorem 2.1, restriction to the holomorphic diagonal, and subtraction of the constant $\log\rho!$, the function

$$u_{\boldsymbol{\alpha}}(x,\eta) := \log\|F_{\boldsymbol{\alpha},\eta}(x)\|^2$$

is psh or identically $-\infty$ on $D_0\times G$. Therefore so is

$$U_{N,\rho} = \log\left(\sum_{\boldsymbol{\alpha}\in\mathfrak{I}_{N,\rho}} \mathrm{e}^{u_{\boldsymbol{\alpha}}}\right).$$

Lemma 3.1 gives

$$\dim_{\mathbb{C}}\operatorname{span}\{\ell_{\alpha,\eta}(x) : |\alpha|\le N\} = \operatorname{rk}(j_x^N|_{H_\eta}) = \dim_{\mathbb{C}} E_N^I(x,\eta).$$

Now $F_{N,\rho,\eta}(x) = 0$ exactly when $\operatorname{span}\{\ell_{\alpha,\eta}(x) : |\alpha|\le N\}$ has dimension less than ρ. This proves (3.5); the definition of $\Omega_{N,\rho}$ gives (3.6).

Fix $\eta\in G$. Each $x\mapsto \ell_{\alpha,\eta}(x)$ is holomorphic because for each $f\in H_\eta$, the function $\ell_{\alpha,\eta}(x)(f) = D_x^\alpha f$ is holomorphic in x. So $F_{N,\rho,\eta}$ is also holomorphic.

It remains to show that $\{U_{N,\rho}(\cdot,\eta) = -\infty\}$ is analytic. Fix $x_0\in D_0$. By coherence, on a neighborhood $O_{x_0}\ni x_0$ choose generators $f_1,\dots,f_c$ of I_η. The moving-center vectors

$$j_x^N\big((z-x)^\gamma f_i(z)\big), \qquad 1\le i\le c, \quad |\gamma|\le N,$$

form the columns of a finite matrix holomorphic in $x\in O_{x_0}$. By the same Taylor-polynomial argument as in (3.3), the image of this matrix is $E_N^I(x,\eta)$. Hence (3.5) identifies $\{U_{N,\rho}(\cdot,\eta) = -\infty\}\cap O_{x_0}$ with the locus where this matrix has rank $<\rho$. It is analytic. Such neighborhoods O_{x_0} cover D_0. □

4. The analytic Bertini theorem

We continue to use the notations in Section 3.

Krull's intersection theorem gives

$$\bigcap_{r\ge 1}(I_{\eta,x} + \mathfrak{m}_x^r) = I_{\eta,x}$$

and hence

$$I_{\eta,x}\ne J_{\eta,x} \iff E_N^I(x,\eta)\ne E_N^J(x,\eta) \text{ for some } N\in\mathbb{N}. \tag{4.1}$$

For each $N\in\mathbb{N}$ and $1\le\rho\le\nu_N$, choose a countable family $\mathcal{Q}_{N,\rho}$ of product polydiscs

$$Q = W_Q\times G_Q \Subset \Omega_{N,\rho}$$

that cover $\Omega_{N,\rho}$.

Let $\operatorname{pr}\colon D_0\times G\to G$ be the base projection and let

$$\mathcal{B}(D_0,G) := \{\eta\in G : I_{\eta,x}\ne J_{\eta,x} \text{ for some } x\in D_0\}. \tag{4.2}$$

By (4.1) and the inclusion $E_N^I(x,\eta)\subseteq E_N^J(x,\eta)$, a stalk difference is equivalent to

$$\dim_{\mathbb{C}} E_N^I(x,\eta) < \rho \le \dim_{\mathbb{C}} E_N^J(x,\eta)$$

for some N and ρ. Hence (3.6) gives

$$\mathcal{B}(D_0, G) = \bigcup_{N\in\mathbb{N}} \bigcup_{\rho=1}^{\nu_N} \bigcup_{Q\in\mathcal{Q}_{N,\rho}} \mathrm{pr}(Q \cap \{U_{N,\rho} = -\infty\}). \tag{4.3}$$

Fix N, ρ, and $Q = W_Q \times G_Q \in \mathcal{Q}_{N,\rho}$. On Q, define

$$\mathcal{H}_\eta := \mathcal{H}_{N,\rho,\eta}, \qquad F_\eta(w) := F_{N,\rho,\eta}(w), \qquad U(w,\eta) := U_{N,\rho}(w,\eta).$$

Then

$$\begin{aligned} U \in \mathrm{PSH}(W_Q \times G_Q) \cup \{-\infty\}, \qquad F_\eta \colon W_Q \to \mathcal{H}_\eta \text{ is holomorphic,} \\ U(w,\eta) = \log \|F_\eta(w)\|^2_{\mathcal{H}_\eta}. \end{aligned} \tag{4.4}$$

Because $Q \Subset D_0 \times G$, these definitions remain valid on a neighborhood of $\overline{W_Q} \times \overline{G_Q}$. By Proposition 3.2, the following set is analytic in W_Q:

$$Z_\eta := F_\eta^{-1}(0) = \{w \in W_Q : U(w,\eta) = -\infty\}.$$

Define the corresponding parameter locus by

$$\mathcal{P}_U := \{\eta \in G_Q : Z_\eta \neq \varnothing\}.$$

Put $G_{\mathrm{good}} := G_Q \cap G_{\mathrm{ae}}$. Recall that G_{ae} is defined in (2.3). The set G_{good} is conull in G_Q by (2.3), and the inclusion $W_Q \times G_Q \subseteq \Omega_{N,\rho}$ together with $E_N^I(w,\eta) = E_N^J(w,\eta)$ for $w \in W_Q$ and $\eta \in G_{\mathrm{good}}$ gives

$$Z_\eta = \varnothing \qquad (\eta \in G_{\mathrm{good}}).$$

In particular $U \not\equiv -\infty$.

It remains to prove that $\mathcal{P}_U$ is pluripolar.

We wish to apply Theorem 2.3 to F_η. However, the lower bound assumption (2.4) is not automatically met. We therefore perturb U first.

Write

$$\mathbb{Q}(\mathrm{i}) := \{r + \mathrm{i}s : r, s \in \mathbb{Q}\}.$$

A *rational polydisc* in $\mathbb{C}^m$ is a polydisc with center in $\mathbb{Q}(\mathrm{i})^m$ and positive rational polyradii.

For $0 \le q \le m$, a *rational affine map with q components* is a map

$$a \colon \mathbb{C}^m \longrightarrow \mathbb{C}^q, \qquad a(w) := Aw + b, \quad A \in \mathrm{Mat}_{q,m}(\mathbb{Q}(\mathrm{i})), \quad b \in \mathbb{Q}(\mathrm{i})^q.$$

Let Λ_Q be the countable family

$$\lambda = (q, a, B, M), \quad 0 \le q \le m, \quad \overline{2B} \Subset W_Q, \quad M \in \mathbb{Z}_{\ge 1},$$

where a has q components and $B \subseteq \mathbb{C}^m$ is a rational polydisc.

For $\lambda = (q, a, B, M) \in \Lambda_Q$ and $\eta \in G_Q$, put

$$\begin{aligned} \mathcal{K}_\eta^\lambda &:= \mathcal{H}_\eta \oplus \mathbb{C}^q, \\ F_\eta^\lambda(w) &:= (F_\eta(w), a(w)), \\ U^\lambda(w,\eta) &:= \log \|F_\eta^\lambda(w)\|^2_{\mathcal{K}_\eta^\lambda} = \log\left(\mathrm{e}^{U(w,\eta)} + |a(w)|^2\right), \end{aligned}$$

and define

$$V_\lambda := \left\{\eta \in G_Q : \min_{w\in\partial B} \|F_\eta^\lambda(w)\|_{\mathcal{K}_\eta^\lambda} > \mathrm{e}^{-M/2}\right\}, \quad Z_\lambda := \left\{\eta \in V_\lambda : \left(F_\eta^\lambda\right)^{-1}(0) \cap B \neq \varnothing\right\}.$$

Proposition 4.1. *For every $\lambda = (q, a, B, M) \in \Lambda_Q$,*

$$U^\lambda \in \mathrm{PSH}(W_Q \times G_Q), \qquad \sup_{\overline{2B}\times G_Q} U^\lambda < \infty. \tag{4.5}$$

The set V_λ is $\mathcal{F}$-open, and

$$U^\lambda > -M \quad \text{on } \partial B \times V_\lambda.$$

Moreover, F_η^λ is nowhere zero on W_Q for $\eta \in G_{\mathrm{good}}$, and

$$\mathcal{P}_U = \bigcup_{\lambda\in\Lambda_Q} Z_\lambda. \tag{4.6}$$

Proof. The psh assertion is immediate since both U and $\log|a|^2$ are psh. By the neighborhood extension noted after (4.4), U is psh on a neighborhood of $\overline{W_Q}\times\overline{G_Q}$, hence bounded above there; the affine map a is bounded on $\overline{2B}$. This gives the upper bound (4.5). The first component of $F^\lambda_\eta=(F_\eta,a)$ also shows that F^λ_η is nowhere zero for $\eta\in G_{\mathrm{good}}$.

We next prove that V_λ is $\mathcal{F}$-open. Put

$$\beta(\eta):=\min_{w\in\partial B}\|F^\lambda_\eta(w)\|_{\mathcal{K}^\lambda_\eta}.$$

Fix $c\geq 0$ and $\eta_0\in G_Q$ with $\beta_0:=\beta(\eta_0)>c$. We shall construct an $\mathcal{F}$-neighborhood of η_0 where $\beta>c$. Put $\tau:=(\beta_0-c)/3$. Local upper bounds for U^λ and Cauchy estimates imply that the maps F^λ_η are equicontinuous in $w\in\partial B$, uniformly for η in some neighborhood O of η_0. Choose a finite net $\{w_j\}\subseteq\partial B$ fine enough that, for every $\eta\in O$ and $w\in\partial B$, some j satisfies

$$\|F^\lambda_\eta(w)-F^\lambda_\eta(w_j)\|_{\mathcal{K}^\lambda_\eta}<\tau.$$

Then

$$O\cap\bigcap_j\Big\{\eta:U^\lambda(w_j,\eta)>2\log(\beta_0-\tau)\Big\}$$

is an $\mathcal{F}$-open neighborhood of η_0, because $U^\lambda(w_j,\cdot)$ is psh or $-\infty$ for each j. On this neighborhood the triangle inequality gives $\beta>\beta_0-2\tau>c$. Thus $V_\lambda=\{\beta>\mathrm{e}^{-M/2}\}$ is $\mathcal{F}$-open. Its definition gives $U^\lambda>-M$ on $\partial B\times V_\lambda$.

It remains to prove (4.6). A zero of F^λ_η has $F_\eta=0$ in its first component, so the right side is contained in $\mathcal{P}_U$.

Conversely, fix $\eta\in\mathcal{P}_U$. Give $Z_\eta=F_\eta^{-1}(0)$ its reduced structure, choose $p\in(Z_\eta)_{\mathrm{reg}}$, and put $q=\dim_p Z_\eta$.

By dimension counting, we can find a full-rank $q\times m$ matrix A and a q-vector b so that $Z_\eta\cap\{Aw+b=0\}$ consists of a single point in a fixed neighborhood of p. After a small perturbation, we may assume that all entries of A and b are in $\mathbb{Q}(\mathrm{i})$. Set

$$a(w):=Aw+b\in\mathbb{C}^q.$$

If p_a is the resulting intersection point, choose a rational polydisc $B\ni p_a$ so small that

$$Z_\eta\cap a^{-1}(0)\cap\overline{2B}=\{p_a\},\quad \overline{2B}\Subset W_Q.$$

The number $\min_{w\in\partial B}\|(F_\eta(w),a(w))\|_{\mathcal{H}_\eta\oplus\mathbb{C}^q}$ is positive, so choose $M\in\mathbb{Z}_{\geq 1}$ with $\mathrm{e}^{-M/2}$ smaller than it. Then $\lambda=(q,a,B,M)\in\Lambda_Q$ and $\eta\in Z_\lambda$. This proves the reverse inclusion. □

We can finally prove the main theorem.

Proof of Theorem 1.1. For each $\lambda=(q,a,B,M)\in\Lambda_Q$, apply Theorem 2.3 with

$$W=W_Q,\quad G=G_Q,\quad (\mathcal{K}_\eta,F_\eta,U,V,G_0)=(\mathcal{K}^\lambda_\eta,F^\lambda_\eta,U^\lambda,V_\lambda,G_{\mathrm{good}}).$$

All its hypotheses are supplied by Proposition 4.1. Thus every Z_λ is pluripolar. Then (4.6) shows that $\mathcal{P}_U$ is pluripolar. Repeating this for the countably many boxes in (4.3) shows that $\mathcal{B}(D_0,G)$ defined in (4.2) is pluripolar.

Finally, choose polydiscs, indexed by $\ell\in\mathbb{N}$, such that

$$D_{0,\ell}\Subset D_{1,\ell}\Subset D_{2,\ell}\Subset\Delta^m_z,\qquad G_\ell\Subset\Delta^n_\eta,$$

and the sets $D_{0,\ell}\times G_\ell$ cover X. The set $P:=\bigcup_\ell\mathcal{B}(D_{0,\ell},G_\ell)$ is pluripolar. If $\eta\notin P$ and $x\in\Delta^m_z$, choose ℓ with $(x,\eta)\in D_{0,\ell}\times G_\ell$. Then $\eta\notin\mathcal{B}(D_{0,\ell},G_\ell)$, so $I_{\eta,x}=J_{\eta,x}$. Thus the two ideal sheaves agree on X_η. □

5. Analytic Bertini theorem for holomorphic maps

Complex manifolds are assumed σ-compact. Recall that a subset P of a complex manifold Y is *pluripolar* if, for every $y \in Y$, there is an open neighborhood U of y and a psh function φ on U such that $\varphi|_{P\cap U} \equiv -\infty$. Some authors prefer to call these sets locally pluripolar sets. Recall that a subset A of a complex manifold Y is *negligible* if A is contained in a countable union of locally closed submanifolds of Y with empty interiors.

We first recall a well-known generic smoothness theorem:

Theorem 5.1. *Let $f\colon Y \to Z$ be a holomorphic map between complex manifolds, and let $C_f \subseteq Y$ be the non-smooth locus of f. Then $f(C_f)$ is negligible.*

In particular, $f(C_f)$ is pluripolar.

Proof. We may assume that Z is connected. Put $n = \dim Z$. Then

$$C_f = \{y \in Y : \mathrm{rk}_{\mathbb{C}}\, \mathrm{d}f_y < n\}.$$

In particular, it is a closed analytic subset.

No local holomorphic section of f over a nonempty open subset of Z has image contained in C_f. In fact, if σ were such a section, then

$$\mathrm{d}f_{\sigma(z)} \circ \mathrm{d}\sigma_z = \mathrm{id},$$

contrary to the definition of C_f. Applying [BS76, Lemma V.4.4] to $f|_{C_f}$ shows that $f(C_f)$ is negligible. □

Proof of Corollary 1.2. Let C_f be the locus in Theorem 5.1.

By definition of smoothness, any point in $Y \setminus C_f$ admits an open neighborhood on which f is isomorphic to the projection of a product chart. Choose a countable family of such product charts covering $Y \setminus C_f$. Applying Theorem 1.1 to each chart, we find that the exceptional locus of each product chart is pluripolar. Taking their union together with $f(C_f)$, we obtain the desired pluripolar exceptional locus. □

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