Periodic data in degenerating families and skew products

Abstract

A degenerating cubic family has one divergent fixed-point multiplier and three bounded ones, so it violates both alternatives of Favre's multiplier dichotomy as stated for every period. We also construct a polynomial skew product with a critical two-cycle whose one-step degree ratio is nonintegral and whose return ratio is integral. Every point in its invariant set over $\mathbb{C}((z))$ is rigid of multiplicity dividing four, and multiplicity four occurs. The proof confines all even-step preimages of zero in a rescaled coordinate to one quartic Puiseux field.

1 Fixed-point multipliers of a degenerating cubic

Let (f_t) be a degenerating family of rational maps of degree d. For x fixed by f_t^n, write $m_n(t,x) = |(f_t^n)'(x)|^{1/n}$. In [Fav25, Theorem A, p. 2], the first alternative bounds $m_n(t,x)$ uniformly over n, x and $0 < |t| \le 1/2$. The second asserts that there is a $\lambda > 0$ such that for every $\varepsilon > 0$ and every $n \ge 1$, some $C_{\varepsilon,n} > 0$ satisfies

$$\frac{1}{d^n}\#\{x \in \mathrm{Fix}(f_t^n) : \max\{1, m_n(t,x)\} \ge C_{\varepsilon,n}|t|^{-\lambda}\} \;\ge\; 1-\varepsilon \tag{1.1}$$

for all $0 < |t| \le 1/2$.

Theorem 1.1. *The cubic family $f_t(x) = x^3 + t^{-1}x^2$, defined for $0 < |t| < 1$, degenerates as $t \to 0$. Its fixed-point multipliers disprove both alternatives of [Fav25, Theorem A].*

Proof. The homogeneous lift $[tX^3 + X^2Y : tY^3]$ extends across $t = 0$ as a projective coefficient vector. For $t \neq 0$ the two forms have no common projective zero, so the family has degree three. For $0 < |t| < 1/2$ its fixed points are 0, ∞ and

$$x_\pm(t) = \frac{-1 \pm \sqrt{1+4t^2}}{2t},$$

where the square root equals 1 at $t = 0$. The multipliers at 0 and ∞ are zero. At a nonzero fixed point, $x^2 + t^{-1}x = 1$ gives $f_t'(x) = 3 - t^{-1}x$. The two nonzero finite fixed points satisfy $x_+ + x_- = -t^{-1}$, so the sum of all four multipliers is $6 + t^{-2}$. Expanding the square root gives

$$f_t'(x_+) = 2 + O(t^2), \qquad f_t'(x_-) = t^{-2} + 4 + O(t^2).$$

The class map is algebraic for $t \neq 0$ and extends at 0 to the projective compactification of the moduli space by properness. The sum of the fixed-point multipliers is a regular function on the degree-three moduli space [Fav25, p. 3]. Its pole forces the limiting class to lie on the boundary.

The multiplier at x_- is unbounded, so the first alternative fails at $n = 1$. In (1.1) take $n = 1$ and $\varepsilon = 1/2$. For every choice of $\lambda > 0$ and $C_{1/2,1} > 0$, the threshold $C_{1/2,1}|t|^{-\lambda}$ tends to infinity. For each sufficiently small nonzero t, at most one of the four fixed points reaches it. The left-hand side is at most $1/3$, which is less than $1 - \varepsilon = 1/2$. □

The proof of the dichotomy invokes [Fav25, Theorem 1.2, p. 6] for a count that holds for sufficiently large n.

2 A periodic critical branch with bounded multiplicities

Put $K = \mathbb{C}((z))$, with its z-adic absolute value normalized by $|z| = \mathrm{e}^{-1}$, and let L be the completion of its Puiseux algebraic closure. Let $f(z, w) = (z^d, P(z, w))$ be a monic polynomial skew product of transverse degree $2 \le c < d$ and let $f_\diamond$ be its semilinear action. For $F = K$ or L, write $\mathcal{K}_F$ for the set of points of the open unit disk in the Berkovich affine line over F whose forward orbit under $f_\diamond$ fails to converge to the unit Gauss point. Thus $\mathcal{K}_K$ is the set in [DFR25, equation (4), p. 3]. A rigid point over K is represented by evaluation at a Puiseux series w, and its multiplicity is $[K(w) : K]$ [DFR25, pp. 9–10].

Let $e(a)$ denote the transverse local degree at a Puiseux branch a. In the notation of [DFR25, p. 25], $e(a) = 1 + J(a)$, so membership in $\mathcal{C}^+$ means that $e(a)$ does not divide d. Under the hypotheses that $\mathcal{K}_K$ is nonsingleton and contains a periodic branch in $\mathcal{C}^+$, the conclusions of [DFR25, Theorem C(iii), p. 5, and Theorem 5.1, p. 31] are the existence of nonrigid points and unbounded multiplicity among rigid points.

Theorem 2.1. *For the monic skew product*

$$f(z, w) = (z^6, P(z, w)), \qquad P(z, w) = w^5 - (z + z^2)w^4 + z^6,$$

the branches 0 *and* z *form a two-cycle in* $\mathcal{K}_K$*, with transverse local degrees* 4 *and* 1*, respectively. Every point of* $\mathcal{K}_L$ *is evaluation at a series in* $z\mathbb{C}[[z^{1/4}]]$*. Every point of* $\mathcal{K}_K$ *is rigid of multiplicity dividing four, and multiplicity four occurs.*

2.1 The return equations

Set $s = w/z$ and let

$$R(z, s) = \frac{P(z, zs)}{z^6} = 1 + \frac{s^4(s - 1 - z)}{z}.$$

Choose $z^{1/4}$ in the Puiseux field, and put $E = \mathbb{C}((z^{1/4}))$, $A = \mathbb{C}[[z^{1/4}]]$ and $O = \mathbb{C}[[z]]$.

Lemma 2.2. *For every* $a \in A$*, all roots of*

$$R(z^6, R(z, s)) = a(z^{36}) \tag{2.1}$$

belong to A*.*

Proof. For $t \in zO$ with $t \neq 0$ and $y \in A$, the equation $R(t, u) = y$ is equivalent to

$$u^4(1 + t - u) = t(1 - y). \tag{2.2}$$

The monic polynomial has integral coefficients, so its roots are integral. Reduction gives $u^4(u-1)$, and Hensel's lemma gives the simple root congruent to 1 in A, or in O if $y \in O$.

The four roots congruent to 0, counted with multiplicity, are obtained from the fourth roots of $t(1 - y)$ by inverting the coordinate $u \mapsto u(1 + t - u)^{1/4}$. The binomial series defines the fourth root of the unit with residue 1. The resulting series in u has unit linear coefficient, and the inverse series has integral coefficients. Both series converge for arguments of absolute value less than 1. If $t(1 - y) = 0$, the four small roots coincide at zero.

Write $b = a(z^{36}) \in O$. First solve (2.2) with $t = z^6$, $y = b$, and unknown $u = R(z, s)$. The simple root lies in O. The small roots lie in A. Indeed, an element of O is zero or a nonnegative integral power of z times a unit with a fourth root in O. For the second equation take $t = z$ and $y = u$. The simple root now lies in A. If u is the first simple root, then $z(1 - u) \in O$. If u is a small root, then $z(1 - u)$ is z times a unit of A. In either case all fourth roots of $z(1 - u)$ lie in A, so the small roots of the second equation also lie in A. □

Define $R_0(z,s) = s$ and $R_{n+1}(z,s) = R(z^{6^n}, R_n(z,s))$. Thus $f^n(z,zs) = (z^{6^n}, z^{6^n} R_n(z,s))$. The identity

$$R_{2n+2}(z,s) = R_{2n}(z^{36}, R_2(z,s))$$

and Lemma 2.2 imply that every root of $R_{2n}(z,s) = 0$ lies in A. Indeed the root of R_0 is zero. At each induction step, substituting z^{36} in the factorization of R_{2n} replaces each root $a(z)$ by $a(z^{36})$. The degree and leading coefficient of R_n in s are

$$\deg_s R_n = 5^n, \qquad \mathrm{lc}_s R_n = z^{-(6^n - 5^n)}. \tag{2.3}$$

The degree multiplies by five at each step, and the exponent of the inverse power of z satisfies $b_{n+1} = 6^n + 5b_n$, $b_0 = 0$.

2.2 The invariant set

Proof of Theorem 2.1. The example has the form in [DFR25, equations (1)–(2)] with $d = 6$ and $c = 5$. In fact $P = w^5 + zh$ with $h = -(1+z)w^4 + z^5$ and $h(0,0) = 0$. The base and transverse equations are monic, so f is finite. Its differential at the origin is zero, and it pulls back the divisor $z = 0$ to six times that divisor. Its restriction to this line is $w \mapsto w^5$. The equations $P(z,0) = z^6$ and $P(z,z) = 0$ give the semilinear cycle $0 \mapsto z \mapsto 0$. Since

$$\partial_w P = w^3(5w - 4z - 4z^2),$$

the two local degrees are 4 and 1. The cycle lies in the open disk and stays away from the Gauss point. Its two evaluations are distinct over K.

For a polynomial Q in the transverse coordinate, the semilinear action is given by

$$|Q|_{f_\diamond(x)} = |Q(z^6, P(z,w))|_x^{1/6},$$

where $z \mapsto z^6$ acts on the coefficients. Over L, first apply the substitution on Puiseux series with a fixed compatible choice of roots of z. It raises the absolute value to the sixth power and so extends to the completion. It agrees with the action of [DFR25, equation (3), p. 3] on Puiseux points. The coefficient norms of P give $|P|_x < 1$ whenever $|w|_x < 1$, so the open unit disk is forward invariant. On the annulus $|z| < |w|_x < 1$, the term w^5 strictly dominates the other two terms of P. The radii under iteration therefore tend increasingly to 1 by the rule $r \mapsto r^{5/6}$. For each nonzero test polynomial, once the radius is close enough to 1 from below, the term of least exponent among those with maximal coefficient norm strictly dominates. Its norm tends to the Gauss norm of the polynomial. This proves convergence to the Gauss point in the pointwise topology of seminorms.

On the other hand, if every iterate satisfies $|w| \le |z|$, the coordinate w itself precludes Gauss convergence. By the iterate formula for f^n, in coordinate s one has

$$|w|_{f_\diamond^n(x)} = |z|\,|R_n(z,s)|_x^{1/6^n}.$$

Hence the survivor condition is $|R_n|_x \le 1$ for all $n \ge 0$.

Let x be a survivor over F, where F is K or L, and view s in its completed residue field. For $F = K$, adjoin a fourth root of z and use the unique extension of the absolute value to this finite extension. This puts E in the resulting valued field, since $T^4 - z$ is Eisenstein over K. For $F = L$ the completed residue field already contains E. The survivor inequalities hold in these fields because all R_n have coefficients in K. Factor R_{2n} over E. All 5^{2n} roots lie in A, counted with multiplicity. By (2.3) and the bound $|R_{2n}|_x \le 1$, the product of the 5^{2n} factors $|s - a|$ is at most $|z|^{6^{2n} - 5^{2n}}$. Taking the geometric mean gives a root $r_n \in A$ with

$$|s - r_n| \le |z|^{(6/5)^{2n} - 1}.$$

The ultrametric inequality bounds $|r_n - r_m|$ by the larger of the two bounds for n and m. Thus (r_n) is Cauchy in A, which is complete. Its limit r satisfies $s = r$ in the valued field. Over L the point is therefore evaluation at $w = zr$. Over K it is evaluation at the same algebraic element, and $[K(zr) : K]$ divides $[E : K] = 4$.

Finally, the four small roots of $R(z, s) = 0$ are obtained by inverting $s \mapsto s(1 + z - s)^{1/4}$ at the fourth roots of z. Each has the form $s = \xi z^{1/4} + O(z^{1/2})$ with $\xi^4 = 1$, so $w = zs$ has z-adic order $5/4$. The z-adic orders of $1, w, w^2, w^3$ have distinct fractional parts. In any nonzero linear combination over K, the nonzero summands therefore have distinct orders, so the one of least order cannot cancel. This proves $[K(w) : K] = 4$. Each such branch maps to 0 and lies in the invariant set. □

The invariant set contains the distinct points 0 and z. Since $4 \nmid 6$, the branch 0 belongs to $\mathcal{C}^+$ and is periodic. The example therefore meets the stated hypotheses and violates both conclusions. For a cycle of period p with successive transverse local degrees $e_0, \dots, e_{p-1}$, the return has base degree d^p. Transverse local degrees multiply under composition, giving $\prod_{j=0}^{p-1} e_j$. The relevant ratio in the periodic-to-fixed reduction [DFR25, p. 32] is therefore $d^p / \prod_{j=0}^{p-1} e_j$. In this example the one-step ratio is $6/4$, while the return ratio is $36/(4 \cdot 1) = 9$.

References

[DFR25] R. Dujardin, C. Favre, and M. Ruggiero. *Polynomial skew products with small relative degree*. July 12, 2025. arXiv: 2507.09197v1 [math.DS].

[Fav25] C. Favre. *Blow-up of multipliers in meromorphic families of rational maps*. Apr. 28, 2025. arXiv: 2504.20284v1 [math.DS].