A Counterexample to the Global Kähler-Current Conjecture for Weak Kähler–Ricci Flows

Abstract

We disprove the global Kähler-current part of the higher-dimensional geometric-smoothing conjecture formulated by Di Nezza, Guedj, and Lu. We construct a compact Kähler surface and singular initial datum for which the maximal weak fixed-background Kähler–Ricci flow fails to dominate any positive multiple of the background form at every time in a genuine Arnold-multiplicity jumping interval. The surface is the blow-up of the projective plane at one point, and the initial current is purely divisorial with unequal weights. An exact formula for the flow reduces the obstruction to a positive current with zero Lelong numbers in a nef and big class of degree zero on the exceptional curve.

2020 Mathematics Subject Classification. Primary 53E30; Secondary 32U15, 32U40.

Keywords. Kähler–Ricci flow, Kähler currents, geometric smoothing, Arnold multiplicity, nef and big classes

1 Introduction

The fixed-background Kähler–Ricci flow evolves singular positive currents within a fixed Kähler class. For purely divisorial initial data, an exact formula identifies the maximal weak flow in terms of a residual Monge–Ampère current [DGL26, Theorem 3.4 and Proposition 3.6]. Regularization with control of Lelong numbers provides a second ingredient for testing whether that current admits a global positive lower bound [Xia26, Theorem 1.6.2].

Definition 1.1. Let q be a quasi-plurisubharmonic function on a complex manifold M. Its local Arnold multiplicity at $x \in M$ is

$$\lambda(q, x) = \left(\sup \{c > 0 \mid e^{-cq} \text{ is locally integrable at } x\}\right)^{-1},$$

where the reciprocal of $+\infty$ is zero. Local integrability is measured against a smooth positive volume form. The jumping values are the thresholds at which the sets

$$A_s(q) = \{x \in M \mid \lambda(q, x) \geq s\}, \qquad s > 0,$$

change. We include zero as the terminal endpoint. The open intervals between consecutive jumping values, with this terminal endpoint included, are the jumping intervals.

Di Nezza, Guedj, and Lu formulate a higher-dimensional geometric-smoothing conjecture in [DGL26, p. 2]. For the decreasing sequence of jumping times of the analytic sets determined by $\lambda(q_0, \cdot)$, its global Kähler-current part asserts that the corresponding maximal weak fixed-background flow

$$V_t = a + \mathrm{dd^c} q_t, \qquad V_t^n = e^{\partial_t q_t} a^n, \qquad n = \dim_{\mathbb{C}} M,$$

is a Kähler current on all of M throughout each jumping interval; equivalently, in the present notation, $V_t \geq \delta_\ell a$ on the ℓ-th interval for some interval-dependent constant $\delta_\ell > 0$. The next theorem disproves this assertion in complex dimension two. In fact, it disproves even the weaker pointwise-in-time statement in which the lower-bound constant is allowed to depend on t: throughout one jumping interval, no positive lower-bound constant exists at all. Section 2 recalls the notation and the precise meaning of the maximal weak solution.

Our example is the graph of the pencil of lines through a point of the projective plane. Throughout the paper we use

$$\mathrm{dd^c} = \frac{i}{2\pi}\partial\bar{\partial}.$$

Theorem 1.2. *Let*

$$X = \left\{([Z_0 : Z_1 : Z_2], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; Z_1 v = Z_2 u\right\},$$

and let $b\colon X \to \mathbb{P}^2$ *and* $f\colon X \to \mathbb{P}^1$ *be the two projections. Put*

$$p = [1 : 0 : 0], \qquad E = b^{-1}(p), \qquad F = f^{-1}([1 : 0]).$$

Let $\omega_{\mathrm{FS},k}$ *be the Fubini–Study form on* $\mathbb{P}^k$*, normalized to have integral one on a projective line, and set*

$$\beta = b^*\omega_{\mathrm{FS},2}, \qquad \gamma = f^*\omega_{\mathrm{FS},1}, \qquad \omega = \beta + \gamma.$$

Equip

$$\mathcal{O}(F) = f^*\mathcal{O}_{\mathbb{P}^1}(1), \qquad \mathcal{O}(E) = b^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1)$$

with the induced Fubini–Study metrics. Their curvature forms are γ *and* $\beta - \gamma$*, respectively. Let* σ_F, σ_E *be their canonical defining sections, write*

$$r_F = |\sigma_F|^2, \qquad r_E = |\sigma_E|^2, \qquad \phi_0 = 2\log r_F + \log r_E,$$

and let $W_t = \omega + \mathrm{dd^c}\phi_t$ *be the maximal weak fixed-background flow with initial potential* ϕ_0 *and volume form* ω^2*. Then the following statements hold.*

1. *The surface* X *is the blow-up of* $\mathbb{P}^2$ *at* p*, the form* ω *is Kähler, and*

$$\phi_0 \in \mathrm{PSH}(X, \omega), \qquad \omega + \mathrm{dd^c}\phi_0 = 2[F] + [E].$$

2. *There exists* $\rho \in \mathcal{E}^1(X, \beta)$ *such that*

$$K = \beta + \mathrm{dd^c}\rho \geq 0, \qquad \langle K^2 \rangle = \frac{e^\rho}{r_F r_E}\omega^2, \qquad \nu(\rho, x) = 0 \quad (x \in X).$$

Here $\mathcal{E}^1$ *and the non-pluripolar product are specified in Definition 2.2, and* ν *denotes the Lelong number. For every* $0 < t < 1$*, the exact flow and potential are*

$$W_t = (2 - t)[F] + (1 - t)[E] + tK, \tag{1.1}$$
$$\phi_t = (2 - t)\log r_F + (1 - t)\log r_E + t\rho + 2(t\log t - t). \tag{1.2}$$

They solve

$$W_t^2 = e^{\partial_t \phi_t}\omega^2$$

in the maximal weak sense.

3. *The positive values of* $\lambda(\phi_0, \cdot)$ *are exactly* 2 *and* 1*. The positive jumping times are* 2 *and* 1*, with terminal endpoint* 0*. In particular,* $(0, 1)$ *is a jumping interval.*

4. *For every fixed* $t \in (0, 1)$*, there is no* $\delta > 0$ *such that*

$$W_t \geq \delta\omega \quad \text{on } X.$$

This applies, in particular, to the fixed interior time $t = \frac{1}{2}$*.*

Corollary 1.3. *The global Kähler-current assertion in the higher-dimensional conjecture of Di Nezza, Guedj, and Lu [DGL26, p. 2] is false, already for a projective surface with purely divisorial initial data.*

Proof. By Theorem 1.2, the interval $(0, 1)$ is a jumping interval, but W_t is not a Kähler current on X for any $t \in (0, 1)$. In particular, the single fixed interior time $t = \frac{1}{2}$ already contradicts the asserted global lower bound. □

This result concerns only the global Kähler-current part of the conjecture. We make no claim here about the separate prediction that the flow metric is complete on the corresponding Zariski open stratum, or about the conjectural exact linear evolution of local Arnold multiplicities.

The mechanism is as follows. The unequal divisor weights produce a Kähler initial class, while the reduced divisor class is nef and big and has degree zero on the exceptional curve. Proposition 2.3 identifies the maximal flow, and a direct integrability calculation determines the jumping interval. If the flow had a global positive lower bound, then its residual current would inherit such a bound. The residual current has zero Lelong numbers, so Theorem 2.5 would produce locally bounded approximants whose restrictions to the exceptional curve have negative total mass, a contradiction.

2 Preliminaries

We fix the conventions and recall the two external results used in the proof of Theorem 1.2. We follow the terminology for positive currents and singular potentials in [Xia26, Chapters 1–3] and [DGL26].

Notation and conventions. Let M be a compact Kähler manifold of complex dimension n, and let a be a Kähler form on M. All positivity statements concern real $(1, 1)$-forms or currents. A current T dominates a current S if $T - S$ pairs nonnegatively with every smooth strongly positive test form of complementary bidegree. We write $[D]$ for the current of integration over a divisor D, and $\{T\}$ for the real $(1, 1)$-cohomology class of a closed form or current T. For a Hermitian line bundle with curvature Θ_h and a nonzero holomorphic section σ, our normalization gives

$$\mathrm{dd^c} \log |\sigma|_h^2 = [\mathrm{div}(\sigma)] - \Theta_h. \tag{2.1}$$

The maximal weak fixed-background flow is the solution constructed as the decreasing limit of smooth approximating flows [DGL26, Section 1.1.2].

Definition 2.1. Let θ be a smooth closed real $(1, 1)$-form on M. A quasi-plurisubharmonic function is locally the sum of a plurisubharmonic function and a smooth function. We write $\mathrm{PSH}(M, \theta)$ for the upper semicontinuous, locally integrable functions q satisfying

$$\theta + \mathrm{dd^c} q \geq 0.$$

The class $\{\theta\}$ is nef if, for every $\epsilon > 0$, it admits a smooth representative bounded below by $-\epsilon a$. It is big if it contains a closed current dominating a positive multiple of a Kähler form. A potential $v \in \mathrm{PSH}(M, \theta)$ has minimal singularities if, for every $u \in \mathrm{PSH}(M, \theta)$, there is a constant C such that $u \leq v + C$.

Definition 2.2. Non-pluripolar products are formed by bounded-potential truncations and give no mass to pluripolar sets. We denote the non-pluripolar Monge–Ampère measure of $\theta + \mathrm{dd^c} q$ by

$$\langle (\theta + \mathrm{dd^c} q)^n \rangle .$$

The class $\mathcal{E}^1(M, \theta)$ consists of the θ-plurisubharmonic potentials of full non-pluripolar Monge–Ampère mass and finite pluripotential energy.

The following proposition is the identity-resolution specialization of [DGL26, Theorem 3.4 and Proposition 3.6].

Proposition 2.3. *Let (M, a) be a compact Kähler manifold of complex dimension n, and let dV be a smooth positive volume form. Let $D_1, \ldots, D_N$ be the components of a simple normal crossings divisor, with coefficients $m_j > 0$. Choose defining sections σ_j and smooth Hermitian metrics with curvatures Θ_j. Assume that*

$$a = \sum_{j=1}^{N} m_j \Theta_j, \qquad \theta = \sum_{j=1}^{N} \Theta_j, \qquad \{\theta\} \textit{ is nef and big.}$$

Put

$$|\sigma|^2 = \prod_{j=1}^{N} |\sigma_j|^2, \qquad q_0 = \sum_{j=1}^{N} m_j \log |\sigma_j|^2.$$

There exists $\rho \in \mathcal{E}^1(M, \theta)$ satisfying

$$\langle (\theta + \mathrm{dd^c} \rho)^n \rangle = e^{\rho} |\sigma|^{-2}\, dV.$$

Its Lelong numbers agree with those of a θ-plurisubharmonic potential with minimal singularities. It is smooth on a dense Zariski open subset.

For $0 < t < \min_j m_j$, the maximal weak fixed-background flow from q_0, with volume form dV, is

$$\begin{aligned} a + \mathrm{dd^c} q_t &= \sum_{j=1}^{N} (m_j - t)[D_j] + t(\theta + \mathrm{dd^c} \rho), \\ q_t &= \sum_{j=1}^{N} (m_j - t) \log |\sigma_j|^2 + t\rho + n(t \log t - t). \end{aligned}$$

All discrepancy numbers are zero in this specialization. The identification concerns the maximal weak solution described above; the comparison argument in [DGL26, Proposition 3.6 and its proof] establishes this maximality globally.

We use the following notion from [Xia26, Definition 1.6.1].

Definition 2.4. A quasi-plurisubharmonic function has analytic singularities if locally it can be written as

$$q = c \log \left(\sum_{j=1}^{N} |h_j|^2 \right) + v,$$

where $c \in \mathbb{Q}_{>0}$, the functions h_j are holomorphic, and v is locally bounded. No smoothness of v is assumed.

We also use Demailly's regularization theorem in the form stated in [Xia26, Theorem 1.6.2].

Theorem 2.5. *Let M be a compact Kähler manifold, let a be a Kähler form, let θ be a smooth closed real $(1,1)$-form, and let $q \in \mathrm{PSH}(M, \theta)$. There exist quasi-plurisubharmonic functions q_m with analytic singularities and nonnegative numbers $a_m \downarrow 0$ such that $q_m \downarrow q$ and*

$$\theta + a_m a + \mathrm{dd^c} q_m \geq 0 \qquad (m \geq 1).$$

3 Proof of the counterexample

The goal of this section is to prove Theorem 1.2 by identifying the maximal flow and then testing a hypothetical lower bound on the exceptional curve.

Proof. We retain the notation of the theorem.

Step 1. Incidence geometry and divisor classes. We identify the blow-up and compute the initial current.

Away from p, the incidence equation determines $[u:v]=[Z_1:Z_2]$, so b is an isomorphism there. Over p, its fiber is the whole $\mathbb{P}^1$. On the chart $Z_0\neq 0$, $u\neq 0$, put

$$z=\frac{Z_1}{Z_0},\qquad w=\frac{v}{u}.$$

The incidence equation becomes

$$\frac{Z_2}{Z_0}=zw.$$

Thus z,w are coordinates, and

$$E=\{z=0\},\qquad F=\{w=0\}$$

on this chart. On the chart $Z_0\neq 0$, $v\neq 0$, the coordinates

$$z'=\frac{Z_2}{Z_0},\qquad w'=\frac{u}{v},\qquad \frac{Z_1}{Z_0}=z'w'$$

cover the remaining part of E. These are the blow-up charts of the pencil $[Z_1:Z_2]$.

Together with the isomorphism away from p, the charts show that X is smooth. The incidence description realizes X as a closed projective surface in $\mathbb{P}^2\times\mathbb{P}^1$, so it is compact and projective. The curves E and F are smooth and meet transversely at one point. In particular, their union is a simple normal crossings divisor.

The map (b,f) is the displayed inclusion in the product. Hence $\omega=\beta+\gamma$ is the restriction of its product Kähler form and is Kähler. Since b is constant on E,

$$\beta|_E=0.$$

The map $f|_E$ is an isomorphism onto $\mathbb{P}^1$, so the Fubini–Study normalization gives

$$\int_E\beta=0,\qquad \int_E\gamma=1,\qquad \int_E\omega=1. \tag{3.1}$$

Since F is a fiber of f, its line bundle is $f^*\mathcal{O}_{\mathbb{P}^1}(1)$. The local quotients

$$\frac{Z_1}{u}\quad (u\neq 0),\qquad \frac{Z_2}{v}\quad (v\neq 0)$$

agree on their overlap by the incidence equation. They define a section of $b^*\mathcal{O}_{\mathbb{P}^2}(1)\otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1)$. The coordinate descriptions show that its zero divisor is exactly E, with multiplicity one. This proves the asserted description of $\mathcal{O}(E)$.

For the induced metrics, the curvature forms therefore satisfy

$$\Theta_F=\gamma,\qquad \Theta_E=\beta-\gamma,\qquad 2\Theta_F+\Theta_E=\omega,\qquad \Theta_F+\Theta_E=\beta. \tag{3.2}$$

Applying (2.1) to the two defining sections gives

$$\begin{aligned}\omega+\mathrm{dd^c}(2\log r_F+\log r_E)&=\omega+2([F]-\gamma)+[E]-(\beta-\gamma)\\&=2[F]+[E].\end{aligned} \tag{3.3}$$

Each logarithm is locally integrable and smooth away from its divisor; we extend it across the divisor by the value $-\infty$. Consequently ϕ_0 is an ω-plurisubharmonic potential with simple normal crossings analytic singularities.

Step 2. The reduced class. We show that $\{\beta\}$ is nef and big and contains no Kähler form.

The pullback form β is semipositive. It itself provides the smooth representatives required for nefness in Definition 2.1. By (2.1) and (3.2),

$$\{[E]\} = \{\beta - \gamma\}.$$

It follows that

$$\left\{\frac{1}{2}\omega + \frac{1}{2}[E]\right\} = \{\beta\}, \qquad \frac{1}{2}\omega + \frac{1}{2}[E] \geq \frac{1}{2}\omega. \tag{3.4}$$

This current proves that the class is big.

Every smooth closed representative of $\{\beta\}$ has integral zero on E, by (3.1). A Kähler form has strictly positive integral on the compact complex curve E. Thus $\{\beta\}$ contains no Kähler form.

Step 3. The compact maximal flow. We apply Proposition 2.3 and determine the Lelong numbers of its residual potential.

Take

$$M = X, \qquad a = \omega, \qquad dV = \omega^2, \qquad (D_1, D_2) = (F, E), \qquad (m_1, m_2) = (2, 1).$$

The surface is compact Kähler, and the two divisors have simple normal crossings by Step 1. The weighted curvature identity in (3.2) is precisely $a = 2\Theta_F + \Theta_E$. The reduced curvature is $\theta = \Theta_F + \Theta_E = \beta$, whose class is nef and big by Step 2. Thus all hypotheses of Proposition 2.3 hold.

The log resolution in this application is the identity map of X. The blow-down b serves to describe the geometry; the application uses no nontrivial resolution. Both discrepancy numbers are therefore zero, and the first time endpoint is

$$\min(2, 1) = 1.$$

Proposition 2.3 gives a potential $\rho \in \mathcal{E}^1(X, \beta)$ and a global positive closed current

$$K = \beta + \mathrm{dd^c}\rho$$

satisfying

$$\langle K^2 \rangle = \frac{e^{\rho}}{r_F r_E}\,\omega^2. \tag{3.5}$$

The constant zero is β-plurisubharmonic because $\beta \geq 0$. Every β-plurisubharmonic function on the compact surface is bounded above. Hence zero has minimal singularities, in the sense of Definition 2.1. The Lelong-number assertion in Proposition 2.3 yields

$$\nu(\rho, x) = 0 \qquad \text{for every } x \in X. \tag{3.6}$$

Since β is smooth, K has the same Lelong numbers.

The flow and potential formulas in Proposition 2.3 now give (1.1) and (1.2). We also check the scalar equation with the chosen normalization. There is a dense open set on which ρ is smooth and both defining sections are nonzero. On that set,

$$W_t = tK, \qquad \partial_t\phi_t = -\log r_F - \log r_E + \rho + 2\log t.$$

Using (3.5), we obtain

$$W_t^2 = t^2 K^2 = t^2 \frac{e^{\rho}}{r_F r_E}\,\omega^2 = e^{\partial_t \phi_t}\omega^2.$$

For $0 < t < 1$, all three coefficients in (1.1) are positive, so the formula defines a global positive current.

At the level of potentials,

$$\phi_t - \phi_0 = t(\rho - \log r_F - \log r_E) + 2(t\log t - t).$$

The three fixed functions in parentheses are integrable on X, and $t\log t \to 0$ as $t \downarrow 0$. Therefore $\phi_t \to \phi_0$ in $L^1(X)$. These computations verify the equation and the initial limit with our conventions. The identification with the compact maximal weak flow follows from Proposition 2.3 and its global comparison argument.

Step 4. The exact jumping interval. We compute every local Arnold multiplicity of the initial potential.

At a point of $F \setminus E$, choose a transverse coordinate ζ. Then

$$\phi_0 = 2\log|\zeta|^2 + O(1).$$

For $c > 0$, the transverse radial integral governing local integrability of $e^{-c\phi_0}$ is

$$\int_0^\epsilon R^{1-4c}\, dR.$$

It is finite exactly when $c < \frac{1}{2}$. Thus $\lambda(\phi_0, x) = 2$ on $F \setminus E$.

At a point of $E \setminus F$, the singular part of ϕ_0 is $\log|\zeta|^2$. The corresponding radial integral is

$$\int_0^\epsilon R^{1-2c}\, dR,$$

which is finite exactly when $c < 1$. Hence $\lambda(\phi_0, x) = 1$ on $E \setminus F$.

At the crossing point, use the coordinates z, w from Step 1. The initial potential has the form

$$\phi_0 = \log|z|^2 + 2\log|w|^2 + O(1).$$

The product radial integral is

$$\left(\int_0^\epsilon R^{1-2c}\, dR\right)\left(\int_0^\epsilon S^{1-4c}\, dS\right).$$

It is finite exactly when both $c < 1$ and $c < \frac{1}{2}$. Therefore the Arnold multiplicity at $E \cap F$ is 2.

Off $E \cup F$, the potential is smooth, so its local Arnold multiplicity is zero. We have proved

$$\lambda(\phi_0, x) = \begin{cases} 2, & x \in F, \\ 1, & x \in E \setminus F, \\ 0, & x \in X \setminus (E \cup F). \end{cases} \tag{3.7}$$

Consequently, the threshold sets from Definition 1.1 are

$$A_s(\phi_0) = \begin{cases} \varnothing, & s > 2, \\ F, & 1 < s \le 2, \\ F \cup E, & 0 < s \le 1. \end{cases} \tag{3.8}$$

The positive jumping times are exactly 2 and 1. With terminal endpoint zero, this gives the jumping interval $(0, 1)$, whose interior contains $t = \frac{1}{2}$.

Step 5. Failure at every fixed interior time. We show that any global positive lower bound forces a positive measure of negative mass on E.

Fix $t \in (0,1)$. Suppose that there exists $\delta > 0$ such that

$$W_t \geq \delta\omega \quad \text{on } X.$$

Set

$$c = \frac{\delta}{t} > 0, \qquad A = F \cup E.$$

On $X \setminus A$, the divisor currents in (1.1) vanish. It follows that

$$K \geq c\omega \quad \text{on } X \setminus A. \tag{3.9}$$

We first extend (3.9) across A. A positive $(1,1)$-current has locally finite measure coefficients. Restricting those coefficients to a Borel set preserves positivity of the resulting matrix-valued measure. Thus $\mathbf{1}_A K$ is positive. The coefficient measures of the smooth form ω give zero mass to A, since A is a union of complex curves in a complex surface. Locally, the coefficient measures therefore decompose as

$$K - c\omega = \mathbf{1}_{X\setminus A}(K - c\omega) + \mathbf{1}_A K. \tag{3.10}$$

The first term is positive by (3.9), and the second is positive by the Borel restriction property. Hence $K - c\omega$ is positive on all of X. It is closed because both K and ω are closed.

Apply Theorem 2.5 on the projective surface X, with

$$\theta = \beta - c\omega, \qquad q = \rho, \qquad a = \omega.$$

The positive closed current to which it applies is

$$\theta + \mathrm{dd^c} q = K - c\omega.$$

By (3.6), the theorem gives quasi-plurisubharmonic functions q_m with analytic singularities and nonnegative numbers $a_m \downarrow 0$ such that

$$\beta + (a_m - c)\omega + \mathrm{dd^c} q_m \geq 0 \tag{3.11}$$

and $q_m \downarrow \rho$. In particular, $q_m \geq \rho$, and monotonicity of Lelong numbers with respect to singularity type gives

$$0 \leq \nu(q_m, x) \leq \nu(\rho, x) = 0 \qquad (x \in X).$$

Thus every q_m has zero Lelong numbers everywhere.

Each q_m is locally bounded. To see this, use its local expression from Definition 2.4. If all the functions h_j vanish at a point, the logarithmic term has positive Lelong number there. The locally bounded remainder does not change this number, contradicting $\nu(q_m, x) = 0$. Thus the functions h_j have no common zero. The logarithm is then locally bounded. This argument uses no smoothness of that remainder.

Choose m large enough that $a_m < c$. The current in (3.11) restricts to a positive current on E. For completeness, let U be a coordinate neighborhood in X, and choose a smooth local potential h for the closed form $\beta + (a_m - c)\omega$. Then

$$\mathrm{dd^c}(h + q_m) \geq 0 \quad \text{on } U.$$

The function $h + q_m$ is plurisubharmonic and locally bounded. Its restriction to the complex curve $E \cap U$ is therefore finite and subharmonic. Taking its distributional $\mathrm{dd^c}$ defines a positive measure on $E \cap U$. These local restrictions give the global positive measure

$$\beta|_E + (a_m - c)\omega|_E + \mathrm{dd}^{\mathrm{c}}_E(q_m|_E) \tag{3.12}$$

on E, where $\mathrm{dd}^{\mathrm{c}}_E$ uses the same normalization as $\mathrm{dd^c}$.

A positive measure has nonnegative total mass. On the compact boundaryless curve E, Stokes' theorem for distributions gives

$$\int_E \mathrm{dd}^{\mathrm{c}}_E(q_m|_E) = 0.$$

The mass of (3.12) is therefore, by (3.1),

$$\int_E \beta + (a_m - c) \int_E \omega = 0 + (a_m - c) < 0.$$

This is a contradiction.

The time $t \in (0,1)$ was arbitrary. Thus the single compact maximal weak flow constructed above fails to dominate every positive multiple of ω at each fixed time of the genuine jumping interval $(0,1)$. This proves the theorem and gives the announced counterexample to the global Kähler-current assertion of Di Nezza, Guedj, and Lu. □

References

[DGL26] E. Di Nezza, V. Guedj, and C. H. Lu. *Geometric smoothing by the Kähler–Ricci Flow.* 2026. arXiv: 2607.08325. URL: https://arxiv.org/abs/2607.08325v1.

[Xia26] M. Xia. *Singularities in Global Pluripotential Theory. Lectures at Zhejiang University.* July 18, 2026. URL: https://mingchenxia.github.io/Lectures/SGPT_final.pdf.