Uniform Green Functions and Sobolev Geometry in Families of Singular Kähler Varieties

Abstract

We prove uniform estimates for Green's functions, diameters, volumes of balls, and Sobolev inequalities for Kähler metrics with bounded potentials on the fibres of a proper family of normal irreducible compact Kähler spaces, assuming only a uniform L^p bound, $p > 1$, on the normalized volume densities. The estimates hold on every fibre, singular fibres included, and no uniform bound on the Kähler potentials is assumed. Green's functions are defined with poles in the regular locus of the metric, while the diameter and non-collapsing estimates hold at every point of the metric completion; the Sobolev inequalities hold in the full subcritical range allowed by the gradient estimate for the Green's function. Two examples—a reducible conic degeneration and a non-normal nodal cubic—show that irreducibility cannot be dropped, and that a Green's function with pole at a singular point is not canonically defined.

2020 Mathematics Subject Classification. 32Q15, 32U40, 58J05, 53C23

Keywords. singular Kähler varieties, Green functions, Friedrichs extension, Sobolev inequality, metric completion, non-collapsing

1 Introduction

Uniform estimates for Green's functions connect pluripotential control of volume densities to diameter bounds, non-collapsing, and Sobolev inequalities. For smooth fibres this programme was carried out by Guo, Phong, Song and Sturm for a fixed manifold, and by Guedj and Tô [GT25] for families with varying complex structure. The purpose of this paper is to extend it to families whose fibres may all be singular, without assuming a Poincaré inequality, compactness of the resolvent, or a uniform bound on the Kähler potentials.

1.1 Standing assumptions

Throughout the paper we work under the following hypotheses, referred to as (H).

The family. Let $\mathbb{D} = \{t \in \mathbf{C} : |t| < 1\}$ and let $\pi\colon \mathcal{X} \to \mathbb{D}$ be a proper surjective holomorphic map from an irreducible reduced Kähler complex space. Assume that every fibre $X_t = \pi^{-1}(t)$ is reduced, irreducible, normal and of pure dimension $n \geq 1$. Fix a Kähler form β on $\mathcal{X}$ and constants

$$A > 0, \qquad B \geq 1, \qquad 1 < p < \infty, \qquad 0 < b_- \leq b_+ < \infty.$$

Write $\beta_t = \beta|_{X_t}$ and assume that

$$b_- \leq V_{\beta,t} := \int_{X_t} \beta_t^n \leq b_+ \quad \text{for } |t| < \tfrac{3}{4}, \qquad \nu_t := \frac{\beta_t^n}{V_{\beta,t}}.$$

Measures on a fibre are always extended by zero from its regular locus.

The metric. For $|t| < \frac{1}{2}$, let θ_t be a smooth closed real $(1,1)$-form on X_t (locally the restriction of a smooth ambient form) with $\theta_t \le A\beta_t$, whose class is Kähler: $\theta_t + dd^c a_t$ is a Kähler form for some smooth real function a_t on X_t. Let U_t be a bounded θ_t-plurisubharmonic function and let $Z_t \subset X_t$ be a proper closed analytic subset containing $X_{t,\mathrm{sing}}$. Set

$$T_t = \theta_t + dd^c U_t, \qquad M_t = X_t \setminus Z_t, \qquad \omega_t = T_t|_{M_t},$$

and assume that (M_t, ω_t) is a smooth Kähler manifold. No bound on $\|U_t\|_{L^\infty}$ is assumed.

The density. For $|t| < \frac{1}{2}$ put

$$V_t = \int_{M_t} \omega_t^n, \qquad \mu_t = \frac{\omega_t^n}{V_t} = f_t\,\nu_t, \qquad \|f_t\|_{L^p(\nu_t)} \le B.$$

The conventions for d^c, the Laplacian Δ_t, the Riemannian metric g_t, the Dirichlet form $\mathcal{E}_t$, its domain W_t, and the Friedrichs operator $L_t = -\Delta_t$ are fixed in Section 2; in particular $\mathcal{E}_t(u,u) = \frac{1}{2}\int_{M_t} |\nabla u|^2_{g_t}\,d\mu_t$. All constants below depend only on the *family data*

$$(\mathcal{X}, \pi, \beta), \quad n, \quad A, \quad B, \quad p, \quad b_-, \quad b_+,$$

and on the displayed exponents, and never on t, U_t or Z_t.

Theorem A. *Assume* (H). *The following assertions hold for every $|t| < \frac{1}{2}$, with constants depending only on the family data and the displayed exponents.*

(1) Energy domain and bounded inverse. The manifold M_t is connected, $0 < V_t < \infty$,

$$W_t = \{u \in W^{1,2}_{\mathrm{loc}}(M_t, \mathbf{R}) : \ \|u\|^2_{L^2(\mu_t)} + \mathcal{E}_t(u,u) < \infty\},$$

and $\ker L_t = \mathbf{R}$*. There is $C_{\mathrm{inv}} > 0$ such that for every real $F \in L^\infty(\mu_t)$ with $\int F\,d\mu_t = 0$ there is a unique $u \in \mathcal{D}(L_t)$ with $L_t u = F$ and $\int u\,d\mu_t = 0$; it satisfies $\|u\|_{L^\infty} \le C_{\mathrm{inv}}\|F\|_{L^\infty}$.*

(2) Green's functions. For every $x \in M_t$ there is a unique function $G_t(x,\cdot) \in L^1(\mu_t)$ such that

$$\int_{M_t} G_t(x,y)\,d\mu_t(y) = 0,$$

$$\int_{M_t} G_t(x,y)\,\Delta_t\eta(y)\,d\mu_t(y) = n\Big(\eta(x) - \int_{M_t} \eta\,d\mu_t\Big) \quad (\eta \in C^\infty_c(M_t)),$$

and $G_t(x,\cdot) - P_x \in W_t$, where P_x is a compactly supported cutoff of a local Dirichlet Green's function with pole x (Section 5); the last condition does not depend on the choice of P_x. The function $(x,y) \mapsto G_t(x,y)$ is symmetric and smooth off the diagonal of $M_t \times M_t$, and $G_t(x,y) \to -\infty$ as $y \to x$.

(3) Uniform kernel estimates. There is $C_0 \ge 0$ such that $G_t(x,y) \le C_0$ for $x \ne y$. Put $\mathcal{H}_t(x,y) := C_0 + 1 - G_t(x,y) \ge 1$. For every $0 < a < \frac{n}{n-1}$ and every $0 < s < \frac{2n}{2n-1}$ there are constants C_a, D_s with

$$\int_{M_t} |G_t(x,y)|^a\,d\mu_t(y) \le C_a, \qquad \int_{M_t} |\nabla_y G_t(x,y)|^s_{g_t}\,d\mu_t(y) \le D_s,$$

where for $n = 1$ the first threshold is $+\infty$ and the second is 2. For every $b > 0$,

$$\int_{M_t} \frac{|\nabla_y G_t(x,y)|^2_{g_t}}{\mathcal{H}_t(x,y)^{1+b}}\,d\mu_t(y) \le \frac{2n}{b}.$$

(4) *Compact completion and volumes of balls. Let d_t be the Riemannian distance of g_t, let $(\widehat{M}_t, \widehat{d}_t)$ be the metric completion of (M_t, d_t), and let $\widehat{\mu}_t$ be the push-forward of μ_t. Then $\widehat{M}_t$ is compact, M_t is open in $\widehat{M}_t$, $\widehat{\mu}_t(\widehat{M}_t \setminus M_t) = 0$, and $\operatorname{diam}(\widehat{M}_t, \widehat{d}_t) \le C_d$. For every $0 < \delta < 1$ there is $C_\delta \ge 1$ such that for every $z \in \widehat{M}_t$ and every $R > 0$,*

$$\widehat{\mu}_t(B_{\widehat{d}_t}(z, R)) \ge C_\delta^{-1} \min\{1, R^{2n+\delta}\}.$$

(5) *Sobolev inequalities. For every $1 < r < \frac{n}{n-1}$ ($1 < r < \infty$ when $n = 1$) there is S_r such that every $u \in W_t$ satisfies*

$$\Big(\int_{M_t} \big|u - {\textstyle\fint} u \, d\mu_t\big|^{2r} d\mu_t\Big)^{1/r} \le S_r \int_{M_t} |\nabla u|^2_{g_t} \, d\mu_t.$$

If $\Omega \subset \widehat{M}_t$ is open with $\widehat{\mu}_t(\widehat{M}_t \setminus \Omega) > 0$, and $W_{0,t}(\Omega)$ denotes the closure of $C_c^\infty(\Omega \cap M_t, \mathbf{R})$ in W_t, then every $u \in W_{0,t}(\Omega)$ satisfies

$$\Big(\int_{\Omega\cap M_t} |u|^{2r} \, d\mu_t\Big)^{1/r} \le 2S_r\Big(1 + \frac{\widehat{\mu}_t(\Omega)}{\widehat{\mu}_t(\widehat{M}_t \setminus \Omega)}\Big) \int_{\Omega\cap M_t} |\nabla u|^2_{g_t} \, d\mu_t.$$

Functions in $W^{1,2}_{\mathrm{loc}}(\Omega \cap M_t)$ with finite L^2 norm and energy that vanish outside a compact subset of Ω belong to $W_{0,t}(\Omega)$ after extension by zero.

(6) *The added boundary has zero capacity. For $E \subset \widehat{M}_t$ let $\operatorname{Cap}_{1,t}(E)$ be the infimum of $\|v\|^2_{L^2(\mu_t)} + \mathcal{E}_t(v, v)$ over all $v \in W_t$ that extend continuously to $\widehat{M}_t$ and satisfy $v \ge 1$ on a neighbourhood of E in $\widehat{M}_t$ ($\inf \varnothing = +\infty$). Then $\operatorname{Cap}_{1,t}(\widehat{M}_t \setminus M_t) = 0$.*

Three comments on the statement are in order. First, Green's functions are only defined for poles in M_t; the metric statements in part (4) nevertheless hold at every point of the completion, because they are obtained from Green's functions with poles in a dense set. The nodal example of Section 8 shows that this distinction is unavoidable: at a non-normal point no Green's function is singled out by mean normalization and integrability conditions. Second, no identification of $\widehat{M}_t$ with X_t is claimed. Third, the theorem is stated for every fibre over the inner disc, singular or not, and its constants do not depend on the fibre.

1.2 Strategy

The proof has four steps, corresponding to Sections 3 to 5 and 7.

Zero-energy cutoffs (Section 3). On a log resolution of X_t, the current T_t^{n-1} has a closed positive extension, and logarithmic truncations of a defining section of the exceptional divisor have energy tending to zero. This identifies the Friedrichs domain W_t with the maximal finite-energy Sobolev space and shows $\ker L_t = \mathbf{R}$.

The bounded inverse (Section 4). A bounded mean-zero solution u of $L_t u = F$ is normalized by $\|u\|_{L^\infty}$ and compared with the solution of an auxiliary twisted Monge–Ampère equation. The maximum principle, in the form of a positive-part test in W_t, gives a one-sided bound; the uniform Skoda integrability of [DGG23] (normalized with the supremum–mean comparison of [Ou22]) and the envelope estimate of [DGG23] control the oscillation of the auxiliary potential and yield $\|u\|_{L^\infty} \le C_{\mathrm{inv}}\|F\|_{L^\infty}$. Coercive resolvents then give existence.

Green's functions and their estimates (Sections 5 and 6). A local Dirichlet Green's function is glued to a bounded global correction. A weak mean-value inequality, again obtained from an auxiliary Monge–Ampère equation, bounds the positive part of the Green's function; a nonlinear bootstrap yields all moments of order $< \frac{n}{n-1}$, and testing the equation against a negative power of the shifted kernel gives weighted energy control and, by Hölder's inequality, gradient bounds in L^s for $s < \frac{2n}{2n-1}$.

Metric geometry and Sobolev inequalities (*Section 7*). Applying the Green representation to truncated distance functions gives diameter and volume bounds for balls centred in M_t, which pass to the completion by density. A weighted Cauchy–Schwarz inequality and Minkowski's integral inequality give the Sobolev inequality, and the cutoffs of Section 3, extended by zero, show that the added boundary has zero capacity.

1.3 Relation to the literature

For smooth fibres (and a possibly singular central fibre that is not itself estimated), [GT25, Theorems A and B] contains the Green upper bound, the moment and gradient estimates, and the diameter and non-collapsing bounds of Theorem A, under a cohomological version of the bound $\theta_t \le A\beta_t$ (see [GT25, Setting 1.5, Definition 1.6]). On a fixed compact normal Kähler space, [GPSS23, Theorems 3.1–3.4] prove compactness of the metric completion, Sobolev inequalities and non-collapsing under an entropy hypothesis with exponent larger than n, and [Vu24, Corollary 1.5] proves diameter and non-collapsing bounds under an $f|\log f|^{p_0}$ hypothesis. The zero-energy cutoff construction of Section 3 is close in spirit to [GPSS23, Lemma 8.2]. Theorem A differs from these results in that all fibres may be singular and the complex structure varies, in that only an L^p density bound is assumed, and in that the Green's functions are constructed and estimated on the singular fibres themselves.

Remark 1.1. The Sobolev inequality [GT25, Theorem 2.6] is printed with the range $1 < r < \frac{2n}{n-1}$, both in arXiv:2405.17232v2 and in the published version. Its proof chooses $\beta > 0$ with $(1+\beta)r < \frac{n}{n-1}$, and the scaling of the embedding $W^{1,2} \subset L^{2n/(n-1)}$ in real dimension $2n$ shows that $2r \le \frac{2n}{n-1}$ is necessary for any such inequality when $n \ge 2$. The range $1 < r < \frac{n}{n-1}$ of part (5) is therefore the one supported by the argument.

2 Preliminaries

2.1 Conventions

All complex spaces are reduced. A Kähler form on a complex space is locally the restriction of $\mathrm{dd^c}$ of a smooth strictly plurisubharmonic function on an ambient open subset of $\mathbf{C}^N$. We use

$$\mathrm{d^c} = \frac{i}{2}(\bar\partial - \partial), \qquad \mathrm{dd^c} = i\partial\bar\partial,$$

and Bedford–Taylor products for currents with bounded local potentials; these are closed, positive, and put no mass on proper analytic subsets, see [EGZ09, §5.2 and Proposition 6.1] for the behaviour under resolutions.

Let (M, ω) be a Kähler manifold of dimension n with finite volume $V = \int_M \omega^n$, let $\mu = \omega^n/V$, and let $g(v, w) = \omega(v, Jw)$ be the Riemannian metric. For real $u, v \in W^{1,2}_{\mathrm{loc}}(M)$ we write

$$\Gamma(u,v)\,d\mu = \frac{n}{V}\,du \wedge \mathrm{d^c}v \wedge \omega^{n-1} = \frac{1}{2}\langle \nabla u, \nabla v\rangle_g\,d\mu, \qquad \Gamma(u) = \Gamma(u,u) = \tfrac{1}{2}|\nabla u|_g^2,$$

(the second equality is checked in coordinates in which ω is standard at a point, using $du \wedge \mathrm{d^c}v = \frac{i}{2}(\partial u \wedge \bar\partial v + \partial v \wedge \bar\partial u)$), and

$$\mathcal{E}(u,v) = \int_M \Gamma(u,v)\,d\mu, \qquad \Delta u = n\,\frac{\mathrm{dd^c}u \wedge \omega^{n-1}}{\omega^n} = \tfrac{1}{2}\Delta_g u.$$

With these normalizations, $dd^c u \geq -\omega$ implies $\Delta u \geq -n$, and for $u \in C^2(M)$, $\eta \in C^\infty_c(M)$ Stokes' formula reads $\mathcal{E}(u,\eta) = -\int_M \eta\, \Delta u\, d\mu$. For $u \in W^{1,2}_{\mathrm{loc}}(M)$ and a locally integrable function h we say that $\Delta u \geq h$ *weakly* if

$$-\mathcal{E}(u,\eta) \geq \int_M h\eta\, d\mu \qquad \text{for all } 0 \leq \eta \in C^\infty_c(M);$$

by mollification the inequality then holds for all nonnegative compactly supported $\eta \in W^{1,2}(M) \cap L^\infty(M)$. Weak equalities are defined analogously.

Let W be the closure of $C^\infty_c(M,\mathbf{R})$ for the norm $\|u\|^2_W = \|u\|^2_{L^2(\mu)} + \mathcal{E}(u,u)$ (the form is closable, see Lemma 2.2), and let L be the Friedrichs operator of $\mathcal{E}$:

$$u \in \mathcal{D}(L),\ Lu = F \quad \Longleftrightarrow \quad u \in W, \quad \mathcal{E}(u,v) = \int_M Fv\, d\mu \ \ (v \in W).$$

Thus $L = -\Delta$ on smooth compactly supported functions, and L is a nonnegative self-adjoint operator on $L^2(\mu)$. On a fibre of the family we index everything by t: $\Delta_t, \mathcal{E}_t, W_t, L_t$.

2.2 Sobolev calculus

We use the following standard facts, see [GT01, Lemmas 7.6 and 7.7] and [EG15, §3.1.2 and §4.2.2].

Lemma 2.1. *Let (M,g) be a Riemannian manifold with distance d.*

(1) If $u\colon M \to \mathbf{R}$ is locally Lipschitz for d, then $u \in W^{1,\infty}_{\mathrm{loc}}(M)$ and $|\nabla u|_g \leq \mathrm{Lip}_d(u)$ almost everywhere, where $\mathrm{Lip}_d(u)$ is the Lipschitz constant of u for d.

(2) If $u \in W^{1,2}_{\mathrm{loc}}(M)$ and $\Phi\colon \mathbf{R} \to \mathbf{R}$ is Lipschitz and piecewise C^1, then $\Phi(u) \in W^{1,2}_{\mathrm{loc}}(M)$ and $d\Phi(u) = \Phi'(u)\, du$ almost everywhere, with the convention that the right side vanishes on level sets of u where Φ' is undefined; indeed $du = 0$ almost everywhere on every level set of u. In particular, for $T_a(s) = \max\{-a, \min\{s,a\}\}$,

$$dT_a(u) = \mathbf{1}_{\{|u|<a\}}\, du, \qquad du^+ = \mathbf{1}_{\{u>0\}}\, du.$$

The next lemma isolates the consequences of the existence of cutoff functions with small energy. In the situation of Theorem A these will be provided by Proposition 3.1.

Lemma 2.2 (Energy domain)**.** *Let (M,ω) be a connected Kähler manifold with $0 < V < \infty$, and suppose there are $\chi_j \in C^\infty_c(M)$, $0 \leq \chi_j \leq 1$, with*

$$\chi_j \to 1 \ \textit{pointwise}, \qquad \mathcal{E}(\chi_j,\chi_j) \to 0. \tag{CUT}$$

Then the following hold.

(1) The form $\mathcal{E}$ on $C^\infty_c(M)$ is closable, and $W = \widetilde{W} := \{u \in W^{1,2}_{\mathrm{loc}}(M,\mathbf{R}) : \|u\|^2_{L^2(\mu)} + \mathcal{E}(u,u) < \infty\}$.

(2) Every nonnegative $u \in W$ is a limit in W of nonnegative functions in $C^\infty_c(M)$; every $u \in W \cap L^\infty$ is a limit in W of functions in $C^\infty_c(M)$ bounded by $\|u\|_{L^\infty}$.

(3) Constants belong to $\mathcal{D}(L)$ with $L1 = 0$, and $\ker L = \mathbf{R}$. Consequently $\int_M Lu\, d\mu = 0$ for all $u \in \mathcal{D}(L)$.

(4) If $w \in W$, $h \in L^2(\mu)$ and $\Delta w \geq h$ weakly, then $-\mathcal{E}(w,\eta) \geq \int_M h\eta\, d\mu$ for every nonnegative $\eta \in W$.

(5) *If $\psi \in W^{1,2}_{\mathrm{loc}}(M) \cap L^\infty(M)$ satisfies $\Delta\psi \geq -n$ weakly, then $\mathcal{E}(\psi,\psi) \leq 2n \operatorname{osc}_M \psi$; in particular $\psi \in W$.*

Proof. (1) If $q_k \in C_c^\infty(M)$, $q_k \to 0$ in $L^2(\mu)$ and dq_k is Cauchy for the energy, then the L^2_{loc}-limit of dq_k is the distributional differential of 0, hence vanishes; this is closability. The space $\widetilde{W}$ is complete, since an $\mathcal{E}$-Cauchy sequence converging in L^2 converges in $W^{1,2}_{\mathrm{loc}}$ to a function with the limiting differential. Hence $W \subset \widetilde{W}$. Conversely let $u \in \widetilde{W}$. By Lemma 2.1 and dominated convergence, $T_a(u) \to u$ in $\widetilde{W}$ as $a \to \infty$, so we may assume u bounded. Then $\chi_j u \to u$ in L^2 and

$$\mathcal{E}(\chi_j u - u, \chi_j u - u) \leq 2\int_M |\chi_j - 1|^2 \Gamma(u)\, d\mu + 2\|u\|_{L^\infty}^2 \mathcal{E}(\chi_j, \chi_j) \to 0$$

by dominated convergence and (CUT). Each $\chi_j u$ has compact support in M and is approximated in $W^{1,2}$ by functions in $C_c^\infty(M)$ via a finite atlas, a partition of unity and mollification; on the relevant compact sets g and μ are uniformly equivalent to their Euclidean counterparts. Hence $u \in W$.

(2) Truncation, multiplication by χ_j and mollification preserve nonnegativity and L^∞ bounds.

(3) Since $\mu(M) = 1$, $1 \in \widetilde{W} = W$, and $\mathcal{E}(1, v) = 0$ for all $v \in W$, so $1 \in \mathcal{D}(L)$ and $L1 = 0$. If $Lu = 0$ then $\mathcal{E}(u,u) = 0$, so $du = 0$ almost everywhere; mollification in charts shows that u is locally constant, hence constant since M is connected. Finally $\int_M Lu\, d\mu = \mathcal{E}(u, 1) = 0$.

(4) Both sides are continuous in η for the norm of W (the right side because $h \in L^2$), so the inequality passes from nonnegative test functions in $C_c^\infty(M)$ to their limits, which by (2) include all nonnegative $\eta \in W$.

(5) Let $a = \operatorname{ess\,inf} \psi$, $Q = \operatorname{osc}_M \psi$ and $\chi \in C_c^\infty(M)$, $0 \leq \chi \leq 1$. Testing $\Delta\psi \geq -n$ with $\chi^2(\psi - a) \geq 0$ and using $d(\chi^2(\psi - a)) = \chi^2 d\psi + 2\chi(\psi - a)d\chi$,

$$\begin{aligned}\int_M \chi^2 \Gamma(\psi)\, d\mu &\leq n\int_M \chi^2(\psi - a)\, d\mu - 2\int_M \chi(\psi - a)\Gamma(\psi, \chi)\, d\mu \\ &\leq nQ + 2Q\Big(\int_M \chi^2\Gamma(\psi)\, d\mu\Big)^{1/2} \mathcal{E}(\chi,\chi)^{1/2}.\end{aligned}$$

With $2ab \leq a^2/2 + 2b^2$ this gives $\int_M \chi^2\Gamma(\psi)\, d\mu \leq 2nQ + 4Q^2\mathcal{E}(\chi,\chi)$. Taking $\chi = \chi_j$ and using Fatou's lemma yields $\mathcal{E}(\psi,\psi) \leq 2nQ$, and $\psi \in \widetilde{W} = W$ by (1). ☐

Lemma 2.3 (Convex compositions)**.** *Let (M, ω) be as in Lemma 2.2, let $u \in W \cap L^\infty(M)$, let $h \in L^\infty(M)$, and assume $\Delta u \geq h$ weakly. If $\Phi \in C^2(\mathbf{R})$ is convex and nondecreasing with Φ' bounded, then $\Phi(u) \in W$ and*

$$\Delta\Phi(u) \geq \Phi'(u)\, h \quad \textit{weakly.}$$

If moreover $h \geq -n$ almost everywhere on $\{u > 0\}$ (for instance if $\Delta u \geq -n$ weakly), then $\Delta u^+ \geq -n$ weakly.

Proof. By Lemma 2.1, $\Phi(u) \in \widetilde{W} = W$. For $0 \leq \eta \in C_c^\infty(M)$, the function $\Phi'(u)\eta$ is a nonnegative, bounded, compactly supported element of $W^{1,2}(M)$ with $d(\Phi'(u)\eta) = \Phi'(u)d\eta + \eta\Phi''(u)du$, hence an admissible test function, and

$$-\mathcal{E}(\Phi(u), \eta) = -\int_M \Phi'(u)\Gamma(u,\eta)\, d\mu = -\mathcal{E}(u, \Phi'(u)\eta) + \int_M \eta\, \Phi''(u)\Gamma(u)\, d\mu \geq \int_M h\, \Phi'(u)\eta\, d\mu.$$

For the positive part choose convex $\Phi_k \in C^2(\mathbf{R})$ with $\Phi_k = 0$ on $(-\infty, 0]$, $0 \leq \Phi_k' \leq 1$, and $\Phi_k'(s) \to 1$ for every $s > 0$ (e.g. $\Phi_k(s) = k^{-1}\Phi_1(ks)$). Since $\Phi_k'(u) = 0$ on $\{u \leq 0\}$ and $h \geq -n$ on $\{u > 0\}$, we get $-\mathcal{E}(\Phi_k(u), \eta) \geq \int_M h\Phi_k'(u)\eta\, d\mu \geq -n\int_M \Phi_k'(u)\eta\, d\mu \geq -n\int_M \eta\, d\mu$, and the left side converges to $-\mathcal{E}(u^+, \eta)$ because $d\Phi_k(u) = \Phi_k'(u)du \to \mathbf{1}_{\{u>0\}}du = du^+$ in L^2 by dominated convergence (recall $du = 0$ a.e. on $\{u = 0\}$). ☐

3 Zero-energy cutoffs on singular Kähler spaces

The singular boundary of M_t is invisible to the Dirichlet form because the $(n-1)$-st power of the metric current extends to a closed positive current on a resolution.

Proposition 3.1. *Let X be an irreducible normal compact Kähler space of pure dimension $n \geq 1$, let $Z \subsetneq X$ be a closed analytic subset containing X_{sing}, and let T be a closed positive $(1,1)$-current on X with bounded plurisubharmonic local potentials such that $\omega = T|_M$ is a Kähler form on $M = X \setminus Z$. Then M is connected, $0 < V = \int_M \omega^n < \infty$, and there are $\eta_j \in C_c^\infty(M, \mathbf{R})$ with $0 \leq \eta_j \leq 1$ satisfying (CUT), i.e.*

$$\eta_j \to 1 \text{ pointwise on } M, \qquad \mathcal{E}(\eta_j, \eta_j) = \frac{n}{V} \int_M d\eta_j \wedge \mathrm{d^c}\eta_j \wedge \omega^{n-1} \to 0.$$

In particular all conclusions of Lemma 2.2 hold for (M, ω).

More generally, the same conclusions hold whenever $M = Y \setminus D$ for a connected compact complex manifold Y and a (possibly empty) simple normal crossing divisor D, provided ω is a Kähler form on M with $0 < V < \infty$ and ω^{n-1} is the restriction to M of a closed positive $(n-1, n-1)$-current S on Y.

Proof. The general statement. Since D is a proper analytic subset of the connected manifold Y, its complement M is connected. If $D = \varnothing$ take $\eta_j = 1$. Otherwise let s be a holomorphic section of $\mathcal{O}_Y(D)$ with divisor D, and fix a smooth Hermitian metric $\|\cdot\|$ on $\mathcal{O}_Y(D)$, rescaled so that

$$\rho := \log \|s\| \leq -1 \quad \text{on } M.$$

On M, $\mathrm{dd^c}\rho$ equals minus one half of the curvature form of the metric, so there is a smooth positive $(1,1)$-form ϖ on Y with $\mathrm{dd^c}\rho \geq -\varpi$ on M.

Fix a smooth nondecreasing function $\gamma \colon \mathbf{R} \to [0,1]$ with $\gamma = 0$ on $(-\infty, 0]$ and $\gamma = 1$ on $[1, \infty)$, and let $F(\tau) = \int_0^\tau \gamma(\sigma)\, d\sigma$. Then F is smooth, convex and nondecreasing, $0 \leq F(\tau) \leq \tau^+$, and $F(\tau) = \tau - c_F$ for $\tau \geq 1$, where $c_F = \int_0^1 (1-\gamma) \in [0,1]$. For $j \geq 2$ set

$$u_j = -j + F(\rho + j), \qquad \eta_j = 1 + \frac{u_j}{j}.$$

Since $\rho \to -\infty$ along D, $u_j \equiv -j$ near D, so u_j and η_j extend smoothly to Y, η_j vanishes near D, and $\operatorname{supp} \eta_j$ is a compact subset of M. Moreover $-j \leq u_j \leq 0$ (because $\rho + j \leq j - 1$), hence $0 \leq \eta_j \leq 1$, and at every point of M one has $u_j = \rho - c_F$ for j large, so $\eta_j \to 1$. Finally,

$$\mathrm{dd^c} u_j = \gamma(\rho + j)\, \mathrm{dd^c}\rho + \gamma'(\rho + j)\, d\rho \wedge \mathrm{d^c}\rho \geq -\varpi \quad \text{on } M,$$

and the same inequality holds on Y because u_j is constant near D.

Since S is a positive current on the compact manifold Y, $\int_Y \varpi \wedge S < \infty$. As u_j is smooth on Y and S is closed, Stokes' formula for currents gives $\int_Y \mathrm{dd^c} u_j \wedge S = 0$ and $\int_Y du_j \wedge \mathrm{d^c} u_j \wedge S = \int_Y (-u_j)\, \mathrm{dd^c} u_j \wedge S$. Since $(\mathrm{dd^c} u_j + \varpi) \wedge S$ is a positive measure and $0 \leq -u_j \leq j$,

$$\int_Y du_j \wedge \mathrm{d^c} u_j \wedge S = \int_Y (-u_j)(\mathrm{dd^c} u_j + \varpi) \wedge S - \int_Y (-u_j)\, \varpi \wedge S \leq j \int_Y (\mathrm{dd^c} u_j + \varpi) \wedge S = j \int_Y \varpi \wedge S.$$

Because $d\eta_j = j^{-1} du_j$ vanishes near D and $S|_M = \omega^{n-1}$,

$$\mathcal{E}(\eta_j, \eta_j) = \frac{n}{Vj^2} \int_Y du_j \wedge \mathrm{d^c} u_j \wedge S \leq \frac{n}{Vj} \int_Y \varpi \wedge S \longrightarrow 0.$$

Note that only smooth forms were wedged with the current S; no integration by parts involving the singular metric was used. The last assertion follows from Lemma 2.2.

The singular space. Let $\varrho\colon Y \to X$ be a log resolution of the ideal of Z as in [EGZ09, §6.1]: Y is a compact Kähler manifold, $D = (\varrho^{-1}(Z))_{\mathrm{red}}$ is a simple normal crossing divisor, and $\varrho\colon Y \setminus D \to M$ is a biholomorphism. Since X is irreducible, Y is connected. Pulling back the bounded local potentials of T gives a closed positive current ϱ^*T on Y with bounded local potentials, and

$$S := (\varrho^*T)^{n-1} \qquad (S = 1 \text{ if } n = 1)$$

is a closed positive current on Y whose restriction to $Y \setminus D \cong M$ is ω^{n-1}. The Bedford–Taylor measure $(\varrho^*T)^n$ has finite mass on the compact manifold Y and puts no mass on D [EGZ09, Proposition 6.1], so $V = \int_{Y\setminus D}(\varrho^*T)^n < \infty$. Since ω is a Kähler form on the nonempty open set M, $V > 0$. The general statement applies. □

Combined with Lemma 2.2, Proposition 3.1 proves the first half of part (1) of Theorem A: for every fibre, M_t is connected, $0 < V_t < \infty$, W_t is the maximal finite-energy Sobolev space and $\ker L_t = \mathbf{R}$. No density bound and no uniformity in t is needed for this.

4 The uniform bounded inverse

The L^∞ estimate for L^{-1} is obtained by comparison with an auxiliary Monge–Ampère equation; no Poincaré inequality is needed. We first isolate the analytic part of the argument.

4.1 A comparison principle

Proposition 4.1. *Let (M, ω) be a connected Kähler manifold with $0 < V < \infty$ satisfying (CUT), and let $K \geq 0$. Assume:*

(MA$_K$) *For every $\epsilon \in (0, 1]$ and every real $v \in \mathcal{D}(L) \cap L^\infty(\mu)$ with*

$$\int_M v\, d\mu = 0, \qquad \|v\|_{L^\infty} \leq 1, \qquad \|Lv\|_{L^\infty} \leq \frac{\epsilon}{16},$$

there is a bounded $\psi \in W^{1,2}_{\mathrm{loc}}(M)$ such that, with

$$h := -\frac{Lv}{n\epsilon} \qquad \text{and} \qquad R := e^{(\epsilon\psi - v)/4}\,(1 + h)^n,$$

one has $\int_M R\, d\mu = 1$, $\Delta\psi \geq n(R^{1/n} - 1)$ weakly on M, and

$$R \leq 2^n \quad \Longrightarrow \quad \operatorname{osc}_M \psi \leq K.$$

Then every real $F \in L^\infty(\mu)$ with $\int_M F\, d\mu = 0$ admits a unique $u \in \mathcal{D}(L)$ with $Lu = F$ and $\int_M u\, d\mu = 0$, and

$$\|u\|_{L^\infty} \leq C_K \|F\|_{L^\infty}, \qquad C_K := 16 \max\{1, 2K\}.$$

The same bound holds for every mean-zero $u \in \mathcal{D}(L)$ with $Lu \in L^\infty$.

Note that $\|h\|_{L^\infty} \leq \frac{1}{16n}$, so $1 + h \geq \frac{15}{16}$ and R is positive and bounded whenever ψ is.

Proof. Step 1: comparison. Let ϵ, v, h, ψ, R be as in (MA$_K$). Since $R > 0$, $\Delta\psi \geq -n$ weakly, so $\psi \in W$ by Lemma 2.2(5), and by Lemma 2.2(4) the inequality $-\mathcal{E}(\psi, \eta) \geq n\int(R^{1/n} - 1)\eta\, d\mu$ holds for all nonnegative $\eta \in W$. Put $w = \epsilon\psi - v \in W$. Since $-\mathcal{E}(v, \eta) = -\int(Lv)\eta\, d\mu = n\epsilon\int h\eta\, d\mu$ and $R^{1/n} = e^{w/(4n)}(1 + h)$,

$$-\mathcal{E}(w, \eta) \geq n\epsilon \int_M ((1 + h)e^{w/(4n)} - 1 - h)\eta\, d\mu = n\epsilon \int_M (1 + h)(e^{w/(4n)} - 1)\eta\, d\mu$$

for all nonnegative $\eta \in W$. Testing with $\eta = w^+ \in W$ and using $\mathcal{E}(w, w^+) = \mathcal{E}(w^+, w^+) \geq 0$ (Lemma 2.1),

$$0 \geq -\mathcal{E}(w^+, w^+) \geq n\epsilon \int_M (1+h)(e^{w/(4n)} - 1)w^+ \, d\mu \geq 0.$$

The integrand is strictly positive on $\{w > 0\}$, so $w^+ = 0$, i.e.

$$\epsilon\psi \leq v \quad \text{almost everywhere.} \tag{4.1}$$

Consequently $R \leq (1+h)^n \leq (1 + \frac{1}{16n})^n \leq e^{1/16} \leq 2^n$, and (MA_K) gives $\operatorname{osc}_M \psi \leq K$.

Step 2: the a priori bound. Let $u \in \mathcal{D}(L) \cap L^\infty$ have mean zero and $\|Lu\|_{L^\infty} \leq \frac{1}{16}$, and put $N = \|u\|_{L^\infty}$. We claim $N \leq \max\{1, 2K\}$. If $N < 1$ there is nothing to prove. Otherwise let $\epsilon = N^{-1} \in (0, 1]$ and $v = \epsilon u$, so that $\|v\|_{L^\infty} = 1$ and $\|Lv\|_{L^\infty} \leq \epsilon/16$. Let ψ, R be given by (MA_K), and let $m = \operatorname{ess\,sup}_M \epsilon\psi$. From $\int_M R \, d\mu = 1$, $\epsilon\psi \leq m$ and $(1+h)^n \leq e^{1/16}$,

$$1 \leq e^{m/4} \, e^{1/16} \int_M e^{-v/4} \, d\mu.$$

Since $|v/4| \leq \frac{1}{4}$, $\int v \, d\mu = 0$ and $\int v^2 \, d\mu \leq 1$, the inequality $e^x \leq 1 + x + x^2$ for $|x| \leq 1$ gives $\int_M e^{-v/4} \, d\mu \leq 1 + \frac{1}{16}$. Taking logarithms and using $\log(1+s) \leq s$,

$$0 \leq \frac{m}{4} + \frac{1}{16} + \frac{1}{16}, \qquad \text{i.e.} \qquad m \geq -\frac{1}{2}.$$

By Step 1, $\operatorname{osc}_M \epsilon\psi \leq \epsilon K$, hence $\epsilon\psi \geq m - \epsilon K \geq -\frac{1}{2} - \epsilon K$, and (4.1) gives $v \geq -\frac{1}{2} - \epsilon K$. Applying the same argument to $-u$ gives $v \leq \frac{1}{2} + \epsilon K$. Thus $1 = \|v\|_{L^\infty} \leq \frac{1}{2} + K/N$, i.e. $N \leq 2K$, which proves the claim. Applying the claim to $u/(16\|Lu\|_{L^\infty})$ yields

$$\|u\|_{L^\infty} \leq 16 \max\{1, 2K\} \, \|Lu\|_{L^\infty}$$

for every mean-zero $u \in \mathcal{D}(L) \cap L^\infty$ with bounded Lu (if $Lu = 0$ then u is a mean-zero constant, hence 0, by Lemma 2.2(3)).

Step 3: existence. Let $F \in L^\infty(\mu)$ be real with $\int F \, d\mu = 0$, and put $F_0 = \|F\|_{L^\infty}$. For $\lambda > 0$ the bilinear form $\mathcal{E}(q, z) + \lambda \int qz \, d\mu$ is an inner product on the Hilbert space W equivalent to the one defining $\|\cdot\|_W$, so by the Riesz representation theorem there is a unique $u_\lambda \in W$ with

$$\mathcal{E}(u_\lambda, z) + \lambda \int_M u_\lambda z \, d\mu = \int_M F z \, d\mu \qquad (z \in W).$$

Thus $u_\lambda \in \mathcal{D}(L)$ and $Lu_\lambda = F - \lambda u_\lambda$; testing with $z = 1$ shows $\int u_\lambda \, d\mu = 0$. Testing with $z = (u_\lambda - F_0/\lambda)^+$ and with $z = (-u_\lambda - F_0/\lambda)^+$ gives $\|u_\lambda\|_{L^\infty} \leq F_0/\lambda$, hence $\|Lu_\lambda\|_{L^\infty} \leq 2F_0$. By Step 2 and the test $z = u_\lambda$,

$$\|u_\lambda\|_{L^\infty} \leq 2C_K F_0, \qquad \mathcal{E}(u_\lambda, u_\lambda) \leq \int_M F u_\lambda \, d\mu \leq 2C_K F_0^2.$$

Hence $(u_{1/j})_j$ is bounded in the Hilbert space W and, after passing to a subsequence, converges weakly in W to some u. Weak convergence in L^2 gives $\int u \, d\mu = 0$ and $\|u\|_{L^\infty} \leq 2C_K F_0$, and for every $z \in W$,

$$\mathcal{E}(u, z) = \lim_j \mathcal{E}(u_{1/j}, z) = \lim_j \Big(\int_M F z \, d\mu - \tfrac{1}{j} \int_M u_{1/j} z \, d\mu \Big) = \int_M F z \, d\mu.$$

So $u \in \mathcal{D}(L)$, $Lu = F$, and Step 2 gives $\|u\|_{L^\infty} \leq C_K F_0$. Two mean-zero solutions differ by an element of $\ker L = \mathbf{R}$ with mean zero, hence coincide. $\square$

4.2 Uniform Monge–Ampère estimates on a fibre

The hypothesis (MA_K) is supplied by pluripotential theory. We collect the required inputs in a form that will also be used in Section 6. Throughout this subsection we fix $|t| < \frac{1}{2}$ and drop the index t, writing $X, \beta, \nu, \theta, U, Z, M, \omega, V, \mu, f$.

We use the following results.

(i) *Supremum versus mean* [Ou22, Corollary 4.8]: there is $C_{\mathrm{sm}} \geq 0$, depending only on $(\mathcal{X}, \pi, \beta)$ and A, such that for every $|t| < \frac{1}{2}$ and every $q \in \mathrm{PSH}(X_t, A\beta_t)$,

$$\sup_{X_t} q - C_{\mathrm{sm}} \leq \frac{1}{V_{\beta,t}} \int_{X_t} q\, \beta_t^n \leq \sup_{X_t} q.$$

The statement in [Ou22] is for a proper family of reduced irreducible fibres over the disc and a fixed smooth closed $(1,1)$-form on $\mathcal{X}$, which is our situation with the form $A\beta$.

(ii) *Uniform Skoda integrability* [DGG23, Setting 2.1, Corollary 2.4 and Theorem 2.9]: there are $a_0 > 0$ and $C_1, C_2 > 0$, depending only on $(\mathcal{X}, \pi, \beta)$ and A, such that for all $|t| < \frac{1}{2}$ and all $q \in \mathrm{PSH}(X_t, A\beta_t)$ with $\sup_{X_t} q = 0$,

$$\int_{X_t} e^{-a_0 q}\, \beta_t^n \leq C_1 \exp\Big(-C_2 \int_{X_t} q\, \beta_t^n\Big).$$

(iii) *The envelope estimate* [DGG23, Theorem 1.9]: let Y be a compact Kähler manifold, θ' a smooth closed real $(1,1)$-form whose class is big, $V_{\theta'} = \sup\{u \in \mathrm{PSH}(Y, \theta') : u \leq 0\}^*$, and ν' a probability measure on Y such that $\int_Y e^{-a_0(q - \sup_Y q)}\, d\nu' \leq H_0$ for all $q \in \mathrm{PSH}(Y, \theta')$. For every $q_0 > 1$ and $J > 0$ there is a constant $\mathcal{M}(n, q_0, a_0, H_0, J)$ such that every $w \in \mathrm{PSH}(Y, \theta')$ with minimal singularities solving $(\theta' + dd^c w)^n = \mathrm{vol}(\theta')\, F\, \nu'$, where $F \geq 0$, $\int F\, d\nu' = 1$ and $\|F\|_{L^{q_0}(\nu')} \leq J$, satisfies

$$-\mathcal{M}(n, q_0, a_0, H_0, J) \leq w - \sup_Y w - V_{\theta'} \leq 0.$$

(iv) *Existence* [LWZ24, Theorems 1.1 and 1.2]: let (Y, σ) be a compact Hermitian manifold and θ' a smooth real $(1,1)$-form admitting a bounded θ'-psh function and with $\int_Y (\theta')^n > 0$. If $0 \leq G \in L^{q_0}(\sigma^n)$, $q_0 > 1$, is not identically zero, then for every $c > 0$ the equation $(\theta' + dd^c w)^n = e^{cw} G \sigma^n$ has a bounded θ'-psh solution, and if $\int_Y G\sigma^n = \int_Y (\theta')^n$ the equation $(\theta' + dd^c w)^n = G\sigma^n$ has a bounded θ'-psh solution.

(v) *The mixed Monge–Ampère inequality* [Din09, Theorem 1.2]: if a, b are bounded psh functions on an open subset of $\mathbf{C}^n$ with $(dd^c a)^n \geq F\, d\Lambda$ and $(dd^c b)^n \geq G\, d\Lambda$ (Λ the Lebesgue measure), then $(dd^c a)^k \wedge (dd^c b)^{n-k} \geq F^{k/n} G^{(n-k)/n}\, d\Lambda$ for $0 \leq k \leq n$.

Proposition 4.2 (Monge–Ampère toolbox)**.** *Assume* (H) *and fix* $|t| < \frac{1}{2}$. *There are constants* $a_0 > 0$ *and* $H_0 \geq 1$, *depending only on the family data, with the following properties.*

(1) For every $q \in \mathrm{PSH}(X, A\beta)$, $\int_X e^{-a_0(q - \sup_X q)}\, d\nu \leq H_0$.

(2) Let $\varrho \colon Y \to X$ *be a log resolution as in Proposition 3.1, isomorphic over* M, *with* $D_Y = (\varrho^{-1}(Z))_{\mathrm{red}}$, *and put* $\theta' = \varrho^*\theta$, $U' = U \circ \varrho$, $\nu' = \varrho^*\beta^n / V_\beta$. *Then* ν' *is a probability measure with* $\nu'(D_Y) = 0$, *the class of* θ' *is nef and big,* $\int_Y (\theta')^n = V$, *the envelope* $V_{\theta'}$ *is bounded, every bounded* θ'*-psh function has minimal singularities, and* $\int_Y e^{-a_0(q - \sup_Y q)}\, d\nu' \leq H_0$ *for every* $q \in \mathrm{PSH}(Y, \theta')$.

(3) *Let R be a bounded measurable function on M with $R \geq 0$, $\int_M R\,d\mu = 1$; or, more generally, let $R = e^{c\psi} R_0$ be prescribed implicitly with $c > 0$ and R_0 bounded, positive and bounded away from zero. Then there is a bounded θ'-psh function w on Y such that $\psi := w \circ \varrho^{-1} - U$ on M is bounded, belongs to $W^{1,2}_{\mathrm{loc}}(M)$, and satisfies*

$$(\omega + \mathrm{dd^c}\psi)^n = R\,\omega^n \text{ on } M, \qquad \int_M R\,d\mu = 1, \qquad \Delta\psi \geq n(R^{1/n} - 1) \text{ weakly on } M.$$

In particular $\psi \in W$, and $-\mathcal{E}(\psi, \eta) \geq n \int_M (R^{1/n} - 1)\eta\,d\mu$ for all nonnegative $\eta \in W$.

(4) *In the situation of (3), if $\|Rf\|_{L^{q_0}(\nu)} \leq J$ for some $q_0 \in (1, p]$, then*

$$\operatorname{osc}_M \psi \leq m_1 + \mathcal{M}(n, q_0, a_0, H_0, J), \qquad m_1 := \mathcal{M}(n, p, a_0, H_0, B).$$

(5) *In the situation of (3), if $\sup_M \psi = 0$ then $\int_M e^{-a_0\psi}\,d\nu \leq H_0 e^{a_0 m_1}$.*

Proof. (1) Let $q \in \mathrm{PSH}(X, A\beta)$ with $\sup q = 0$. By (i), $-\int_X q\,\beta^n \leq C_{\mathrm{sm}} V_\beta \leq C_{\mathrm{sm}} b_+$, so (ii) and $V_\beta \geq b_-$ give

$$\int_X e^{-a_0 q}\,d\nu \leq \frac{C_1 e^{C_2 C_{\mathrm{sm}} b_+}}{b_-} =: H_0,$$

after increasing H_0 to make it ≥ 1.

(2) The form $\varrho^*\beta^n$ is smooth and semipositive on Y, so ν' is a probability measure not charging D_Y. By hypothesis $\kappa = \theta + \mathrm{dd^c}\phi_\kappa$ is a Kähler form on X for some smooth function ϕ_κ; its pull-back κ' is smooth, semipositive, positive off D_Y, and $\int_Y (\kappa')^n = \int_X \kappa^n > 0$. Hence $\{\theta'\} = \{\kappa'\}$ is nef with positive volume, thus big by [DP04, Theorem 2.12]. The bounded θ'-psh function $U' - \sup U'$ is a competitor for $V_{\theta'}$, so $U' - \sup U' \leq V_{\theta'} \leq 0$; in particular $V_{\theta'}$ is bounded and every bounded θ'-psh function has minimal singularities. The identity

$$(\theta' + \mathrm{dd^c}U')^n - (\theta')^n = \mathrm{dd^c}\Big[U' \sum_{j=0}^{n-1} (\theta' + \mathrm{dd^c}U')^j \wedge (\theta')^{n-1-j}\Big]$$

integrates to zero over the compact manifold Y (Stokes' formula for currents with bounded potentials), and $(\theta' + \mathrm{dd^c}U')^n$ puts no mass on D_Y, so $\int_Y (\theta')^n = \int_M \omega^n = V$.

For the integrability statement, let $b \in \mathrm{PSH}(Y, \theta')$ be bounded. Its restriction to $Y \setminus D_Y \cong M$ is a bounded θ-psh function on $X \setminus Z$; since X is normal and θ has smooth local potentials, it extends to a bounded θ-psh function b_X on X [EGZ09, §5.2], and $b_X \circ \varrho = b$ because both sides are θ'-psh and agree off the proper analytic subset D_Y. As $\theta \leq A\beta$, $b_X \in \mathrm{PSH}(X, A\beta)$, and (1) together with $\nu' = \varrho^*\nu$, $\nu'(D_Y) = 0$ gives $\int_Y e^{-a_0(b - \sup_Y b)}\,d\nu' \leq H_0$. For arbitrary $b \in \mathrm{PSH}(Y, \theta')$, the functions $b_j = \max\{b, U' - j\}$ are bounded, θ'-psh, decrease to b, and satisfy $\sup_Y b_j = \sup_Y b$ for j large; monotone convergence concludes.

(3) Write $\nu' = g\,\sigma^n$ for a Kähler form σ on Y and a smooth function $g \geq 0$, and put $f' = f \circ \varrho$. Then

$$\int_Y (f'g)^p\,\sigma^n \leq \|g\|_{L^\infty}^{p-1} \int_Y (f')^p\,d\nu' \leq \|g\|_{L^\infty}^{p-1} B^p,$$

so $V f' g \in L^p(\sigma^n)$, and $(\theta' + \mathrm{dd^c}U')^n = V f' g\,\sigma^n$ on Y (both sides put no mass on D_Y). In the first case, $G := V\,(R \circ \varrho) f' g \in L^p(\sigma^n)$ is nonnegative with $\int_Y G\sigma^n = V \int_M R\,d\mu = V = \int_Y (\theta')^n$, and (iv) gives a bounded θ'-psh w with $(\theta' + \mathrm{dd^c}w)^n = G\sigma^n$. In the second case $G := e^{-cU'} V (R_0 \circ \varrho) f' g \in L^p(\sigma^n)$ and (iv) gives a bounded θ'-psh w with $(\theta' + \mathrm{dd^c}w)^n = e^{cw} G\sigma^n$. In both cases, on $M \cong Y \setminus D_Y$ we have $\omega + \mathrm{dd^c}\psi = \theta + \mathrm{dd^c}w$, hence $(\omega + \mathrm{dd^c}\psi)^n = R\,\omega^n$ with $R = e^{c\psi} R_0$ in the second case, and

$$\int_M R\,d\mu = \frac{1}{V} \int_{Y \setminus D_Y} (\theta' + \mathrm{dd^c}w)^n = \frac{1}{V} \int_Y (\theta')^n = 1,$$

by the same Stokes argument as in (2).

Local regularity: on a coordinate ball $\Omega \Subset M$ let φ be a smooth strictly psh potential of ω. Then $\varphi+\psi$ is bounded and psh. If α is a smooth psh function on a larger ball and $\zeta \in C_c^\infty$, testing $\Delta_{\mathrm{Euc}}\alpha \geq 0$ against $\zeta^2(\alpha - \inf\alpha)$ gives $\int \zeta^2|\nabla\alpha|^2 \leq 4(\operatorname{osc}\alpha)^2 \int |\nabla\zeta|^2$; applied to mollifications of $\varphi+\psi$ this shows $\psi \in W^{1,2}_{\mathrm{loc}}(M)$. Writing $\omega^n = G_0\, d\Lambda$ on Ω and applying (v) to $a = \varphi+\psi$, $b = \varphi$, $k = 1$, $F = RG_0$, $G = G_0$, we get

$$(\omega + \mathrm{dd^c}\psi) \wedge \omega^{n-1} \geq R^{1/n}\,\omega^n \quad \text{on } \Omega.$$

Subtracting ω^n, multiplying by n/V, and integrating against $0 \leq \eta \in C_c^\infty(\Omega)$ (with an integration by parts justified by mollifying ψ) gives $-\mathcal{E}(\psi,\eta) \geq n\int(R^{1/n}-1)\eta\, d\mu$, i.e. $\Delta\psi \geq n(R^{1/n}-1)$ weakly. The remaining assertions follow from Lemma 2.2(5) and (4).

(4) The bounded θ'-psh functions U' and $w = \varrho^*(U+\psi)$ have Monge–Ampère measures $Vf'\nu'$ and $V(Rf)'\nu'$ with densities of $L^p(\nu')$- and $L^{q_0}(\nu')$-norm at most B and J respectively, and $\mathrm{vol}(\theta') = V$. By (2) and (iii),

$$-m_1 \leq U' - \sup_Y U' - V_{\theta'} \leq 0, \qquad -\mathcal{M}(n, q_0, a_0, H_0, J) \leq w - \sup_Y w - V_{\theta'} \leq 0.$$

Subtracting, $w - U'$ differs from the constant $\sup w - \sup U'$ by a function with values in $[-\mathcal{M}(n,q_0,a_0,H_0,J), m_1]$, so $\operatorname{osc}_M \psi = \operatorname{osc}_Y(w-U') \leq m_1 + \mathcal{M}(n,q_0,a_0,H_0,J)$.

(5) Normalize $\sup_Y U' = 0$ (this does not change ψ) and put $w = U' + \varrho^*\psi$, a bounded θ'-psh function. Since $w - \sup_Y w \leq 0$ is θ'-psh, $w - \sup_Y w \leq V_{\theta'}$, while $U' - V_{\theta'} \geq -m_1$ by (iii) applied to U'. Hence, on M,

$$\psi = w - U' = (w - \sup_Y w - V_{\theta'}) - (U' - V_{\theta'}) + \sup_Y w \leq m_1 + \sup_Y w,$$

and $\sup\psi = 0$ gives $\sup_Y w \geq -m_1$. As $U' \leq 0$, $\psi \geq w = (w - \sup_Y w) + \sup_Y w \geq (w - \sup_Y w) - m_1$, so by (2)

$$\int_M e^{-a_0\psi}\, d\nu \leq e^{a_0 m_1}\int_Y e^{-a_0(w - \sup_Y w)}\, d\nu' \leq H_0 e^{a_0 m_1}. \qquad \square$$

4.3 The uniform inverse

Theorem 4.3. *Assume* (H). *With* $m_1 = \mathcal{M}(n,p,a_0,H_0,B)$ *and* $m_2 = \mathcal{M}(n,p,a_0,H_0,2^nB)$, *every fibre* (M_t,ω_t), $|t| < \frac{1}{2}$, *satisfies* (MA_K) *with* $K = m_1 + m_2$. *Consequently part (1) of Theorem A holds with*

$$C_{\mathrm{inv}} = 16\max\{1, 2(m_1+m_2)\},$$

a constant depending only on the family data.

Proof. Fix a fibre and drop the index t. Let $\epsilon \in (0,1]$ and v be as in (MA_K), and put $h = -Lv/(n\epsilon)$, $c = \epsilon/4$ and $R_0 = e^{-v/4}(1+h)^n$; R_0 is bounded and bounded away from zero since $\|v\|_{L^\infty} \leq 1$ and $1 + h \geq \frac{15}{16}$. Proposition 4.2(3) (second case) provides a bounded $\psi \in W^{1,2}_{\mathrm{loc}}(M)$ with

$$(\omega + \mathrm{dd^c}\psi)^n = R\,\omega^n, \qquad R = e^{(\epsilon\psi - v)/4}(1+h)^n, \qquad \int_M R\, d\mu = 1, \qquad \Delta\psi \geq n(R^{1/n}-1) \text{ weakly.}$$

If $R \leq 2^n$, then $\|Rf\|_{L^p(\nu)} \leq 2^nB$, and Proposition 4.2(4) with $q_0 = p$ gives $\operatorname{osc}_M\psi \leq m_1 + m_2 = K$. This is (MA_K). Proposition 3.1 supplies (CUT), and Proposition 4.1 gives the conclusion. $\square$

We write L_t^{-1} for the resulting mean-zero inverse: for $F \in L^\infty(\mu_t)$ with $\int F\, d\mu_t = 0$, $L_t^{-1}F$ is the unique mean-zero $u \in \mathcal{D}(L_t)$ with $L_t u = F$; it satisfies $\|L_t^{-1}F\|_{L^\infty} \leq C_{\mathrm{inv}}\|F\|_{L^\infty}$, and by elliptic regularity $L_t^{-1}F \in W^{2,r}_{\mathrm{loc}}(M_t)$ for all $r < \infty$, so it has a continuous representative, which we always use.

5 Green's functions

Proposition 5.1. *Let (M,ω) be a connected Kähler manifold with $0<V<\infty$ satisfying (CUT), and suppose that the mean-zero inverse L^{-1} exists on $L^\infty_0(\mu) := \{F \in L^\infty(\mu) : \int F\,d\mu = 0\}$ with $\|L^{-1}F\|_{L^\infty} \le C\|F\|_{L^\infty}$. For $x\in M$ let P_x be a compactly supported cutoff of a local Dirichlet Green's function with pole x as in Step 1 below. Then there is a unique $G_x \in L^1(\mu)$ such that*

$$\int_M G_x\,d\mu = 0, \qquad \int_M G_x\,\Delta\eta\,d\mu = n\eta(x) - n\int_M \eta\,d\mu \quad (\eta\in C^\infty_c(M)), \qquad G_x - P_x \in W, \tag{5.1}$$

and the last condition does not depend on the choice of P_x. Moreover:

(1) $G_x = P_x + b_x$ with $b_x \in W\cap L^\infty(M)$; G_x is smooth on $M\setminus\{x\}$, $G_x(y)\to-\infty$ as $y\to x$, $\Delta G_x = -n$ on $M\setminus\{x\}$, and $G_x^+ \in W\cap L^\infty$ with $\Delta G_x^+ \ge -n$ weakly on M.

(2) For every $F\in L^\infty_0(\mu)$, $\int_M G_x F\,d\mu = -n\,(L^{-1}F)(x)$; equivalently, if $u\in\mathcal{D}(L)$ and $Lu\in L^\infty$, then $u(x) - \int_M u\,d\mu = -\frac{1}{n}\int_M G_x\,Lu\,d\mu$ for the continuous representative of u.

(3) $\int_M |G_x|\,d\mu \le 2nC$ for every $x\in M$.

(4) The function $(x,y)\mapsto G_x(y)$ is jointly continuous on $\{x\ne y\}$, $G_x(y) = G_y(x)$ for $x\ne y$, and $(x,y)\mapsto G_x(y)$ is smooth on $\{x\ne y\}$.

(5) If q is bounded and Lipschitz for the Riemannian distance of g, then $q\in W$, the integral $\mathcal{E}(G_x,q) = \int_M \Gamma(G_x,q)\,d\mu$ converges absolutely, and

$$q(x) - \int_M q\,d\mu = -\frac{1}{n}\,\mathcal{E}(G_x,q) = -\frac{1}{2n}\int_M \langle\nabla G_x,\nabla q\rangle_g\,d\mu.$$

Proof. Step 1: the local parametrix. Let $\Omega \Subset M$ be a smooth coordinate ball containing x, and let $m=2n$ be the real dimension. The Dirichlet Green's function of Δ on Ω (with respect to μ) exists, is symmetric and smooth off the diagonal [Aub98, Theorems 4.17 and 3.54]; we normalize it as Q_x with

$$\Delta Q_x = n\delta_x \text{ in } \mathcal{D}'(\Omega), \qquad Q_x<0, \qquad Q_x(y)\to-\infty\ (y\to x),$$

where δ_x is the Dirac mass relative to μ. Near x, $|Q_x|$ is bounded by a multiple of $d(x,\cdot)^{2-m}$ (of $|\log d(x,\cdot)|$ if $m=2$) and $|\nabla Q_x|$ by a multiple of $d(x,\cdot)^{1-m}$; in particular $Q_x\in L^1_{\mathrm{loc}}$ and $\nabla Q_x \in L^1_{\mathrm{loc}}$. Let $\zeta\in C^\infty_c(\Omega)$ equal 1 near x, and let $P_x = \zeta Q_x$, extended by zero to M. Then

$$\Delta P_x = n\delta_x + F_x \text{ in } \mathcal{D}'(M), \qquad F_x = Q_x\Delta\zeta + \langle\nabla\zeta,\nabla Q_x\rangle_g \in C^\infty_c(M\setminus\{x\}).$$

Testing this identity against a function in $C^\infty_c(M)$ equal to 1 near $\operatorname{supp} P_x$ gives $\int_M F_x\,d\mu = -n$, so $n + F_x \in L^\infty_0(\mu)$.

Step 2: construction. Put $b_x = L^{-1}(n+F_x) - \int_M P_x\,d\mu$ and $G_x = P_x + b_x$. Then $b_x\in W\cap L^\infty$, b_x is smooth (elliptic regularity, Lb_x being smooth), $\Delta b_x = -n - F_x$, hence $\Delta G_x = n\delta_x - n$ in $\mathcal{D}'(M)$ and $\int_M G_x\,d\mu = 0$; this is (5.1).

Step 3: uniqueness and the positive part. If P'_x is another admissible parametrix, then $P_x - P'_x$ is compactly supported with $\Delta(P_x - P'_x) = F_x - F'_x$ smooth, hence smooth by elliptic regularity, hence in W; so the energy condition in (5.1) is independent of P_x. If G'_x also satisfies (5.1), then $G_x - G'_x \in W$ is weakly harmonic with mean zero, hence zero by Lemma 2.2(3).

Let $\beta_x = \|b_x\|_{L^\infty}$ and let N be a small ball around x on which $P_x < -\beta_x - 1$, so that $G_x<-1$ on N. Choose $\vartheta\in C^\infty_c(N)$ with $0\le\vartheta\le1$, $\vartheta=1$ near x, and put $\widetilde{P}_x = (1-\vartheta)P_x - \vartheta(\beta_x+1)$; this is smooth, compactly supported, agrees with P_x outside a compact subset of N, and $\widetilde{G}_x := \widetilde{P}_x + b_x \le -1$ on N. Then $\widetilde{G}_x \in W\cap L^\infty$, $G_x^+ = \widetilde{G}_x^+$, so $G_x^+\in W\cap L^\infty$ by Lemma 2.1;

and $\widetilde{G}_x$ is smooth with $\Delta\widetilde{G}_x = -n$ outside N, so $\Delta\widetilde{G}_x \geq h := \Delta\widetilde{G}_x$ weakly with $h \in L^\infty$ and $h = -n$ on $\{\widetilde{G}_x > 0\} \subset M \setminus N$. Lemma 2.3 gives $\Delta G_x^+ \geq -n$ weakly. This proves (1).

Step 4: representation and the L^1 bound. Let $F \in L_0^\infty(\mu)$ and $u = L^{-1}F \in W^{2,r}_{\mathrm{loc}}$, $r > m$. Approximating u near $\operatorname{supp} P_x$ by smooth functions converging in $W^{2,r}$ and uniformly, the identity $\Delta P_x = n\delta_x + F_x$ tested against u yields (recall $\Delta u = -F$)

$$-\int_M P_x F\, d\mu = \int_M P_x \Delta u\, d\mu = nu(x) + \int_M F_x u\, d\mu.$$

Since u, $L^{-1}(n + F_x) \in \mathcal{D}(L)$, the symmetry of $\mathcal{E}$ gives

$$\int_M (L^{-1}(n + F_x))F\, d\mu = \mathcal{E}\big(u, L^{-1}(n + F_x)\big) = \int_M (n + F_x)u\, d\mu = \int_M F_x u\, d\mu,$$

the last step because u has mean zero. Adding, and using $\int F\, d\mu = 0$ to discard the constant in b_x, we get $\int_M G_x F\, d\mu = -nu(x)$. For $u \in \mathcal{D}(L)$ with bounded Lu, Lemma 2.2(3) gives $\int Lu\, d\mu = 0$, and uniqueness of the mean-zero solution identifies $u - \int u\, d\mu = L^{-1}(Lu)$; this is (2). Choosing $F = \operatorname{sgn} G_x - \int_M \operatorname{sgn} G_x\, d\mu$, which has $\|F\|_{L^\infty} \leq 2$, gives $\int_M |G_x|\, d\mu = -n\,(L^{-1}F)(x) \leq 2nC$, which is (3).

Step 5: joint continuity and symmetry. For x in a ball $B \Subset \Omega$, use the same ζ. The Dirichlet Green's function $Q_x(y)$ depends smoothly on (x, y) off the diagonal, and its singularity is uniformly integrable on small balls, so $x \mapsto P_x$ is continuous into $L^1(\mu)$ and $x \mapsto F_x$ is continuous into $L^\infty(\mu)$. By the inverse bound, $x \mapsto b_x$ is continuous into $L^\infty(\mu)$, i.e. the continuous functions b_x depend uniformly continuously on x; thus $(x, y) \mapsto G_x(y) = P_x(y) + b_x(y)$ is jointly continuous off the diagonal.

For bounded real ϕ, χ with mean-zero parts ϕ_0, χ_0, (3) justifies Fubini in

$$\begin{aligned}\iint \phi(x)G_x(y)\chi(y)\, d\mu(y)\, d\mu(x) &= \int_M \phi(x)\big(-n\, L^{-1}\chi_0(x)\big)\, d\mu(x) \\ &= -n \int_M \phi_0\, L^{-1}\chi_0\, d\mu = -n\, \mathcal{E}\big(L^{-1}\phi_0, L^{-1}\chi_0\big),\end{aligned}$$

where we used $\int_M G_x\, d\mu = 0$ and (2). The last expression is symmetric in (ϕ, χ). Taking indicator functions shows $G_x(y) = G_y(x)$ for almost every (x, y), and joint continuity upgrades this to all $x \neq y$.

For joint smoothness, let $\varphi \in C_c^\infty(M \times M)$. Applying (5.1) to $\eta = \varphi(x, \cdot)$ for each x, integrating in x, and using Fubini (justified by (3)) gives

$$\iint G_x(y)\, \Delta_y \varphi(x, y)\, d\mu(y)\, d\mu(x) = n \int_M \varphi(x, x)\, d\mu(x) - n \iint \varphi\, d\mu\, d\mu,$$

and by symmetry the same identity holds with Δ_x in place of Δ_y. Thus $G \in L^1(M \times M)$ satisfies $(\Delta_x + \Delta_y)G = 2n(\delta_{\mathrm{diag}} - 1)$ in $\mathcal{D}'(M \times M)$, where δ_{diag} is the measure $\varphi \mapsto \int \varphi(x, x)\, d\mu(x)$. The right side is smooth on $\{x \neq y\}$ and $\Delta_x + \Delta_y$ is an elliptic operator with smooth coefficients on $M \times M$, so G is smooth on $\{x \neq y\}$ by elliptic regularity. This proves (4).

Step 6: Lipschitz functions. Let q be bounded and d-Lipschitz. By Lemma 2.1, $|\nabla q|_g \leq \mathrm{Lip}_d(q)$ almost everywhere, so $q \in \widetilde{W} = W$. The pairing $\Gamma(G_x, q)$ is integrable: $\Gamma(b_x, q)$ by Cauchy–Schwarz and $\Gamma(P_x, q)$ because $\nabla P_x \in L^1$ and $\nabla q \in L^\infty$. Mollifying q near $\operatorname{supp} P_x$ with uniformly bounded gradients and using $\Delta P_x = n\delta_x + F_x$, we get $\mathcal{E}(P_x, q) = -nq(x) - \int_M F_x q\, d\mu$, while $\mathcal{E}(b_x, q) = \int_M (n + F_x)q\, d\mu$ because $b_x - \mathrm{const} = L^{-1}(n + F_x)$. Adding gives $\mathcal{E}(G_x, q) = -nq(x) + n\int_M q\, d\mu$, and $\mathcal{E}(G_x, q) = \frac{1}{2}\int \langle \nabla G_x, \nabla q\rangle_g\, d\mu$ by definition. This is (5). □

By Theorem 4.3, Proposition 5.1 applies to every fibre with $C = C_{\mathrm{inv}}$. We write $G_t(x, y) = G_x(y)$ on the fibre M_t. This proves part (2) of Theorem A.

6 Uniform estimates for Green's functions

Throughout this section we assume (H), fix a fibre and drop the index t; all constants depend only on the family data and on the displayed exponents. We write $\mu(u) = \int_M u\, d\mu$, and we use Proposition 4.2 freely.

6.1 A weak mean-value inequality

Lemma 6.1. *There is a constant C_{mv} such that every $v \in W \cap L^\infty(M)$ with $\Delta v \geq -n$ weakly satisfies*

$$\operatorname*{ess\,sup}_M v \leq C_{\mathrm{mv}}(1 + \mu(v^+)).$$

Proof. By Lemma 2.3, $v^+ \in W \cap L^\infty$ and $\Delta v^+ \geq -n$ weakly, so we may assume $v \geq 0$. Put

$$m = \mu(v), \qquad \alpha = \frac{n}{n+1}, \qquad c_n = \Big(1 + \frac{2}{\alpha}\Big)^n, \qquad \kappa = c_n(1+m).$$

Apply Proposition 4.2(3) with $R = (1+v)/(1+m)$, and normalize the solution so that $\sup_M \psi = -1$. Then $\psi \in W$ is bounded, $-\psi \geq 1$, and $\Delta\psi \geq n(R^{1/n} - 1)$ weakly. Let $\Phi \in C^2(\mathbf{R})$ be convex, nondecreasing, with bounded derivative, and equal to $s \mapsto -\kappa(-s)^\alpha$ on the range of ψ (this function is convex and increasing on $(-\infty, -1]$). By Lemma 2.3,

$$w := 1 + v + \Phi(\psi) \in W, \qquad \Delta w \geq n\Big[-1 + \kappa\alpha(-\psi)^{\alpha-1}(R^{1/n} - 1)\Big] \quad \text{weakly}.$$

On $\{w > 0\}$ we have $1 + v > \kappa(-\psi)^\alpha$, hence $R^{1/n} > \kappa^{1/n}(1+m)^{-1/n}(-\psi)^{\alpha/n}$. Using $\alpha - 1 + \alpha/n = 0$ and $(-\psi)^{\alpha-1} \leq 1$, the bracket is bounded below on $\{w > 0\}$ by

$$\alpha\kappa^{1+1/n}(1+m)^{-1/n} - \alpha\kappa - 1 = \alpha c_n(1+m)(c_n^{1/n} - 1) - 1 = 2c_n(1+m) - 1 \geq 1.$$

Testing with $\eta = w^+ \in W$ (Lemma 2.2(4)) gives $0 \geq -\mathcal{E}(w^+, w^+) \geq n\mu(w^+)$, so $w^+ = 0$, i.e.

$$1 + v \leq c_n(1+m)(-\psi)^\alpha. \tag{6.1}$$

It remains to bound $-\psi$ from above. Let $q_0 = (p+1)/2 \in (1, p)$. By (6.1), $Rf \leq c_n(-\psi)^\alpha f$, so Hölder's inequality with exponents p/q_0 and $p/(p - q_0)$ gives

$$\int_M (Rf)^{q_0}\, d\nu \leq c_n^{q_0}\Big(\int_M f^p\, d\nu\Big)^{q_0/p}\Big(\int_M (-\psi)^{\alpha q_0 p/(p-q_0)}\, d\nu\Big)^{(p-q_0)/p}.$$

Since $\psi + 1$ has supremum 0 and solves the same equation, Proposition 4.2(5) gives $\int_M e^{-a_0\psi}\, d\nu \leq H_0 e^{a_0(m_1+1)}$, which bounds every polynomial moment of $-\psi$ with respect to ν. Hence $\|Rf\|_{L^{q_0}(\nu)} \leq J$ for a constant J depending only on the family data, and Proposition 4.2(4) gives $\operatorname{osc}_M \psi \leq M' := m_1 + \mathcal{M}(n, q_0, a_0, H_0, J)$. Thus $-\psi \leq 1 + M'$, and (6.1) yields $1 + v \leq c_n(1+M')^\alpha(1+m)$. Take $C_{\mathrm{mv}} = c_n(1+M')^\alpha$. □

Corollary 6.2. *With $C_0 := C_{\mathrm{mv}}(1 + nC_{\mathrm{inv}})$, one has $G_x(y) \leq C_0$ for all $x \neq y$ in M.*

Proof. By Proposition 5.1(1), $G_x^+ \in W \cap L^\infty$ and $\Delta G_x^+ \geq -n$ weakly, and by (3) and $\mu(G_x) = 0$, $\mu(G_x^+) = \frac{1}{2}\mu(|G_x|) \leq nC_{\mathrm{inv}}$. Lemma 6.1 gives $\operatorname{ess\,sup} G_x^+ \leq C_0$, and continuity of G_x on $M \setminus \{x\}$ gives the pointwise bound. □

From now on we write

$$\mathcal{H}_x := C_0 + 1 - G_x \geq 1, \qquad \mu(\mathcal{H}_x) = C_0 + 1,$$

and note that $\mathcal{H}_x(y) = \mathcal{H}_y(x)$ for $x \neq y$.

6.2 Moments

Lemma 6.3 (Bootstrap)**.** *Let $a \geq 1$ and suppose $\sup_{x \in M} \mu(\mathcal{H}_x^a) \leq J_0$. Then for every $0 < \beta_0 < a/n$ there is J_1, depending only on a, β_0, J_0 and the family data, such that $\sup_{x \in M} \mu(\mathcal{H}_x^{1+\beta_0}) \leq J_1$.*

Proof. Fix x and, for $k \geq 1$, put

$$Q_k = \min\{\mathcal{H}_x, k\}^{\beta_0}, \qquad A_k = \mu(Q_k), \qquad D_k = \mu(Q_k^n)^{1/n}.$$

Then $1 \leq A_k \leq \mu(\mathcal{H}_x^{\beta_0}) \leq J_0^{\beta_0/a}$ and $1 \leq D_k \leq \mu(\mathcal{H}_x^{n\beta_0})^{1/n} \leq J_0^{\beta_0/a}$. Let

$$u_k = -nL^{-1}\Big(\frac{Q_k}{A_k} - 1\Big), \qquad \text{so that} \qquad \Delta u_k = n\Big(\frac{Q_k}{A_k} - 1\Big), \qquad u_k(y) = \int_M G_y(z)\frac{Q_k(z)}{A_k}\,d\mu(z)$$

by Proposition 5.1(2) and $\mu(G_y) = 0$. Symmetry, Fubini and (3) give $\mu(|u_k|) \leq 2nC_{\mathrm{inv}}$.

Next let $R_k = (Q_k/D_k)^n$, a bounded probability density for μ. Choose $q \in (1, p)$ so close to 1 that $d := n\beta_0 q(p-1)/(p-q) < a$. Since $d\mu = f\,d\nu$, $D_k \geq 1$, and Hölder's inequality with exponents $(p-1)/(q-1)$ and $(p-1)/(p-q)$,

$$\int_M (R_k f)^q\,d\nu \leq \int_M \mathcal{H}_x^{n\beta_0 q} f^{q-1}\,d\mu \leq \Big(\int_M f^p\,d\nu\Big)^{\frac{q-1}{p-1}}\Big(\int_M \mathcal{H}_x^d\,d\mu\Big)^{\frac{p-q}{p-1}} \leq B^{\frac{p(q-1)}{p-1}} J_0^{\frac{d}{a}\cdot\frac{p-q}{p-1}} =: J^q,$$

where we used $\int_M f^{p-1}\,d\mu = \int_M f^p\,d\nu$ and $\mu(\mathcal{H}_x^d) \leq \mu(\mathcal{H}_x^a)^{d/a}$; the bound J is independent of k and x. By Proposition 4.2(3) and (4) there is $\psi_k \in W$ with $\sup \psi_k = 0$,

$$-M_0 \leq \psi_k \leq 0, \qquad \Delta\psi_k \geq n\Big(\frac{Q_k}{D_k} - 1\Big) \text{ weakly}, \qquad M_0 := m_1 + \mathcal{M}(n, q, a_0, H_0, J).$$

Put $z_k = \psi_k - u_k/D_k \in W \cap L^\infty$. Since $A_k \geq 1$,

$$\Delta z_k \geq n\Big[\frac{Q_k}{D_k}\Big(1 - \frac{1}{A_k}\Big) - 1 + \frac{1}{D_k}\Big] \geq -n \quad \text{weakly}, \qquad \mu(z_k^+) \leq M_0 + 2nC_{\mathrm{inv}}.$$

Lemma 6.1 gives $z_k \leq C_{\mathrm{mv}}(1 + M_0 + 2nC_{\mathrm{inv}})$ almost everywhere, hence $u_k \geq -D_k(M_0 + C_{\mathrm{mv}}(1 + M_0 + 2nC_{\mathrm{inv}})) \geq -J_1'$ almost everywhere, with J_1' independent of k, x; by continuity of u_k the bound holds everywhere. Evaluating the representation of u_k at $y = x$ and writing $G_x = C_0 + 1 - \mathcal{H}_x$,

$$-J_1' \leq u_k(x) = C_0 + 1 - \frac{\mu(\mathcal{H}_x Q_k)}{A_k}, \qquad \text{so} \qquad \mu(\mathcal{H}_x Q_k) \leq (C_0 + 1 + J_1')A_k \leq (C_0 + 1 + J_1')J_0^{\beta_0/a}.$$

Letting $k \to \infty$, monotone convergence gives $\mu(\mathcal{H}_x^{1+\beta_0}) \leq (C_0 + 1 + J_1')J_0^{\beta_0/a} =: J_1$. □

Corollary 6.4. *For every $0 < a < \frac{n}{n-1}$ ($0 < a < \infty$ if $n = 1$) there is C_a with $\sup_{x \in M} \mu(|G_x|^a) \leq \sup_{x \in M} \mu(\mathcal{H}_x^a) \leq C_a$.*

Proof. The first inequality holds because $|G_x| \leq \mathcal{H}_x$ (as $G_x \leq C_0$ and $\mathcal{H}_x \geq 1$). Fix $\gamma \in (0, 1)$ and put $\alpha_0 = 1$, $\alpha_{j+1} = 1 + \gamma\alpha_j/n$. The hypothesis of Lemma 6.3 holds for α_0 with $J_0 = C_0 + 1$, and the lemma with $\beta_0 = \gamma\alpha_j/n < \alpha_j/n$ passes from α_j to α_{j+1}. The sequence α_j increases to $n/(n-\gamma)$ (to $1/(1-\gamma)$ if $n = 1$), which exceeds any prescribed $a < n/(n-1)$ for γ close to 1. Finitely many steps therefore reach any such a; intermediate exponents are handled by Hölder's inequality. □

6.3 Weighted energy and gradient estimates

Lemma 6.5. *For every $b > 0$ and every $x \in M$,*

$$\int_M \frac{|\nabla G_x|_g^2}{\mathcal{H}_x^{1+b}}\, d\mu \le \frac{2n}{b}.$$

Proof. Let $\varsigma_j \ge 0$ be smooth probability densities (for μ) supported in coordinate balls shrinking to x, and put

$$v_j = -nL^{-1}(\varsigma_j - 1), \qquad \text{so that} \qquad \Delta v_j = n(\varsigma_j - 1), \quad v_j(y) = \int_M G_y(z)\varsigma_j(z)\, d\mu(z), \quad \mu(v_j) = 0,$$

and $v_j \le C_0$ by Corollary 6.2. Let $\mathcal{H}_j = C_0 + 1 - v_j \ge 1$. The function $\mathcal{H}_j^{-b}$ is a bounded Lipschitz function of $v_j \in W \cap L^\infty$, so it belongs to W with $d(\mathcal{H}_j^{-b}) = b\,\mathcal{H}_j^{-b-1} dv_j$. Since $v_j \in \mathcal{D}(L)$,

$$b \int_M \frac{\Gamma(v_j)}{\mathcal{H}_j^{1+b}}\, d\mu = \mathcal{E}(v_j, \mathcal{H}_j^{-b}) = \int_M (Lv_j)\, \mathcal{H}_j^{-b}\, d\mu = n \int_M (1 - \varsigma_j)\mathcal{H}_j^{-b}\, d\mu \le n,$$

that is, $\int_M |\nabla v_j|^2 \mathcal{H}_j^{-1-b}\, d\mu \le 2n/b$. By joint continuity of the kernel and symmetry, $v_j(y) - G_x(y) = \int_M (G_z(y) - G_x(y))\varsigma_j(z)\, d\mu(z) \to 0$ locally uniformly on $M \setminus \{x\}$. On any $\Omega \Subset M \setminus \{x\}$ the functions $v_j - G_x$ are harmonic for j large, so the Caccioppoli inequality gives $\nabla v_j \to \nabla G_x$ in $L^2(\Omega)$. Along a subsequence the convergence is almost everywhere, and Fatou's lemma gives the claim. $\square$

Lemma 6.6. *For every $0 < s < \frac{2n}{2n-1}$ ($0 < s < 2$ if $n = 1$) there is a constant D_s such that $\int_M |\nabla G_x|_g^s\, d\mu \le D_s$ for every $x \in M$.*

Proof. Since $\frac{s}{2-s} < \frac{n}{n-1}$ is equivalent to $s < \frac{2n}{2n-1}$, we may choose $b > 0$ with $r := \frac{s(1+b)}{2-s} < \frac{n}{n-1}$ (any $b > 0$ if $n = 1$). Hölder's inequality with exponents $2/s$ and $2/(2-s)$ gives

$$\int_M |\nabla G_x|^s\, d\mu = \int_M \frac{|\nabla G_x|^s}{\mathcal{H}_x^{(1+b)s/2}}\, \mathcal{H}_x^{(1+b)s/2}\, d\mu \le \Big(\int_M \frac{|\nabla G_x|^2}{\mathcal{H}_x^{1+b}}\, d\mu\Big)^{s/2} \Big(\int_M \mathcal{H}_x^r\, d\mu\Big)^{(2-s)/2},$$

and both factors are bounded by Lemma 6.5 and Corollary 6.4. $\square$

Theorem 6.7. *Assume* (H). *Then part (3) of Theorem A holds: $G_t(x, y) \le C_0$ for $x \ne y$, the moments of $G_t(x, \cdot)$ of every order $a < \frac{n}{n-1}$, the L^s norms of $\nabla_y G_t(x, \cdot)$ for every $s < \frac{2n}{2n-1}$, and the weighted energies $\int |\nabla_y G_t|^2 \mathcal{H}_t^{-1-b}\, d\mu_t \le 2n/b$ are bounded uniformly in t and x.*

Proof. Combine Corollaries 6.2 and 6.4 and Lemmas 6.5 and 6.6; all constants depend only on the family data and the exponents. $\square$

7 Metric completion and Sobolev inequalities

7.1 From Green's functions to the metric completion

Proposition 7.1. *Let (N, h) be a connected Riemannian manifold with distance d, let σ be a probability measure on N with smooth positive density, and let $S \subset N$ be dense. Suppose that for some $q > 1$, $\Lambda > 0$,*

$$\Big| v(x) - \int_N v\, d\sigma \Big| \le \Lambda \Big(\int_N |\nabla v|_h^q\, d\sigma \Big)^{1/q} \tag{7.1}$$

for every bounded d-Lipschitz function v and every $x \in S$. Then the metric completion $(\overline{N}, \overline{d})$ is compact, $\operatorname{diam}(\overline{N}) \le 2\Lambda$, *and for every $z \in \overline{N}$ and $R > 0$,*

$$\overline{\sigma}(B_{\overline{d}}(z, R)) \ge \min\Big\{\frac{1}{2}, \Big(\frac{R}{2\Lambda}\Big)^q\Big\},$$

where $\overline{\sigma}$ is the push-forward of σ.

In particular, if (M, ω) is as in Proposition 5.1 and $\|\nabla G_x\|_{L^s(\mu)} \le K_\nabla$ for some $s > 1$ and all x in a dense subset of M, the conclusions hold for (M, g, μ) with $q = \frac{s}{s-1}$ and $\Lambda = \frac{K_\nabla}{2n}$.

Proof. Diameter. Let $x, y \in S$ with $D = d(x, y) > 0$, and let $v_x = \min\{d(x, \cdot), D\}$, $v_y = \min\{d(y, \cdot), D\}$. These are bounded and 1-Lipschitz, so $|\nabla v_x|_h, |\nabla v_y|_h \le 1$ almost everywhere (Lemma 2.1) and (7.1) at the centres gives $\int v_x \, d\sigma \le \Lambda$, $\int v_y \, d\sigma \le \Lambda$. The triangle inequality gives $v_x + v_y \ge D$, hence $D \le 2\Lambda$. By density, $\operatorname{diam}(\overline{N}) \le 2\Lambda$.

Balls centred in S. Fix $x \in S$, $R > 0$, and put $m = \sigma(B_d(x, R))$, $v = \min\{d(x, \cdot), R\}$. By Lemma 2.1, $|\nabla v|_h \le 1$ almost everywhere, $\nabla v = 0$ almost everywhere on the level set $\{d(x, \cdot) = R\}$ and outside the closed ball, so $\int |\nabla v|^q \, d\sigma \le m$. Since $v(x) = 0$ and $v = R$ outside $B_d(x, R)$,

$$R(1 - m) \le \int_N v \, d\sigma \le \Lambda m^{1/q}.$$

If $m < \frac{1}{2}$ this gives $m \ge (R/2\Lambda)^q$.

Completion and compactness. Let $z \in \overline{N}$ and $R > 0$; choose $x_j \in S$ with $\overline{d}(x_j, z) = \epsilon_j \to 0$. Then $B_d(x_j, R - \epsilon_j) \subset B_{\overline{d}}(z, R) \cap N$, and the bound at x_j with radius $R - \epsilon_j$ passes to the limit. An ϵ-separated subset of $\overline{N}$ has pairwise disjoint balls of radius $\epsilon/3$, each of measure at least $\min\{\frac{1}{2}, (\epsilon/6\Lambda)^q\}$, hence has bounded cardinality; so $\overline{N}$ is totally bounded, and being complete, compact.

The Kähler case. For a bounded d-Lipschitz v and x in the dense set, Proposition 5.1(5) and Hölder's inequality give

$$\Big|v(x) - \int_M v \, d\mu\Big| = \frac{1}{2n}\Big|\int_M \langle \nabla G_x, \nabla v \rangle_g \, d\mu\Big| \le \frac{K_\nabla}{2n}\Big(\int_M |\nabla v|_g^q \, d\mu\Big)^{1/q}, \qquad q = \frac{s}{s-1}. \qquad \square$$

7.2 Sobolev inequalities

Proposition 7.2. *Assume* (H) *and fix a fibre. For every $1 < r < \frac{n}{n-1}$ ($1 < r < \infty$ if $n = 1$) there is S_r, depending only on r and the family data, such that*

$$\|u - \mu(u)\|_{L^{2r}(\mu)}^2 \le S_r \int_M |\nabla u|_g^2 \, d\mu \qquad (u \in W). \tag{7.2}$$

If $\Omega \subset \widehat{M}$ is open with $b_0 := \widehat{\mu}(\widehat{M} \setminus \Omega) > 0$, $a := \widehat{\mu}(\Omega)$, and $u \in W_0(\Omega)$ (the closure of $C_c^\infty(\Omega \cap M)$ in W), then

$$\|u\|_{L^{2r}(\Omega \cap M, \mu)}^2 \le 2S_r\Big(1 + \frac{a}{b_0}\Big) \int_{\Omega \cap M} |\nabla u|_g^2 \, d\mu. \tag{7.3}$$

Every $u \in W^{1,2}_{\mathrm{loc}}(\Omega \cap M)$ with $\int_{\Omega \cap M}(u^2 + |\nabla u|^2) \, d\mu < \infty$ that vanishes outside a compact subset of Ω belongs to $W_0(\Omega)$ after extension by zero.

Proof. The global inequality on $C_c^\infty(M)$. Choose $b > 0$ with $(1 + b)r < \frac{n}{n-1}$ (any $b > 0$ if $n = 1$). Let $u \in C_c^\infty(M, \mathbf{R})$; it is bounded and d-Lipschitz, so Proposition 5.1(5) and the Cauchy–Schwarz inequality with weight $\mathcal{H}_x^{1+b}$, together with Lemma 6.5, give for every $x \in M$

$$|u(x) - \mu(u)|^2 \le \frac{1}{4n^2}\Big(\int_M \frac{|\nabla_y G_x(y)|^2}{\mathcal{H}_x(y)^{1+b}} \, d\mu(y)\Big)\Big(\int_M \mathcal{H}_x(y)^{1+b} |\nabla u(y)|^2 \, d\mu(y)\Big)$$

$$\leq \frac{1}{2nb} \int_M \mathcal{H}_x(y)^{1+b} |\nabla u(y)|^2 \, d\mu(y).$$

Taking the $L^r(d\mu(x))$ norm and applying Minkowski's integral inequality (first with $\mathcal{H}_x$ truncated at a finite level, then letting the level tend to infinity by monotone convergence),

$$\|u - \mu(u)\|^2_{L^{2r}(\mu)} \leq \frac{1}{2nb} \int_M \Big(\int_M \mathcal{H}_x(y)^{r(1+b)} \, d\mu(x) \Big)^{1/r} |\nabla u(y)|^2 \, d\mu(y).$$

By symmetry $\mathcal{H}_x(y) = \mathcal{H}_y(x)$, so the inner integral is $\mu(\mathcal{H}_y^{r(1+b)})$, which is bounded uniformly in y by Corollary 6.4 and the choice of b. This gives (7.2) on $C_c^\infty(M)$ with $S_r = \frac{1}{2nb} \sup_y \mu(\mathcal{H}_y^{r(1+b)})^{1/r}$.

Extension to W. If $u \in W$ and $u_j \in C_c^\infty(M)$ converge to u in W, then (7.2) applied to $u_j - u_k$ shows that $u_j - \mu(u_j)$ is Cauchy in $L^{2r}(\mu)$; its limit is $u - \mu(u)$ by L^2-convergence, and (7.2) passes to the limit.

The local inequality. Let $u \in W_0(\Omega)$. Its approximants vanish on $M \setminus \Omega$, so $u = 0$ and $\nabla u = 0$ almost everywhere on $M \setminus \Omega$; in particular $\int_M |\nabla u|^2 \, d\mu = \int_{\Omega \cap M} |\nabla u|^2 \, d\mu$. Put $m = \mu(u)$. If $a = 0$ then $u = 0$. Otherwise, by the Cauchy–Schwarz inequality and $\int_M u^2 \, d\mu = \int_M (u - m)^2 \, d\mu + m^2$,

$$m^2 = \Big(\int_{\Omega \cap M} u \, d\mu \Big)^2 \leq a \int_M u^2 \, d\mu = a \Big(\int_M (u - m)^2 \, d\mu + m^2 \Big),$$

so, using $1 - a = b_0$ and $\mu(M) = 1$,

$$m^2 \leq \frac{a}{b_0} \int_M (u - m)^2 \, d\mu \leq \frac{a}{b_0} \|u - m\|^2_{L^{2r}(\mu)}.$$

The triangle inequality in $L^{2r}(\Omega \cap M, \mu)$, $(A + B)^2 \leq 2A^2 + 2B^2$, and $a^{1/r} \leq 1$ give

$$\|u\|^2_{L^{2r}(\Omega \cap M)} \leq 2\|u - m\|^2_{L^{2r}(\mu)} + 2a^{1/r} m^2 \leq 2\Big(1 + \frac{a}{b_0}\Big) \|u - m\|^2_{L^{2r}(\mu)} \leq 2S_r \Big(1 + \frac{a}{b_0}\Big) \int_{\Omega \cap M} |\nabla u|^2 \, d\mu.$$

Zero extension. Let u be as in the last assertion, with $u = 0$ outside the compact set $K \subset \Omega$. Its extension by zero is in $W^{1,2}_{\mathrm{loc}}(M)$ (it vanishes on the open set $M \setminus K$) with finite norm, hence in W by Lemma 2.2. The approximation procedure of Lemma 2.2 (truncation, multiplication by η_j, and mollification in charts) produces functions in $C_c^\infty(\Omega \cap M)$, because η_j has compact support in M, the supports stay in $K \cap \operatorname{supp} \eta_j$, and mollification enlarges supports by an arbitrarily small amount. Hence the extension lies in $W_0(\Omega)$. $\square$

7.3 Completion estimates and the capacity of the boundary

Proposition 7.3. *Assume* (H) *and fix a fibre. Then:*

(1) $\widehat{M}$ is compact, $\operatorname{diam}(\widehat{M}, \widehat{d}) \leq C_d$, and for every $0 < \delta < 1$ there is C_δ with $\widehat{\mu}(B_{\widehat{d}}(z, R)) \geq C_\delta^{-1} \min\{1, R^{2n+\delta}\}$ for all $z \in \widehat{M}$, $R > 0$.

(2) M is open in $\widehat{M}$, and $\widehat{\mu}(\widehat{M} \setminus M) = 0$.

(3) $\operatorname{Cap}_1(\widehat{M} \setminus M) = 0$.

Proof. (1) Given $\delta \in (0, 1)$, let $q = 2n + \delta$ and $s = q/(q - 1) < \frac{2n}{2n-1}$. Lemma 6.6 provides $K_\nabla = D_s^{1/s}$, and Proposition 7.1 gives compactness, $\operatorname{diam} \leq 2\Lambda_\delta$ with $\Lambda_\delta = K_\nabla / 2n$, and $\widehat{\mu}(B(z, R)) \geq \min\{\frac{1}{2}, (R/2\Lambda_\delta)^{2n+\delta}\} \geq C_\delta^{-1} \min\{1, R^{2n+\delta}\}$. Take $C_d = 2\Lambda_{1/2}$.

(2) Each point of M has a relatively compact coordinate ball on which d is comparable to the Euclidean distance; a smaller closed ball is complete for d, hence closed in $\widehat{M}$, and its interior is a $\widehat{d}$-neighbourhood of the point contained in M. The measure statement holds by definition of $\widehat{\mu}$.

(3) Let η_j be the cutoffs of Proposition 3.1 and $v_j = 1 - \eta_j \in W$. Since $\operatorname{supp} \eta_j$ is compact in M, hence closed in $\widehat{M}$, the function v_j extends continuously by 1 to $\widehat{M} \setminus M$ and equals 1 on the open set $\widehat{M} \setminus \operatorname{supp} \eta_j \supset \widehat{M} \setminus M$. Dominated convergence and (CUT) give $\|v_j\|^2_{L^2(\mu)} + \mathcal{E}(v_j, v_j) = \|1 - \eta_j\|^2_{L^2(\mu)} + \mathcal{E}(\eta_j, \eta_j) \to 0$. □

Proof of Theorem A

Proposition 3.1 and Lemma 2.2 give the first half of part (1), and Theorem 4.3 the bounded inverse. Proposition 5.1 gives part (2), Theorem 6.7 gives part (3), Proposition 7.3(1)–(2) give part (4), Proposition 7.2 gives part (5), and Proposition 7.3(3) gives part (6). All constants depend only on the family data and the displayed exponents. □

8 Two examples

The hypotheses of Theorem A address two distinct phenomena: a reducible fibre changes the kernel of the Friedrichs operator, and at a non-normal point the Green's function is not determined by the natural normalizations. In this section $n = 1$; to keep the formulas classical we use the Riemannian Laplacian Δ_g (so $\Delta_g = a^{-1}(\partial_\xi^2 + \partial_\zeta^2)$ for $g = a(d\xi^2 + d\zeta^2)$), unnormalized area measures, and the corresponding Dirichlet form $\int \langle \nabla u, \nabla v\rangle_g \, dA_g$. The phenomena exhibited are insensitive to these normalizations.

8.1 A reducible conic degeneration

Proposition 8.1. *Let $[Z_0 : Z_1 : Z_2]$ be homogeneous coordinates on $\mathbf{P}^2$, $x = Z_0/Z_2$, $y = Z_1/Z_2$, and let $\Omega = \frac{i}{2}\partial\bar{\partial}\log(1 + |x|^2 + |y|^2)$ be the Fubini–Study form. Consider the family*

$$\mathcal{C} = \{(Z, t) \in \mathbf{P}^2 \times \mathbb{D} : \ Z_0 Z_1 = t Z_2^2\} \longrightarrow \mathbb{D},$$

with fibres C_t, and give C_t^{reg} the metric g_t induced by Ω and the area measure μ_t. Then $\mathcal{C}$ is smooth, the projection is proper with connected fibres, C_t is a smooth conic for $t \neq 0$, and C_0 is the union of two lines meeting at $o = [0 : 0 : 1]$. All fibres have area $\mu_t(C_t^{\mathrm{reg}}) = 2\pi$; taking the metric itself as reference form, the normalized volume density is identically 1 and the potential is 0. Nevertheless:

1. *On $M = C_0^{\mathrm{reg}} = M_1 \sqcup M_2$, let $\mathcal{D}_0$ be the closure of $C_c^\infty(M)$ for $\int_M (u^2 + |\nabla u|^2_{g_0})\, d\mu_0$, and let A_0 be the Friedrichs operator of $\mathcal{E}_0(u, v) = \int_M \langle \nabla u, \nabla v\rangle_{g_0} \, d\mu_0$ on $\mathcal{D}_0$. Then $\ker A_0 = \operatorname{span}\{\mathbf{1}_{M_1}, \mathbf{1}_{M_2}\}$. The bounded mean-zero function $f_0 = \mathbf{1}_{M_1} - \mathbf{1}_{M_2}$ is not in the range of A_0, and for $y \in M$ there is no $G_y \in L^1(\mu_0)$ with $\int_M G_y A_0 \phi \, d\mu_0 = \phi(y) - \frac{1}{2\pi}\int_M \phi \, d\mu_0$ for all bounded $\phi \in \operatorname{Dom}(A_0) \cap C^\infty(M)$ with bounded $A_0\phi$.*

2. *For $0 < t < e^{-4}$ put $\ell = \log(1/t)$ and $A_t = -\Delta_{g_t}$. There are $f_t, U_t \in C^\infty(C_t)$ with $|f_t| \le 1$, $\int f_t \, d\mu_t = \int U_t \, d\mu_t = 0$, $A_t U_t = f_t$ and $U_t(p) \ge \ell/48$ at $p = [1 : 0 : 0]$. Consequently every $\Gamma_{t,p} \in L^1(\mu_t)$ with $\int \Gamma_{t,p} A_t \phi \, d\mu_t = \phi(p) - \frac{1}{2\pi}\int \phi \, d\mu_t$ for all $\phi \in C^\infty(C_t)$ satisfies $\|\Gamma_{t,p}\|_{L^1(\mu_t)} \ge \ell/48$, and the Poincaré constant*

$$\lambda_P(t) = \inf\Big\{\frac{\int_{C_t} |\nabla v|^2_{g_t} \, d\mu_t}{\int_{C_t} v^2 \, d\mu_t} : \ 0 \neq v \in C^\infty(C_t, \mathbf{R}), \ \int_{C_t} v \, d\mu_t = 0\Big\}$$

   tends to 0 as $t \downarrow 0$.

Thus connected reducible fibres admit no uniform inverse, Green, Poincaré, or Sobolev estimate, even with fixed area, fixed reference form, density 1 and zero potential.

Proof. Geometry. For $t \neq 0$ the gradient $(Z_1, Z_0, -2tZ_2)$ of the defining polynomial does not vanish on $\mathbf{P}^2$, so C_t is smooth; near $(o, 0)$ the total space is smooth because $\partial_t(xy - t) = -1$. Properness follows from compactness of $\mathbf{P}^2$, and for $t \neq 0$ the map $[u : v] \mapsto [u^2 : tv^2 : uv]$ is an isomorphism $\mathbf{P}^1 \to C_t$, so the fibres are connected. In the chart $x \mapsto (x, t/x)$ write $w = \log|x|$, $\vartheta = \arg x$ and $h_t(w) = \log(1 + e^{2w} + |t|^2 e^{-2w})$; then

$$g_t = b_t(w)\,(dw^2 + d\vartheta^2), \qquad d\mu_t = b_t(w)\,dw\,d\vartheta, \qquad b_t = \tfrac{1}{4}h_t''.$$

Hence $\mu_t(C_t) = \frac{2\pi}{4}\big(h_t'(+\infty) - h_t'(-\infty)\big) = \frac{\pi}{2}\cdot 4 = 2\pi$, and C_0 consists of two lines of area π each.

(1). Let $\chi\colon \mathbf{R} \to [0,1]$ be smooth, 0 on $(-\infty, 0]$ and 1 on $[1, \infty)$, and in a coordinate z centred at o on the line M_i put $\chi_\epsilon(z) = \chi\big((\log|z| - 2\log\epsilon)/\log(1/\epsilon)\big)$, which vanishes for $|z| \le \epsilon^2$ and equals 1 for $|z| \ge \epsilon$. The function equal to χ_ϵ on M_i and 0 on the other component lies in $C_c^\infty(M)$, converges to $\mathbf{1}_{M_i}$ in $L^2(\mu_0)$, and by conformal invariance of the Dirichlet energy in dimension two,

$$\mathcal{E}_0(\chi_\epsilon, \chi_\epsilon) = 2\pi \int_{\epsilon^2}^{\epsilon} \Big|\frac{\chi'(\cdot)}{r\log(1/\epsilon)}\Big|^2 r\,dr = \frac{2\pi}{\log(1/\epsilon)} \int_0^1 |\chi'|^2 \to 0.$$

Thus $\mathbf{1}_{M_i} \in \mathcal{D}_0$ with zero energy, so $\mathbf{1}_{M_i} \in \mathrm{Dom}(A_0)$ and $A_0\mathbf{1}_{M_i} = 0$. Conversely a kernel element has zero energy, hence is constant on each component. If $u \in \mathcal{D}_0$ solved $\mathcal{E}_0(u, v) = \int f_0 v\,d\mu_0$ for all $v \in \mathcal{D}_0$, testing with $v = \mathbf{1}_{M_1}$ would give $0 = \pi$. If $y \in M_1$ and G_y were as in (1), testing with $\phi = \mathbf{1}_{M_2}$ (which is bounded, smooth on M, in $\mathrm{Dom}(A_0)$, with $A_0\phi = 0$) would give $0 = \mathbf{1}_{M_2}(y) - \frac{1}{2} = -\frac{1}{2}$; for $y \in M_2$ use $\mathbf{1}_{M_1}$.

(2): the Poincaré constant. Let $\eta\colon \mathbf{R} \to [-1, 1]$ be smooth, odd, nondecreasing, equal to ± 1 on $\pm[1, \infty)$, let $J = \int_{-1}^{1} |\eta'|^2$, and set $f_t(w, \vartheta) = \eta(4w/\ell + 2)$. The involution $Z_0 \leftrightarrow Z_1$ acts as $w \mapsto -\ell - w$, preserves b_t and reverses f_t, so $\int f_t\,d\mu_t = 0$. Conformal invariance gives $\int |\nabla f_t|_{g_t}^2\,d\mu_t = 2\pi \int (4/\ell)^2 |\eta'(4w/\ell + 2)|^2\,dw = 8\pi J/\ell$. On the annulus $1 \le |x| \le 2$ we have $w \ge 0 > -\ell/4$, so $f_t = 1$ there; the area density of the immersion $F_t(x) = (x, t/x)$ for Ω is

$$a_t = \frac{(1 + |F_t|^2)|F_t'|^2 - |\langle F_t', F_t\rangle|^2}{(1 + |F_t|^2)^2} \ge \frac{|F_t'|^2}{(1 + |F_t|^2)^2} \ge \frac{1}{36}$$

on the annulus (as $|F_t'|^2 = 1 + |t|^2|x|^{-4} \ge 1$ and $1 + |F_t|^2 \le 6$ there), so $\int f_t^2\,d\mu_t \ge \frac{3\pi}{36} = \frac{\pi}{12}$ and $\lambda_P(t) \le \frac{8\pi J/\ell}{\pi/12} = \frac{96J}{\ell} \to 0$.

(2): the solution U_t. Let $w_0 = -\ell/2$, $k_t = b_t f_t$, $Q_t(w) = \int_w^\infty k_t$, and $U_t(w) = \int_{w_0}^w Q_t$. Then $U_t'' = -b_t f_t$, i.e. $-\Delta_{g_t} U_t = f_t$. The function k_t is odd about w_0, so Q_t is even and U_t is odd about w_0; hence $\int U_t\,d\mu_t = 0$. For $w \ge -\ell/4$, $f_t = 1$ gives $Q_t = \frac{1}{4}(2 - h_t')$ and

$$U_t(w) = c_t + \frac{w}{2} - \frac{h_t(w)}{4} = c_t - \frac{1}{4}\log(1 + e^{-2w} + |t|^2 e^{-4w}),$$

which is smooth at p in the coordinate $1/x$; by oddness U_t is smooth at the other pole too. Since $f_t \ge 0$ for $w \ge w_0$, Tonelli's theorem gives

$$U_t(p) = \int_{w_0}^\infty (w - w_0)\,b_t(w) f_t(w)\,dw \ge \frac{\ell}{2}\int_0^{\log 2} b_t(w)\,dw = \frac{\ell}{2}\cdot\frac{\mu_t(\{1 \le |x| \le 2\})}{2\pi} \ge \frac{\ell}{48}.$$

Testing the defining identity of $\Gamma_{t,p}$ with $\phi = U_t$ gives $\int \Gamma_{t,p} f_t\,d\mu_t = U_t(p)$, and $|f_t| \le 1$ yields the L^1 lower bound. Replacing μ_t by $\mu_t/2\pi$ multiplies $\Gamma_{t,p}$ by 2π and leaves its L^1 norm with respect to the normalized measure unchanged, so the conclusion does not depend on this normalization. $\square$

8.2 A non-normal nodal cubic

Proposition 8.2. *Let $X_{\mathrm{nd}} = \{Y^2Z = X^2(X+Z)\} \subset \mathbf{P}^2$, with node $p_{\mathrm{nd}} = [0:0:1]$, and let $\nu_{\mathrm{nd}} \colon \mathbf{P}^1 \to X_{\mathrm{nd}}$, $[u:v] \mapsto [(u^2-v^2)v : u(u^2-v^2) : v^3]$, be its normalization. Then X_{nd} is irreducible, p_{nd} is its only singular point, and ν_{nd} identifies $M_{\mathrm{nd}} = X_{\mathrm{nd}} \setminus \{p_{\mathrm{nd}}\}$ with $\mathbf{P}^1 \setminus \{1,-1\}$ (coordinate $z = u/v$). Equip M_{nd} with the Fubini–Study metric $g = |dz|^2/(1+|z|^2)^2$, area form $\omega_{\mathrm{nd}} = \frac{i}{2}\partial\bar\partial \log(1+|z|^2)$ and area measure μ_{nd}. Then ω_{nd} has smooth strictly plurisubharmonic ambient local potentials on X_{nd}, $\mu_{\mathrm{nd}}(M_{\mathrm{nd}}) = \pi$, the metric completion of M_{nd} is the round sphere of radius $\frac{1}{2}$, and the two completion points over the node are antipodal, at distance $\pi/2$. The Friedrichs kernel is $\mathbf{R}$, every smooth function on the sphere restricts to an element of the operator domain, and the operator is $A_{\mathrm{nd}} = -(1+|z|^2)^2(\partial^2_{\operatorname{Re} z} + \partial^2_{\operatorname{Im} z})$.*

Call $G \in L^2(\mu_{\mathrm{nd}}) \cap C^\infty(M_{\mathrm{nd}})$ an ambient-test Green's function at the node *if $\int G\, d\mu_{\mathrm{nd}} = 0$ and*

$$\int_{M_{\mathrm{nd}}} G\, A_{\mathrm{nd}}(\Psi|_{M_{\mathrm{nd}}})\, d\mu_{\mathrm{nd}} = \Psi(p_{\mathrm{nd}}) - \frac{1}{\pi}\int_{X_{\mathrm{nd}}} \Psi\, d\mu_{\mathrm{nd}} \qquad \text{for all real } \Psi \in C^\infty(\mathbf{P}^2).$$

For $c \in \{1,-1\}$ let

$$H_c(z) = -\frac{1}{4\pi}\log\frac{|z-c|^2}{(1+|z|^2)(1+|c|^2)}, \qquad \Gamma_c = H_c - \frac{1}{\pi}\int H_c\, d\mu_{\mathrm{nd}}.$$

The ambient-test Green's functions at the node are exactly $G_s = s\Gamma_1 + (1-s)\Gamma_{-1}$, $s \in \mathbf{R}$. Each G_s lies in $L^q(\mu_{\mathrm{nd}})$ for every $q < \infty$ and has gradient in $L^r(\mu_{\mathrm{nd}})$ for every $r < 2$; G_s is bounded below if and only if $0 \le s \le 1$. Thus mean normalization, lower bounds, and subcritical integrability do not single out a Green's function with pole at the node.

Proof. The curve. In the chart $Z = 1$ the equation is $y^2 = x^2(x+1)$; it is irreducible because $x^2(x+1)$ is not a square, and Z does not divide the homogeneous equation. The only affine singular point is $(0,0)$, and the point at infinity $[0:1:0]$ is smooth. The map $z \mapsto (x,y) = (z^2-1, z(z^2-1))$ has inverse $z = y/x$ off the node, sends $z = \pm 1$ to the node, and extends to an isomorphism $\mathbf{P}^1 \to X_{\mathrm{nd}}$ over the point at infinity. Near the node the two branches are smooth and transverse, so there are ambient coordinates (q,r) in which they are $\{r = 0\}$ (with $q = z-1$) and $\{q = 0\}$ (with $r = z+1$); the strictly psh function $P(q,r) = \log(1+|1+q|^2) + \log(1+|r-1|^2) - \log 2$ restricts to $\log(1+|z|^2)$ on both branches. At smooth points a local potential is obtained by extending $\log(1+|z|^2)$ constantly in a transverse direction and adding the squared modulus of the transverse coordinate. The area is $2\pi\int_0^\infty r(1+r^2)^{-2}\,dr = \pi$, and $z \mapsto (\operatorname{Re} z, \operatorname{Im} z, \frac{1}{2}(|z|^2-1))/(1+|z|^2)$ is an isometry onto the round sphere of radius $\frac{1}{2}$ minus two antipodal points. Detouring a minimizing arc around a puncture disc of radius ϵ costs $O(\epsilon)$, so the intrinsic distance of M_{nd} is the round distance; the completion is the sphere and the punctures are at distance $\pi/2$.

The Friedrichs domain. Logarithmic cutoffs κ_ϵ as in the proof of Proposition 8.1 at the two punctures have energy $2\pi\int_0^1 |\kappa'|^2/\log(1/\epsilon) \to 0$, so for smooth ϕ on the sphere $\chi_\epsilon\phi \to \phi$ in the form norm and $\phi|_{M_{\mathrm{nd}}}$ lies in the form domain; integration by parts against compactly supported tests and density put it in the operator domain with $A_{\mathrm{nd}}\phi$ the negative round Laplacian. Kernel elements have zero energy and M_{nd} is connected, so the kernel is $\mathbf{R}$. Note that the smooth sphere function $\operatorname{Re} z/(1+|z|^2)$ takes different values at the two punctures, so elements of the operator domain need not have a common limit over the node.

The functions Γ_c. In the coordinate $\zeta = 1/z$ the argument of the logarithm in H_c is $|1-c\zeta|^2/((1+|\zeta|^2)(1+|c|^2))$, so H_c is smooth at infinity. From $\Delta_{\mathrm{Euc}}\log|z-c|^2 = 4\pi\delta_c$ and $\Delta_{\mathrm{Euc}}\log(1+|z|^2) = 4(1+|z|^2)^{-2}$ we get, for every smooth ϕ on the sphere,

$$\int \Gamma_c\, A_{\mathrm{nd}}\phi\, d\mu_{\mathrm{nd}} = -\int \Gamma_c\, \Delta_{\mathrm{Euc}}\phi\, dA = \phi(c) - \frac{1}{\pi}\int \phi\, d\mu_{\mathrm{nd}}.$$

The isometry $z \mapsto -z$ exchanges H_1 and H_{-1}, so their means agree and $\Gamma_1 - \Gamma_{-1} = -\frac{1}{4\pi}\log(|z-1|^2/|z+1|^2) \neq 0$. For $\Psi \in C^\infty(\mathbf{P}^2)$, the pull-back $\phi = \Psi \circ \nu_{\mathrm{nd}}$ is smooth on the sphere with $\phi(1) = \phi(-1) = \Psi(p_{\mathrm{nd}})$; hence every G_s is an ambient-test Green's function at the node.

Classification. Let G be an ambient-test Green's function and $h = G - \Gamma_{-1}$. Every function in $C_c^\infty(M_{\mathrm{nd}})$ is the restriction of an ambient smooth function (extend normally in submanifold charts and use a partition of unity), so h is harmonic on the twice-punctured sphere. On a punctured disc around $c = \pm 1$ expand h in Fourier modes: $c_0 = B_0 + D_0 \log r$ and $c_k = B_k r^{|k|} + D_k r^{-|k|}$ for $k \neq 0$. Since $h \in L^2$, $D_k = 0$ for $k \neq 0$, and the remaining series extends smoothly across the puncture (its coefficients decay rapidly, being those of smooth boundary data). Hence $A_{\mathrm{nd}} h = t_1 \delta_1 + t_{-1}\delta_{-1}$ on the sphere, and testing with 1 gives $t_1 + t_{-1} = 0$. Subtracting $t_1(\Gamma_1 - \Gamma_{-1})$ removes both logarithmic singularities, leaving a smooth harmonic function on the sphere with mean zero, which vanishes. So $G = t_1\Gamma_1 + (1 - t_1)\Gamma_{-1}$.

Integrability and lower bounds. Each Γ_c is continuous except at c, where it tends to $+\infty$. Hence G_s is bounded below for $0 \leq s \leq 1$, while for $s < 0$ it tends to $-\infty$ at 1 and for $s > 1$ at -1. Near a puncture $|G_s| = O(1 + |\log r|)$ and $|\nabla G_s| = O(1 + r^{-1})$, so $G_s \in L^q$ for all $q < \infty$ and $\nabla G_s \in L^r$ for all $r < 2$. □

The conic example leaves open a theory with componentwise mean subtraction. The nodal example shows that a Green's function with pole at a non-normal point requires a choice of branch; it does not show that normality is necessary for the regular-centre estimates of Theorem A.

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