MIXED VOLUMES OF OKOUNKOV BODIES

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Abstract. Let X be a connected smooth projective variety of dimension n. We first prove that the movable product of any big line bundles $L_1, \dots, L_n$ equals $n!$ times the supremum of the mixed volumes of their Okounkov bodies over all valuations.

1. Introduction

Let X be a connected smooth projective variety of dimension n. For a big line bundle L and a surjective valuation

$$\nu\colon \mathbb{C}(X)^\times \longrightarrow \mathbb{Z}^n_{\mathrm{lex}}$$

that is trivial on $\mathbb{C}^\times$, write $\Delta_\nu(L) \subseteq \mathbb{R}^n$ for the associated Okounkov body. Its Euclidean volume recovers the volume, or equivalently the top movable self-intersection, of L:

$$n! \operatorname{vol} \Delta_\nu(L) = \operatorname{vol}(L) = \langle L^n \rangle. \tag{1.1}$$

Okounkov bodies depend only on numerical classes [LM09, Proposition 4.1]. Conversely, Jow proved that two big divisors on a normal complex projective variety are numerically equivalent if their Okounkov bodies agree for every admissible flag [Jow10, Theorem A]. Thus the collection of all Okounkov bodies is a faithful numerical invariant.

The first conclusion of this paper is that this collection also recovers mixed movable products.

Theorem 1.1. *Let $L_1, \dots, L_n$ be big line bundles on X, then*

$$\langle L_1, \dots, L_n \rangle = n! \sup_\nu \operatorname{vol}\Big(\Delta_\nu(L_1), \dots, \Delta_\nu(L_n)\Big), \tag{1.2}$$

where ν ranges over the valuations above.

Here vol denotes mixed volume, normalized by $\operatorname{vol}(K, \dots, K) = \operatorname{vol}(K)$. For ample line bundles, Wilms proved that a suitable single admissible flag realizes the mixed intersection number [Wil25, Theorem 1.1].

We now pass to its singular-metric refinement. A Hermitian big line bundle (L, h) will mean a big line bundle endowed with a singular psh metric with positive volume (in the sense of Cao). In [Xia25], the author introduced the partial Okounkov body $\Delta_\nu(L, h) \subseteq \mathbb{R}^n$ extending the usual Okounkov body construction. The volume theorem [Xia25, Theorem A] gives

$$n! \operatorname{vol} \Delta_\nu(L, h) = \operatorname{vol} \mathrm{dd}^{\mathrm{c}} h, \tag{1.3}$$

where the volume on the right-hand side is taken in the sense of Cao [Cao14]. Moreover, the family of partial Okounkov bodies over all rank-n valuations determines the $\mathcal{I}$-singularity type of h [Xia25, Theorem B], a metric analogue of Jow's theorem.

A formula of this type was first proposed in [Xia25, Conjecture 8.8]. We prove it here.

Theorem 1.2. *Let X be a connected smooth projective variety of dimension n. For $1 \le i \le n$, let L_i be a big line bundle and let h_i be a psh metric on L_i whose volume is positive. Then*

$$\operatorname{vol}\left(\mathrm{dd}^{\mathrm{c}} h_1, \dots, \mathrm{dd}^{\mathrm{c}} h_n\right) = n! \sup_\nu \operatorname{vol}\Big(\Delta_\nu(L_1, h_1), \dots, \Delta_\nu(L_n, h_n)\Big). \tag{1.4}$$

Here ν runs over all valuations as above.

Again, the mixed volume is taken in the sense of Cao, as shown in [Xia26b], it coincides with the mixed volume in the sense of Darvas–Xia as recalled in Section 2. Taking each h_i to have minimal singularities reduces Theorem 1.2 to Theorem 1.1, while its diagonal specialization is (1.3).

The proof of the $\leq$ direction in (1.4) can be easily reduced to Wilms' theorem [Wil25, Theorem 1.1]. Conversely, the $\geq$ direction can be reduced to the $\geq$ direction in (1.2). We shall apply a suitable toric degeneration to further reduce to the toric case, where the results are essentially known.

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2. Preliminaries

We follow the language and notation of [Xia26a]. All products of closed positive currents are non-pluripolar products. We use $\mathrm{dd^c}=(2\pi)^{-1}\mathrm{i}\partial\bar{\partial}$, and $\mathbb{N}=\{0,1,2,\dots\}$. The notation $\mathbb{Z}^n_{\mathrm{lex}}$ means $\mathbb{Z}^n$ with the lexicographic order, and $e_1,\dots,e_n$ denotes its standard basis. A superscript $^\times$ means that zero is omitted, and $\operatorname{Conv} A$ denotes the convex hull of a subset $A\subseteq\mathbb{R}^n$. We use $\mathbb{P}(V)$ for the space of lines in a vector space V.

2.1. **Singularities and partial Okounkov bodies.** Fix a smooth Hermitian metric h^0 on a big line bundle L and write

$$\theta\coloneqq c_1(L,h^0),\qquad h=h^0\mathrm{e}^{-\varphi},\qquad \theta_\varphi\coloneqq\theta+\mathrm{dd^c}\varphi,$$

where $\mathrm{PSH}(X,\theta)$ is the set θ-psh functions, and

$$\mathrm{PSH}(X,\theta)_{>0}\coloneqq\left\{u\in\mathrm{PSH}(X,\theta):\int_X\theta_u^n>0\right\}.$$

For $u,v\in\mathrm{PSH}(X,\theta)$, write $u\preceq v$ if $u\leq v+O(1)$. The model envelope is

$$P_\theta[u]\coloneqq\sup{}^*\{v\in\mathrm{PSH}(X,\theta):v\leq 0,\ v\preceq u\},$$

where $\sup^*$ denotes the upper-semicontinuous regularization of the pointwise supremum. A positive-mass potential u is a model potential if $P_\theta[u]=u$. Two positive-mass potentials are P-equivalent if their model envelopes agree. They are $\mathcal{I}$-equivalent if $\mathcal{I}(\lambda u)=\mathcal{I}(\lambda v)$ for every $\lambda>0$, where $\mathcal{I}(\lambda u)$ is the multiplier ideal sheaf. The $\mathcal{I}$-model envelope is

$$P_\theta[u]_{\mathcal{I}}\coloneqq\sup{}^*\{v\in\mathrm{PSH}(X,\theta):v\leq 0,\ v\text{ is }\mathcal{I}\text{-equivalent to }u\}.$$

A potential $\varphi\in\mathrm{PSH}(X,\theta)_{>0}$ is called $\mathcal{I}$-good if

$$P_\theta[\varphi]=P_\theta[\varphi]_{\mathcal{I}}.$$

The corresponding volume is

$$\operatorname{vol}\theta_\varphi\coloneqq\int_X\theta^n_{P_\theta[\varphi]_{\mathcal{I}}}.$$

More generally, if $L_1,\dots,L_n$ are big line bundles with θ_i being a closed smooth real $(1,1)$-form in $c_1(L_i)$ and $\varphi_i\in\mathrm{PSH}(X,\theta_i)_{>0}$, we write

$$\operatorname{vol}\left(\theta_1+\mathrm{dd^c}\varphi_1,\dots,\theta_n+\mathrm{dd^c}\varphi_n\right)=\int_X\left(\theta_1+\mathrm{dd^c}P_{\theta_1}[\varphi_1]_{\mathcal{I}}\right)\wedge\dots\wedge\left(\theta_n+\mathrm{dd^c}P_{\theta_n}[\varphi_n]_{\mathcal{I}}\right).$$

The mixed volume defined here coincides with that defined by Cao [Cao14].

By [Xia26a, Theorem 7.1.1], $\mathcal{I}$-goodness is equivalent both to the mass–volume identity

$$\int_X\theta_\varphi^n=\operatorname{vol}\theta_\varphi$$

and to the existence of potentials φ_j with analytic singularities such that

$$\varphi_j\xrightarrow{d_S}\varphi.$$

Here d_S is the pseudometric on singularity types introduced in [DDNL21], see also [Xia26a, Definition 6.2.1]. The approximants may be chosen so that every θ_{φ_j} is a Kähler current.

Following [Xia26a, Definition 1.6.1], a potential has analytic singularities if locally it is of the form

$$c \log(|f_1|^2 + \cdots + |f_N|^2) + O(1), \qquad c \in \mathbb{Q}_{>0},$$

where the f_j are holomorphic and $O(1)$ is locally bounded.

Let $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n_{\text{lex}}$ be surjective and trivial on $\mathbb{C}^\times$. Its center exists on X by properness. Choose a rational section τ of L which is regular and nowhere vanishing near the center. For a section s of L^k, set $\nu_\tau(s) := \nu(s/\tau^k)$, and suppress the subscript τ below. For every $k \geq 1$ for which the space below is non-zero, set

$$\Delta_{\nu,k}(\theta, \varphi) := \operatorname{Conv}\left\{ k^{-1}\nu(s) : s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi))^\times \right\}. \tag{2.1}$$

Let $\Gamma(\theta, \varphi) \subseteq \mathbb{Z}^n \times \mathbb{N}$ consist of $(0,0)$ and all pairs $(\nu(s), k)$ occurring above. By [Xia26a, Theorem 10.3.1], $\Gamma(\theta, \varphi)$ is an almost semigroup. Using the continuous Okounkov-body map Δ of [Xia26a, Theorem 10.1.2], define $\Delta_\nu(\theta, \varphi) = \Delta_\nu(L, h) := \Delta(\Gamma(\theta, \varphi))$. Changing the rational trivialization translates all the bodies in (2.1); none of the mixed volumes below is affected.

We record the properties used in the proof. We write $\varphi \preceq_{\mathcal{I}} \psi$ when $\mathcal{I}(\lambda\varphi) \subseteq \mathcal{I}(\lambda\psi)$ for every $\lambda > 0$. Let $\mathcal{K}_n$ be the space of non-empty compact convex subsets of $\mathbb{R}^n$, and let d_{Haus} be the Hausdorff distance on $\mathcal{K}_n$. We write $\operatorname{vol}(K)$ for the Euclidean n-volume of $K \in \mathcal{K}_n$. First,

$$\operatorname{vol} \Delta_\nu(\theta, \varphi) = \frac{1}{n!} \operatorname{vol} \theta_\varphi \tag{2.2}$$

by [Xia26a, Corollary 10.3.2]. The right-hand side is independent of ν. Second, the map

$$\Delta_\nu(\theta, \cdot)\colon (\mathrm{PSH}(X, \theta)_{>0}, d_S) \longrightarrow (\mathcal{K}_n, d_{\text{Haus}})$$

is continuous by [Xia26a, Theorem 10.3.2]. Third, for another Hermitian big line bundle (L', h'), singularity comparison and tensor products give

$$\varphi \preceq_{\mathcal{I}} \psi \Longrightarrow \Delta_\nu(\theta, \varphi) \subseteq \Delta_\nu(\theta, \psi), \tag{2.3}$$

$$\Delta_\nu(L, h) + \Delta_\nu(L', h') \subseteq \Delta_\nu(L \otimes L', h \otimes h'), \tag{2.4}$$

by [Xia26a, Propositions 10.3.8 and 10.3.10]. In particular, the partial body is contained in the ordinary Okounkov body $\Delta_\nu(L)$, obtained from the same construction using the complete linear series $\mathrm{H}^0(X, L^k)$. The latter satisfies

$$\operatorname{vol} \Delta_\nu(L) = \frac{1}{n!} \operatorname{vol}(L) \tag{2.5}$$

by [Xia26a, Corollary 10.3.1], where $\operatorname{vol}(L) := \lim_{k\to\infty} n! k^{-n} h^0(X, L^k)$.

We will also use basic facts about model envelopes. By [Xia26a, Corollary 3.1.2], $P_\theta[\varphi]$ is a model potential. The equality $d_S(\varphi, P_\theta[\varphi]) = 0$ follows from [Xia26a, Proposition 6.2.2], and P-equivalent potentials are $\mathcal{I}$-equivalent by [Xia26a, Proposition 3.2.9]. Consequently, replacing a potential by its model envelope changes neither its d_S-class nor its partial Okounkov body. If φ has analytic singularities, then so does $P_\theta[\varphi]$ by [Xia26a, Proposition 3.2.10].

Lemma 2.1. *Let $u_j, u \in \mathrm{PSH}(X, \theta)_{>0}$ be model potentials such that $u_j \xrightarrow{d_S} u$. The sequence $(u_j)_j$ has a further subsequence, still indexed by j, for which there are increasing model potentials η_j and decreasing model potentials ψ_j satisfying*

$$\eta_j \leq u_j \leq \psi_j, \qquad \eta_j \leq u \leq \psi_j, \qquad \eta_j \xrightarrow{d_S} u, \qquad \psi_j \xrightarrow{d_S} u.$$

Proof. By [Xia26a, Corollary 6.2.5], the masses $\int_X \theta^n_{u_j}$ have a positive lower bound after finitely many terms. So we can apply [Xia26a, Proposition 6.2.3] (and then take the P-envelope) to obtain η_j and ψ_j. Only the inequality $\eta_j \leq u \leq \psi_j$ needs an extra proof. It follows from $\eta_j \leq u_j \leq \psi_j$ and [Xia26a, Proposition 6.2.5]. □

2.2. **Mixed volumes and movable intersections.** We retain the mixed-volume normalization fixed in Theorem 1.2. Mixed volume is symmetric, multilinear for Minkowski addition, monotone, translation invariant, and continuous for the Hausdorff topology; see [Sch14, Section 5.1].

For big classes $\alpha_1,\dots,\alpha_n \in \mathrm{H}^{1,1}(X,\mathbb{R})$, we write $\langle \alpha_1,\dots,\alpha_n\rangle$ for their movable intersection number [BDPP13]. If $\mu\colon Y \to X$ is a modification with Y smooth and

$$\mu^*\alpha_i = \beta_i + \{E_i\}, \qquad \beta \text{ nef}, \quad E_i \text{ is an effective } \mathbb{R}\text{-divisor},$$

then

$$\beta_1\cdots\beta_n \le \langle \alpha_1,\dots,\alpha_n\rangle. \tag{2.6}$$

For nef α_i, the movable and ordinary intersection numbers agree.

3. An algebraic upper bound

Lemma 3.1. *Let $F/\mathbb{C}$ be a finitely generated field of transcendence degree n, let X be an integral projective model with $\mathbb{C}(X) = F$, and let ν be a surjective valuation which is trivial on $\mathbb{C}^\times$, written as*

$$\nu\colon F^\times \longrightarrow \mathbb{Z}^n_{\mathrm{lex}}.$$

Let $W_1,\dots,W_n \subset F$ be non-zero finite-dimensional vector spaces. Evaluation of rational functions defines

$$\Phi_i\colon X \dashrightarrow \mathbb{P}(W_i^\vee), \qquad x \longmapsto [\mathrm{ev}_{i,x}], \qquad \mathrm{ev}_{i,x}(f) \coloneqq f(x).$$

This map is defined wherever all elements of W_i are regular and $\mathrm{ev}_{i,x} \neq 0$. Put

$$\Phi \coloneqq (\Phi_1,\dots,\Phi_n)\colon X \dashrightarrow \prod_{i=1}^n \mathbb{P}(W_i^\vee),$$

and let Y be the Zariski closure of its image. Assume that Φ is birational onto Y and that

$$\Lambda \coloneqq \operatorname{span}_{\mathbb{Z}}\{\nu(f)-\nu(g) : f,g \in W_i^\times,\ 1\le i\le n\} = \mathbb{Z}^n.$$

Put $P_i \coloneqq \operatorname{Conv}\nu(W_i^\times)$. If π_i is the projection of the ambient product onto $\mathbb{P}(W_i^\vee)$, put

$$H_i \coloneqq c_1\big(\pi_i^*\mathcal{O}_{\mathbb{P}(W_i^\vee)}(1)\big).$$

Then

$$n!\operatorname{vol}(P_1,\dots,P_n) \le (H_1\cdots H_n\cdot Y). \tag{3.1}$$

Proof. For $a\in\mathbb{Z}^n$, let

$$\mathcal{F}_{\succeq a} \coloneqq \{0\}\cup\{f\in F^\times : \nu(f)\succeq a\}, \qquad \mathcal{F}_{\succ a} \coloneqq \{0\}\cup\{f\in F^\times : \nu(f)\succ a\},$$

where $\succeq$ is the lexicographic order. The associated graded ring and the initial form of $f\in F^\times$ are

$$\operatorname{gr}_\nu F \coloneqq \bigoplus_{a\in\mathbb{Z}^n} \mathcal{F}_{\succeq a}/\mathcal{F}_{\succ a}, \qquad \operatorname{in}_\nu(f) \coloneqq f \bmod \mathcal{F}_{\succ \nu(f)}.$$

The quotient $\mathcal{F}_{\succeq a}/\mathcal{F}_{\succ a}$ is called the leaf of ν at a. Since ν has rational rank n, Abhyankar's inequality forces its residue field to be $\mathbb{C}$; hence every non-zero graded piece is one-dimensional over $\mathbb{C}$. Equivalently, every non-zero finite-dimensional subspace $V\subset F$ satisfies

$$\#\nu(V^\times) = \dim_{\mathbb{C}} V;$$

compare [LM09, Lemma 1.3] and [Xia26a, Section 10.3].

Put $r_i \coloneqq \dim_{\mathbb{C}} W_i - 1$. Choose one $f_{ij}\in W_i^\times$ above each value in $\nu(W_i^\times)$ and set

$$a_{ij} \coloneqq \nu(f_{ij}), \qquad 0\le j\le r_i.$$

The a_{ij} are pairwise distinct, so the valuation axiom makes the f_{ij} linearly independent. They therefore form a basis of W_i, and

$$\nu(W_i^\times) = \{a_{i0},\dots,a_{ir_i}\}.$$

The initial degeneration. Choose $u_\ell \in F^\times$ with $\nu(u_\ell) = e_\ell$. One-dimensionality of the leaves gives a non-canonical graded isomorphism. More precisely, for $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, write $u^a := u_1^{a_1} \cdots u_n^{a_n}$, and write

$$\mathbb{C}[\mathbb{Z}^n] := \bigoplus_{a \in \mathbb{Z}^n} \mathbb{C}\chi^a, \qquad \chi^a \chi^{a'} = \chi^{a+a'},$$

for the group algebra of $\mathbb{Z}^n$. Then

$$\mathbb{C}[\mathbb{Z}^n] \xrightarrow{\sim} \operatorname{gr}_\nu F, \qquad \chi^a \longmapsto \operatorname{in}_\nu(u^a).$$

After rescaling the f_{ij}, which merely changes the homogeneous coordinates, we may and do assume that

$$\operatorname{in}_\nu(f_{ij}) = \chi^{a_{ij}}.$$

Let x_{ij} be the homogeneous coordinate on $\mathbb{P}(W_i^\vee)$ dual to f_{ij}. In these coordinates,

$$\Phi_i(x) = [f_{i0}(x) : \cdots : f_{ir_i}(x)].$$

Let $\epsilon_1, \dots, \epsilon_n$ be the standard basis of the multidegree lattice $\mathbb{Z}^n$. The multihomogeneous coordinate ring of the ambient product is

$$S := \mathbb{C}[x_{ij} : 1 \le i \le n,\ 0 \le j \le r_i], \qquad \deg x_{ij} := \epsilon_i.$$

Let $I \subseteq S$ be the multihomogeneous prime ideal of Y. Equivalently, a multihomogeneous polynomial G belongs to I exactly when $G(f_{ij}) = 0$ in F.

Recall that each a_{ij} is a column vector in $\mathbb{Z}^n$. Put $N := \sum_{i=1}^n (r_i + 1)$, and define A to be the $n \times N$ matrix whose column indexed by (i,j) is a_{ij}. Equivalently,

$$\mathsf{A} := \left[a_{10} \ \cdots \ a_{1r_1} \mid \cdots \mid a_{n0} \ \cdots \ a_{nr_n}\right] \in \mathbb{Z}^{n \times N}.$$

For $\alpha = (\alpha_{ij}) \in \mathbb{N}^N$, write $x^\alpha := \prod_{i,j} x_{ij}^{\alpha_{ij}}$, and define

$$\operatorname{wt}_{\mathsf{A}}(x^\alpha) := \sum_{i,j} \alpha_{ij} a_{ij} \in \mathbb{Z}^n.$$

These weights are compared lexicographically. If $0 \ne G = \sum_\alpha \gamma_\alpha x^\alpha$, let $\operatorname{in}_{\mathsf{A}}(G)$ be the sum of the terms of least A-weight, and put

$$Q := \operatorname{in}_{\mathsf{A}}(I) := (\operatorname{in}_{\mathsf{A}}(G) : 0 \ne G \in I).$$

Let $t_1, \dots, t_n$ be algebraically independent variables of multidegrees $\deg t_i = \epsilon_i$, and define

$$J := \ker\left(S \longrightarrow \mathbb{C}[\mathbb{Z}^n][t_1, \dots, t_n], \quad x_{ij} \longmapsto \chi^{a_{ij}} t_i\right).$$

If $G \in I$ is multihomogeneous, then $G(f_{ij}) = 0$. Taking initial forms of the terms of least valuation in this equality gives $\operatorname{in}_{\mathsf{A}}(G) \in J$. Since I is multihomogeneous, the initial forms of its multihomogeneous elements generate Q; hence

$$Q \subseteq J.$$

This is the argument of [KM19, Lemma 2.16].

By [KM19, Proposition 8.10], there is a rational weight vector $\omega = (\omega_{ij})$ such that $Q = \operatorname{in}_\omega(I)$, where in_ω is defined by taking terms of least ω-weight. After clearing denominators, the ω_{ij} may be taken integral. We spell out the resulting Rees degeneration. Put

$$\operatorname{wt}_\omega(x^\alpha) := \sum_{i,j} \alpha_{ij}\omega_{ij}, \qquad F_\omega^p S := \operatorname{span}_{\mathbb{C}}\{x^\alpha : \operatorname{wt}_\omega(x^\alpha) \ge p\} \quad (p \in \mathbb{Z}).$$

Let $R := S/I$, let $F_\omega^p R$ be the image of $F_\omega^p S$ in R, and let s be an indeterminate. The Rees algebra of this descending filtration is

$$\mathcal{R}_\omega(R) := \bigoplus_{p \in \mathbb{Z}} F_\omega^p R\, s^{-p} \subseteq R[s, s^{-1}].$$

It is the $\mathbb{C}[s]$-algebra generated by s and the elements $\overline{x}_{ij}s^{-\omega_{ij}}$, where $\overline{x}_{ij}$ is the image of x_{ij} in R. In particular, it is finitely generated and, being a subalgebra of $R[s,s^{-1}]$, is torsion-free over the principal ideal domain $\mathbb{C}[s]$; it is therefore flat over $\mathbb{C}[s]$. Its fibers are

$$\mathcal{R}_\omega(R)/(s-1) \simeq S/I, \qquad \mathcal{R}_\omega(R)/s\mathcal{R}_\omega(R) \simeq \operatorname{gr}_{F_\omega} R \simeq S/\operatorname{in}_\omega(I) = S/Q.$$

Here $\operatorname{gr}_{F_\omega} R := \bigoplus_{p\in\mathbb{Z}} F^p_\omega R/F^{p+1}_\omega R$ is the associated graded ring. Thus $\operatorname{Spec}\mathcal{R}_\omega(R) \to \mathbb{A}^1 := \operatorname{Spec}\mathbb{C}[s]$ is the desired flat degeneration. Giving s multidegree zero makes this family multigraded. More explicitly, each fixed multidegree piece is generated over $\mathbb{C}[s]$ by the finitely many monomials of that block multidegree in the generators $\overline{x}_{ij}s^{-\omega_{ij}}$. It is torsion-free and hence free over $\mathbb{C}[s]$. Its two fibers therefore have the same dimension, so the fibers of the Rees family have the same multigraded Hilbert function.

The toric component. Define the affine semigroup

$$\Gamma_{\mathsf{A}} := \left\{ \sum_{i,j} m_{ij}(\epsilon_i, a_{ij}) : m_{ij} \in \mathbb{N} \right\} \subseteq \mathbb{N}^n \times \mathbb{Z}^n.$$

The image of the homomorphism defining J is exactly $\mathbb{C}[\Gamma_{\mathsf{A}}]$. Hence $S/J \simeq \mathbb{C}[\Gamma_{\mathsf{A}}]$, where $\mathbb{C}[\Gamma_{\mathsf{A}}]$ is the semigroup algebra of Γ_{A}. Projection to the first factor gives an exact sequence

$$0 \longrightarrow \{0\} \oplus \Lambda \longrightarrow \operatorname{span}_{\mathbb{Z}} \Gamma_{\mathsf{A}} \longrightarrow \mathbb{Z}^n \longrightarrow 0.$$

Indeed, the kernel is generated by the differences $(0, a_{ij} - a_{ik})$ with fixed i. Therefore

$$\dim S/J = \operatorname{rank}(\operatorname{span}_{\mathbb{Z}} \Gamma_{\mathsf{A}}) = n + \operatorname{rank}\Lambda = 2n.$$

Birationality gives $\dim S/I = \dim Y + n = 2n$, while preservation of the multigraded Hilbert function gives $\dim S/Q = 2n$. Since $Q \subseteq J$ and J is prime, J is a minimal prime over Q.

It remains to check that this component survives multiprojective saturation. Set

$$\mathfrak{m}_i := (x_{i0}, \ldots, x_{ir_i}), \qquad \mathfrak{B} := \mathfrak{m}_1 \cdots \mathfrak{m}_n.$$

Thus $\mathfrak{B}$ is the multigraded irrelevant ideal. For an ideal $K \subseteq S$, write

$$(K : \mathfrak{B}^m) := \{g \in S : g\mathfrak{B}^m \subseteq K\}, \qquad K : \mathfrak{B}^\infty := \bigcup_{m\geq 0} (K : \mathfrak{B}^m).$$

Set $b := x_{10}\cdots x_{n0}$. Its image under the homomorphism defining J is

$$\chi^{\sum_i a_{i0}} t_1 \cdots t_n \neq 0,$$

so $b \notin J$. If $g \in Q : \mathfrak{B}^\infty$, then $gb^m \in Q \subseteq J$ for some m, and the primality of J gives $g \in J$. Thus

$$Q \subseteq Q : \mathfrak{B}^\infty \subseteq J, \qquad J : \mathfrak{B}^\infty = J.$$

The quotients by the two outer ideals have dimension $2n$, so the middle quotient has the same dimension. The same dimension argument shows that J is a minimal prime over $Q : \mathfrak{B}^\infty$.

We now make the multiprojective convention explicit. For $\mathbf{j} = (j_1, \ldots, j_n)$ with $0 \leq j_i \leq r_i$, put

$$b_{\mathbf{j}} := x_{1j_1} \cdots x_{nj_n}.$$

If $K \subseteq S$ is multihomogeneous, then $\operatorname{MultiProj}(S/K)$ is the closed subscheme of $\prod_i \mathbb{P}(W_i^\vee)$ obtained by gluing the affine schemes

$$\operatorname{Spec}(((S/K)_{b_{\mathbf{j}}})_0),$$

where the subscript 0 denotes the part of multidegree $(0, \ldots, 0)$. Equivalently, its multihomogeneous ideal is the saturation $K : \mathfrak{B}^\infty$; in particular, replacing K by $K : \mathfrak{B}^\infty$ does not change $\operatorname{MultiProj}(S/K)$. Consequently

$$Z := \operatorname{MultiProj}(S/J)$$

is a top-dimensional irreducible component of $Y_0 := \mathrm{MultiProj}(S/Q)$. Saturation does not change sufficiently positive multidegrees, so flatness shows that Y and Y_0 have the same multigraded Hilbert polynomial and hence the same mixed degree. The fundamental cycle of Y_0 is

$$[Y_0] = \sum_W \ell(\mathcal{O}_{Y_0,\eta_W})[W],$$

where W runs over the n-dimensional irreducible components and η_W is the generic point of W. In particular, Z occurs with multiplicity $m_Z := \ell(\mathcal{O}_{Y_0,\eta_Z}) \geq 1$. Since the H_i are globally generated, their mixed degree on every component is non-negative. Therefore

$$(H_1 \cdots H_n \cdot Y) = (H_1 \cdots H_n \cdot Y_0) \geq m_Z(H_1 \cdots H_n \cdot Z) \geq (H_1 \cdots H_n \cdot Z). \tag{3.2}$$

The mixed degree of the toric component. Let $T := (\mathbb{C}^\times)^n$. For $z = (z_1, \dots, z_n) \in T$ and $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, write $z^a := z_1^{a_1} \cdots z_n^{a_n}$. The variety Z is the closure of the image of

$$\tau \colon T \longrightarrow \prod_{i=1}^n \mathbb{P}(W_i^\vee), \qquad z \longmapsto ([z^{a_{i0}} : \cdots : z^{a_{ir_i}}])_{i=1}^n .$$

On the dense torus of the ambient product, contained in the chart $x_{10} \cdots x_{n0} \neq 0$, the map is defined by the characters $z^{a_{ij}-a_{i0}}$. Their exponents generate $\Lambda = \mathbb{Z}^n$, so τ is a closed immersion of T into that torus. We identify T with its image $\tau(T)$, which is dense in Z; in particular, τ is birational onto Z. The exponent polytope of its i-th coordinate map is

$$\mathrm{Conv}\{a_{i0}, \dots, a_{ir_i}\} = P_i.$$

For completeness, choose general hyperplanes in the n projective factors. Their pullbacks to T are general Laurent polynomials with Newton polytopes $P_1, \dots, P_n$. Since the boundary $Z \setminus T$ has dimension at most $n-1$ and each hyperplane system is basepoint-free, the incidence variety of n-tuples whose intersection meets the boundary has dimension strictly smaller than the parameter space. Thus a general n-tuple has no common point on the boundary. The Bernstein–Kushnirenko theorem, or equivalently the toric mixed-intersection formula [Ber75; Ful93], therefore gives

$$(H_1 \cdots H_n \cdot Z) = n! \,\mathrm{vol}(P_1, \dots, P_n).$$

Together with (3.2), this proves (3.1). □

Proposition 3.2. *Let X be a smooth connected projective variety of dimension n, let $D_1, \dots, D_n$ be big $\mathbb{Q}$-divisors, and let $\nu \colon \mathbb{C}(X)^\times \to \mathbb{Z}^n_{\mathrm{lex}}$ be surjective and trivial on $\mathbb{C}^\times$. Then*

$$n! \,\mathrm{vol}\Big(\Delta_\nu(D_1), \dots, \Delta_\nu(D_n)\Big) \leq \langle D_1, \dots, D_n \rangle.$$

Here $\Delta_\nu(D)$ for a big $\mathbb{Q}$-divisor is defined by homogeneity after clearing denominators.

Proof. By homogeneity, assume that all D_i are integral. Let x_ν be the center of ν on X. For each i, choose a non-zero rational section τ_i of $\mathcal{O}_X(D_i)$ which is regular and nowhere vanishing near x_ν. For every integer $k \geq 1$, put

$$W_{i,k} := \mathrm{H}^0(X, \mathcal{O}_X(kD_i))\tau_i^{-k} \subset \mathbb{C}(X).$$

Whenever $W_{i,k} \neq 0$, put

$$P_{i,k} := \mathrm{Conv}\, \nu(W_{i,k}^\times).$$

Let Φ_k be the joint evaluation map associated with the $W_{i,k}$, as in Lemma 3.1:

$$\Phi_k = (\Phi_{1,k}, \dots, \Phi_{n,k}) \colon X \dashrightarrow \prod_{i=1}^n \mathbb{P}(W_{i,k}^\vee).$$

Under multiplication by τ_i^k, its i-th component is the Kodaira map of $|kD_i|$.

The proof of [Xia26a, Proposition 10.3.1] supplies an integer $p_0 > 0$ and $s_0, s_1, \dots, s_n \in W_{1,p_0}^\times$ with

$$\nu(s_j) - \nu(s_0) = e_j, \qquad 1 \leq j \leq n.$$

Choose a sufficiently divisible multiple $p = rp_0$ such that every map defined by $|pD_i|$ is birational. Set

$$\widetilde{s}_0 := s_0^r, \qquad \widetilde{s}_j := s_j s_0^{r-1} \quad (1 \leq j \leq n).$$

These elements lie in $W_{1,p}^\times$ and satisfy $\nu(\widetilde{s}_j) - \nu(\widetilde{s}_0) = e_j$. If $k = mp$, then $\widetilde{s}_0^m, \widetilde{s}_j\widetilde{s}_0^{m-1} \in W_{1,k}^\times$, so

$$\operatorname{span}_{\mathbb{Z}}\{\nu(f) - \nu(g) : f, g \in W_{i,k}^\times,\ 1 \leq i \leq n\} = \mathbb{Z}^n.$$

Moreover, the image of the multiplication map

$$\operatorname{Sym}^m W_{i,p} \longrightarrow W_{i,k}$$

is a linear subsystem defining the m-th Veronese of the birational map associated with $|pD_i|$. Hence $\Phi_{i,k}$, and therefore Φ_k, is birational onto its image.

Let Y_k be the closure of the image of Φ_k, and let $H_{i,k}$ be the restriction to Y_k of the hyperplane class from the i-th factor. Let $\mu_k \colon X_k \to X$ be a common resolution of the base ideals of the systems $|kD_i|$, with X_k smooth, and write

$$\mu_k^*(kD_i) = M_{i,k} + F_{i,k}, \qquad F_{i,k} \geq 0,$$

where $M_{i,k}$ is basepoint-free and $F_{i,k}$ is the effective fixed part. The induced morphism $X_k \to Y_k$ is birational and pulls $H_{i,k}$ back to $M_{i,k}$. Hence the projection formula and Lemma 3.1 give

$$\begin{aligned} n!\operatorname{vol}(P_{1,k}, \dots, P_{n,k}) &\leq (H_{1,k} \cdots H_{n,k} \cdot Y_k) \\ &= M_{1,k} \cdots M_{n,k} \\ &\leq k^n \langle D_1, \dots, D_n \rangle. \end{aligned}$$

The last inequality is (2.6), since $\mu_k^* D_i - k^{-1}M_{i,k} = k^{-1}F_{i,k} \geq 0$.

Finally, [Xia26a, Proposition 10.3.4] gives

$$k^{-1}P_{i,k} = \Delta_{\nu,k}(D_i) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(D_i).$$

This holds in particular along $k \in p\mathbb{Z}_{>0}$. Divide the preceding inequality by k^n and use continuity of mixed volume. □

4. Uniformity in the valuation

We first isolate a convex-geometric compactness statement. It is important that neither the position nor the shape of the bodies is fixed. Recall that $\mathcal{K}_n$ is the set of convex bodies in $\mathbb{R}^n$.

Lemma 4.1. *For $j \in \mathbb{Z}_{>0}$ and $1 \leq i \leq n$, let $A_{i,j}, C_{i,j}, D_{i,j}, B_{i,j} \in \mathcal{K}_n$ satisfy*

$$A_{i,j} \subseteq C_{i,j}, D_{i,j} \subseteq B_{i,j}.$$

Suppose that there are constants $v, M > 0$, independent of i and j, such that

$$\begin{aligned} \operatorname{vol}(A_{i,j}) &\geq v, \\ \operatorname{vol}(B_{1,j} + \cdots + B_{n,j}) &\leq M, \\ \operatorname{vol}(B_{i,j}) - \operatorname{vol}(A_{i,j}) &\longrightarrow 0 \quad (1 \leq i \leq n). \end{aligned}$$

Then

$$|\operatorname{vol}(C_{1,j}, \dots, C_{n,j}) - \operatorname{vol}(D_{1,j}, \dots, D_{n,j})| \longrightarrow 0.$$

We refer to [Sch14] for John's theorem and Blaschke selection theorem used in the proof below.

Proof. Translate each i-th quadruple by the same vector so that $0 \in A_{i,j}$, and set $S_j := B_{1,j} + \cdots + B_{n,j}$. These translations do not change mixed volumes. Every body is now contained in S_j, and

$$v \leq \operatorname{vol}(S_j) \leq M.$$

Let $\mathbb{B}^n \subseteq \mathbb{R}^n$ be the closed Euclidean unit ball. Choose an affine isomorphism $\mathcal{T}_j(x) := L_j x + b_j$, with $L_j \in \mathrm{GL}_n(\mathbb{R})$ and $b_j \in \mathbb{R}^n$, which sends the John ellipsoid of S_j to $\mathbb{B}^n$. Writing $\kappa_n := \mathrm{vol}(\mathbb{B}^n)$, John's theorem gives

$$\mathbb{B}^n \subseteq \mathcal{T}_j(S_j) \subseteq n\mathbb{B}^n, \qquad \frac{\kappa_n}{M} \leq |\det L_j| \leq \frac{n^n \kappa_n}{v}.$$

Thus the transformed bodies lie in one fixed compact set, the transformed $A_{i,j}$ have volume bounded away from zero, and their transformed volume gaps tend to zero.

Starting with any subsequence, Blaschke's selection theorem gives a further subsequence along which all $4n$ transformed bodies converge. At the limit,

$$\widehat{A}_i \subseteq \widehat{C}_i, \widehat{D}_i \subseteq \widehat{B}_i, \qquad \mathrm{vol}(\widehat{A}_i) = \mathrm{vol}(\widehat{B}_i) > 0,$$

where the hats denote the respective Hausdorff limits. Hence $\widehat{A}_i = \widehat{C}_i = \widehat{D}_i = \widehat{B}_i$. Continuity of mixed volume now makes the transformed mixed-volume difference tend to zero. Finally,

$$\mathrm{vol}\Big(\mathcal{T}_j(K_1), \dots, \mathcal{T}_j(K_n)\Big) = |\det L_j|\, \mathrm{vol}(K_1, \dots, K_n)$$

for arbitrary $K_1, \dots, K_n \in \mathcal{K}_n$; the affine translations are harmless. The lower bound for $|\det L_j|$ gives the same conclusion before transformation. Since this works for every subsequence, the full sequence converges. □

The next result upgrades fixed-valuation continuity only at the level of mixed-volume values. It does not assert valuation-uniform Hausdorff convergence of the bodies themselves.

Proposition 4.2. *For $1 \leq i \leq n$, fix a big line bundle L_i, a smooth reference metric with curvature θ_i, and potentials $\varphi_{i,j}, \varphi_i \in \mathrm{PSH}(X, \theta_i)_{>0}$ such that*

$$\varphi_{i,j} \xrightarrow{d_S} \varphi_i.$$

Then

$$\sup_\nu \left| \mathrm{vol}\Big(\Delta_\nu(\theta_1, \varphi_{1,j}), \dots, \Delta_\nu(\theta_n, \varphi_{n,j})\Big) - \mathrm{vol}\Big(\Delta_\nu(\theta_1, \varphi_1), \dots, \Delta_\nu(\theta_n, \varphi_n)\Big) \right| \longrightarrow 0, \tag{4.1}$$

where the supremum is over all surjective valuations $\mathbb{C}(X)^\times \to \mathbb{Z}^n_{\mathrm{lex}}$ that are trivial on $\mathbb{C}^\times$.

Proof. Replace every $\varphi_{i,j}$ and φ_i by its model envelope. As recalled in Section 2, this changes neither its d_S-class nor its partial Okounkov body.

Suppose that (4.1) fails. Then there are a subsequence, still indexed by j, valuations ν_j, and a constant $\delta > 0$ for which the absolute difference in (4.1), evaluated at ν_j, is at least δ.

Apply Lemma 2.1 to this subsequence. After passing to a further subsequence, there are increasing model potentials $\eta_{i,j}$ and decreasing model potentials $\psi_{i,j}$ such that

$$\eta_{i,j} \leq \varphi_{i,j} \leq \psi_{i,j}, \qquad \eta_{i,j} \xrightarrow{d_S} \varphi_i, \quad \psi_{i,j} \xrightarrow{d_S} \varphi_i. \tag{4.2}$$

For every i, j, put

$$A_{i,j} := \Delta_{\nu_j}(\theta_i, \eta_{i,j}), \qquad B_{i,j} := \Delta_{\nu_j}(\theta_i, \psi_{i,j}),$$
$$C_{i,j} := \Delta_{\nu_j}(\theta_i, \varphi_{i,j}), \qquad D_{i,j} := \Delta_{\nu_j}(\theta_i, \varphi_i).$$

By (2.3) and (4.2),

$$A_{i,j} \subseteq C_{i,j}, D_{i,j} \subseteq B_{i,j}.$$

The numerical bounds in Lemma 4.1 do not depend on ν_j. Indeed, (2.2) and the d_S-continuity of the volume [Xia26a, Corollary 7.3.1] give

$$\mathrm{vol}(B_{i,j}) - \mathrm{vol}(A_{i,j}) = \frac{1}{n!}\Big(\mathrm{vol}(\theta_i + \mathrm{dd^c}\psi_{i,j}) - \mathrm{vol}(\theta_i + \mathrm{dd^c}\eta_{i,j})\Big) \longrightarrow 0,$$
$$\mathrm{vol}(A_{i,j}) = \frac{1}{n!}\,\mathrm{vol}(\theta_i + \mathrm{dd^c}\eta_{i,j}) \longrightarrow \frac{1}{n!}\,\mathrm{vol}(\theta_i + \mathrm{dd^c}\varphi_i) > 0.$$

Furthermore, (2.4) and (2.3) yield

$$B_{1,j}+\cdots+B_{n,j}\subseteq\Delta_{\nu_j}\left(\sum_i\theta_i+\mathrm{dd^c}\psi_{i,j}\right)\subseteq\Delta_{\nu_j}(L_1\otimes\cdots\otimes L_n).$$

By (2.5), the last body has Euclidean volume $\mathrm{vol}(L_1\otimes\cdots\otimes L_n)/n!$, independently of j and ν_j.

All assumptions of Lemma 4.1 are therefore satisfied. It forces the mixed-volume difference at ν_j to tend to zero, contradicting its lower bound δ. □

5. Proof of the mixed-volume formula

Proposition 5.1. *The conclusion of Theorem 1.2 holds if each h_i has analytic singularities and every $c_1(L_i,h_i)$ is a Kähler current.*

Proof. Take a projective log resolution $\pi\colon Y\to X$, with Y smooth and projective, for which

$$\pi^*c_1(L_i,h_i)=[D_i]+R_i,$$

where D_i is an effective $\mathbb{Q}$-divisor and R_i is a closed positive current with locally bounded potentials; see [Xia26a, Definition 1.6.2 and Theorem 1.6.1]. Put $\alpha_i:=\{R_i\}\in\mathrm{H}^{1,1}(Y,\mathbb{R})$. The class α_i is nef by [Xia26a, Lemma 1.6.1]. Then

$$\int_X c_1(L_1,h_1)\wedge\cdots\wedge c_1(L_n,h_n)=\int_Y R_1\wedge\cdots\wedge R_n=\alpha_1\cap\cdots\cap\alpha_n. \tag{5.1}$$

Here we used [Xia26a, Proposition 2.4.2 and Proposition 2.2.1(1), (3), and (4)]. Note that

$$\alpha_i^n=\int_X c_1(L_i,h_i)^n>0,$$

Thus $\alpha_i^n>0$, so the nef class α_i is big.

For a $\mathbb{Q}$-Cartier divisor D and a valuation ν centered at $y_\nu\in Y$, choose $m>0$ such that mD is Cartier and let g be a local equation of mD at y_ν. Set

$$\nu(D):=m^{-1}\nu(g)\in\mathbb{Q}^n.$$

For every valuation ν, the log-resolution formula [Xia26a, Remark 10.3.1, Equation (10.22)] reads

$$\Delta_\nu(L_i,h_i)=\Delta_\nu(\alpha_i)+\nu(D_i). \tag{5.2}$$

Translation invariance, Proposition 3.2, and nefness give

$$n!\,\mathrm{vol}\Big(\Delta_\nu(L_1,h_1),\dots,\Delta_\nu(L_n,h_n)\Big)\le\langle\alpha_1,\dots,\alpha_n\rangle=\alpha_1\cap\cdots\cap\alpha_n. \tag{5.3}$$

It remains to prove the opposite inequality after taking the supremum. Fix an ample $\mathbb{Q}$-divisor H on Y. Choose one $c\in\mathbb{Q}_{>0}$ so large that every $c\alpha_i-\{H\}$ is big, and choose effective $\mathbb{Q}$-divisors E_i in the cohomology class $c\alpha_i-\{H\}$. For $\epsilon\in\mathbb{Q}_{>0}$ set

$$\beta_i:=\alpha_i+\epsilon\{H\}.$$

The class β_i is ample and

$$\frac{1}{1+c\epsilon}\beta_i+\frac{\epsilon}{1+c\epsilon}\{E_i\}=\alpha_i. \tag{5.4}$$

By [Wil25, Theorem 1.1], there is an admissible flag

$$Y_\bullet\colon Y=Y_0\supset Y_1\supset\cdots\supset Y_n=\{y\},$$

where each Y_r is irreducible of codimension r in Y and is nonsingular at y. Let ν_ϵ be the valuation defined by this flag. It satisfies

$$n!\,\mathrm{vol}\Big(\Delta_{\nu_\epsilon}(\beta_1),\dots,\Delta_{\nu_\epsilon}(\beta_n)\Big)=\beta_1\cap\cdots\cap\beta_n. \tag{5.5}$$

The flag valuation is surjective and trivial on $\mathbb{C}^\times$; since $\mathbb{C}(Y)=\mathbb{C}(X)$, it is among the valuations in Theorem 1.2. Then by (5.4), we have

$$\frac{1}{1+c\epsilon}\Delta_{\nu_\epsilon}(\beta_i)+\frac{\epsilon}{1+c\epsilon}\nu_\epsilon(E_i)\subseteq\Delta_{\nu_\epsilon}(\alpha_i).$$

Using (5.5), we find

$$\sup_\nu \operatorname{vol}\Big(\Delta_\nu(\alpha_1),\dots,\Delta_\nu(\alpha_n)\Big) \geq \frac{1}{n!(1+c\epsilon)^n}\beta_1\cdots\beta_n. \tag{5.6}$$

Letting $\epsilon \to 0+$ in (5.6), and then using the translations in (5.2), proves the reverse of (5.3). Together with (5.1), this proves the proposition. □

Proof of Theorem 1.2. Choose smooth Hermitian metrics h_i^0 on L_i and write

$$\theta_i := c_1\left(L_i, h_i^0\right), \qquad h_i = h_i^0 \mathrm{e}^{-\varphi_i}.$$

We may replace φ_i by $P_{\theta_i}[\varphi_i]_{\mathcal{I}}$ and assume that φ_i is $\mathcal{I}$-good. [Xia26a, Theorem 7.1.1] gives potentials $\psi_{i,j}$ with analytic singularities such that

$$\psi_{i,j} \xrightarrow{d_S} \varphi_i, \qquad \theta_{i,\psi_{i,j}} \text{ is a Kähler current.}$$

By Proposition 5.1, for every j,

$$\int_X \theta_{1,\psi_{1,j}} \wedge \cdots \wedge \theta_{n,\psi_{n,j}} = n! \sup_\nu \operatorname{vol}\Big(\Delta_\nu(\theta_1,\psi_{1,j}),\dots,\Delta_\nu(\theta_n,\psi_{n,j})\Big). \tag{5.7}$$

By [Xia26a, Theorem 6.2.1], the left-hand side converges to

$$\int_X c_1(L_1,h_1)\wedge\cdots\wedge c_1(L_n,h_n).$$

By Proposition 4.2, the right-hand side converges to

$$n! \sup_\nu \operatorname{vol}\Big(\Delta_\nu(L_1,h_1),\dots,\Delta_\nu(L_n,h_n)\Big).$$

Taking limits in (5.7) proves (1.4). □

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