# Non-Archimedean Pluripotential Theory over a Trivially Valued Field

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# Preface

The book studies plurisubharmonic functions on spaces over a trivially valued field. A valuation supplies a point of such a space, and a degeneration supplies a function on it. Intersections on a degeneration then give a measure. We begin with the projective line, where the measure can also be computed from slopes along valuation rays.

The intended reader knows the language of schemes, line bundles, and projective morphisms, together with the basic intersection theory of divisors. Further results of commutative algebra and algebraic geometry are cited from the Stacks Project. The transcendental part assumes the pluripotential theory of quasi-plurisubharmonic functions on compact Kähler manifolds developed in *Foundations of Pluripotential Theory* [Xia26a], together with the facts about complex spaces and modifications collected in [Xia26a, Appendix A], and the theory of singularity types, model potentials and test curves developed in *Singularities in Global Pluripotential Theory* [Xia26b]. The first of these books also supplies the analytic facts about multiplier ideals and the $L^2$ estimates that Appendix B uses in characteristic zero.

The book has six parts. The preliminaries develop analytic points and domains, divisorial valuations, Radon measures and convexity, test configurations, plurisubharmonic potentials in real numerical classes, and the homogeneous potentials that record their singularities. The second part introduces tropical charts, real forms with their positivity and Stokes formula, and local Monge–Ampère products. The third part proves the envelope theorem for integral unibranch projective varieties over an algebraically closed field, in every characteristic. The global energy theory then treats mixed measures and energy, orthogonality of envelopes and the variational solution of the Monge–Ampère equation, and, on integral regular projective curves, the agreement of the global measure with the local curvature current. The transcendental part constructs the space of ideal semivaluations of a compact complex space. On connected compact Kähler manifolds it solves the Monge–Ampère equation in Kähler classes, in nef classes of positive volume and, relative to a minimal singularity type, in big classes, and it realizes compatible families of $\mathcal{I}$-model test curves in pseudoeffective classes as functions on the valuation space. The final part solves the equation explicitly on integral regular projective curves, and by convex roofs

on polarized toric varieties for measures carried by the Gauss section. On integral projective varieties over an algebraically closed field and on integral regular projective curves, it identifies the expected vanishing order of a valuation of linear growth with the energy of its Dirac mass. Valuative thresholds of stability, for which these invariants are inputs, are not treated.

Monotonicity is meant in the weak sense: a decreasing function or net is one that never increases, and the strict notion is called strictly decreasing; likewise for increasing. The trivial valuation of a function field, viewed as a point of an analytification, is written $v_0$.

We allow arbitrary trivially valued fields in the preliminary constructions. Later hypotheses on the field, singularities, or positivity are stated where the arguments use them. Results from the two books above and from the Stacks Project are cited with precise locators, and the appendices state the foundational theorems they import, such as resolution of singularities and the existence of Hilbert schemes, with references. Apart from standard background in topology, analysis and algebra, the other results used in the main text are proved in the book. Worked examples include their calculations and proofs.

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# Introduction

## The projective line

Let $k$ be a field with the trivial absolute value. The analytification of $\mathbb{P}^1_k$ has one point $v_0$ for the trivial valuation of $k(\mathbb{P}^1)$. For every closed point $p$ there is a ray of points $x_{p,t} = t \operatorname{ord}_p$, $0 < t < \infty$, ending at the point $x_{p,\infty}$ that evaluates functions at $p$. By Theorem 1.4.1 these are all the points, and the space is the one-point compactification of the disjoint union of the rays, a star with one branch for every closed point.

A section $s$ of $O(1)$ vanishing at a rational point $p$ has log norm $\log |s|(x_{p,t}) = -t$ on the $p$-ray and zero on all the other rays. Maxima of shifted log norms of sections of $O(m)$, divided by $m$, are the Fubini–Study potentials. On each ray they are convex, piecewise affine and decreasing. Their decreasing limits, other than the constant $-\infty$, are the plurisubharmonic potentials of the class of $O(1)$ (Definition 6.2.1 and Proposition 6.4.1). They are exactly the functions finite at $v_0$ that are convex and decreasing on each finite ray, take at each endpoint $x_{p,\infty}$ the limit along the ray, and have total initial slope at least $-1$ when the slopes are weighted by residue degrees (Theorem 20.4.1 with $V = 1$). The Monge–Ampère measure of a continuous such potential, and more generally of one of finite energy, is the Dirac mass at $v_0$ plus the distributional Laplacian along the rays (Theorem 20.5.1 and Corollary 20.5.1). For a probability measure of finite energy, the normalized solution of the Monge–Ampère equation is, on the ray of $p$, minus the integral up to each radius of the mass of the measure beyond that radius, divided by the residue degree of $p$. A probability measure has finite energy exactly when the squares of these masses, divided by the residue degrees, have finite integral over the union of the rays. The energy of the solution is then $-1/2$ times this integral, which is the degree-weighted Dirichlet integral of its slopes (Theorem 31.2.1).

Many constructions of the book can already be seen on this star. Blowing up the point $(p, 0)$ of the central fibre of $\mathbb{P}^1 \times \mathbb{A}^1$, or more generally the flag ideal generated by $\mathfrak{m}_p^b$ and $\tau^a$, where $\tau$ is the coordinate of $\mathbb{A}^1$, creates a central component whose valuation is $x_{p,a/b}$ (Section 1.5 and Lemma 20.3.1; see Chapter 5 for the general construction). Its multiplicity and its intersection degrees give the atomic measures

of Chapter 17. The coordinate $-\log|x|$ of a local equation $x$ of $p$ identifies the $p$-ray with a half-line. Real forms in this coordinate compute the same measures locally (Chapters 10 and 20). Over $\mathbb{C}$, the transcendental construction of Chapter 25 evaluates the multiplier ideals of singular metrics at divisorial valuations, here the points $x_{p,t}$ with $0 < t < \infty$, and so produces functions on the rays. Finally, the order of vanishing at a rational point $p$ filters the sections of $O(m)$. Its average normalized value is $1/2$, which is the energy of the Dirac mass at $\mathrm{ord}_p$ (Corollary 34.4.1).

## The main results

The first part develops semivaluations, affinoid algebras of arbitrary radii, divisorial valuations, Radon measures, test configurations and plurisubharmonic potentials in real numerical classes. Psh potentials are defined as decreasing limits of Fubini–Study potentials whose classes may vary, and they are determined by their values at divisorial points (Theorem 6.3.1).

The second part introduces tropical charts, in which log norms of units serve as coordinates. In these coordinates the tropicalization of a pure-dimensional closed subscheme of a split torus is a balanced weighted fan (Theorem 8.7.1). A moment of an affinoid space of dimension at most $n$ into an $n$-dimensional annular box has finite fibres over the Gauss points (Corollary 9.3.1). At each point of these fibres in the relative interior of the moment, it restricts to a finite free chart with that point alone over the Gauss point (Theorem 9.4.1). Real forms on these charts carry a notion of positivity and a Stokes formula, and they define local Monge–Ampère products on balanced charts (Chapters 10 and 11).

The third part proves the *envelope property*: the upper semicontinuous regularization of the pointwise supremum of a nonempty uniformly bounded-above family of psh potentials is psh. For an integral projective variety over an algebraically closed trivially valued field it holds in every real numerical class exactly when the variety is unibranch (Corollary 16.5.1 and Proposition 12.3.1). The proof works in every characteristic. It starts from smooth projective varieties, where Frobenius coefficient ideals or multiplier ideals give decreasing approximations, regularizes locally on affine charts using projective space, and descends along the Stein factorization of a smooth alteration and through the normalization.

The fourth part constructs mixed Monge–Ampère measures and the energy. On an integral projective variety, in every ample class with the envelope property, in particular on unibranch varieties over an algebraically closed field and on integral regular projective curves, it proves the orthogonality of envelopes (Theorem 18.3.1) and solves the normalized Monge–Ampère equation for every probability measure of finite energy (Theorem 19.6.1). Over an arbitrary trivially valued field and in a real ample class, solvability for all these measures is equivalent to the envelope property (Corollary 19.7.1). On integral regular projective curves the global measure of a continuous or finite-energy potential agrees with the curvature current of the local theory (Theorem 20.5.1 and Corollary 20.5.1).

The fifth part works on compact complex spaces. Coherent ideal sheaves give a space of semivaluations, and flag ideals on the product with the projective line give model functions and cohomological measures. In a Kähler class on a connected compact Kähler manifold, the Monge–Ampère equation has a unique normalized solution for every probability measure of finite energy (Theorem 23.6.1), and the normalized finite-energy potentials correspond order-isomorphically, with equal energy, to the normalized finite-energy $\mathcal{I}$-model test curves with left limit zero (Theorem 25.10.1). On smooth connected complex projective varieties, divisorial evaluation identifies the compatible families of $\mathcal{I}$-model test curves order-isomorphically with the algebraic psh potentials of the first part, in every pseudoeffective class spanned by line bundles (Theorem 25.7.1). The variational theory extends to nef classes of positive volume and to every big class on a connected compact Kähler manifold, relative to a minimal singularity type (Theorems 27.4.1 and 30.3.2).

The last part computes solutions on curves and toric varieties and identifies the energy of a Dirac mass in the class of an ample line bundle with the expected order of vanishing, along the valuation, of sections of the powers of that line bundle, for valuations of linear growth on integral projective varieties over an algebraically closed field and on integral regular projective curves (Theorem 34.3.2).

## Historical development

Berkovich constructed analytic spaces over every complete non-Archimedean field as spaces of multiplicative seminorms, and his construction includes the trivial absolute value [Ber90, Introduction, pp. 1 and 5, and § 3.5]. Over a trivially valued field, Thuillier interpreted toroidal embeddings analytically and showed, over a perfect ground field, that the homotopy type of the dual complex of the boundary of a resolution depends only on the original pair [Thu07, Théorème 4.8]. Spaces of real valuations became a tool in complex analysis and birational geometry, for instance through the valuative tree of Favre and Jonsson [FJ04], a space of valuations on formal power series in two variables, and through the analysis of asymptotic invariants of graded sequences of ideals by Jonsson and Mustaţă [JM12]. Boucksom, Küronya, Maclean and Szemberg introduced valuations of linear growth in their study of vanishing sequences [Bou+15].

In arbitrary dimension, Chambert-Loir defined measures on Berkovich spaces by intersecting metrized line bundles on models and used them to prove equidistribution [CL06]. Chambert-Loir and Ducros developed real differential forms and currents on Berkovich spaces [CLD25]. Gubler and Künnemann introduced tropical $\delta$-forms and recovered these measures from them [GK17]. Boucksom, Favre and Jonsson developed singular semipositive metrics over discretely valued fields of residue characteristic zero [BFJ16], and solved the non-Archimedean Monge–Ampère equation by a variational method, under an algebraicity assumption and a condition on the support of the measure [BFJ15]. Their continuity of envelopes rests on multiplier ideals. Gubler, Jell, Künnemann and Martin used test ideals in positive characteris-

tic [Gub+19]. Together with Burgos Gil they proved that the volume is differentiable with the Monge–Ampère measure as derivative, which removed the algebraicity assumption in residue characteristic zero [BG+20].

Over a trivially valued field, Boucksom and Jonsson developed singular semipositive metrics, finite-energy pluripotential theory and the Monge–Ampère equation on projective varieties [BJ22]. Test configurations give model functions over the trivial valuation, which connects the theory with K-stability. Boucksom, Hisamoto and Jonsson expressed invariants of uniform K-stability through non-Archimedean functionals of test configurations [BHJ17], and Berman, Boucksom and Jonsson gave a variational proof of a version of the Yau–Tian–Donaldson conjecture [BBJ21]. Blum and Jonsson studied the valuative thresholds defined through expected vanishing orders [BJ20].

The transcendental side starts from rays of potentials on a compact Kähler manifold. Ross and Witt Nyström related analytic test configurations to geodesic rays [RWN14]. Darvas and Xia characterized the rays approximable by test configurations and the closure of algebraic singularity types [DX22], and Darvas, Xia and Zhang constructed non-Archimedean metrics of pseudoeffective classes from maximal test curves [DXZ24]. Mesquita-Piccione constructed the analogue of the analytification for complex spaces [MP25], and with Witt Nyström proved a transcendental Calabi–Yau theorem [MPWN25]. The pluripotential background is in [Xia26a], and the theory of singularity types and test curves is in [Xia26b].

The book develops the valuative, model, tropical and transcendental approaches side by side and compares them where their domains overlap. The source notes in Appendix G trace individual results to their sources and record where a proof here differs from the original.

# Part I
# Preliminaries

To construct a measure on an analytification, we need both a supply of test functions and a topology on which to integrate them. The first chapter constructs analytic points and proves compactness for proper schemes. The second supplies the analytic algebras that let us work on domains and extend the ground field.

Divisorial valuations enter through normalized blowups in the third chapter. Their density makes it possible to approximate inequalities at an arbitrary analytic point by inequalities at divisor orders. The next chapter develops the directed limits and Radon measures that we will use on analytifications, and the convex analysis that we will use on tropical charts and polytopes. Three projective chapters supply the model functions and approximation results needed for the envelope theorem.

# Chapter 1
# Semivaluations and analytic points

For each closed point $p$ of the projective line, the order of vanishing at $p$ can be multiplied by any positive real number. These multiples form a ray in the analytification. Its finite points measure vanishing on rational functions, and its endpoint records evaluation at $p$. All rays start at the trivial valuation of the function field. We will construct this space and see how a function on one ray gives a calculation with both slopes and intersection degrees.

Throughout this chapter $k$ is a field with the trivial absolute value $|a|_k = 1$ for $a \neq 0$ and $|0|_k = 0$. We allow arbitrary characteristic and use $-\log 0 = +\infty$. In this chapter an algebraic analytification means only its underlying topological space. Chapter 2 constructs the analytic chart algebras and their spectra.

We construct analytic points by gluing affine spectra and use the valuative criterion to assign centres on a proper scheme. The compactness theorem in Theorem 1.3.1 provides the space on which our global measures live, and the centres of Section 1.3 are used in the following chapters to evaluate coherent ideals. On a proper regular curve every point can be described by its place on a single ray, by Theorem 1.4.1. On the projective line this turns the calculation on a blowup into a calculation of the two slopes of a function.

## 1.1 Semivaluations on an affine scheme

**Definition 1.1.1** Let $A$ be a $k$-algebra. A *$k$-semivaluation* on $A$ is a map $v\colon A \to \mathbb{R} \cup \{+\infty\}$ satisfying

$$v(0) = +\infty, \qquad v(a) = 0 \quad (a \in k^\times),$$
$$v(fg) = v(f) + v(g), \qquad v(f+g) \geq \min\{v(f), v(g)\}.$$

Addition of a finite value to $+\infty$ gives $+\infty$, and $(+\infty)+(+\infty) = +\infty$. The corresponding multiplicative seminorm is $|f|_v = \exp(-v(f))$. The set of all such semivaluations

is denoted $(\operatorname{Spec} A)^{\mathrm{an}}$. It has the topology of pointwise evaluation of the functions $v \mapsto |f|_v$ for $f \in A$.

These conditions give the ultrametric triangle inequality

$$|f+g|_v \leq \max\{|f|_v, |g|_v\}.$$

If the two values $v(f)$ and $v(g)$ differ, applying the semivaluation inequality also to $f = (f+g) - g$ or to $g = (f+g) - f$ gives equality in the minimum formula. By induction, a finite sum with a unique term of smallest finite valuation has that valuation.

**Proposition 1.1.1** *A semivaluation on $A$ consists of a prime $\mathfrak{p} \subset A$ and a real-valued valuation on $\kappa(\mathfrak{p})$ that is trivial on $k$. Here a valuation on a field is finite on the nonzero elements. The prime $\mathfrak{p} = \{f \in A \mid v(f) = +\infty\}$ is the* support *of $v$.*

***Proof*** The semivaluation inequalities show that the indicated set is an ideal. It is proper since $v(1) = 0$, and it is prime by multiplicativity. If $f-g$ belongs to this ideal, applying the inequality to $f = g + (f-g)$ and to $g = f - (f-g)$ gives $v(f) = v(g)$. The induced semivaluation on the domain $A/\mathfrak{p}$ is finite away from zero. It extends uniquely to its fraction field by $v(f/g) = v(f) - v(g)$ for $g \neq 0$. Cross-multiplication proves well-definedness, and writing a sum over a common denominator gives the inequality for sums. Restriction from a valuation on $\kappa(\mathfrak{p})$, with value $+\infty$ on the kernel, is the inverse construction. □

The completion of this valued residue field is denoted $\mathcal{H}(v)$. Its norm extends the original one by continuity. A Cauchy sequence with nonzero limit has eventually constant positive norm by the ultrametric inequality, so the inverses of its terms are eventually defined and form a Cauchy sequence. This shows that the completion is still a field.

A closed subscheme defined by an ideal $I \subset A$ yields the closed subspace

$$(\operatorname{Spec}(A/I))^{\mathrm{an}} = \{v \in (\operatorname{Spec} A)^{\mathrm{an}} \mid |f|_v = 0 \text{ for all } f \in I\}.$$

The topology agrees with the subspace topology because every element of $A/I$ lifts to $A$. In particular, passing to the reduction leaves the space of points and its topology unchanged. For $f \in A$, the principal open $D(f)$ gives the open subspace

$$D(f)^{\mathrm{an}} = \{v \in (\operatorname{Spec} A)^{\mathrm{an}} \mid |f|_v > 0\}.$$

Indeed, on this open the seminorm has the unique extension $|a/f^m|_v = |a|_v/|f|_v^m$, and these evaluations are continuous. Thus the support map to $\operatorname{Spec} A$ with its Zariski topology is continuous. Each scheme point $\xi$ has a distinguished point above it obtained by giving $\kappa(\xi)$ the trivial valuation. We denote this semivaluation by $v_{\xi,\mathrm{triv}}$.

**Lemma 1.1.1** *Every real-valued valuation on an algebraic extension of $k$ that is trivial on $k$ is trivial.*

***Proof*** For a nonzero algebraic element $a$, choose a polynomial relation with nonzero constant and leading coefficients. If $v(a) > 0$, the constant term has the unique smallest valuation in that relation. If $v(a) < 0$, the leading term does. Both possibilities contradict the fact that the sum is zero. □

## 1.2 Gluing and compactness

For a $k$-scheme $X$ locally of finite type, the points of $X^{\mathrm{an}}$ are pairs $(\xi, v)$ where $\xi \in X$ and $v$ is a valuation of $\kappa(\xi)$ trivial on $k$. To glue the affine topologies, consider two affine opens $W \subset U$. Principal opens of $U$ contained in $W$ cover $W$. Each such open $D(f)$ is also principal in $W$, defined by $f|_W$. The localization formula gives the same topology on $D(f)^{\mathrm{an}}$ in both charts. This proves that $W^{\mathrm{an}}$ embeds as the open subset of $U^{\mathrm{an}}$ with support in $W$. Covering the intersection of any two affine charts by affine opens inside the intersection now gives agreement on the overlap. In particular, $U^{\mathrm{an}}$ is open in $X^{\mathrm{an}}$ for every Zariski open $U$.

A $k$-morphism $g\colon X \to Y$ induces a continuous map $g^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$. It sends a pair with support $\xi$ to the restriction of its valuation along $\kappa(g(\xi)) \hookrightarrow \kappa(\xi)$. Locally on source and target this just evaluates the pullbacks of regular functions. Restriction also proves compatibility with composition. For later use, multiplication by $s > 0$ defines a continuous action $v \mapsto sv$ on $X^{\mathrm{an}}$. On an affine the seminorm is raised to the power $s$, and $(s, r) \mapsto r^s$ is continuous on $\mathbb{R}_{>0} \times [0, \infty)$, including at $r = 0$.

**Lemma 1.2.1** *Let $f_1, \ldots, f_m$ generate $A$ as a $k$-algebra. For positive real numbers $R_1, \ldots, R_m$, the space*

$$B(A; f_i, R_i) = \{v \in (\operatorname{Spec} A)^{\mathrm{an}} \mid |f_i|_v \leq R_i \text{ for } 1 \leq i \leq m\}$$

*is compact Hausdorff.*

***Proof*** If $A = 0$, the set is empty. Otherwise, for each $f \in A$ choose a polynomial expression in the $f_i$. A monomial with multi-index $\alpha$ has seminorm at most $\prod_i R_i^{\alpha_i}$ when its coefficient is nonzero. Thus the chosen expression gives a finite bound $M_f$ on $|f|_v$ for $v \in B(A; f_i, R_i)$. In the compact Hausdorff product $\prod_{f \in A}[0, M_f]$, the conditions $|0| = 0$, $|a| = 1$ for $a \in k^\times$, multiplicativity, the ultrametric inequalities and the bounds on the generators are closed conditions. They cut out exactly the displayed space with its evaluation topology. □

**Proposition 1.2.1** *If $X$ is a separated scheme locally of finite type over $k$, then $X^{\mathrm{an}}$ is Hausdorff and locally compact.*

***Proof*** Each affine chart is Hausdorff for the evaluation topology. A point in a finite-type affine chart has a compact neighborhood there by Lemma 1.2.1, taking each $R_i$ strictly larger than the generator's value at the point.

To separate points $x \in U^{\mathrm{an}}$ and $y \in V^{\mathrm{an}}$ for finite-type affine opens $U, V$, consider $W = U \cap V$. The diagonal of the separated scheme is closed, so $W$ is a closed

subscheme of $U \times_k V$. The images of generators for $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ therefore generate $\mathcal{O}_X(W)$. Choose compact generator-bounded neighborhoods $K_U, K_V$ of $x, y$ in their respective charts. The set of points in $W^{\mathrm{an}}$ whose restrictions belong to $K_U, K_V$ is compact by Lemma 1.2.1 for the combined generators. Its image in $U^{\mathrm{an}} \times V^{\mathrm{an}}$ is compact and closed. If $x \neq y$ as points of $X^{\mathrm{an}}$, the pair $(x, y)$ lies outside this image. Take a product neighborhood disjoint from the image and intersect its factors with the interiors of $K_U, K_V$ in the two charts. These open neighborhoods have empty intersection in $X^{\mathrm{an}}$, since a common point would belong to $W^{\mathrm{an}}$ and give a pair in the image. If $W$ is empty, the two charts already separate the points. □

## 1.3 Centres

Let $x = (\xi, v) \in X^{\mathrm{an}}$. The valuation ring of $v$ is

$$R_v = \{a \in \kappa(\xi) \mid v(a) \geq 0\}, \qquad \mathfrak{m}_v = \{a \in \kappa(\xi) \mid v(a) > 0\}.$$

A *centre* of $x$ on $X$ is the image of the closed point of $\operatorname{Spec} R_v$ under a $k$-morphism $\operatorname{Spec} R_v \to X$ extending the given map $\operatorname{Spec} \kappa(\xi) \to X$. The centre is a specialization of $\xi$. If $X$ is separated such an extension is unique when it exists, by the valuative criterion for separatedness [Stacks, Tag 01KZ].

For an affine open $U = \operatorname{Spec} A$, the point $x$ has a centre in $U$ if and only if the support belongs to $U$ and $v(f) \geq 0$ for every $f \in A$. Indeed, a morphism from $\operatorname{Spec} R_v$ with closed point in $U$ factors through $U$, since an open subset of the spectrum of a local ring containing its closed point is the whole spectrum. Conversely the inequalities say exactly that the map $A \to \kappa(\xi)$ factors through $R_v$. The inverse image of $\mathfrak{m}_v$ in $A$ is then the prime of the centre. For any scheme point $\xi$, the distinguished point $v_{\xi,\mathrm{triv}}$ has support and centre both equal to $\xi$.

**Theorem 1.3.1** *If $X$ is proper over $k$, every point of $X^{\mathrm{an}}$ has a unique centre and $X^{\mathrm{an}}$ is compact Hausdorff.*

***Proof*** The valuative criterion for properness [Stacks, Tag 0BX5] supplies the extension over $R_v$. Fix a finite affine cover of $X$ and finite algebra generators $f_i$ on each affine. By the criterion above, the points whose centre lies in one such affine $U$ form the generator-bounded subset $B(\mathcal{O}_X(U); f_i, 1)$. In fact the ultrametric inequality and the trivial norm on $k$ extend the bound from the generators to every regular function on the affine. These compact subsets cover $X^{\mathrm{an}}$ since every point has a centre. Hausdorffness follows from Proposition 1.2.1. □

A centre also lets us evaluate a coherent ideal at an analytic point. Suppose $X$ is proper, $\mathfrak{a}$ is a coherent ideal, and $c \in X$ is the centre of $x$. The homomorphism $\mathcal{O}_{X,c} \to R_v$ lets us set $v(\mathfrak{a}) = \min_j v(a_j)$ for any finite generating family $a_j$ of $\mathfrak{a}_c$, with $v(\mathfrak{a}) = +\infty$ for the zero stalk. Every linear combination has value at least this minimum, because the coefficients have nonnegative value. This proves independence of the generators. In particular a unit at the centre has valuation zero.

## 1.4 The rays of a proper regular curve

Let $C$ be an integral regular proper curve over $k$, with generic point $\eta$. For a closed point $p$, let $\mathrm{ord}_p$ be the valuation of the discrete valuation ring $\mathcal{O}_{C,p}$, normalized by value 1 on a uniformizer. Write

$$v_0 = v_{\eta,\mathrm{triv}}, \qquad x_{p,t} = t\,\mathrm{ord}_p \quad (0 < t < \infty), \qquad x_{p,\infty} = v_{p,\mathrm{triv}}.$$

All points $x_{p,t}$ with $t > 0$ have centre $p$. Their support is $\eta$ for finite $t$ and $p$ for $t = \infty$.

**Theorem 1.4.1** *The points of $C^{\mathrm{an}}$ are $v_0$ and the $x_{p,t}$ above, each occurring exactly once. The complement of $v_0$ is the disjoint union of the rays $(0, \infty]$ indexed by closed points $p$. Its one-point compactification is $C^{\mathrm{an}}$. In particular, a basis of neighborhoods of $v_0$ is obtained by keeping the full ray for all but finitely many $p$ and keeping only $0 < t < \varepsilon_p$ for those finitely many rays.*

***Proof*** If the support is a closed point $p$, its residue field is finite over $k$, and Lemma 1.1.1 gives the point $x_{p,\infty}$. Otherwise the support is $\eta$. By Theorem 1.3.1 the point has a unique centre on the proper curve $C$. If the centre is also $\eta$, every nonzero element of $k(C)$ is a unit at the centre, giving $v_0$. If the centre is a closed point $p$, let $z$ be a uniformizer there. Every element of $k(C)^\times$ is $uz^m$ with $u$ a unit and $m \in \mathbb{Z}$. Since $v(u) = 0$ and $v(z) > 0$, the valuation is $v(z)\,\mathrm{ord}_p$. Conversely these multiples have valuation ring $\mathcal{O}_{C,p}$ and centre $p$. This gives the asserted bijection.

The disjoint union of the locally compact rays has the stated neighborhood basis in its one-point compactification. A compact subset of the disjoint union meets finitely many rays, and on each it is bounded away from $t = 0$.

To check continuity of the bijection on a ray, choose a nonempty affine $U \subset C$ containing $p$. For a nonzero regular function $f$ on $U$, its seminorm at finite $t$ is $\exp(-t\,\mathrm{ord}_p(f))$, and the value at the endpoint is its limit as $t \to \infty$. If $\mathrm{ord}_p(f) = 0$, this limit is 1. If the order is positive, the limit is 0. The zero function has constant value zero. At $v_0$, work in any nonempty affine $U$. Its complement in $C$ consists of finitely many closed points. A basic neighborhood of $v_0$ inside $U^{\mathrm{an}}$ imposes conditions on finitely many functions. Every nonzero such function has nonzero order at only finitely many closed points of $C$. On all other rays with $p \in U$, the evaluations equal those at $v_0$. On the finitely many exceptional rays, the evaluations on their finite points tend to those at $v_0$ as $t \to 0$. Shortening these rays also excludes the endpoints outside $U$. This proves continuity at $v_0$. The domain is compact and the target is Hausdorff by Theorem 1.3.1, so the bijection is a homeomorphism. □

On $\mathbb{P}^1_k$, the rays are indexed by the monic irreducible polynomials in $k[z]$ and by the point at infinity. For example, along the ray of the rational point $p = \{z = 0\}$, the analytic value of $z$ is $e^{-t}$ at $x_{p,t}$. Along the infinity ray, the value of $1/z$ is $e^{-t}$. Both tend to 1 towards the common origin in their respective charts.

**Corollary 1.4.1** *The space $C^{\mathrm{an}}$ is first countable at $v_0$ if and only if the set of closed points of $C$ is countable.*

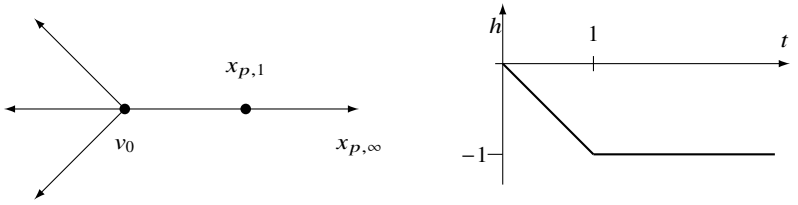
***Proof*** For a countable set of rays the basic neighborhoods with rational cutoffs form a countable family. Conversely, in any proposed countable neighborhood basis, choose a basic neighborhood inside each member of that basis. The union of the finite sets of rays shortened by these choices is countable. If some ray remains outside this union, every member of the proposed basis contains it in full. The neighborhood obtained by shortening that ray then contains no member of the basis. □

## 1.5 A slope and intersection calculation

Take $C = \mathbb{P}^1_k$, a $k$-rational point $p$, and let $\mathfrak{m}_p$ denote its ideal sheaf on $C$. The function

$$h(x) = -\min\{v(\mathfrak{m}_p), 1\} \quad (x = (\xi, v))$$

is zero on all other rays and at $v_0$. On the $p$-ray it is $-\min\{t, 1\}$, with value $-1$ at $x_{p,\infty}$. The ray description makes its continuity immediate. Figure 1.1 shows the ray and the profile of $h$ on it.




Analytic rays Profile on the $p$-ray

**Fig. 1.1** The other rays have value zero. On the $p$-ray, the slope is $-1$ up to $x_{p,1}$ and zero beyond it.

For the slope calculation, take a tree formed by $v_0$ and finitely many initial segments $0 \le t \le t_p$ of rays. Use all breakpoints of the function as vertices and extend it constantly along each omitted tail. At each vertex put a point mass with weight the sum of the outgoing slopes there, measured in $t$. On the ray of a closed point $q$ each slope is multiplied by $[\kappa(q) : k]$ (see (20.3)); here $p$ is $k$-rational, so no such factor enters. The only nonzero terms for $h$ come from $v_0$ and $x_{p,1}$. The first is $-1$, from the direction towards $p$. The second is 1, from the direction towards $v_0$; the opposite direction contributes zero. With $\delta_x$ the unit point mass at $x$, the slope measure is therefore

$$\Delta h = \delta_{x_{p,1}} - \delta_{v_0}. \tag{1.1}$$

The same difference arises from a blowup. Let $\tau$ be the coordinate on $\mathbb{A}^1_k$ and $\pi\colon C \to C \times \mathbb{A}^1_k$ the blowup of $(p,0)$, with exceptional divisor $E$. The ideal of the blown-up point $(p,0)$ pulls back to $\mathcal{O}_C(-E)$. Near $(p,0)$ this ideal is $(z,\tau)$ for a uniformizer $z$ at $p$. For $x = (\xi, v)$ with centre $p$, extend $v$ to the polynomial ring in $\tau$ by the rule

$$w\left(\sum_i a_i \tau^i\right) = \min_i\{v(a_i) + i\}.$$

Here coefficients from the local ring are evaluated in $\kappa(\xi)$, and the minimum for the zero polynomial is $+\infty$. The ultrametric inequality holds coefficientwise. To check multiplicativity for two nonzero polynomials over the residue field, pick the least index attaining the minimum in each. In the coefficient with index equal to their sum, their product contributes the unique term of least value after adding the index. Indeed, any other pair of indices with that sum has at least one index smaller than its chosen minimum index, and so gives larger weighted value. The ultrametric inequality gives the lower bound on all other coefficients, proving multiplicativity. Thus the negative of the minimum of $w(z)$ and $w(\tau)$ is $h(x)$. The same rule defines $w$ for every $x$. If the centre of $x$ on $C$ is not $p$, the centre of $w$ lies over it, the ideal of $(p,0)$ is the unit ideal there, and its value is $0 = h(x)$.

The central fibre of $C$ has components $E$ and the strict transform $F$ of $C \times \{0\}$, both with multiplicity 1. In the chart $\tau = zu$, these components are given by $z = 0$ and $u = 0$ and meet transversely. In the other blowup chart the equation of the central fibre is $\tau = 0$. The map $F \to C$ is an isomorphism, as is seen by using $z$ as coordinate on $u = 0$ near its one point above $p$. Put

$$\mathcal{L} = \pi^* \operatorname{pr}_1^* \mathcal{O}_C(1) \otimes \mathcal{O}_C(-E).$$

The pullback factor has degree zero on $E$. The restriction $\mathcal{O}_C(-E)|_E$ is the tautological $\mathcal{O}_{\mathbb{P}^1}(1)$ of the blowup construction. Hence $\deg(\mathcal{L}|_E) = 1$. The transverse intersection with $E$ subtracts one from the pullback degree on $F$, so $\deg(\mathcal{L}|_F) = 0$.

With central-fibre multiplicity 1, the valuations attached to these components (Section 5.3, where $v_E = b_E^{-1} \operatorname{ord}_E|_{k(C)}$ for a component of multiplicity $b_E$) are the restrictions of $\operatorname{ord}_E$ and $\operatorname{ord}_F$ to $k(C)$. They are $x_{p,1}$ and $v_0$, respectively. The first restriction follows from $\operatorname{ord}_E(z) = 1$ and the second from the fact that $F$ dominates $C$. Putting each component degree at its valuation gives the measure $\delta_{x_{p,1}}$ (the model measure (17.1); compare Example 17.3.1). On the trivial product the single component has degree 1 and valuation $v_0$, so the corresponding measure is $\delta_{v_0}$. Their difference is exactly (1.1).

# Chapter 2
# Analytic domains and scalar extension

On a ray of the projective line, a bound on the order of vanishing becomes a bound on the absolute value of a coordinate. Completing polynomials for this bound yields a ring of analytic functions. The real number in the bound becomes part of the ring's norm. For example, over a trivially valued field the rings of functions on two different closed discs of radius less than one are both $k[[T]]$, but the restriction between them changes the norm.

We begin over an arbitrary complete ultrametric field $K$. This lets us use the same constructions after a scalar extension. When we compute along the rays of $\mathbb{P}^1_k$, the field $k$ again has its trivial absolute value.

The spectral radius formula in Theorem 2.2.1 relates the norm on an algebra to the values at its analytic points. We then identify the spectra of rational domain algebras and develop two changes of algebra that will be needed locally. The fibres of a finite map are spectra of finite algebras over completed residue fields (Theorem 2.3.1), and complete scalar extension is surjective on spectra even when the extension is transcendental (Proposition 2.4.1). These constructions explain the different ways in which the boundary of an annulus appears in Section 2.6, where Proposition 2.6.1 computes the analytic boundaries of discs and annuli.

## 2.1 Tate algebras of arbitrary radii

For a tuple $r = (r_1, \ldots, r_n)$ of positive real numbers and $\alpha \in \mathbb{N}^n$, put $r^\alpha = \prod_i r_i^{\alpha_i}$. Define $K\{r^{-1}T\}$ to consist of the series

$$F = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \qquad |a_\alpha| r^\alpha \longrightarrow 0 \quad \text{as } \alpha_1 + \cdots + \alpha_n \longrightarrow \infty.$$

Its *Gauss norm* is $\|F\|_r = \max_\alpha |a_\alpha| r^\alpha$, with value zero for the zero series. The convergence condition means that for every $\varepsilon > 0$ only finitely many terms have weight at least $\varepsilon$. In particular the maximum for a nonzero series is attained.

**Proposition 2.1.1** *The algebra $K\{r^{-1}T\}$ is the completion of the polynomial ring for the Gauss norm. Its Gauss norm is multiplicative.*

***Proof*** The ultrametric inequality is coefficientwise. On polynomials, each term contributing to a coefficient of a product has weight bounded by the product of the two Gauss norms. This proves submultiplicativity. For nonzero factors, choose the lexicographically largest exponent attaining the maximum in each factor. The sum of these two exponents has exactly one contribution of maximal possible weight. Indeed, a second pair with the same sum would have one exponent lexicographically larger than its chosen one. Since this contribution is the unique term of maximal weight, the ultrametric inequality is an equality, so the coefficient at the sum has weight exactly $\|F\|_r\|G\|_r$.

The space of series is complete. For a Cauchy sequence of series, the coefficients converge in $K$, and the uniform Gauss bounds on differences pass to the limits of the coefficients. Approximating by one series in the sequence shows that the limit has only finitely many terms above any positive threshold. Polynomials are dense by truncation. Multiplication and its norm identity therefore extend to this completion.□

A homomorphism of normed $K$-algebras is *bounded* if $\|\phi(a)\| \leq C\|a\|$ for a fixed constant $C$ and all $a$. Suppose $B$ is a complete normed $K$-algebra with a submultiplicative ultrametric norm. For $b_i \in B$, a bounded map from $K\{r^{-1}T\}$ to $B$ sending $T_i$ to $b_i$ exists if and only if each sequence $\left(r_i^{-m}\|b_i^m\|\right)_{m\geq 0}$ is bounded. Necessity follows by applying the map to $T_i^m$. For sufficiency, submultiplicativity bounds $\|b^\alpha\|$ by $Cr^\alpha$ with $C$ the product of the individual bounds. Substitution on polynomials then extends uniquely by completeness. In particular $\|b_i\| \leq r_i$ suffices.

The same series construction over a complete normed algebra $A$ gives $A\{r^{-1}T\}$. It has a submultiplicative norm and the corresponding substitution rule for bounded $A$-algebra maps, where the bound for the map on $A$ enters the estimate.

**Theorem 2.1.1** *Every submodule of $K\{r^{-1}T\}^d$ is finitely generated and closed for the maximum of the d Gauss norms. In particular a Tate algebra is Noetherian and all its ideals are closed.*

***Proof*** Write $P = K\{r^{-1}T\}$ and let $e_1, \ldots, e_d$ be the standard basis of $P^d$. Order the pairs $(\alpha, i)$ by lexicographic order of $\alpha$, then by the index $i$. This is a well-order compatible with shifting $\alpha$ by any fixed element of $\mathbb{N}^n$. The leading monomial of a nonzero vector is the largest pair among its terms of maximum weight. Multiplication by $T^\beta$ shifts this leading pair by $\beta$.

Let $M \subset P^d$ be a submodule. For each $i$, the set of exponents for leading monomials in the $i$-th component is upward closed for coordinatewise comparison. It is a finite union of translates of $\mathbb{N}^n$. One way to see this is that every infinite sequence in $\mathbb{N}^n$ has an infinite coordinatewise increasing subsequence, by successively extracting one for each coordinate. The minimal elements are therefore finite in number, and every element lies above one of them. Choose a finite list $G_1, \ldots, G_m \in M$ realizing all these minimal leading pairs.

We divide a vector into a sum of multiples of the $G_j$ and a remainder. For a single term $bT^\gamma e_i$ whose pair is divisible by a leading pair of some $G_j$ in that component,

use the first such $G_j$ in our fixed list. There is a unique scalar monomial $Q$ such that the leading term of $QG_j$ is $bT^\gamma e_i$. Write the original term as $QG_j$ plus the negatives of the remaining terms of $QG_j$. Record $Q$ in the quotient for $G_j$, and apply the procedure to the remaining terms. A term with indivisible pair goes into the remainder.

Here the division can be performed on each original term separately, with convergent sums. Indeed, in any replacement all terms have weight at most the replaced weight. Children of equal weight have strictly smaller pairs in the well-order. Among the terms of every $G_j$ whose weight is strictly below its maximum, the ratio to that maximum is bounded by a constant $c < 1$, common to the finite list. This follows from the convergence condition on the series. If there are no such terms, take any $c \in (0, 1)$. Every replacement weight decrease thus has ratio at most $c$. For a fixed positive threshold $\varepsilon$, there are finitely many original terms of weight at least $\varepsilon$, and each term has only finitely many children above that threshold. An infinite branch above the threshold would have only finitely many strict weight decreases, followed by an infinite strictly decreasing sequence of pairs, which is impossible. A finitely branching tree with infinitely many vertices would have an infinite branch, by repeatedly choosing a child with infinitely many descendants. Hence only finitely many replacement terms have weight above any positive threshold.

The quotient terms for $G_j$ have weight equal to the replaced weight divided by $\|G_j\|$. The quotient series and remainder consequently converge, and recording the replacements gives

$$F = \sum_{j=1}^{m} Q_j G_j + R(F), \qquad \|R(F)\| \leq \|F\|.$$

To justify the identity, perform all replacements above a positive threshold first. The unreplaced part then has norm at most that threshold, and the recorded partial series converge as the threshold tends to zero. The chosen rule depends only on the exponent and the component of each term. Division of separate monomials and summation therefore make the remainder $R$ a bounded $K$-linear map. Its support contains only indivisible pairs. If $F \in M$, then the displayed identity puts $R(F)$ in $M$, where a nonzero vector must have divisible leading pair. Thus $R(F) = 0$. Conversely $R(F) = 0$ expresses $F$ as a sum of multiples of the $G_j$. This proves finite generation and identifies $M$ with the closed kernel of $R$. □

A *$K$-affinoid algebra* is a Banach $K$-algebra of the form $K\{r^{-1}T\}/I$, with its quotient norm or an equivalent Banach algebra norm. We take the norm to be ultrametric, submultiplicative and normalized by $\|1\| = 1$ for a nonzero algebra. It is *strictly $K$-affinoid* if it admits such a presentation with all radii equal to one. A quotient by a closed linear subspace of a Banach space is complete. This follows by lifting successive differences in a rapidly Cauchy subsequence by vectors with norms tending to zero. The sum of the lifted differences converges and gives a lift of the quotient limit. Thus Theorem 2.1.1 ensures that these quotient norms give Banach algebras.

If $A$ is $K$-affinoid, $A\{s^{-1}Y\}$ is again $K$-affinoid. For a presentation $P \twoheadrightarrow A$ with quotient norm, lift each coefficient of a series in $A\{s^{-1}Y\}$ to $P$ with norm at most twice its quotient norm. This gives a preimage in $P\{s^{-1}Y\}$, which is a Tate algebra in the combined variables. The kernel is the ideal generated by the kernel of $P \to A$. Indeed every series in the kernel has all coefficients there, and its finite truncations lie in this extended ideal, which is closed by Theorem 2.1.1. The lifting bound shows that the quotient norm is equivalent to the series norm. Similarly, submodules of finite free $A$-modules are closed, by lifting them to finite free $P$-modules and using the quotient topology.

## 2.2 Spectra and inequality domains

For a Banach $K$-algebra $A$, its *affinoid spectrum* $\mathcal{M}(A)$ is the set of bounded multiplicative ultrametric seminorms extending $|\cdot|$ on $K$, with the topology of pointwise evaluation. We write $|f(x)|$ for the value of $f$ at $x \in \mathcal{M}(A)$. Every bounded seminorm is dominated by the norm on $A$. In fact $|f(x)|^m \le C\|f^m\| \le C\|f\|^m$, and taking $m$-th roots and letting $m \to \infty$ proves the claim. In particular $\mathcal{M}(A)$ is unchanged by replacing the norm with an equivalent norm. It is compact Hausdorff by the closed equations and inequalities in $\prod_{f\in A}[0, \|f\|]$, as in Lemma 1.2.1. A bounded algebra map $A \to B$ induces a continuous map $\mathcal{M}(B) \to \mathcal{M}(A)$ by restriction.

The kernel of $x$ is prime by the argument of Proposition 1.1.1. The fraction field of $A/\ker(x)$ has the induced absolute value, and its completion is denoted $\mathcal{H}(x)$. Thus $f(x)$ also denotes the image of $f$ in $\mathcal{H}(x)$. Conversely any bounded map to a complete valued extension of $K$ gives a point by composing with its absolute value.

**Theorem 2.2.1** *Let $B$ be a nonzero commutative ultrametric Banach $K$-algebra. Then $\mathcal{M}(B)$ is nonempty, and for $b \in B$,*

$$\max_{x\in\mathcal{M}(B)} |b(x)| = \lim_{n\to\infty} \|b^n\|^{1/n} = \inf_{n\ge 1} \|b^n\|^{1/n}. \tag{2.1}$$

***Proof*** For a submultiplicative ultrametric $K$-seminorm $p$ with $p(1) \neq 0$, write $\rho_p(b) = \inf_{n\ge1} p(b^n)^{1/n}$. The quotient and remainder on dividing $n$ by a fixed positive integer show that the infimum is the limit. More precisely, for any $u > \rho_p(b)$ choose $m$ with $p(b^m) < u^m$. Writing powers as $b^{qm+j}$ with $0 \le j < m$ then gives $p(b^n) \le C_b u^n$ with $C_b$ independent of $n$. If $u$ is larger than both $\rho_p(b)$ and $\rho_p(c)$, expand $(b+c)^n$ by the binomial formula. Its terms have seminorm at most $C_bC_cu^n$, since the binomial coefficients have absolute value at most one. Taking roots and then decreasing $u$ proves the ultrametric inequality for $\rho_p$. Submultiplicativity follows by taking limits for $(bc)^n = b^nc^n$. Hence $\rho_p$ is such a seminorm, bounded by $p$. It is power multiplicative, and $\rho_p(1) = 1$ since $p(1) \ge 1$ by submultiplicativity.

Choose a minimal submultiplicative ultrametric $K$-seminorm $p \le \|\cdot\|$ with $p(1) = 1$. This is possible by Zorn's lemma. The set of such seminorms is nonempty,

since it contains $\rho_{\|\cdot\|}$ by the first paragraph. The pointwise infimum along a decreasing chain satisfies the seminorm and submultiplicativity conditions, since values on finitely many elements can be approximated at the same member of the chain. Minimality implies $p = \rho_p$ and in particular $p(b^n) = p(b)^n$. For $b$ with $p(b) > 0$, the sequence $p(b^n c)/p(b)^n$ is decreasing in $n$. Its limit defines another seminorm bounded by $p$ with value 1 at 1. Submultiplicativity follows by estimating $p(b^{2n}cd) \leq p(b^n c)p(b^n d)$ and taking limits. Minimality again makes this seminorm equal to $p$. Shifting $n$ by one in the limit gives $p(bc) = p(b)p(c)$. The equality also holds for $p(b) = 0$ by submultiplicativity. Thus $p$ is a point of $\mathcal{M}(B)$.

For the radius formula, the maximum does not exceed $\rho_{\|\cdot\|}(b)$, by domination and taking roots. If $0 < r < \rho_{\|\cdot\|}(b)$, then $B\{rU\}/\overline{(bU-1)}$ is a nonzero Banach algebra. Otherwise a multiple of $bU - 1$ approximates 1 in norm to within less than one, and is invertible by the geometric series. This would make $bU - 1$ invertible in $B\{rU\}$. Its formal inverse is $-\sum_{n\geq 0} b^n U^n$, which fails the convergence condition at radius $1/r$. A point of the nonzero quotient has $|b| \geq r$. Since $r$ can tend to $\rho_{\|\cdot\|}(b)$, this gives the other inequality. When the infimum is zero, the first inequality already gives equality. □

If $B$ is affinoid, every maximal ideal is closed and its Banach field quotient has a point by Theorem 2.2.1. Thus an element of $B$ that vanishes at no spectral point is a unit. Moreover, a bound $\sup_x |b(x)| < s$ implies that $\left(s^{-n}\|b^n\|\right)_{n\geq 0}$ is bounded, by the radius formula. At equality the formula gives a bound on the limit of the roots, and controlling the powers themselves requires more information about the affinoid norm (see Proposition 2.5.1).

**Proposition 2.2.1** *Suppose $k$ is trivially valued and $A_0 = k[T_1, \ldots, T_n]/I_0$. Write $t_i$ for the image of $T_i$ in $A_0$. There is a homeomorphism*

$$\mathcal{M}\big(k\{r^{-1}T\}/I_0 k\{r^{-1}T\}\big) \longrightarrow B(A_0; t_i, r_i)$$

*induced by restriction to $A_0$. It identifies the two completed residue fields at each point.*

***Proof*** Any seminorm on the left sends $T_i$ to a value at most $r_i$. Conversely, evaluate a polynomial at a point of the set on the right in its completed residue field from Proposition 1.1.1. The bounds give a contractive homomorphism for the Gauss norm. It extends to the Tate algebra, kills $I_0$, and then kills the ideal generated by $I_0$. The underlying polynomial image is dense, so the extension is unique and the two completed residue fields agree. The resulting continuous bijection is a homeomorphism, since the source is compact and the target is Hausdorff. □

A *rational subset* of $\mathcal{M}(A)$, for $A$ an affinoid algebra, is a subset

$$V = \{x \mid |f_i(x)| \leq s_i |g(x)| \text{ for every } i\}, \qquad s_i > 0, \qquad (g, f_1, \ldots, f_m) = A. \tag{2.2}$$

The last condition is on the ideal generated by these elements. It implies that $g(x) \neq 0$ for all $x \in V$. The basic cases $g = 1$ and $f_1 = 1$ give an upper and a positive lower

bound on the value of a function, respectively. The algebra of the given rational presentation is

$$A_V = A\{s^{-1}U\}/(gU_i - f_i)_{1\le i\le m}. \tag{2.3}$$

The expression $g^{-1}$ belongs to this algebra, because a relation expressing 1 in the ideal $(g, f_1, \ldots, f_m)$ becomes a multiple of $g$ in $A_V$.

**Proposition 2.2.2** *Restriction identifies $\mathcal{M}(A_V)$ homeomorphically with $V$ and gives an isometric identification of the completed residue fields at corresponding points.*

***Proof*** An element of $\mathcal{M}(A_V)$ satisfies $f_i(x) = g(x)U_i(x)$ and $|U_i(x)| \le s_i$. For the converse, substitute $f_i(x)/g(x)$ for $U_i$ in the complete field $\mathcal{H}(x)$ for $x \in V$. The inequalities make this substitution contractive, and it kills all relations. Any extension must take those values on $U_i$ in its own residue field. The resulting seminorm on polynomials in the $U_i$ is forced, because $g(x)$ can be used as a common denominator. Density gives uniqueness. The image of $A_V$ in $\mathcal{H}(x)$ contains the image of $A$, so the induced isometric embedding of completed residue fields is surjective. The homeomorphism follows from compactness and Hausdorffness. □

Rational subsets form a basis of closed neighborhoods in the following sense. If $x \in \mathcal{M}(A)$ and $W$ is an open neighborhood of $x$, there is a rational $V \subset W$ in whose inequalities all signs are strict at $x$. To see this, first use finitely many evaluations to specify an open neighborhood inside $W$. Shrink the upper and lower bounds on their values to closed bounds with strict inequalities at $x$. For a positive lower bound on $|b|$, use $|1| \le c|b|$. Products of the resulting systems give a single rational system. More explicitly, to intersect two systems with denominators $g, h$ and numerators $f_i, l_j$, take denominator $gh$ and numerators $f_i h, g l_j, f_i l_j$, with the original constants for the first two lists and the products of constants for the last. They generate the product of the unit ideals of the two systems. Both $g, h$ are nonzero wherever the combined system holds, and division then recovers the original systems. The strict inequalities are preserved at $x$.

Over $k$ trivially valued, when $r_i = 1$ for all $i$ the Tate algebra is the polynomial ring with the trivial norm. The quotient norm on any nonzero quotient is again trivial. For such a quotient $A$, the spectral seminorm $\rho(f) = \sup_{x\in\mathcal{M}(A)} |f(x)|$ is zero on nilpotents and equal to one on every other element. Indeed, a nonnilpotent survives in some prime quotient, and the trivial norm there gives a point of $\mathcal{M}(A)$. Consequently the degree-one spectral reduction

$$\{f \mid \rho(f) \le 1\}/\{f \mid \rho(f) < 1\}$$

is $A_{\mathrm{red}}$. For example, the degree-one reduction of $k[\varepsilon]/(\varepsilon^2)$ is $k$.

## 2.3 Finite maps and their fibres

We first recall how a complete field's absolute value behaves on a finite extension. We give the argument since the fields that occur in the fibres below need not be trivially valued.

**Lemma 2.3.1** *If $F$ is a complete ultrametric field and $E/F$ is finite, the absolute value of $F$ extends uniquely to an ultrametric absolute value on $E$. For that value $E$ is complete.*

***Proof*** Let $R$ be the valuation ring of $F$. Among local subrings of $E$ dominating $R$, choose a maximal one $S$ by the union-of-chains argument. Domination means that the maximal ideal of the larger ring contracts to the maximal ideal of the smaller ring. Order the local subrings by domination. Write $\mathfrak{m}$ for the maximal ideal of $S$. For $z \in E^\times$, at least one of $\mathfrak{m}S[z]$ and $\mathfrak{m}S[z^{-1}]$ is proper. If the first were the whole ring, an equation $1 = a_0 + a_1 z + \cdots + a_n z^n$ with $a_i \in \mathfrak{m}$ would give a monic equation for $z^{-1}$ with all lower coefficients in $\mathfrak{m}$, since $1 - a_0$ is a unit. If the second were also the whole ring, the same argument would apply to $z$. In the integral extension $S[z, z^{-1}]$ a prime over $\mathfrak{m}$ would then contain both $z$ and $z^{-1}$, which is impossible. Localizing whichever ring has proper extended ideal at a maximal ideal over $\mathfrak{m}$ gives a local ring dominating $S$. By maximality $z$ or $z^{-1}$ already belongs to $S$. Thus $S$ is a valuation ring. Its restriction to $F$ is $R$: if $b \in F^\times \setminus R$, then $b^{-1}$ belongs to the maximal ideal of $R$, hence of $S$, and $b$ cannot be in $S$.

Embed the value group of $F$ in $\mathbb{R}$ via $-\log|\cdot|$. For the valuation of $S$, a polynomial relation over $F$ for $z \in E^\times$ has at least two terms of smallest value. This implies that a positive integer multiple of the value of $z$ lies in the value group of $F$. The real embedding therefore extends to the value group of $S$ by dividing such a multiple by that integer. This is well-defined, additive and order-preserving, since the ordered group is torsion-free. Exponentiating with a minus sign gives the extension of the absolute value.

Every ultrametric vector norm on a finite-dimensional $F$-vector space is equivalent to a maximum norm on coordinates. To see this, argue by induction on dimension. A hyperplane spanned by all but one basis vector is complete in its induced norm by induction and hence closed. If the last vector is $e$, its distance $d$ to the hyperplane is positive. For a vector $u + ae$, with $u$ in the hyperplane, this gives $\|u + ae\| \geq |a|d$. The ultrametric bound on $\|u\|$ and the inductive estimate give bounds on the other coordinates. The reverse estimate follows by the triangle inequality. This proves completeness of every such norm. In particular two extending absolute values on $E$ are equivalent. Applying their comparison to each power of an element and taking roots proves equality. □

**Proposition 2.3.1** *Let $A$ be $K$-affinoid and let $B$ be an $A$-algebra finite as an $A$-module. Give $B$ the Banach topology of a quotient of a finite free $A$-module. It admits a submultiplicative norm for this topology and is $K$-affinoid. The topology is independent of the finite generating system.*

***Proof*** The zero algebra causes no difficulty. Otherwise choose module generators $b_0 = 1, b_1, \ldots, b_d$. The kernel of $A^{d+1} \to B$ is closed by Theorem 2.1.1 and its quotient consequence for $A$. Two generating systems give equivalent quotient norms by expressing each finite list in terms of the other. Choose a real number $R \geq 1$ bounding the norms of all coefficients in chosen expressions for $b_i b_j$ in the generators. Assign weight 1 to $b_0$ and weight $R$ to the other generators in the free module norm. If both indices are positive, the chosen representative of $b_i b_j$ has free module norm at most $R^2$. Products with $b_0 = 1$ already obey the desired bound. Expanding products in the generators and taking infima gives a submultiplicative quotient norm on $B$, with $\|1\| = 1$ since it is at most one and a nonzero submultiplicative norm cannot give 1 norm less than one.

Substituting $b_i$ for $T_i$ defines a contractive surjection

$$A\{R^{-1}T_1, \ldots, R^{-1}T_d\} \longrightarrow B.$$

The polynomial $\sum_{i=0}^{d} a_i T_i$, with $T_0 = 1$, has the same norm as the free module vector $(a_i)_{0 \leq i \leq d}$. Taking infima over such lifts shows that the surjection induces the chosen quotient norm. The source is $K$-affinoid by the series construction above, so $B$ is $K$-affinoid as well. □

For a finite module $M$ over a $K$-affinoid algebra $A$, a *quotient module norm* is the quotient norm of a surjection $A^d \to M$, where $A^d$ carries the maximum of the norms of the coordinates. It is a complete norm, since the kernel is closed by the end of Section 2.1, and it satisfies $\|am\| \leq \|a\|\|m\|$ for $a \in A$ and $m \in M$.

**Lemma 2.3.2** *Let $A$ be $K$-affinoid and let $M$ be a finite $A$-module.*

(1) *Any two quotient module norms on $M$ are equivalent.*
(2) *For every submodule $N \subset M$, the norm induced on $N$ by a quotient module norm on $M$ is equivalent to a quotient module norm on $N$. In particular an injective $A$-linear map of finite $A$-modules is a homeomorphism onto its image for the quotient module topologies.*
(3) *Let $A \to A'$ be a bounded homomorphism of $K$-affinoid algebras such that $A'$ is a finite $A$-module whose norm is equivalent to a quotient module norm over $A$. If $M$ is a finite $A'$-module, its quotient module norms over $A'$ and over $A$ are equivalent.*

***Proof*** For (1), express each generating list in terms of the other, as in the proof of Proposition 2.3.1.

For (2), equivalent norms on $A$ give equivalent quotient module norms, so we may assume that $A$ has the quotient norm of a presentation $P \to A$ by a Tate algebra. We may assume $N \neq 0$. Then the quotient module norm of $A^d \to M$ is also the quotient norm of $P^d \to M$. Let $\widetilde{N}$ be the inverse image of $N$ in $P^d$, and let $G_1, \ldots, G_m$ be the finite list of nonzero elements of $\widetilde{N}$ chosen in the proof of Theorem 2.1.1. Every $u \in \widetilde{N}$ has remainder zero there, so $u = \sum_j Q_j G_j$. Each quotient term recorded for $G_j$ has weight equal to a replaced weight, which is at most $\|u\|$, divided by $\|G_j\|$. Hence $\|Q_j\| \leq \|u\|/\|G_j\|$. The images $g_j$ of the $G_j$ generate $N$. A nonzero $n \in N$ has a lift

$u \in \widetilde{N}$ with $\|u\| \leq 2\|n\|$, and the images of the $Q_j$ express $n$ in the $g_j$ with coefficients of norm at most $2\|n\|/\min_j \|G_j\|$. Conversely $\left\|\sum_j a_j g_j\right\| \leq \max_j \|a_j\|\|g_j\|$. With (1) this proves the first assertion. An injective $A$-linear map carries a quotient module norm of its source to a quotient module norm of its image, for the images of the generators, which gives the second assertion.

For (3), let $m_i$ generate $M$ over $A'$ and let $a'_j$ generate $A'$ over $A$; by (1) we may take the norm of $A'$ to be the quotient module norm for the $a'_j$, and we choose expressions within a factor 2 of the infimum. The products $a'_j m_i$ generate $M$ over $A$. Given $m = \sum_i \beta_i m_i$ with $\beta_i \in A'$, the norm equivalence for $A'$ writes $\beta_i = \sum_j c_{ij} a'_j$ with $c_{ij} \in A$ and $\|c_{ij}\| \leq C\|\beta_i\|$ for a constant $C$. Conversely an expression $m = \sum_{i,j} c_{ij} a'_j m_i$ gives coefficients $\beta_i = \sum_j c_{ij} a'_j$ of norm bounded by a constant times $\max_j \|c_{ij}\|$, because $A \to A'$ is bounded. Taking infima over such expressions proves the equivalence. □

A map of affinoid spaces is *finite* if it is induced by a finite algebra with the quotient module topology of Proposition 2.3.1. The topology matters in this definition. For a trivially valued $k$ and $0 < r < s < 1$, the restriction $k\{s^{-1}T\} \to k\{r^{-1}T\}$ is the identity on the underlying ring $k[[T]]$. It is bounded, but its inverse is unbounded because the ratio of the norms of $T^n$ is $(s/r)^n$. The inclusion of the smaller closed disc is therefore not finite as a map of affinoid spaces.

**Theorem 2.3.1** *Let $\mathcal{M}(B) \to \mathcal{M}(A)$ be finite and let $x \in \mathcal{M}(A)$. Put*

$$D_x = B \otimes_A \mathcal{H}(x).$$

*Its maximal ideals index the points above $x$. At the point indexed by $\mathfrak{n}$, the completed residue field is $D_x/\mathfrak{n}$ with its unique extending absolute value. In particular the fibre is finite, and the map is surjective if $A \to B$ is injective.*

***Proof*** A point $y$ over $x$ gives an isometric map $\mathcal{H}(x) \to \mathcal{H}(y)$ and an evaluation $B \to \mathcal{H}(y)$. The induced homomorphism from $D_x$ to $\mathcal{H}(y)$ has maximal kernel. Indeed, the image is an integral domain finite-dimensional over $\mathcal{H}(x)$, hence a field. Conversely a field quotient $E = D_x/\mathfrak{n}$ has the complete absolute value of Lemma 2.3.1. Pull this absolute value back to $B$. It is bounded for the quotient module topology. For fixed module generators $b_i$, the evaluation of $\sum a_i b_i$ has absolute value at most $\max_i \|a_i\||b_i|$, where the last values are in $E$. Taking infima gives the requisite bound. The closure in $E$ of the fraction field of the image of $B$ contains $\mathcal{H}(x)$, since the restrictions to $A$ and its fractions are dense there. It also contains $B$'s image, so is all of $E$. These correspondences are inverse by uniqueness of the field absolute value and density of the fractions of $A$ in $\mathcal{H}(x)$. The finite-dimensional algebra $D_x$ is Artinian and has finitely many maximal ideals.

If $A \to B$ is injective, there is a prime of $B$ over $\ker(x)$ by lying over for the integral extension. Thus $B \otimes_A \kappa(\ker(x)) \neq 0$. Extension to the field $\mathcal{H}(x)$ preserves this nonzero algebra, so the fibre is nonempty. □

## 2.4 Extending scalars

Let $L/K$ be a complete isometric field extension. For an affinoid presentation $A = K\{r^{-1}T\}/I$, scalar extension has the presentation

$$A_L = A\widehat{\otimes}_K L = L\{r^{-1}T\}/IL\{r^{-1}T\}. \tag{2.4}$$

Here the completed tensor product is the Hausdorff completion for the seminorm

$$\|z\|_\pi = \inf_{z=\sum_j a_j\otimes\lambda_j} \max_j \|a_j\||\lambda_j|.$$

The seminorm is submultiplicative and ultrametric by expansion of products and sums. To see the presentation formula, first take $P = K\{r^{-1}T\}$. On $K[T]\otimes_K L = L[T]$ the seminorm agrees with the Gauss norm. One inequality follows from the coefficients $T^\alpha\otimes\lambda_\alpha$. The other follows by taking coefficients in any tensor expression, even one in $P\otimes_K L$. Truncating the $P$-factors in a finite tensor expression approximates it in $\|\cdot\|_\pi$, so this identifies the completion of $P\otimes_K L$ with $L\{r^{-1}T\}$. The algebraic quotient $P\otimes_K L\to A\otimes_K L$ induces the quotient seminorm because each coefficient in a tensor expression can be lifted with norm arbitrarily close to its quotient norm. Completing a quotient in this way gives the quotient of the completion by the closure of the kernel's image: the quotient map is dense, and the infima defining the quotient norm are unchanged by closing the subspace. Here that closure is the ideal generated by $I$ in $L\{r^{-1}T\}$. Indeed, the dense algebra $P\otimes_K L$ has $I\otimes_K L$ as an ideal, and its closure is an ideal containing $I$. Conversely the extended ideal contains $I\otimes_K L$ and is closed by Theorem 2.1.1. This proves (2.4) and makes it independent of the chosen presentation.

The map $\mathcal{M}(A_L)\to\mathcal{M}(A)$ restricts a seminorm to $A$. If $L/K$ is finite, the norm equivalence on finite-dimensional spaces used in Lemma 2.3.1 identifies $A_L$ with $A\otimes_K L$ as a Banach $A$-module, with no completion needed. Explicitly, for any $K$-basis of $L$ the coordinate maps are bounded, so the projective tensor norm is equivalent to the maximum norm on a finite direct sum of copies of $A$. This is already complete. The map is then surjective by Theorem 2.3.1, and its fibres are read from $\mathcal{H}(x)\otimes_K L$.

**Corollary 2.4.1** *A finite purely inseparable extension $L/K$ induces a homeomorphism $\mathcal{M}(A_L)\to\mathcal{M}(A)$. If $L\neq K$ embeds over $K$ in $\mathcal{H}(x)$, the fibre algebra at $x$ has a nonzero nilpotent.*

***Proof*** For $L = K$ the map is the identity. Otherwise $K$ has positive characteristic $p$. Choose $e$ so that every element of $L$ has its $p^e$-th power in $K$. In $D = \mathcal{H}(x)\otimes_K L$ every element has its $p^e$-th power in $\mathcal{H}(x)$ by the Frobenius identity. An element is therefore a unit or nilpotent, according as that power is nonzero or zero. The reduced algebra is a field, so there is exactly one point over $x$ by Theorem 2.3.1. The map is continuous from a compact space to a Hausdorff space, hence a homeomorphism.

If $a \in L \setminus K$ and $a' \in \mathcal{H}(x)$ is its image under the embedding, the element $1 \otimes a - a' \otimes 1$ has $p^e$-th power zero. It is nonzero because $1, a$ are linearly independent over $K$, so $1 \otimes 1, 1 \otimes a$ are linearly independent over $\mathcal{H}(x)$. □

Rational presentations commute with this scalar extension. In fact after extending scalars the relations in (2.3) are the same relations for the images of $f_i, g$ over $L$, by the quotient description in (2.4). Thus the new rational subset is the inverse image of the old one.

**Proposition 2.4.1** *For every complete isometric extension $L/K$, the restriction $\mathcal{M}(A_L) \to \mathcal{M}(A)$ is surjective.*

***Proof*** Let $x \in \mathcal{M}(A)$. The completed projective tensor $\mathcal{H}(x)\widehat{\otimes}_K L$ is a nonzero Banach $K$-algebra. It suffices to show that $1 \otimes 1$ has projective norm one. Consider a finite expression $1 \otimes 1 = \sum_i a_i \otimes \lambda_i$ and the $K$-subspace $E$ of $\mathcal{H}(x)$ spanned by 1 and the $a_i$. For any $c > 1$ there is a $K$-linear functional $\varphi : E \to K$ with $\varphi(1) = 1$ and norm at most $c$. To construct it, start with the functional on $K \cdot 1$. Given a functional on a hyperplane $H$ of the next larger finite-dimensional subspace, choose $b \notin H$ and let $d > 0$ be its distance to $H$, which is closed by finite-dimensional norm equivalence. Replace $b$ by $b' \in b + H$ with $\|b'\| \leq c_0 d$ for a prescribed $c_0 > 1$, and set the extending functional to zero on $b'$. For $u \in H$ and $a \in K$, if $\|u\| > |a|\|b'\|$ then $\|u + ab'\| = \|u\|$. Otherwise $\|u\| \leq |a|\|b'\| \leq c_0\|u + ab'\|$. The new norm bound is at most $c_0$ times the old. Choose the finitely many factors $c_0$ with product at most $c$. Applying $\varphi \otimes 1$ to the tensor identity gives

$$1 \leq c \max_i \|a_i\| \, |\lambda_i|.$$

The identity can be read in $E \otimes_K L$ because tensoring vector spaces preserves injections. Letting $c$ decrease to one and taking the infimum proves the projective norm assertion. The maps from $\mathcal{H}(x)$ and $L$ are isometric. The projective norm bounds each field norm from above, and applying this bound to the inverse gives the reverse bound by submultiplicativity.

The evaluation $A \to \mathcal{H}(x)$ and the scalar map induce a bounded map from $A_L$ to this nonzero completed tensor algebra. A point of its spectrum exists by Theorem 2.2.1. Its restriction to the isometrically embedded fields $\mathcal{H}(x)$ and $L$ is their given absolute value. Indeed, a bounded multiplicative seminorm on a valued field agrees with the field norm by taking powers, then applying the same bound to inverses. Its pullback to $A_L$ therefore lies over $x$. □

### 2.4.1 Gauss extensions

Fix a finite tuple $s = (s_1, \dots, s_\ell)$ of positive real numbers. On the Laurent polynomial ring $K[Z_1^{\pm 1}, \dots, Z_\ell^{\pm 1}]$ use the norm

$$\left\| \sum_{\nu \in \mathbb{Z}^\ell} a_\nu Z^\nu \right\|_s = \max_\nu |a_\nu| s^\nu .$$

Its completion consists of the Laurent series whose weights tend to zero away from finite subsets. The proof of Proposition 2.1.1 applies to Laurent polynomials, since the finite sets of maximal-weight monomials still have lexicographically largest elements. It then extends to the Laurent series by completion.

Call $s$ *$K$-free* if for $\nu \neq 0$ the number $s^\nu$ does not belong to $|K^\times|$. If $s$ is $K$-free, a nonzero Laurent series has a unique dominant monomial. Dividing by that monomial writes the series as a unit times $1 + h$, where $\|h\|_s < 1$. The geometric series in $-h$ inverts $1 + h$. Thus the completion is a field, denoted $K_s$, and its absolute values on nonzero elements form exactly $|K^\times| \prod_i s_i^{\mathbb{Z}}$. We call $K_s/K$ a Gauss extension.

**Proposition 2.4.2** *For $s$ $K$-free and $A$ $K$-affinoid, the map $\mathcal{M}(A_{K_s}) \to \mathcal{M}(A)$ has a continuous section. On coefficients from $A$ and Laurent monomials from $K_s$, it is given by the Gauss formula $|\sum_\nu a_\nu Z^\nu| = \max_\nu |a_\nu(x)| s^\nu$.*

***Proof*** For $x \in \mathcal{M}(A)$, form the completed Gauss Laurent algebra over $\mathcal{H}(x)$ at radii $s$. It has a multiplicative norm whether or not $s$ remains free over $\mathcal{H}(x)$. Evaluation $A \to \mathcal{H}(x)$ and the isometric coefficient map from $K_s$ define a bounded map into this algebra, first on $A \otimes_K K_s$, then on its completion. The norm there gives a point lifting $x$. For continuity, use a representative $G \in K_s\{r^{-1}T\}$ of an element of $A_{K_s}$. Expanding each coefficient of $G$ as a Laurent series and collecting gives

$$G = \sum_{\nu \in \mathbb{Z}^\ell} G_\nu(T) Z^\nu, \qquad \|G_\nu\|_r s^\nu \longrightarrow 0 \quad \text{away from finite subsets.}$$

Indeed, in the double series only finitely many terms are above any positive threshold. The norm at the lift is $\max_\nu |G_\nu(x)| s^\nu$. The maxima over finite sets converge uniformly because $|G_\nu(x)| \leq \|G_\nu\|_r$. Each finite maximum is continuous, proving the assertion. □

A finite collection of radii can be made strict after a Gauss extension. Choose a maximal $\mathbb{Q}$-linearly independent subcollection of their logarithms in $\mathbb{R}/\mathbb{Q}\log|K^\times|$. The radii $s$ of this subcollection are $K$-free. For each original radius $r_i$, some positive power $r_i^{m_i}$ is $|c_i|$ for $c_i \in K_s^\times$, by clearing denominators in the $\mathbb{Q}$-linear relation. If this field is trivially valued, then every $r_i = 1$. Otherwise choose $d_i \in K_s^\times$ with $|d_i| \geq r_i$. The substitution

$$Y_i \longmapsto T_i^{m_i}/c_i, \qquad X_i \longmapsto T_i/d_i$$

from a unit-radius Tate algebra to $K_s\{r^{-1}T\}$ is bounded. It is surjective: group the exponents of any series by their remainders $e_i$ modulo $m_i$ to write it as a finite sum of $T^e H_e(T_i^{m_i}/c_i)$, with $H_e$ in the unit-radius Tate algebra in the $Y_i$. The lift $d^e X^e H_e(Y)$ has norm bounded by $(\prod_i(|d_i|/r_i)^{e_i})$ times the norm of that group of terms. Only finitely many $e$ occur, which gives equivalence with the quotient norm. Consequently the extended algebra in (2.4) is strictly $K_s$-affinoid.

## 2.5 Domains, the G-topology and boundaries

An *affinoid domain* in an affinoid space $X = \mathcal{M}(A)$ is a subset $V$ represented by a bounded map $A \to A_V$ of affinoid algebras. Representability means that this map identifies $\mathcal{M}(A_V)$ with $V$ and that every bounded map $A \to B$ to an affinoid algebra whose spectral image is contained in $V$ factors uniquely through $A_V$. For the presentation (2.3), a factorization exists as soon as the image of $g$ is invertible and the powers of each $f_i/g$ satisfy the bounds for substitution at radius $s_i$. It is unique by density. This applies in particular to maps to complete valued fields, as in Proposition 2.2.2. The image of $g$ vanishes at no spectral point of $B$, so it is a unit by Theorem 2.2.1 and the closedness of maximal ideals. The same radius theorem gives all the power bounds when the suprema of the values of $f_i/g$ are strictly below the prescribed radii. At equality, the factorization uses the power bound of Proposition 2.5.1.

**Lemma 2.5.1** *Let $B$ be affinoid, let $N$ be its nilradical, and take $a \in B$ and $s > 0$. If the sequence $s^{-m}\|\bar{a}^m\|_{B/N}$ is bounded for the quotient norm, so is $s^{-m}\|a^m\|_B$.*

***Proof*** The Noetherian property implies $N^d = 0$ for some $d$, since $N$ is generated by finitely many nilpotents. The powers $N^j$ are closed. The successive quotient maps $N^j \to N^j/N^{j+1}$ admit bounded $K$-linear sections. In fact, present $B = P/I$ with $P$ a Tate algebra. For the inverse image $J$ of $N^{j+1}$ in $P$, the remainder map in Theorem 2.1.1 provides a bounded section $P/J \to P$. The remainder has norm bounded by the infimum of norms of representatives. Composing with $P \to B$ gives a section $B/N^{j+1} \to B$, whose restriction over $N^j/N^{j+1}$ lands in $N^j$. This splits the finite filtration as a direct sum of Banach spaces.

Multiplication by $a$ preserves the filtration. In the split coordinates it is block triangular, with diagonal blocks the induced multiplications on $N^j/N^{j+1}$. On these quotients $N$ acts by zero. The $m$-th power of each diagonal block has operator norm at most $\|\bar{a}^m\|_{B/N}$, since multiplication can be computed with any representative of $\bar{a}^m$. Thus each diagonal block satisfies a bound $Cs^m$ with $C \geq 1$. Let $D$ bound the norms of the off-diagonal blocks. In the block expansion of the $m$-th power of the full multiplication map, a term contains at most $d-1$ off-diagonal blocks. If it contains $l$ such blocks, its norm is at most $C^d D^l s^{m-l}$, by grouping consecutive diagonal blocks into powers. The ultrametric inequality bounds each matrix entry by the maximum of these bounds. It follows that the full $m$-th power is uniformly bounded after division by $s^m$, first in the split norm and hence in the equivalent norm of $B$. Apply it to 1 to obtain the assertion. □

For a strictly affinoid target over a trivially valued field, the needed norm theorem has an elementary form. The description of the degree-one spectral reduction at the end of Section 2.2 detects every prime and hence detects units. A quotient $f_i/g$ whose supremum absolute value is less than one is nilpotent. All elements have norm at most one for the strict quotient norm. These observations give the required power bounds for every radius in this case.

For an affinoid algebra $C$ over a complete ultrametric field, write $\rho_C(c) = \sup_{x\in\mathcal{M}(C)} |c(x)|$ for $c \in C$. We first compare $\rho_C$ with the norm of a reduced strictly affinoid algebra, and then show that scalar extension preserves norms and spectral suprema.

**Lemma 2.5.2** *Let $K$ be a complete ultrametric field of characteristic zero, or of characteristic $p > 0$ with $[K : K^p]$ finite. If $C$ is a reduced strictly $K$-affinoid algebra, there is a constant $M$ such that $\|c\| \le M\rho_C(c)$ for every $c \in C$.*

***Proof*** The zero algebra is immediate. Finite modules carry their quotient module norms, which are compared by Lemma 2.3.2.

*Strict normalization.* An integral strictly affinoid algebra $C$ is finite over an embedded unit-radius Tate algebra $P = K\{T_1, \dots, T_d\}$. The module topology from this finite presentation is the affinoid topology of $C$. We explain the construction by induction on the number of variables in a strict presentation. If the kernel is nonzero, choose a nonzero element in it and divide by a scalar coefficient of maximal absolute value. Call the resulting series $f$, of norm one. Its reduction modulo coefficients of absolute value less than one is a nonzero polynomial over the residue field of $K$. For the current last variable $T_e$, apply substitutions $T_i \mapsto T_i + T_e^{N_i}$ for $i < e$, with positive integers $N_i$. They are isometric automorphisms, since both these substitutions and their inverses are contractive. Choose the $N_i$ so that the weighted exponents $\alpha_e + \sum_{i<e} N_i\alpha_i$ of the finitely many monomials surviving reduction are distinct. For instance, successive powers of a sufficiently large integer separate the finitely many exponent vectors. The transformed reduced polynomial then has a unique term of highest $T_e$-degree, and that term is $c\,T_e^m$ with $c$ a nonzero scalar. With lexicographic order comparing $T_e$ first, the leading monomial of the transformed series is $T_e^m$. Any lexicographic order is a well-order compatible with multiplication by monomials, so the replacement argument of Theorem 2.1.1 and its remainder bound apply unchanged. Here $m > 0$. Indeed $f$ lies in the kernel of the surjection onto a nonzero domain, so it is not a unit, while a series of norm one with reduction a nonzero constant is a unit by the geometric series.

Divide by this single series using the monomial replacement argument from Theorem 2.1.1. The convergence and norm bound for the remainder in that argument apply to any finite list of nonzero divisors. In this case the remainder has $T_e$-degree less than $m$. The powers $1, T_e, \dots, T_e^{m-1}$ therefore generate $C$ as a module over the image of the Tate algebra in the other variables. The norm bound on the remainder also bounds the coefficient lifts in that generating list. Thus the finite module quotient norm is equivalent to the given norm. The image is a domain with the quotient strict presentation, so induction applied to it, together with part (3) of Lemma 2.3.2

for the composed finite presentations, proves this normalization. If the kernel is zero, the original Tate algebra already serves as $P$.

*A separable finite field extension.* Suppose $C$ is finite over $P$ as above, and put $F = \mathrm{Frac}(P)$ and $E = \mathrm{Frac}(C)$. Assume first that $E/F$ is separable. Use the Gauss point of $\mathcal{M}(P)$, whose absolute value on $F$ is the Gauss norm on $P$, multiplicative by Proposition 2.1.1, extended by division. Let $H$ be the completion of that valued field. The finitely many embeddings $\sigma$ of $E$ over $F$ into an algebraic closure of $H$ have images in one finite extension $H'$ of $H$. Its unique extending absolute value from Lemma 2.3.1 restricts to the unique one on each $\sigma(E)H$. Composing with the $\sigma$ gives points of $\mathcal{M}(C)$, since the evaluations are bounded by writing elements in a finite $P$-module generating system and taking the quotient norm. The trace is the sum of the conjugates $\sigma(c)$ in the single valued field $H'$, so the ultrametric inequality gives

$$\left|\mathrm{Tr}_{E/F}(c)\right|_{\mathrm{Gauss}} \leq \rho_C(c) \qquad (c \in C).$$

Choose an $F$-basis $e_j$ of $E$ from $C$ and its dual basis $e_j^*$ for the trace pairing. There are nonzero $a, t \in P$ such that $aC \subset \sum_j Pe_j$ and $te_j^* \in C$ for all $j$. For the first statement clear the denominators of the coordinates of finitely many module generators of $C$. For the second, localizing $C$ at $P \setminus \{0\}$ gives the finite-dimensional domain $E$ over $F$, so one can also clear these denominators in $P$. If $ac = \sum_j f_j e_j$, then

$$|t|_{\mathrm{Gauss}}|f_j|_{\mathrm{Gauss}} = \left|\mathrm{Tr}_{E/F}(te_j^* ac)\right|_{\mathrm{Gauss}} \leq \rho_C(te_j^* a)\rho_C(c).$$

Multiplication by $a$ is an injective map of finite $P$-modules from $C$ into the free module $\sum_j Pe_j$. Its inverse on the image is bounded by part (2) of Lemma 2.3.2. The displayed inequality therefore bounds $\|c\|$ by a fixed constant times $\rho_C(c)$.

*Removing the separability hypothesis.* In characteristic zero the preceding case suffices. In characteristic $p > 0$ the degree $[K : K^p]$ is finite by hypothesis. There is a power $q$ of $p$ such that a finite generating family $u_j$ of $E/F$ has the property that all $u_j^q$ are separable over $F$. This follows by writing each irreducible polynomial as a separable polynomial in a suitable $p$-power of its variable. Here $[K^{1/q} : K] = [K : K^q]$ is finite, since $[K : K^p]$ is, and $K^{1/q}$ is complete by Lemma 2.3.1. The Tate algebra

$$P' = K^{1/q}\{T_1^{1/q}, \ldots, T_d^{1/q}\}$$

is finite over $P$, with its finite module topology. Indeed expansion by exponent classes modulo $q$ and by a finite $K$-basis of $K^{1/q}$ gives a finite module basis and a norm comparison. Hence $P'$ is $K$-affinoid by Proposition 2.3.1. Its fraction field is $F^{1/q}$, since taking $q$-th powers is an isomorphism from $P'$ to $P$, with each absolute value raised to the $q$-th power. Work in an algebraic closure of $F$. The extension $EF^{1/q}/F^{1/q}$ is separable. For each $u_j$, take the $q$-th roots of the coefficients of a separable polynomial for $u_j^q$ over $F$. The new polynomial has root $u_j$, and its roots are distinct since their $q$-th powers are distinct.

Let $C' = P'[C]$ in $EF^{1/q}$. It is finite over $P'$ by finiteness of $C$ over $P$ and is an integral affinoid algebra with fraction field $EF^{1/q}$. It is also finite as a $P$-module, with the same topology by part (3) of Lemma 2.3.2, and hence $K$-affinoid by Propo-

sition 2.3.1. The embedding $C \to C'$ is bounded in both directions on its image by part (2) of that lemma over $P$. The separable case over $P'$, with $K^{1/q}$ in place of $K$, gives a bound of the norm on $C'$ by a constant times $\rho_{C'}$. Restriction of spectral points along the bounded map $C \to C'$ gives $\rho_{C'}(c) \leq \rho_C(c)$. Thus $\|c\|$ is bounded by a fixed multiple of $\rho_C(c)$.

For the reduced algebra $C$ of the lemma, with minimal primes $\mathfrak{p}_i$, apply the domain result to $C/\mathfrak{p}_i$. The injection $C \to \bigoplus_i C/\mathfrak{p}_i$ is an embedding of finite $C$-modules with their quotient topologies, by Lemma 2.3.2. The supremum $\rho_C$ is the maximum of the suprema on these quotients, because each prime kernel of a seminorm contains a minimal prime. □

**Lemma 2.5.3** *Let $K$ be a complete ultrametric field, let $L/K$ be a complete isometric extension, and let $B$ be $K$-affinoid. The map $B \to B_L$, $b \mapsto b \otimes 1$, is isometric for the projective tensor norm on $B_L = B\widehat{\otimes}_K L$, and $\rho_{B_L}(b \otimes 1) = \rho_B(b)$ for every $b \in B$.*

***Proof*** For an expression $b \otimes 1 = \sum_i b_i \otimes \lambda_i$, use a $K$-linear functional of norm at most $c > 1$ on the span of the $\lambda_i$ and 1, taking 1 to 1. Such a functional was constructed in the proof of Proposition 2.4.1 for a subspace of a completed residue field. The construction applies verbatim to any finite-dimensional $K$-subspace of $L$ containing 1. Applying it to the tensor identity gives $\|b\| \leq c \max_i \|b_i\| |\lambda_i|$. Let $c$ decrease to one and take the infimum. The reverse inequality comes from $b \otimes 1$ itself. Also $\rho_{B_L}(b \otimes 1) = \rho_B(b)$ by surjectivity of the map on spectra in Proposition 2.4.1. □

**Proposition 2.5.1** *Let $B$ be an affinoid algebra over a complete ultrametric field $K$. If $b \in B$, $s > 0$ and $\sup_{x \in \mathcal{M}(B)} |b(x)| \leq s$, then the sequence $\left(s^{-n}\|b^n\|_B\right)_{n \geq 0}$ is bounded. Consequently, for every $K$-affinoid algebra $A$, rational data $g, f_1, \ldots, f_m$ with radii as in (2.2), and bounded map from $A$ to an affinoid algebra whose spectral image lies in the set (2.2), there is a unique bounded $A$-algebra factorization through the algebra (2.3).*

***Proof*** Choose a complete isometric extension $L/K$ such that $B_L$ is strict and, in characteristic $p > 0$, $[L : L^p]$ is finite. In characteristic zero use the Gauss extension in the closing paragraph of Section 2.4.1, which makes the presentation (2.4) of $B_L$ strict. In characteristic $p > 0$ first pass to the completion of the perfect closure of $K$. The absolute values extend compatibly to each finite subextension by uniqueness. The completion $K'$ is perfect because the $p$-th roots of a Cauchy sequence in the perfect closure are again Cauchy. Now use a $K'$-free tuple of $\ell$ radii strictifying the presentation (2.4) over $K'$, as in Section 2.4.1, and let $L$ be the resulting Gauss field. Grouping Laurent exponents modulo $p$ shows that $L$ is spanned over $L^p$ by the finitely many monomials with exponents between 0 and $p - 1$ in each variable. Indeed the coefficients have $p$-th roots in $K'$, and taking these roots together with division of exponents preserves the Laurent convergence after taking $p$-th roots of the weights. Hence $[L : L^p]$ is finite.

The reduced quotient $B_L/N$, with $N$ the nilradical of $B_L$, is still strict. Write $\bar{b}$ for the image of $b \otimes 1$ in it. Every point of $\mathcal{M}(B_L/N)$ restricts to a point of $\mathcal{M}(B_L)$,

so $\rho_{B_L/N}(\bar{b}) \leq \rho_{B_L}(b \otimes 1) = \rho_B(b) \leq s$ by Lemma 2.5.3. Lemma 2.5.2, applied over $L$ to $B_L/N$ with its quotient norm, gives a constant $M$ with $\|\bar{b}^n\| \leq M\rho(\bar{b}^n) = M\rho(\bar{b})^n \leq Ms^n$, since the spectral seminorm is power multiplicative. By Lemma 2.5.1 the bound after division by $s^n$ lifts to $B_L$. Since $B \to B_L$ is isometric by Lemma 2.5.3, this gives the desired bound in $B$.

Finally, let $A \to B$ be a bounded map of affinoid algebras whose spectral image lies in the set $V$ of (2.2). The image of $g$ vanishes at no point of $\mathcal{M}(B)$, so it is a unit by spectral detection of units in Section 2.2. Each $f_i/g$ has absolute value at most $s_i$ on $\mathcal{M}(B)$, so the first assertion gives the power bounds at radius $s_i$. By the substitution rule for $A\{s^{-1}U\}$ in Section 2.1, sending $U_i$ to $f_i/g$ defines a bounded $A$-algebra map $A\{s^{-1}U\} \to B$. It kills the relations $gU_i - f_i$, so it factors boundedly through the algebra in (2.3). The factorization is unique by density, as noted before Lemma 2.5.1. This is the claimed universal factorization. □

**Corollary 2.5.1** *Let $B$ be an affinoid algebra over a complete ultrametric field $K$. If $b \in B$ satisfies $|b(x)| = 0$ for every $x \in \mathcal{M}(B)$, then $b$ is nilpotent.*

***Proof*** Choose the complete isometric extension $L/K$ of the proof of Proposition 2.5.1, so that $B_L$ is strictly $L$-affinoid and, in characteristic $p > 0$, $[L : L^p]$ is finite. By Lemma 2.5.3, $\rho_{B_L}(b \otimes 1) = \rho_B(b) = 0$. Let $N$ be the nilradical of $B_L$. Every point of $\mathcal{M}(B_L/N)$ restricts to a point of $\mathcal{M}(B_L)$, so the image of $b \otimes 1$ in the reduced strictly affinoid algebra $B_L/N$, with its quotient norm, also has spectral supremum zero. Lemma 2.5.2, applied over $L$, shows that this image is zero. Hence $b^n \otimes 1 = (b \otimes 1)^n = 0$ for some $n \geq 1$. Since $B \to B_L$ is isometric by Lemma 2.5.3, $b^n = 0$. □

Affinoid charts can meet along domains that are closed in one or both charts. The relevant covers allow a point to lie on these seams. A family $(V_j)_j$ of subsets of a topological space $V$ is a *G-cover* if for each $x \in V$ a finite subfamily, each of whose members contains $x$, has union a neighborhood of $x$ in $V$. For a finite closed cover, the complement of the union of the members omitting $x$ is a neighborhood covered by the remaining members. Thus every finite closed cover is a G-cover.

An *affinoid atlas* on a Hausdorff space $X$ consists of subsets $X_i$ with homeomorphisms $\mathcal{M}(A_i) \cong X_i$, the charts, whose images G-cover $X$, together with gluing data for each pair of charts. For charts $X_i, X_j$ these are a family of *common affinoids*: subsets of $X_i \cap X_j$ that are affinoid domains of both charts, that G-cover $X_i \cap X_j$, and whose family is stable under finite intersections. For each common affinoid $C$ an isomorphism between its algebras in the two charts is given, inducing the identity of $C$. These identifications commute with restriction to smaller common affinoids, and on a subset that is a common affinoid for each pair among three charts they satisfy the cocycle condition. For such a space, an affinoid domain means an affinoid domain of some chart.

In a Berkovich analytic space, an *analytic domain* is a subset G-covered, in its induced topology, by the affinoid domains it contains. These are the objects of its G-topology, with inclusions as arrows and G-covers as covering families. A space is *good* if each point has an affinoid neighborhood. A G-cover by affinoids alone

allows a neighborhood to be covered by several charts without any one chart being a neighborhood of the point. A space is *strictly analytic* if it has an atlas of strictly affinoid charts.

In Appendix E we prove that every affinoid domain has a finite cover by rational domains of its ambient affinoid space (Theorem E.2.1). Exactness for rational covers then gives the structure sheaf on the G-topology of an affinoid space (Theorem E.3.1). We do not glue these sheaves along an affinoid atlas.

We use the following boundary conventions for affinoid charts. For a bounded $A$-algebra $B$, an admissible presentation over $A$ is a surjection $A\{s^{-1}U\} \to B$ for which the given norm on $B$ is equivalent to the quotient norm. A point $y \in \mathcal{M}(B)$ is in the *relative interior* over $\mathcal{M}(A)$ if some admissible presentation over $A$ has $|U_i(y)| < s_i$ for every $i$, where $U_i(y)$ means the image of the variable evaluated at $y$. The *relative boundary* is the complement of this interior. For $A = K$ we call it the analytic boundary of $\mathcal{M}(B)$. For maps of good spaces the criterion is used on affinoid neighborhoods in source and target. The relative criterion is about the existence of a presentation, so an equality $|U_i(y)| = s_i$ in one presentation alone proves no boundary statement.

Finite affinoid maps have empty relative boundary. Indeed, take the finite generating list and the admissible presentation in Proposition 2.3.1, then increase all the chosen radii strictly. It is still an admissible presentation. The lift by a finite linear combination of generators bounds its quotient norm by a constant times the given norm. The bounds on the values of the generators are strictly below the enlarged radii at every point.

## 2.6 Discs and annuli

For a radius $R > 0$, $\mathcal{M}(K\{R^{-1}T\})$ is the closed disc of radius $R$. This terminology follows from restriction to $K[T]$, as in Proposition 2.2.1 when $K = k$. The Gauss norm defines a point $\eta_R$ of the disc, at which every function attains its maximum absolute value.

For $0 < r \le R$, the closed annulus is defined by $r \le |T| \le R$. Its affinoid algebra is $B_{r,R} = K\{R^{-1}T, rU\}/(TU-1)$. We identify it with the Laurent series $\sum_{n\in\mathbb{Z}} a_n T^n$ such that $|a_n| \max(r^n, R^n) \to 0$ as $|n| \to \infty$, with norm

$$\|f\|_{r,R} = \max\{\|f\|_r, \|f\|_R\}, \qquad \|f\|_s = \max_n |a_n| s^n. \tag{2.5}$$

To verify this description, substitute $U = T^{-1}$. A monomial $T^iU^j$ has norm $R^i r^{-j}$, which is at least the annulus norm of $T^{i-j}$. Let $M$ be $T^{i-j}$ as a monomial in $T$ if $i \ge j$, and $U^{j-i}$ otherwise. With $m = \min(i, j)$, one has

$$T^iU^j - M = (TU-1)M \sum_{l=0}^{m-1} (TU)^l.$$

The last factor, including $M$, has Gauss norm at most $R^i r^{-j}$, since $R/r \geq 1$. Thus this rewriting extends to series by convergence. Every Laurent series on the right has a norm-preserving lift by using nonnegative powers of $T$ and positive powers of $U$. These two facts identify the kernel and the quotient norm. The spectral identification with the indicated inequalities then follows from Proposition 2.2.2. The two endpoint Gauss points $\eta_r, \eta_R$ give (2.5), so every function attains its maximum absolute value at one of them.

From now on $K = k$ is trivially valued. On the ray of $T = 0$ in $\mathbb{P}^1_k$, $\eta_s = x_{0,-\log s}$ for $0 < s < 1$ in the notation of Theorem 1.4.1. If $0 < r < R < 1$, the annulus is exactly the segment of that ray where $-\log R \leq t \leq -\log r$. Its functions are the formal Laurent series $k((T))$: the radius $r$ condition permits only finitely many negative powers, and positive powers converge for every $s < 1$. The norm of a nonzero series with first exponent $m$ at the point $\eta_s$ is $s^m$, since the coefficients are trivially valued. In particular

$$\log |f(x_{0,t})| = -mt. \tag{2.6}$$

We can compute on the open interval $r < |T| < R$ using the explicit Laurent charts. Closed subannuli give a neighborhood basis there, with restriction maps given by the same formal Laurent series. Compatibility on overlaps therefore means equality of the Laurent series. A function given locally in these charts has one series throughout the connected open interval. Conversely every element of $k((T))$ gives such a function, and (2.6) holds at every point. All of these computations take place on the Laurent charts. The rational factorization at equality follows from Proposition 2.5.1, and the structure sheaf and its compatibility with these Laurent charts follow from the gluing theorem in Appendix E.

The *Shilov boundary* of an affinoid, when it exists, is the smallest closed subset where the supremum absolute value of every element is attained. For a closed disc it is $\{\eta_R\}$. To see the necessity of this point, let $C$ be a closed subset omitting $\eta_R$. At each point of $C$ some function has value strictly smaller than its Gauss norm, since the seminorm differs from the Gauss point. By compactness finitely many such strict inequalities cover $C$. Their product is strictly smaller than its Gauss norm at every point of $C$, whereas it has that norm at $\eta_R$. For an annulus $0 < r < R < 1$ over $k$, the Shilov boundary is $\{\eta_r, \eta_R\}$. The two points suffice by (2.5). The functions $T^{-1}$ and $T$ attain their maxima uniquely at $\eta_r$ and $\eta_R$, respectively, by the ray description.

**Proposition 2.6.1** *Over the trivially valued $k$, the analytic boundary of the closed disc of any radius $R > 0$ is $\{\eta_R\}$. The analytic boundary of the closed annulus $0 < r < R < 1$ is $\{\eta_r, \eta_R\}$. In the explicit Laurent charts of the open interval $r < |T| < R$, every point has a chart neighborhood in which it is interior.*

***Proof*** A point with $|T| < R$ is interior in the disc presentation. If $R \neq 1$, the only point in the disc with $|T| = R$ is $\eta_R$, since each nonzero polynomial has a unique term of maximal weight at that radius. For $R = 1$, let $x \neq \eta_1$ have $|T(x)| = 1$. There is a monic polynomial $f$ of positive degree with $|f(x)| < 1$, since some polynomial has different value at the two points. Choose $S > 1$ and use the presentation with generators $T$ and $f(T)$ and radii $S$ and 1. Its source is the polynomial ring in two variables, with those Gauss weights. Modulo the relation identifying the second variable

with $f(T)$, each polynomial in $T$ is a linear combination of $1, T, \dots, T^{\deg f-1}$ over $k[f(T)]$. The resulting lift has norm at most $S^{\deg f-1}$ for each nonzero polynomial, so the presentation is admissible for the trivial norm on $k[T]$. Both inequalities at $x$ are now strict. Similarly, a point with $r < |T| < R$ on the annulus is interior by the two generators $T, U$ with $TU = 1$.

It remains to exclude the indicated Gauss points from the interior. The same argument treats the disc point at $s = R$ and each annulus endpoint $s \in \{r, R\}$. Suppose a presentation with images $b_1, \dots, b_d$ of radii $q_1, \dots, q_d$ were strict at $\eta_s$. Choose $\theta < 1$ with $|b_i(\eta_s)| \le \theta q_i$ for all $i$. Admissibility gives a constant $C$ such that any element $h$ in the chart algebra has a lift $H$ of norm at most $C\|h\|$. Truncating $H$ to total degree less than $N$ and evaluating at the $b_i$ gives an element $h_N$ in the fixed finite-dimensional $k$-span of the $b^\alpha$ with $|\alpha| < N$, satisfying

$$|(h - h_N)(\eta_s)| \le C\theta^N \|h\|.$$

Use monomials $h = T^m$ with $m \to +\infty$ for $s = R$, and $m \to -\infty$ for $s = r$ in the annulus. Then $\|h\| = s^m = |h(\eta_s)|$. Choose $N$ so that $C\theta^N < 1$. The inequality forces $|h_N(\eta_s)| = s^m$ for infinitely many $m$. If $s \neq 1$, this is impossible because an ultrametric norm on a finite-dimensional space over a trivially valued field has only finitely many different positive values. Vectors with distinct norms are linearly independent, by the unique maximum inequality.

For the remaining case of $s = 1$ on the disc, the norm is trivial and the inequality forces $h = h_N$. Again this is impossible because infinitely many monomials are linearly independent. This proves the closed-disc and closed-annulus assertions. On the open annulus, around every point one can choose a smaller closed subannulus on which it satisfies both bounds strictly. This is the interior criterion on a Laurent chart neighborhood. □

# Chapter 3
# Valuations, models and intersection

An ideal gives a function of a centred valuation by evaluating its local generators. The integral closure of the ideal gives the same function. On the normalized blowup, the ideal is invertible, and orders along finitely many prime divisors decide which functions belong to its integral closure. This is the first use of a model to reduce a comparison of ideals to a finite set of valuations.

Throughout this chapter, $k$ is trivially valued, $X$ is an integral separated scheme of finite type over $k$, and $F = k(X)$. When we say that a valuation of $F$ has a centre on $X$, we regard it as its associated point of $X^{\mathrm{an}}$, which has generic support. For $x = (\xi, v) \in X^{\mathrm{an}}$ we write $v_x = v$ and $\mathrm{supp}(x) = \xi$.

The first two sections characterize divisor orders among field valuations and use them to test integral dependence of ideals (Theorems 3.1.1 and 3.2.1). We then construct valuations with several independent monomial weights. The density theorem, Theorem 3.4.1, returns to single divisor orders: it finds such an order satisfying any finite set of strict inequalities realized at an analytic point. The final section shows that intersection degrees of pulled-back classes can be computed on whichever common model is convenient. In Chapter 5 the divisorial characterization identifies the valuations attached to the special fibre of a model.

## 3.1 Divisorial valuations and rational rank

Let $Y \to X$ be a proper birational morphism from a normal integral scheme. For a prime divisor $E$ on $Y$, the local ring at its generic point is a discrete valuation ring. Denote its order of vanishing by $\mathrm{ord}_E$, normalized so that a uniformizer has value one. A *divisorial valuation* over $X$ is $c\,\mathrm{ord}_E$ for such a divisor and $c > 0$. It has a centre on $X$ given by the image of the generic point of $E$. We will specify when the multipliers are required to be rational.

For a real valuation $v$ of $F$, let $\Gamma_v = v(F^\times)$, let $R_v$ be its valuation ring, and let $\kappa_v$ be the residue field of $R_v$. The *rational rank* of $v$ is $\dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q})$.

**Proposition 3.1.1** *For every real valuation $v$ of $F$ trivial on $k$,*

$$\dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}) + \operatorname{trdeg}_k \kappa_v \;\leq\; \operatorname{trdeg}_k F.$$

***Proof*** Choose elements $z_1, \dots, z_a$ of $F^\times$ whose values are $\mathbb{Q}$-linearly independent. Choose $u_1, \dots, u_b \in R_v^\times$ whose residues are algebraically independent over $k$. Any nonzero polynomial in the $u_j$ with coefficients in $k$ has nonzero residue and therefore value zero. In a polynomial in the two families, group the terms by the exponent $\alpha$ of $z^\alpha$. The nonzero groups have pairwise distinct values $\sum_i \alpha_i v(z_i)$. A group of least value cannot cancel. Thus the $a + b$ elements are algebraically independent over $k$. Taking all finite choices of $a, b$ proves the inequality. □

**Theorem 3.1.1** *Let $v$ be a nontrivial real valuation of $F$ trivial on $k$ and having a centre on $X$. Then $v$ is divisorial over $X$ if and only if* $\operatorname{trdeg}_k \kappa_v = \dim X - 1$.

***Proof*** For the order at a prime divisor of a normal model, the residue field is the function field of that divisor, of transcendence degree $\dim X - 1$. Scaling has no effect on the valuation ring or residue field.

Conversely, write $n = \dim X$. Choose $n-1$ units $u_j$ with algebraically independent residues and $z \in F^\times$ with $v(z) > 0$. The argument of Proposition 3.1.1 shows that $F$ is finite over $k(u_1, \dots, u_{n-1}, z)$. Let $B$ be the integral closure of $k[u_1, \dots, u_{n-1}, z]$ in $F$. It is finite over the polynomial ring by finiteness of normalization in a finite function field extension [Stacks, Tag 0AVK]. The valuation ring contains $B$, since it contains the polynomial ring and is integrally closed. Its maximal ideal contracts to $(z)$ in the polynomial ring, because a polynomial has nonzero constant term as a polynomial in $z$ exactly when it has value zero. The centre on $\operatorname{Spec} B$ therefore has height one. Indeed primes in a strict chain contract to distinct primes by incomparability for integral extensions, and the prime lies above the nonzero height-one prime $(z)$. By normality the localization there is a discrete valuation ring. Because $v$ is nonnegative on this local ring and is positive on its uniformizer, it is a positive multiple of its order.

To put this divisor on a proper model of $X$, embed the affine $k$-scheme $\operatorname{Spec} B$ as an open subset of its projective closure $\overline{Y}$. The rational map $\operatorname{Spec} B \dashrightarrow X$ is defined near the height-one point just found. Indeed the centre of $v$ on $X$ has an affine neighborhood whose coordinate functions lie in $R_v$, which by the DVR calculation is the local ring at that point of $\operatorname{Spec} B$. Finitely many such functions are regular on a common neighborhood there and determine the map. Take the reduced closure of the generic graph in $X \times_k \overline{Y}$. It is proper and birational over $X$. Over the indicated open in $\operatorname{Spec} B$ it is the graph of a morphism, by separatedness of $X$, hence isomorphic to that open. Its normalization is finite [Stacks, Tag 0BXR] and leaves this normal open unchanged. The height-one point persists there, yielding the required divisor over $X$. □

## 3.2 Integral closure of an ideal and its Rees valuations

For an ideal $I$ in a ring $A$, an element $f$ is *integral over* $I$ if there is a monic equation

$$f^d + a_1 f^{d-1} + \cdots + a_d = 0, \qquad a_j \in I^j.$$

The integral closure $\overline{I}$ is the set of such elements. To see that they form an ideal, note that $f$ is integral over $I$ exactly when $fT$ is integral over the graded algebra $A[IT] \subset A[T]$. One implication follows by multiplying the equation by $T^d$. For the converse take the homogeneous component of degree $d$ in an integral equation for $fT$. The integral elements form a subring, so the degree-one part gives the assertion.

Suppose now $X = \operatorname{Spec} A$ and $I \neq 0$. For a point with centre on this affine, evaluate the generators of the ideal in its valuation ring, as in Section 1.3. Blow up $I$ and normalize. Blowing up is projective [Stacks, Tag 02NS] and an isomorphism away from $V(I)$ [Stacks, Tag 02OS], and normalization is finite and birational [Stacks, Tag 0BXR], so the resulting morphism $\pi\colon Y \to X$ is proper and birational. The pullback ideal is $\mathcal{O}_Y(-D)$ for an effective Cartier divisor $D$. The orders $\operatorname{ord}_E$ for the prime components $E$ of $D$ are called the *Rees valuations* of $I$. The definition uses the normalized blowup, and the components are those of the support of $D$, including a component over a divisor of $X$ along which the ideal already vanishes.

**Theorem 3.2.1** *For $f \in A$, the following conditions are equivalent.*

(1) $f \in \overline{I}$.
(2) *For every Rees valuation $v$ of $I$, $v(f) \geq v(I)$.*
(3) *For every $x \in X^{\mathrm{an}}$ with centre on $X$, $v_x(f) \geq v_x(I)$, with the value $+\infty$ allowed on both sides.*

*The same Rees valuations test $\overline{I^m}$ by the inequalities $v(f) \geq mv(I)$ for each positive integer $m$.*

***Proof*** For the implication from integrality, take the defining equation. If $v_x(f) < v_x(I)$, then $v_x(f)$ is finite. The first term has finite value $d\,v_x(f)$ and every other term has strictly larger value, since $v_x(a_j) \geq j\,v_x(I)$. This argument also applies when $v_x(I) = +\infty$. The first term cannot cancel. This proves $1 \Rightarrow 3$, and the Rees valuations are centred on $X$, so $3 \Rightarrow 2$.

For $2 \Rightarrow 1$, let $a_1, \ldots, a_q$ be nonzero generators of $I$. The normalized blowup has charts given by normalizing $A[I/a_i]$ in $F$. On the chart corresponding to $a_i$, the rational function $f/a_i$ has no pole along a prime divisor. In fact $a_i$ generates the pullback ideal there. Where it has positive order, the divisor is a component of $D$ and the assertion is precisely the assumed inequality. Elsewhere $a_i$ has order zero and $f$ is regular. The normal coordinate ring is the intersection of its height-one localizations [Stacks, Tag 031T]. It contains $f/a_i$, so $f/a_i$ is integral over $A[I/a_i]$.

We explain how these chartwise equations give an equation over $I$. A monic equation of degree $d_i$ for $f/a_i$ over $A[I/a_i]$, after multiplying by $a_i^{d_i}$, has lower coefficients of the form $a_i^j b_j$, with $b_j \in A[I/a_i]$. Choose $M_i \geq 1$ such that

$a_i^{M_i+j} b_j \in I^{M_i+j}$ for all those coefficients. This is possible because each $b_j$ is a finite sum of terms in $a_i^{-h} I^h$. For $d \geq \max_i d_i$, multiplying the equation also by $f^{d-d_i}$ gives

$$a_i^{M_i} f^d \in \sum_{j=1}^{d} I^{M_i+j} f^{d-j} \quad \text{for every } i.$$

For $M > \sum_i (M_i - 1)$, every product of $M$ of the generators of $I$ contains $a_i^{M_i}$ for some $i$. Use the indicated relation and then multiply by the remaining factors. Summing these inclusions gives

$$I^M f^d \subset \sum_{j=1}^{d} I^{M+j} f^{d-j}.$$

Put $J = (I, f)$ and $P = I^M J^{d-1}$. The last inclusion yields $fP \subset IP$. The only term in the expansion of $fP$ not already in $IP$ by its factors from $I$ is $I^M f^d$. Choose a finite generating list for the nonzero ideal $P$. Multiplication by $f$ can be expressed on that list by a matrix with entries in $I$. The determinant of $f$ times the identity minus this matrix annihilates $P$, by the adjugate identity. Since $A$ is a domain and $P \neq 0$, the determinant is zero in $A$. Its coefficient at degree lower by $j$ belongs to $I^j$, as required.

Finally the blowup of $I^m$ has the same charts as the blowup of $I$. On the chart of $a_i^m$ the ring is $A[I^m/a_i^m] = A[I/a_i]$, since $a_j/a_i = (a_j a_i^{m-1})/a_i^m$. To see that these charts cover the blowup of $I^m$, suppose a product of $m$ generators is outside a homogeneous prime. Then every corresponding $m$-th power is also outside that prime, by taking the $m$-th power of the product. The pullback divisor becomes $mD$ on the common normalization, so the same prime components test membership in the integral closure of $I^m$. □

In particular $I$ and $\overline{I}$ have the same evaluation at every centred analytic point. The inequality for integral elements bounds all their values from below by $v_x(I)$, and inclusion of $I$ in $\overline{I}$ gives the reverse bound. Conversely, if two nonzero ideals have the same evaluation at every centred analytic point, Theorem 3.2.1 shows that they have the same integral closure.

## 3.3 Monomial weights on a regular model

Let $Y \to X$ be a normal proper birational model and let $R = \mathcal{O}_{Y,\xi}$ be regular at a point $\xi$ of codimension $d > 0$. Fix a regular system of parameters $z_1, \ldots, z_d$ and positive real weights $w_1, \ldots, w_d$. The completion of $R$ is a formal power series ring in these parameters over a coefficient field lifting its residue field [Stacks, Tag 0C0S, part (1)]. We expand elements of the completion with respect to this coefficient field. This field need not contain the image of $k$, and we do not need it to. The composite $k \to R \to \kappa(\xi)$ is injective, so $\kappa(\xi)$ is an extension of $k$.

For a nonzero series set

$$v_w\left(\sum_\alpha a_\alpha z^\alpha\right) = \min_{a_\alpha \neq 0} \sum_{i=1}^{d} w_i \alpha_i . \tag{3.1}$$

The minimum is attained, since for every bound there are only finitely many exponents below it.

The rule gives a valuation on the fraction field of the completion. Addition satisfies the inequality term by term. For multiplicativity, take the two polynomials formed by the terms of least weight in the factors. Their product is nonzero because the polynomial ring over the residue field is a domain. All other products of terms have larger weight. This proves equality for multiplication. The map from $R$ to its completion is injective by Krull intersection, so restriction and division define $v_w$ on $F$. Its centre on $Y$ is $\xi$, since the series of positive weight are precisely those in the maximal ideal of the completion. In particular the valuation is trivial on $k$.

The value is independent of the chosen coefficient field. For a series of value $c$, choose an integer $N$ with $N \min_i w_i > c$ and truncate modulo the $N$-th power of the maximal ideal. Changing a representative of a nonzero residue coefficient changes it by an element of the maximal ideal, so every correction to a term has strictly larger weight than that term. The coefficients of weight $c$ therefore survive, and the discarded part has weight larger than $c$ in either expansion. The same argument shows invariance under replacing each $z_i$ by a unit times $z_i$, as is useful when the $z_i$ define components of a simple normal crossings divisor. In that setting the valuations obtained from positive weights at generic points of strata are called *quasi-monomial valuations* on the given model.

**Proposition 3.3.1** *Let $\ell$ be the rational rank of the weights. The valuation in* (3.1) *has rational rank $\ell$ and residue transcendence degree* $\dim X - \ell$ *over $k$. In particular it satisfies equality in Proposition 3.1.1.*

***Proof*** Every element of $F^\times$ is a fraction of elements of $R$. Each of their values is a sum of the $w_i$ with nonnegative integer coefficients, and the $z_i$ attain the values $w_i$. Thus the value group is $\sum_i \mathbb{Z} w_i$. The reduction map on $R$ gives an embedding $\kappa(\xi) \to \kappa_{v_w}$, since $v_w$ is zero on its units and positive on its maximal ideal.

Choose $\mathbb{Z}$-linearly independent vectors $\beta_1, \dots, \beta_{d-\ell}$ in the kernel of $\mathbb{Z}^d \to \mathbb{R}$, $\beta \mapsto w \cdot \beta$, and take the residues of the monomials $z^{\beta_j}$. They are algebraically independent over $\kappa(\xi)$. Indeed, a polynomial in them lifts to a finite sum of distinct Laurent monomials of weight zero, with unit coefficients in $R$ lifting the corresponding nonzero coefficients in $\kappa(\xi)$. Multiplying by a monomial clears the negative exponents. The constant terms of the lifted coefficients contribute distinct monomials of the same weight, while their other terms have larger weight. The lift of the polynomial therefore has value zero, so its residue is nonzero. As $\operatorname{trdeg}_k \kappa(\xi) = \dim X - d$, this gives the lower bound $\dim X - \ell$ for the residue transcendence degree. The opposite bound follows from Proposition 3.1.1. □

**Proposition 3.3.2** *Allow weights $w_i \geq 0$. Formula (3.1) defines a continuous map from the closed orthant of weights to $X^{\mathrm{an}}$. If $J = \{i : w_i > 0\}$, the valuation for $w$ is the one on $R_P$ using just the $z_i$ with $i \in J$, where $P = (z_i : i \in J)$. The all-zero weights give the trivial valuation.*

***Proof*** Group the formal expansion by the exponents of the positive-weight variables. Its coefficients are now series in the zero-weight variables, over a domain, so the least-weight product calculation remains valid. The centre in $\operatorname{Spec} R$ is $P$, since the elements of positive value are those in $P\widehat{R}$, and $P\widehat{R} \cap R = P$. The ideal $P$ is prime, as is seen from the formal power series quotient. The same holds for the ideal generated by the $z_i$ with $i$ in any subset of $J$, and distinct subsets give distinct ideals. A maximal chain of subsets of $J$ thus gives a chain of primes of length $|J|$ ending at $P$. Since $P$ has $|J|$ generators, Krull's height theorem [Stacks, Tag 0BBZ] shows that $P$ has height exactly $|J|$. Hence $R_P$ is regular with regular system of parameters $(z_i)_{i \in J}$, and the construction above applies to it.

To compare with $R_P$, fix a real threshold $\lambda > 0$ and let $I_\lambda$ be the ideal generated by the monomials in the positive-weight variables of weight at least $\lambda$. In the completion, a series belongs to $I_\lambda \widehat{R}$ exactly when its value is at least $\lambda$. The quotient of $\widehat{R}$ by this ideal is a finite free module over the power series ring in the zero-weight variables, with basis the surviving monomials. For an element of $R \setminus P$, multiplication on the associated graded module for the positive-variable ideal is injective. On each graded piece it is multiplication by the nonzero image of that element modulo the positive variables, on a free module over the zero-variable power series ring. Since the filtration is finite, multiplication on the quotient is injective as well. By contraction from the completion, $I_\lambda R_P \cap R = I_\lambda$. Thus the membership test for $v_w \geq \lambda$ is the same in $R$ and in $R_P$ with its positive weights, by faithful flatness of the two completions. Varying $\lambda$ and taking fractions proves equality of valuations.

For continuity on a nonzero element of $R$, take weights $w'$ tending to $w$. For every $\varepsilon > 0$, for $w'$ close to $w$ all its coordinates satisfy $w'_i \geq (1 - \varepsilon) w_i$. This bounds the minimum for $w'$ from below by $1 - \varepsilon$ times the minimum for $w$. A single exponent attaining the minimum for $w$ gives an upper bound converging to that minimum. The same reasoning works at zero weights, and fractions give continuity on $F^\times$. All the points have generic support, so this implies continuity in the evaluation topology of $X^{\mathrm{an}}$. □

*Example 3.3.1* Let $A = k[x, y]$ and $I = (x^2, y^3)$. Give $x$ weight 3 and $y$ weight 2 in the local ring at the origin. The resulting valuation takes value 6 on $I$. The integral closure $\overline{I}$ consists of the polynomials all of whose monomials $x^i y^j$ satisfy $3i+2j \geq 6$. The necessity follows from the integral equation and the monomial valuation. For sufficiency, choose an integer $a$ with $0 \leq a \leq 6$, $a \leq 3i$ and $6 - a \leq 2j$. Such an integer exists precisely when $3i + 2j \geq 6$. Then $(x^i y^j)^6$ is divisible by $(x^2)^a (y^3)^{6-a}$ and belongs to $I^6$. This gives an integral equation over $I$ for $x^i y^j$. Since integral closure is an ideal, sums of the indicated monomials belong to it. In particular $xy^2$ belongs to $\overline{I}$ and $xy$ does not.

## 3.4 Approximating points by divisorial valuations

Divisorial valuations with rational scale are already dense on a proper integral $X$ of positive dimension. We prove this by realizing finitely many strict inequalities on an auxiliary affine model with one extra variable. No resolution is needed.

**Lemma 3.4.1** *Let $F \subset L$ be a field extension, let $w$ be a valuation of $L$, and let $v = w|_F$. For their residue fields,*

$$\operatorname{trdeg}_{\kappa_v} \kappa_w \leq \operatorname{trdeg}_F L.$$

***Proof*** Lift finitely many algebraically independent residue classes to units of the valuation ring of $w$. A polynomial relation over $F$ among the lifts would give a relation over $\kappa_v$ by dividing by a coefficient of least value and reducing. At least one coefficient is still nonzero there. This contradicts the algebraic independence. □

**Theorem 3.4.1** *Let $X$ be proper, integral, and of positive dimension over $k$. Then the valuations $c\,\mathrm{ord}_E$ over $X$ with $c \in \mathbb{Q}_{>0}$ are dense in $X^{\mathrm{an}}$.*

***Proof*** Take $x \in X^{\mathrm{an}}$ and an open neighborhood of $x$. Choose an affine chart $\operatorname{Spec} A$ containing its centre, hence its support. In that chart a smaller neighborhood of $x$ is defined by finitely many inequalities

$$m_i v(f_i) > n_i, \qquad m'_j v(g_j) < n'_j, \qquad m_i, m'_j \in \mathbb{Z}_{>0}, \quad n_i, n'_j \in \mathbb{Z}.$$

Indeed, rational cutoffs on the valuation give a basis for the topology of finitely many absolute-value evaluations. Conditions involving the zero function can be omitted, and each $g_j$ has nonzero image at the support of $x$. All functions in $A$ have nonnegative value at $x$.

Form the finitely generated integral $k$-algebra

$$C = A\left[\tau,\ f_i^{m_i}/\tau^{n_i},\ \tau^{n'_j}/g_j^{m'_j}\right] \subset F(\tau).$$

The evaluation $A \to \kappa(\operatorname{supp}(x))$ extends to a map on $C$ into $\kappa(\operatorname{supp}(x))(\tau)$, by localizing at the $g_j$ and $\tau$. Give this field the Gauss valuation extending $v_x$ with value 1 on $\tau$, constructed by the minimum formula of Section 1.5. The induced semivaluation on $C$ is nonnegative. Its centre is a prime $\mathfrak{p}$ containing $\tau$ and all the displayed fractions, since each of them has strictly positive value.

Blow up $\mathfrak{p}$ on $\operatorname{Spec} C$ and normalize. Some component $E$ of the divisor of the pullback ideal has centre $\mathfrak{p}$. The proper birational map is surjective, and a point above $\mathfrak{p}$ lies in the support of that divisor. Any component containing that point maps into $V(\mathfrak{p})$ and its image contains the generic point. Thus $\mathrm{ord}_E$ is strictly positive on $\tau$ and the displayed fractions. The normalized restriction

$$v' = \frac{\mathrm{ord}_E|_F}{\mathrm{ord}_E(\tau)}$$

satisfies all the strict inequalities and hence lies in the chosen neighborhood.

If $v'$ is nontrivial, then it is divisorial over $X$. The residue field of $\mathrm{ord}_E$ has transcendence degree $\dim X$ over $k$, since $E$ is a divisor of a model of the $(\dim X + 1)$-dimensional field $F(\tau)$. By Lemma 3.4.1 its residue field has transcendence degree at most 1 over $\kappa_{v'}$. Thus $\mathrm{trdeg}_k\, \kappa_{v'}$ is at least $\dim X - 1$, and the reverse inequality is Proposition 3.1.1. Apply Theorem 3.1.1, using properness for the existence of its centre. The nonzero value group of the restriction $\mathrm{ord}_E|_F$ is a subgroup of $\mathbb{Z}$. Consequently its generator after division by $\mathrm{ord}_E(\tau)$ is rational, which is the scale of $v'$ relative to its divisorial order.

If $v'$ is trivial, the neighborhood contains the trivial valuation with generic support. For any fixed prime divisor on a normal model of $X$, the positive rational multiples of its order converge to the trivial valuation as the multipliers tend to zero, by evaluation of every nonzero rational function. A prime divisor exists since $\dim X > 0$. Hence the neighborhood meets the same set in this case as well. □

## 3.5 Intersection numbers on models

A model in a birational problem is a proper birational morphism $Y \to X$. Two models of an integral separated $X$ admit a common proper model given by the reduced closure of the point over the generic point in their fibre product. Its projections to both models are proper and birational. Normalization gives a normal common model when needed, by finiteness of normalization over $k$. This construction lets us compare divisors pulled back from different models without requiring a resolution of singularities.

For a projective $k$-scheme $Z$, the degree of a Cartier class on an integral curve $C \subset Z$ means the degree over $k$ of its restriction to $C$. A Cartier class is *nef* if these degrees are nonnegative for every such curve. For a projective morphism $Z \to S$, the analogous condition on curves in fibres defines *relative nefness*. In this relative setting a curve in a fibre is taken over the residue field of that fibre. For real classes, extend the degrees by linearity. Two Cartier classes on $Z$ are numerically equivalent if they have the same degrees on all integral curves of $Z$. Write $N^1(Z)_{\mathbb{R}}$ for the quotient of the real Cartier class space by this relation.

We use the ordinary Cartier intersection action on cycles [Stacks, Tag 02SN]. The first Chern actions descend to Chow groups and commute [Stacks, Tags 02TI and 02TJ]. If $Z$ is proper over $k$ and of pure dimension $n$, the number $(L_1 \cdots L_n \cdot [Z])$ is the degree over $k$ of the zero-cycle class obtained by applying the first Chern classes of $L_1, \ldots, L_n$ to the fundamental cycle of $Z$. The coefficient of a component in the fundamental cycle is the length of the local ring at its generic point. At the other end of the construction, the degree of a closed point $z$ as a zero-cycle is $[\kappa(z) : k]$. Both multiplicities are relevant even when one works with the same space of semivaluations after reduction.

Let $q : Z' \to Z$ be a proper dominant generically finite morphism of integral proper $k$-schemes of dimension $n$, and let $e = [k(Z') : k(Z)]$. Then

$$(q^*L_1 \cdots q^*L_n \cdot [Z']) = e(L_1 \cdots L_n \cdot [Z]). \tag{3.2}$$

Indeed $q_*[Z'] = e[Z]$ by the definition of proper pushforward. Apply the projection formula for the first Chern class action [Stacks, Tag 02SU] once for each $L_i$ and take degrees. The same argument applied to a cycle whose proper image has dimension less than $n$ gives degree zero for the product of $n$ pullback classes, since the pushforward of that cycle is zero. For a common model $e = 1$ by birationality, so an intersection of pullback classes has the same degree on every dominating model.

# Chapter 4
# Limits, measures and convexity

Take an integral regular projective curve with uncountably many closed points and consider the indicator of the origin of its valuation rays. It is upper semicontinuous, and a continuous majorant is greater than one half on a neighborhood of the origin. By Theorem 1.4.1, that neighborhood contains all but finitely many entire rays. A countable family of such majorants therefore cannot recover the indicator as its infimum. Directed families of continuous majorants will give this approximation on any compact Hausdorff space.

On a compact Hausdorff space a monotone net of continuous functions converging pointwise to a continuous function converges uniformly (apply Proposition 4.1.1 to the net or to its negative). For integration we can instead use increasing nonnegative lower semicontinuous functions or decreasing upper semicontinuous functions bounded above (Theorem 4.2.1). We then construct Radon measures from positive functionals and study their weak convergence. The chapter closes with three approximation tools. A separating lattice approximates continuous obstacles, convex duality reconstructs a function from its affine minorants, and the volume of a Minkowski sum of polytopes is a polynomial in the scaling factors. The lattice statement applies to the model functions of Chapter 5, and the two convex constructions will enter the toric calculations.

## 4.1 Semicontinuity and directed approximation

A *directed set* is a nonempty preordered set in which any two elements have a common upper bound. A family indexed by a directed set is a *net*. It converges to a point if it eventually lies in every neighborhood of that point. A net of functions is increasing if $f_i \le f_j$ whenever $i \le j$, and decreasing if the inequalities are reversed.

A function $u : X \to [-\infty, +\infty]$ on a topological space is upper semicontinuous if $\{u < a\}$ is open for every real $a$, and lower semicontinuous if $\{u > a\}$ is open for every real $a$. Pointwise infima of upper semicontinuous functions are upper semicontinuous. Pointwise suprema of lower semicontinuous functions are lower

semicontinuous. For any extended real function $h$, put

$$h^*(x) = \inf_{U \ni x} \sup_{y \in U} h(y), \tag{4.1}$$

with $U$ ranging over open neighborhoods. This is the least upper semicontinuous majorant. Indeed, if $h^*(x) < a$ then some such $U$ has supremum less than $a$, and $h^* < a$ on $U$. Every upper semicontinuous majorant of $h$ majorizes the displayed infimum, by applying its semicontinuity at $x$.

**Lemma 4.1.1** *Let $X$ be compact Hausdorff and let $u : X \to [-\infty, +\infty)$ be upper semicontinuous. The continuous real-valued majorants of $u$ form a downward directed family with pointwise infimum $u$.*

***Proof*** Compactness applied to the open sets $\{u < m\}$ for positive integers $m$ gives a finite constant upper bound. For any point $x$ and any real number $a > u(x)$, the open set $U = \{u < a\}$ contains $x$. Choose a constant $C \geq a$ majorizing $u$ and a continuous function $b : X \to [0, 1]$ with $b(x) = 0$ and $b = 1$ on $X \setminus U$. Such a function exists by normality and Urysohn's lemma. Then $a + (C - a)b$ is a continuous majorant taking value $a$ at $x$. This works for every real $a$ when $u(x) = -\infty$. The pointwise infimum is therefore $u$. The minimum of two continuous majorants is again a continuous majorant, so the family is directed under the reverse of the usual pointwise order. □

**Lemma 4.1.2** *Let $(U_i)_i$ be an increasing net of open subsets of a space. If a compact subset $K$ is contained in $\bigcup_i U_i$, then $K \subset U_i$ for some $i$.*

***Proof*** A finite subfamily covers $K$. An upper bound of its finitely many indices gives the assertion. □

**Proposition 4.1.1** *Let $X$ be compact and $f_i$ a decreasing net of upper semicontinuous functions on $X$ converging pointwise to a continuous real-valued function $f$. Then for every $\varepsilon > 0$, all sufficiently late indices satisfy $f \leq f_i < f + \varepsilon$ on $X$.*

***Proof*** The sets $U_i = \{f_i - f < \varepsilon\}$ are open and increase to all of $X$. Apply Lemma 4.1.2 to the whole space. The lower inequality holds by pointwise monotonicity. □

Applying Proposition 4.1.1 to the net $-f_i$ gives the increasing version. If $f_i$ is an increasing net of lower semicontinuous functions on a compact space converging pointwise to a continuous real-valued function $f$, then for every $\varepsilon > 0$, all sufficiently late indices satisfy $f - \varepsilon < f_i \leq f$ on $X$.

## 4.2 Radon measures and integrals of nets

On a compact Hausdorff space we use finite positive Borel measures that are outer regular on Borel sets and inner regular by compact sets on open sets. We call them

Radon measures. The integral of any extended real function considered below is defined in the usual sense for Borel functions. For a nonnegative function it may take the value $+\infty$.

**Lemma 4.2.1** *For a finite positive Radon measure $\mu$ and an increasing net of open sets $U_i$,*

$$\mu\Big(\bigcup_i U_i\Big) = \sup_i \mu(U_i).$$

***Proof*** By Lemma 4.1.2, each compact subset of the open union lies inside one $U_i$. Take suprema over compact subsets and use inner regularity. The reverse inequality follows from inclusion. □

**Theorem 4.2.1** *Let $X$ be compact Hausdorff and $\mu$ a finite positive Radon measure. If $g_i : X \to [0, +\infty]$ is an increasing net of lower semicontinuous functions, then*

$$\int_X \sup_i g_i \, \mathrm{d}\mu = \sup_i \int_X g_i \, \mathrm{d}\mu.$$

*If $f_i : X \to [-\infty, +\infty)$ is a decreasing net of upper semicontinuous functions, then*

$$\int_X \inf_i f_i \, \mathrm{d}\mu = \inf_i \int_X f_i \, \mathrm{d}\mu.$$

***Proof*** Put $g = \sup_i g_i$, which is lower semicontinuous. For $t \geq 0$, Lemma 4.2.1 gives $\mu\{g > t\} = \sup_i \mu\{g_i > t\}$. The integral of $g$ is the integral of the decreasing function $t \mapsto \mu\{g > t\}$ over $[0, +\infty)$, by the layer-cake formula for nonnegative functions. It is thus the supremum of the lower sums

$$\sum_{j=1}^{m} (t_j - t_{j-1}) \mu\{g > t_j\}, \qquad 0 = t_0 < t_1 < \cdots < t_m < \infty.$$

For any fixed finite list of these thresholds, directedness gives a common index for which the superlevel measures of $g_i$ approximate all the superlevel measures of $g$ from below. Hence the lower sum is bounded by $\sup_i \int g_i \, \mathrm{d}\mu$. Taking the supremum of sums proves the first equality, including the case of infinite integral. For the second equality, fix an index $i_0$ and restrict to the cofinal tail $i \geq i_0$. The compactness argument at the start of the proof of Lemma 4.1.1 gives a finite constant $C \geq f_{i_0}$, hence $C \geq f_i$ on the tail. The functions $C - f_i$ on this tail form an increasing net of nonnegative lower semicontinuous functions. Apply the first equality and subtract from $C\mu(X)$. □

*Example 4.2.1* An upper semicontinuity or lower semicontinuity hypothesis in the preceding net arguments cannot simply be omitted. For Lebesgue measure on $[0, 1]$, direct the finite subsets by inclusion and use their indicator functions. The net increases to the constant function one, while every integral is zero. The indicators of finite closed sets are upper semicontinuous, so this example also explains the direction of monotonicity paired with semicontinuity in Theorem 4.2.1.

## 4.3 Weak convergence on a compact space

For a compact Hausdorff space $X$, write $C(X)$ for the continuous real-valued functions, and $\mathcal{P}(X)$ for the Radon probability measures. A net of finite positive Radon measures $\mu_i$ converges *weakly* to $\mu$ if $\int_X f \, \mathrm{d}\mu_i \to \int_X f \, \mathrm{d}\mu$ for all $f \in C(X)$. The same evaluations define the weak topology. The compactness of $\mathcal{P}(X)$ for this topology, proved in Corollary 4.3.1, does not require $X$ to be metrizable.

**Theorem 4.3.1** *Let $L : C(X) \to \mathbb{R}$ be a positive linear functional, where $X$ is compact Hausdorff. There is a unique finite positive Radon measure $\mu$ such that $L(f) = \int_X f \, \mathrm{d}\mu$ for every $f \in C(X)$.*

***Proof*** For $U$ open, let $m(U)$ be the supremum of $L(f)$ over $0 \le f \le 1$ with compact support in $U$. It is monotone, with $m(\varnothing) = 0$ and $m(X) = L(1)$. To see countable subadditivity on open sets, take such an $f$ for a union of open sets $U_j$. A finite subfamily covers its compact support. By normality there are finitely many functions $b_j$ with $0 \le b_j \le 1$ and support in the chosen $U_j$, whose sum is positive on the support of $f$. Construct them by covering the support with finitely many smaller open sets whose closures lie in members of the cover, choosing for each closure a further open set whose closure still lies in the same member, and applying Urysohn's lemma to the first closure and the complement of that open set. Then $f_j = f b_j / \sum b_j$, extended by zero where $\sum b_j = 0$, is continuous and admissible in $U_j$, and $\sum f_j = f$. Thus $L(f) \le \sum m(U_j)$. If two open sets are disjoint, admissible functions there have sum at most one, giving additivity for the union.

Set

$$\mu^*(E) = \inf_{\substack{U \supset E \\ U \text{ open}}} m(U).$$

This is an outer measure, by countable subadditivity on opens and the choice of open approximations to a countable list of sets. For open $U$ we have $\mu^*(U) = m(U)$. Moreover

$$m(U) = \sup_{K \subset U,\ K \text{ compact}} \mu^*(K).$$

In fact every admissible $f$ for $U$, with support $K$, satisfies $L(f) \le m(V)$ for every open $V \supset K$. The reverse inequality follows from monotonicity.

Every open $U$ is measurable for $\mu^*$. To verify the nontrivial inequality of the outer measure criterion, fix $E \subset X$ and $\varepsilon > 0$. Choose $V$ open containing $E$ with $m(V) < \mu^*(E) + \varepsilon$, and choose compact $K \subset V \cap U$ with $\mu^*(K) > m(V \cap U) - \varepsilon$. There is an open $W$ with $K \subset W \subset \overline{W} \subset V \cap U$. The disjoint opens $W$ and $V \setminus \overline{W}$ lie in $V$, and the latter contains $E \setminus U$. Additivity on disjoint opens therefore gives

$$\mu^*(E) + 2\varepsilon > m(V \cap U) + \mu^*(E \setminus U) \ge \mu^*(E \cap U) + \mu^*(E \setminus U).$$

Let $\varepsilon$ decrease to zero. Thus the Borel restriction $\mu$ of $\mu^*$ is a finite measure. It is outer regular by definition and inner regular on opens by the compact approximation already shown. For an arbitrary Borel set, outer regularity on its complement and finiteness give approximation from within by compact sets.

For $f \in C(X)$ nonnegative, use admissible functions $b_j$ for the sets $\{f > t_j\}$ where $0 = t_0 < t_1 < \cdots < t_m$. The inequality $\sum_j (t_j - t_{j-1}) b_j \le f$ holds pointwise, because only thresholds below $f(x)$ can contribute at $x$. Hence $L(f)$ is at least the corresponding lower sum for the layer-cake integral of $f$, by taking suprema of $L(b_j)$. Taking the supremum of these sums gives $L(f) \ge \int f \, \mathrm{d}\mu$. Apply the same inequality to $\|f\|_\infty - f$ and use $L(1) = \mu(X)$ to get equality. Linearity handles arbitrary $f$. Any representing Radon measure has values $m(U)$ on opens, by compact inner regularity and by Urysohn's lemma applied to a compact $K \subset U$ and the complement of an open set $W$ with $K \subset W \subset \overline{W} \subset U$. Outer regularity then gives uniqueness on Borel sets. □

**Corollary 4.3.1** *The space $\mathcal{P}(X)$ is compact Hausdorff for the weak topology.*

***Proof*** In the product $\prod_{f \in C(X)} [-\|f\|_\infty, \|f\|_\infty]$, the coordinates of a probability measure satisfy closed conditions of linearity, positivity, and value one on the constant one. By Theorem 4.3.1 these conditions characterize the image of $\mathcal{P}(X)$. The weak topology is its subspace topology, giving the result. □

**Proposition 4.3.1** *Suppose $\mu_i$ converges weakly to $\mu$. If $u : X \to (-\infty, +\infty]$ is lower semicontinuous and bounded below, then*

$$\int_X u \, \mathrm{d}\mu \le \liminf_i \int_X u \, \mathrm{d}\mu_i .$$

*For an upper semicontinuous function with a finite upper bound, the upper limit yields the reverse inequality.*

***Proof*** The continuous minorants of $u$ are upward directed with supremum $u$, by applying Lemma 4.1.1 to $-u$. If $c$ is a finite lower bound for $u$, replacing any minorant $v$ by $\max\{v, c\}$ gives a cofinal subfamily. For any member $v$ of this family, weak convergence and $v \le u$ give $\int v \, \mathrm{d}\mu \le \liminf_i \int u \, \mathrm{d}\mu_i$. Apply Theorem 4.2.1 to the nonnegative minorants $v - c$ and add $c\mu(X)$ to obtain the desired bound. Negating the function gives the upper limit statement. □

**Corollary 4.3.2** *If $D \subset X$ is Borel and $\mu(\overline{D} \setminus D^\circ) = 0$, then $\mu_i(D) \to \mu(D)$ under weak convergence.*

***Proof*** Apply the lower and upper bounds to the indicators of $D^\circ$ and $\overline{D}$, respectively. □

## 4.4 A lattice of continuous functions

The following lattice form of the Stone–Weierstrass theorem gives the uniform density of model functions in Theorem 5.4.1.

**Lemma 4.4.1** *Let $X$ be compact Hausdorff. Let $\mathcal{L} \subset C(X)$ be a real vector subspace containing the constants, closed under finite maxima and minima, and separating points. Then $\mathcal{L}$ is dense in $C(X)$ for the uniform norm.*

***Proof*** Fix $f \in C(X)$ and $\varepsilon > 0$. For $x, y \in X$, there is an $h_{x,y} \in \mathcal{L}$ taking the values $f(x)$ and $f(y)$ there. For distinct points take a separating function and adjust by a scalar and a constant. For equal points the constant suffices. Fix $x$ and let $y$ vary. The opens $\{h_{x,y} < f + \varepsilon\}$ cover $X$. Take the minimum $h_x$ of a finite subfamily corresponding to a cover. Then $h_x < f + \varepsilon$ on $X$ and $h_x(x) = f(x)$. Now the opens $\{h_x > f - \varepsilon\}$ cover $X$. Taking the maximum of finitely many corresponding $h_x$ gives the required approximant. □

**Corollary 4.4.1** *Under the hypotheses of Lemma 4.4.1, for any continuous $h$ the majorants of $h$ in $\mathcal{L}$ form a downward directed net decreasing uniformly to $h$.*

***Proof*** The family is directed by finite minima. Approximate $h$ within $\varepsilon/2$ by an element of $\mathcal{L}$ and add $\varepsilon/2$. Every late term in the reverse pointwise order lies between $h$ and this majorant. □

The lattice approximation also works for a $\mathbb{Q}$-vector subspace of $C(X)$ containing the rational constants, closed under finite maxima and minima, and separating points. For two distinct points, first choose real coefficients interpolating the values and then approximate the two coefficients by rationals. Take the two endpoint errors less than $\varepsilon/2$. For equal points take a rational constant within $\varepsilon/2$ of $f(x)$. In the proof of Lemma 4.4.1, the finite minimum then lies above $f(x) - \varepsilon$ at $x$ and below $f + \varepsilon$ everywhere. The finite maximum gives the same uniform approximation. Adding a positive rational constant after a sufficiently close approximation gives the directed majorants in Corollary 4.4.1 with rational coefficients as well.

## 4.5 Affine minorants and convex duality

Let $V$ be a finite-dimensional real vector space, with dual $V^*$ and pairing $\langle \cdot, \cdot \rangle$. A convex function $\varphi : V \to (-\infty, +\infty]$ is proper if it is finite somewhere. Its Legendre transform is

$$\varphi^\vee(p) = \sup_{x \in V} \big( \langle p, x \rangle - \varphi(x) \big) \qquad (p \in V^*).$$

The supremum is taken over the finite locus of $\varphi$. Thus the transform is a supremum of affine continuous functions and is lower semicontinuous and convex. The transform of a function on $V^*$ is defined by the same formula, using the identification $V^{**} = V$.

**Theorem 4.5.1** *A proper lower semicontinuous convex function on $V$ is the supremum of its real affine minorants. In particular $\varphi^{\vee\vee} = \varphi$.*

***Proof*** The epigraph of $\varphi$ is a nonempty closed convex set in $V \times \mathbb{R}$. For $t < \varphi(x)$, separate the point $(x, t)$ strictly from the epigraph. Such a separating hyperplane is

obtained by taking a closest point of the closed convex set in a Euclidean norm and using the vector joining them. Write the inequality as

$$\langle a, y\rangle + bs \geq c > \langle a, x\rangle + bt \qquad \text{for } s \geq \varphi(y).$$

Since the epigraph contains upward vertical rays, $b \geq 0$. If $b > 0$, the affine function $(c - \langle a, y\rangle)/b$ is a minorant whose value at $x$ exceeds $t$.

There is at least one affine minorant $\ell_0$, by using this argument with $x$ in the finite locus. In that case $b$ cannot vanish because the epigraph contains a point over $x$. If the separation at the original point has $b = 0$, then $\langle a, y\rangle \geq c$ on the finite locus and $c > \langle a, x\rangle$. Thus $\ell_0(y) + M(c - \langle a, y\rangle)$ is a minorant for any $M > 0$, and for large $M$ it exceeds $t$ at $x$. This proves the assertion about the supremum, including where $\varphi(x) = +\infty$. For a given slope $p$, the largest possible affine minorant of that slope has constant term $-\varphi^\vee(p)$ when that value is finite. Taking the supremum over slopes gives the double transform. □

For $x$ in the finite locus, its *subdifferential* $\partial\varphi(x)$ is the set of $p \in V^*$ such that

$$\varphi(y) \geq \varphi(x) + \langle p, y - x\rangle \qquad \text{for every } y \in V.$$

Equivalently $\varphi^\vee(p) = \langle p, x\rangle - \varphi(x)$. This follows by evaluating the supremum at $x$ and using the defining inequality. For a finite maximum of affine functions, the subdifferential at a point is the convex hull of the slopes of the terms attaining the maximum. Every slope in the hull gives a supporting affine function, by taking convex combinations of the active terms. For a slope outside the hull, strict separation gives a direction in which its slope exceeds all the active slopes. Along a sufficiently short segment in that direction only active terms can attain the maximum, because the remaining finitely many terms have a strict gap at the point. The proposed slope therefore fails the supporting inequality.

## 4.6 Mixed volumes of polytopes

Fix a Euclidean structure and its Lebesgue measure on an $n$-dimensional space $V$. For nonempty compact polytopes $P_1, \ldots, P_m$, put $P(\lambda) = \sum_i \lambda_i P_i$, where $\lambda_i \geq 0$ and the sum is the Minkowski sum. The support function is $h_P(u) = \max_{x \in P}\langle u, x\rangle$. Write $P^u$ for the face attaining this maximum.

**Theorem 4.6.1** *The function* $\mathrm{vol}_n(P(\lambda))$ *is a polynomial in the* $\lambda_i$ *with nonnegative coefficients. Every monomial with a nonzero coefficient has total degree n.*

***Proof*** In dimension zero every nonempty polytope is a point of volume one. Assume $n \geq 1$ and argue by induction on the dimension. Translate each $P_i$ so it contains zero, which changes the sum only by a translation. Assume first that all weights are positive. The direction space of the affine span of the sum is the sum of the direction spaces of the $P_i$. If it is proper, the volume is identically zero. Otherwise, for any

unit normal $u$ the face of the sum is $\sum_i \lambda_i P_i^u$, by maximizing independently in the factors. Its dimension is independent of the positive weights by the same assertion about direction spaces. Thus the finite list of unit facet normals is independent of these weights.

The volume is the sum of the volumes of the pyramids with apex zero and bases the facets, including zero volume for a facet through zero. Indeed, each ray from zero through an interior point has a last point in the polytope. This terminal point belongs to a unique facet except on the cones over faces of smaller dimension. Consequently

$$n \operatorname{vol}_n(P(\lambda)) = \sum_u h_{P(\lambda)}(u) \operatorname{vol}_{n-1}(P(\lambda)^u).$$

The height is $\sum_i \lambda_i h_{P_i}(u)$, with nonnegative coefficients. To compute the facet volume, translate the summand faces $P_i^u$ separately into $u^\perp$. Their Minkowski sum differs from $P(\lambda)^u$ by a translation, so induction on dimension shows that the facet volume is a homogeneous polynomial of degree $n-1$ with nonnegative coefficients. This proves the formula for positive weights.

If some weights vanish, approximate them from above by adding a common positive number tending to zero to all the weights. The sums decrease since the $P_i$ contain zero. Their intersection is the sum at the limiting weights, by compactness of the product of the $P_i$. Finiteness and continuity from above of Lebesgue measure show that the polynomial formula still holds. □

For $n$ polytopes, define their mixed volume $\operatorname{MV}(P_1, \ldots, P_n)$ as $1/n!$ times the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial of Theorem 4.6.1. It is symmetric, and in each factor it is additive with respect to Minkowski sum and homogeneous under dilations by nonnegative reals. For additivity in the first factor, expand the polynomial in separate weights for $P_1$ and $Q_1$, then set the two weights equal. Only terms of total degree one in those two weights contribute to the coefficient in question. If $P_1 = \cdots = P_n = Q$, then $P(\lambda) = (\sum_i \lambda_i)Q$ by convexity of $Q$, so the polynomial is $(\sum_i \lambda_i)^n \operatorname{vol}_n(Q)$ and $\operatorname{MV}(Q, \ldots, Q) = \operatorname{vol}_n(Q)$. For the two coordinate unit segments in the plane, the sum is a rectangle of area $\lambda_1 \lambda_2$. Their mixed volume is $1/2$.

# Chapter 5
# Test configurations and model potentials

For an ideal $\mathfrak{a}$ on $X$, the function $-\min\{v(\mathfrak{a}), 1\}$ is continuous even at valuations that assign infinite value to $\mathfrak{a}$. It is the potential of the blowup of $(\mathfrak{a}, t)$ on the product with the affine line. This description explains the change of slope in the projective-line example of Section 1.5, and gives a divisor on which to perform the intersection calculations of the global theory.

Throughout the chapter, $X$ is a nonempty reduced projective scheme over the trivially valued field $k$. We allow components of different dimensions and use the function field of each component separately.

We first extend a valuation on $X$ to the product with the affine line by giving the parameter $t$ value one. Conversely, the normalized orders along the components of the special fibre of a model integrally closed along the puncture are such extensions of a finite set of valuations on $X$ (Proposition 5.3.1). This leads to the order test and uniform density in Theorem 5.4.1. When the line bundle of a model is relatively semiample, its potential has a description in terms of sections on $X$. This is the Fubini–Study correspondence of Theorem 5.5.1. At the end we compute the effect of changing the base parameter and return to the blowup of $(\mathfrak{a}, t)$ for a smooth centre in Example 5.6.1. The order test and the Fubini–Study potentials are the starting point for the psh potentials of Chapter 6.

## 5.1 The Gauss lift

Write $S = \mathbb{A}^1_k$, with coordinate $t$, and put $S^\times = S \setminus \{0\}$. The projection from $X \times S$ admits a section on analytic points given by the relation $w(t) = 1$ and the formula

$$\sigma(v)\left(\sum_j f_j t^j\right) = \min_j\bigl(v(f_j) + j\bigr). \tag{5.1}$$

Here the coefficients are taken in the residue field of the support of $v$. One first uses the formula for Laurent polynomials and then extends it to their field of fractions. To

check multiplicativity, take the largest index attaining the minimum in each factor. In the coefficient of the product indexed by the sum of these indices, the term given by those two indices is the unique one of smallest weighted value. The ultrametric inequality then gives equality for the product. Formula (5.1) also gives the ultrametric inequality for sums. The resulting valuation extends $v$ on the residue field and depends continuously on $v$ as a point of $(X \times S^{\times})^{\mathrm{an}}$. On an affine chart this is the continuity of the finite minimum in (5.1), including the case when every term has value $+\infty$.

## 5.2 Marked models

**Definition 5.2.1** A *test configuration* is a flat projective morphism $\mathcal{X} \to S$ with a $\mathbb{G}_m$-action lifting multiplication on $S$ and an equivariant marking $\mathcal{X}|_{S^{\times}} \simeq X \times S^{\times}$, such that $\mathcal{X}$ admits a relatively ample linearized invertible sheaf. The action on the marked product is trivial on the $X$-factor. Such an auxiliary sheaf is not part of the data. A morphism of models respects the action and the marking.

A test configuration for a rational line bundle $L$ on $X$ also carries a linearized rational line bundle $\mathcal{L}$ and an equivariant identification of its restriction with the pullback of $L$. Identifications of rational line bundles are taken after clearing denominators, with two identifications agreeing if they agree after a further tensor power. We call the pair relatively ample or relatively semiample when $\mathcal{L}$ has the corresponding property over $S$.

The scheme $\mathcal{X}$ is reduced. On each affine open its coordinate ring embeds in its localization at $t$ by flatness, and that localization is reduced by the marking. Two test configurations have a common dominating one. Take the scheme-theoretic closure of the diagonal marked product in their fibre product over $S$. The closure is $\mathbb{G}_m$-stable, since the diagonal marked product is stable and scheme-theoretic closure commutes with the flat base changes along the action and projection maps [Stacks, Tag 081I]. Its rings embed in their localizations and hence have no $k[t]$-torsion. They are therefore flat over the principal ideal domain $k[t]$. The tensor product of the two pulled-back auxiliary ample sheaves is relatively ample on the closure and is linearized. This also gives common dominance with the product model. The compactification $\overline{\mathcal{X}} \to \mathbb{P}^1_k$, obtained by gluing in the product over $\mathbb{P}^1_k \setminus \{0\}$, is projective. The auxiliary sheaf extends by gluing, since a linearized invertible sheaf on the punctured product is pulled back from the fibre at $t = 1$. Its extension is relatively ample on the resulting open cover of the base.

The valuation $\sigma(v)$ has a unique centre on $\mathcal{X}$, in the special fibre, by the valuative criterion for the proper morphism $\mathcal{X} \to S$. Equivalently, this is its centre on the compactification. Centres are compatible with morphisms of models. Thus a coherent ideal can be evaluated by taking the minimum of the values of local generators. For an ideal containing a power of $t$ the value is finite. The same holds for the value of a local equation of a *vertical* Cartier divisor, that is, a Cartier divisor supported on

the special fibre, allowing an equation of the form $f/t^a$. Its local equation is a unit after inverting $t$, so its value and the value of its inverse are both finite. The values computed in this way are independent of generators because every local unit at the centre has value zero.

**Definition 5.2.2** A test configuration $\mathcal{X}$ is *integrally closed along the puncture* if, for every affine open $\operatorname{Spec} A \subseteq \mathcal{X}$, the ring $A$ is integrally closed in $A[t^{-1}]$.

**Lemma 5.2.1** *Every test configuration has a finite dominating test configuration that is integrally closed along the puncture. On an affine chart with ring $A$, its ring is the integral closure of $A$ in $A[t^{-1}]$.*

***Proof*** The ring in the statement is the intersection of the finite normalization of $A$ in its total ring of fractions [Stacks, Tag 0BXR] with $A[t^{-1}]$. It is a finite $A$-module by Noetherianity. Integral closure commutes with localization, so the local constructions glue. They are unchanged on the puncture, and the resulting model is flat by the torsion criterion used above. To lift the action, adjoin an invertible indeterminate $s$ to these localizations. The normalization of $A[s, s^{-1}]$ is the Laurent polynomial ring over the normalization of $A$, since the latter ring is normal and the extension is integral and birational. Taking the intersection with $A[t^{-1}][s, s^{-1}]$ commutes with forming the Laurent polynomial ring by comparison of coefficients. The isomorphism $(s, x) \mapsto (s, sx)$ on $\mathbb{G}_m \times \mathcal{X}$ thus lifts. The action identities hold by their restrictions to the schematically dense marked open. The pullback of the auxiliary ample sheaf is relatively ample along the finite morphism. □

On a model integrally closed along the puncture, membership in the structure sheaf has a direct valuative test. The notation $\operatorname{ord}_E$ below refers to order at the generic point of a component $E$ of the special fibre.

**Proposition 5.2.1** *If $\mathcal{X}$ is integrally closed along the puncture, its local rings at the generic points of the special fibre are discrete valuation rings. For an affine open $U = \operatorname{Spec} A$ and $g \in A[t^{-1}]$, membership in $A$ is equivalent to $\operatorname{ord}_E(g) \geq 0$ for each such generic point in $U$, where the order of zero is $+\infty$. In particular, a vertical rational Cartier divisor is effective if and only if its coefficients at these generic points are nonnegative.*

***Proof*** Let $\mathfrak{p}$ be an associated prime of $A/t^mA$, for some $m \geq 1$, and let $R = A_{\mathfrak{p}}$, with maximal ideal $\mathfrak{m}$. There is an $h = f/t^m \in R[t^{-1}] \setminus R$ with $h\mathfrak{m} \subset R$. If $h\mathfrak{m} \subset \mathfrak{m}$, the determinant trick applied to the finite $R$-module $\mathfrak{m}$ gives a monic integral dependence relation for $h$. Indeed $\mathfrak{m}$ contains the nonzerodivisor $t$, so an annihilator of the module in $R[t^{-1}]$ is zero. This contradicts integral closure in $R[t^{-1}]$. Otherwise $ha$ is a unit for some $a \in \mathfrak{m}$. Then $h^{-1} = a/(ha)$ lies in $\mathfrak{m}$. The containment $h\mathfrak{m} \subset R$ now gives $\mathfrak{m} = h^{-1}R$. The principal ideal theorem and the nonzerodivisor $t \in \mathfrak{m}$ give $\dim R = 1$. Thus $R$ is a one-dimensional regular local ring, hence a discrete valuation ring. The prime $\mathfrak{p}$ is minimal over $(t)$, and every minimal prime over $(t)$ occurs among the associated primes.

Write $g = f/t^m$, with $f \in A$. If the image of $f$ in $A/t^m A$ is nonzero, its cyclic submodule has an associated prime $\mathfrak{p}$. The image remains nonzero after localization there, so $g \notin A_{\mathfrak{p}}$. This proves the membership assertion. After clearing denominators on a rational Cartier divisor, apply this assertion to its local equations and use their invertibility away from $t = 0$. □

The proof uses only that the affine rings are Noetherian, that $t$ is a nonzerodivisor on them, and that each such ring $A$ is integrally closed in $A[t^{-1}]$. It therefore applies verbatim to any Noetherian scheme over $S$ with these properties, for example $\mathbb{G}_m \times \mathcal{X}$ in the proof of Theorem 5.4.1.

For later use, a proper morphism $p : \mathcal{Y} \to \mathcal{X}$ of test configurations with $\mathcal{X}$ integrally closed along the puncture satisfies $p_* \mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$. Indeed the direct image is coherent [Stacks, Tag 02O5] and is a subsheaf of regular functions on the puncture by schematic density. Each affine ring of sections of this direct image is finite over the base ring and therefore integral in the punctured ring. Also, any morphism of test configurations from a model integrally closed along the puncture lifts to the modification of Lemma 5.2.1 of its target. On an affine chart every integral element pulls back to a function on the puncture integral over each affine open of the source lying over the chart, so it pulls back regularly.

## 5.3 The valuations of the special fibre

Let $\mathcal{X}$ be integrally closed along the puncture. Write $b_E = \mathrm{ord}_E(t)$ for a component $E$ of the special fibre. This positive integer is its scheme-theoretic multiplicity in that fibre. The generic point of $E$ belongs to a unique component of $\mathcal{X}$, corresponding to a component $X_i$ of $X$. Via the marking its function field is $k(X_i)(t)$. Define

$$v_E = b_E^{-1} \, \mathrm{ord}_E\big|_{k(X_i)} .$$

**Proposition 5.3.1** *One has $b_E^{-1} \, \mathrm{ord}_E = \sigma(v_E)$. The valuation $v_E$ is either trivial on $k(X_i)$ or a positive rational multiple of a prime divisor order on a proper normal birational model of $X_i$.*

***Proof*** The action preserves $E$, since the image of the integral scheme $\mathbb{G}_m \times E$ lies in the special fibre and contains $E$. Adjoin an indeterminate $s$, pass to $k(s)$, and apply the action by $s$. At the base extension of the generic point of $E$, the DVR extends to $R[s]_{\mathfrak{m}R[s]}$, where $R$ is the original DVR and $\mathfrak{m}$ its maximal ideal. The order of a polynomial in $s$ is the minimum of the orders of its coefficients. This follows by dividing out the minimum power of a uniformizer and reducing to a nonzero polynomial over the residue field. Invariance under the action sends $\sum_j f_j t^j$ to $\sum_j f_j t^j s^j$. Comparing orders gives (5.1) for $b_E^{-1} \, \mathrm{ord}_E$.

Put $n_i = \dim X_i$. The residue field of $\mathrm{ord}_E$ has transcendence degree $n_i$ over $k$, since $E$ has codimension one in a component of dimension $n_i + 1$. By Lemma 3.4.1,

restriction from $k(X_i)(t)$ to $k(X_i)$ can lower residue transcendence degree by at most one. If the restriction is nontrivial, its value group is a nonzero subgroup of $b_E^{-1}\mathbb{Z}$. The Abhyankar inequality of Proposition 3.1.1 gives the reverse bound $n_i - 1$ on residue transcendence degree. The restriction has a centre on the proper scheme $X_i$. The divisorial criterion of Theorem 3.1.1 therefore applies, with rational scale. □

## 5.4 Flag ideals and piecewise linear functions

A *flag ideal* on $X \times S$ is a coherent homogeneous ideal containing $t^N$ for some $N \geq 0$. We also use fractional flag ideals, obtained by multiplication by $t^{-a}$ for an integer $a \geq 0$. As a graded subsheaf of $\mathcal{O}_X[t, t^{-1}]$, a fractional flag ideal $\mathcal{I}$ has coefficient ideals $\mathfrak{a}_j \subset \mathcal{O}_X$ with $\mathfrak{a}_j \subset \mathfrak{a}_{j+1}$, zero for $j$ sufficiently small and $\mathcal{O}_X$ for $j$ sufficiently large. It is generated by the terms $\mathfrak{a}_j t^j$ up through the first term with coefficient ideal $\mathcal{O}_X$. Conversely such a finite list defines a fractional flag ideal. Define

$$\varphi_{\mathcal{I}}(v) = -\sigma(v)(\mathcal{I}) = \max_j \bigl(-v(\mathfrak{a}_j) - j\bigr). \tag{5.2}$$

Here ideal values on $X$ are taken at the centre of $v$. On the compact set of points with a centre in a given affine chart, the value of an ideal is the finite minimum of the values of its generators. The maximum in (5.2) includes a finite constant, so it gives a finite continuous function. A finite affine cover of $X$ gives a finite closed cover by these compact sets, as in the proof of Theorem 1.3.1, and proves continuity everywhere.

For a vertical Cartier divisor $D$ on a model, write $\varphi_D(v) = \sigma(v)(D)$, using the value of a local equation. Extend this definition to rational divisors by clearing denominators. Evaluation is compatible with pullback and additive in $D$. On the blowup $\mu$ of a flag ideal, $\mathcal{I}\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{X}(D)$ for a vertical Cartier divisor $D$ and $\varphi_D = \varphi_{\mathcal{I}}$. For integral $\mathcal{I}$ the divisor $-D$ is effective. The blowup is a test configuration. Its Rees algebra embeds in a polynomial ring over the ring of the product on every affine chart. The blowup charts thus have no $k[t]$-torsion and reduce to the marked product away from zero. Since $\mathcal{I}$ is homogeneous, its pullbacks under the action and projection maps $\mathbb{G}_m \times X \times S \to X \times S$ agree. These maps are flat, and blowing up commutes with flat base change [Stacks, Tag 0805]. Hence the action lifts to the blowup, with the action identities checked on the schematically dense marked open, and the tautological sheaf is linearized. It is ample relative to the product. Tensoring a suitable power with a pullback of a high power of an ample sheaf on $X$ gives an auxiliary ample sheaf over $S$.

**Theorem 5.4.1** *The functions $\varphi_D$ are precisely the functions $m^{-1}(\varphi_{\mathcal{I}} - \varphi_{\mathcal{J}})$ with $m$ a positive integer and $\mathcal{I}, \mathcal{J}$ fractional flag ideals. They form a rational vector lattice $\mathrm{PL}_{\mathbb{Q}}(X)$, uniformly dense in $C(X^{\mathrm{an}})$. For two rational vertical Cartier divisors on a common test configuration integrally closed along the puncture, their potentials are ordered if and only if the divisors are ordered. In fact it suffices to test the potential inequality at the finitely many $v_E$ on that model.*

***Proof*** The difference of flag ideal potentials is represented on a common model of their blowups. For the converse, take a common dominating model that maps by $\mu$ to the product and is integrally closed along the puncture. On this model every vertical Cartier divisor is invariant. Indeed the action and projection pull it back to two vertical divisors on $\mathbb{G}_m \times \mathcal{X}$. Their orders at each generic point of $\mathbb{G}_m \times E$ agree, since the automorphism $(s, x) \mapsto (s, sx)$ preserves $\mathbb{G}_m \times E$. Integral closure along the puncture persists after adjoining $s, s^{-1}$, as in Lemma 5.2.1. The pole criterion of Proposition 5.2.1, in the generality noted after its proof, applied in both directions identifies the two divisors. Equality of their fractional invertible sheaves gives the linearization compatible with the marking, as the action identities hold inside the ring of functions on the puncture.

Choose an auxiliary ample linearized invertible sheaf $\mathcal{A}$. Its restriction to the puncture is the pullback of a line bundle $A$ on $X$. This gives a vertical Cartier divisor $D_0$ with $\mathcal{O}(D_0) = \mathcal{A} \otimes \mu^* \operatorname{pr}_X^* A^{-1}$, using the induced trivialization off zero. This sheaf is $\mu$-ample. For an integral vertical $D$, the sheaves $\mathcal{O}(D + jD_0)$ and $\mathcal{O}(jD_0)$ are generated relative to $\mu$ for all sufficiently large $j$ by the relative generation property of ample sheaves [Stacks, Tags 01VK and 01QA]. For either divisor $B$ the direct image $\mu_* \mathcal{O}(B)$ embeds via the marking in $\mathcal{O}_{X \times S}[t^{-1}]$ and agrees with the product ring on the puncture. Coherence and a finite affine cover imply the bounds by $t^{-a}\mathcal{O}_{X \times S}$ from above and $t^N \mathcal{O}_{X \times S}$ from below for some $a, N$. The direct image is equivariant, hence homogeneous, and is a fractional flag ideal. Relative generation identifies its pullback ideal with $\mathcal{O}(B)$. Subtract the corresponding two potentials and then allow a rational denominator.

Multiplication of ideals adds their functions, and the sum of ideals takes their maximum. In the formula for a difference, pass to a common denominator and use

$$\max(\varphi_{\mathcal{I}} - \varphi_{\mathcal{J}}, \varphi_{\mathcal{K}} - \varphi_{\mathcal{H}}) = \varphi_{\mathcal{I}\mathcal{H} + \mathcal{K}\mathcal{J}} - \varphi_{\mathcal{J}\mathcal{H}}.$$

The lattice contains the rational constants by using powers of $t$. If two points have distinct centres, choose the ideal of the closure of one of the centres that does not contain the other. Its values at the points are strictly positive and zero respectively. If their centres are equal, some element of the local ring at the common centre has different values, since the restrictions to the local ring determine the corresponding seminorms on an affine chart. The principal ideal of such an element on an affine neighborhood extends to a coherent ideal on $X$ by scheme-theoretic closure. In both cases the flag $(\mathfrak{a}, t^N)$ separates the two points for sufficiently large $N$, by (5.2). The uniform closure is a real vector lattice, since rational scalar multiplication approximates real scalar multiplication in the uniform norm. It contains the constants and separates points, so density follows from Lemma 4.4.1.

Finally the potential of an effective vertical divisor is nonnegative, as a regular local equation has nonnegative value at any centred valuation. Conversely, at $v_E$ the potential is the coefficient of the divisor at $E$ divided by $b_E$, by Proposition 5.3.1. The pole criterion of Proposition 5.2.1 proves the order assertion. □

**Proposition 5.4.1** *Every positively rationally scaled divisorial valuation of a component $X_i$ of $X$ occurs as a $v_E$ on a test configuration integrally closed along the puncture. The trivial valuation of each component occurs on the product model.*

***Proof*** Write the valuation as $v = (\ell/m)\,\mathrm{ord}_F$, with positive integers $\ell, m$, and let $z$ be a uniformizer for $\mathrm{ord}_F$. Choose $n_i - 1$ elements of $k(X_i)$ of value zero with algebraically independent residues. Their residues under $w = \sigma(v)$, together with the residue of $z^m/t^\ell$, are algebraically independent over $k$. Indeed a nonzero polynomial in these residues can be lifted to a polynomial in $z^m/t^\ell$ with coefficients of value zero whenever the coefficient polynomial is nonzero. Its terms have distinct powers of $t$ and have Gauss value zero by (5.1).

On an affine open of $X$ containing the centre of $v$, write the resulting $n_i$ homogeneous fractions as $a_j/b$ with common homogeneous denominator $b$. The homogeneous polynomials here can be taken in $\mathcal{O}_X[t]$, by clearing denominators on the component and lifting coefficients. Choose $N > w(b)$. The ideal generated by $b, a_1, \ldots, a_{n_i}, t^N$ is a flag ideal on this affine chart. Extend its finitely many coefficient ideals by scheme-theoretic closure, and include $t^N$, to obtain a flag ideal on the product. One can alternatively extend just the finitely many homogeneous generators using the closures of their principal coefficient ideals; the two extensions agree over the chart, which is all that is used. On the chart of the blowup indexed by $b$, the valuation has a centre, since $w(b)$ is the minimum value among the generators. Its residue field contains the independent residues just constructed. Pass to the finite modification integrally closed along the puncture. The centre there lies on the component with function field $k(X_i)(t)$, has residue transcendence degree at least $n_i$, and lies over $t = 0$. Thus it has codimension one on that component. In the local ring $R$ at this centre, the minimal prime of that component is a closed point of $\mathrm{Spec}\, R[t^{-1}]$. In fact the only prime properly containing it in $R$ is the maximal ideal, which contains $t$. It is an entire irreducible component of $\mathrm{Spec}\, R[t^{-1}]$, disjoint from the other components. The idempotent of this open and closed point lies in $R$ by integral closure along the puncture. Since $R$ is local, the idempotent is one. Every minimal prime of $R$ avoids $t$, so no other component of the total space passes through this centre. It is therefore a generic point of the special fibre of the model. The DVR at that point is dominated by $w$. A valuation dominating a DVR in its own field is a positive multiple of its order, since units have value zero. Normalize by $w(t) = 1$ to get $v = v_E$. The final assertion follows on the product by taking the order of $t$ at a generic point of a component of $X$. The product is integrally closed along the puncture. In fact on a reduced affine chart with ring $A$, a Laurent polynomial with negative lowest exponent cannot be integral over $A[t]$. In a monic relation its highest power has a nonzero term of strictly lower exponent than every other term, since the lowest coefficient has nonzero powers in the reduced ring $A$. □

## 5.5 Fubini–Study potentials

Let $L$ be a rational line bundle on $X$. If $mL$ is integral and $s$ is a section of $mL$, put $\log|s|(v) = -v(s/e)$, with $e$ a local frame near the centre of $v$. The right-hand side is independent of the frame, since the ratio of two frames is a local unit. At a point where the support is in the zero scheme of $s$, the log norm is $-\infty$. This norm is the trivial model norm on $mL$.

**Definition 5.5.1** A *rational Fubini–Study potential* for $L$ is a function

$$\varphi = m^{-1} \max_j \bigl(\log|s_j| + a_j\bigr) \tag{5.3}$$

for a finite family of sections $s_j \in H^0(X, mL)$ without common zeros and weights $a_j \in \mathbb{Q}$. Here $m$ is a positive integer with $mL$ integral.

A marked test configuration for $L$ defines a potential even when its line bundle is not semiample. Pull it back to a common dominating model with the product, and write its line bundle in additive notation as $\mu^* \operatorname{pr}_X^* L + D$, with the vertical rational Cartier divisor $D$ given by the marking. The model potential is $\varphi_D$, which is independent of the common model by pullback. On a common model integrally closed along the puncture, equality of the potentials gives equality of the marked rational line bundles by Theorem 5.4.1. This identifies equivalence under common pullback with equality of potentials.

**Theorem 5.5.1** *The model potentials of relatively semiample test configurations for $L$ are precisely the rational Fubini–Study potentials for $L$. If $L$ is ample, every such potential is represented by a unique relatively ample test configuration for $L$ integrally closed along the puncture, up to marked equivariant isomorphism.*

***Proof*** Raise the sections in (5.3) to a common power to clear the denominators of the weights and multiply $m$ by the same power. The sections $t^{-a_j} s_j$ of $\operatorname{pr}_X^*(mL)$ on the puncture generate a fractional ideal after tensoring their image by $\operatorname{pr}_X^*(-mL)$. It is a fractional flag ideal. In fact the sections generate on the puncture by the condition on common zeros, and multiplication by some power of $t$ makes them all regular on the product. On the blowup of that ideal, write $\mathcal{I}\mathcal{O} = \mathcal{O}(D)$. The sections generate $\mu^* \operatorname{pr}_X^*(mL) + D$ over $S$. Its potential divided by $m$ is precisely (5.3).

Conversely take a sufficiently divisible positive power $m\mathcal{L}$ of a relatively semiample test configuration. Since $S$ is affine, there are finitely many global sections generating it. Each section decomposes into a finite sum of homogeneous sections for the algebraic $\mathbb{G}_m$-action. For example, the action map applied to the section is a section over $\mathbb{G}_m \times \mathcal{X}$, hence a Laurent polynomial with section coefficients. The group law makes the coefficients homogeneous. Replacing the generators by their homogeneous terms still gives a generating system. Under the marking the homogeneous sections are $t^{-a_j} s_j$, with $a_j \in \mathbb{Z}$ and $s_j \in H^0(X, mL)$. They generate off zero, so the $s_j$ have no common zeros. After pullback to a common model with the product

the sections generate the ideal of the marked divisor in a frame of $mL$. Evaluating its generators gives the formula (5.3).

Assume now that $L$ is ample. Starting with the semiample configuration, we first make it integrally closed along the puncture. Choose $m$ also large enough that $mL$ is very ample. We can require the finite homogeneous generating family of $m\mathcal{L}$ to restrict, up to powers of $t$, to a spanning set for $H^0(X, mL)$. Indeed any section on the puncture extends to the model after multiplication by a sufficiently high power of $t$, as is seen on a finite affine trivializing cover. These generators define an equivariant morphism to a projective space over $S$, with $m\mathcal{L}$ the pullback of $\mathcal{O}(1)$. Its scheme-theoretic image is the closure of the image of the marked puncture, by schematic density in the source. It is $\mathbb{G}_m$-stable, since the morphism is equivariant and scheme-theoretic image commutes with the flat base changes along the action and projection maps [Stacks, Tag 081I]. The image over the puncture is isomorphic to the marked product by very ampleness, and the closure is flat by the torsion criterion. With $\mathcal{O}(1)$ linearized by the weights of the homogeneous generators, it is therefore a test configuration. After its finite modification integrally closed along the puncture, the pullback of $\mathcal{O}(1)/m$ is relatively ample and the map from the source lifts. The marked line bundle pulls back to $\mathcal{L}$, which gives the required potential.

For uniqueness, take a common dominating model of two such relatively ample configurations. Equality of the potentials identifies their pullbacks as marked rational line bundles on its modification integrally closed along the puncture. The proper direct image identity following Proposition 5.2.1 shows, by the projection formula for an invertible sheaf, that the two configurations have the same marked section ring of a sufficiently divisible common tensor power. A projective scheme with an ample invertible sheaf over an affine base is recovered as the relative Proj of this ring. The canonical open immersion has dense image [Stacks, Tag 01Q1] and is proper over the affine base, because the Proj is separated over that base. It is therefore an isomorphism. The line bundle powers are recovered as the tautological sheaves [Stacks, Tag 01PZ]. The marked graded identification gives an equivariant isomorphism of the pairs. □

## 5.6 Scaling and deformation to the normal cone

For $c > 0$, write $cv$ for multiplication of all the values of $v$ by $c$, keeping its infinite values. On a continuous function $\varphi$ on $X^{\mathrm{an}}$, define

$$(c \cdot \varphi)(v) = c\varphi(c^{-1}v). \tag{5.4}$$

This gives an action of the positive reals. For an integer $d \geq 1$ it is induced on model potentials by the base change $t = u^d$. Indeed the Gauss lift in $u$ restricts on the puncture to the Gauss lift with $t$-weight $d$. That valuation is $d$ times the lift of $d^{-1}v$ with $t$-weight one. The centre on the base-changed model maps to the centre of this restriction, which gives (5.4) by evaluation. The action on the model uses the $d$-th

power on the old action and multiplication on $u$. No separability is required for this base change. Formula (5.4) preserves $\mathrm{PL}_{\mathbb{Q}}(X)$ for rational $c > 0$. In addition to the integer action just proved, for $c = d^{-1}$ formula (5.2) gives $c \cdot \varphi_I = d^{-1}\varphi_{\mathcal{J}}$, where $\mathcal{J}$ has coefficient ideals $\mathfrak{a}_j^d$ at the same exponents $j$. It preserves rational Fubini–Study potentials for rational $c > 0$, as it multiplies the weights in (5.3) by $c$ and leaves the section terms unchanged.

*Example 5.6.1 (Deformation to the normal cone)* Let $X$ be smooth integral, and let $Z \subsetneq X$ be a nonempty smooth integral closed subscheme with ideal $\mathfrak{a}$. Blow up $(\mathfrak{a}, t)$ on $X \times S$. The special fibre consists of the proper transform $E_0$ of $X \times \{0\}$, which is the blowup of $X$ along $Z$, and the exceptional divisor $E_Z$, a projective bundle over $Z$ with fibres $\mathbb{P}^{\operatorname{codim}_X Z}$. Both occur with multiplicity one. To see the multiplicities, use local generators $z_1, \dots, z_r$ of the ideal of $Z$ from a regular system of parameters, with $t$ the additional transverse parameter. On the $z_i$-chart one has $t = z_i(t/z_i)$, with $z_i = 0$ defining the exceptional divisor and $t/z_i = 0$ defining the proper transform. On the $t$-chart only the exceptional divisor appears, given by $t = 0$. The projective-bundle description follows by reducing the Rees algebra to the symmetric algebra of the conormal bundle of $Z \times \{0\}$.

The function of the ideal is

$$\varphi(v) = -\min\{v(\mathfrak{a}), 1\}.$$

At $E_0$ the normalized valuation restricts trivially on $k(X)$. At $E_Z$ it restricts to the order of the exceptional divisor of the blowup of $X$ along $Z$, or to $\operatorname{ord}_Z$ if $Z$ is a divisor. Indeed, at the generic point of $Z$ the order of a regular function is the degree of its initial form in the associated graded ring of the ideal of $Z$, and the order of a quotient is the difference. This degree is unchanged after adjoining the parameter $t$ to the parameters of $X$, which proves the restriction formula. The potential is zero at $v_{E_0}$ and $-1$ at $v_{E_Z}$. Let $\mu$ be the blowup morphism to $X \times S$. For ample $L$, the configuration with line bundle $\mu^* \operatorname{pr}_X^*(mL) - E_Z$ is semiample for large $m$. Sections of $mL \otimes \mathfrak{a}$ together with $t$ times sections of $mL$ generate it on the blowup. Taking $X = \mathbb{P}^1_k$ and $Z$ a rational point recovers the function in Section 1.5.

# Chapter 6
# Plurisubharmonic potentials in real classes

The section of $O(1)$ vanishing at a rational point of $\mathbb{P}^1_k$ has log norm $-r$ at the point $r\,\mathrm{ord}_p$ of the corresponding ray. The log norm is $-\infty$ at its endpoint. The Fubini–Study potentials of Chapter 5 allow us to replace this log norm by $\max\{-r, -b\}$, and to decrease to the log norm by letting $b$ tend to infinity. To include a class that has sections with an unavoidable common zero locus, we will also allow approximants with poles and let the numerical classes of their line bundles vary.

Throughout, $X$ is a nonempty reduced projective scheme over a trivially valued field $k$. The numerical space $N^1(X)_{\mathbb{R}}$ uses the convention of Section 3.5. We use its usual finite-dimensional topology. In particular the ample cone is open, the nef cone is its closure, and a rational line bundle is ample if its numerical class is ample, by Proposition A.1.1. Write $[L]$ for the numerical class of a rational line bundle $L$.

We first show how lists of sections give decreasing approximations in a real class. In a nef class the approximating functions can be chosen finite, and this allows us to iterate decreasing limits in Theorem 6.2.2. The order theorem in Theorem 6.3.1 then shows that divisorial values determine the whole function, in every numerical class; this order theorem is used throughout Chapter 7. For a rational ample class we recover decreasing approximation in a fixed line bundle (Proposition 6.4.1). The chapter ends with the logarithmic wells of Example 6.5.1, which obstruct a general argument for iteration when the inner approximants have poles.

## 6.1 Finite families of sections and their poles

For an integral closed subscheme $W \subseteq X$, let $v_{W,0}$ be the trivial valuation of its function field, viewed as an analytic point of $X$. This is the point $v_{\eta_W,\mathrm{triv}}$ of Section 1.1, where $\eta_W$ is the generic point of $W$. For every irreducible component $X_i$ we use its reduced structure and write $v_{i,0} = v_{X_i,0}$.

**Definition 6.1.1** For a rational line bundle $L$, let $\mathcal{H}^{\mathrm{gf}}(L)$ denote the functions

$$f = m^{-1} \max_j (\log |s_j| + \lambda_j), \qquad s_j \in H^0(X, mL), \quad \lambda_j \in \mathbb{R}, \tag{6.1}$$

where $m$ is a positive integer with $mL$ integral, the family is finite, and on each component some $s_j$ is generically nonzero. The superscript gf refers to generic finiteness. Put $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ for the same construction with rational weights. Write $\mathcal{H}_{\mathbb{Q}}(L)$ for its everywhere finite part, as in Definition 5.5.1.

Using local frames as in Section 5.5, (6.1) is continuous as a function with values in $[-\infty, +\infty)$. Equivalently its exponential is continuous. Its pole set is $Z^{\mathrm{an}}$, with $Z$ the reduced common zero locus of the listed sections. If $v$ lies in $W^{\mathrm{an}}$, each $s_j$ vanishing identically on $W$ has value zero in the multiplicative norm at $v$. For the other sections, $\log |s_j|(v) \leq 0$ since the section is measured in the local ring at the centre. It follows that

$$\sup_{W^{\mathrm{an}}} f = f(v_{W,0}), \tag{6.2}$$

including the case $f|_{W^{\mathrm{an}}} \equiv -\infty$. In particular, the condition of generic finiteness of a Fubini–Study function is equivalent to finiteness at each $v_{i,0}$, or to finiteness on every point with support the generic point of a component.

Let $\mathrm{Psef}(X)$ be the closed numerical cone generated by effective Cartier divisors. If $\mathcal{H}^{\mathrm{gf}}(L) \neq \varnothing$, then $[L]$ is pseudoeffective. In fact a sufficiently divisible power of $L$ has a regular section. To see this, take a function (6.1) and extend to an algebraic closure of $k$. For each geometric irreducible component, some listed section is generically nonzero. Indeed on any original component where it is nonzero, the injectivity of multiplication by the section persists on that component after field extension, by flatness. Any geometric component is a component of the base extension of some original component. Over the algebraic closure, the linear combinations of the $s_j$ that vanish generically on a given component form a proper linear subspace. The algebraic closure is infinite, so a finite union of such subspaces misses some linear combination. If $k$ is infinite, one can instead choose a linear combination over $k$ avoiding the finitely many original components. If $k$ is finite, the coefficients of the chosen combination lie in a finite Galois extension. The product of its conjugates is invariant and descends to a section of a power of $mL$ over $k$. On the extension it is nonzero at every generic point. Its descent is therefore nonzero at the generic point of every component, hence is a nonzerodivisor since $X$ is reduced. Its divisor represents a positive integral multiple of $[L]$.

**Definition 6.1.2** For an integral line bundle $B$, its *base ideal* is the image of the evaluation map tensored by $B^{-1}$,

$$H^0(X, B) \otimes_k B^{-1} \longrightarrow \mathcal{O}_X.$$

We denote the ideal by $\mathfrak{b}(B)$ and its zero set by $\mathrm{Bs}(B)$. For a rational line bundle $L$, its *stable base locus* $\mathbf{B}(L)$ is the intersection of $\mathrm{Bs}(mL)$ over the positive $m$ for which $mL$ is integral.

The product of sections gives $\mathfrak{b}(mL)\mathfrak{b}(nL) \subseteq \mathfrak{b}((m+n)L)$. In particular the base loci decrease along a sequence of factorially divisible multiples. Since $X$ is

Noetherian and these multiples are cofinal for divisibility, the sequence stabilizes at $\mathbf{B}(L)$. Once a base locus equals $\mathbf{B}(L)$, it does so for every multiple of that index. The argument with regular sections above shows that some multiple of $L$ admits a regular section exactly when $\mathbf{B}(L)$ contains no component. Semiampleness is equivalent to $\mathbf{B}(L) = \varnothing$.

Suppose a basis of $H^0(X, mL)$ is generically nonzero on every component. With zero weights it gives

$$-\frac{1}{m}v(\mathfrak{b}(mL)).$$

The formula follows by evaluating the generators in a local frame. The function has pole set $\mathrm{Bs}(mL)^{\mathrm{an}}$, and every function in $\mathcal{H}^{\mathrm{gf}}(L)$ has poles on $\mathbf{B}(L)^{\mathrm{an}}$. Thus one can read the common obstruction to finiteness directly from the section algebra.

All the following operations can be checked on the families of sections. A tensor power of all sections, with the same multiplication of the weights and of $m$, leaves the function unchanged. After choosing a common $m$, maxima in the same line bundle correspond to union of the two families. The sum of functions for $L$ and $L'$ corresponds to pairwise products of sections of $mL$ and $mL'$, with weights added. At each generic point the product of nonzero sections is nonzero, so the sum is generically finite. For a positive rational $a$, multiplication by $a$ in the ordinary sense takes $\mathcal{H}^{\mathrm{gf}}(L)$ onto $\mathcal{H}^{\mathrm{gf}}(aL)$, as is seen by clearing the denominators and taking tensor powers. These assertions also hold with rational weights. Addition of a constant acts within $\mathcal{H}^{\mathrm{gf}}(L)$, with a rational constant in the rational case. The scaling action of (5.4) acts by multiplying all the weights, without the section terms.

When $A$ is an ample rational line bundle, globally generating sections of a power of $A$ with weights zero give the function zero. The sum rule yields

$$\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L) \subseteq \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L + A) \tag{6.3}$$

and the analogous statement for real weights. In fact, for a fixed positive power, the zero-weight sections give zero because for any centre at least one section is a local frame. This argument will allow us to perturb a class without perturbing its approximating function.

## 6.2 Decreasing approximation in varying classes

**Definition 6.2.1** A function $u : X^{\mathrm{an}} \to [-\infty, +\infty)$ is *$\theta$-plurisubharmonic*, or $\theta$-psh, for $\theta \in N^1(X)_{\mathbb{R}}$, if it is finite at each $v_{i,0}$ and is the pointwise limit of a decreasing net

$$u_\alpha \in \mathcal{H}^{\mathrm{gf}}(L_\alpha), \qquad [L_\alpha] \longrightarrow \theta.$$

We denote this space by $\mathrm{PSH}(X, \theta)$, or $\mathrm{PSH}(\theta)$. Every such function is upper semicontinuous as an infimum of continuous extended-real-valued functions.

Equation (6.2) passes to an infimum in the following simple way. At any point of $W^{\mathrm{an}}$, the inequality $u_\alpha(v) \le u_\alpha(v_{W,0})$ holds for every index, so

$$\sup_{W^{\mathrm{an}}} u = u(v_{W,0}). \tag{6.4}$$

Thus the finiteness condition in Definition 6.2.1 is equivalent to requiring a function not identically $-\infty$ on each component. Taking the trivial points of the components also gives a finite maximum of $u$ on $X^{\mathrm{an}}$. All the classes of the approximants are pseudoeffective, so $\theta$ is pseudoeffective whenever $\mathrm{PSH}(\theta)$ is nonempty.

**Proposition 6.2.1** *In Definition 6.2.1 we can require rational weights and $[L_\alpha] - \theta$ ample. If $\theta$ is nef, the approximants can be taken in $\mathcal{H}_{\mathbb{Q}}(L_\alpha)$ with $L_\alpha$ ample.*

***Proof*** Fix a decreasing net $(f_i)_{i\in I}$ as in Definition 6.2.1. For each $i$ and small rational $\varepsilon > 0$, round the weights of $f_i$ upwards so the rational weight function $r$ satisfies

$$f_i + \varepsilon \le r \le f_i + 2\varepsilon.$$

The pole sets of $f_i$ and $r$ are equal. Order the triples $(i, \varepsilon, r)$ by increasing $i$, decreasing $\varepsilon$ and decreasing $r$ pointwise. Given two triples, take a common later index and choose $2\varepsilon$ less than both previous margins. Since the original net decreases, the new rational function is bounded above by both previous rational functions. The same inequalities hold at any point where a previous function is $-\infty$, since all later functions are $-\infty$ there. This gives a directed set. The resulting net converges pointwise to $u$, and its classes converge to $\theta$ because later triples have later original indices. In particular this replacement only uses comparisons to shifted functions from a single monotone net.

Fix an ample integral line bundle $A$. Choose positive rational numbers $e_i$ such that $e_i \to 0$ and $[L_i] + e_i[A] - \theta$ is ample. Such numbers exist because a neighborhood of $[A]$ lies in the ample cone: a fixed multiple of the norm of $[L_i] - \theta$, with an additional arbitrarily small positive rational margin, can be used to choose $e_i$. More explicitly, if a ball of radius $\delta$ around $[A]$ lies in that cone, then for nonzero difference one can require, after rounding, a rational $e_i$ between $2\|[L_i] - \theta\|/\delta$ and $3\|[L_i] - \theta\|/\delta$. To treat all indices uniformly, write $I$ for the index set of the rational net, replace it by $I \times \mathbb{Z}_{>0}$ with the product order, keep the function $f_i$ at $(i, j)$, and choose a rational $e_{(i,j)}$ with

$$2\|[L_i] - \theta\|/\delta + 1/j \le e_{(i,j)} \le 3\|[L_i] - \theta\|/\delta + 2/j.$$

Then $[L_i] + e_{(i,j)}[A] - \theta$ is ample, $e_{(i,j)} \to 0$ along the product net, and by (6.3) the function $f_i$ is unchanged after replacing $L_i$ by $L_i + e_{(i,j)}A$. This gives the first assertion.

If $\theta$ is nef, the resulting line bundles are ample. On the product directed set with positive integer $b$, replace the approximants by $\max\{f_i, -b\}$. The constant $-b$ comes from a generating family of a power of the ample $L_i$, with suitable weights. These maxima are rational and finite and form a decreasing net with the same pointwise limit and class limit. □

**Theorem 6.2.1** *For real classes $\theta, \eta$ and $a > 0$, we have*

$$\mathrm{PSH}(\theta) + \mathrm{PSH}(\eta) \subseteq \mathrm{PSH}(\theta + \eta), \qquad a\,\mathrm{PSH}(\theta) = \mathrm{PSH}(a\theta).$$

*Each space $\mathrm{PSH}(\theta)$ is convex and closed under finite maxima, translation by constants, and the action (5.4). It is contained in $\mathrm{PSH}(\eta)$ when $\eta - \theta$ is nef. For a morphism $q : Y \to X$ of reduced projective schemes, the pullback of a $\theta$-psh function is $q^*\theta$-psh whenever it is finite at the trivial point of each component of $Y$.*

***Proof*** Use the product directed set and the sum rule for Fubini–Study functions to prove the sum assertion. The limit is finite at every $v_{i,0}$. Positive rational multiplication follows at once from the corresponding rule for section families. For a real scalar $a > 0$, pass to the tail starting at any fixed approximant. Subtract a constant making all the remaining approximants nonpositive, which is possible because that fixed approximant is bounded above. Multiply these approximants by an increasing sequence of positive rational numbers tending to $a$, on the product directed set. The new net is decreasing by nonpositivity and its classes converge to $a\theta$. Adding back the appropriate constant proves the inclusion, and using $a^{-1}$ proves equality. The two assertions imply convexity. The translation and scaling actions follow directly from the corresponding formulas for section weights.

For the maximum choose approximants $f_i, g_j$ for the same class $\theta$, using rational line bundles $L_i, M_j$. Fix an ample integral line bundle $A$. On the product directed set, the differences of their numerical classes converge to zero. By openness of the ample cone choose small rational $e_{i,j} > 0$ tending to zero such that both differences

$$(L_i + e_{i,j}A) - L_i, \qquad (L_i + e_{i,j}A) - M_j$$

are ample, using the same choice mechanism as in Proposition 6.2.1. Thus the functions $f_i, g_j$ both come from section families in $L_i + e_{i,j}A$. Their maximum comes from the union of those families after clearing denominators. It decreases pointwise to the desired maximum, and the numerical limit is still $\theta$.

The zero function belongs to $\mathrm{PSH}(\eta-\theta)$ if $\eta-\theta$ is nef, by taking zero-weight generating families in rational ample classes converging to $\eta - \theta$. The inclusion follows from the sum rule. For pullback, $q^*\theta$ is induced by pullback of Cartier classes. It is well defined on numerical classes: for a Cartier divisor $D$ on $X$ and an integral curve $C \subseteq Y$, the degree of $q^*D$ on $C$ is $[k(C) : k(q(C))]$ times the degree of $D$ on $q(C)$ when $q(C)$ is a curve, and zero when $q(C)$ is a point, by (3.2) with $n = 1$ and the argument following it. Thus $q^*$ is a linear, hence continuous, map $N^1(X)_{\mathbb{R}} \to N^1(Y)_{\mathbb{R}}$. The log norm of a pulled-back section is the pullback of its log norm by compatibility of centres. For each approximant the pulled-back families are generically nonzero on every component, since they are bounded below by the limit at the trivial points. The pulled-back numerical classes tend to $q^*\theta$, which gives the result. □

### 6.2.1 The decreasing-limit theorem in nef classes

The definition uses one decreasing net of functions given by sections. If the terms of a second decreasing net are already psh, each comes with its own approximating net. A comparison on the compact space can combine these nets when the inner approximants are finite. The finiteness hypothesis matters, as Example 6.5.1 shows. If an inner approximant and the function against which it is compared both have poles at a point, adding a constant can leave an equality $-\infty = -\infty$ there. Continuity does not turn it into a strict inequality on a neighborhood.

**Lemma 6.2.1** *Let $K$ be compact Hausdorff and $T$ a metric space. For each $t \in T$, let $\mathcal{F}_t$ be a family of finite continuous functions on $K$, closed under adding positive rational constants. Suppose that $g_i$ decreases pointwise to $g$, that $t_i \to t$ in $T$, and that each $g_i$ is the pointwise limit of a decreasing net $f_{ij}$ with $f_{ij} \in \mathcal{F}_{t_{ij}}$ and $t_{ij} \to t_i$. The limits may take the value $-\infty$. Then $g$ is the pointwise limit of a decreasing net of functions in $\mathcal{F}_{s_\alpha}$ with $s_\alpha \to t$.*

***Proof*** A triple $\alpha = (i, j, \delta)$, with $\delta > 0$ rational, gives the function $F_\alpha = f_{ij} + \delta$ and the number $e_\alpha = d(t_{ij}, t_i) + \delta$. Order triples by increasing $i$, decreasing $F_\alpha$ pointwise, and decreasing $e_\alpha$. To dominate two triples choose a common later $i$ and choose $\delta$ strictly smaller than both previous positive shifts. The inequalities $g_i + \delta < F_{\alpha_1}, F_{\alpha_2}$ hold throughout $K$. This is a strict comparison because both right-hand sides are finite. For fixed $i$ and $\delta$, the open sets where $f_{ij} + \delta$ satisfies both strict comparisons increase to a cover as $j$ increases. By Lemma 4.1.2, one of these sets covers $K$. We can increase $j$ further so $d(t_{ij}, t_i) + \delta$ is smaller than both previous budgets. This proves directedness.

Given any $x \in K$ and real number $b > g(x)$, first choose $i$ with $g_i(x) < b$, then choose $j$ and $\delta$ so $f_{ij}(x) + \delta < b$. Every term majorizes $g$. This proves the pointwise limit, also when $g(x) = -\infty$. For the class limit, given $\varepsilon > 0$ choose $i$ so $d(t_{i'}, t) < \varepsilon/2$ for all $i' \geq i$. Then choose $j, \delta$ with $e_\alpha < \varepsilon/2$. For every later triple, its class is within $\varepsilon$ of $t$ by the triangle inequality. □

**Theorem 6.2.2** *Suppose $\theta_\alpha \to \theta$, with $\theta$ nef, and $u_\alpha \in \mathrm{PSH}(\theta_\alpha)$ decreases pointwise to $u$. If $u$ is finite at the trivial valuation $v_{i,0}$ of every component $X_i$ of $X$, then $u \in \mathrm{PSH}(\theta)$.*

***Proof*** Fix an ample $A$. By openness of the ample cone we can choose a vanishing net $\varepsilon_\alpha > 0$ such that $\theta_\alpha + \varepsilon_\alpha[A] - \theta$ is ample, repeating indices if necessary as in Proposition 6.2.1. The functions $u_\alpha$ are then psh in the ample classes $\theta_\alpha + \varepsilon_\alpha[A]$ by Theorem 6.2.1. For these perturbed classes, Proposition 6.2.1 gives decreasing inner nets of finite continuous functions. Apply Lemma 6.2.1 on the compact Hausdorff space $X^{\mathrm{an}}$ (Theorem 1.3.1) to the families of finite Fubini–Study functions with the prescribed numerical classes. The result is exactly the required approximation of $u$.□

If all $u_\alpha$ are everywhere finite, then so are their Fubini–Study approximants, whose sections therefore have no common zero. Their line bundles are then semi-

ample, so the classes $\theta_\alpha$, and hence $\theta$, are nef without further assumption. The finiteness of $u$ at the points $v_{i,0}$ cannot be dropped: the constant functions $-m$, for positive integers $m$, lie in $\mathrm{PSH}(0)$ and decrease to $-\infty$.

Let $u \leq 0$ be $\theta$-psh for a nef class $\theta$. The truncations $\max\{u, -b\}$ are bounded and decrease to $u$ as $b$ increases. The $\theta$-psh property of each truncation follows from the maximum rule and the constant functions in a nef class. For a bounded $\theta$-psh function in any class, the defining approximants are everywhere finite, so their line bundles are semiample and $\theta$ is nef. Conversely, a nef class contains every constant potential.

## 6.3 Order on the valuation space

The scaling of a single valuation yields an ordinary convex function of one variable. For $v \in X^{\mathrm{an}}$, let $0v = v_{\xi,\mathrm{triv}}$ denote the trivial valuation of its support $\xi$. For $c > 0$ use the action on points of Section 5.6. Each nonzero section at the support of $v$ contributes an affine function of $c$ to (6.1). Its slope is nonpositive because the section is evaluated at the centre. Terms vanishing at the support are $-\infty$ for all $c \geq 0$. Thus $c \mapsto f(cv)$ is convex and decreasing, allowing the identically $-\infty$ case. The same inequalities pass to a decreasing limit $u$. As $c \to +\infty$, the points $cv$ tend to $v_{Z,0}$, where $Z$ is the closure of the centre of $v$. In an affine chart around that centre, the scaled values tend to $+\infty$ precisely on its prime ideal and are zero on the other elements. Since the support of $v$ specializes to its centre, the sections nonzero at the latter have log norm zero at $v$, which gives $u(cv) \geq u(v_{Z,0})$. Upper semicontinuity gives the reverse inequality on the limit. This proves convergence of the ray values to the maximum on $Z^{\mathrm{an}}$ from (6.4).

Let the *divisorial set* $\mathcal{D}(X)$ consist of the trivial valuations $v_{i,0}$ and the positively rationally scaled divisorial valuations of the components of $X$. They are viewed as points of $X^{\mathrm{an}}$. By Propositions 5.3.1 and 5.4.1 this is precisely the set of points arising from the special fibres of the models used in Chapter 5.

**Lemma 6.3.1** *For every $h \in \mathrm{PL}_{\mathbb{Q}}(X)$ there is a finite $\Sigma \subset \mathcal{D}(X)$ such that for every $u$ in any $\mathrm{PSH}(\theta)$,*

$$\sup_{X^{\mathrm{an}}}(u - h) = \max_{\Sigma}(u - h).$$

***Proof*** Write $h = a^{-1}(\varphi_{\mathcal{I}} - \varphi_{\mathcal{J}})$, by Theorem 5.4.1, for an integer $a \geq 1$ and fractional flag ideals $\mathcal{I}, \mathcal{J}$. Choose $\Sigma$ to be the special-fibre valuations on the blowup of $\mathcal{I}$ after taking its integral closure along the puncture. Consider first an inequality $\varphi_{\mathcal{K}} \leq \varphi_{\mathcal{I}^r}$ on $\Sigma$, where $r \geq 1$. On this fixed model $\mathcal{I}^r\mathcal{O}$ is invertible. Divide a local generator of the pullback of $\mathcal{K}$ by a local generator of this invertible ideal. The quotient is regular after inverting $t$, and its order is nonnegative at every special-fibre generic point by the inequality on $\Sigma$. By Proposition 5.2.1 it is regular. Hence $\mathcal{K}\mathcal{O} \subseteq \mathcal{I}^r\mathcal{O}$, giving the same potential inequality everywhere by evaluation.

Every finite rational Fubini–Study function $f$ is a positive rational multiple of a flag ideal function by the construction in Theorem 5.5.1. After multiplying by a

common denominator, an inequality $f \leq h + c$ with rational $c$ is of the preceding form, by adding the potential of a power of $\mathcal{J}$ to the left and combining products of ideals and powers of $t$. The power of $\mathcal{I}$ on the right has the same test model for every choice of $f$ and denominator. Thus $\sup(f - h) = \max_\Sigma(f - h)$ by testing rational bounds.

For a general $u$, choose an ample class $\omega$ with $\omega - \theta$ nef. By Theorem 6.2.1 we have $u \in \mathrm{PSH}(\omega)$. By Proposition 6.2.1 it is the decreasing limit of finite rational Fubini–Study functions. The inequalities $f_i(v) - h(v) \leq \max_\Sigma(f_i - h)$ pass to the limit since a finite maximum commutes with decreasing limits on a directed set. This gives the required equality. □

**Lemma 6.3.2** *Let $X$ be integral of dimension $n$ and $v \in \mathcal{D}(X)$. There is a linear, hence continuous, function $C_v : N^1(X)_\mathbb{R} \to \mathbb{R}$ such that*

$$v(s)/m \leq C_v([L])$$

*for every rational line bundle $L$, every positive integer $m$ with $mL$ integral, and every nonzero $s \in H^0(X, mL)$.*

***Proof*** By Proposition 5.4.1, whose models are constructed over the product, $v = v_E$ for a component $E$ of the special fibre of a test configuration integrally closed along the puncture, with a morphism to the product. Compactify the test configuration over $\mathbb{P}^1_k$, write $\mu$ for its map to the product, and take an ample integral line bundle $B$ on this compactification. For any nonzero section $s$ of $mL$, the pulled-back divisor of $s$ is effective, and its coefficient at $E$ is $b_E v(s)$. All degrees of the form $(B^n \cdot [E'])$ are positive for prime divisors on this integral projective scheme of dimension $n+1$, by ampleness. Taking degrees of the effective divisor therefore yields

$$b_E v(s)(B^n \cdot [E]) \leq m(\mu^* \operatorname{pr}_X^* L \cdot B^n \cdot [\overline{X}]).$$

Let $C_v([L])$ be the right-hand side divided by $m b_E(B^n \cdot [E])$. Proper pushforward of the curve class and the projection formula identify it with the degree of $L$ on a fixed curve class in $X$, divided by the positive number $b_E(B^n \cdot [E])$. This expression is linear on $N^1(X)_\mathbb{R}$. □

**Theorem 6.3.1** *A $\theta$-psh function $u$ is finite at every point of $\mathcal{D}(X)$. If $h : X^{\mathrm{an}} \to [-\infty, +\infty)$ is upper semicontinuous, the inequality $u \leq h$ on $\mathcal{D}(X)$ implies the inequality everywhere. In particular, $u$ is determined by its restriction to $\mathcal{D}(X)$.*

***Proof*** For finiteness restrict to the component supporting the divisorial point in question. The restriction is psh by Theorem 6.2.1. We can therefore suppose $X$ integral. The approximants for $u$ have classes converging to $\theta$, so continuity of the function $C_v$ of Lemma 6.3.2 gives, on a tail of the approximants, a common upper bound $C$ for $v(s)/m$ over their nonzero sections $s$. Choosing a section attaining the Fubini–Study value at $v_{X,0}$ gives $f_i(v) \geq f_i(v_{X,0}) - C$. Pass to the limit and use finiteness at $v_{X,0}$.

For the inequality, first take a rational PL obstacle and apply Lemma 6.3.1. For a continuous obstacle, uniform density of $\mathrm{PL}_{\mathbb{Q}}(X)$ (Theorem 5.4.1) gives rational PL majorants arbitrarily close to it from above, by adding a small rational constant to uniform approximants. The same inequality then follows for the continuous obstacle. Finally an upper semicontinuous obstacle is the pointwise infimum of its finite continuous majorants by Lemma 4.1.1. Applying the continuous case to each majorant proves the theorem. □

**Corollary 6.3.1** *On a connected $X$, the functions in* $\mathrm{PSH}(0)$ *are precisely the constants.*

***Proof*** First suppose $X$ integral. Subtract the value at its trivial valuation to arrange that $u \leq 0$ with maximum zero. The sequence $mu$, for positive integers $m$, consists of 0-psh functions and decreases to a function whose value at the trivial valuation is zero. By Theorem 6.2.2 the limit is 0-psh and hence is finite on $\mathcal{D}(X)$ by Theorem 6.3.1. It follows that $u = 0$ on $\mathcal{D}(X)$, since any negative value would give an infinite negative limit there. Determination by $\mathcal{D}(X)$ gives $u = 0$. In general restrict to the components. The constants on meeting components agree, since a point in their intersection gives an analytic point in both. Connectedness of $X$ forces all the constants to agree. □

**Definition 6.3.1** The *weak topology* on $\mathrm{PSH}(\theta)$ is pointwise convergence on $\mathcal{D}(X)$.

This topology is Hausdorff by Theorem 6.3.1. The functional $u \mapsto \sup u$ is continuous because it is the maximum of the evaluations at the finitely many $v_{i,0}$. More generally, for continuous $h$ the functional $u \mapsto \sup(u - h)$ is continuous. For rational PL $h$ it is the maximum of finitely many evaluations by Lemma 6.3.1. Replacing $h$ by $h'$ changes this functional by at most $\|h - h'\|_\infty$ at every $u$, so uniform density of $\mathrm{PL}_{\mathbb{Q}}(X)$ (Theorem 5.4.1) gives the continuity for all continuous $h$.

## 6.4 Approximation with a fixed ample line bundle

**Proposition 6.4.1** *Let $L$ be an ample rational line bundle. Every $[L]$-psh function is a pointwise decreasing limit of functions in $\mathcal{H}_{\mathbb{Q}}(L)$. For a finite continuous $[L]$-psh function the convergence is uniform. Conversely, any uniform limit of functions in $\mathcal{H}_{\mathbb{Q}}(L)$ is continuous and $[L]$-psh.*

***Proof*** Subtract a rational constant so that $u \leq 0$. Choose the finite rational Fubini–Study approximants $f_i$ of Proposition 6.2.1, with classes $[L_i] \to [L]$. For each rational $t > 1$, eventually $tL - L_i$ is ample, by convergence of classes and openness of the ample cone. The function $t^{-1} f_i$ then belongs to $\mathcal{H}_{\mathbb{Q}}(L)$, by ordinary rational multiplication and (6.3). This tail decreases pointwise to $t^{-1}u$. Along a sequence of rational $t$

decreasing to one, the functions $t^{-1}u$ decrease to $u$. Apply Lemma 6.2.1 with the parameter space a single point and the family $\mathcal{H}_{\mathbb{Q}}(L)$, then restore the rational constant. Uniform convergence in the finite continuous case follows from Proposition 4.1.1.

Conversely take a sequence selected from any uniformly convergent net, with $\|f_j - u\|_\infty \leq 2^{-j}$. For instance $g_j = f_j + 3 \cdot 2^{-j}$ is decreasing, since

$$u + 2 \cdot 2^{-j} \leq g_j \leq u + 4 \cdot 2^{-j}.$$

The right-hand bound at index $j+1$ is the left-hand bound at index $j$. The functions $g_j$ remain in $\mathcal{H}_{\mathbb{Q}}(L)$ and have limit $u$, which proves the assertion. □

*Example 6.4.1 (A log norm on the projective line)* Let $s$ be a section of $\mathcal{O}(1)$ with unique zero at a rational point $p$. For a section $s'$ not vanishing at $p$, $\log|s|$ is the decreasing limit of the rational Fubini–Study potentials $\max\{\log|s|, \log|s'| - b\}$ as positive integers $b$ tend to infinity. On the ray from $p$, the potential is $\max\{-r, -b\}$, with the limiting value $-\infty$ at the endpoint. On the other rays $\log|s| = 0$, so all the approximants are zero. The function $\log|s|$ is therefore $[\mathcal{O}(1)]$-psh and has a pole at exactly the point supported at $p$.

### 6.4.1 Approximation in a fixed real ample class

**Proposition 6.4.2** *Let $\theta$ be a real ample class. Every $\theta$-psh potential is a decreasing limit of continuous $\theta$-psh potentials of the form $cf + b$, where $0 < c < 1$, $b \in \mathbb{R}$, and $f$ is a finite rational Fubini–Study potential. The convergence is uniform when the limit is finite and continuous.*

***Proof*** Subtract a constant to arrange $u \leq 0$ and take finite rational Fubini–Study majorants $f_j$ with classes $\eta_j \to \theta$ as in Proposition 6.2.1. For fixed $0 < c < 1$, the difference $\theta - c\eta_j$ is ample eventually. Thus $cf_j$ is $\theta$-psh by Theorem 6.2.1, and this tail decreases to $cu$. Let $c$ increase to one so that $cu$ decreases to $u$. Apply Lemma 6.2.1 with the parameter space a single point and the family of continuous $\theta$-psh functions of the indicated form, which is closed under the required constant shifts. Restore the subtracted constant. Uniform convergence for finite continuous limits follows from Proposition 4.1.1. □

## 6.5 An obstruction to iterating decreasing limits

The restriction to finite inner approximants in Lemma 6.2.1 can be tested on a single compact interval. The following example also shows why the proof of Theorem 6.2.2 first makes the classes ample.

*Example 6.5.1 (Disjoint logarithmic wells)* For $i \geq 1$ put $p_i = 2^{-i}$, $r_i = 2^{-i-2}$ and

$$\ell_i(x) = \min\{0, \log(|x - p_i|/r_i)\}, \qquad 0 \leq x \leq 1.$$

The closed intervals $[p_i - r_i, p_i + r_i]$ are disjoint. With $n, j$ positive integers, let

$$f_{n,j} = \sum_{i=1}^{n} \max\{2\ell_i, (1 + 1/n)\ell_i - j\}.$$

These functions are continuous with values in $[-\infty, 0]$. Let $\mathcal{F}$ contain exactly their translates by real constants. For fixed $n$, the functions $f_{n,j}$ decrease to $g_n = 2\sum_{i=1}^{n} \ell_i$, and $g_n$ decreases as $n$ increases to $g = 2\sum_{i\geq 1} \ell_i$. At each point at most one summand is nonzero.

Suppose a decreasing net $f_{n_\alpha, j_\alpha} + c_\alpha$ converged to $g$. Near $p_1$, the coefficient of $\log|x - p_1|$ in $f_{n,j}$ is $1 + 1/n$. For $n > 1$ the second term of the maximum dominates sufficiently close to $p_1$. For $n = 1$ the first term dominates and has the same coefficient 2. Therefore increasing $n_\alpha$ at a later index would increase the function arbitrarily close to $p_1$, regardless of the constant translates. The integers $n_\alpha$ form a decreasing net and are constant, say $N$, on a tail beginning at an index where the minimum is attained. But on that tail the functions have the same value $c_\alpha$ at 1 and $p_{N+1}$, whereas $g(1) = 0$ and $g(p_{N+1}) = -\infty$. Thus the limit after two decreasing approximations need not arise from one decreasing net in the original family.

The example concerns an arbitrary family of continuous extended-real-valued functions. In the case of psh potentials outside the nef cone, a proof of decreasing closure needs additional geometric information about the approximating families.

# Chapter 7
# Homogeneous potentials and singularities

On the ray of a closed point $p$ of $\mathbb{P}^1_k$, the log norm of a section with a simple zero at $p$ is $-r$ at $r\,\mathrm{ord}_p$. This function records one vanishing order. The truncated function $\max\{-r,-h\}$ records the same order near the origin, together with another piece of data, the height at which the function becomes constant. Taking the largest homogeneous minorant after subtracting a level recovers the part of the original function visible at that level.

We work over a trivially valued field and use the notation of Chapter 6. Homogeneous approximation and the decreasing-limit theorem apply to reduced projective schemes. Birational descriptions use an integral scheme, with normality where the argument invokes divisor orders. We first construct the homogeneous slices and recover the original potential from them (Theorems 7.1.1 and 7.1.2), and prove that homogeneous potentials are closed under decreasing limits in varying classes (Theorem 7.2.1). We then describe homogeneous functions using ideals and divisors on birational models. We also introduce valuations of linear growth, at which every $\theta$-psh function is finite (Section 7.4). In the last sections we prove division by a fixed fraction of a divisor (Proposition 7.5.1) and show how increasing-limit results for homogeneous functions enter the envelope problem (Theorem 7.6.1).

Later, Corollary 7.6.1 enters the proof of Theorem 16.5.1, and Section 16.6 combines Theorems 7.1.1, 7.1.2 and 7.2.1 with Lemma 7.6.2. The section count (7.1) is used in Chapter 18, and linear growth in Chapter 34.

## 7.1 Taking homogeneous minorants

We use the scaling action of (5.4),

$$(t\cdot u)(v) = tu(t^{-1}v), \qquad t>0.$$

A function is *homogeneous* if $u(tv) = tu(v)$. Write $\mathrm{PSH}^{\mathrm{hom}}(X,\theta)$ for the homogeneous $\theta$-psh functions. Homogeneity forces their values at the trivial points of the

components to be zero. The maximum principle (6.4) then shows that they are non-positive. Write $\mathcal{H}^{\mathrm{hom}}(L)$ for the functions

$$m^{-1} \max_j \log |s_j|, \qquad s_j \in H^0(X, mL),$$

with the sections generically nonzero on every component as a family. These are the zero-weight Fubini–Study functions.

**Definition 7.1.1** For a function $u : X^{\mathrm{an}} \to [-\infty, +\infty)$ and $\lambda \in \mathbb{R}$, its homogeneous slice at level $\lambda$ is

$$\widehat{u}_\lambda = \inf_{t>0} \bigl(t \cdot u - t\lambda\bigr).$$

Thus $\widehat{u}_0$ is the homogenization of $u$.

Changing variables in the infimum proves homogeneity. The term $t = 1$ gives $\widehat{u}_\lambda + \lambda \le u$. Conversely, if a homogeneous $q$ satisfies $q + \lambda \le u$, scaling gives $q \le t \cdot u - t\lambda$ for every $t > 0$. Thus $\widehat{u}_\lambda$ is the largest such $q$. The infimum preserves upper semicontinuity, and slices are monotone in $u$ and decreasing in $\lambda$. For decreasing $u_i$ with limit $u$, one has

$$\widehat{u}_\lambda = \inf_i \widehat{(u_i)}_\lambda$$

by commuting the two infima. At a trivial point $v_{W,0}$, the slice equals zero if $u(v_{W,0}) \ge \lambda$ and $-\infty$ otherwise.

**Lemma 7.1.1** *Let $f \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$, and suppose $f(v_{i,0}) \ge 0$ on each component. Then $\widehat{f_0} \in \mathcal{H}^{\mathrm{hom}}(L)$.*

***Proof*** Choose the sections and rational weights $a_j$ of (6.1). At a point $v$, put $x_j = \log |s_j|(v)$. The required value multiplied by $m$ is

$$\inf_{t>0} \max_j (x_j + t a_j).$$

We can discard indices with $x_j = -\infty$ in the calculation. For finite $x_j$, a number $c$ bounds all the lines at some $t \ge 0$ precisely under the following conditions. Each index $h$ with $a_h \ge 0$ requires $c \ge x_h$. For each $a_h > 0 > a_l$, the upper bound $(c - x_h)/a_h$ for $t$ has to be at least the lower bound $(x_l - c)/(-a_l)$. The second condition is

$$c \ge \frac{-a_l x_h + a_h x_l}{a_h - a_l}.$$

These inequalities are sufficient because the maximum of the finitely many lower bounds, including zero, can be used for $t$. Using $t = 0$ in the calculation leaves the infimum unchanged. If all the surviving slopes are negative, the infimum is $-\infty$ as $t \to \infty$. The value is therefore the maximum of the displayed rational convex combinations and of the $x_h$ with $a_h \ge 0$. We use the convention that a combination with a positive coefficient of $-\infty$ equals $-\infty$, and an empty maximum equals $-\infty$. With that convention the same finite formula applies at every point.

Each rational convex combination is a normalized log norm of a product of powers of sections of $mL$. Take one common denominator for the coefficients of the combinations. The resulting products and powers of single sections all belong to the same tensor power, and give a zero-weight Fubini–Study presentation. At each $v_{i,0}$ the homogenization equals zero by the hypothesis. The family of products is therefore generically nonzero on all components. □

**Theorem 7.1.1** *Let $u \in \mathrm{PSH}(X, \theta)$ and $b_i = u(v_{i,0})$. For $\lambda \leq \min_i b_i$, the slice $\widehat{u}_\lambda$ is homogeneous $\theta$-psh and is a decreasing limit of functions in $\mathcal{H}^{\mathrm{hom}}(L_j)$ with $[L_j] \to \theta$. In particular every homogeneous $\theta$-psh function has such an approximation. If $\theta = [L]$ for an ample rational line bundle $L$, one can use $L_j = L$.*

***Proof*** Apply Proposition 6.2.1 to $u - \lambda$. Every approximant is nonnegative at each trivial point of a component. Its homogenization is a zero-weight Fubini–Study function by Lemma 7.1.1. The homogenizations decrease to $\widehat{u}_\lambda$. This limit equals zero at the trivial points, so the definition of $\theta$-psh applies. For an ample rational line bundle use Proposition 6.4.1 on $u - \lambda$ instead. Taking $\lambda = 0$ when $u$ is homogeneous gives the approximation of $u$ itself. □

**Theorem 7.1.2** *Let $u \in \mathrm{PSH}(X, \theta)$ on a reduced projective scheme, and put $b_i = u(v_{i,0})$ and $b = \max_i b_i$. Pointwise on $X^{\mathrm{an}}$,*

$$u = \sup_{\lambda < b} (\widehat{u}_\lambda + \lambda).$$

*For an integral $X$ all the slices in this formula belong to $\mathrm{PSH}^{\mathrm{hom}}(X, \theta)$. On a component $X_i$, one can use the levels $\lambda < b_i$ and slices in $\mathrm{PSH}^{\mathrm{hom}}(X_i, \theta|_{X_i})$.*

***Proof*** Fix $v$, and set $g(t) = tu(t^{-1}v)$ for $t > 0$. The function $g$ is convex. Indeed it is the pointwise infimum of a decreasing net of convex functions of $t$, obtained by taking the maximum of the finitely many affine functions $m^{-1}(\log |s_j|(v) + ta_j)$ for each Fubini–Study approximant. A convex function on an open interval with values in $[-\infty, +\infty)$ is either everywhere finite or identically $-\infty$. If it takes the value $-\infty$ at an interior point, every other point lies between that point and an interior point farther away, which forces the value there to be $-\infty$ as well. In the identically $-\infty$ case every slice at $v$ is $-\infty$.

Otherwise take a slope $a$ of a supporting line of $g$ at 1. Such a slope exists between the left and right derivatives. Since $g(t) \leq tb$, the supporting line implies $a \leq b$ by letting $t \to \infty$. If $a < b$, its intercept is $\widehat{u}_a(v) = g(1) - a$, giving the desired equality. If $a = b$, then for each $\lambda < b$ the supporting line gives $\widehat{u}_\lambda(v) \geq g(1) - b$. Letting $\lambda$ increase to $b$ gives the required lower bound for the supremum. The upper bound holds for every slice by its defining inequality. Restriction to an irreducible component is compatible with scaling and with taking the slice, so the componentwise statement follows as well. □

*Example 7.1.1* For $p$ a rational point of $\mathbb{P}^1_k$, let $u(r\,\mathrm{ord}_p) = \max\{-r, -h\}$ for $h > 0$, and $u = 0$ on the other rays. Use the limiting value at each endpoint. This is the real-weight Fubini–Study truncation of Section 6.4. Here $\sup u = 0$, so the level bound $b$ of Theorem 7.1.2 is zero, and for $\lambda \le 0$ the slice is

$$\widehat{u}_\lambda(r\,\mathrm{ord}_p) = -r\max\{0, 1 + \lambda/h\} \qquad (0 \le r < \infty).$$

Indeed the infimum is that of $\max\{-r, -ht\} - t\lambda$, whose breakpoint is $t = r/h$. If $\lambda \le -h$, the infimum is zero as $t$ tends to zero. The slice at the endpoint is zero for $\lambda \le -h$ and is $-\infty$ for $-h < \lambda \le 0$, as follows by taking the infimum of $t(-h - \lambda)$. Thus a bounded potential can have slices with poles. On the ray the reconstruction gives the larger of the contribution at $\lambda = -h$ and the limit of the contributions as $\lambda$ increases to zero.

## 7.2 Decreasing limits of homogeneous potentials

The obstruction in Example 6.5.1 used poles at which adding a constant does not improve a pointwise comparison. For a homogeneous approximant one can first compare on a compact level set. Its pole locus is algebraic, so a second comparison can be made at the finitely many generic points of that locus.

**Lemma 7.2.1** *Let $p_j$ be a decreasing net of zero-weight Fubini–Study functions with limit $p$. Let $q$ be another zero-weight Fubini–Study function, and suppose $p \le q$. For every $0 < c < 1$, one has $p_j \le cq$ eventually.*

***Proof*** The set $\{q = -1\}$ is compact by continuity of $q$ with extended real values. On this set the open subsets $\{p_j < -c\}$ increase to a cover. Some member covers it by Lemma 4.1.2. By homogeneity the resulting inequality extends to $\{-\infty < q < 0\}$, by scaling each point to level $-1$. It holds where $q = 0$ by nonpositivity of $p_j$.

The set $\{q = -\infty\}$ is $Z^{\mathrm{an}}$ for the common zero locus $Z$ of the sections for $q$. At the trivial point of each irreducible component $W$ of $Z$, every $p_j$ has value either zero or $-\infty$. Its limit is $-\infty$, so it is $-\infty$ eventually. The sections giving that $p_j$ then vanish on $W$, hence $p_j = -\infty$ on $W^{\mathrm{an}}$. A common index works for the finitely many $W$, and it can also satisfy the compact level-set comparison. □

**Theorem 7.2.1** *If $u_i \in \mathrm{PSH}^{\mathrm{hom}}(X, \theta_i)$ decreases pointwise and $\theta_i \to \theta$, then its limit is in $\mathrm{PSH}^{\mathrm{hom}}(X, \theta)$.*

***Proof*** Choose homogeneous approximants $p_{i,j} \downarrow u_i$ with classes $\eta_{i,j} \to \theta_i$ using Theorem 7.1.1. For each triple $\alpha = (i, j, c)$ with $c$ rational and $0 < c < 1$, put

$$F_\alpha = cp_{i,j}, \qquad e_\alpha = \|\eta_{i,j} - \theta_i\| + 1 - c,$$

using a norm on the finite-dimensional numerical space. The function $F_\alpha$ is zero-weight Fubini–Study in the class $c\eta_{i,j}$. It dominates the desired limit by nonpositivity

of $p_{i,j}$. Order the triples by increasing $i$, decreasing $F_\alpha$ pointwise, and decreasing $e_\alpha$.

Given two triples, take an index $i$ beyond both their first indices. Choose $c$ larger than their two coefficients and close enough to one to make $1 - c$ smaller than both errors. For each of the old indices $(i_k, j_k, c_k)$, $u_i \le p_{i_k,j_k}$. Apply Lemma 7.2.1 with ratio $c_k/c$ to arrange that $cp_{i,j} \le c_k p_{i_k,j_k}$. Increasing $j$ also makes $e_\alpha$ smaller than the two old errors. Thus the triples are directed. Arbitrarily late choices of $i$ and arbitrarily small error show that $c\eta_{i,j} \to \theta$. For each $v$, one can choose $i$ and $j$ with $p_{i,j}(v)$ arbitrarily close from above to the desired infimum and then let $c$ approach one. This proves pointwise convergence, including at points where the desired value is $-\infty$. The limit is zero at the trivial points and homogeneous, so it satisfies the definition. □

## 7.3 Ideals and divisors on birational models

For a coherent ideal $\mathfrak{a}$ that is generically the unit ideal on every component, put $\ell_\mathfrak{a}(v) = -v(\mathfrak{a})$. Each nonempty finite family of local generators gives a finite maximum of log norms. Hence $\ell_\mathfrak{a}$ is homogeneous and continuous with extended real values. Its poles are exactly $V(\mathfrak{a})^{\mathrm{an}}$. In particular, if the sections in a zero-weight presentation generate $\mathfrak{a} \otimes mL$, the potential is $m^{-1}\ell_\mathfrak{a}$. For two such ideals,

$$\ell_{\mathfrak{a}\mathfrak{b}} = \ell_\mathfrak{a} + \ell_\mathfrak{b}, \qquad \ell_{\mathfrak{a}+\mathfrak{b}} = \max\{\ell_\mathfrak{a}, \ell_\mathfrak{b}\}.$$

On an integral $X$, the normalized blowup of $\mathfrak{a}$ has an effective Cartier divisor $F$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. For a point with generic support on $X$, the valuation lifts to its centre on $Y$, where $\ell_\mathfrak{a}(v) = -v(F)$. In particular integral closure of the ideal preserves its function by Theorem 3.2.1.

For the rest of this section, assume that $X$ is integral and normal. Consider the normal projective birational models $\pi : Y \to X$, ordered by domination. A normalized graph closure gives a common dominating model for every finite collection.

**Definition 7.3.1** A rational *Cartier b-divisor* $B$ over $X$ is given by a rational Cartier divisor $B_Y$ on one such model. Two representatives give the same b-divisor when their pullbacks coincide on a common dominating model. We call a model carrying a representative a determination of $B$. Its trace on an arbitrary model is the Weil divisor obtained by pushing forward from a common dominating determination.

The trace is independent of the common determination. A proper birational morphism between normal models has pushforward of the pullback of a Cartier divisor equal to that divisor as a Weil divisor. At a prime divisor of the target its valuation ring is a DVR, and the birational map has the same local DVR at the strict transform. All other primes of the pullback contribute zero to the pushforward. Functoriality of proper pushforward then proves the assertion. For a valuation with generic support, define

$$\ell_B(v) = v(B_Y)$$

at its centre on a determination. The value is unchanged by pullback. It is finite on these valuations, though a difference of two ideal functions need not have a defined value at a common pole on the full analytification.

**Proposition 7.3.1** *The map $B \mapsto \ell_B$ is injective and additive. Its image on valuations of generic support consists of the functions*

$$m^{-1}(\ell_{\mathfrak{a}} - \ell_{\mathfrak{b}})$$

*with $m$ a positive integer and $\mathfrak{a}, \mathfrak{b}$ nonzero coherent ideals. For rational Cartier b-divisors $B, B'$ with a common determination $Y$, one has $\ell_B \le \ell_{B'}$ on $\mathcal{D}(X)$ if and only if $B_Y \le B'_Y$; in that case $\ell_B \le \ell_{B'}$ on all valuations of generic support. If $B_Y \le 0$ and $B_Y$ is semiample relative to $X$, then $\ell_B = m^{-1}\ell_{\mathfrak{a}}$ for a nonzero coherent ideal. Conversely each such ideal function comes from a relatively semiample nonpositive Cartier b-divisor.*

***Proof*** On a common determination, the values at the orders of prime divisors are the coefficients of the two Cartier divisors. A Cartier divisor on a normal scheme is effective if and only if its Weil coefficients are nonnegative, since regularity of a rational local equation can be checked at height one. This proves injectivity and the assertion about order. An effective divisor has nonnegative value at every centred valuation, giving the comparison on all valuations of generic support.

An ideal on $X$ gives a nonpositive representative $-F$ on its normalized blowup, and $\mathcal{O}_Y(-F)$ is generated relative to $X$. A difference of such b-divisors gives the displayed difference of functions. For the converse, clear the denominator of $B_Y$ and choose a relatively ample Cartier divisor $H$ on $Y$. By taking large multiples of $H$, write the resulting integral Cartier divisor as a difference of two relatively generated Cartier divisors. Their pushforward invertible sheaves are coherent fractional ideals of $\mathcal{O}_X$ in $k(X)$. By relative generation the extended fractional ideals on $Y$ equal the corresponding invertible sheaves. The two fractional ideals can be multiplied by the same nonzero coherent ideal to make them integral. Indeed the local ideals of common denominators, consisting of the elements sending both fractional ideals into $\mathcal{O}_X$, glue and are nonzero and coherent by finite generation. Multiplication by the common ideal leaves the difference of the two values unchanged. Taking account of the initial denominator proves the assertion.

If $B_Y \le 0$ and $mB_Y$ is Cartier and relatively generated, $\pi_*\mathcal{O}_Y(mB_Y)$ is an ideal in $\pi_*\mathcal{O}_Y = \mathcal{O}_X$. For the last equality use proper pushforward coherence. On each affine open the pushforward is a finite algebra inside $k(X)$, so it is integral over the normal coordinate ring and equals it. Relative generation identifies the pullback ideal with $\mathcal{O}_Y(mB_Y)$. This proves the assertion with a single ideal. □

A zero-weight Fubini–Study function for $L$ describes the movable part of a finite family of sections. If the family generates $\mathfrak{a} \otimes mL$, so that the function is $m^{-1}\ell_{\mathfrak{a}}$, take on the normalized blowup $\pi : Y \to X$ of $\mathfrak{a}$ the divisor $B_Y = -m^{-1}F$, where $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Then $m(\pi^*L + B_Y)$ is globally generated by the pulled-back family with its common factor removed.

**Proposition 7.3.2** *For a rational line bundle $L$ and a rational Cartier b-divisor $B$, the function $\ell_B$ on valuations with generic support is the restriction of a function in $\mathcal{H}^{\hom}(L)$ if and only if $B_Y \le 0$ and $\pi^*L + B_Y$ is semiample on a determination.*

***Proof*** The preceding blowup construction proves necessity, since $\ell_B$ determines $B$. For sufficiency, take a divisible power $m$ for which $m(\pi^*L + B_Y)$ is integral and globally generated. Since $B_Y \le 0$, multiplication by the canonical section of $-mB_Y$ maps the generators into $H^0(Y, m\pi^*L) = H^0(X, mL)$. This equality follows from the proper birational pushforward identity and the projection formula. At every centre on $Y$ the residual family generates, so the maximum of its local log norms is zero in a local frame of the generated bundle. The original family therefore has normalized log norm $v(B_Y)$ on valuations of generic support. The family is generically nonzero and defines the desired function. □

There is also a section count attached to this description. Let $X$ have dimension $n$. If $M = \pi^*L + B_Y$ is semiample, its volume is

$$\operatorname{vol}(M) = \lim_{j\to\infty} \frac{n!}{(jd)^n} \dim_k H^0(Y, jdM) = (M^n \cdot [Y]), \tag{7.1}$$

where $d > 0$ is an integer such that $dM$ is integral and generated. To verify the limit, use its morphism $f : Y \to Z$ onto the projective image with $dM = f^*\mathcal{O}_Z(1)$. The sheaf $f_*\mathcal{O}_Y$ is coherent. Push it forward along the closed immersion of $Z$ into projective space. Its section dimensions equal its Hilbert polynomial for large $j$ [Stacks, Tag 08AE]. If $\dim Z < n$, its degree is less than $n$ [Stacks, Tag 0BEM] and the top intersection is zero by the projection formula. If $\dim Z = n$, the generic rank is $[k(Y) : k(Z)]$ and the leading coefficient is that rank times $\deg Z/n!$ [Stacks, Tags 0BEN and 0BJ8]. The projection formula of Section 3.5 gives the same top intersection for $dM$. This count and its limit are unchanged on a dominating normal model because proper birational pushforward preserves the structure sheaf. Thus the volume of the movable bundle in Proposition 7.3.2 is independent of the choice of determination.

For a prime divisor $E$ on a normal projective model and $\lambda \ge 0$, the subspace

$$\{s \in H^0(X, mL) : \operatorname{ord}_E(s) \ge m\lambda\}$$

can be computed on that model by an ordinary divisor condition. For a rational $\lambda$ with $\lambda E$ rational Cartier, and sufficiently divisible $m$, it is $H^0(Y, m(\pi^*L - \lambda E))$. Indeed a pulled-back section already has nonnegative divisor, and the one extra requirement is the indicated order at $E$. Regularity after division is again checked in codimension one. When $\pi^*L - \lambda E$ is semiample the formula above computes the leading coefficient of this count. For example, for $L = \mathcal{O}(1)$ and a rational point $p$ on $\mathbb{P}^1_k$, the dimension of the subspace is $\max\{0, m + 1 - \lceil m\lambda \rceil\}$. Dividing by $m$ and passing to the limit gives the function $\max\{0, 1 - \lambda\}$.

## 7.4 Valuations of linear growth

Assume $X$ is integral and projective. Fix an ample line bundle $A$. For a valuation $v$ with generic support, put

$$T_A(v) = \sup_{m \geq 1} \sup_{0 \neq s \in H^0(X, mA)} \frac{v(s)}{m}.$$

Values of $m$ with $H^0(X, mA) = 0$, for which the inner supremum is over the empty set, are omitted. The valuation has *linear growth* if $T_A(v) < \infty$.

The condition is independent of the ample line bundle. For another ample $A'$, choose $d$ so that $dA - A'$ is generated. A generating section nonvanishing at the centre of $v$, raised to the power $m$ and multiplied by any $s \in H^0(X, mA')$, gives a section of $mdA$ with the same value as $s$. Hence $T_{A'}(v) \leq dT_A(v)$, and interchanging the line bundles proves independence. Scaling multiplies $T_A$ by the same positive factor. By Lemma 6.3.2 with $L = A$, every divisorial valuation has linear growth. The trivial valuation has $T_A = 0$.

If $u$ is $\theta$-psh, then $u(v) > -\infty$ for every valuation of linear growth. Indeed, choose approximants with rational line bundles $L_i$ and bounded numerical classes, restricting to a tail. Choose an integer $d$ with $d[A] - [L_i]$ ample throughout that tail, using openness about $[A]$ and boundedness of the classes. For a nonzero section $s$ of $mL_i$, pass to a power that makes the corresponding power of $dA - L_i$ integral and generated. Multiplying the power of $s$ by a section of that bundle nonvanishing at the centre yields $v(s)/m \leq dT_A(v)$. In the formula for each approximant choose a section giving its value at the trivial point. The bound on its order and passage to the limit give

$$u(v) \geq u(v_{X,0}) - dT_A(v).$$

This estimate is one reason to distinguish linear growth from the weaker requirement of generic support. The following example shows that the two conditions differ.

*Example 7.4.1* On $\mathbb{A}^2_k \subset \mathbb{P}^2_k$, take coordinates $x, y$ and a sequence of strictly increasing positive integers $N_j$ with $N_{j+1}/N_j \to \infty$. The map

$$x \longmapsto x, \qquad y \longmapsto h(x) = \sum_{j \geq 1} x^{N_j}$$

embeds $k(x, y)$ in $k((x))$. Indeed, if a nonzero polynomial $P(x, y)$ vanished at $y = h(x)$, its value at the truncation $h_j = \sum_{i \leq j} x^{N_i}$ would have order at least $N_{j+1}$. But the degree of $P(x, h_j(x))$ is at most $\deg_x P + N_j \deg_y P$, which is smaller than $N_{j+1}$ for all large $j$. Thus $P(x, y)$, viewed over $k(x)$, would have infinitely many distinct roots $h_j$. Pullback of $\mathrm{ord}_x$ gives a valuation of generic support centred at the origin. The polynomial $y - h_j(x)$ is the affine expression of a section of $O(N_j)$ and has value $N_{j+1}$. Hence this valuation does not have linear growth.

## 7.5 Division by an effective divisor

Assume now that $X$ is integral, normal and projective. For an effective rational Cartier divisor $E$, let $e_E = -v(E)$, interpreted as the normalized log norm of the canonical section of a Cartier multiple of $E$. It is zero-weight Fubini–Study of class $[E]$.

**Proposition 7.5.1** *Let $\psi \in \mathrm{PSH}^{\mathrm{hom}}(X, \eta)$ satisfy $\psi \leq e_E$. For each fixed rational $0 < c < 1$, there is a unique $q_c \in \mathrm{PSH}^{\mathrm{hom}}(X, \eta - c[E])$ such that $\psi = ce_E + q_c$.*

***Proof*** If $E = 0$, take $q_c = \psi$. Otherwise write $E = \sum_l a_l E_l$, where the $E_l$ are its prime components and $a_l > 0$. Choose zero-weight approximants $p_j \downarrow \psi$, of rational line bundles $L_j$ with $[L_j] \to \eta$, using Theorem 7.1.1. For every $E_l$ we have

$$\lim_j p_j(\mathrm{ord}_{E_l}) \leq -a_l < -ca_l.$$

One tail therefore satisfies all the inequalities $p_j(\mathrm{ord}_{E_l}) \leq -ca_l$. Write such an approximant as $m^{-1} \max_h \log |s_h|$, using a power with $mcE$ integral and Cartier. Each section has order at least $mca_l$ along every $E_l$. After division by the canonical section of $mcE$ it is regular, since normality tests regularity of a rational section in codimension one. The resulting family is still generically nonzero and yields $q_j \in \mathcal{H}^{\mathrm{hom}}(L_j - cE)$, with $p_j = ce_E + q_j$.

The $q_j$ decrease on $\mathcal{D}(X)$, where $e_E$ is finite. Theorem 6.3.1 gives their monotonicity everywhere. Their numerical classes tend to $\eta - c[E]$, and their limit equals zero at the trivial point. This gives the desired $q_c$. The equality passes to the limit using addition with $-\infty$, which applies at the poles of $e_E$. Their values on $\mathcal{D}(X)$ also give uniqueness. □

The coefficient in this proof is fixed before taking the decreasing approximants. Subtracting $c_j e_E$ with $c_j$ increasing need not preserve their order. For instance, take $E$ to be a point on $\mathbb{P}^1_k$, $p_j = (1 - 1/(2j))e_E$ and $c_j = 1 - 1/j$ for $j \geq 2$. Then $p_j$ decreases to $e_E$, whereas $p_j - c_j e_E = e_E/(2j)$ strictly increases in $j$ at every point $r\,\mathrm{ord}_E$ with $0 < r < \infty$, where $e_E = -r < 0$.

**Theorem 7.5.1** *Suppose that in every ample real class on $X$, an increasing net of homogeneous psh functions admits a homogeneous psh function in that class with the pointwise supremum as its values on $\mathcal{D}(X)$. If $\psi \in \mathrm{PSH}^{\mathrm{hom}}(X, \theta + [E])$ satisfies $\psi \leq e_E$, then there exists a unique $q \in \mathrm{PSH}^{\mathrm{hom}}(X, \theta)$ with $\psi = e_E + q$.*

***Proof*** It suffices to construct $q$ with values $w = \psi - e_E$ on $\mathcal{D}(X)$. There $w \leq 0$. Choose a very ample line bundle $A$ with $\theta + [A]$ ample. If $A_l$ are the divisors of nonzero generating sections of $A$, then $\max_l e_{A_l} = 0$, since at least one section is a unit at every centre.

Fix a rational $0 < \varepsilon < 1$ and an index $l$. Apply Proposition 7.5.1 to $\psi + e_{A_l}$ and $E$. For rational $c < 1$ sufficiently close to one this gives homogeneous functions with values $w + e_{A_l} + (1-c)e_E$ on $\mathcal{D}(X)$, in the class $\theta + [A] + (1-c)[E]$. Those functions

are also psh in the fixed ample class $\theta + (1+\varepsilon)[A]$, since the difference $\varepsilon[A] - (1-c)[E]$ is ample. They increase as $c$ increases, by the order theorem and $e_E \le 0$. The hypothesis gives a homogeneous function $\chi_l$ in that fixed ample class with values $w + e_{A_l}$ on $\mathcal{D}(X)$. In particular $\chi_l \le e_{A_l}$ everywhere. Apply Proposition 7.5.1 now to $\chi_l$ and $A_l$, with coefficient $1-\varepsilon$. The divided function is homogeneous psh in $\theta + 2\varepsilon[A]$, with divisorial values $w + \varepsilon e_{A_l}$.

The maximum of these finitely many divided functions is homogeneous psh in $\theta + 2\varepsilon[A]$, with values $w$ on $\mathcal{D}(X)$. The function is independent of $\varepsilon$, by Theorem 6.3.1. Apply Theorem 7.2.1 to this constant family with numerical classes tending to $\theta$. The resulting function $q$ is in the required class. The equality $\psi = e_E + q$ follows by determination, since both sides are psh in $\theta + [E]$ and agree on $\mathcal{D}(X)$. The same comparison gives uniqueness. □

## 7.6 From homogeneous limits to envelopes

We keep $X$ integral and projective for this section. Normality will enter only when invoking division to change the numerical class. A useful extension of the homogeneous decreasing-limit argument allows finite maxima after shifts, provided that one homogeneous function bounds every homogeneous constituent from above.

**Lemma 7.6.1** *Fix $Q \in \mathrm{PSH}^{\mathrm{hom}}(X,\theta)$. For each positive integer $n$, let*

$$U_n = \max\left\{Q - n,\ \max_a (q_{n,a} + c_{n,a})\right\},$$

*where the family is finite, $q_{n,a} \in \mathrm{PSH}^{\mathrm{hom}}(X,\theta)$, $q_{n,a} \le Q$, and $c_{n,a} \in \mathbb{R}$. If $U_n$ decreases pointwise to $u$ and $u(v_{X,0}) > -\infty$, then $u$ is $\theta$-psh.*

***Proof*** For fixed $n$, include $q_{n,0} = Q$ and $c_{n,0} = -n$ in the list. Choose decreasing homogeneous approximants $p_{n,a,j}$ for each constituent and use the product of the finitely many index sets. By adding small ample differences, as in Theorem 6.2.1, promote the families at each index to one rational line bundle of class $\eta_{n,j} \to \theta$. The promotion multiplies sections by a generating family with weights zero, so it preserves the homogeneous functions. Put

$$F_{n,j} = \max_a (p_{n,a,j} + c_{n,a}), \qquad R_{n,j} = \max_a p_{n,a,j}.$$

Then $F_{n,j} \downarrow U_n$ and $R_{n,j} \ge Q$. For rational $0 < b < 1$ and real $\delta > 0$,

$$G(v) = F_{n,j}(bv) + \delta$$

is Fubini–Study of class $b\eta_{n,j}$. In fact it is the maximum of $bp_{n,a,j} + c_{n,a} + \delta$ and rational multiplication by $b$ changes the common line bundle by that factor. Nonpositivity of the homogeneous terms gives $G \ge U_n$.

Consider two such choices $G_k$, made with $(n_k, j_k, b_k, \delta_k)$ for $k = 1, 2$, and write $R_k = R_{n_k,j_k}$. Choose $n \geq \max n_k$, rational $b > \max b_k$ below one, and $0 < \delta < \min \delta_k$. Pick $\beta_k$ between $b_k$ and $b$. Since every $q_{n,a} \leq Q \leq R_k$, Lemma 7.2.1 gives on a common tail in $j$ that $bp_{n,a,j} \leq \beta_k R_k$ for all $a, k$. The finite lists of constants then give a finite $C \geq 0$, independent of $j$, such that

$$G \leq \beta_k R_k + C, \qquad G_k \geq b_k R_k - C.$$

Since $\beta_k > b_k$, this proves $G \leq G_k$ on $\{R_k < -M\}$ for a sufficiently large $M$. It includes $R_k = -\infty$, where every term in $G$ equals $-\infty$.

On the compact set $\{R_k \geq -M\}$, the function $G_k$ is finite and continuous. Since $U_n$ decreases along each scaling ray and $U_n \leq U_{n_k}$, we have

$$U_n(bv) \leq U_{n_k}(b_k v) \leq F_{n_k,j_k}(b_k v).$$

Thus $G$ decreases as $j$ increases to a limit strictly below $G_k$ on this compact set, since $\delta < \delta_k$. The open sets $\{G < G_k\}$ exhaust it there. Choosing a common tail gives $G \leq G_k$ for both old choices.

Order the choices by increasing $n$, decreasing $G$ pointwise, and decreasing the positive error $\|\eta_{n,j} - \theta\| + 1 - b + \delta$. The preceding argument permits arbitrarily large $n$ and arbitrarily small error in finding a common successor, by choosing $b$ close to one, then $\delta$ small and $j$ late. Hence this is a directed preorder with $b\eta_{n,j} \to \theta$. Its functions have infimum $u$, since one can choose $n$ and $j$ first at any specified point, then let $b$ increase to one and $\delta$ decrease to zero. A term with value $-\infty$ remains $-\infty$ under this change. Finiteness at the trivial point now gives the defining approximation of $u$. □

**Lemma 7.6.2** *Let $q_\lambda$, $\lambda < 0$, be a family in $\mathrm{PSH}^{\mathrm{hom}}(X, \theta)$, decreasing in $\lambda$, with a common upper bound $Q$ in that class. Then*

$$U = \sup_{\lambda<0}(q_\lambda + \lambda)$$

*belongs to* $\mathrm{PSH}(X, \theta)$.

***Proof*** Partition $[-n, 0]$ into subintervals of length $2^{-n}$. If $A_n$ is the list of left endpoints, put

$$U_n = \max\left\{Q - n,\ \max_{a\in A_n}(q_a + a + 2^{-n})\right\}.$$

Upon subdivision of an old interval the left endpoint can only increase and the right endpoint can only decrease. The contribution therefore decreases. New intervals to the left of $-n$ contribute at most $Q - n$. Moreover,

$$U \leq U_n \leq \max\{Q - n, U + 2^{-n}\}$$

pointwise. For the first bound use $q_\lambda \leq Q$ to the left of $-n$ and the left-endpoint term of the interval containing $\lambda$ elsewhere. The second bound uses the definition of $U$ at

each left endpoint. Since $Q \le 0$, the bounds give $U_n \downarrow U$, including at points where $U = -\infty$. At the trivial point $U = 0$. The conclusion follows from Lemma 7.6.1. □

**Theorem 7.6.1** *Suppose that for every increasing net in* $\mathrm{PSH}^{\mathrm{hom}}(X,\theta)$ *there is a function in* $\mathrm{PSH}^{\mathrm{hom}}(X,\theta)$ *whose values on* $\mathcal{D}(X)$ *are the pointwise supremum. For every uniformly bounded-above increasing net* $u_i \in \mathrm{PSH}(X,\theta)$*, the upper semicontinuous regularization of its supremum is* $\theta$*-psh and agrees with* $\sup_i u_i$ *on* $\mathcal{D}(X)$*.*

***Proof*** Subtract a constant so that the numbers $b_i = \sup_{X^{\mathrm{an}}} u_i$, which are increasing, have supremum zero. For each $\lambda < 0$, the indices with $b_i > \lambda$ form a cofinal tail. The slices $\widehat{(u_i)}_\lambda$ are homogeneous psh there and increase with $i$. The hypothesis gives $q_\lambda$ in the homogeneous class with their supremum as its divisorial values. By order determination these functions decrease in $\lambda$. Writing $f = \sup_i u_i$ on $\mathcal{D}(X)$, reconstruction and cofinality give

$$f(v) = \sup_{\lambda<0}(q_\lambda(v) + \lambda), \qquad v \in \mathcal{D}(X).$$

The hypothesis applied to $q_{-n}$ gives a homogeneous $Q$ dominating every $q_\lambda$, first on $\mathcal{D}(X)$ and then everywhere.

Now $U = \sup_{\lambda<0}(q_\lambda + \lambda)$ is psh by Lemma 7.6.2, and it agrees with $f$ on $\mathcal{D}(X)$.

Finally every $u_i \le U$ by determination. Since $U$ is upper semicontinuous this gives $(\sup_i u_i)^* \le U$. The reverse inequality follows by applying determination to $U$ against the upper semicontinuous obstacle $(\sup_i u_i)^*$, using the divisorial equality just obtained. □

**Corollary 7.6.1** *On a normal integral projective* $X$*, the ample-class hypothesis of Theorem 7.5.1 implies the conclusion of Theorem 7.6.1 in every real numerical class.*

***Proof*** For a given class $\theta$, choose an effective ample Cartier divisor $E$ with $\theta + [E]$ ample, by taking a section of a sufficiently high very ample power. If $u_i$ are increasing and homogeneous in $\theta$, then $u_i + e_E$ increase in $\theta + [E]$. The hypothesis gives a homogeneous $\psi$ in that ample class with their supremum as its divisorial values. It satisfies $\psi \le e_E$ on $\mathcal{D}(X)$, hence everywhere. Theorem 7.5.1 gives $\psi = e_E + q$ in the required sense, so $q$ has the divisorial values $\sup_i u_i$. Apply Theorem 7.6.1. □

# Part II
# Tropical charts and local products

Log norms of units give coordinates on tropical charts. The first chapter of this part constructs these coordinates on algebraic tori and proves the balancing of the resulting weighted fans. The second studies the fibres of a finite analytic chart over the Gauss section and builds finite free charts at arbitrary weights. Real forms on these charts give a notion of positivity and a Stokes formula, and the last chapter uses them to define local Monge–Ampère products and to approximate bounded potentials on a fixed chart. The separation of the lifts of a Gauss point, which refines the chart construction, is in an appendix.

The envelope theorem of the next part does not depend on this part: its affine regularization uses locally convex piecewise linear functions introduced in that part. The comparison of local and global measures on curves at the end of the global energy theory uses the tropical weights of the first chapter of this part to integrate real forms on the valuation rays. The toric chapters of the last part compare their model measures with the local products of the last chapter.

# Chapter 8
# Tropical coordinates

In the torus with coordinates $x, y$, the equation $1 + x + y = 0$ gives three rays in the plane of log norms. By the ultrametric inequality, at each point of the curve the least of the three values $0$, $-\log|x|$ and $-\log|y|$ is attained at least twice. To see that every point on the rays is attained, we use an affinoid algebra in which the norms of both coordinates are prescribed. This argument also gives an initial-ideal criterion for any ideal in a Laurent polynomial ring (Theorem 8.3.1). The primitive vectors of the three rays sum to zero.

Throughout this chapter $k$ is a trivially valued field. We use additive coordinates $-\log|f|$ and allow arbitrary real radii, as in Chapter 2. In particular the coordinate image of an analytic chart need not be cut out by inequalities with rational constants.

For a closed subscheme $Z$ of a split torus, the initial ideals are constant on the relative interiors of the cones of a finite rational fan. If $Z$ is pure of dimension $n$, its tropicalization is the support of a subfan whose maximal cones have dimension $n$ (Theorem 8.6.1). Weighting each such cone by the dimension of its transverse initial algebra makes the tropicalization balanced (Theorem 8.7.1), as for the three rays above. The proof reduces to curves. For a curve, Lemma 8.7.2 expresses the weights through the orders of the characters at the closed points of the normalized projective closures of its components, counted with lengths and residue degrees.

Chapter 9 studies finite maps to the annular domains of Section 8.4 along their Gauss points. Chapter 10 integrates real forms on weighted polyhedral complexes, of which balanced tropicalizations are examples, and compares these weights with the degrees of Gauss fibres. Chapter 20 uses Lemma 8.7.2 and its branch decomposition on the valuation rays of a curve.

## 8.1 Moments and their images

Let $U$ be an analytic domain with invertible analytic functions $f_1, \dots, f_r$. They define a moment map to the analytic torus with coordinate functions $T_1, \dots, T_r$. Its tropicalization is the continuous map

$$\operatorname{trop}_f : U \longrightarrow \mathbb{R}^r, \qquad x \longmapsto (-\log|f_1(x)|, \ldots, -\log|f_r(x)|).$$

On each affinoid chart, the norms of the units and their inverses give a closed annular box containing the image. Bounded substitution into that box constructs the map on the chart. On affinoid refinements of an overlap, compatible units give the same bounded substitutions into a common enlarged annular box. Equality on Laurent polynomials extends to the completed algebra of the box, so the maps agree as analytic morphisms. The image of each affinoid chart is compact. An algebraic map to a torus gives a moment on the analytification by evaluating its coordinate functions.

A change of moment given by monomials $g_j = c_j \prod_i f_i^{a_{ji}}$ induces $\operatorname{trop}_g = A \operatorname{trop}_f$, where $A = (a_{ji})$ and $c_j \in k^\times$. For two lists of units on the same domain, concatenating the lists gives a moment refining both. Its coordinate projections recover the two given maps. This refinement is a statement about coordinates. The dimension and integration weights of the image require further arguments.

## 8.2 A torus fibre at real weight

Write $R = k[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$. For $w \in \mathbb{R}^r$ define a norm by

$$\left\| \sum_a c_a T^a \right\|_w = \max_{c_a \neq 0} \exp(-\langle a, w \rangle)$$

and let $B_w$ be the completion. Here $a$ ranges over $\mathbb{Z}^r$. The norm is multiplicative. Indeed, for each nonzero Laurent polynomial its terms of least weight form a nonzero homogeneous Laurent polynomial for the real grading $\deg T^a = \langle a, w \rangle$. For a product the product of these terms is nonzero because $R$ is a domain, and all remaining terms have strictly greater weight. The same argument applies to any nonzero series in the completion. Its weighted term norms tend to zero off finite sets, so its terms of largest norm form a nonempty finite set. We write $\operatorname{in}_w(g)$ for the least-weight part of a nonzero polynomial or series.

Put $r_i = \exp(-w_i)$. The algebra $B_w$ is the quotient of the Tate algebra with variables $T_i, S_i$ of radii $r_i, r_i^{-1}$ by the closed ideal generated by $T_i S_i - 1$. In fact substitution $S_i = T_i^{-1}$ is contractive since it preserves the weight of each monomial. Each Laurent monomial lifts to the monomial $\prod_i T_i^{\max(a_i,0)} S_i^{\max(-a_i,0)}$ with the same norm. This linear lifting extends to the completed series and is a right inverse. On polynomials the difference from the lifted substitution belongs to the displayed ideal. By continuity the same holds with its closure for series. This proves the claimed quotient with its norm.

A point of $\mathcal{M}(B_w)$ has $|T_i| = r_i$, by the bounds on $T_i$ and its inverse. Conversely a point of the torus with these values is bounded by the Laurent Gauss norm and extends uniquely to the completion. Thus this spectrum identifies with the fibre of

the tropicalization map at $w$. In particular the Gauss norm gives a point of the fibre over every $w$.

The Gauss point alone is a boundary for $\mathcal{M}(B_w)$, in the sense that every function attains its supremum there. Every closed boundary contains that point. Otherwise, for each point of the boundary some function has value strictly less than its Gauss norm. The strict comparison sets cover the boundary, so finitely many suffice by compactness. The product of the corresponding functions has value strictly below its Gauss norm throughout the boundary, because the norm is multiplicative. This contradicts the boundary property.

**Proposition 8.2.1** *An element of $B_w$ is a unit if and only if its least-weight part is a single nonzero monomial.*

***Proof*** Write the element with a single leading monomial as $cT^a(1+b)$. All terms of $b$ have norm less than one, and their norms tend to zero off finite sets. Hence $\|b\|_w < 1$, so the geometric series gives the inverse. Conversely the least-weight parts of an element and its inverse multiply to one. Laurent polynomial units over a field are monomials. For instance, choose a real linear functional separating the distinct exponents in the two finite supports. Both the largest and smallest resulting weights add under multiplication. The unit product has width zero, forcing each support to have width zero and hence to be a singleton. □

The ordinary reduction of $B_w$ can be read off from its series. Write

$$B_w^\circ = \{g : \|g\|_w \leq 1\}, \qquad B_w^{\circ\circ} = \{g : \|g\|_w < 1\}, \qquad L_w = \ker\langle -, w\rangle.$$

For an element of $B_w^\circ$, the terms of weight zero form a finite sum. Taking this sum is a ring homomorphism to the group algebra $k[L_w]$, since weights are nonnegative. It is surjective by lifting those Laurent polynomials, and its kernel consists of the elements with norm strictly below one. Thus the ordinary reduction is $k[L_w]$. In the graded reduction, retain the quotient $\{g : \|g\|_w \leq e^{-s}\}/\{g : \|g\|_w < e^{-s}\}$ for every $s \in \mathbb{R}$. The same coefficient argument identifies it with the space of Laurent polynomials homogeneous of weight $s$. Multiplication adds weights and gives the real-graded ring $R$. This distinction matters when a weight has no nontrivial integral relation, since then the ordinary reduction is just $k$.

## 8.3 Initial ideals

For an ideal $I \subset R$, define $\mathrm{in}_w(I)$ as the homogeneous ideal of the real-graded ring $R$ generated by all $\mathrm{in}_w(f)$ for nonzero $f \in I$. Let $Z$ be the closed subscheme of the torus defined by $I$.

**Theorem 8.3.1** *A weight $w$ occurs in the image of $Z^{\mathrm{an}}$ if and only if $\mathrm{in}_w(I)$ is a proper ideal in $R$. Equivalently, no nonzero $f \in I$ has a unique term of least weight.*

***Proof*** Every point of the fibre on $Z^{\mathrm{an}}$ extends to the spectrum of $B_w/\overline{IB_w}$, and conversely every point of this quotient gives a point of the fibre by restriction. By the nonemptiness theorem for affinoid spectra in Theorem 2.2.1, the fibre is nonempty precisely when $\overline{IB_w}$ is proper. If an element of $I$ has a unique least-weight term, it is a unit in $B_w$ by Proposition 8.2.1. In the other direction, if $1 \in \overline{IB_w}$, there is a finite sum $\sum_l b_l f_l$ within distance less than one of 1, with $b_l \in B_w$ and $f_l \in I$. By density of Laurent polynomials and continuity of multiplication, the $b_l$ can be replaced by Laurent polynomials with the same strict error bound. The resulting element of $I$ has least-weight part 1.

It remains to check the equivalence with the homogeneous ideal criterion. If the ideal contains a monomial, it contains 1 because each Laurent monomial is a unit. Express 1 as a finite sum $\sum_l h_l \operatorname{in}_w(f_l)$. By taking the component of degree zero, we can take each $h_l$ homogeneous of degree opposite to that of $\operatorname{in}_w(f_l)$, omitting zero terms. Then $\sum_l h_l f_l$ has degree-zero part exactly one and its other terms have strictly positive weight. This gives an element with a unique least-weight term. The converse is immediate since a homogeneous monomial is a unit in $R$. □

For a nonzero Laurent polynomial $f$, multiplicativity shows that $\operatorname{in}_w((f))$ is generated by $\operatorname{in}_w(f)$. It follows that the tropicalization of a Laurent hypersurface is the union of the polyhedral loci where at least two terms of its equation have least weight. For $1 + x + y$, these are the rays generated by $(1, 0)$, $(0, 1)$, and $(-1, -1)$. The sum of these primitive integral vectors is zero. This calculation is the first instance of the balancing identity for tropical curves.

## 8.4 Domains defined by monomial inequalities

Let $P \subset \mathbb{R}^r$ be a compact polytope given by inequalities $\langle a_l, w\rangle \geq b_l$ with $a_l \in \mathbb{Z}^r$ and $b_l \in \mathbb{R}$. Choose a bounded product of closed annuli whose tropical image contains $P$. Inside that product, the inverse image of $P$ is the finite intersection of the affinoid domains $\{|T^{a_l}| \leq \exp(-b_l)\}$. These are obtained by adjoining disc variables of radii $\exp(-b_l)$ and identifying them with the given monomials. The identification at the level of spectra follows by adjoining the coordinates successively, using the rational domain construction of Chapter 2. The inverse image is compact and its image is all of $P$, since it contains the Gauss point over each $w \in P$. For a closed algebraic subscheme of the torus, intersecting with this affinoid domain restricts the initial-ideal criterion to weights in $P$.

## 8.5 Initial degenerations

Let $M = \mathbb{Z}^r$ be the character lattice of the split torus, let $N = \operatorname{Hom}(M, \mathbb{Z})$, and write its coordinate ring as $R = k[M]$. The tropicalization of $Z = \operatorname{Spec}(R/I)$ is denoted by $\operatorname{Trop}(Z) \subset N_{\mathbb{R}}$. For $w \in N_{\mathbb{R}}$ put

$$Z_w = \operatorname{Spec}\bigl(R/\operatorname{in}_w(I)\bigr).$$

This is the *initial degeneration* at $w$.

There is a useful distinction between the filtration coming from a presentation and the one defined by spectral norms. Give the affinoid algebra $A_w = B_w/\overline{IB_w}$ the quotient filtration, whose piece of weight at least $s$ is the image of $\{b \in B_w : \|b\|_w \leq e^{-s}\}$. This is a filtration by images. Its associated graded ring is $R/\operatorname{in}_w(I)$. Indeed, the kernel of the map from the graded ring of $B_w$ consists of initials of elements of $\overline{IB_w}$. To see this, subtract from a representative of a kernel element a lift from a strictly higher filtration piece. The difference lies in $\overline{IB_w}$ and has the given initial. Each nonzero element of $\overline{IB_w}$ can be approximated by an element of $I$ with strictly smaller error norm, so the initial does not change. This proves the description without any condition on the value group generated by $w$. For example, at weight zero the quotient for $(T-1)^2$ retains the nonzero nilpotent $T-1$ in degree zero. Its spectral norm vanishes.

For rational $w$, choose a positive integral multiple lying in $N$, which has the same initial ideal. We describe a flat family with special fibre $Z_w$. Take $w \in N$ and let $\tau$ be a parameter. In $R[\tau, \tau^{-1}]$ apply the automorphism $T^a \mapsto \tau^{\langle a,w\rangle}T^a$ to $I$. Let $\mathcal{I} \subset R[\tau]$ be the contraction of the resulting ideal. The quotient embeds in a Laurent polynomial ring over $R/I$ by the inverse automorphism, so it is torsion-free over $k[\tau]$, and hence flat. In each nonzero fibre the family is obtained from $Z$ by translating the coordinates.

The reduction of $\mathcal{I}$ at $\tau = 0$ is precisely $\operatorname{in}_w(I)$. One inclusion follows by substituting in $f \in I$ and dividing by the least power of $\tau$ that occurs. For the other, write any $g \in \mathcal{I}$, after the inverse automorphism, as $\sum_l \tau^l f_l$ with $f_l \in I$. For a fixed Laurent monomial in $g$, distinct indices $l$ have distinct $\tau$-exponents. Consequently each summand after substitution back is itself regular at $\tau = 0$. Its constant coefficient is either zero or $\operatorname{in}_w(f_l)$. The special fibre is therefore $Z_w$.

If $Z$ is nonempty and pure of dimension $n$, every nonempty $Z_w$ for rational $w$ is pure of dimension $n$. Each irreducible component of the flat family meets the punctured base, since $\tau$ is a nonzerodivisor. Those components have dimension $n+1$ by their restriction to the puncture. On each component meeting the special fibre, that fibre is locally the vanishing of the nonzero function $\tau$. The principal ideal theorem [Stacks, Tag 00KV] and the equal length of all maximal chains of primes in a finite type domain over a field [Stacks, Tag 00OS] give dimension $n$ for all its components. The support of the entire special fibre is the union of these intersections. This argument concerns the dimensions of irreducible components and permits embedded associated primes.

## 8.6 A finite fan of weights

**Theorem 8.6.1** *There is a finite rational polyhedral fan in $N_{\mathbb{R}}$ such that $\mathrm{in}_w(I)$ is constant on the relative interior of each cone. The tropicalization is the support of a subfan. If $Z$ is pure of dimension $n$, then every maximal cone of that subfan has dimension $n$.*

We first construct the fan and then prove purity.

***Proof (Finiteness)*** Let $S = k[X_0, \ldots, X_r]$, with its total degree grading. Replace $I$ by the homogeneous ideal $J$ whose degree pieces are the homogeneous polynomials mapping into $I$ under $X_0 \mapsto 1$, $X_i \mapsto T_i$. Extend $w$ to $S$ by assigning weight zero to $X_0$. The initial ideal of $J$ maps to $\mathrm{in}_w(I)$ after dehomogenizing and inverting the $T_i$. Indeed every Laurent polynomial in $I$ can be cleared of denominators and homogenized, and the initial operation then recovers its Laurent initial up to a monomial. Conversely, on any homogeneous polynomial dehomogenizing preserves the partition into weight pieces. The same reasoning works for the Laurent image of any homogeneous ideal in $S$. For this, a polynomial in the Laurent image can first be multiplied by a monomial to clear the denominators in an expression in dehomogenized generators. Homogenizing that expression after multiplying terms by suitable powers of $X_0$ gives a homogeneous lift.

Consider the space of degree-compatible monomial orders. It is a closed subspace of the product, over total degrees, of the finite sets of all orders of the monomials of that degree. The closed conditions express compatibility with multiplication by monomials. Total degree takes priority, so each such order is a well-order. In particular this space is compact. For a fixed order, select a finite homogeneous family $G \subset J$ whose leading monomials generate the leading monomial ideal. For all orders that agree on the leading terms of this finite family the same family works. In fact the monomial ideal generated by those terms is then contained in the new leading monomial ideal. Both ideals have the same number of monomials in every degree, namely $\dim_k J_d$ in degree $d$, as follows by row elimination in the ordered monomial basis. A monomial multiple of a leading term is itself a leading term in the appropriate degree. Thus the two ideals coincide. Finitely many of these open sets cover the space of orders, and the union of their families is a finite set $G_*$.

Given $w$, refine the comparison by weight to a monomial order, taking the lowest weight first among monomials of the same total degree and breaking ties by a fixed monomial order. Here the first term is the leading term. Homogeneous division by a working family for this order writes each homogeneous $f \in J$ as a sum of monomial multiples of the family, all with least weight at least that of $f$. Indeed each division subtracts a multiple with the leading monomial of the current remainder and strictly decreases that monomial. Taking the terms of least weight then proves that the weight initial ideal is generated by $\mathrm{in}_w(g)$ for $g \in G_*$. The signs of the finitely many comparisons between weights of terms in each $g$ cut $N_{\mathbb{R}}$ into a rational polyhedral fan on whose relative interiors all those initials are constant. Include the coordinate hyperplanes so that the cones are pointed.

The set $\mathrm{Trop}(Z)$ is closed. By Theorem 8.3.1 it is the intersection, over nonzero $f \in I$, of the closed loci where the minimum of the term weights of $f$ is attained

more than once. It follows from the constancy just proved that the cones with relative interiors in the tropicalization form a subfan. □

For a cone $\sigma$ of this fan put $N_\sigma = N \cap \operatorname{span}_{\mathbb{R}}(\sigma)$ and $M_\sigma = M \cap N_\sigma^{\perp}$. For $w$ in its relative interior, dehomogenize the initial form $\operatorname{in}_w(g)$ of any $g \in G_*$. The exponents of any two resulting terms differ by an element of $M_\sigma$. Thus the Laurent initial ideal is generated by polynomials in $k[M_\sigma]$, after monomial factors are removed. There is a canonical algebra, the *transverse initial algebra*,

$$C_\sigma = k[M_\sigma] / \bigl(\operatorname{in}_w(I) \cap k[M_\sigma]\bigr).$$

Writing $d = \dim \sigma$, a splitting of the saturated sublattice $M_\sigma \subset M$ identifies the initial algebra with

$$R/\operatorname{in}_w(I) \simeq C_\sigma[Z_1^{\pm 1}, \ldots, Z_d^{\pm 1}]. \tag{8.1}$$

In particular, for $Z$ pure of dimension $n$ and $\sigma \subset \operatorname{Trop}(Z)$, all components of $\operatorname{Spec}(C_\sigma)$ have dimension $n - d$. This follows from the dimension result for rational weights in Section 8.5, applied at a rational weight in the relative interior, and it proves $d \le n$.

**Lemma 8.6.1** *For any $w, \xi \in N_{\mathbb{R}}$, for all sufficiently small positive $\varepsilon$ one has*

$$\operatorname{in}_{w+\varepsilon\xi}(I) = \operatorname{in}_\xi\bigl(\operatorname{in}_w(I)\bigr).$$

***Proof*** In the homogeneous polynomial ring, the finite set $G_*$ shows that the left-hand ideal is contained in the right-hand ideal. On the finite supports the perturbed weight compares first by $w$, then by $\xi$ in the case of a tie. Both ideals have the same Hilbert function as $J$. For any real weight on a homogeneous ideal this follows by taking the associated graded of each finite-dimensional degree piece with the weight filtration. The degree pieces of the initial ideal are exactly the spaces of leading weight parts obtained this way, since a monomial multiple of an initial is the initial of the corresponding multiple. Thus the inclusion is an equality. Passing to Laurent images gives the assertion by the dehomogenization argument above. □

***Proof (Purity in Theorem 8.6.1)*** Suppose $\sigma$ is maximal and $d < n$. Choose a point $w$ in its relative interior. In (8.1) the algebra $C_\sigma$ has a positive-dimensional integral component. At least one of its generating characters is transcendental over $k$. On a normal projective compactification of this component the character must have a pole along a divisor. Otherwise the codimension-one regularity criterion for a normal scheme [Stacks, Tag 031T], applied on an affine open cover of the compactification, would make it a global regular function, hence algebraic over $k$. The order at a pole gives a nonzero weight on $M_\sigma$. Extend the valuation to the additional Laurent variables by giving them Gauss weight zero. This is a point of $Z_w^{\mathrm{an}}$ of weight $\xi \notin \operatorname{span}\sigma$. By Lemma 8.6.1 and Theorem 8.3.1, $w + \varepsilon\xi$ lies in $\operatorname{Trop}(Z)$ for small $\varepsilon > 0$. In a finite fan the cones sufficiently near $w$ contain $w$, hence contain $\sigma$ as a face. The perturbed point lies outside $\sigma$, contradicting maximality. □

## 8.7 Weights and balancing

Assume $Z$ is pure of dimension $n$. On an $n$-dimensional cone $\sigma \subset \mathrm{Trop}(Z)$ give the tropicalization the weight

$$m_\sigma = \dim_k C_\sigma.$$

This is a positive integer since $C_\sigma$ is a nonzero zero-dimensional algebra of finite type. Equivalently it is the sum of its local lengths multiplied by residue degrees. The definition is independent of a splitting in (8.1) and remains the same on the top-dimensional cones of any rational subdivision.

We will prove balancing by reducing it to the branches of a *Laurent curve*, that is, a pure one-dimensional closed subscheme of a split torus. The calculation for a curve also explains the residue degrees in the definition.

**Lemma 8.7.1** *Let $t$ be the standard affine coordinate on $\mathbb{P}^1_k$, so that $k(\mathbb{P}^1_k) = k(t)$, and let $C \to \mathbb{P}^1_k$ be a finite dominant map from a normal integral curve; put $E = k(C)$. For $a = 0$ or $\infty$, let $t_a = t$ or $t^{-1}$, respectively. Put $F_a = k((t_a))$. There is a decomposition*

$$E \otimes_{k(t)} F_a \simeq \prod_{P \mapsto a} E_P, \qquad E_P = \mathrm{Frac}\, \widehat{O}_{C,P}. \tag{8.2}$$

*The factor over $P$ has degree $e_P[\kappa(P) : k]$, where $e_P = \mathrm{ord}_P(t_a)$. For $s > 0$ and each point $P$ over $a$, the unique absolute value on $E_P$ extending $|t_a| = \exp(-s)$ on $F_a$ restricts on $E^\times$ to $\exp(-s\, \mathrm{ord}_P / e_P)$.*

***Proof*** The inverse image of the affine $t_a$-line has algebra $D$, finite and torsion-free over $k[t_a]$, hence free. The radical of $(t_a)$ in $D$ is the intersection of the maximal ideals $\mathfrak{m}_P$ over $a$. Their product is that intersection, and a power of it lies in $(t_a)$, since $D/(t_a)$ is Artinian. The powers of $(t_a)$ and the products of the powers of $\mathfrak{m}_P$ are therefore cofinal. The Chinese remainder theorem identifies the $t_a$-adic completion with the product of the $\mathfrak{m}_P$-adic completions. In each quotient by $\mathfrak{m}_P^m$ all elements outside $\mathfrak{m}_P$ are units, so this factor is the completion of $O_{C,P}$.

A one-dimensional Noetherian local normal domain is a discrete valuation ring [Stacks, Tag 00PD]. Its completion is again a discrete valuation ring: the residue field is unchanged and every nonzero completed element is a power of a uniformizer times a unit, by its first nonzero graded term. As $D$ is finite free, its $t_a$-adic completion is $D \otimes_{k[t_a]} k[[t_a]]$. Inverting $t_a$ gives (8.2). Each completed local factor is finite and torsion-free over $k[[t_a]]$, so its rank is the dimension of its quotient by $t_a$. This quotient has a filtration of length $e_P$ with each successive quotient $\kappa(P)$. This gives the stated degree. The normalized order on the completed field gives the indicated absolute value, and uniqueness of the extension follows from Lemma 2.3.1. □

**Lemma 8.7.2** *Let $Z$ be a pure one-dimensional closed subscheme of a split torus. For each irreducible component $Z_i$, let $l_i$ be the length of its generic local ring in $Z$ and let $\overline{Z}_i$ be the normalization of a projective closure of its reduced scheme. For a closed point $P$ of $\overline{Z}_i$, form the vector $\lambda_P \in N$ of the orders of the characters at $P$. If $u$ is the primitive generator of a tropical ray, its weight is*

$$m_u = \sum_{\lambda_P = e_P u,\ e_P > 0} l_i \, e_P [\kappa(P) : k].$$

*The sum ranges over the indicated points of all the $\overline{Z}_i$.*

***Proof*** A pure one-dimensional rational fan has its nonzero cones in rays. At $u$, split the character lattice by $M = M_u \oplus \mathbb{Z}a_0$ with $\langle a_0, u\rangle = 1$. Write $z = T^{a_0}$, and choose coordinates $y_j$ for $M_u$. With $|z| = e^{-1}$, the Laurent Gauss algebra at $u$ is $B_u = F\langle y_1^{\pm 1}, \dots, y_{r-1}^{\pm 1}\rangle$, where $F = k((z))$. This follows by collecting terms in $z$ in the defining completion. Its norm has discrete nonzero values. For $A_u = B_u/\overline{IB_u}$, the quotient norm is attained on every coset, by discreteness and closedness. Its unit ball is complete in the $z$-adic topology, and its reduction modulo $z$ is $C_u$, by the quotient-filtration calculation of Section 8.5. In fact multiplication by $z$ shifts the norm by exactly $|z|$, and the degree-zero part of (8.1) at $u$ is $C_u$.

Choose lifts of a $k$-basis of $C_u$ in the unit ball. Every element of that ball has a unique expansion over $k[[z]]$ in these lifts. Construct its coefficients by successive reduction modulo $z$, division by $z$, and completeness. Independence follows by dividing a hypothetical relation by the smallest power of $z$ among its coefficients and reducing. Thus $\dim_F A_u = m_u$.

On the other hand let

$$A_{\mathrm{alg}} = F \otimes_{k[z,z^{-1}]} (R/I).$$

This is finite-dimensional over $F$. Components on which $z$ is algebraic over $k$ disappear at the generic point of the $z$-line, and the other components give zero-dimensional generic fibres, by the transcendence degree formula. The natural map $A_{\mathrm{alg}} \to A_u$ is surjective, since its image contains the dense Laurent subalgebra in the quotient and both spaces are finite dimensional over $F$. We identify exactly the surviving local Artinian factors of $A_{\mathrm{alg}}$.

A factor whose residue field gives $|y_j| = 1$ for every $j$ is a bounded quotient of the Laurent Gauss algebra. Indeed $y_j$ and its inverse in this factor satisfy monic equations over $k[[z]]$. Such equations follow by first using the minimal polynomial in the residue field and then taking a sufficiently high power to kill the nilpotent error. The polynomial has integral coefficients because all conjugates have the same absolute value over the complete field. The recurrence gives uniformly bounded powers, and products over finitely many $j$ remain uniformly bounded. Thus Laurent substitution extends to the completion. It factors through $A_u$ and restricts to the projection of $A_{\mathrm{alg}}$. This shows that every indicated factor survives entirely. Conversely a nonzero quotient of a factor has its residue field and hence a point, which must have $|y_j| = 1$ by the Laurent Gauss inequalities. This removes all other factors.

For a component $Z_i$ on which $z$ is transcendental, the rational function $z$ on $\overline{Z}_i$ defines a finite map to $\mathbb{P}^1_z$. It extends across closed points by the discrete valuation rings of the normal curve and is finite by properness and quasi-finiteness [Stacks, Tag 02LS]. Lemma 8.7.1 identifies the factors of $F \otimes_{k(z)} k(Z_i)$ with the points $P$ above zero. The factor dimension is $e_P[\kappa(P) : k]$.

The factor at $P$ survives precisely when all $y_j$ have order zero there. Since $z$ then has positive order, this is $\lambda_P = e_P u$, with $e_P = \operatorname{ord}_P(z)$. Before completion the zero-dimensional generic algebra over $k(z)$ has a composition series with $l_i$ factors isomorphic to $k(Z_i)$ for each dominating component. Tensoring it with $F$ is exact. Projection onto the surviving factors is also exact, and on the composition factors it picks precisely the field factors just computed. Points of components on which $z$ is algebraic over $k$ do not occur in the sum, since there $\operatorname{ord}_P(z) = 0$, whereas $\langle e_P u, a_0 \rangle = e_P > 0$. Hence the dimension of $A_u$ is the asserted sum. □

**Theorem 8.7.1** *Let $Z$ be pure of dimension $n \geq 1$. Give $\operatorname{Trop}(Z)$ the subfan structure of Theorem 8.6.1, or a rational subdivision of it, with the weights $m_\sigma$ on its $n$-dimensional cones. For a cone $\tau \subset \operatorname{Trop}(Z)$ of this fan of dimension $n-1$, these weights satisfy*

$$\sum_{\sigma \supset \tau} m_\sigma u_{\sigma/\tau} = 0 \qquad \text{in } N/N_\tau,$$

*where $\sigma$ ranges over the $n$-dimensional cones containing $\tau$ and $u_{\sigma/\tau}$ is the primitive generator of the outgoing ray modulo $N_\tau$.*

***Proof*** For a Laurent curve, sum the formula of Lemma 8.7.2 over the rays. Every nonzero $\lambda_P$ belongs to a tropical ray, since the order at $P$ is a valuation on the function field of the component. The resulting sum is

$$\sum_{i,P} l_i[\kappa(P) : k]\lambda_P.$$

In each character coordinate it is zero, by the following degree-zero property applied to every character on every $\overline{Z}_i$. Let $C$ be a normal projective integral curve over $k$ and $h \in k(C)^\times$. Then $\sum_P [\kappa(P) : k] \operatorname{ord}_P(h) = 0$, the sum ranging over the closed points of $C$. Indeed, if $h$ is algebraic over $k$, it is a unit at every $P$. Otherwise $h$ defines a finite map $C \to \mathbb{P}^1_k$, as $z$ does in the proof of Lemma 8.7.2. By Lemma 8.7.1 at $a = 0$ and $a = \infty$, both $\sum_{P \mapsto 0} e_P[\kappa(P) : k]$ and $\sum_{P \mapsto \infty} e_P[\kappa(P) : k]$ equal $[k(C) : k(h)]$. Since $\operatorname{ord}_P(h)$ is $e_P$ over 0, $-e_P$ over $\infty$, and zero elsewhere, the two sums cancel.

For general $n$, choose $w$ in the relative interior of $\tau$. At that weight (8.1) expresses the initial degeneration as a product of a torus of dimension $n-1$ and the Laurent curve $\operatorname{Spec}(C_\tau)$ with character lattice $M_\tau$. By Lemma 8.6.1 and the initial-ideal criterion, the directions $\xi$ for which $w + \varepsilon\xi$ is tropical for all small $\varepsilon > 0$ form the tropicalization of $Z_w$. Its quotient by the span of $\tau$ is exactly the tropicalization of that Laurent curve. This follows either from the torus factor or by applying the initial-ideal criterion to polynomials in $k[M_\tau]$. For a cone $\sigma$ containing $\tau$, a direction gives points of $\sigma$ near $w$ precisely when it lies in $\sigma + \operatorname{span}\tau$. In one direction subtract $w$

and divide by $\varepsilon$. In the other, write the direction as $a+b$, with $a \in \sigma$ and $b \in \operatorname{span}\tau$, and use $w + \varepsilon b \in \tau$ for small $\varepsilon$. If the projected direction is in the interior of the outgoing ray, take $a$ in the relative interior of $\sigma$ to see that the nearby point is in that relative interior. Thus the quotient rays correspond to the outgoing cones in the formula. At a relative interior direction of one of these rays, taking the initial in $k[M_\tau]$ and removing the ray's one-dimensional torus factor gives $C_\sigma$, by the same iterated-initial identity. Thus the multiplicity is the weight of the corresponding curve ray, with primitive generators taken in the lattice dual to $M_\tau$, namely $N/N_\tau$. Apply the curve case. □

*Example 8.7.1 (A binomial and its tropical weight)* Let $a \neq b$ in the character lattice $M$ of a split torus and let $c \in k^\times$. Write $a - b = q\mu$, where $q > 0$ is an integer and $\mu$ is primitive. The tropicalization of $T^a - cT^b = 0$ is the hyperplane $\langle \mu, w\rangle = 0$, with weight $q$.

Indeed, the initial-ideal criterion gives precisely the hyperplane on which the two terms have equal weight. Extend $\mu$ to a lattice basis and put $z = T^\mu$. After dividing by $T^b$, the equation is $z^q - c$. At a top-dimensional cone $\sigma$ in the hyperplane, the transverse initial algebra $C_\sigma$ defining the weight $m_\sigma$ is $k[z, z^{-1}]/(z^q - c)$. The $q$ powers from 1 to $z^{q-1}$ form a basis by polynomial division and $z^{-1} = c^{-1}z^{q-1}$. Thus its dimension is $q$, including when the characteristic divides $q$.

# Chapter 9
# Gauss fibres and finite charts

Over the one-dimensional torus with coordinate $t$, the equation $z^2 - z - t = 0$ defines a finite cover of rank two. Over the Gauss point of weight $w > 0$ it has two points, each of local degree one. For $w \leq 0$ it has a single point of local degree two. As $w \to 0$, the log norms of $z$ at all these points tend to zero, and the points converge to the unique point over weight zero (Example 9.1.1). To integrate over such fibres, we weight each point by the dimension of its local fibre algebra.

The general statements about finite maps and affinoid dimension are proved over a complete ultrametric field. For moments, $k$ is trivially valued, $Q \subset N_{\mathbb{R}}$ is a compact full-dimensional box, $T_Q$ is the closed product of annuli over $Q$ from Section 8.4, and $\eta_w$ is its Gauss point of weight $w$. Relative interiors are those of Section 2.5.

For a finite locally free map the weighted fibres vary continuously (Proposition 9.1.1). Over $T_Q$, the joint log norms of finitely many units on the source carry the weighted Gauss fibres to finitely many affine graphs over the cells of a finite subdivision of $Q$, with constant integer weights (Theorem 9.1.1). To treat a moment of an affinoid space of dimension at most $n$ into an $n$-dimensional box, we prove the dimension formula of Theorem 9.3.1 for affinoid spaces with arbitrary real radii. Every fibre over a Gauss point is then finite (Corollary 9.3.1), and a relative interior point isolated in its fibre has a finite rational neighborhood (Proposition 9.2.1). Theorem 9.4.1 gives a finite free chart with a single point over the Gauss point at each relative interior point. If the weight lies in the interior of $Q$ and its coordinates are linearly independent over $\mathbb{Q}$, the finite free chart can be chosen over a smaller annular box (Corollary 9.2.1). Otherwise the target may need additional inequalities (Example 9.4.1). The rank of the chart is the intrinsic degree of Proposition 9.4.1, and Corollary 9.4.1 gives the graph description on the chart. Finally, Example 9.5.1 shows that two lifts of the same tropical point can have projected images of different local dimensions, so the projected rank depends on a source neighborhood.

Chapter 10 integrates through finite torus projections with these weighted fibres and compares their degrees with the weights of Chapter 8. Chapter 11 computes local products in finite torus coordinates. Appendix F describes the Gauss preimage of a single chart and constructs units that separate its points.

## 9.1 Degrees over the Gauss section

A finite map can have fewer analytic points in its fibre than its algebraic degree. To integrate over these fibres, we weight each point by the dimension of its local algebra. We first show that weighted sums of continuous functions vary continuously, even when points merge.

**Proposition 9.1.1** *Let $A \to B$ be a finite locally free homomorphism of affinoid algebras over a complete ultrametric field. Write $p : \mathcal{M}(B) \to \mathcal{M}(A)$. For a point $x \in \mathcal{M}(A)$, factor the fibre algebra $B \otimes_A \mathcal{H}(x)$ into its local Artinian factors $D_y$, indexed by $y \in p^{-1}(x)$ as in Theorem 2.3.1. Put $d_p(y) = \dim_{\mathcal{H}(x)} D_y$. For any real continuous function $\chi$ on $\mathcal{M}(B)$ the sum*

$$p_{\#}\chi(x) = \sum_{y \in p^{-1}(x)} d_p(y)\chi(y)$$

*is continuous. The sum of the weights is the module rank at $x$.*

*If $p^{-1}(x)$ is partitioned among finitely many pairwise disjoint open sets in $\mathcal{M}(B)$, then all nearby fibres lie in their union and the total weight in each set is locally constant near $x$.*

***Proof*** The rank statement follows by summing the dimensions of the local factors. Fix $b \in B$. Near $x$ use a principal open on which $B$ is free as an $A$-module. It gives an open neighborhood in $\mathcal{M}(A)$, since the element inverted has nonzero norm there. In this basis the characteristic polynomial of multiplication by $b$ has continuous coefficient norms. At a spectral point $x'$ its specialization is the characteristic polynomial on the fibre over $\mathcal{H}(x')$.

On each factor $D_y$ filter by powers of its maximal ideal. The graded pieces are vector spaces over $\mathcal{H}(y)$, and $b$ acts on them by its value in that field. After passing to a splitting field, all conjugates of that value have the same absolute value, by uniqueness of extension of the complete absolute value. The absolute values of the roots of the specialized characteristic polynomial are therefore $|b(y)|$, each counted $d_p(y)$ times. This counts inseparable roots with their polynomial multiplicities.

Write the specialized polynomial as $\sum_i a_i Z^i$ and its root absolute values in increasing order as $s_1, \ldots, s_d$. For every $r > 0$, the multiplicative Gauss norm at radius $r$ gives

$$\max_i |a_i| r^i = \prod_{j=1}^{d} \max\{r, s_j\}. \tag{9.1}$$

On taking logarithms and using $\log r$ as variable, the derivative jumps by the multiplicity at each positive root absolute value. The leftmost slope counts the zeros among the $s_j$. Thus the ordered list is determined by the left hand side for all $r > 0$. All $s_j$ are bounded by the supremum of $|b|$ on $\mathcal{M}(B)$. For a net of spectral points converging to $x$, any convergent subnet of the lists therefore has the list over $x$ as its limit, by (9.1). Compactness of the space of bounded lists gives continuity. In particular $p_{\#}|b|$, the sum of the entries, is continuous.

The real span of the functions $|b|$ is a unital algebra, since $|b||b'| = |bb'|$. These functions separate seminorms. For the density argument their uniform closure is a real vector lattice: approximate the real absolute value on any bounded interval by polynomials, and use the maximum and minimum formulas in terms of the absolute value of the difference. For that approximation on $[-R, R]$, expand $\sqrt{1-z}$ with $z = 1 - u^2/R^2$. The coefficients after the constant term have one sign, and their absolute values sum to one by taking $z \uparrow 1$ in the binomial series. Thus the series converges uniformly for $0 \leq z \leq 1$ and gives polynomial approximations to $|u|/R$. Apply Lemma 4.4.1. Since the rank is bounded on the compact base, the trace of a uniform limit is the uniform limit of the traces. This proves the claimed continuity.

For the last assertion, choose smaller open neighborhoods of the respective sets of fibre points, with closures inside the given open sets. Compact Hausdorff separation gives functions with values in $[0, 1]$ equal to one on the respective closures and zero outside the given open sets. The closed complement of the union of the smaller open neighborhoods is avoided by all fibres over a neighborhood of $x$, since the finite map between the compact spectra is closed. There the traces of these functions are the respective integer weights. They are constant after a further shrinking, by continuity. □

The weights also describe extension of scalars in the fibre. For a complete extension $F'/\mathcal{H}(x)$, each $D_y \otimes_{\mathcal{H}(x)} F'$ splits into local factors. Because it has dimension $d_p(y)$ over $F'$, the new weights over $y$ sum to $d_p(y)$. The factors over different $y$'s are separated by the unchanged orthogonal idempotents of the original product.

**Lemma 9.1.1** *Let $D$ be a finite local algebra over a field $F$ with residue field $E$ and length $\ell$ as a $D$-module. For any field extension $L/F$, the local factors of $D \otimes_F L$ correspond to the local factors $E_j$ of $E \otimes_F L$. The factor corresponding to $E_j$ has $L$-dimension $\ell \dim_L E_j$.*

***Proof*** The extension of the maximal ideal of $D$ to $D \otimes_F L$ is nilpotent and has quotient $E \otimes_F L$, so it preserves the indexing by maximal ideals. A composition series for $D$ as a $D$-module has $\ell$ subquotients all isomorphic to $E$. It remains a filtration after tensoring with $L$, and projection onto each local factor is exact. Each projected subquotient is $E_j$, giving the claimed dimension. □

Now suppose the ground field $k$ is trivially valued. Let $M$ be the character lattice of a split torus, with dual $N$, and let $Q \subset N_{\mathbb{R}}$ be a compact full-dimensional box in lattice coordinates. Write $T_Q$ for the closed product of annuli with log norm coordinates in $Q$, as in Section 8.4, and $\eta_w$ for its Gauss point of weight $w$. This is a continuous section, since on Laurent polynomials the seminorm is the maximum of the weighted terms, and the polynomials are dense in the annular algebra. The uniform norm bound on the box gives continuity for all elements of that algebra by approximation.

For a finite locally free map $p : Y \to T_Q$ of affinoid spaces the compact set

$$S_p = p^{-1}(\{\eta_w : w \in Q\})$$

retains the individual points of each fibre. Compactness follows from continuity of the Gauss section and closedness of its image. Put

$$K_w = \sum_{p(y)=\eta_w} d_p(y)\delta_y. \tag{9.2}$$

Proposition 9.1.1 shows that $w \mapsto K_w$ is continuous for integration of continuous tests on $Y$; its total mass is the rank of the finite locally free algebra at $\eta_w$.

**Lemma 9.1.2** *Let $p : Y \to X$ be finite locally free of constant rank between affinoid spaces. Let $Q$ be a compact full-dimensional polytope and $\sigma : Q \to X$ a continuous map. Let $g_1, \dots, g_l$ be analytic units on $Y$, with joint log norm map $g_{\mathrm{trop}}$. For each coefficient $a$ of the characteristic polynomial of multiplication by any $g_j$, suppose either $|a(\sigma(w))| = 0$ on $Q$, or $\log|a(\sigma(w))|$ is finite and piecewise affine with rational linear parts on a finite polyhedral subdivision. Then the joint log norm image of the weighted fibre over $\sigma(w)$ is given by finitely many affine graphs with constant positive integer weights over the relative interiors of full-dimensional cells of a subdivision. The graphs have rational linear parts. At a point of a lower-dimensional face, the weighted image is the limit from any one incident full-dimensional cell, obtained by extending that cell's graphs continuously and adding weights where their values agree; the limits from different incident cells coincide.*

***Proof*** For each $g_j$, take the characteristic polynomial of its multiplication on the finite locally free module defining $Y$. It is defined over the base algebra: locally the determinant is invariant under change of basis, so its coefficients glue on the principal opens where the module is free. The polynomial is monic of degree equal to the rank. The constant coefficient is invertible because multiplication by $g_j$ is invertible. At each $w$ the logarithms of the root norms are the breakpoints in $\log r$ in (9.1). Each breakpoint is an intersection of two lines $\log|a_i(\sigma(w))| + i\log r$ with different indices and nonzero coefficients at $w$. On a common finite polyhedral subdivision for the coefficient norms, each possible breakpoint is affine with rational linear part. Taking the negative of these breakpoints gives the coordinate candidates for $g_{\mathrm{trop}}$. There are no infinite coordinates since the $g_j$ are units.

Refine by the equality hyperplanes of the finitely many coordinate candidates. On each open full cell, the possible tuples are then distinct affine graphs, after identifying tuples that agree on the whole cell. Fix a point of the cell. Around its distinct candidate tuples take disjoint neighborhoods in $\mathbb{R}^l$. For nearby $w$, a continuous cutoff equal to one near one candidate and zero near the others reads exactly the weight of the chosen graph, with zero if it is unoccupied. Its integral is continuous in $w$ by Proposition 9.1.1 applied to the cutoff composed with $g_{\mathrm{trop}}$ and restricted along $\sigma$. The weight is integer-valued, so it is locally constant, and it is constant on the open cell by connectedness. Discard the zero-weight graphs. The continuous depen-

dence on $w$ also identifies the weighted image at a face point with this limit from any incident full cell, so these limits coincide. □

**Theorem 9.1.1** *Let $k$ be trivially valued, $Q \subset N_{\mathbb{R}}$ a compact full-dimensional box, and $p : Y \to T_Q$ a finite locally free map of affinoid spaces with weighted Gauss fibres $K_w$ as in (9.2). Let $g_1, \dots, g_l$ be analytic units on $Y$. Write $g_{\mathrm{trop}}$ for their log norm coordinates. There is a finite polyhedral subdivision of $Q$ with the following property. On each full-dimensional cell, the measures $(g_{\mathrm{trop}})_* K_w$ are sums of point masses of constant positive integer weights at finitely many affine functions of $w$. The linear parts of these functions are rational. On a face take the limiting affine values with the same weights, adding them wherever values coincide.*

***Proof*** Every nonzero element $a$ of the annular algebra has a Laurent expansion $\sum_{m \in M} a_m T^m$ converging uniformly on $Q$. This description follows by taking the completion of Laurent polynomials for the maximum of their Gauss norms at the vertices of $Q$. For a box this is the usual annular presentation: reduce products of positive and negative powers in each variable to one Laurent monomial. Its norm is the product of the minimal available weights for that exponent. Conversely lift a Laurent monomial using that choice of positive or negative powers. The two maps extend to the completions and give inverse maps after imposing the annular relations. For any one nonzero term of $a$, its norm has a positive minimum on $Q$. The terms below this bound uniformly on $Q$ cannot affect the Gauss norms. The remaining terms form a finite set. Thus $\log |a(\eta_w)|$ is the maximum of finitely many affine functions with integral linear parts, and is finite on $Q$. In particular the annular algebra is a domain, since each of these Gauss seminorms has zero kernel.

The finite locally free module has constant rank because the annular algebra is a domain. Apply Lemma 9.1.2 to the Gauss section $w \mapsto \eta_w$, using the coefficient calculation just proved. □

*Example 9.1.1 (Two branches with one limiting point)* Over the one-dimensional torus with coordinate $t$, consider the rank-two algebra given by $z^2 - z - t = 0$. The element $z$ is a unit since $z(z-1) = t$. For $w = -\log |t| > 0$, the map $u \mapsto u^2 - t$ on the Gauss field at $\eta_w$ preserves the closed ball of radius $|t|$ and has Lipschitz constant at most $|t| < 1$. Iteration gives a root $z_0$ there, and the equation shows $|z_0| = |t|$. The other root is $1 - z_0$, of norm one. The roots are distinct and each local factor has dimension one.

For $w < 0$, any root has norm $|t|^{1/2}$, since $|t| > 1$ forces the quadratic term to dominate the linear term. The value group of the Gauss field is $|t|^{\mathbb{Z}}$. This holds on $k(t)$ by the monomial norm and remains true on completion since the norm of a sequence converging to a nonzero element is eventually constant. Thus the quadratic is irreducible over that field. The only point has weight two. At $w = 0$, the Gauss field is $k(t)$ with trivial valuation. A rational root would have a pole at infinity of order one half, by the equation and the order-one pole of $t$. This is impossible; again the only point has weight two. As $w \to 0$, the two weight-one values $0, w$ on the positive side and the weight-two value $w/2$ on the negative side all tend to zero. By compactness over a box about zero, the points converge to the unique point over $\eta_0$.

## 9.2 Finite neighborhoods of isolated fibre points

A finite morphism supplies both a finite fibre and an admissible presentation with strict bounds. Conversely, an isolated fibre point with strict bounds in a relative presentation can be put in a finite chart.

**Proposition 9.2.1** *Let $f : \mathcal{M}(B) \to \mathcal{M}(A)$ be a morphism of affinoid spaces over a complete ultrametric field. Suppose $y \in \mathcal{M}(B)$ is isolated in the fibre over $x = f(y)$ and is in the relative interior of $f$. Inside any prescribed ordinary neighborhoods of $x$ and $y$, there are rational neighborhoods $U$ and $V$ of those points such that $f : V \to U$ is finite and $f^{-1}(x) \cap V = \{y\}$.*

***Proof*** Put $Y = \mathcal{M}(B)$ and $X = \mathcal{M}(A)$. Since $y$ is in the relative interior, some admissible presentation $A\{r^{-1}T\} \to B$ over $A$ has $|T_i(y)| < r_i$ for all $i$. By isolation there is an open neighborhood of $y$ meeting the fibre only at $y$. Inside its intersection with the prescribed neighborhood choose a closed Laurent neighborhood $W$ of $y$. More precisely, use finitely many upper bounds $|b_j| \le s_j$ and positive lower bounds $|c_l| \ge t_l$, each strict at $y$. Such systems give the neighborhood basis of Section 2.2.

The algebra of $W$ is obtained by adjoining variables with values $b_j, c_l^{-1}$ of radii $s_j, t_l^{-1}$, respectively. Together with the original $T_i$ they give an admissible presentation over $A$. Indeed successive Tate extensions and quotients can be combined in one Tate algebra by lifting each coefficient with a uniform factor in its quotient norm, as in Theorem 2.1.1 and its quotient consequences. All the generator bounds are strict at the sole point $y$ of $W \cap f^{-1}(x)$. The closed subset of $W$ where at least one bound is an equality is compact and its image misses $x$. Choose a rational neighborhood $U$ of $x$ in the prescribed target neighborhood disjoint from that image. The domain $V = W \cap f^{-1}(U)$ is rational in $Y$ by pullback and intersection. It is a neighborhood of $y$, since both $W$ and $U$ are neighborhoods of the corresponding points. The relative presentation for $W$ base changes to an admissible one over $A_U$, using the combined quotient in the proof of Proposition E.1.3. On compact $V$ every generator has spectral supremum strictly below its radius. Apply Lemma E.2.1 to obtain finiteness over $U$, with the quotient module topology. The fibre condition is preserved by construction. □

### 9.2.1 Freeness near the Gauss section

For the rest of this section, $k$ is trivially valued. Use a closed full-dimensional annular box $Q$ with algebra $A_Q$ and Gauss points $\eta_w$, as in Section 9.1. Call a weight $w$ *independent* if its coordinates in the lattice basis are linearly independent over $\mathbb{Q}$, and *dependent* otherwise. At an independent weight $w$, the algebra of the exact norm domain is the field $F_w = \mathcal{H}(\eta_w)$ and its residue field is $k$. In fact its algebra $B_w$ from Section 8.2 is the Laurent completion in the weight-$w$ norm. Every nonzero series has a unique largest term. After dividing by that monomial the remaining terms have norm strictly below one, so the geometric series gives an inverse. In an element of

norm one the largest term has exponent zero and gives its residue. In particular the exact norm fibre in $T_Q$ at an independent weight consists of $\eta_w$ alone.

The kernel of the Gauss norm on $A_Q$ is zero for every $w \in Q$, by the Laurent formula used in Theorem 9.1.1. We will also use the following elementary form of generic freeness. For a finite module over an integral domain $A_Q$, take elements giving a basis over $\mathrm{Frac}(A_Q)$. The map of the free module on those elements is injective since it is so over the fraction field. Its cokernel is finite torsion, so some nonzero $h \in A_Q$ annihilates it, by taking a product of annihilators of generators. The original module becomes free after inverting $h$.

Under a bounded map $A_Q \to A'$ to another affinoid algebra, a finite affinoid algebra $B$ base changes to the ordinary tensor product $B \otimes_{A_Q} A'$ with the quotient module topology. Indeed it is again a finite $A'$-algebra by Proposition 2.3.1. The natural map from $B$ is bounded by estimating on a finite list of module generators and taking the infimum of the norms of lifts. For bounded compatible maps from $B, A'$ to an affinoid algebra, the induced map from the tensor product is bounded by the same estimate. This gives the universal property of the affinoid base change.

**Proposition 9.2.2** *A finite map $Y \to T_Q$ is finite free after restriction to a rational domain of $T_Q$ containing the entire Gauss section in its interior relative to $T_Q$. The dimension of its algebra after tensoring with any $\mathcal{H}(\eta_w)$ is therefore the same free rank. At an independent weight $w \in Q^\circ$, the rational domain contains the inverse image of a smaller annular box about $w$, on which the map is finite free.*

***Proof*** Apply generic freeness to the finite algebra as a module and choose $h$ as above. Its value on the Gauss section has positive minimum, since it is continuous and never zero there. The domain $\{|h| \geq c\}$ for $c$ strictly between zero and the minimum has the required interior. Its algebra is an $A_Q[1/h]$-algebra by the rational presentation, so the module basis remains a free basis after this restriction. This also proves the rank statement, because restriction does not change the completed residue fields. For independent $w$, the exact norm fibre at $w$ lies in the interior of that domain. The tropical image of the closed complement of its interior is compact and misses $w$. A smaller closed box about $w$ inside $Q^\circ$ avoids this image. Base change to that box preserves the free basis and the finite module topology. □

**Corollary 9.2.1** *Suppose $f : \mathcal{M}(B) \to T_Q$ has an isolated fibre point $y$ in the relative interior over $\eta_w$, where $w$ is independent and lies in $Q^\circ$. Then any ordinary neighborhood of $y$ contains an affinoid neighborhood on which $f$ is finite free over a smaller annular box about $w$, with $y$ as the sole lift of $\eta_w$ in that neighborhood.*

***Proof*** Apply Proposition 9.2.1. Its target neighborhood $U$ contains the sole point of the norm fibre at $w$ in its interior. Compactness of the tropical image of the complement of the interior gives a smaller closed annular box about $w$ with inverse image contained in $U$. Restrict the finite map to that box and use Proposition 9.2.2 for a further annular restriction. Restricting only the base after the initial choice of the source neighborhood preserves the indicated lift over $\eta_w$. □

## 9.3 Affinoid dimension and fibres of a moment

The construction in Proposition 9.2.1 requires fibre isolation. We prove it over Gauss points when the source dimension is at most that of the target torus. We work over a complete ultrametric field $K$, specializing to trivially valued fields for moments.

### 9.3.1 Dimension of a strict affinoid algebra

Write $P_m = K\{X_1, \ldots, X_m\}$ for the Tate algebra with all radii one.

**Lemma 9.3.1** *For a nonzero strictly $K$-affinoid algebra $B$ there is a finite injective map $P_d \to B$, with $B$ given its quotient module topology. One has $d = \dim_{\mathrm{Krull}} B$.*

***Proof*** Choose a quotient presentation $B = P_m/I$. We construct the finite injective map by induction on $m$, with the case $I = 0$ given by the identity. For $I \neq 0$ choose $0 \neq f \in I$ and scale it by a coefficient of largest norm, so that $\|f\| = 1$. Its reduction in the polynomial ring over the residue field is nonzero. For $i < m$ substitute $X_i + X_m^{e_i}$ for $X_i$, with positive integers $e_i$. This is an automorphism of $P_m$ with the inverse given by subtraction, by bounded substitution. Choose the $e_i$ so that the exponents $\alpha_m + \sum_{i<m} e_i\alpha_i$ distinguish the finitely many monomials of the reduction of $f$. For example, successively large powers of an integer greater than all the exponents of those monomials suffice. The largest resulting exponent $D$ occurs in a single monomial, and its coefficient as a polynomial in $X_m$ is a nonzero constant. All terms selecting at least one $X_i$ from a translated variable have lower $X_m$-degree than that monomial's largest exponent.

Put $R = P_{m-1}$. The coefficient of $X_m^D$ in the transformed $f$ is a unit of $R$ with norm-one inverse, since its reduction is a nonzero constant. Normalize it to one. The result is a monic polynomial $P$ in $X_m$, of degree $D$ with coefficient norms at most one, plus a tail $E$ of Gauss norm strictly less than one. The strict inequality follows from the convergence of the series and the vanishing of all higher coefficients in the reduction. For a proper ideal $I$, the case $D = 0$ cannot occur: the resulting series would be a unit.

Polynomial division by monic $P$ gives linear maps $g \mapsto q(g), r(g)$, where $g = Pq(g) + r(g)$ and $r(g)$ has degree less than $D$. Eliminating one highest-degree term at a time shows that both maps have norm at most one, since every coefficient of $P$ has norm at most one. They extend by density to $R\{X_m\}$. Modulo $f = P + E$, the class of $g$ equals that of $r(g) - Eq(g)$. The map $g \mapsto -Eq(g)$ has norm strictly less than one.

Iteration gives a representative in the span of $1, X_m, \ldots, X_m^{D-1}$ with coefficient norm at most $\|g\|$. The error after iteration tends to zero, and the series of remainders converges in the finite free $R$-module of these polynomials. This gives a finite $R$-module presentation of $B$ with bounded lifts for its given quotient norm.

Replace $R$ by $R/(I \cap R)$. Its map to $B$ is injective and bounded for the quotient norm, and the module presentation still gives the quotient module topology. Apply

the induction hypothesis to that quotient. The composite finite module presentation is admissible: lift coefficients in the two presentations successively and use the products of their finite generator lists. This proves the asserted finite injective map with $d < m$ when $I \neq 0$.

The ring $P_m$ has dimension $m$. The coordinate prime ideals give a chain of length $m$. For the upper bound, induct on $m$ and use the construction just proved for a nonzero prime quotient. It is finite injective over $P_d$ with $d < m$, with the same Krull dimension by the prime chain properties of integral extensions. A chain in $P_m$ thus has length at most $m$, since $P_m$ is a domain and its quotient by the first nonzero prime has dimension at most $m-1$. The same prime chain properties give the dimension of $B$ from its finite injection over $P_d$. □

**Lemma 9.3.2** *For a strictly $K$-affinoid $B$ and any complete extension $L/K$, the two Krull dimensions of $B$ and $B_L$ agree.*

***Proof*** The zero algebra stays zero under base change. Take the injection $P_d \to B$ of Lemma 9.3.1. The map on spectra is surjective by Theorem 2.3.1. The Gauss norm on $P_d$ is itself a point, so the norm in $B$ bounds the norm in $P_d$ from below. The injection is therefore a bounded embedding with bounded inverse on its image.

We check that such an embedding stays injective under completed scalar extension. Equip the subspace with the induced norm, equivalent here to the given norm. For an algebraic tensor in the subspace and any finite expression for it in the ambient space, take a finite-dimensional ambient subspace containing the vectors of both expressions. Its intersection with the given subspace admits a linear projection of norm at most any prescribed $c > 1$. Indeed a projection onto a hyperplane has this bound by taking a complement vector within factor $c$ of its distance to the hyperplane, as in the norm argument of Proposition 2.4.1. Successive hyperplane projections give the same bound by choosing their factors with product at most $c$. Apply the projection to the ambient tensor expression and take the infimum over all such expressions and $c \downarrow 1$. This proves equality of the two projective tensor seminorms on the subspace. Upon Hausdorff completion it proves the assertion.

The base change of the finite algebra is the ordinary finite tensor $B \otimes_{P_d} (P_d)_L$ with the finite module topology. To see this, note that the algebra is finite over the $L$-affinoid $(P_d)_L$, so it is affinoid by Proposition 2.3.1. The map from $B$ is bounded because its generators admit bounded coefficient lifts over $P_d$. The map from $(P_d)_L$ is bounded by the definition of the finite module topology. Conversely, bounded compatible maps from these two algebras induce a bounded map from their tensor by estimating on module generators and taking the infimum over lifts. By (2.4), these are exactly the bounded maps from $B_L$ among $L$-affinoid algebras. The two algebras therefore agree. The base-changed injection is now finite injective over the unit Tate algebra in $d$ variables over $L$. Its dimension is $d$ by the prime chain argument of Lemma 9.3.1. □

### 9.3.2 Dimension from valued fields

For a valued field extension $E/K$, put

$$d_K(E) = \operatorname{trdeg}_{\widetilde{K}} \widetilde{E} + \dim_{\mathbb{Q}}\bigl(\mathbb{Q}\log|E^\times|/\mathbb{Q}\log|K^\times|\bigr).$$

Here $\widetilde{K}, \widetilde{E}$ are the ordinary residue fields for the absolute values. Write $d_K(x) = d_K(\mathcal{H}(x))$ for an analytic point.

**Lemma 9.3.3** *For valued field extensions $K \subset E \subset F$ one has $d_K(F) = d_K(E) + d_E(F)$, and the completion $\widehat{E}$ of $E$ satisfies $d_K(\widehat{E}) = d_K(E)$. Moreover $d_K(E) \leq \operatorname{trdeg}_K E$. If $E$ is the completion of $K(Z_1, \ldots, Z_m)$ in a Gauss norm of arbitrary positive radii, then $d_K(E) = m$.*

***Proof*** Additivity is the tower formula for transcendence degrees and the dimension formula for a sequence of vector subspaces of rationalized value groups. Completion preserves the value group and residue field: approximate a nonzero element to strictly smaller error than its absolute value, or a unit to error less than one to preserve its residue.

For the bound, choose finitely many elements whose log norms are linearly independent modulo $\mathbb{Q}\log|K^\times|$, and units whose residues are algebraically independent over $\widetilde{K}$. The chosen elements together are algebraically independent over $K$. In any nonzero polynomial in the residue lifts its value has norm equal to the largest coefficient norm, by dividing out by one such coefficient and using algebraic independence after reduction. In a polynomial also involving the elements with independent norms, collect the terms by the powers of those elements. The nonzero coefficient polynomials have norms in $|K^\times|$, so the distinct remaining monomials have distinct norms. A unique largest term prevents cancellation. This proves the inequality by taking arbitrarily large finite independent lists.

For the Gauss field at radii $s_1, \ldots, s_m$, let $r$ be the rank of the classes of $\log s_i$ modulo $\mathbb{Q}\log|K^\times|$. These classes contribute $r$ to the value group rank. Choose $m-r$ linearly independent integral relations among them and multiply each relation by an integer so that the corresponding product of radii equals the norm of some $a_j \in K^\times$. For the resulting exponent vectors $u_j$, the monomials $Z^{u_j}/a_j$ have norm one. Their residues are algebraically independent: a polynomial in them with lifted coefficients of norm at most one and some norm-one coefficient has Gauss norm one, since all its Laurent exponents are distinct. Thus $d_K \geq m$. The opposite bound follows from the preceding inequality and invariance under completion. □

**Theorem 9.3.1** *For a nonempty $K$-affinoid spectrum $X = \mathcal{M}(A)$, the Krull dimension of $A_L$ is the same for all complete extensions $L/K$ making $A_L$ strictly affinoid. This defines the analytic dimension of $X$. It satisfies*

$$\dim X = \max_{x \in X} d_K(x). \tag{9.3}$$

*This dimension is invariant under complete scalar extension. A zero-dimensional affinoid spectrum has finite algebra over the ground field and is a finite set.*

***Proof*** First suppose $A$ is strict. For a point of the unit $d$-polydisc, its completed residue field is the completion of the field generated by the values of the coordinates over $K$, by polynomial density and taking fractions. Its invariant $d_K$ is at most $d$ by Lemma 9.3.3 and is $d$ at the Gauss point. Apply Lemma 9.3.1. Its map on spectra is surjective and the extension of completed residue fields at each point is finite by Theorem 2.3.1. Such an extension contributes zero to $d_K$ by Lemma 9.3.3. This proves (9.3) and attainment in the strict case.

For general $A$, use a $K$-free Gauss extension $K_s$ making $A_{K_s}$ strict, as in Section 2.4.1, and let $\ell$ be the length of $s$. For a lift $x'$ of $x$ over $K_s$, the field $\mathcal{H}(x')$ contains isometric copies of $\mathcal{H}(x)$ and $K_s$. It is the completion of the field they generate: the finite sums of products from the two images are dense in the image of the affinoid algebra, and its fractions give the completed field. Since $K_s$ is the completion of the rational function field in the $\ell$ Gauss variables, $\mathcal{H}(x')$ is the completion of the field obtained from $\mathcal{H}(x)$ by adjoining their values. Hence $d_{\mathcal{H}(x)}(\mathcal{H}(x')) \le \ell$. At the specific Gauss lift of Proposition 2.4.2 the field is the completion of $\mathcal{H}(x)(Z_1, \dots, Z_\ell)$ for the Gauss norm of radii $s$, so equality holds. This follows by realizing the lift in the fraction field of the Gauss Laurent algebra over $\mathcal{H}(x)$ and using the same density argument. By tower additivity and $d_K(K_s) = \ell$, we obtain

$$d_{K_s}(x') \le d_K(x), \qquad d_{K_s}(x') = d_K(x) \text{ at the Gauss lift.}$$

Thus the maximum on $X$ equals the attained maximum on $\mathcal{M}(A_{K_s})$, which is its Krull dimension.

For any complete extension $L/K$, form the complete field $E$ obtained as the Gauss completion of $L(Z_1, \dots, Z_\ell)$ at radii $s$. It contains isometric copies of $L$ and $K_s$. Choose a maximal $L$-free subtuple $t$ of the radii $s$ modulo rationalized values of $L$; the field $E$ also contains the corresponding $L_t$. The presentation radii of $A$ have logarithms in the span of those of $s$ modulo rationalized values of $K$, hence in the span of those of $t$ modulo rationalized values of $L$. The strictification argument of Section 2.4.1 therefore also makes $(A_L)_{L_t}$ strict. The strict algebras $A_{K_s}$ and $A_{L_t}$ both extend to $A_E$: the quotient description (2.4) makes successive extensions agree. By Lemma 9.3.2 their Krull dimensions are equal. This proves dimension invariance and independence of the choice of a strictifying extension, including when $A_L$ is already strict.

If the dimension is zero, $A_{K_s}$ is finite over $K_s$ by Lemma 9.3.1. Every finite-dimensional $K$-subspace of $A$ remains embedded after completed tensoring, by the norm argument of Lemma 9.3.2. Its extended dimension is its original dimension by

equivalence of finite-dimensional norms. Thus the dimensions of these subspaces are bounded by $\dim_{K_s} A_{K_s}$, so $A$ is finite over $K$. Its spectrum is finite by Theorem 2.3.1. □

### 9.3.3 Isolation over Gauss points

**Proposition 9.3.1** *Let $f : Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ be a morphism of $K$-affinoid spaces. At $x \in X$ the analytic fibre $Y_x$ over $\mathcal{H}(x)$, if nonempty, satisfies*

$$\dim_{\mathcal{H}(x)} Y_x \leq \dim_K Y - d_K(x).$$

*Its spectrum identifies homeomorphically with the topological fibre.*

***Proof*** Use the affinoid algebra $B\widehat{\otimes}_A\mathcal{H}(x)$, equivalently the relative Tate presentation of $B$ with coefficients extended to $\mathcal{H}(x)$, as in the combined presentation of Proposition E.1.3. At each $y$ over $x$ the maps from $B$ and $\mathcal{H}(x)$ to $\mathcal{H}(y)$ agree on $A$ and give an evaluation of the fibre algebra by bounded substitution. Conversely a spectral point of the fibre algebra restricts to a point $y$ over $x$. Its completed residue field contains the canonical $\mathcal{H}(y)$, hence also the evaluation of $\mathcal{H}(x)$ induced by $x$. The fibre algebra is generated densely by its polynomials with coefficients in $\mathcal{H}(x)$ and the images of the relative generators of $B$. Their evaluations lie in the copy of $\mathcal{H}(y)$, so the two completed residue fields are equal. The same dense algebra forces uniqueness of the point above $y$, and compactness gives the homeomorphism.

At every fibre point the tower formula gives $d_K(y) = d_K(x) + d_{\mathcal{H}(x)}(y)$. Apply (9.3) to $Y$ and the nonempty fibre to obtain the inequality. □

**Corollary 9.3.1** *Let $Y$ be a $k$-affinoid space of dimension at most $n$ over a trivially valued field, and let its moment map take values in an $n$-dimensional annular box. Every fibre over a Gauss point is finite. At a point of the relative interior over a Gauss point there is a finite rational neighborhood as in Proposition 9.2.1. If the Gauss weight is in the box interior and its coordinates are independent over $\mathbb{Q}$, the source neighborhood is finite free over a smaller annular box as in Corollary 9.2.1.*

***Proof*** The completed residue field of any Gauss point of the box is the Gauss completion of the rational function field in $n$ torus coordinates. Its invariant is $n$ by Lemma 9.3.3, including at dependent weights. A nonempty fibre has dimension zero by Proposition 9.3.1, and is finite by Theorem 9.3.1. The two neighborhood assertions follow by applying Proposition 9.2.1 and Corollary 9.2.1 to each indicated isolated point. □

The dimension bound passes to affinoid domains because their completed residue fields agree, by Proposition E.1.3. Consequently the fibre assertion applies on the affinoid charts of an analytic space all of whose affinoid charts have dimension at most $n$. A finite family of such charts whose union contains an ordinary neighborhood of

a point meets a Gauss fibre in a finite set. Every singleton meets each affinoid chart in a closed subset, so the singleton is closed in the analytic topology. Removing the other points of the finite set leaves a neighborhood isolating the given fibre point. For a point with an ordinary affinoid neighborhood on which the map is interior, the corollary gives the finite neighborhood there. The local discreteness statement only uses the affinoid cover.

## 9.4 Finite free neighborhoods at Gauss points

At a dependent weight, several points of the torus can have the same coordinate norms. To obtain a finite free chart near the Gauss point, we allow additional inequalities in the target. We first establish the local algebra needed for this restriction.

### 9.4.1 Reduced Laurent domains

**Lemma 9.4.1** *Let $A$ be affinoid over a complete ultrametric field $K$, and let $L/K$ be complete. Tensoring with $A_L$ is exact on finite $A$-modules. For a prime $\mathfrak{p}'$ of $A_L$ with contraction $\mathfrak{p}$, the map $A_{\mathfrak{p}} \to (A_L)_{\mathfrak{p}'}$ is injective.*

***Proof*** Give each finite module its quotient norm from a finite free presentation. These norms are equivalent for different generating sets by expressing one list in terms of another. Write a presentation as $P^d/E$ for a Tate presentation $P \to A$. The remainder in Theorem 2.1.1 is a bounded $K$-linear section of $P^d \to P^d/E$. If a submodule of $P^d/E$ has inverse image $G$, the same division for $G$ bounds the coefficients in its generators by a fixed multiple of the norm of the dividend. Apply this to a lift in $G$ of an element of $G/E$, with norm at most twice the quotient norm in $P^d/E$. This proves that the finite module topology of $G/E$ is its induced topology. The remainder section for $G$ then splits the quotient $P^d/E \to P^d/G$ by a bounded $K$-linear map. Thus short exact sequences of finite modules split as Banach $K$-spaces.

Completed scalar extension preserves such split sequences, since both maps of a bounded direct sum decomposition extend to completed tensor products. For $P^d$ the extended space is $P_L^d$, by the series argument of (2.4). The image of $E\widehat{\otimes}_K L$ is the $P_L$-submodule generated by $E$. It is contained in that submodule by closedness. Conversely it contains all multiples by $P_L$ of elements of $E$, by approximation of coefficients and the closedness of the split image. It follows that the extension of $P^d/E$ is $P_L^d/EP_L$, which is also its ordinary tensor product with $A_L$ over $A$. This proves exactness.

If $a \in A$ maps to zero in $(A_L)_{\mathfrak{p}'}$, some $h \notin \mathfrak{p}'$ in $A_L$ kills its image. By exactness applied to multiplication by $a$, its annihilator in $A_L$ is generated by the annihilator in $A$. The latter ideal cannot be contained in $\mathfrak{p}$. Hence $a$ is zero in $A_{\mathfrak{p}}$. The same conclusion holds for a fraction with denominator outside $\mathfrak{p}$. □

**Lemma 9.4.2** *Let $T_Q$ be a closed product of annuli in a split torus over $K$. A domain in $T_Q$ cut out by finitely many inequalities $|a_i| \le r_i$ and $|b_j| \ge s_j$, with $r_i, s_j > 0$ and $a_i, b_j$ analytic on $T_Q$, has reduced algebra after every complete extension of $K$.*

***Proof*** The presentation uses coordinates $T_l, T_l^{-1}$ with their annular bounds, together with variables $U_i, V_j$ of radii $r_i, s_j^{-1}$, respectively. With $S_l$ denoting the inverse-coordinate variables, its equations are

$$T_l S_l = 1, \qquad U_i = a_i(T, S), \qquad V_j b_j(T, S) = 1.$$

Here the functions on the right are any Tate lifts of the given annular functions. The presentation follows by imposing the inequalities by bounded substitution. It has the same form after a complete scalar extension by (2.4). It suffices to prove reducedness after a further complete extension whose value group contains all presentation radii. Such a field is the completion of a rational function field in variables of those Gauss norms. Injectivity into the extended algebra follows from the bounded remainder section for a Tate presentation, which extends to the new field, and the injectivity of coefficient extension on Tate series.

We may therefore assume the radii are values in $K$, so that rescaling makes the presentation strict. Let $A$ be the domain algebra. For a maximal ideal $\mathfrak{m}$ of $A$, the field $F = A/\mathfrak{m}$ is finite over $K$. Indeed, the quotient is strict, and finite normalization in Lemma 9.3.1 forces its normalizing Tate algebra to have dimension zero. The quotient map to $F$, with its complete absolute value, is bounded by finite-dimensional norm equivalence. It gives an $F$-rational point of $\mathcal{M}(A_F)$ by evaluation. Let $\mathfrak{n}$ be the corresponding maximal ideal of $A_F$. Its contraction is $\mathfrak{m}$.

At a rational point of the presenting polydisc, all coordinates have norms at most their radii. Translation to their values is a bounded substitution with bounded inverse. After translating, the quotient of the Tate algebra by the $q$-th power of the point ideal is the polynomial algebra in the translated variables truncated at total degree $q$. One sees this by separating the monomials of degree less than $q$. Each higher monomial is divisible by a monomial of degree $q$, and grouping by a choice of those finitely many divisors gives convergent series.

In these quotients the equations first eliminate $S_l$ as the formal inverse of $T_l$ at its nonzero value. They then eliminate $U_i$ by substitution and $V_j$ by inversion, since $b_j$ has nonzero value at the point. More explicitly, after imposing the equations the translated eliminated variables are uniquely the indicated truncated series in the translated $T_l$, of constant term zero. Conversely these substitutions satisfy the equations and respect the truncated point ideal, giving inverse maps between the quotients. This works at every $q$ and commutes with truncation. The completion of $(A_F)_{\mathfrak{n}}$ is consequently the formal power series ring over $F$ in the translated torus coordinates. The ring maps injectively to its completion by Krull intersection [Stacks, Tag 00IP], so it is reduced. Now Lemma 9.4.1 embeds $A_{\mathfrak{m}}$ into this reduced local ring. This holds for every maximal ideal, and an element zero in all maximal localizations is zero, since its annihilator lies in no maximal ideal. Thus $A$ is reduced, proving also the claim after arbitrary complete extensions. □

**Lemma 9.4.3** *Suppose $0 \in Q$ and $k$ is trivially valued. The annular algebra of $T_Q$ is $k[M]$, the Laurent polynomial ring of the character lattice. A Laurent domain of $T_Q$ whose bounds are all strict at $\eta_0$ has algebra $k[M, 1/h]$, where $h$ is the product of the functions given positive lower bounds.*

***Proof*** Since $Q$ contains zero, all coordinate and inverse-coordinate radii of its presentation are at least one. The weight of every nonzero Tate monomial is then at least one, so a convergent series with coefficients in $k$ is a polynomial. Imposing the inverse-coordinate relations gives $k[M]$. All its nonzero elements have norm one at $\eta_0$. In the Laurent presentation, omit upper bounds on the zero function. Strictness means that every other upper radius is greater than one, as is the inverse radius used for each lower bound. The additional Tate variables therefore still give a polynomial algebra. The relations eliminate the variables for upper bounds by substitution and invert precisely the functions in the lower bounds, giving the stated localization. □

**Lemma 9.4.4** *Suppose $A$ is a $K$-affinoid algebra that becomes strict and reduced after a $K$-free Gauss extension. If $x \in \mathcal{M}(A)$ satisfies $d_K(x) = \dim \mathcal{M}(A)$, then $A_{\ker(x)}$ is a field.*

***Proof*** Put $L = K_s$ for the specified extension and let $x'$ be the Gauss lift of $x$. By the lift computation in Theorem 9.3.1,

$$d_L(x') = d_K(x) = \dim_{\mathrm{Krull}} A_L.$$

Put $\mathfrak{p}' = \ker(x')$. The strict algebra $A_L/\mathfrak{p}'$ contains the point $x'$ with the same completed residue field, so its dimension is at least $d_L(x')$. If $\mathfrak{p}'$ were strictly above a prime, a chain of maximal length in the quotient could be extended by a prime below it, contradicting equality of the two dimensions. Hence $\mathfrak{p}'$ is minimal. The localization $(A_L)_{\mathfrak{p}'}$ is zero-dimensional and reduced, so its unique prime is zero and it is a field. By Lemma 9.4.1, $A_{\ker(x)}$ injects into it. Its maximal ideal maps to zero, since $\mathfrak{p}'$ contracts to $\ker(x)$. This proves the assertion. □

### 9.4.2 Interior moment charts and their degrees

**Theorem 9.4.1** *Let $k$ be trivially valued, $Q$ an $n$-dimensional annular box and $f : Y \to T_Q$ a moment of an affinoid space of dimension at most $n$. At any point $y$ in the relative interior of $f$ over a Gauss point $\eta_w$, there are affinoid neighborhoods $V$ of $y$ and $U$ of $\eta_w$ such that $f : V \to U$ is finite free and $V \cap f^{-1}(\eta_w) = \{y\}$. They can be taken inside any prescribed ordinary neighborhoods of the two points.*

***Proof*** By Corollary 9.3.1 the fibre is finite. Apply Proposition 9.2.1 to obtain a finite map with a single point over $\eta_w$, then restrict the base to a Laurent neighborhood $U_0$ of $\eta_w$ inside the rational target domain. Such neighborhoods form a basis by the definition of the spectrum topology. The map over $U_0$ is finite by the base-change

paragraph preceding Proposition 9.2.2, whose argument applies verbatim over any affinoid base, here the algebra of the rational target domain. The coordinate values generate a dense subfield of each completed residue field of the box, by Laurent polynomial density and taking fractions. Thus $d_k \le n$ on the box by Lemma 9.3.3. The domain preserves completed residue fields and contains $\eta_w$ with $d_k(\eta_w) = n$, so $\dim U_0 = n$. Its algebra $A$ satisfies Lemma 9.4.4 by Lemma 9.4.2. Write $\mathfrak{p} = \ker(\eta_w)$ on $A$. The finite module of the source is free over the field $A_{\mathfrak{p}}$. Lift a basis after clearing denominators to obtain a map $A^d$ to the finite module inducing an isomorphism at $\mathfrak{p}$. Its kernel and cokernel are finite and both vanish after inverting one element $h \notin \mathfrak{p}$: take the product of a choice of annihilators of their generators outside $\mathfrak{p}$. Restrict to the domain $|h| \ge c$ in $U_0$, for $0 < c < |h(\eta_w)|$. This gives the required neighborhoods, since its algebra inverts $h$. □

For $f : Y \to T_Q$ as in Theorem 9.4.1, define the *Gauss set* of $f$ as the inverse image of the Gauss section of $T_Q$. When $f$ is finite locally free, it is the set $S_f$ of Section 9.1. For $y$ in this set, put

$$d_f(y) = \dim_{\mathcal{H}(f(y))} \big(\mathcal{O}(Y_{f(y)})\big)_y, \tag{9.4}$$

where the subscript is the local factor of the zero-dimensional fibre algebra. That algebra is finite over $\mathcal{H}(f(y))$ by Proposition 9.3.1 and Theorem 9.3.1.

**Proposition 9.4.1** *The degree* (9.4) *is unchanged by affinoid domain restriction in the source and the target, for each retained point. At relative-interior points it is the rank of the single-point chart in Theorem 9.4.1. On a relatively open subset of the inverse image of any subset of the Gauss section, contained in the relative interior and on which $f$ is injective, the degree is locally constant. If all points over a Gauss point $\eta_w$ are in the relative interior, the measures $\sum_{f(y)=\eta_v} d_f(y)\delta_y$, the analogue of* (9.2)*, vary weakly continuously in $v$ near $w$.*

***Proof*** A target domain preserves the field $\mathcal{H}(f(y))$ and the fibre algebra by base change. For a source domain $V \subset Y$, choose an ambient rational domain $W \subset V$ containing $y$, by Theorem E.2.1. Its pullback to $V$ is the same domain by the domain property. The rational equations commute with the relative Tate presentation of the fibre over $\mathcal{H}(f(y))$. Thus the fibre of $W$ is a rational domain in the fibres of both $Y$ and $V$.

In a finite algebra over a complete field the domain on a subset of the spectrum is the product of the corresponding local factors. For the domain property, the idempotent of the omitted factors must vanish under every map whose spectral image avoids those factors. Indeed its complement then vanishes nowhere and is a unit by Theorem 2.2.1. Thus the map factors uniquely through that product. Comparing both fibres to the fibre of $W$ proves invariance of the factor, including its nilpotents. The rank assertion follows from the one-point fibre description of the chart.

At a point of the specified relatively open subset, choose the finite free source neighborhood small enough that its intersection with the restricted Gauss set is contained in that subset. On this intersection each fibre has a single point, by injectivity. Its degree is the same free rank, so the degree is locally constant.

For the final assertion, choose the finite free charts around the finitely many points of the given fibre with pairwise disjoint source neighborhoods. Intersect their target neighborhoods, then shrink again to avoid the image of the closed complement of the union of the interiors of the source neighborhoods. That image is compact and misses the given Gauss point. After an affinoid restriction there, all fibres are accounted for by the disjoint finite free charts. Their weighted sums are continuous by Proposition 9.1.1 and restriction invariance. Their sum is the desired weighted sum. In the empty-fibre case compactness of $Y$ gives a neighborhood where it remains empty. □

### 9.4.3 Approximating functions at a Gauss point

**Lemma 9.4.5** *Let $U$ be an affinoid neighborhood of $\eta_w$ in $T_Q$, and let $a_1, \dots, a_l$ be analytic functions on $U$. There is a Laurent neighborhood $U' \subset U$ of $\eta_w$ such that for each $j$ one has the following alternatives. If $|a_j(\eta_w)| = 0$, then $a_j = 0$ on $U'$. Otherwise there is a rational function $R_j \in \mathrm{Frac}(k[M])^\times$, analytic and invertible on $U'$, with*

$$|a_j - R_j| < |R_j| \quad \text{on } U'. \tag{9.5}$$

***Proof*** First replace $U$ by a Laurent neighborhood inside it. The localization of its algebra at the kernel of $\eta_w$ is a field by Lemmas 9.4.2 and 9.4.4: the point has invariant $n$ and the domain has dimension $n$, as in the proof of Theorem 9.4.1. If $a_j$ vanishes at the point, some element nonzero there kills $a_j$; invert that element on a smaller neighborhood.

If $a_j$ has nonzero value at the point, choose $R_j$ as in (9.5) at $\eta_w$ by density of $\mathrm{Frac}(k[M])$ in $\mathcal{H}(\eta_w)$. The numerator and denominator are nonzero at the Gauss point, so $R_j$ is an analytic unit after a rational restriction. Continuity preserves the strict inequality on an ordinary neighborhood. Choose $U'$ inside the intersection of these finitely many neighborhoods. □

**Corollary 9.4.1** *Let $V \to U$ be a chart of Theorem 9.4.1 at $y$ over $\eta_w$. For any finite list of analytic units on $V$, there is a compact box $Q' \subset Q$ containing a neighborhood of $w$ relative to $Q$, with $\eta_v \in U$ for $v \in Q'$, on which the joint log norm images of the degree-weighted Gauss fibres of $V$ have the graph description of Lemma 9.1.2.*

***Proof*** Apply Lemma 9.4.5 to the finitely many characteristic coefficients. The map remains finite free by base change. Each coefficient is zero on the smaller Gauss neighborhood or has the same norm there as a Laurent rational function. The logarithm of the norm of such a function is the difference of two finite maxima of affine functions with integral linear parts, given by its numerator and denominator.

Take $Q'$ inside these finitely many weight neighborhoods. On its common finite subdivision the coefficient hypotheses of Lemma 9.1.2 are satisfied. Apply that lemma. By restriction invariance the weights are the original local degrees, and the fibres agree with the original chart over the part of the Gauss section in $Q'$. □

*Example 9.4.1 (A rational target neighborhood at weight zero)* Take $0 < c < 1$ and the identity map from the Laurent domain $\{|t-1| \geq c\}$ of a one-dimensional closed annular box containing weight zero in its interior. At $\eta_0$ this is an isomorphism onto a rational neighborhood of the Gauss point, of degree one. No full annular neighborhood of weight zero lies in its image.

Indeed the Gauss norm of $t-1$ at weight zero is one, so $\eta_0$ is in the ordinary interior of the indicated domain. On the domain the function $t-1$ is a unit. Along the Gauss section its log norm coordinate is $\min\{w, 0\}$, since $|t-1|(\eta_w) = \max\{\exp(-w), 1\}$. The identity has fibre algebra $\mathcal{H}(\eta_w)$ at each of these points, of dimension one. The rational point $t = 1$ also has weight zero and is excluded by $|t-1| \geq c$. Thus any annular box whose weights contain zero also contains a point missing from the image.

## 9.5 Source neighborhoods and projected rank

Two analytic points can have the same moment coordinates and give different local dimensions after projection. The distinction already occurs on a smooth very affine surface over a trivially valued field.

*Example 9.5.1 (Two lifts of one tropical point)* Let $X$ be the closed subvariety of the three-dimensional split torus with equation $(t-1)(s-1) = u$. Write $\tau$ for the moment given by $(t, s, u)$, and let $p$ be the projection to the two-dimensional torus with coordinates $(t, u)$. Denote the corresponding log norm maps by $\tau_{\mathrm{trop}}$ and $p_{\mathrm{trop}}$. For $b > 0$, the point $(0, 0, b)$ in the image of $\tau_{\mathrm{trop}}$ admits distinct lifts $x_t, x_s$ with the following properties.

(1) The point $p(x_t)$ is the Gauss point of weight $(0, b)$. In every neighborhood of $x_t$, the image under $p_{\mathrm{trop}}$ contains a neighborhood of $(0, b)$.
(2) The point $p(x_s)$ has $|t-1| = e^{-b}$ and is distinct from that Gauss point. On a neighborhood of $x_s$, $p_{\mathrm{trop}}$ has image in the line whose first coordinate is zero.

In particular $(0, 0, b)$ lies in the closure of the open two-dimensional cells of the image of $\tau_{\mathrm{trop}}$ on which projection to the first and third coordinates is injective, and $x_s$ lies over it although $p(x_s)$ is not a Gauss point. Hence the full inverse image of that closure under $\tau_{\mathrm{trop}}$ can contain points outside the preimage of the target Gauss section.

***Proof*** Eliminating $u$ identifies $X$ with the open subset of the $(t, s)$-torus on which $t-1$ and $s-1$ are invertible. In particular it is smooth. Also $p$ is an isomorphism onto an open subset, by the inverse formula $s = 1+u/(t-1)$ with the denominators and the resulting unit $s$ required to be nonzero. On $k(t, u)$ give $t, u$ the Gauss norm with log norm coordinates $(0, b)$. This means the maximum of the weighted terms on Laurent polynomials and its multiplicative extension to fractions. It defines a valuation $x_t$ of $k(X)$, since $s = 1 + u/(t-1)$. At $x_t$, the Gauss norm of $t-1$ is one, and the norm of $u$ is $e^{-b}$. Thus

$$|s-1|(x_t) = e^{-b}, \qquad |s|(x_t) = 1.$$

The same construction starting with $s, u$ gives a point $x_s$, at which $|s-1| = 1$ and $|t-1| = e^{-b}$. Both points map to $(0, 0, b)$ under $\tau_{\text{trop}}$.

For every choice of the log norm $\lambda$ of $t$ and the log norm $c > 0$ of $u$, use the Gauss valuation of these weights on $k(t, u)$. Since the norm of $t-1$ is $\max(1, e^{-\lambda})$, the norm of $s-1$ is less than one. This gives a continuous section of $p_{\text{trop}}$ over $\mathbb{R}\times\mathbb{R}_{>0}$. Continuity follows by writing each regular function as a rational function of $t, u$: the weighted maximum is continuous for Laurent polynomials, and denominators are nonzero everywhere on the section. It passes through $x_t$. This proves the asserted two-dimensional image at that lift. Near $x_s$, the strict inequality $|t-1| < 1$ defines an open neighborhood and forces $|t| = 1$ throughout it. This proves the assertion about the other lift.

To determine the global image in the half-space $c > 0$, note that if $|t| \neq 1$, then $|t-1| \geq 1$ and the defining equation forces $|s-1| < 1$, hence $|s| = 1$. The same implication holds with $s, t$ exchanged.

The two Gauss sections just used show that every point of the two open half-planes

$$\{(\lambda, 0, c) : c > 0\} \quad \text{and} \quad \{(0, \lambda, c) : c > 0\}$$

occurs. Projecting to the first and third coordinates is injective of rank two on the first, and has rank one on the second. Their intersection belongs to the closure of the open two-dimensional cells on which this projection is injective. The point $x_s$ over that intersection shows the claimed failure for the full inverse image. □

# Chapter 10
# Real forms on tropical charts

On the real line the function $\max(0, x)$ has a second derivative concentrated at the origin. In tropical coordinates a piecewise affine function has the same type of second derivative, supported on its facets. We first define the two real exterior derivatives and integration on a weighted polyhedral complex. Stokes' formula then gives the facet contributions, including their lattice normalization.

The first four sections take place on real affine spaces with a lattice and on finite rational polyhedral complexes in them; no ground field enters. The balanced algebraic tropicalizations of Theorem 8.7.1 are examples of such complexes. If an $n$-dimensional complex is balanced at every $(n-1)$-dimensional face meeting an open set, the integrals of $d'\alpha$ and $d''\beta$ vanish for forms of bidegrees $(n-1, n)$ and $(n, n-1)$ with compact support in that set (Theorem 10.3.1). For a continuous piecewise affine function $f$ on a convex open set of a lattice affine space, $d'd''f$ is the sum of the facet integrations weighted by the jumps of the slope, measured with primitive lattice normals (Proposition 10.4.1).

The remaining sections integrate on analytic spaces over a trivially valued field $k$, through the finite charts of Chapter 9. For a finite locally free map to an annular box, the calibration (10.1) pairs a top form in the base coordinates with the degree-weighted Gauss fibres, and Proposition 10.5.1 writes it as a sum over affine graphs. For two finite projections of a common moment whose log norm coordinates are related on a rational plane by a linear map $F$, the Gauss fibre degrees at independent weights differ by the factor $|\det F|$ (Theorem 10.6.1), and the two integrals agree after pullback of top forms by $F$ (Corollary 10.6.1). On a top cell of an algebraic tropicalization, the Gauss fibre degrees of a projection sum to its projection index times the weight of Section 8.7 (Theorem 10.7.1), and Theorem 10.8.1 compares two charts across a common subdivision. Chapter 11 uses this calibration to define local products in finite torus coordinates, and Chapter 20 evaluates it on the valuation rays of a curve.

## 10.1 The two exterior derivatives

Let $V$ be a real vector space of dimension $r$ and $U \subset V$ open. A smooth form of bidegree $(p, q)$ on $U$ is a smooth function with values in $\bigwedge^p V^* \otimes \bigwedge^q V^*$. We regard it as the corresponding part of the exterior algebra on two copies of $V^*$. In affine coordinates we write the two copies of $\mathrm{d}x_i$ as $\mathrm{d}'x_i, \mathrm{d}''x_i$. Both types of one-forms anticommute, so the exterior sign is determined by total degree. The operators $d'$ and $d''$ are the graded derivations defined by

$$d'f = \sum_i \partial_i f \, \mathrm{d}'x_i, \qquad d''f = \sum_i \partial_i f \, \mathrm{d}''x_i$$

on functions and by zero on constant forms. The formula is invariant under affine coordinate changes.

There are identities

$$(d')^2 = (d'')^2 = 0, \qquad d'd'' = -d''d'.$$

Indeed second partial derivatives commute, whereas the constant one-forms anticommute. An affine map pulls a form back using its derivative in both copies of the cotangent space. The chain rule gives commutation of pullback with both derivatives.

The involution $J$ exchanges the two copies in the exterior algebra. Thus $Jd' = d''J$ and $J^2 = 1$. The form $d'd''f$ of a real smooth function $f$ is

$$d'd''f = \sum_{i,j} \partial_i \partial_j f \, \mathrm{d}'x_i \wedge \mathrm{d}''x_j.$$

A $(1,1)$-form $\alpha = \sum_{i,j} a_{ij} \, \mathrm{d}'x_i \wedge \mathrm{d}''x_j$ is *symmetric* if the matrix $(a_{ij})$ is symmetric, equivalently $J\alpha = -\alpha$, and *positive* if this matrix is positive semidefinite. These definitions are coordinate independent, since a change of coordinates changes the matrix by congruence. In particular $d'd''f$ is positive on a convex open set exactly when $f$ is convex, by restricting to affine lines.

We will also use strongly positive forms. At a point, these are nonnegative combinations of

$$\alpha_1 \wedge J\alpha_1 \wedge \cdots \wedge \alpha_p \wedge J\alpha_p$$

with real $(1,0)$-forms $\alpha_i$. For $p = 1$ this cone is exactly the cone of symmetric positive semidefinite matrices, by their decomposition into rank-one matrices. Wedges of strongly positive forms are strongly positive. The expression of top bidegree in affine coordinates is

$$\mathrm{vol}_V = \mathrm{d}'x_1 \wedge \mathrm{d}''x_1 \wedge \cdots \wedge \mathrm{d}'x_r \wedge \mathrm{d}''x_r.$$

A strongly positive form of bidegree $(r, r)$ is a nonnegative multiple of this form. Indeed each product of paired one-forms multiplies it by the square of a determinant.

## 10.2 Integration on a rational polyhedron

Let $N$ be a lattice in $V$. A polyhedron in $V$ is *rational* if the tangent space $\operatorname{span}(\sigma - \sigma)$ of each of its faces $\sigma$ is spanned by vectors of $N$. For a rational polyhedron $\sigma$ of dimension $n$, let $N_\sigma = N \cap \operatorname{span}(\sigma - \sigma)$. Choose affine coordinates on the tangent affine space from a basis of $N_\sigma$, and use Lebesgue measure for which that basis spans a parallelepiped of volume one. Restrict the form to that affine space in both copies. Its integral on $\sigma$ in bidegree $(n, n)$ is the integral of its coefficient relative to the paired volume form. A change of lattice basis has determinant $\pm 1$, so it preserves the measure and multiplies the paired form by one. In particular the integral needs no orientation.

For a finite pure $n$-dimensional complex of rational polyhedra, restrict a smooth ambient form to the tangent space of every face. This defines a form on the complex, with two representatives identified when these restrictions agree. The derivatives descend to this identification: equality on the relative interior of a face implies equality for the tangential derivatives there, and continuity gives equality at the boundary of the face. Locally defined forms of this kind are allowed on open subsets of the support. On a compact set finitely many ambient representatives can be combined using a smooth ambient partition of unity. This also allows a compactly supported form to be represented by an ambient form with compact support near its support.

If $m_\sigma$ is a real weight on each top-dimensional face, define

$$\int_C \beta = \sum_{\dim \sigma = n} m_\sigma \int_\sigma \beta$$

for a compactly supported form $\beta$ of bidegree $(n, n)$. Here $C$ denotes the weighted complex. The integral is invariant under subdivisions with inherited weights because faces of smaller dimension have zero $n$-dimensional measure. If the weights are nonnegative it is nonnegative on strongly positive top-degree forms.

## 10.3 Stokes' formula and the balancing condition

For a facet $\tau$ of a top-dimensional face $\sigma$, choose $v_{\sigma,\tau} \in N_\sigma$ lifting the primitive inward generator in $N_\sigma / N_\tau$. Write $\iota_{v''}$ for insertion of this vector in the second copy of the tangent space. For an ambient form $\alpha$ of bidegree $(n-1, n)$, its insertion restricts to a form on $\tau$. Changing the lift by a tangent vector of $\tau$ has no effect. Indeed one would then evaluate the second $n$-fold exterior factor on $n$ vectors in an $(n-1)$-dimensional space.

**Lemma 10.3.1** *On a single $n$-dimensional rational polyhedron $\sigma$, for a smooth compactly supported ambient $(n-1, n)$-form $\alpha$ one has*

$$\int_\sigma d'\alpha = -\sum_{\tau \text{ facet of } \sigma} \int_\tau \iota_{v''_{\sigma,\tau}} \alpha\big|_\tau \,.$$

***Proof*** Hold the second exterior factor of top degree fixed and apply ordinary Stokes to the first exterior factor. To compute the sign and normalization on a facet, start the lattice basis of $N_\sigma$ with $\nu_{\sigma,\tau}$, followed by a basis of $N_\tau$. The first coordinate increases inward. The component of $\alpha$ contributing at the facet is the coefficient of

$$\mathrm{d}''x_1 \wedge \mathrm{d}'x_2 \wedge \mathrm{d}''x_2 \wedge \cdots \wedge \mathrm{d}'x_n \wedge \mathrm{d}''x_n.$$

Insertion of the inward vector in the second copy returns that coefficient in the facet volume form. Differentiating it in the first coordinate and integrating has the negative boundary sign at the inward end. The remaining coordinates use the lattice measure on the facet. Ordinary Stokes for polyhedra with compact support yields the formula, since intersections of facets have zero boundary measure. □

**Definition 10.3.1** A weighted complex $C$ as in Section 10.2 is *balanced* at an $(n-1)$-dimensional face $\tau$ if

$$\sum_{\sigma \supset \tau} m_\sigma \nu_{\sigma,\tau} \in \operatorname{span}(\tau - \tau),$$

the sum running over the top-dimensional faces containing $\tau$; equivalently, this sum vanishes in $(N/N_\tau) \otimes \mathbb{R}$. The condition does not depend on the lifts $\nu_{\sigma,\tau}$. The complex is *balanced* on an open set $U \subset V$ if it is balanced at every $(n-1)$-dimensional face meeting $U$, and *balanced* if this holds for $U = V$.

If an $(n-1)$-dimensional face lies in exactly one top face and that face has nonzero weight, the complex is not balanced there. Thus if $C$ is balanced on $U$, no $(n-1)$-dimensional face meeting $U$ lies in exactly one top face of nonzero weight.

**Theorem 10.3.1** *Let $C$ be a weighted complex balanced on an open set $U \subset V$. Then*

$$\int_C d'\alpha = 0, \qquad \int_C d''\beta = 0$$

*for forms $\alpha, \beta$ on $C$ of bidegrees $(n-1, n)$ and $(n, n-1)$, respectively, with compact support in $U$.*

***Proof*** Represent $\alpha$ by an ambient form with compact support in $U$, as in Section 10.2. For $d'$ sum the boundary formula over the top faces. For each facet $\tau$ of a top face $\sigma$, the lattice $N_\sigma/N_\tau$ is saturated in $N/N_\tau$, since its quotient $N/N_\sigma$ is torsion-free. The inward primitive generator is therefore primitive in the ambient quotient. At a fixed facet, the sum inserts $\sum_{\sigma \supset \tau} m_\sigma \nu_{\sigma,\tau}$. This vector is tangent to $\tau$ by balancing, and the restriction of its insertion vanishes by the dimension of the second exterior factor. Facets not meeting $U$ contribute zero by support. The second identity follows by applying the first to $J\beta$, since $J$ acts on the paired top volume by $(-1)^n$. □

## 10.4 Currents and corner measures

On an $n$-dimensional complex a current of bidegree $(p,q)$ acts on compactly supported forms of bidegree $(n-p,n-q)$. It is a continuous linear functional for smooth test convergence on a common compact support. In a fixed ambient presentation this means continuity on smooth ambient tests, and the functional must vanish on tests that restrict to zero on the complex. The sign convention is

$$(d'T)(\gamma) = (-1)^{p+q+1}T(d'\gamma), \qquad (d''T)(\gamma) = (-1)^{p+q+1}T(d''\gamma).$$

On a balanced complex this agrees with differentiation of smooth forms viewed as currents by integration of a wedge product, by Theorem 10.3.1. The integration current of the complex is closed under both operators. A function integrable on compact subsets also gives a current by multiplication before integration.

We compute a second derivative on a convex open set in a lattice affine space of dimension $n$. Let $f$ be continuous and affine on the cells of a finite rational polyhedral subdivision of this set. For an internal facet $\tau$, choose a primitive lattice normal modulo $N_\tau$, directed from the minus side to the plus side. Denote a lift by $\nu_\tau$ and the two linear parts of $f$ by $a_-$ and $a_+$. Their restrictions to the tangent space of $\tau$ are equal by continuity. The number $s_\tau = (a_+ - a_-)(\nu_\tau)$ is independent of the lift and of the simultaneous reversal of the normal and the designation of the sides.

**Proposition 10.4.1** *For every compactly supported smooth $(n-1,n-1)$-test $\gamma$,*

$$(d'd''f)(\gamma) = \sum_\tau s_\tau \int_\tau \gamma.$$

*The numbers $s_\tau$ are nonnegative exactly when $f$ is convex.*

***Proof*** Integrate by parts on the cells. For the first derivative $d''f$, the boundary terms from adjacent cells cancel because the traces of $f$ are equal. Thus $d''f$ is represented on each open cell by $Ja_\sigma$, where $a_\sigma$ is the linear part. In the next derivative the sign in the dual convention is plus. On each cell the form $Ja_\sigma$ is constant, and

$$d'\left(Ja_\sigma \wedge \gamma\right) = -Ja_\sigma \wedge d'\gamma.$$

The facet formula therefore inserts the inward vector into $Ja_\sigma \wedge \gamma$ with a positive sign. Take opposite lifts on the two sides of $\tau$. The parts that insert into $\gamma$ cancel because the tangential restrictions of the linear parts agree. The remaining terms insert into $Ja_\sigma$ and give precisely $s_\tau$ times the restriction of $\gamma$. This proves the identity.

Convexity implies nonnegative jumps by restricting to a transverse segment through the relative interior of each facet. Conversely a generic segment meets only relative interiors of facets and has increasing slopes when all jumps are nonnegative. Approximate any segment by such segments and use continuity. □

For $f(x) = \max(0,x)$ on the line the formula gives the unit point mass at zero. In higher dimension the jump is measured using the primitive lattice normal, rather

than its Euclidean length. This is the normalization used when combining corner measures and the weights of a tropicalization.

## 10.5 Integration through a finite torus projection

Consider a finite locally free map $p : Y \to T_Q$ as in Section 9.1. We integrate a top form in the coordinates of this projection, with continuous tests on $Y$. Write a smooth $(n,n)$-form $\Omega$ on $Q^\circ$ as $a(w) \bigwedge_{i=1}^n (\mathrm{d}' w_i \wedge \mathrm{d}'' w_i)$ in lattice coordinates. For a real continuous test $\chi$ compactly supported over $Q^\circ$, put

$$I_p(\chi, \Omega) = \int_{Q^\circ} K_w(\chi) a(w)\, \mathrm{d}w, \tag{10.1}$$

with lattice measure $\mathrm{d}w$. This is finite: the projected support is compact in $Q^\circ$, and $K_w(\chi)$ is continuous with absolute value bounded by $d\|\chi\|_\infty$, where $d$ is the rank over the Gauss section. We call (10.1) the *calibration* of $p$.

**Proposition 10.5.1** *Suppose additional coordinates are given by analytic units $g_1, \dots, g_l$ on $Y$. For a test $\chi(y) = b(\mathrm{trop}(p(y)), g_{\mathrm{trop}}(y))$, with $b$ continuous on $Q \times \mathbb{R}^l$ and the projection of its support to $Q$ compact in $Q^\circ$, the integral in (10.1) is the sum of the integrals over the affine graphs of Theorem 9.1.1, with graph weights in the projected lattice coordinates given by their fibre degrees. Refining the moment by additional units preserves this formula.*

***Proof*** On a full-dimensional open cell of $Q$, let $\sigma_i(w) = (w, \alpha_i(w))$ be the affine graphs of Theorem 9.1.1, with weights $d_i$. The trace of $\chi$ there is exactly $\sum_i d_i b(\sigma_i(w))$. The faces of lower dimension have zero lattice measure, so this gives the formula after integration. In terms of the lattice in the graph direction, its weight is $d_i/e_i$, where $e_i$ is the index of the projection of that lattice into the base lattice. Indeed projection has Jacobian $e_i$ for lattice measures and pulls a top form back with factor $e_i^2$ from the two copies of the differential, so integration of the pullback with weight $d_i/e_i$ gives exactly $d_i$ times integration in the base coordinates.

The combined moment for two lists of units gives a common refinement. Summing the projected fibre degrees of the refined graphs over an original graph recovers its projected degree, since the sums count the same fibre points with the same local dimensions. Thus dropping the added coordinates preserves the integral. If a cell in the full tropical image projects with rank less than $n$, its restriction of the pulled-back form is zero, because its first copy already receives a wedge of $n$ dependent one-forms. Off the countable union of rational hyperplanes in the base, the fibre of the tropical coordinates on $T_Q$ is the singleton $\eta_w$. In fact, every Laurent polynomial then has a unique largest nonzero term, and polynomial density determines the seminorm on the box. This is why the formula for the projected top form uses only the Gauss fibres. □

The inverse image of a smaller closed box is again finite locally free over that box, by tensoring the finite projective module with the restricted annular algebra. The analytic points of this tensor product are exactly the inverse image: evaluation at a point of $Y$ over the smaller box extends from the base to its smaller annular presentation by the bounded substitution property. Conversely the tensor product restricts to such a point, and its finite module generators with the base algebra determine the seminorm in the complete residue field. Over $\eta_w$ the fibre algebra after this restriction is the old one, since the base restriction has the same completed residue field at the Gauss point. Consequently (10.1) agrees for tests compactly supported over the interior of the smaller box.

**Proposition 10.5.2** *Let $q : T \to T'$ be a finite surjective homomorphism of split tori and $F : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ its invertible linear map on log norm coordinates. Consider a closed box $Q'$ in the target with $F^{-1}(Q') \subset Q$, and restrict $Y$ to its inverse image. For tests supported over the interior of $Q'$ one has*

$$I_{q\circ p}(\chi, \Omega) = I_p(\chi, F^*\Omega).$$

*Here the left hand side uses the composite finite locally free algebra over $T'_{Q'}$, and the right hand side integrates on $F^{-1}(Q'^{\circ})$.*

***Proof*** The characters of $T'$ pull back to a sublattice $M' \subset M$ of index $h$, and the Laurent ring of $T$ is a free module of rank $h$ over the Laurent ring of $T'$, with basis indexed by coset representatives in $M/M'$. The inverse image of $T'_{Q'}$ has the algebra obtained by tensoring this finite free presentation with the target box algebra. The points of that algebra are the points of $T$ whose projected norms lie in $Q'$: evaluate the Laurent coordinates at the point and the base series by bounded substitution on the box. This identification also follows from the monomial inequalities defining the inverse image and their affinoid universal property. The domain lies in $T_Q$, by $F^{-1}(Q') \subset Q$. Restricting the algebra $B$ of $Y$ by tensor product makes a finite locally free algebra over the inverse image domain. The composite is finite locally free since a finite projective module over a finite free algebra is finite projective over the base.

Over the Gauss point $\eta'_{Fw}$ this first finite free algebra is the single Gauss field $\mathcal{H}(\eta_w)$, of degree $h$ over $\mathcal{H}(\eta'_{Fw})$. To check this, write Laurent polynomials in $M$ as sums over cosets of $M'$. In the Gauss norm the norm of the sum is the maximum of the norms of the terms, since different cosets have disjoint monomial supports. The same holds for coefficients in the fraction field of the Laurent ring of $M'$ by clearing a common denominator. Hence the weighted maximum norm on the finite free vector space extends the Gauss norm isometrically to its completion. Its multiplication remains multiplicative by density. The completed algebra is thus a domain and, being finite-dimensional over the completed base field, a field. This proves the Gauss field assertion and the uniqueness of the point in the fibre.

The fibre algebra for $Y$ over $\eta'_{Fw}$ has the same local factors as the fibre over $\eta_w$. Measured over the smaller field their dimensions are multiplied by $h$. Thus the trace for the composite is $hK_w(\chi)$. Also $|\det F| = h$, since both lattices have dual bases. Changing variables in the target integral introduces a second factor of $h$. The pullback

$F^*\Omega$ on the right introduces $(\det F)^2 = h^2$ through the two differential copies. The two integrals are equal. □

## 10.6 Comparing two finite projections of a moment

Two projections of a common moment need not factor through one another on the analytic source. Their logarithmic coordinates can still be linearly related on one cell. We compare the two degrees at a common lift of a generic weight. Throughout this section $k$ is trivially valued and the dimension $n$ of a projected torus is positive.

### 10.6.1 Changing the map on an exact Gauss fibre

Let $w$ be an independent weight, that is, one with coordinates linearly independent over $\mathbb{Q}$, and use the Gauss field $F_w$ of Section 9.2, the algebra of the exact norm domain. Its residue field is $k$.

**Lemma 10.6.1** *Let $D$ be a nonzero local Artinian affinoid $k$-algebra. Suppose two bounded $k$-algebra maps $i, i' : F_w \to D$ make $D$ finite-dimensional over $F_w$. Then* $\dim_i D = \dim_{i'} D$.

***Proof*** We first reduce to algebraically closed $k$. Its algebraic closure $\overline{k}$ with the trivial norm is complete. By (2.4) and the exact annular presentation, $(F_w)_{\overline{k}}$ is the Gauss field of weight $w$ over $\overline{k}$. For either structure map $i$, the norm $\sup_{a\neq 0} \|i(a)b\|/|a|$ on $D$ is equivalent to the given norm by boundedness and the term $a = 1$. It is homogeneous over $F_w$. The finite-dimensional norm comparison of Lemma 2.3.1 therefore gives a bounded basis isomorphism and bounded inverse in either structure. Completing after scalar extension identifies $D_{\overline{k}}$ with the corresponding base change over $F_w$. Thus the two dimensions are unchanged, and decompose as the sums over the same local factors of $D_{\overline{k}}$. It suffices to compare on each factor. We henceforth take $k$ algebraically closed and $D$ local.

Write $y$ for the unique point of the spectrum of $D$. The residue field of $\mathcal{H}(y)$ is finite over $k$. In fact, lifts of linearly independent residue classes are linearly independent over $F_w$: normalize a linear relation by a coefficient of greatest norm and reduce it in the residue field. This bounds the residue degree by $[\mathcal{H}(y) : F_w]$, which is finite. Since $k$ is algebraically closed, the residue field is $k$. If $T_i$ are the Laurent coordinates, the quotient $i'(T_i)/i(T_i)$ has norm one at $y$. Its nonzero residue is some $\lambda_i \in k^\times$. The substitution $T_i \mapsto \lambda_i T_i$ is an isometric automorphism of $F_w$. Up to composing one map with such an automorphism, we have

$$i'(T_i) = i(T_i)(1 + u_i), \qquad |u_i(y)| < 1.$$

Choose $R > 1$ and positive real $s_i > |u_i(y)|$ with $Rs_i < 1$ for each $i$. Write $A = F_w\{R^{-1}z\}$ and $B = D\{R^{-1}z\}$ for the algebras on the parameter disc. In addition

to the original map $i : A \to B$, define a map $j : A \to B$ by

$$j(z) = z, \qquad j(T_i) = i(T_i)(1 + zu_i).$$

For every spectral point of $B$ the restriction to $D$ is $y$, so $|zu_i| < 1$. The factors $1+zu_i$ are units, by the spectral radius formula and the geometric series. They have absolute value one at all spectral points, so the new coordinates and their inverses have the prescribed exact radii. The annular presentation and Proposition 2.5.1 give bounded substitution for $j$.

We verify that $B$ is a finite $A$-module for $j$. Choose an $F_w$-basis $b_l$ of $D$ for $i$, and radii $t_l > |b_l(y)|$. Over $A$ for $j$, introduce a Tate algebra $\widetilde{A} = A\{s^{-1}U, t^{-1}W\}$ and substitute $U_i \mapsto u_i$, $W_l \mapsto b_l$. This bounded map $\widetilde{A} \to B$ admits the following lifts. The original map from $A$ lifts to $\widetilde{A}$ by sending $T_i$ to $T_i(1 + zU_i)^{-1}$ and fixing $z$. In $\widetilde{A}$ the factors $1 + zU_i$ and their inverses have norm at most one, since $Rs_i < 1$. Thus the lift is bounded in the annular presentation, and its composition with substitution is the map induced by $i$. Every element of $B$ has an expansion $\sum_l b_l i(a_l)$, $a_l \in A$, with the coefficient norms bounded by a fixed multiple of its norm. This follows coefficient by coefficient in $z$ from the norm equivalence for the basis of $D$. Lifting this sum using $W_l$ and the lifted $i(a_l)$ gives a uniform norm bound for the lifts to $\widetilde{A}$. Thus $\widetilde{A} \to B$ is admissible. All its relative images have spectral radii strictly below their presentation radii, so Lemma E.2.1 makes $B$ finite over $A$ for $j$.

The ring $A$ is a principal ideal domain. In the division proof of Theorem 2.1.1 for one variable and one component, a single element of smallest leading exponent in a nonzero ideal suffices for the division. Its remainder for every element of the ideal vanishes, since a nonzero remainder in the ideal would have a smaller leading exponent. The ring is a domain since it is contained in $F_w[[z]]$. The finite $A$-module $B$ for $j$ is torsion-free. For $0 \neq a = \sum_l a_l z^l \in A$, let $l_0$ be the first nonzero index. Modulo $z$ the map $j$ is $i$, so $j(a)$ has leading coefficient $i(a_{l_0})$ at index $l_0$. This coefficient is a unit of $D$. In $D[[z]]$ the series $j(a)$ is therefore a power of $z$ times a unit and is a nonzerodivisor. This also proves injectivity of multiplication by $j(a)$ on its subring $B$. It follows from the structure theorem for finite modules over a principal ideal domain that $B$ is free over $A$.

Specialize the free module at $z = 0$ and $z = 1$. In both cases evaluation on $B$ has kernel $(z - a)B$, where $a = 0$ or $a = 1$. Indeed a convergent series minus its value at $a$ is divisible by $z - a$: division of each difference $z^m - a^m$ gives a polynomial of norm at most $R^{m-1}$, and the coefficient series after this division is summable in $B$. The same holds in $A$. Thus the two specialized modules are $D$ for $i$ and $D$ for $i'$. Both have the rank of the free module, which proves the equality. □

### 10.6.2 Two projections on a common cell

Let $\mu : X \to T$ be a moment on an affinoid space. Let $\varphi : T \to T_1$ and $\psi : T \to T_2$ be homomorphisms to split $n$-tori. Suppose $p = \varphi \circ \mu$ and $q = \psi \circ \mu$ restrict to finite

locally free maps on affinoid domains $Y_p, Y_q$ of $X$, respectively, to affinoid domains of the two tori. Write $\varphi_{\mathrm{trop}}, \psi_{\mathrm{trop}}$ for the lattice maps. In a rational $n$-plane $L$ in the log norm space of $T$, suppose both lattice maps have full rank. Put

$$F = (\psi_{\mathrm{trop}}|_L)(\varphi_{\mathrm{trop}}|_L)^{-1}.$$

This is a rational invertible linear map, not to be confused with the Gauss fields $F_w$. The full-dimensional graphs of Theorem 9.1.1 lie in rational linear planes: in its proof the coefficient log norms are piecewise linear with zero constant terms because the valuation on $k$ is trivial.

**Theorem 10.6.1** *For $y \in Y_p \cap Y_q$ whose moment weight lies in $L$, assume $w = \mathrm{trop}(p(y))$ has coordinates linearly independent over $\mathbb{Q}$. Then the two Gauss fibre degrees satisfy*

$$d_q(y) = |\det F|\, d_p(y). \tag{10.2}$$

*In particular the degree divided by the projection index of the lattice of $L$ is independent of the projection at these points.*

***Proof*** The $q$-weight is $w' = Fw$, also with independent coordinates. Thus the points of either base at the given weights are just the respective Gauss points. The common source domain $V$ of those moment coordinates is obtained by intersecting $Y_p, Y_q$ and imposing the exact norms of the units of $\mu$ at $y$. These equalities define rational domains, using both an upper and a lower bound for each unit. A finite intersection of affinoid domains is an affinoid domain, by Proposition E.1.4. Every point of $V$ has $p$-weight $w$ and $q$-weight $w'$, so the maps from $V$ to the two bases factor through the exact norm domains at these weights, whose algebras are $F_w$ and $F_{w'}$. Thus $V$ lies in the Gauss fibres $p^{-1}(\eta_w)$ and $q^{-1}(\eta_{w'})$ and maps to them. An affinoid map to either fibre is determined by its composite to $X$, since Laurent polynomials are dense in $F_w$ and $F_{w'}$. The domain universal property of $V$ in $X$ therefore makes $V$ an affinoid domain of each of the two Gauss fibres. A domain in a finite-dimensional fibre algebra simply selects the corresponding local factors. To see this, note that the product decomposition of the Artinian algebra realizes each subset of its spectrum by projection onto a product of factors. This projection has the domain property: the omitted idempotent must map to zero, because it vanishes at every point of the target spectrum and its image is idempotent. Uniqueness in the domain property then proves the claim. In particular at $y$ the two local fibre algebras identify with the same local algebra $D$, retaining any nilpotents. It has structure maps from $F_w$ and $F_{w'}$ with dimensions $d_p(y)$ and $d_q(y)$, respectively.

Choose an integer $a > 0$ for which $aF$ is integral. The isogenies with lattice matrices $aF$ and $aI$ on the two projected tori send $w$ and $w'$ to the same weight $v = aFw$. They give two structure maps $F_v \to D$. By the Gauss field computation in Proposition 10.5.2, the intermediate field degrees are $|\det(aF)|$ and $a^n$. The weight $v$ still has independent coordinates. Applying Lemma 10.6.1 and multiplying dimensions in the two towers gives $|\det(aF)| d_p(y) = a^n d_q(y)$. This is (10.2). If $\Lambda = L \cap N$ is the lattice in $L$, with $N$ the ambient cocharacter lattice, the ratio of the indices

of $\psi_{\mathrm{trop}}(\Lambda)$ and $\varphi_{\mathrm{trop}}(\Lambda)$ in the two base lattices is also $|\det F|$, by computing both indices in the same basis of $\Lambda$. □

When the target domains are closed annular boxes $(T_1)_{Q_1}$ and $(T_2)_{Q_2}$, we compare the two calibrations of Section 10.5 on the part of the source above a cell of $L$. Let $\Delta \subset L$ be a relatively open polyhedral cell with $\varphi_{\mathrm{trop}}(\Delta) \subset Q_1^\circ$ and $\psi_{\mathrm{trop}}(\Delta) \subset Q_2^\circ$. Let $\chi$ be a continuous test on $X$ such that $\mathrm{trop} \circ p$ maps $Y_p \cap \operatorname{supp} \chi$ into $Q_1^\circ$ and $\mathrm{trop} \circ q$ maps $Y_q \cap \operatorname{supp} \chi$ into $Q_2^\circ$. Let $\Omega = a(v) \bigwedge_{i=1}^n (\mathrm{d}' v_i \wedge \mathrm{d}'' v_i)$ be a smooth paired top form on $Q_2^\circ$ in lattice coordinates, so that $F^*\Omega = (\det F)^2 a(Fw) \bigwedge_{i=1}^n (\mathrm{d}' w_i \wedge \mathrm{d}'' w_i)$ on $F^{-1}(Q_2^\circ)$. Define the restricted calibrations

$$\begin{aligned} I_p^\Delta(\chi, F^*\Omega) &= \int_{Q_1^\circ \cap F^{-1}(Q_2^\circ)} \sum_{y \in E_w} d_p(y) \chi(y)\, (\det F)^2 a(Fw)\, \mathrm{d}w, \\ I_q^\Delta(\chi, \Omega) &= \int_{Q_2^\circ} \sum_{y \in E'_v} d_q(y) \chi(y)\, a(v)\, \mathrm{d}v, \end{aligned} \tag{10.3}$$

where $E_w$ is the set of $y \in Y_p \cap Y_q$ with $p(y) = \eta_w$ and $\mathrm{trop}\, \mu(y) \in \Delta$, and $E'_v$ is the set of $y \in Y_p \cap Y_q$ with $q(y) = \eta_v$ and $\mathrm{trop}\, \mu(y) \in \Delta$.

**Corollary 10.6.1** *In this situation both integrands in* (10.3) *are measurable, bounded and compactly supported in the domains of integration, and*

$$I_p^\Delta(\chi, F^*\Omega) = I_q^\Delta(\chi, \Omega).$$

*In particular, if $\chi$ has compact support in an ordinary open subset of $X$ contained in $Y_p \cap Y_q$ with moment weights in $\Delta$, then $I_p(\chi, F^*\Omega) = I_q(\chi, \Omega)$ for the calibrations* (10.1)*. Here, as in Proposition 10.5.2, the left-hand side is the integral* (10.1) *taken over $Q_1^\circ \cap F^{-1}(Q_2^\circ)$, on which $K_w(\chi)$ has compact support.*

***Proof*** The sums in (10.3) are bounded by the ranks times $\|\chi\|_\infty$. A point counted with $\chi(y) \neq 0$ has $p$-weight in the compact image of $Y_p \cap \operatorname{supp} \chi$ in $Q_1^\circ$, and $q$-weight, equal to $F$ of its $p$-weight, in the compact image of $Y_q \cap \operatorname{supp} \chi$ in $Q_2^\circ$. This gives the compact supports. The sums are measurable in the base weight. Indeed $Y_q$ is a finite union of rational domains of $X$ by Theorem E.2.1. Each such domain is the zero locus of the continuous nonnegative maximum of the violations of its finitely many inequalities and zero. A finite union is obtained by taking the minimum of those functions. The indicator of a zero locus of $b \geq 0$ is the limit of $\max(0, 1 - jb)$. The indicator of a relatively open polyhedral cell in Euclidean space is also a bounded pointwise limit of continuous functions, using distance to its closure and to the relative boundary. Multiply these approximants, composed with $\mathrm{trop}\, \mu$ for the cell, with the test and apply Proposition 9.1.1. The corresponding argument works for $Y_p$.

The set where the coordinates of $w$ are dependent is a countable union of proper hyperplanes and has lattice measure zero. Off that set the two restricted fibres $E_w$ and $E'_{Fw}$ have the same points: both base points are Gauss, and $\Delta$ projects injectively to each base. By (10.2), the $q$-sum at $Fw$ equals $|\det F|$ times the $p$-sum at $w$. Changing

variables from $Fw$ to $w$ gives another factor of $|\det F|$. This is the factor $(\det F)^2$ of $F^*\Omega$, as in Section 10.5, and proves equality of the integrals. A test with compact support in the indicated ordinary open subset satisfies the support conditions above, since $\operatorname{trop} \circ p = \varphi_{\mathrm{trop}} \circ \operatorname{trop} \mu$ and similarly for $q$. Every fibre point with nonzero test lies in $Y_p \cap Y_q$ and has moment weight in $\Delta$. The restricted sums are then the full sums of (10.1), after extension of the test by zero to $X$. In particular the first paragraph shows that $K_w(\chi)$ has compact support in $Q_1^\circ \cap F^{-1}(Q_2^\circ)$. □

## 10.7 Algebraic tropical weights and Gauss fibre degrees

Let $Z$ be a pure $n$-dimensional closed subscheme of a split torus $T$, with $n \geq 1$ and $k$ trivially valued. Use the character lattice $M$, the ideal $I$ of $Z$ in $k[M]$, and the fan of Theorem 8.6.1. For a top-dimensional cone $\sigma$ put $L = \operatorname{span}_{\mathbb{R}} \sigma$, $N_\sigma = N \cap L$, and $M_\sigma = M \cap L^\perp$. Recall the transverse initial algebra $C_\sigma$ of (8.1) and its dimension $m_\sigma$.

Choose a splitting $M = M_\sigma \oplus M'$. Use $z_1, \ldots, z_n$ for Laurent coordinates of $M'$, and $y_1, \ldots, y_l$ for those of $M_\sigma$. The $z_i$ give a projection $p_0$ of $T$ to a split $n$-torus, which induces an isomorphism of $N_\sigma$ with the target lattice. The $y_j$ have log norm zero on $L$.

**Lemma 10.7.1** *Suppose $\xi \in \sigma^\circ$, where $\sigma^\circ$ denotes the relative interior, and $w = \operatorname{trop}(p_0)(\xi)$ has coordinates linearly independent over $\mathbb{Q}$. Let $A_\xi = B_\xi / \overline{IB_\xi}$ be the algebra of the exact norm domain of $Z$ at $\xi$. Then*

$$\dim_{F_w} A_\xi = m_\sigma.$$

*The dimension counts the full algebra, with its nilpotents.*

***Proof*** Regrouping Laurent terms by their $y$-exponents gives an isometric identification of $B_\xi$ with the completion of $F_w[y_1^{\pm 1}, \ldots, y_l^{\pm 1}]$ at unit radii. Indeed the coefficients for a fixed $y$-exponent form the Laurent series in the $z_i$ for $F_w$, and their Gauss norms tend to zero off finite sets of $y$-exponents. These two convergence conditions are equivalent to the original one for $B_\xi$, and give the same maximum norm. Equivalently use the unit-radius Tate algebra over $F_w$ in the $y_j$ and inverse variables, imposing the inverse relations. This presentation is isometric on the quotient: reduce each monomial to its Laurent exponent and lift each Laurent monomial using its positive and negative parts. Both maps are norm-bounded with bound one and extend to the completions.

Thus $A_\xi$ has the quotient norm of a unit-radius Tate algebra over $F_w$. In the division proof of Theorem 2.1.1 the remainder is a linear map with norm at most one. Its image is the space of convergent series supported on the monomials indivisible by the chosen leading monomials. In fact the remainder is the identity on that space. The quotient norm of a class equals the norm of its remainder, since every representative has norm at least that of the remainder and the remainder is itself a representative.

The quotient is therefore isometric, as a vector space over $F_w$, to the space of such coefficient families with norms tending to zero off finite sets.

The degree-zero part of the quotient filtration on $A_\xi$ is $C_\sigma$. This follows from Section 8.5 and (8.1), since the only degree-zero monomials in the $z_i$ at $\xi$ are constants. The quotient filtration is also the filtration by the quotient norm just computed: the monomial lift to the unit Tate presentation preserves the norm, and the remainder attains it. The unit ball modulo its elements of norm less than one has a basis over $k$ given by all the remainder monomials. Indeed $F_w$ has residue field $k$ and only finitely many coefficients of any convergent series can have norm one. If all coefficients have norm less than one, their maximum is strictly less than one. Since $C_\sigma$ has dimension $m_\sigma$, there are exactly $m_\sigma$ remainder monomials. The isometric quotient description gives the asserted dimension over $F_w$. □

**Theorem 10.7.1** *Fix a point of $\sigma^\circ$, and let $p$ be a torus projection of rank $n$ on $L$, with projection index $e_p$ on $N_\sigma$. There are affinoid neighborhoods of the entire source over that weight on which $p_0$ and $p$ are finite over annular boxes, with no source weights outside $\sigma^\circ$. For nearby weights $\xi$ in $\sigma^\circ$, write $w = \mathrm{trop}(p_0)(\xi)$ and $v = \mathrm{trop}(p)(\xi)$. The sums of local Gauss fibre degrees over these neighborhoods are $m_\sigma$ for $p_0$ and $e_p m_\sigma$ for $p$. As measures on the common source at $\xi$, the two Gauss fibre sums agree after division by their projection indices. This includes dependent weights.*

***Proof*** In a sufficiently small ambient neighborhood of the fixed weight, the tropicalization equals the intersection with $L$. Indeed no other cone of the finite tropical subfan contains the fixed weight, and $\sigma^\circ$ is open in $L$. Choose within that neighborhood a closed product of a $z$-box $P$ about the projected weight and a $y$-box with zero in its interior. Let $Y_0$ be the intersection of the corresponding polyannulus with $Z^{\mathrm{an}}$. Its affinoid presentation is obtained by imposing $I$ on the annular algebra, as in Section 8.4. The defining inequalities are strict at the fixed weight, so $Y_0$ is a neighborhood of every point with that weight. All points of $Y_0$ have $|y_j| = 1$. Over $A_P$, its relative presentation uses the $y_j$ and their inverses with radii strictly greater than one, modulo the inverse equations and $I$. Thus $Y_0 \to T_P$ is finite by Lemma E.2.1. It is finite free on a rational neighborhood of the entire target Gauss section by Proposition 9.2.2.

At an independent weight $w \in P^\circ$, the $p_0$-fibre over $\eta_w$ is the affinoid exact norm domain over the unique $\xi$ in $L$ with $p_0$-weight $w$. Indeed the base norm domain at $w$ has algebra $F_w$, and all the transverse norm equalities already hold on the spectral fibre. Imposing those equalities by rational localization therefore leaves its algebra unchanged, by the domain universal property. By Lemma 10.7.1 that fibre has dimension $m_\sigma$. The free rank near the Gauss section is consequently $m_\sigma$, giving the same degree sum there at every weight.

On $L$ write $v = Fw$ for the second projected weight as a function of the first. Because $p_0$ identifies $N_\sigma$ with its target lattice, $e_p = |\det F|$. Choose a closed box $Q$ about the fixed $p$-weight inside $F(P^\circ)$, and put $Y_p = Y_0 \cap p^{-1}(T_Q)$. The absolute presentation of $Y_0$ has variables for the $z_i, y_j$ and their inverses. An admissible relative

presentation of $Y_p$ over $A_Q$ uses the same variables and relations, with the additional relations identifying the target coordinates with the monomials that define $p$. This is the combined quotient for an affinoid base change. In this presentation the bounds for $y_j$ and their inverses remain strict. The bounds for the $z_i$ and their inverses are also strict, since all their weights lie in $F^{-1}(Q) \subset P^\circ$. Compactness gives strict spectral bounds, so $p$ is finite over $T_Q$. Again it is finite free on a rational neighborhood of the Gauss section.

Let $K^0_w$ and $K^p_v$ denote the degree-weighted Gauss fibre sums for $Y_0$ and $Y_p$, the latter viewed as measures on $Y_0$. For any continuous test on $Y_0$, their integrations vary continuously in the base weight. This is Proposition 9.1.1 applied over the finite free rational neighborhoods just found; their restrictions preserve the completed residue fields and hence the fibre algebras. At independent weights $v \in Q^\circ$, both projections can be restricted to finite free annular boxes by Proposition 9.2.2. For $w = F^{-1}v$, every point of the $p_0$-fibre at $\eta_w$ lies in $Y_p$ and has $p$-image $\eta_v$, since that is the only base point of weight $v$. The reverse inclusion follows using independence of $w$. All these points have the same moment weight $\xi$. Apply Theorem 10.6.1 to the two finite free restrictions. It follows that $K^p_v = e_p K^0_{F^{-1}v}$. Independent weights of $Q^\circ$ are dense in $Q$. The continuity for each test extends this equality to all $v \in Q$, giving the assertion including the total degree for $p$. □

At an independent weight, each individual Gauss lift in this construction has a finite free annular neighborhood as in Corollary 9.2.1. Indeed over a smaller box the map is already finite and has empty relative boundary by Section 2.5. Its fibres are finite. The equality of measures in Theorem 10.7.1 also identifies the sets of Gauss lifts in the two charts at dependent weights, since every local degree is strictly positive. To compute the calibration on the top cell, a test pulled back from the cell has the same value at all the lifts of a weight, whose degrees sum to $e_p m_\sigma$. Projection of the cell lattice has Jacobian $e_p$ and pullback of a paired top form contributes $e_p^2$. Thus the cell weight is $m_\sigma$, as in Section 8.7. This comparison is local on the relative interior of the cell.

## 10.8 Calibration across a subdivision

Let $X$ be affinoid and let $p$ and $q$ be $n$-tuples of analytic units on $X$. Suppose they give finite locally free maps from affinoid domains $Y_p$ and $Y_q$ to affinoid domains of the respective split tori. Take full-dimensional compact weight boxes $B_p, B_q$ whose Gauss sections lie in the respective targets. Write $S_p, S_q$ for the preimages in $Y_p, Y_q$ of these compact Gauss sections. Write $p_{\mathrm{trop}}, q_{\mathrm{trop}}$ for the log norm maps, and put

$$\mu_{\mathrm{trop}} = (p_{\mathrm{trop}}, q_{\mathrm{trop}}, g_{\mathrm{trop}}),$$

where $g$ is any finite tuple of auxiliary units on $X$. We work on an ordinary open subset $W$ of $X$ contained in $Y_p \cap Y_q$, with both projected weights in the interiors of the chosen boxes.

Use (10.1) to define the calibration over each box. This formula also applies to the affinoid target domains just described. Indeed Proposition 9.1.1 gives the continuous trace on each Gauss section, and a test in $C_c(W)$, extended by zero to $X$, has projected support compactly contained in each open box. The degrees are taken in the finite fibre algebras over the completed Gauss fields.

For closed annular targets, Theorem 9.1.1 describes the joint images of $S_p$ and $S_q$ by closed polyhedral graphs. For the Laurent neighborhoods of Theorem 9.4.1, use the boxes given by Corollary 9.4.1 for the chosen tuple of units. In either case all graph directions lie in rational linear planes: the Gauss norm of each nonzero coefficient is the norm of a Laurent polynomial or a Laurent rational function on the chosen box, so the coefficient pieces have zero constant term. The root norm formulas preserve this property. The following comparison applies whenever we have these graph descriptions.

**Theorem 10.8.1** *Take a common polyhedral subdivision of $\mu_{\mathrm{trop}}(S_p) \cup \mu_{\mathrm{trop}}(S_q)$ in which each of the graphs is a union of cells. Let $\Omega_p, \Omega_q$ be smooth paired top forms on the interiors of the two weight boxes. Suppose their pullbacks by the respective projections to each open $n$-cell agree at every point of that cell reached by $W \cap (S_p \cup S_q)$. Then for every $\chi \in C_c(W)$,*

$$I_p(\chi, \Omega_p) = I_q(\chi, \Omega_q). \tag{10.4}$$

***Proof*** Discard the images of the lower-dimensional cells in each base, which have lattice measure zero. For a full-dimensional cell consider the part of each trace whose joint weight lies in the cell. These restricted traces are measurable by the indicator approximation in Corollary 10.6.1. At a cell on which the $q$-projection has rank less than $n$, the $q$-contribution is zero because the $q$-image of the cell has lattice measure zero. Moreover the pullback of $\Omega_q$ vanishes on the cell. Thus the $p$-contribution also vanishes there: if $p$ has full rank the hypothesis forces its top coefficient to vanish where the trace test is nonzero, and otherwise the cell projects to a set of measure zero. The argument with $p$ and $q$ interchanged treats the other cells with deficient rank.

On a cell with both projections of full rank, let $F$ be their linear transition in its rational plane, as in Theorem 10.6.1. We can restrict to weights with independent coordinates over $\mathbb{Q}$, again excluding a set of lattice measure zero. A point counted by one trace with nonzero test is in $W$ and has weight in the other box. Its weight in the other base also has independent coordinates, so its image there is the Gauss point. Consequently the same analytic points with nonzero test occur in the two sums. The $q$-sum at $Fw$ is the $p$-sum at $w$ multiplied by $|\det F|$, by (10.2). Changing the base variable contributes a second factor of $|\det F|$. This gives the factor $(\det F)^2$ in the pullback of the two differential copies of $\Omega_q$. The coefficient hypothesis therefore identifies the two integrals over the cell. Summing the finite list of cell contributions proves (10.4). □

**Corollary 10.8.1** *Assume that $\mu_{\mathrm{trop}}$ is injective on each of $S_p, S_q$. Suppose smooth functions $H_i, G_i$ on the interiors of $B_p, B_q$ satisfy $H_i \circ p_{\mathrm{trop}} = G_i \circ q_{\mathrm{trop}}$ on $W$ for $1 \leq i \leq n$. Then the calibrations of $\bigwedge_i d'd''H_i$ and $\bigwedge_i d'd''G_i$ agree on $W$.*

***Proof*** On each of the two compact Gauss preimages the joint log norm map is a homeomorphism onto its image, by injectivity and compactness. If a point of an open top cell has a lift in $W \cap S_p$, that whole cell is contained in the joint image of $S_p$. Therefore the two expressions for each function agree on a relative neighborhood of the point in the cell, because $W$ is open. The same applies to a lift in $W \cap S_q$. The restrictions of the projections to the plane of the cell are linear, so differentiation there commutes with their pullbacks. Taking the wedge of the resulting equalities gives the hypothesis of Theorem 10.8.1. □

# Chapter 11
# Local products on balanced charts

For $f_i(x) = \max(0, x_i)$ on $\mathbb{R}^n$, the corner measures meet at the origin, and their mixed product is the unit point mass there. Replacing the coordinate functions by independent integral linear functions multiplies the mass by the index of the lattice they generate (Proposition 11.7.1). Before computing this example, we construct the mixed product by approximation on balanced charts and extend its definition to bounded potentials obtained by local decreasing limits.

Throughout the chapter $C$ is a finite pure $n$-dimensional rational polyhedral complex with positive weights in a lattice affine space. We work in a region of its support in which balancing holds at every facet, and use the paired volume and differentiation conventions of Chapter 10.

For continuous inputs that are locally uniform limits of smooth functions with positive Hessian, Theorem 11.2.1 constructs the mixed product as a closed current, positive on strongly positive tests, local, and continuous under locally uniform convergence. Theorem 11.4.1 defines ordered products of locally bounded members of the sheaf generated by smooth psh functions under decreasing limits, and shows that they agree with the previous products on directly locally smooth-approximable inputs. Propositions 11.3.1 and 11.5.1 and Theorem 11.6.1 describe this sheaf and its decreasing smooth approximations on a flat chart, along an affine spine, and at an isolated intersection of subspaces. Theorem 11.8.1 and Corollary 11.8.1 compute the products of convex polyhedral functions by volumes and mixed volumes of their subgradient polytopes. The last sections transfer the products to analytic spaces through a finite locally free map to an annular box (Theorem 11.9.1) and compare the measures obtained from analytic coordinate changes, from two projections of a moment, and from overlapping charts (Corollaries 11.10.1 and 11.10.2 and Theorem 11.11.1). Chapter 32 uses these products to compute toric measures, and Chapter 33 uses the products in finite torus coordinates on finite covers.

## 11.1 Smooth inputs and local mass bounds

A smooth function on the chart is given locally by ambient smooth representatives. Call its Hessian positive on the chart if $d'd''f$ restricts to a positive $(1,1)$-form on every top face. The condition is independent of the representatives. For $p \le n$ smooth functions with this property, the form

$$T_p = d'd''f_1 \wedge \cdots \wedge d'd''f_p$$

defines a current by integration against tests of bidegree $(n-p, n-p)$. It is closed under both derivatives, and it is nonnegative on strongly positive tests because the wedge is strongly positive on each top face. The empty product acts by integration on $C$.

Fix a constant positive definite ambient $(1,1)$-form $\beta$. One can use the Hessian of half the squared Euclidean norm. Its restriction to each face is positive definite. For a compact set $K$ in the chart region, the *mass* of the smooth product on $K$ means $\int_K T_p \wedge \beta^{n-p}$. We will use the following estimate.

**Proposition 11.1.1** *Given compact sets $K \subset \operatorname{int} K'$ in the chart region, there is a constant $B$ such that*

$$\int_K T_p \wedge \beta^{n-p} \le B \prod_{i=1}^{p} \sup_{K'} |f_i|.$$

*Here the interior is relative to the support of $C$. The constant may depend on the chart, the weights, the sets and $\beta$.*

***Proof*** For smooth products and any compactly supported form $\eta$ of complementary bidegree, Stokes gives

$$\int_C d'd''f \wedge T_{p-1} \wedge \eta = \int_C f\, T_{p-1} \wedge d'd''\eta. \tag{11.1}$$

Indeed apply $d'$ to $d''f \wedge T_{p-1} \wedge \eta$ and then $d''$ to $fT_{p-1} \wedge d'\eta$. The intervening product is closed and has even total degree. The anticommutation of $d'$ and $d''$ gives the displayed sign.

Choose an ambient cutoff $\chi$ of compact support in the given region on $C$, with $0 \le \chi \le 1$ and $\chi = 1$ near $K$. Its support on $C$ can be contained in a compact set in the interior of $K'$. Using $\eta = \chi\beta^{n-p}$ in (11.1), both signs of $d'd''\chi$ are bounded from above by a multiple of $\beta$ on the support. Positivity of the other factors bounds the resulting integral by a constant times $\sup_{K'} |f|$ times the mass of $T_{p-1}$ on that compact set. Iterate the estimate with nested compact neighborhoods in $K'$. The case $p = 0$ is the finite lattice volume of a bounded portion of $C$. □

If $\eta$ is an arbitrary smooth complementary test supported in $K$, then

$$\left| \int_C T_p \wedge \eta \right| \le B_\eta \int_K T_p \wedge \beta^{n-p}.$$

The constant can be taken proportional to the supremum of the coefficient norms of an ambient representative of $\eta$ on $K$. For the pointwise comparison, it is enough to take a decomposable generator of the cone of strongly positive $(p, p)$-forms. Orthonormalize its $p$ first-copy factors for $\beta$; the paired wedge changes by a positive square of a determinant. On the remaining $n - p$ directions the test coefficient is bounded by its norm, while the corresponding coefficient of $\beta^{n-p}$ is $(n - p)!$. Integrating the pointwise comparison gives the result.

## 11.2 Passage to continuous inputs

**Theorem 11.2.1** *Suppose each $f_i$ admits locally uniform approximation near every point by smooth functions with positive Hessian on the chart. On these neighborhoods the smooth mixed products converge as currents, independently of the approximations. The limits agree on overlaps. The product is closed, positive on strongly positive tests, and local in the chart region. It varies continuously under locally uniform convergence of such tuples.*

***Proof*** First take two smooth tuples $f_i, g_i$, with uniform bounds on a compact neighborhood of the support of a test $\eta$. Telescope their wedge products one slot at a time. The term in the $i$-th slot is the integral of $d'd''(f_i - g_i)$ times a mixed product from the other slots and the test. By (11.1) it equals the integral of $f_i - g_i$ against that mixed product times $d'd''\eta$. Bound the latter by the coefficient comparison and Proposition 11.1.1. Thus the difference on a fixed test tends to zero when the differences of the functions do so uniformly on the compact neighborhood, provided their norms there stay bounded.

It follows that the smooth products for any chosen approximations are Cauchy on every test. The estimate between two choices gives independence. The bounds on arbitrary tests pass to the limit, so the limit is a current of order zero. Closedness and positivity pass to the limit by the definitions. Locality follows by using the estimate on compact neighborhoods inside any region where the functions agree. Thus the currents agree on overlaps and define a single current by partition of unity. Symmetry passes from the smooth products as well.

Let $T$ be the product for the last $p-1$ inputs. On the local approximating neighborhoods the smooth products multiplied by the first function converge to $f_1 T$. Indeed, approximate $f_1$ uniformly on the compact support of a test by a smooth function and use the order-zero bound before passing to the current limit. Hence the smooth integration identity gives

$$T_p = d'd''(f_1 T).$$

This identity is local, so it holds for the glued current.

For continuity take a net of input tuples converging locally uniformly to a tuple satisfying the hypotheses. Induct on $p$, starting with the fixed integration current. The products $T_i$ of the last $p - 1$ inputs converge weakly to $T$ by induction. Take a compact neighborhood $K$ of the support of a test and a nonnegative compactly

supported cutoff $\chi$ equal to one near $K$. For tests supported in $K$, the coefficient comparison gives

$$|T_i(\eta)| \le B\|\eta\|_{\infty,K}\, T_i(\chi\beta^{n-p+1})$$

where the norm is that of the coefficients of an ambient representative. The estimate passes to each local continuous product and then to the glued product by a nonnegative partition of unity. It also holds on continuous tests by uniform approximation. The masses on the right are bounded on a tail by weak convergence on the single test $\chi\beta^{n-p+1}$. Thus multiplication by the first inputs converges to $f_1 T$: use uniform convergence for the difference of the inputs, and approximate the fixed continuous test $f_1\eta$ by smooth tests for the convergence of $T_i$. Applying $d'd''$ proves continuity of the $p$-fold products. □

For $p = n$, the order-zero bound extends the positive functional on smooth tests to compactly supported continuous functions, giving a positive Radon measure. A decreasing net of inputs whose limit also satisfies the hypotheses of the theorem converges uniformly on compact sets by Proposition 4.1.1, so the mixed products converge.

The input class in Theorem 11.2.1 is closed under finite maxima. For smooth inputs $g_j$ with positive Hessians use the smooth function

$$H_\delta = \delta \log \sum_j \exp(g_j/\delta).$$

It differs from the maximum by between zero and $\delta \log m$, where $m$ is the number of inputs. In tangent coordinates on a face put

$$p_j = \frac{\exp(g_j/\delta)}{\sum_l \exp(g_l/\delta)}.$$

The Hessian of $H_\delta$ is the weighted sum of the Hessians of $g_j$ plus $1/\delta$ times the covariance matrix of their gradients with weights $p_j$. Both matrices are positive semidefinite. Approximate given continuous inputs on compact neighborhoods by the smooth ones and let $\delta$ tend to zero. This proves the closure on the chart.

## 11.3 Approximation on a flat tropical chart

On an open set in $\mathbb{R}^n$ with its usual affine structure, smooth psh functions have positive semidefinite Hessian. Consider the smallest sheaf of functions with values in $[-\infty, \infty)$ containing those smooth functions and closed under decreasing limits on neighborhoods. More explicitly, a decreasing family is given on a common neighborhood, and its pointwise limit is included there. Intersecting all sheaves with these properties defines the smallest one. We compare it to direct approximation on this chart.

**Proposition 11.3.1** *Every germ in this sheaf is either identically $-\infty$ near its base point or is finite and convex on a convex neighborhood. A finite germ has both decreasing smooth convex approximants and decreasing convex piecewise affine approximants on a single fixed smaller neighborhood. Both sequences converge there locally uniformly.*

***Proof*** Let $C$ be the sheaf of functions whose germs have the two indicated descriptions. It contains the smooth functions with positive semidefinite Hessian, by restriction to line segments. On a convex open neighborhood, a section of $C$ is either finite everywhere or identically $-\infty$: the two sets of points are disjoint and open. In the finite case it is convex along each segment, since a locally convex function on an interval is convex. For the latter fact, subtract the affine function joining any two values. A positive interior maximum of the difference cannot occur unless its maximum set is open on that interval. Indeed a convex function with an interior maximum is constant nearby, by the two secant slope inequalities. The maximum set is also closed, by continuity. The zero endpoint values exclude a positive maximum.

Take a decreasing net of sections on a common neighborhood, and restrict to a convex ball $B$ about the base point. If one section there is $-\infty$, the limit is $-\infty$ on $B$. Otherwise all sections are finite convex on $B$. If their limit is $-\infty$ at $x \in B$, take any other point $y \in B$. There exists $z \in B$ just past $y$ on the line from $x$ to $y$. Write $y = (1-t)x + tz$ for $0 < t < 1$. The convexity inequality bounds the values of the net at $y$ by the same combination at $x, z$. On a tail the values at $z$ stay bounded above by a fixed finite term. Hence the limit at $y$ is $-\infty$. If the limit is finite everywhere, pass the convexity inequality to the limit. A finite convex function is continuous in the interior of its domain. For example, on a small box its values are bounded above by the maximum at the vertices. Reflecting points through the center of the box and using convexity gives a lower bound on a smaller box. The bounds on the secant slopes of lines through a still smaller box then give continuity at the center. This proves that $C$ is closed under the required limits. The generated sheaf is contained in $C$.

For the reverse inclusion, let $f$ be finite convex on a neighborhood of the base point. Take concentric balls $B' \Subset B$ on which it is convex. Convolution with a smooth nonnegative kernel of integral one and small support gives a smooth convex function on $B'$. Uniform errors on $\overline{B'}$ tend to zero by uniform continuity on a compact intermediate ball. Choose the $j$-th error at most $2^{-j-2}$ and add $2^{-j}$ to the approximant. The resulting sequence decreases to $f$ there, since the difference of two successive added constants exceeds the sum of their error bounds. The identically $-\infty$ germ is the limit of the constants $-j$.

For the piecewise affine approximation, first restrict to a closed intermediate ball in $B$ and extend $f$ by $+\infty$ outside that closed ball. The extension is lower semicontinuous and convex. By Theorem 4.5.1, it is the supremum of its affine minorants. On the smaller closed ball, the maximum of finitely many of those minorants approximates $f$ to any desired error, by continuity and a finite subcover. If a lattice basis is fixed, approximate each slope and intercept by rationals. On the smaller ball the resulting change of the finite maximum is bounded by the largest uniform error of the affine terms. The same added constants give a decreasing sequence in this case.□

Since the reverse inclusion uses only decreasing sequences, the same sheaf is generated under local decreasing sequences. Its sections are upper semicontinuous, so on a flat chart it is the sheaf $\mathcal{P}$ of Section 11.4.

On a union of faces, an approximation near a common face must use smooth representatives compatible on all the incident faces.

## 11.4 Products of bounded potentials defined by decreasing limits

We work on an open region $U$ of a balanced polyhedral chart under the standing hypotheses of this chapter. Smooth psh functions are the smooth functions with positive Hessian on every top face. Let $\mathcal{P}$ be the smallest sheaf of upper semicontinuous functions with values in $[-\infty, \infty)$ containing the smooth psh functions and closed under limits of decreasing sequences on a common neighborhood. An infimum of upper semicontinuous functions is upper semicontinuous, so this closure operation stays in the stated range. We construct products for locally bounded members of $\mathcal{P}$ using ordinary smooth tests.

For a $(p, p)$-test form, use the paired convention in which the matrix $A = (A_{IJ})$, indexed by strictly increasing multi-indices $I, J$ of length $p$, represents

$$(-1)^{p(p-1)/2} \sum_{I,J} A_{IJ} \mathrm{d}' x_I \wedge \mathrm{d}'' x_J .$$

The form is symmetric if this matrix is symmetric on the tangent spaces of the top faces, and positive if these matrices are positive semidefinite. A positive current of bidimension $(p, p)$ acts on forms of bidegree $(p, p)$, vanishes on antisymmetric forms, and is nonnegative on positive tests. Its closedness means that both derivatives vanish. These definitions are invariant under changes of tangent coordinates, since the matrix transforms by congruence in the corresponding exterior power.

Fix a smooth cutoff $\chi \geq 0$ of compact support that equals one near a given compact test support $K$. For the constant Euclidean form $\beta$ used in Section 11.1, put $\Theta_p = \chi \beta^p$. On the tangent exterior powers the matrix of $\beta^p$ in orthonormal coordinates is $p!$ times the identity, by expanding the wedge. Thus a symmetric test $\omega$ supported in $K$ satisfies $\pm\omega \leq c\Theta_p$, where $c$ is bounded by a fixed multiple of its ambient coefficient supremum. Restriction to the tangent spaces preserves the matrix inequalities, also when the ambient representative is first symmetrized. Hence for any positive current $T$, its value on $\omega$ is bounded in absolute value by $cT(\Theta_p)$. In particular positivity in this sense gives order zero. A smooth positive $(1, 1)$-form can be wedged with a positive current and the result is positive. Pointwise, decompose its matrix into rank-one terms. For a term $\alpha \wedge J\alpha$, wedging a positive test applies the exterior multiplication by $\alpha$ to both matrix factors. The sign in the paired convention makes this operation a congruence, so the new matrix is positive.

**Lemma 11.4.1** *An ordinary positive current $T$ on $U$ can be multiplied by a locally bounded Borel function. For a sequence $u_j \to u$ pointwise with a common bound on each compact subset, one has $u_j T \to uT$ weakly. If $T_i \to T$ weakly through positive currents and continuous functions $v_i$ converge locally uniformly to $v$, then $v_i T_i \to vT$ weakly.*

***Proof*** For each symmetric test $\omega$, the scalar functional $a \mapsto T(a\omega)$ is bounded by $cT(\Theta_p)\|a\|_\infty$, and the scalar functionals given by $c\Theta_p + \omega$ and $c\Theta_p - \omega$ are positive. Smooth scalar tests are uniformly dense on the union of their compact supports. Indeed, cover that union by small ambient balls on each of which a given continuous function oscillates by less than the desired error. Multiply a choice of its value on each ball by a smooth nonnegative partition of unity near the compact set and sum. Apply the compact-space representation theorem to the two positive functionals. Half the difference of their measures gives the unique signed Radon measure $\lambda^T_\omega$. It is supported on $\operatorname{supp}\omega$, by testing away from that support. For antisymmetric tests use the zero measure. Define

$$(uT)(\omega) = \int u \, \mathrm{d}\lambda^T_\omega .$$

The construction respects the chart test equivalence, because any test restricting to zero has zero scalar functional. The total variation bound implied by $\pm\omega \le c\Theta_p$ is $|\lambda^T_\omega| \le c\lambda^T_{\Theta_p}$. Indeed testing the order inequality against nonnegative scalar functions gives measure domination of both signs. This gives order zero for $uT$ and proves the pointwise convergence assertion by dominated convergence.

For positive currents converging weakly, the quantities $T_i(\Theta_p)$ are uniformly bounded on a tail. Thus the same estimate makes $(v_i - v)T_i$ tend to zero on $\omega$. For $vT_i$, approximate $v$ uniformly on the support of $\omega$ by a smooth scalar test. Weak convergence gives the limit for the smooth multiplier, and the common bound makes the error tend to zero as the uniform approximation improves. This proves convergence also for the product with $v_i$. □

**Theorem 11.4.1** *Let $T$ be an ordinary positive closed current of bidimension $(p, p)$ on $U$, and let $u$ be locally bounded and in $\mathcal{P}$. For $p \ge 1$ the current $D_u T = d'd''(uT)$ is positive and closed. It follows that ordered products with at most $p$ locally bounded members of $\mathcal{P}$ are defined.*

*For $p \ge 2$ and continuous $v \in \mathcal{P}$ one has*

$$D_u D_v T = D_v D_u T .$$

*The products of continuous members are symmetric and continuous under locally uniform convergence of their tuples. With the integration current as base, they agree on directly locally smooth-approximable inputs with the products of Theorem 11.2.1.*

***Proof*** For smooth $u$, closedness of $T$ and the product rules give $D_u T = (d'd''u) \wedge T$, which is positive. The Leibniz rule with a smooth form follows on tests by the

exterior sign and the product rule for smooth forms. In general $D_u T$ is closed by the square-zero and anticommutation identities. To prove positivity, consider all upper semicontinuous functions $u$ for which the assertion holds on every open where $u$ is locally bounded and for every such base current. This condition is local. It is closed under decreasing sequences: if $u_j \downarrow u$ and $u$ is locally bounded, then $u \le u_j \le u_1$. The upper semicontinuous $u_1$ is bounded above on compact subsets, so all $u_j$ are locally bounded with a common compact bound. The multiplier lemma yields $D_{u_j} T \to D_u T$, and positivity passes to the limit on fixed positive tests. The defining minimality of $\mathcal{P}$ proves the assertion. Iteration decreases the bidimension by one at each step.

For symmetry fix continuous $v \in \mathcal{P}$ and work first with smooth $u$. The form $\alpha = d'd''u$ is closed and has even total degree. Hence

$$D_u D_v T = \alpha \wedge d'd''(vT) = d'd''\big(v(\alpha \wedge T)\big) = D_v D_u T.$$

The equality $\alpha \wedge (vT) = v(\alpha \wedge T)$ follows from the measure construction on each smooth test, since scalar multiplication there commutes with wedging by $\alpha$. The condition that the symmetry identity holds locally for every continuous $v$ and positive closed base is again closed under decreasing sequences of $u_j$ on the open where the limit is locally bounded. In fact $D_{u_j} D_v T \to D_u D_v T$ by dominated convergence with the fixed positive base $D_v T$. On the other side $D_{u_j} T \to D_u T$ weakly through positive currents, and continuity of $v$ allows multiplication before differentiating by Lemma 11.4.1. Minimality of $\mathcal{P}$ proves the identity.

For continuous inputs, exchange adjacent operations using this identity and positivity of the remaining inner product. For convergence of products, induct on the number of inputs and multiply the converging positive inner currents by the converging continuous outer functions using Lemma 11.4.1. Differentiation is weakly continuous, which completes the induction. Finally directly locally smooth-approximable inputs belong to $\mathcal{P}$: pass to a subsequence with uniform errors at most $2^{-j-2}$ on a fixed compact neighborhood, and add $2^{-j}$ to make it decreasing there. For the integration current, balanced Stokes gives closedness, and the top-degree matrix criterion gives positivity. Apply the just proved continuity to the smooth approximants. Their smooth products are the wedge products by the smooth Leibniz identity, which proves the comparison. □

## 11.5 Smooth approximation along an affine spine

Let $S, E_1, \dots, E_m$ be finite-dimensional real vector spaces, with $E_i \neq 0$, and put $E = \bigoplus_i E_i$. For an open ball $B \subset S$ and a radius $r > 0$ in fixed Euclidean norms, the local model is

$$X = \bigcup_{i=1}^{m} \big(B \times B_{E_i}(0, r)\big) \subset S \times E.$$

The sets in the union meet along $B\times\{0\}$, the *spine*. Use the induced smooth structure, in which smooth functions have local ambient smooth representatives. A function is smooth psh if on each $B\times B_{E_i}(0,r)$ its Hessian is positive semidefinite in the tangent variables. On this space, take the sheaf generated by smooth psh functions under local decreasing sequences, as in Section 11.3.

**Proposition 11.5.1** *A germ belongs to this sheaf precisely when it is locally identically $-\infty$ or is finite and convex on each member of the union in a neighborhood of the point. Every finite germ of this kind has a decreasing approximation by smooth psh functions on a single fixed smaller neighborhood, with uniform convergence there.*

***Proof*** Off the spine, use the flat-chart result. At a spine point consider functions which locally, on each member of the union, are finite convex or identically $-\infty$. This is a sheaf under restrictions and gluing. On a common product neighborhood of a spine point, the restriction of any section to each connected member of the union is finite everywhere or identically $-\infty$. The two possibilities define disjoint open subsets on that member. In the finite case local convexity gives convexity on its convex domain by the segment argument of Proposition 11.3.1. The types agree across all members since their restrictions agree at the spine point.

Pass to the limit in the convex inequalities along a decreasing sequence on this neighborhood. On each member of the union the result is either finite convex or identically $-\infty$, by the propagation of a $-\infty$ value along segments in the flat-chart proof. Agreement on the spine again forces the same type. Thus this sheaf contains the smooth psh functions and is closed under the required limits. It contains the generated sheaf.

For the converse, consider a finite section near a spine point $(x_0,0)$, with jointly convex restrictions $f_i(x,t)$ for $t\in E_i$ and common trace $f_0(x)$. Use a smaller ambient product $B'\times\prod_i B_{E_i}(0,r')$ whose intersection with each member has closure inside the domain of the section there. All the $f_i$ are uniformly continuous on a slightly larger compact product in their respective variables. Choose a nonnegative smooth kernel $\rho_\delta$ of integral one and radius $\delta$ in the spine variables. Let $b_\delta = f_0 * \rho_\delta$, and let $g_i$ be the product convolution of $f_i$ with $\rho_\delta$ and a nonnegative smooth kernel of integral one and small radius $\varepsilon$ in $E_i$. The spine convolution is the identity when $S=0$. The $g_i$ are smooth and jointly convex in their variables, by averaging translates. Given a tolerance, choose $\delta$ and then $\varepsilon$ small enough that the uniform errors are below that tolerance.

The trace difference $g_i(x,0)-b_\delta(x)$ is the convolution in $x$ of $\rho_\delta$ with the difference between the normal convolution at zero and $f_i(x,0)$. For fixed $\delta$, the latter difference tends uniformly to zero on the required compact spine neighborhood as $\varepsilon\downarrow 0$. Differentiating the spine kernel bounds the Hessian norm of the trace difference by $\|D^2\rho_\delta\|_{L^1}$ times that uniform error. Hence for any $\tau>0$ the Hessians of all trace differences have operator norm at most $\tau$ on $B'$, after shrinking $\varepsilon$. For $S=0$ the trace correction is constant.

On the smaller ambient product define

$$G(x,t_1,\dots,t_m)=b_\delta(x)+\frac{\tau}{2}\|x-x_0\|^2+\sum_i\bigl(g_i(x,t_i)-g_i(x,0)\bigr).$$

On the $i$-th member of the union this is $g_i$ plus a function of $x$ with positive semidefinite Hessian, by the trace bound. Thus it is smooth psh on the union. The trace differences tend to zero in supremum norm, as do the convolution errors and the small quadratic term. This gives uniform smooth psh approximation on the fixed smaller neighborhood. As in the flat case, require errors at most $2^{-j-2}$ and add the constants $2^{-j}$ to make the approximation decreasing. The identically $-\infty$ section is the decreasing limit of constants. Locality of sheaf membership now gives the assertion. □

## 11.6 A common support at an isolated intersection

Let $E_1, \dots, E_m$ be nonzero linear subspaces of a Euclidean space $W$ with $E_i \cap E_j = \{0\}$ for $i \neq j$. We work near 0 in their union, using ambient smooth representatives and testing the Hessian on each subspace as in Section 11.5. The subspaces may be linearly dependent, and then the restrictions of an ambient differential at 0 satisfy relations.

**Theorem 11.6.1** *At 0, a germ belongs to the sheaf generated by smooth psh functions and local decreasing sequences if and only if it is identically $-\infty$ near 0, or it is finite and convex on each subspace near 0 and admits a common supporting affine function there. In the finite case there are smooth psh approximants decreasing uniformly to the germ on a fixed smaller neighborhood.*

***Proof*** Write the support condition as

$$f(v) \geq f(0) + p(v), \qquad v \in E_i \text{ near } 0, \tag{11.2}$$

for some $p \in W^*$ and all $i$. A smooth psh function satisfies the condition with its ambient differential at 0, by the supporting tangent inequality for a differentiable convex function.

We show that the prescribed germs form a sheaf closed under the decreasing limits in question. Away from 0 the condition is the flat one. A section on a small common ball about 0 has either the finite convex type on every subspace or is everywhere $-\infty$. Indeed the type is locally constant on each subspace and agrees at 0. Convexity on the whole ball in each subspace follows from the segment argument of Proposition 11.3.1. For the finite type, any common support for the germ extends to that ball. Apply the convex inequality to 0 and $v$, using the lower bound (11.2) at a small positive multiple of $v$.

For a decreasing sequence $f_j$ on this ball, the flat-chart limit argument on each subspace gives the same dichotomy for the limit, and the types again agree at 0. In the finite case choose common supports $p_j$ of $f_j$, extended by zero on the orthogonal complement of the span of the $E_i$. Choose a basis of that span from vectors in the subspaces. Scale the basis vectors so that they and their negatives lie in the ball. Applying (11.2) at those fixed points bounds all basis evaluations of $p_j$, because $f_j$

has a finite limit at each point used. A subsequence of $p_j$ converges to $p$. Pass to the limit in the supporting inequalities on the ball to obtain a common support for the limit. Locality and closure under the limits prove that every generated psh germ has the stated property.

Conversely let $f$ be finite and let $\ell(v) = f(0) + p(v)$ be a common support. On the $i$-th subspace put $h_i = f|_{E_i} - \ell|_{E_i}$. These functions are convex, nonnegative and zero at 0. The functions $\max(h_i - \varepsilon, 0)$ are convex and identically zero near 0. On a fixed smaller ball in each $E_i$, convolve them with smooth nonnegative kernels of integral one whose radii are smaller than half the radii of the zero neighborhoods. The results $k_i$ are smooth convex functions, still zero near 0. Shrink the kernel radii so that the convolution errors are at most $\varepsilon$ there, by uniform continuity on intermediate compact balls. Thus $\|k_i - h_i\|_\infty \leq 2\varepsilon$ on the fixed smaller balls.

The unit spheres in the $E_i$ are pairwise disjoint compact subsets of the unit sphere of $W$. Choose smooth cutoffs $\chi_i$ on that sphere equal to one on the sphere of $E_i$ and zero near the spheres of the other subspaces. Such cutoffs follow by finite unions of coordinate bumps in disjoint neighborhoods of these compact sets. If $\pi_i$ is orthogonal projection to $E_i$, the formula

$$G(v) = \ell(v) + \sum_i \chi_i(v/\|v\|) k_i(\pi_i v)$$

defines an ambient smooth function on the smaller ball off 0. It is $\ell$ on a neighborhood of 0 since all $k_i$ vanish there. On each subspace it restricts to $\ell + k_i$, which is convex. It is therefore smooth psh on the union and approximates $f$ uniformly there. Take a sequence with errors at most $2^{-j-2}$ and add $2^{-j}$ as in Proposition 11.3.1 to make it decreasing. The identically $-\infty$ germ is obtained by decreasing constants. This proves the equivalence. □

*Example 11.6.1* In $\mathbb{R}^2$ take the full lines of directions $e_1, e_2, e_1 + e_2$. The function zero on the first two and equal to $t$ at $t(e_1 + e_2)$ on the third is continuous and linear on each line. A common support would have $p(e_1) = p(e_2) = 0$ and $p(e_1 + e_2) = 1$: apply (11.2) with both signs of $t$ on every line. Linearity of $p$ excludes these values. By Theorem 11.6.1, the function does not belong to the generated psh sheaf.

**Corollary 11.6.1** *Let $X$ be either the local model of Section 11.5 or a neighborhood of $0$ in the union of the subspaces $E_i$ of this section. Suppose the ambient vector space carries a lattice for which each linear subspace $S \times E_i$, respectively $E_i$, is rational of dimension $n$; call these subspaces the* pages *of $X$. Give each page a positive constant weight. With the integration current of the resulting weighted chart as base, mixed products of locally bounded members of the generated psh sheaf on $X$ agree with those obtained by direct smooth approximation.*

***Proof*** Subdivide by rational hyperplanes containing the page intersections and take the induced weights. Integration is the sum of the weighted integrations on the pages, since their intersections have measure zero in dimension $n$. On each page the two Stokes identities for compactly supported tests follow by subdivision and cancellation of oppositely oriented facets. Hence the integration current is positive and closed. Locally bounded members of the sheaf are finite. By Proposition 11.5.1 and Theorem 11.6.1 they have smooth psh approximants converging uniformly on a common smaller neighborhood. The comparison follows from Theorem 11.4.1. □

## 11.7 Transverse corner functions

On a lattice affine space of dimension $n$, with weight one, let $f_i = \max(0, \ell_i)$ for $1 \le i \le n$, where the $\ell_i$ are affine functions. For a positive $\delta$ use the smooth convex approximation

$$h_\delta(s) = \delta \log(1 + \exp(s/\delta)), \qquad f_{i,\delta} = h_\delta(\ell_i).$$

The approximation is uniform with error at most $\delta \log 2$. The density $h''_\delta$ is nonnegative on the line, has integral one, and concentrates at zero as $\delta$ tends to zero. These assertions follow by differentiating and noting that $h'_\delta$ increases from zero to one.

**Proposition 11.7.1** *Suppose the linear parts of $\ell_i$ are rows of a matrix $L$ in lattice coordinates. If they are independent, their mixed product is a point mass at their common zero, of mass $|\det L|$. If they are dependent, the product is zero.*

***Proof*** Each factor is $h''_\delta(\ell_i) d'\ell_i \wedge d''\ell_i$. Their wedge in paired lattice volume has coefficient

$$(\det L)^2 \prod_i h''_\delta(\ell_i).$$

If $L$ is singular this vanishes identically. Otherwise change variables to the affine coordinates $\ell_1, \dots, \ell_n$. The Lebesgue measure gains the inverse absolute determinant, leaving $|\det L|$ times the product of the probability densities. These probability measures converge to the point mass at the common zero. Theorem 11.2.1 identifies this limit with the mixed product. □

In particular, when the linear parts are integral, the mass in the independent case is the index of the sublattice they generate in the dual lattice.

## 11.8 Subgradients of polyhedral weights

The local calculation for a toric metric takes place in the real vector space of its torus, with no quotient by the lattice. A change of the maximal affine term of a convex function creates a first curvature current. Its top power can have mass only at the points where the active affine terms span all gradient directions.

Let $N$ be a lattice of rank $n > 0$ and let $M$ be its dual. On the single chart $N_{\mathbb{R}}$ of weight one, use the derivatives and lattice integration of Chapter 10. Suppose

$$H(x) = \max_i \big(\langle m_i, x\rangle + c_i\big), \qquad m_i \in M_{\mathbb{R}}, \quad c_i \in \mathbb{R},$$

with finitely many terms. The subgradient polytope $Q_x \subset M_{\mathbb{R}}$ is the convex hull of the $m_i$ whose terms attain the maximum at $x$. This is the subdifferential of $H$ at $x$, by Section 4.5.

**Theorem 11.8.1** *The local product of $n$ copies of $d'd''H$ is*

$$(d'd''H)^n = \sum_{x:\,\dim Q_x = n} n!\,\mathrm{vol}_M(Q_x)\delta_x, \tag{11.3}$$

*where $\mathrm{vol}_M$ gives a fundamental parallelepiped of $M$ volume one.*

***Proof*** At any $x$ there is a neighborhood on which the difference $H(x+y) - H(x)$ equals the support function $\max_{i\in I_x}\langle m_i, y\rangle$ of the active gradients, indexed by $I_x$. If $Q_x$ has dimension $r < n$, subtract one of the active linear forms. The remaining maximum then depends on at most $r$ linear coordinates. Its smooth convex approximations by logarithms of sums of exponentials depend on the same coordinates. Their Hessians thus have rank at most $r$, and the top products vanish near $x$ by Theorem 11.2.1.

There are finitely many $x$ for which $\dim Q_x = n$. Indeed equality of $n+1$ affinely independent terms determines $x$ uniquely. To compute the mass at one of them, translate it to zero and use its active set $I$ to define

$$q(y) = \log \sum_{i\in I} \exp\langle m_i, y\rangle, \qquad q_\varepsilon(y) = \varepsilon q(y/\varepsilon).$$

The functions $q_\varepsilon$ converge uniformly to the support function of $Q_x$. The Hessian of $q$ is the covariance matrix of the gradients $m_i$ with strictly positive weights proportional to $\exp\langle m_i, y\rangle$. The gradients affinely span the dual space, so this Hessian is positive definite.

The gradient of $q$ maps $N_{\mathbb{R}}$ diffeomorphically onto the interior of $Q_x$. It takes values there by the strict positivity of all the weights, and is injective by strict convexity. For every interior $m$, the support function of $Q_x - m$ is bounded below by a positive constant times the norm. Since $q$ is at least the support function of $Q_x$, $q(y) - \langle m, y\rangle$ tends to infinity with the norm of $y$. At its minimum $\nabla q(y) = m$, which proves surjectivity. The positive Hessian gives a smooth inverse.

In lattice coordinates the smooth top form of $q_\varepsilon$ has density $n!\det D^2 q_\varepsilon$, by expanding the paired exterior product. For a smooth test $\chi$ supported in the neighborhood where the active formula holds, change variables using $m = \nabla q(y/\varepsilon)$ to get

$$\int \chi(y)(d'd''q_\varepsilon)^n = n!\int_{\operatorname{int} Q_x} \chi\big(\varepsilon(\nabla q)^{-1}(m)\big)\,\mathrm{d}m.$$

The integral tends to $n!\operatorname{vol}_M(Q_x)\chi(0)$ by dominated convergence. Local uniform convergence and Theorem 11.2.1 identify the limit with the measure of $H$ near $x$. This gives each of the masses in (11.3). □

**Corollary 11.8.1** *Let $H_1,\dots,H_n$ be finite convex polyhedral functions as above, with subgradient polytopes $Q_{j,x}$. The mixed product is supported at the finitely many points $x$ where the sum $Q_{1,x}+\cdots+Q_{n,x}$ has dimension $n$. Its mass there is*

$$n!\,\mathrm{MV}_M(Q_{1,x},\dots,Q_{n,x}),$$

*where the mixed volume is normalized by* $\mathrm{MV}_M(Q,\dots,Q)=\operatorname{vol}_M(Q)$.

***Proof*** For $\lambda_j>0$, the affine terms of $\sum_j \lambda_j H_j$ are obtained by distributing the sums across the maxima. A term attains the maximum at $x$ precisely when each summand is active there. Thus the subgradient there is $\sum_j \lambda_j Q_{j,x}$. The top measure from (11.3) is supported at a fixed finite set for all positive $\lambda_j$. By multilinearity of the local products, which passes from smooth sums to the local uniform limits, the top product is a measure-valued polynomial in the $\lambda_j$. Testing off the fixed finite set shows that all its coefficient measures are supported on that set. In the mass polynomial at $x$, the coefficient of $\lambda_1\cdots\lambda_n$ is $n!$ times the mixed mass. On the other hand the mass polynomial is $n!\operatorname{vol}_M(\sum_j \lambda_j Q_{j,x})$. This volume is a polynomial by Theorem 4.6.1. Its coefficient of $\lambda_1\cdots\lambda_n$ is $n!\,\mathrm{MV}_M(Q_{1,x},\dots,Q_{n,x})$, by the definition of the mixed volume after that theorem, computed with $\operatorname{vol}_M$. This gives the formula. □

For instance, $\max(0,x,y,x+y)$ on $\mathbb{R}^2$ has four active gradients at the origin. Their hull is the unit lattice square and its top measure has mass 2 at the origin. In the decomposition $\max(0,x)+\max(0,y)$, the two pure powers vanish and the two equal mixed terms each have mass one, as in Proposition 11.7.1.

## 11.9 Products in finite torus coordinates

For a finite locally free projection $p : Y \to T_Q$ of Section 10.5, let $H_1,\dots,H_n$ be continuous convex functions on $Q^\circ$. Use $p_{\mathrm{trop}}$ for the composition of $p$ with the base log norm map. We compute the product of the weights $H_i \circ p_{\mathrm{trop}}$ through smooth convex approximation in these same coordinates.

**Theorem 11.9.1** *The calibrated products for smooth convex approximants converging locally uniformly to $H_i$ have a common limit against continuous tests of compact support in $p_{\mathrm{trop}}^{-1}(Q^\circ)$. It is a positive Radon measure determined by*

$$\int \chi \, \mathrm{d}\nu_p(H_1, \ldots, H_n) = \int_{Q^\circ} K_w(\chi)\, d'd''H_1 \wedge \cdots \wedge d'd''H_n. \tag{11.4}$$

*It gives full mass to the Gauss fibres and its pushforward by $p_{\mathrm{trop}}$ is the real mixed product multiplied by the module rank $d$. For polyhedral weights, a real atom of mass $a$ at $w$ has mass $ad_p(y)$ at each $y \in p^{-1}(\eta_w)$.*

***Proof*** For smooth weights use (10.1) with $\Omega = d'd''H_1 \wedge \cdots \wedge d'd''H_n$. For continuous weights, Theorem 11.2.1 gives the real product as the locally weak limit of the smooth ones. The trace $K_w(\chi)$ of a continuous compactly supported test is continuous and has support inside the projected compact support of $\chi$. Thus testing on $Y$ is legitimate in this locally weak limit. Convolution on relatively compact convex subdomains of $Q^\circ$ supplies the approximants, also when the weights are given only on $Q^\circ$.

The positive functional is bounded on each compact test support by $d\|\chi\|_\infty$ times the real product mass on the projection of that support. This gives the Radon measure. Since the traces vanish on tests supported off the relatively closed Gauss fibres, the measure gives zero mass to their complement. For a test pulled back from the base log norms, its trace multiplies it by $d$. This proves the pushforward statement. Formula (11.3) and its mixed version Corollary 11.8.1 compute the real atoms, and (9.2) then gives the individual masses above them. □

The constructions on two nested boxes agree over the interior of the smaller box, by the restriction of the smooth calibration and the locally weak limit. For an isogeny as in Proposition 10.5.2, let the weights in the original base coordinates be $H_i' \circ F$ for continuous convex $H_i'$ in the target coordinates. The two measures agree over the interior of the smaller domain: approximate the $H_i'$ smoothly, pull back by the linear map, and apply that proposition before passing to the limit in (11.4).

## 11.10 Analytic changes of coordinates on the base torus

The isogeny calculation also applies when the new coordinates are analytic units on the original base box. Their leading monomials describe the map on log norms. A uniform estimate on the other terms gives the degree of the analytic map itself.

**Proposition 11.10.1** *Let $Q$ be a full-dimensional closed box on the split $n$-torus as in Section 9.1. Let $g_1, \dots, g_n$ be analytic units on $T_Q$. The functions $-\log|g_i(\eta_w)|$ are affine on $Q$. Assume their linear parts give an invertible map $F : N_{\mathbb{R}} \to \mathbb{R}^n$. For any closed box $R \subset F(Q^\circ)$, the map $g$ from the torus domain over $F^{-1}(R)$ to the target torus over $R$ is finite locally free of rank $h = |\det F|$ in the lattice bases. Above $\eta'_{Fw}$ it has the single point $\eta_w$, with local degree $h$.*

***Proof*** The functions $\log|g_i(\eta_w)|$ and $\log|g_i^{-1}(\eta_w)|$ are convex by the Laurent formula in Theorem 9.1.1. They are negatives of one another, so both are affine. The log norm of any monomial term of $g_i$ is an affine function of $w$ lying below $\log|g_i(\eta_w)|$ throughout the box. If equality holds at an interior point, their slopes coincide. Hence there is a unique such term $c_i T^{m_i}$, and it dominates strictly throughout the interior. Since $k$ is trivially valued, $|c_i| = 1$ and the affine function $-\log|g_i(\eta_w)|$ is $\langle m_i, w\rangle$. Thus the tropical coordinate change is $F$ as stated, with no constant term. The exponents $m_i$ generate a sublattice $M' \subset M$ of index $h$.

Put $P = F^{-1}(R)$. The proof of Proposition 10.5.2 constructs the annular algebra $A$ over $P$ as a finite free module over the target box algebra $A'$ by the monomial map $\phi_i = c_i T^{m_i}$. Including $c_i$ amounts to translating the torus by constants of norm one, which extends to an isometry of the box algebra by substitution. Choose monomials $T^{r_\alpha}$ representing the cosets of $M'$. If $b = \sum_\alpha T^{r_\alpha}\phi^* b_\alpha$, then

$$|b(\eta_w)| = \max_\alpha |T^{r_\alpha}(\eta_w)||b_\alpha(\eta'_{Fw})|. \tag{11.5}$$

This follows from disjoint monomial supports for Laurent polynomials and then by completion. The supremum of this norm over $P$ is equivalent to the finite module norm, since the weights of the finitely many coset representatives have positive lower and finite upper bounds on $P$. In particular it is a complete algebra norm.

Write $g_i = \phi_i(1+u_i)$. On $P$, the Laurent norms of the $u_i$ have a common bound $\delta < 1$. Indeed, every remaining term is strictly smaller than the leading monomial there, and only a finite set of terms can come close to it, by uniform convergence. To verify the same bound on the whole torus domain, use the norm $\|b\| = \sup_{w\in P}|b(\eta_w)|$ just constructed. Each bounded multiplicative seminorm for this norm is at most the norm itself, by applying boundedness to all powers and taking roots. Thus $g$ has the same log norms as $\phi$ at every point of the domain and defines the indicated map to the target box. In particular the induced substitution from $A'$ is bounded, by the annular presentation.

On each Laurent monomial in the target, the ratio between substitution by $g$ and substitution by $\phi$ is a product of integral powers of $1 + u_i$. Its difference from one has norm at most $\delta$, also in each individual Gauss norm over $P$. To see this, note that the elements of the form $1 + u$ with $\|u\| \le \delta$ are closed under multiplication by expansion and under inversion by the geometric series. For a Laurent polynomial $a$ it follows that

$$\left|(g^*a - \phi^*a)(\eta_w)\right| \le \delta\,|a(\eta'_{Fw})|.$$

The bound uses the Gauss maximum of the terms of $a$, so it passes to $A'$ by completion.

On the finite free decomposition for $\phi^*$, define the linear operator

$$S\Big(\sum_{\alpha} T^{r_\alpha}\phi^* b_\alpha\Big) = \sum_{\alpha} T^{r_\alpha} g^* b_\alpha.$$

By the preceding estimate and (11.5), $\|S(b) - b\| \le \delta\|b\|$. The completed algebra norm makes the operator series $\sum_{j\ge 0}(1-S)^j$ converge to the inverse of $S$, as is seen by multiplying its finite partial sums by $S$. Hence the same monomials are a free $A'$-basis for substitution by $g$. The inverse is bounded, so this basis gives the finite free module topology on the source.

The strict inequality at each $\eta_w$ also gives $g(\eta_w) = \eta'_{Fw}$, by taking just the single coefficient $b_\alpha = a$ in the estimate for substitution. More generally $S$ preserves the norm at $\eta_w$ of each nonzero element in the monomial basis. This identity for Laurent polynomial coefficients extends to rational coefficients by clearing a common denominator and dividing by its equal Gauss norms under the two maps. The $h$-dimensional fibre algebra over $\mathcal{H}(\eta'_{Fw})$ therefore embeds in $\mathcal{H}(\eta_w)$. Indeed, the coset norm identity extends isometrically on the completed vector space with that basis. The fibre algebra is a domain and hence a field. By Theorem 2.3.1, it is the completed residue field of the unique lift. □

**Corollary 11.10.1** *Let $p : Y \to T_Q$ be finite locally free and let $g, F, R$ be as in Proposition 11.10.1. On the inverse image of $R^\circ$ in $Y$, the calibration for $g \circ p$ agrees with the calibration for $p$ after pulling back the paired top form by $F$. For continuous convex weights $H'_i$ on $R^\circ$, one has*

$$\nu_{g\circ p}(H'_1, \dots, H'_n) = \nu_p(H'_1 \circ F, \dots, H'_n \circ F).$$

*The equality can be restricted to any open subset of the indicated inverse image. On the original base, the product on $F^{-1}(R^\circ)$ is defined by restriction to smaller boxes.*

***Proof*** Tensor the finite algebra of $Y$ over the source of $g$ with the target Gauss field at $Fw$. The intermediate algebra is the single field $\mathcal{H}(\eta_w)$ of degree $h$, by Proposition 11.10.1. The same local fibre factors are therefore counted with dimensions multiplied by $h$, so the trace is $hK_w$. Changing variables from $Fw$ to $w$ contributes a second factor of $h$. On the original base pulling back a paired top form by $F$ gives the factor $h^2$ from the two differential copies. The integrals agree by (10.1).

Approximate the weights on $R^\circ$ locally uniformly by smooth convex functions. Their pullbacks by $F$ are such approximations in the original coordinates, and the smooth Hessians pull back by the chain rule. Pass to the limit in (11.4). For a test with compact support in an open subset of the inverse image, extension by zero gives a continuous test on the full domain. This gives the restricted equality. □

**Corollary 11.10.2** *Use the two projections of Section 10.6. On an ordinary open subset $U$ of $X$ contained in $Y_p \cap Y_q$ whose moment weights lie in the cell $\Delta$ of Corollary 10.6.1, one has*

$$\nu_q(H_1, \dots, H_n)|_U = \nu_p(H_1 \circ F, \dots, H_n \circ F)|_U$$

*for continuous convex functions $H_i$ on the interior of the $q$-box. On the right the product is formed on smaller $p$-boxes in the inverse image of that interior.*

***Proof*** For smooth inputs the paired real Hessians pull back by the linear chain rule. The mixed products for tests with compact support in $U$ then agree by Corollary 10.6.1. To pass to continuous convex $H_i$, choose smooth convex approximants converging locally uniformly. On every smaller $p$-box on which the images under $F$ are in the $q$-box interior, their pullbacks are smooth convex approximants to $H_i \circ F$. The weak convergence in Theorem 11.9.1 gives the equality on each such open box inverse image in $U$, first testing with compact support there. These open subsets cover $U$. Equality of the Radon measures on them gives equality on $U$, by a finite partition of unity over the support of each compactly supported test. □

*Example 11.10.1 (Degrees two and three on one annulus)* Let $z$ be the coordinate of a split torus and restrict it to a compact log norm interval $Q$ in $(0, +\infty)$. The moment with coordinates $p = z^2$ and $q = z^3(1 + z)$ has log norm image $\{(2w, 3w) : w \in Q\}$. The two maps to the corresponding annuli are finite locally free of ranks two and three. The first rank follows from the monomial basis $1, z$. For the second, extend first to a slightly larger compact interval in $(0, +\infty)$. There $1+z$ is a unit and the dominant term of $q$ is $z^3$. Apply Proposition 11.10.1 with target the interval $3Q$. Both maps have one point above a base Gauss point, of weight two and three, respectively. The normalized graph weight is one in both computations, since $(2, 3)$ is the primitive generator and its two projection indices are two and three. The function $q = z^3 + z^4$ is outside the image of analytic functions of $p$ on its annulus: the Laurent expansion of each such pullback has only even exponents, and the odd coefficient of $z^3$ is nonzero. The comparison on the graph therefore uses the two source maps themselves.

## 11.11 Products at dependent weights on an overlap

Use the two charts, boxes and open set $W$ of Section 10.8. For continuous convex weights in either projected coordinate system, (11.4) defines the mixed measure on $W$ also for Laurent targets. The proof of Theorem 11.9.1 uses the continuous trace on the box, its finite degree bound, and the locally uniform approximation of convex functions on the open weight box. All three apply to these targets.

**Theorem 11.11.1** *Suppose there is an invertible rational linear map $F$ satisfying*

$$q_{\mathrm{trop}}(y) = F p_{\mathrm{trop}}(y) \qquad (y \in W \cap (S_p \cup S_q)).$$

*Then $W \cap S_p = W \cap S_q$, and at every point of this common set*

$$d_q(y) = |\det F|\, d_p(y).$$

*If $H_i, G_i$ are continuous convex functions in the interiors of $B_p, B_q$, respectively, with $H_i \circ p_{\mathrm{trop}} = G_i \circ q_{\mathrm{trop}}$ on the common set for $1 \le i \le n$, then*

$$\nu_p(H_1, \dots, H_n)|_W = \nu_q(G_1, \dots, G_n)|_W. \tag{11.6}$$

***Proof*** Let $y \in W \cap S_p$, and write $w = p_{\mathrm{trop}}(y)$. In any ordinary neighborhood of $y$ inside $W$, choose a nonnegative continuous test on the compact Hausdorff space $X$, supported inside that neighborhood and positive at $y$. Its $p$-trace is positive at $w$, so it is positive in a neighborhood of $w$ in the open box. That neighborhood contains weights with independent coordinates over $\mathbb{Q}$. A lift at any such weight $u$ with positive test has $q$-weight $Fu$, also independent, so its image under $q$ is the Gauss point at $Fu$. It belongs to $W \cap S_q$. Since that set is closed in $W$, it contains $y$. Interchanging the roles and using $F^{-1}$ gives the reverse inclusion.

Set $D = B_p^\circ \cap F^{-1}(B_q^\circ)$ and $h = |\det F|$. If $w \in D$ has independent coordinates, the points in $W$ over the two Gauss points at $w$ and $Fw$ coincide. Their degrees are related by (10.2), applied to the graph plane of $F$ in the joint coordinates of $p, q$. Thus for every $\chi \in C_c(W)$

$$K_{q,Fw}(\chi) = h K_{p,w}(\chi) \qquad (w \in D). \tag{11.7}$$

Here the extension to every $w \in D$ follows from continuity of both traces and density of the independent weights. A test equal to one at a selected point of the common fibre and zero at the other points extracts its individual degree. Such a test exists with support in $W$ by the finiteness of the fibre and compact Hausdorff separation in $X$. This proves the degree formula at dependent weights as well.

Let $A = p_{\mathrm{trop}}(W \cap S_p) \subset D$. It is open in the base weight space. For any weight in $A$, a test supported in $W$ and positive at a selected lift has positive trace throughout a neighborhood of that weight. On $A$ one has $H_i = G_i \circ F$. By locality of the real mixed product in Theorem 11.2.1, its restriction to $A$ is therefore the real mixed product of the $G_i \circ F$ there.

For the real products a linear change of variables gives

$$F_*\big(d'd''(G_1 \circ F) \wedge \cdots \wedge d'd''(G_n \circ F)\big) = h\, d'd''G_1 \wedge \cdots \wedge d'd''G_n$$

on $F(A)$, with the left hand measure restricted to $A$. For smooth functions the two exterior differential copies give the factor $h^2$, and the base lattice measure gives $h^{-1}$. For continuous convex functions approximate locally uniformly by smooth convex functions and apply Theorem 11.2.1 on both sides. The traces of $\chi \in C_c(W)$ have compact supports in $A$ and $F(A)$, respectively, since their supports are contained in

the images of $\operatorname{supp}(\chi) \cap S_p$ and $\operatorname{supp}(\chi) \cap S_q$. The change of variables and (11.7) in the two instances of (11.4) give (11.6). □

*Example 11.11.1 (An atom at a splitting weight)* Suppose $\operatorname{char}(k) \neq 2$. On a closed two-dimensional annular box with characters $z, s$, take $v(z) > 0$ throughout and let the $s$-weight interval contain zero in its interior. Work on a smaller closed box contained in the interior and still meeting $v(s) = 0$ in its interior. Form the finite free cover $b^2 = 1 + s$. Compare the projections

$$p = (z^2, s), \qquad q = (z^3(1+z), s).$$

The first map from the $(z, s)$-box is a monomial isogeny of degree two. The second is a perturbed monomial isogeny of degree three: $z^3(1+z) = z^3 + z^4$, and $|z| < 1$ on the box. The finite freeness statements and the single Gauss lifts on the $(z, s)$-box follow from Propositions 10.5.2 and 11.10.1 after taking corresponding base boxes with the same inverse weight range. The cover by $b$ is free of rank two, since its equation is monic. Thus the two composite ranks are four and six.

At positive $s$-weight, the equation for $b$ splits over the Gauss field, since $1 + s$ has initial part one. At zero $s$-weight, the initial part is $1 + s$. It is not a square in $k(z, s)$ since it has order one at $s = -1$. The power test of Lemma F.3.1 thus gives one lift at zero $s$-weight, of degrees four and six in the two projections.

Throughout the source, the log norm transition is $F(w_1, w_2) = (3w_1/2, w_2)$. Choose $a$ in the interior of the first $q$-weight interval. Use $G_1 = \max(0, w_1 - a)$ and $G_2 = \max(0, w_2)$ in the $q$-coordinates and $H_i = G_i \circ F$ in the $p$-coordinates. The independent-corner formula of Proposition 11.7.1 gives a real mixed atom of mass one at $(a, 0)$ in the $q$-coordinates, and mass $3/2$ at $(2a/3, 0)$ in the $p$-coordinates. Above these weights the common Gauss point receives mass 6 in either analytic measure, by (11.4).

# Part III
# The envelope theorem

For a real numerical class $\theta$, the envelope of a bounded function on an analytification is the supremum of the $\theta$-psh potentials below it, a candidate for its largest $\theta$-psh minorant. The envelope property asks that the upper semicontinuous regularization of the supremum of a nonempty uniformly bounded-above family of $\theta$-psh potentials be $\theta$-psh. The envelope of an upper semicontinuous function that is bounded above and dominates some $\theta$-psh potential is then $\theta$-psh. The first chapter of this part characterizes the property in ample classes, shows that branching obstructs it, and computes envelopes on smooth projective varieties by base ideals, using Frobenius coefficient ideals in positive characteristic and multiplier ideals in characteristic zero.

The next two chapters regularize psh potentials on affine charts and extend the regularization across an exceptional set. Finite descent and the maximum principle on proper fibres then reduce the general case to the smooth one through a smooth alteration. The last chapter proves the envelope theorem for integral unibranch projective varieties over an algebraically closed field, in every characteristic and in every real numerical class, and deduces that decreasing limits of psh potentials in convergent classes are psh in the limit class.

# Chapter 12
# Projective obstacle problems

A bounded function on an analytification has a largest candidate for a psh minorant. One can compute its values from section spaces even before knowing that the candidate is psh. On the projective line, a truncated ray already exhibits the distinction between the obstacle and its envelope (Example 12.1.1).

Throughout, $k$ is trivially valued and $X$ is a reduced projective scheme over $k$. We characterize the envelope property by continuous obstacles in ample classes (Proposition 12.1.1) and by increasing nets on integral schemes (Corollary 12.2.1), show that, for an integral $X$ over an algebraically closed field, a normalization which is not a homeomorphism obstructs it (Proposition 12.3.1), and derive the base-ideal formula needed to establish it (Proposition 12.4.1). The last two sections use this formula on smooth projective varieties, with Frobenius coefficient ideals in positive characteristic (Theorem 12.5.1) and multiplier ideals in characteristic zero (Theorem 12.6.1). The arguments use the construction of projective potentials from Chapter 6. The descent arguments of the following chapters rely on the increasing-net criterion, on agreement at divisorial points (Theorem 12.2.1) and on the smooth case.

## 12.1 The obstacle problem

For a function $f : X^{\mathrm{an}} \to [-\infty, +\infty]$ and a real numerical class $\theta$ put

$$P_\theta(f) = \sup\{u \in \mathrm{PSH}(X, \theta) : u \le f\},$$

with value $-\infty$ when the set is empty. We call $f$ the *obstacle*. A *competitor* is a function $u \in \mathrm{PSH}(X, \theta)$ with $u \le f$. For a function $F$, use $F^*$ for its upper semicontinuous regularization, given by $F^*(x) = \inf_{U \ni x} \sup_U F$ over open neighborhoods. In particular $P_\theta(f)^* \le f$ if $f$ is upper semicontinuous. A class $\theta$ has the *envelope property* if $(\sup_i u_i)^*$ is $\theta$-psh for every nonempty uniformly bounded-above family of $\theta$-psh potentials. Under this property the envelope of an upper semicontinuous

bounded-above obstacle is psh whenever it has a competitor. Indeed its regularization is another competitor.

The operator $P_\theta$ preserves order and commutes with addition of constants. In a nef class constants are psh, so the envelope of a bounded obstacle is bounded between its infimum and its supremum. It follows that

$$\|P_\theta(f) - P_\theta(g)\|_\infty \le \|f - g\|_\infty$$

for bounded obstacles in this case. The same estimate holds for their regularizations.

**Proposition 12.1.1** *In an ample real class $\omega$, the envelope of a bounded lower semicontinuous obstacle is the supremum of the continuous $\omega$-psh potentials below it. If $\omega = [L]$ for a rational ample line bundle $L$, finite rational Fubini–Study potentials for $L$ suffice. The following conditions are equivalent.*

(1) *The class has the envelope property.*
(2) *The envelope of every continuous obstacle is continuous.*
(3) *The envelope of every rational PL obstacle is $\omega$-psh.*

***Proof*** If $u \le f$ is a psh competitor, let $u_j$ decrease to $u$ through continuous psh potentials in the fixed class by Proposition 6.4.2. For a rational ample line bundle use the approximations of Proposition 6.4.1. The increasing open sets $\{u_j < f + \varepsilon\}$ cover the compact space. For sufficiently large $j$, $u_j - \varepsilon$ is a competitor within $\varepsilon$ of $u$ from below in the sense that $u_j - \varepsilon \ge u - \varepsilon$. Use positive rational $\varepsilon$ for the rational approximation. Taking the supremum proves the first assertion. In particular $P_\omega(f)$ is lower semicontinuous. If the envelope property holds and $f$ is continuous, the envelope is psh and hence upper semicontinuous. This proves the second condition. Conversely, if a bounded lower semicontinuous obstacle has a continuous envelope, take the net of finite maxima of its continuous psh competitors. It increases to the envelope and converges uniformly by compactness. Choose approximants with errors at most $2^{-j-2}$ and add the constants $2^{-j}$. The resulting sequence decreases to the same envelope, which is psh by Theorem 6.2.2. This gives the third condition from the second one.

Assume the third condition and write $F = (\sup_i u_i)^*$ for the regularized supremum of a nonempty bounded-above family. All its rational PL majorants form a downward-directed set under pointwise order, since the minimum of two rational PL functions is rational PL. Their infimum is $F$. In fact $F$ is the infimum of its continuous majorants by Lemma 4.1.1, and each continuous majorant can be approximated uniformly from above by rational PL functions. For each such majorant $h$ one has $u_i \le P_\omega(h)$ for all $i$, and therefore $F \le P_\omega(h) \le h$ by upper semicontinuity. As $h$ decreases through the majorants, the middle functions decrease to $F$. At each component's trivial valuation $F$ is bounded below by the value of any one $u_i$. Thus Theorem 6.2.2 gives $F \in \mathrm{PSH}(X, \omega)$. □

The envelope property also passes from all ample real classes to all nef real classes. For a nef $\theta$, fix an ample $\omega$ and use the envelope of the same family in the classes $\theta + \varepsilon\omega$. This gives the same regularized supremum each time. It is $\theta$-psh

by the decreasing-limit theorem for varying nef classes, Theorem 6.2.2. The transfer from ample to arbitrary classes on a normal integral scheme uses the homogeneous argument in Chapter 7.

*Example 12.1.1 (A truncated ray on the projective line)* Let $X = \mathbb{P}^1_k$, let $p$ be a rational point, and put $v_r = r\,\mathrm{ord}_p$ on the ray from the trivial valuation $v_0$ to the point with support $p$. Use $r = +\infty$ for that endpoint. Take $a > d > 0$ with $d$ integral. The obstacle equals $-a\min(r, 1)$ along the ray and zero off the ray. For $\omega = [\mathcal{O}(d)]$, its envelope is

$$P_\omega(f)(v_r) = d - a - d\min(r, 1), \qquad P_\omega(f) = d - a \quad \text{off the ray.}$$

Choose sections $s, s'$ of $\mathcal{O}(1)$ vanishing at distinct rational points, with $s$ vanishing at $p$. The displayed candidate equals $d - a + d\max\{\log|s|, \log|s'| - 1\}$, hence is psh in $\omega$ and lies below the obstacle.

For the reverse inequality, every $\omega$-psh $u$ satisfies $u(v_0) - u(v_1) \le d$. For a finite rational Fubini–Study approximant, use a section attaining its value at $v_0$ and the bound $md$ on the order at $p$ of a nonzero section of $\mathcal{O}(md)$. Decreasing approximation in $\mathcal{H}_{\mathbb{Q}}(\mathcal{O}(d))$ (Proposition 6.4.1) passes the inequality to $u$. If $u \le f$, then $u(v_1) \le -a$, so $\sup u = u(v_0) \le d - a$. For $0 \le r \le 1$, use convexity of $r \mapsto u(v_r)$. This holds for the approximants because they are maxima of affine functions on the finite ray, and it passes to the limit. Thus $u(v_r) \le (1-r)(d-a) - ra$. For $r \ge 1$ use the obstacle bound directly. The formula follows. The contact set is the part of the ray with $r \ge 1$.

## 12.2 Agreement at divisorial points

Suppose now that $X$ is integral. For $w \in \mathcal{D}(X)$ let $C_w$ be the maximum, over a compact neighbourhood of $\theta$ in $N^1(X)_{\mathbb{R}}$, of the linear function of Lemma 6.3.2 for $w$. Then

$$\sup_{X^{\mathrm{an}}} u - C_w \le u(w) \le \sup_{X^{\mathrm{an}}} u \qquad (u \in \mathrm{PSH}(X, \theta)).$$

Indeed, a tail of the Fubini–Study approximants $f_i$ of $u$ has classes in this neighbourhood, so $f_i(w) \ge \sup f_i - C_w$, as in the proof of Theorem 6.3.1. Since $f_i \ge u$ and $f_i(w) \to u(w)$, this gives the lower bound.

**Lemma 12.2.1** *Let $\theta$ be a real numerical class and $v \in \mathcal{D}(X)$. There is a rational PL function $h$, depending on $\theta$ and $v$, such that every $u \in \mathrm{PSH}(X, \theta)$ satisfies*

$$\sup_{X^{\mathrm{an}}}(u - h) = (u - h)(v).$$

***Proof*** Choose a flag ideal $\mathcal{I}$ on the product such that $v = v_E$ on its blowup $\mathcal{X}$ integrally closed along the puncture. This is the choice made in Proposition 5.4.1, and the product suffices at the trivial valuation. Write $\Sigma$ for the special-fibre valuations

on this model, and write $D$ for the vertical Cartier divisor with $\mathcal{I}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(D)$. It is ample over the product by the blowup construction and finiteness of the integral closure along the puncture. Write $\rho : \mathcal{X} \to X \times \mathbb{A}^1$ for the model map; line bundles on $X$ are pulled back to the product along the projection. After adding the pullback of a sufficiently ample line bundle $L$ on $X$, the line bundle $M = \rho^* L \otimes \mathcal{O}_{\mathcal{X}}(D)$ is ample over $\mathbb{A}^1$, by the composition property for relative ampleness [Stacks, Tag 0C4K].

Consider the sheaf $\mathcal{F}$ of functions regular after inverting $t$ with order at least $-1$ at $E$ and order at least zero at each other special-fibre component. It lies between $\mathcal{O}_{\mathcal{X}}$ and $t^{-1}\mathcal{O}_{\mathcal{X}}$ by the vertical pole criterion. It is coherent. On an affine open its sections are a submodule of the finite module $t^{-1}\mathcal{O}$. On a principal open the order conditions at the generic points that remain are unchanged, and a power of the denominator restores the dropped conditions. Thus the submodule localizes, proving coherence on the Noetherian model. Its stalk at the generic point of $E$ is the inverse of the maximal ideal of the DVR there. Indeed, after shrinking around this point, the other conditions are absent, and any element of order $-1$ in the DVR has a representative in $t^{-1}\mathcal{O}$ after such a shrinking.

For $m$ large, $M^m \otimes \mathcal{F}$ is generated over $\mathbb{A}^1$ by global sections by relative Serre generation. Choose one attaining order $-1$ relative to $M^m$ at $E$. On the product over the puncture it is a Laurent polynomial in $t$ with coefficients in $H^0(X, mL)$. The maximum of the log norms of these coefficients, with weights the negatives of the exponents, is the negative Gauss order of the section. At $v$ it equals $m\varphi_{\mathcal{I}}(v) + 1/b_E$. At each other point of $\Sigma$ it is at most $m\varphi_{\mathcal{I}}$, by the imposed order conditions. Adjoin sections generating $mL$ with a common integer weight at most $\min_\Sigma m\varphi_{\mathcal{I}}$. The full maximum $g$ is then finite and is the potential of a fractional flag ideal by Theorem 5.5.1 and its construction at degree $m$.

Set $h = a(m\varphi_{\mathcal{I}} - g)$ for a positive integer $a$. Write $g = \varphi_{\mathcal{J}}$ for the fractional flag ideal $\mathcal{J}$ just obtained, so that $h = \varphi_{\mathcal{I}^{am}} - \varphi_{\mathcal{J}^a}$. In the proof of Lemma 6.3.1 for this presentation, the test set consists of the special-fibre valuations of the blowup of $\mathcal{I}^{am}$, integrally closed along the puncture. This model is $\mathcal{X}$, since an ideal and its powers have the same blowup. Hence the supremum of $u - h$ is attained on $\Sigma$. At $v$, $h$ has value $-a/b_E$, whereas its values at the other points of $\Sigma$ are nonnegative. The uniform bounds on $u(w) - u(v)$ for $w \in \Sigma$ show that for sufficiently large $a$ the maximum over $\Sigma$ is attained at $v$, for all $u$. □

**Theorem 12.2.1** *For any uniformly bounded-above family of $\theta$-psh potentials on a reduced projective scheme, $(\sup_i u_i)^* = \sup_i u_i$ on $\mathcal{D}(X)$.*

***Proof*** Suppose $X$ is integral and fix $v \in \mathcal{D}(X)$. Use Lemma 12.2.1 to choose $h$. Then $u_i \le u_i(v) - h(v) + h$ everywhere. Taking the supremum and regularizing retains the upper bound $\sup_i u_i(v) - h(v) + h$ because $h$ is continuous. Evaluate at $v$ and use the reverse inequality from the definition of regularization.

In general $v$ has support the generic point of one irreducible component. It has a neighborhood disjoint from the other analytic components, since they form a finite union of closed sets. Restrict the potentials to the component containing $v$. Their restrictions are psh by the pullback rule. The regularization at $v$ is unchanged by this restriction, so the integral case applies. □

**Corollary 12.2.1** *For an integral $X$, a real class $\theta$ has the envelope property if and only if every bounded-above increasing net of $\theta$-psh potentials has a $\theta$-psh limit on $\mathcal{D}(X)$. In a nef class these conditions are also equivalent to compactness of the set of $\theta$-psh potentials with supremum zero in the topology of convergence on $\mathcal{D}(X)$.*

***Proof*** The envelope property implies the assertion for increasing nets by Theorem 12.2.1. Conversely, for a bounded-above family $u_i$, take the increasing net of finite maxima and let $u$ be its psh limit on $\mathcal{D}(X)$. The theorem gives equality there with $F = (\sup_i u_i)^*$. Since $u_i \le u$ on $\mathcal{D}(X)$, the inequality holds everywhere by divisorial determination (Theorem 6.3.1). Hence $F \le u$ by upper semicontinuity. The reverse inequality follows by the same order test with upper semicontinuous obstacle $F$.

For the compactness assertion, suppose the envelope property holds and $u_i$ have supremum zero. The bounds at each divisorial point and compactness of the product of the resulting intervals give a subnet converging on $\mathcal{D}(X)$. Along this subnet the regularized suprema of tails are psh and decrease. They are zero at the trivial valuation. Theorem 6.2.2 gives a psh limit there and everywhere. On $\mathcal{D}(X)$ this is the limit of the original subnet by agreement at divisorial points. It has supremum zero by its value at the trivial valuation.

Conversely, let $u_i$ be a bounded-above increasing net. Its suprema increase to a finite constant, and the compactness hypothesis gives a subnet of $u_i - \sup u_i$ converging on $\mathcal{D}(X)$ to a psh potential. Adding the limit of the suprema gives the required psh limit of the full net, since the original net is increasing at each divisorial point.□

## 12.3 An obstruction from normalization

**Proposition 12.3.1** *Let $X$ be integral and projective over an algebraically closed field. If one ample class on $X$ has the envelope property, the finite normalization of $X$ is a homeomorphism.*

***Proof*** Write $\nu : Y \to X$ for the normalization. It is finite since $X$ is of finite type over a field [Stacks, Tags 035S and 0335]. If it is not injective, the complement of the diagonal in $Y \times_X Y$ is a nonempty scheme of finite type over $k$. A closed $k$-point there gives two distinct $k$-points $p, q$ of $Y$ in the same fibre. Their trivial semivaluations have the same image in $X^{\mathrm{an}}$.

Let $\omega$ be an ample class with the envelope property. Choose an ample line bundle $L$ on $X$, replacing it by a power so that $\nu^* L$ is very ample. Choose a section vanishing at $p$ and nonvanishing at $q$, and a finite family of sections generating $\nu^* L$. Give the first section weight zero and the generating family weight $-1$. Their maximum of weighted log norms, multiplied by a small positive rational $\delta$, gives a finite Fubini–Study potential $g$ in class $\delta[\nu^* L]$ that separates $p$ from $q$. Choose $\delta$ so that $\omega - \delta[L]$ is ample.

The conductor $\mathfrak{c}$ of the finite birational map is a nonzero ideal, and $\mathfrak{c}\, \nu_* \mathcal{O}_Y \subseteq \mathcal{O}_X$. Choose an ample power $A$ on $X$ and a nonzero section $s$ of $\mathfrak{c} \otimes A$. If $s_i$ and $\lambda_i$ are the

sections and weights defining $g$, the products $ss_i^m$ descend to sections of $A + mL$ by the conductor condition. Thus for $m \geq 1$ the function

$$u_m = \frac{\delta}{m} \max_i \{\log |ss_i^m| + m\lambda_i\}$$

is a generically finite Fubini–Study potential on $X$ in class $\delta[L] + (\delta/m)[A]$. These classes are dominated by $\omega$ for all sufficiently large $m$. On $Y^{\mathrm{an}}$ we have $\nu^* u_m = g + (\delta/m) \log |\nu^* s| \leq g$. The potentials on $X$ are uniformly bounded above by their values at the trivial valuation there, which here equal $\sup g$. On $\mathcal{D}(Y)$ the function $\log |\nu^* s|$ is finite, so $\nu^* u_m$ increases to $g$ on this set.

The envelope property makes $u = (\sup_m u_m)^*$ $\omega$-psh, taking the supremum over the stated tail. The map $\nu^{\mathrm{an}}$ identifies $\mathcal{D}(Y)$ with $\mathcal{D}(X)$ since the function fields are the same and both schemes are proper. By divisorial agreement, $\nu^* u = g$ there. Both sides are $\nu^*\omega$-psh, so they agree everywhere by divisorial determination. This contradicts the different values of $g$ at $p, q$. The normalization is therefore injective, and finiteness and surjectivity make it a homeomorphism. □

For instance, an integral projective nodal curve over an algebraically closed field has two points over the node in its normalization. No ample class on the curve has the envelope property. The proof constructs an increasing family whose limit on the divisorial set separates the two branches on the normalization.

## 12.4 Envelopes from base ideals

Let $L$ be an ample line bundle on a reduced projective scheme $X$. After scaling a rational PL obstacle and the class $[L]$ by the same positive integer if necessary, we can represent the obstacle as $f = \varphi_D$ for an integral vertical Cartier divisor $D$ on a model $\mathcal{X}$ integrally closed along the puncture. Set $\mathcal{L} = \rho^* L \otimes \mathcal{O}_{\mathcal{X}}(D)$, where, as in Lemma 12.2.1, $\rho : \mathcal{X} \to X \times \mathbb{A}^1$ is the model map and $L$ is pulled back to the product along the projection. Write $\mathfrak{a}_m$ for the image of evaluation of global sections of $\mathcal{L}^m$, tensored by $\mathcal{L}^{-m}$. It is an invariant ideal. For sufficiently large $m$ the sections generate away from the special fibre, since $mL$ is globally generated there and the base of the model is affine. In fact formation of the module of global sections commutes with localizing the base at $t$. Thus $\mathfrak{a}_m$ contains a power of $t$ for every such $m$. Its potential $\varphi_{\mathfrak{a}_m}$ is the negative Gauss value of the ideal evaluated at the centre on $\mathcal{X}$. Equivalently it is the potential of the vertical Cartier divisor that makes the ideal invertible on its blowup.

**Proposition 12.4.1** *With this notation, put $f_m = f + m^{-1}\varphi_{\mathfrak{a}_m}$ for all sufficiently large $m$. Then $f_m$ is a finite rational Fubini–Study potential in class $[L]$, $f_m \leq f$, and*

$$P_{[L]}(f) = \sup_{m \gg 1} f_m = \lim_{m \to \infty} f_m$$

*pointwise. The notation for the supremum permits any tail. The envelope is continuous if and only if the convergence is uniform.*

***Proof*** On the blowup of $\mathfrak{a}_m$, write $\mathfrak{a}_m O = O(D_m)$ with $D_m \leq 0$. After integral closure along the puncture, the pullback of $\mathcal{L}^m \otimes O(D_m)$ to the resulting model is generated over the base by construction. The marked potential divided by $m$ is $f_m$, so Theorem 5.5.1 makes it Fubini–Study. Evaluation of an integral ideal has nonpositive potential, giving $f_m \leq f$. Multiplication of sections gives $\mathfrak{a}_m \mathfrak{a}_l \subseteq \mathfrak{a}_{m+l}$. Hence $m f_m$ is pointwise superadditive on the defined tail. The superadditive limit is the supremum of $f_m$ over any tail. Indeed fix one index $r$ on the tail and write a large integer as $qr + s$ with $s$ on a fixed finite interval of length $r$ on the tail. The lower bound $(qr f_r + s f_s)/(qr + s)$ shows that the limit inferior is at least $f_r$.

For the converse inequality to $\sup f_m \leq P_{[L]}(f)$, take a finite Fubini–Study competitor $u$ for $L$. Represent $u = \varphi_E$ and $f = \varphi_D$ on a common dominating model, writing $D$ for its pullback. The divisor $D - E$ is effective by the vertical pole criterion. For large divisible $m$, the line bundle of $mE$ relative to $mL$ is generated over the base. Its sections are sections of the pullback of $\mathcal{L}^m$, with image ideal $O(m(E - D))$. The pullback of $\mathfrak{a}_m$ is the base ideal of the pulled-back bundle. Indeed any global section upstairs gives a section over the marked puncture downstairs. At each special-fibre DVR downstairs, properness gives a lift of its spectrum to the dominating model, so the section has nonnegative order. The vertical pole criterion makes the section regular downstairs. Every downstairs section also pulls back. It follows that $O(m(E - D)) \subseteq \mathfrak{a}_m O$, and evaluating this inclusion gives $u \leq f_m$. Proposition 12.1.1 shows that these competitors suffice to calculate the envelope.

Continuity follows from uniform convergence. If the envelope is continuous, $f_{r!}$ increases to it for large $r$ by superadditivity, so this subsequence converges uniformly. Fix an index $d$ of that subsequence for which the uniform error is small. Write a large index as $qd + s$ with $s$ in a fixed finite interval on the tail, as above. The lower bound involving $f_d$ and the finitely many bounded $f_s$, together with $f_m \leq P_{[L]}(f)$, gives uniform convergence of the full sequence. □

**Lemma 12.4.1** *On a smooth integral projective scheme $X$, it suffices for the envelope property in all nef real classes to treat rational PL obstacles in ample classes. After scaling the class and obstacle together and translating the obstacle, one may work with an ample line bundle $L$ and $f = \varphi_D$, where $D \leq 0$ is a vertical Cartier divisor on a model $\rho : Y \to S = X \times \mathbb{A}^1$ integrally closed along the puncture. The sections of $r(\rho^* L + D)$ define a graded family of invariant ideals $\mathfrak{I}_r \subset O_S$, with $\mathfrak{I}_r(rL)$ globally generated. For large $r$, these ideals contain a power of $t$ and*

$$P_{[L]}(f) = \sup_{r \gg 1} r^{-1} \varphi_{\mathfrak{I}_r} = \lim_{r \to \infty} r^{-1} \varphi_{\mathfrak{I}_r}.$$

***Proof*** If every rational ample class has the envelope property, take such classes converging to a given nef class from above in the nef order. A bounded-above family that is psh in the nef class is psh in each of the ample classes. Its regularized supremum is therefore psh in every class of the approximating sequence. The constant sequence

of this function is permitted in the decreasing-limit theorem for varying nef classes Theorem 6.2.2. Hence the regularized supremum is psh in the nef class. In the rational ample case Proposition 12.1.1 reduces the assertion to rational PL obstacles. Scale to an integral polarization and an integral vertical divisor. Subtract a constant to arrange $D \leq 0$ by the vertical pole criterion.

The sections on $Y$ belong to $\rho^*(rL)$. They descend to sections of $rL$ on $S$: the map is proper birational with normal target, so regularity at the codimension-one points of $S$ suffices to extend their restrictions. Let $\mathfrak{J}_r$ be their image ideal after trivializing $rL$. Its extension to $Y$ is $\mathfrak{a}_r \mathcal{O}_Y(rD)$, with $\mathfrak{a}_r$ as in Proposition 12.4.1. It is invariant and its products satisfy $\mathfrak{J}_r \mathfrak{J}_s \subset \mathfrak{J}_{r+s}$ by multiplication of sections. For $r$ large, $rL$ is globally generated and one can multiply its sections by a power of $t$ so that their pullbacks vanish along $-rD$. Thus $\mathfrak{J}_r$ contains a power of $t$. The base-ideal formula of Proposition 12.4.1 then gives the equalities. □

## 12.5 Smooth projective varieties in positive characteristic

Frobenius expansions replace the base ideals of high powers by coefficient ideals in a fixed degree. The coefficient ideals give a decreasing sequence along the powers of $p$. The twist that makes them globally generated contributes a numerical class of order $1/m$ at degree $m$.

**Theorem 12.5.1** *Let $X$ be a smooth integral projective scheme over a perfect trivially valued field $k$ of characteristic $p > 0$. Every nef real class on $X$ has the envelope property.*

***Proof*** Use the reduction and notation of Lemma 12.4.1. In particular $Q = P_{[L]}(\varphi_D)$ is the pointwise limit of $r^{-1}\varphi_{\mathfrak{J}_r}$ along the sufficiently large indices.

On a smooth affine chart with $d$ étale coordinates $x_1, \dots, x_d$, each regular function has a unique expansion

$$f = \sum_{0 \leq \beta_i < q} f_\beta^q x^\beta, \qquad q = p^e.$$

Indeed, this is the monomial basis over the subring of $q$-th powers on affine space. The Cartesian square for relative Frobenius under an étale map transports the basis to our chart, using perfectness of $k$ [Stacks, Tag 0EBS]. On $S$ we take product charts of $X$ with the affine line, with $t$ as the last coordinate. For an ideal $I$, write $I^{[q]}$ for the ideal generated by the $q$-th powers of its elements. The ideal generated by the coefficients of generators of $I$ is the smallest $B$ with $I \subset B^{[q]}$. Indeed membership of $f$ in $B^{[q]}$ is equivalent to membership of every $f_\beta$ in $B$, by expanding the coefficients of a representation in $B^{[q]}$. Denote this smallest ideal by $I^{[1/q]}$. Localization commutes with this operation. On an open where a function is inverted, one can present a localized generator with a $q$-th power as denominator, and the coefficients are the original coefficients divided by its $q$-th root. The operation is monotone and

taking roots successively gives the root for the product of the exponents. The last assertion follows from the minimum-ideal characterization and $(B^{[q]})^{[p]} = B^{[pq]}$. The expansion also gives $(I^{[p]})^{[1/pq]} = I^{[1/q]}$.

These operations apply to the ideals $\mathfrak{J}_r$ on the smooth affine charts of $S$. They are independent of the coordinates by the minimum-ideal characterization, and compatible on overlaps by localization. For fixed $m \geq 1$ define

$$\mathfrak{B}_{m,e} = \mathfrak{J}_{mp^e}^{[1/p^e]}, \qquad \mathfrak{B}_m = \bigcup_{e \geq 0} \mathfrak{B}_{m,e}.$$

The ideals in $e$ increase since $\mathfrak{J}_{mp^e}^{[p]} \subset \mathfrak{J}_{mp^{e+1}}$. They stabilize, since $S$ is Noetherian and has a finite affine cover. Composition of roots gives $\mathfrak{B}_{pm}^{[1/p]} = \mathfrak{B}_m$, and therefore $\mathfrak{B}_{pm} \subset \mathfrak{B}_m^{[p]}$. All these ideals are invariant under the action on $t$. For example the expansion of a homogeneous $t$-term has coefficients homogeneous in $t$, so an invariant system of generators gives an invariant coefficient ideal.

For sufficiently large $q = p^e$, the ideal $\mathfrak{J}_{mq}$ contains $t^N$ for some $N$. Since it contains $t^{q\lceil N/q \rceil}$, its Frobenius root contains $t^{\lceil N/q \rceil}$. In particular $\mathfrak{B}_m$ contains a power of $t$.

Put $u_m = m^{-1}\varphi_{\mathfrak{B}_m}$. The inclusion $\mathfrak{B}_{pm} \subset \mathfrak{B}_m^{[p]}$ gives $u_{pm} \leq u_m$. From $\mathfrak{J}_{mp^e} \subset \mathfrak{B}_m^{[p^e]}$ we have $u_m \geq (mp^e)^{-1}\varphi_{\mathfrak{J}_{mp^e}}$ for all sufficiently large $e$. Letting $e$ increase gives $Q \leq u_m \leq 0$.

We claim that $\mathfrak{B}_m(mL + A)$ is globally generated on $S$, where $n = \dim X$, $H$ is a very ample line bundle on $X$, and $A = \omega_X + (n+1)H$. Line bundles on $X$ in this formula are pulled back to $S$. Let $F$ be absolute Frobenius. The Frobenius trace on top forms, with the target twist identified over the perfect field, selects the coefficient of index $(q-1, \ldots, q-1)$ in the expansion relative to the form $\eta = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1}$. This is the computation in [Stacks, proof of Tag 0FW2], iterated $e$ times. Every coefficient of $f$ is the trace coefficient of $x^{q-1-\beta} f \eta$. It follows that the trace maps $F_*^e(I\omega_S)$ onto $I^{[1/q]}\omega_S$ for each ideal $I$.

Choose $e$ large enough for stabilization and for the vanishing used below, with $q = p^e$. By construction $\mathfrak{J}_{mq}(mqL)$ is generated by finitely many global sections. Tensor their evaluation map with $\omega_S(q(n+1)H)$ and push forward by $F^e$. The projection formula and trace then give a surjection from finitely many copies of $F_*^e\omega_S(q(n+1)H)$ onto $\mathfrak{B}_m(mL + A)$. Here $\omega_S$ identifies with the pullback of $\omega_X$ using $\mathrm{d}t$.

To check generation of the source, write $\pi : S \to X$. For a coherent $\mathcal{F}$ on $X$, the sheaf $F_{S,*}^e\pi^*\mathcal{F}$ is the direct sum of $q$ copies of $\pi^* F_{X,*}^e\mathcal{F}$. On a product chart the summand indexed by $0 \leq \ell < q$ sends $st^j$ to $st^{qj+\ell}$. This respects scalar multiplication by the definition of Frobenius pushforward, and every polynomial in $t$ has a unique such decomposition. For $\mathcal{F} = \omega_X(q(n+1)H)$, the projection formula gives

$$H^i\big(X, (F_{X,*}^e\mathcal{F})(-iH)\big) = H^i\big(X, \omega_X(q(n+1-i)H)\big)$$

as abelian groups. For $1 \leq i \leq n$, these groups vanish for $q \gg 1$ by Serre vanishing [Stacks, Tag 0B5U], and for $i > n$ they vanish by dimension. Push the sheaf forward

under the embedding given by $H$ and apply the generation lemma for a 0-regular sheaf on projective space [Stacks, Tag 08A8]. Thus the source of the surjection is globally generated, and the claim follows.

By invariance in $t$, the global generators of $\mathfrak{B}_m(mL + A)$ can be replaced by their homogeneous components $s_j t^{b_j}$. Evaluating in Gauss norms shows that $u_m$ is the everywhere finite Fubini–Study potential $m^{-1} \max_j(\log|s_j| - b_j)$ in class $[L] + [A]/m$.

We next control the coefficient ideals at a divisorial point. Let $v \in \mathcal{D}(X)$ and $w = \sigma(v)$ be its Gauss extension. By Propositions 5.3.1 and 5.4.1, $w$ is a positive rational multiple of a divisor order over $S$, even when $v$ is the trivial point of $X$. There is a finite constant $A_w$ with

$$w(\mathfrak{J}_{mq}^{[1/q]}) \geq q^{-1} w(\mathfrak{J}_{mq}) - A_w.$$

For the proof choose a smooth affine chart of the indicated kind at the centre of $w$. The multiplier $x^{q-1-\beta}$ in the trace formula has nonnegative $w$-value.

For $w = \mathrm{ord}_E$, take a normal model carrying $E$. It is smooth near the generic point of $E$, since it is regular there and $k$ is perfect. If $\xi$ generates its top forms at that point, the pulled back form is $h\xi$, with $a = \mathrm{ord}_E(h) \geq 0$. The trace on rational top forms is the same on both models by localization at their common function field [Stacks, Tag 0FLB]. It preserves regular forms at $E$. For a local function $g$ on $S$, the pulled back form $g\eta$ has order $w(g) + a$. Factoring out the $q$-th power of the uniformizer with exponent $\lfloor (w(g)+a)/q \rfloor$ shows that the trace form has order at least this integer. Its coefficient relative to $\eta$ then has order at least

$$\left\lfloor \frac{w(g)+a}{q} \right\rfloor - a \;\geq\; \frac{w(g)}{q} - (a+1).$$

This gives the coefficient-ideal bound with $A_w = a + 1$; rescale the bound when $w$ is a multiple of the order. The argument applies to every generator of the root ideal.

Apply the bound at the stabilized ideal for $m$ and use the lower estimate for $Q$ from the base-ideal formula. This gives

$$Q(v) \leq u_m(v) \leq Q(v) + A_w/m.$$

The sequence $u_{p^j}$ decreases to an $[L]$-psh potential: its terms are Fubini–Study in classes tending to $[L]$, and its value at the trivial point is bounded below by $Q$. It agrees with $Q$ on divisorial points. Since $Q \leq \varphi_D$, divisorial determination (Theorem 6.3.1) puts the limit below the continuous obstacle everywhere. It is a competitor for $Q$, and the opposite inequality follows from $u_m \geq Q$. Thus $Q$ is psh, as required. □

**Corollary 12.5.1** *Over a perfect trivially valued field of positive characteristic, every nef real class on $\mathbb{P}^n_k$ has the envelope property. For a rational PL obstacle in the class of an ample line bundle, the decreasing approximants above belong to that same class.*

***Proof*** Take $H = \mathcal{O}(1)$ in the proof. Then $\omega_{\mathbb{P}^n} + (n+1)H = 0$. □

## 12.6 Smooth projective varieties in characteristic zero

On the smooth product $X \times \mathbb{A}^1$ in characteristic zero, multiplier ideals replace Frobenius coefficient ideals. Subadditivity gives a decreasing sequence along the powers of two. The generation bound in Appendix B keeps the twist fixed as the exponent varies.

**Theorem 12.6.1** *Let $X$ be a smooth integral projective scheme over a trivially valued field of characteristic zero. Every nef real class on $X$ has the envelope property.*

***Proof*** The reduction in Lemma 12.4.1 gives an ample line bundle $L$, a rational PL obstacle represented by a vertical Cartier divisor $D \leq 0$, and a graded family of invariant ideals $\mathfrak{J}_r$ on $S = X \times \mathbb{A}^1$. For all large $r$ these ideals contain a power of the affine parameter $t$, and $\mathfrak{J}_r(rL)$ is globally generated. Write $Q = P_{[L]}(\varphi_D)$ for the envelope.

For $m \geq 1$, let $\mathfrak{B}_m$ be the stable value of

$$\mathcal{M}(\mathfrak{J}_{mq}, 1/q)$$

for sufficiently large $q$ ordered by divisibility. This family increases: multiplication of sections gives $\mathfrak{J}_{mq}^a \subset \mathfrak{J}_{mqa}$, and taking a power of an ideal multiplies the exponent in its multiplier ideal. Stabilization follows from Noetherianity. For example, take factorial indices, which are cofinal for divisibility, and use the ascending chain condition on a finite affine cover.

Choose $q$ so that the ideal for $\mathfrak{B}_{2m}$ has stabilized at $q$ and the ideal for $\mathfrak{B}_m$ has stabilized at $2q$. Then Corollary B.1.1 gives

$$\mathfrak{B}_{2m} = \mathcal{M}(\mathfrak{J}_{2mq}, 1/q) \subset \mathcal{M}(\mathfrak{J}_{2mq}, 1/(2q))^2 = \mathfrak{B}_m^2.$$

For all sufficiently divisible $q$, the same subadditivity and the inclusion $\mathfrak{J}_{mq} \subset \mathcal{M}(\mathfrak{J}_{mq}, 1)$ give

$$\mathfrak{J}_{mq} \subset \mathcal{M}(\mathfrak{J}_{mq}, 1) \subset \mathfrak{B}_m^q.$$

In particular $\mathfrak{B}_m$ contains a power of $t$. Put $u_m = m^{-1}\varphi_{\mathfrak{B}_m}$. The two inclusions imply $u_{2m} \leq u_m$ and

$$(mq)^{-1}\varphi_{\mathfrak{J}_{mq}} \leq u_m \leq 0.$$

The limit formula of Lemma 12.4.1 shows $Q \leq u_m$.

Let $H$ be very ample and put $A = K_X + (\dim X + 2)H$. Take a divisible stabilization index $q$. By Proposition B.3.1, $\mathfrak{B}_m(mL + A)$ is globally generated on $S$. Since $\mathfrak{J}_{mq}$ is invariant under scaling $t$ and multiplier ideals are intrinsic, $\mathfrak{B}_m$ is invariant under the $k^\times$-action. As $k$ is infinite, each homogeneous part of a polynomial section in the ideal is again in the ideal, by evaluating at finitely many scalings and solving the Vandermonde system. A finite generating family may therefore be chosen of the form $s_j t^{b_j}$. Its Gauss evaluation gives

$$u_m = m^{-1} \max_j \{\log |s_j| - b_j\} \in \mathcal{H}_{\mathbb{Q}}\big(L + A/m\big).$$

It is everywhere finite since the ideal contains a power of $t$.

At a divisorial point $v$ of $X^{\mathrm{an}}$, including the trivial point, the Gauss extension $w = \sigma(v)$ is a nontrivial divisorial valuation of $S$ by Propositions 5.3.1 and 5.4.1. At a stabilized divisible index, Proposition B.1.1 gives

$$w(\mathfrak{B}_m) \geq q^{-1} w(\mathfrak{J}_{mq}) - A_S(w).$$

Since $Q \geq (mq)^{-1} \varphi_{\mathfrak{J}_{mq}}$, we obtain

$$Q(v) \leq u_m(v) \leq Q(v) + m^{-1} A_S(w).$$

The functions $u_{2^j}$ decrease in classes converging to $[L]$ and are bounded below at the trivial point by $Q$, which is finite there. Thus their limit is $[L]$-psh by definition. It agrees with $Q$ on divisorial points, and consequently lies below the continuous obstacle everywhere by divisorial determination (Theorem 6.3.1). The limit is a competitor for $Q$ and majorizes it. Hence the limit equals $Q$, and $Q$ is psh, as required.□

# Chapter 13
# Affine approximation and logarithmic growth

On the affine line, $\max(0, \log|z|)$ grows only along the ray toward infinity. Adding the log norm of a section vanishing at infinity cancels that growth and gives a bounded potential on the projective line. The same construction works on a normal affine variety. One must choose the compactification using the functions and exponents in the affine maximum.

Throughout let $k$ be a trivially valued field and $Y$ an integral normal affine scheme of finite type over $k$. In this chapter a divisorial point of $Y^{\mathrm{an}}$ is the trivial valuation or a positive rational multiple of a divisor order of the function field $k(Y)$. These points have generic support, even when their centres on a proper compactification lie outside $Y$.

The compactification adapted to a given affine maximum is constructed in Proposition 13.2.1. With it, a function of logarithmic growth that is a decreasing limit of locally convex PL functions near every compact set is a decreasing limit of affine Fubini–Study functions (Proposition 13.4.1). The local input is the positivity criterion Theorem 13.3.1. Polynomial hulls then let us glue this property over an affine open cover when $k$ is countable (Proposition 13.10.1), and finite morphisms push regularizable functions forward by a norm (Proposition 13.12.1). The chapter ends with the local potentials of semiample classes and with the reduction of the projective envelope problem to a countable field (Proposition 13.14.1). These results are used in Chapters 14 and 16, and the finite norm also in Chapter 33.

## 13.1 Growth bounds

An *affine Fubini–Study function* is a function

$$g = \max\{\, b_0,\ a_1 \log|f_1| + b_1, \dots, a_r \log|f_r| + b_r \,\},$$

where $a_i \in \mathbb{Q}_{>0}$, $b_i \in \mathbb{Q}$, and the list $f_i \in k[Y]$ contains generators of $k[Y]$ as a $k$-algebra. Discard zero terms. Write $\mathrm{FS}_{\mathrm{aff}}(Y)$ for these functions. We say $u$ has *logarithmic growth* if $u \le g + C$ for some such $g$ and a real constant $C$.

Each $g$ is continuous since the constant term keeps the maximum finite. Its sublevel sets are compact: the algebra generators embed $Y$ as a closed subscheme of an affine space, and a bound on $g$ bounds their absolute values. The sublevel set is closed in the resulting compact polydisc. A sum or a maximum of finitely many such functions has the same form. For a sum expand it as the maximum of all pairwise sums. The constant from either family pairs with each term of the other, so the new list still contains generators. For two terms containing logarithms, clear the denominators and use multiplication of functions.

**Proposition 13.1.1** *If $p : Y' \to Y$ is a morphism of normal integral affine schemes of finite type, $g \in \mathrm{FS}_{\mathrm{aff}}(Y)$ and $g' \in \mathrm{FS}_{\mathrm{aff}}(Y')$, then $g \circ p \le Cg' + D$ for constants $C > 0, D$. If $p$ is finite and dominant, then also $g' \le C'g \circ p + D'$ for some $C' > 0, D'$.*

***Proof*** Shift $g'$ by a constant so that it is nonnegative. The log absolute value of each algebra generator in its presentation is bounded above by a linear function of $g'$ with positive slope. Each pullback of an $f_i$ is a polynomial in these generators. The ultrametric inequality bounds its log norm by the maximum of the log norms of the monomials. The finitely many monomials give the first bound.

For the reverse inequality take a monic integral relation of degree $m$ over $k[Y]$ for each function used in $g'$. At a point where that function $b$ has nonzero value, the ultrametric inequality shows

$$\log|b| \le \max_{i<m} \frac{\log|c_i| \circ p}{m-i} \qquad \text{if} \qquad b^m + \sum_{i<m} c_i b^i = 0.$$

At a zero of $b$ the same inequality holds in the extended order. Bound the log norms of the finitely many coefficients by the polynomial calculation on $Y$. □

## 13.2 Compactification in a fixed class

**Proposition 13.2.1** *For $g_0 \in \mathrm{FS}_{\mathrm{aff}}(Y)$ there are a normal integral projective compactification $P$ of $Y$, an ample line bundle $H$ on $P$, a section $s$ of $H$ whose nonvanishing locus is $Y$, and $\kappa \in \mathbb{Q}_{>0}$ such that $g_0 + \kappa \log|s|$ extends to a finite rational Fubini–Study function on $P^{\mathrm{an}}$. For every $g \in \mathrm{FS}_{\mathrm{aff}}(Y)$ with $g \le g_0$, the function $g + \kappa \log|s|$ extends to a generically finite Fubini–Study function in class $\kappa[H]$.*

***Proof*** Choose $N \ge 1$ so that every $Na_i$ in the presentation of $g_0$ is a positive integer. The elements $h_i = f_i^{Na_i}$ give a finite morphism from $Y$ to their affine image, since each $f_i$ is integral over the algebra they generate. Take the projective closure $Z$ of that image in $\mathbb{P}^r_k$, and normalize it in $k(Y)$. The resulting finite morphism $P \to Z$ is projective, and over the affine chart it recovers $Y$ by normality. This normalization

is finite, since $Z$ is of finite type over $k$ and hence Nagata [Stacks, Tags 035B and 0AVK]. Pull back $\mathcal{O}(1)$ and the homogeneous coordinates to $H$ and $s_0, \dots, s_r$, and set $s = s_0$, $\kappa = 1/N$. Finite surjective pullback preserves ampleness [Stacks, Tag 0B5V], and these sections generate $H$. On $Y$ one has $s_i/s = h_i$. Thus

$$g_0 + \kappa \log|s| = \frac{1}{N} \max_{0 \le i \le r} (\log|s_i| + N b_i)$$

on $Y^{\mathrm{an}}$ and the right side defines the asserted finite function on $P^{\mathrm{an}}$.

For $g = \max\{b'_0, a'_i \log|f'_i| + b'_i\} \le g_0$, take a common denominator $m$ for $\kappa$ and all the $a'_i$. The rational sections $(f'_i)^{m a'_i} s^{m\kappa}$ of $m\kappa H$ are regular on $Y$. If $E$ is a prime divisor in $P \setminus Y$, evaluate $a'_i \log|f'_i| + \kappa \log|s| + b'_i$ at the points $t \operatorname{ord}_E$, which have generic support and hence lie in $Y^{\mathrm{an}}$. The inequality with the bounded function $g_0 + \kappa \log|s|$, as $t \to \infty$, gives $a'_i \operatorname{ord}_E(f'_i) + \kappa \operatorname{ord}_E(s) \ge 0$. The rational sections are therefore regular also along every boundary divisor. They extend by normality. Include $s^{m\kappa}$ for the constant term. The weighted maximum of all these log norms divided by $m$ is the asserted extension, which is finite at the trivial point. □

**Definition 13.2.1** A function $u : Y^{\mathrm{an}} \to [-\infty, \infty)$ is *regularizable* if it is a decreasing limit of a sequence in $\mathrm{FS}_{\mathrm{aff}}(Y)$ and it is finite somewhere. We denote this class by $\mathcal{R}(Y)$.

**Corollary 13.2.1** *A function in $\mathcal{R}(Y)$ has logarithmic growth and is finite at every divisorial point. For a compactification built from the first term in a decreasing presentation, $u + \kappa \log|s|$ on $Y^{\mathrm{an}}$ extends to a $\kappa[H]$-psh potential on $P^{\mathrm{an}}$.*

***Proof*** The first term gives the growth bound. Extend every term after adding $\kappa \log|s|$, using Proposition 13.2.1. Their extensions decrease by Theorem 6.3.1, since all divisorial points of $P^{\mathrm{an}}$ are in $Y^{\mathrm{an}}$. Log norms of global sections of line bundles on a proper scheme are at most zero and equal zero at the trivial valuation for a nonzero section. Thus each extension is bounded above by its value at the trivial point. The limit remains finite there by the value at a point where $u$ is finite. The defining decreasing approximation makes this limit $\kappa[H]$-psh. Theorem 6.3.1 gives finiteness at all divisorial points of $P^{\mathrm{an}}$, hence on $Y^{\mathrm{an}}$. □

*Example 13.2.1* On $Y = \mathbb{G}_m$ with coordinate $z$, use

$$g_0 = \max(0, \log|z|, \log|z^{-1}|).$$

The affine image under $(z, z^{-1})$ has projective closure $X_1 X_2 = X_0^2$. Its parametrization $[a : b] \mapsto [ab : a^2 : b^2]$ gives $P = \mathbb{P}^1$, $H = \mathcal{O}(2)$ and $s = ab$. The section $ab$ vanishes once at each boundary point. On either boundary ray with parameter $r$, its log norm is $-r$ and $g_0 = r$. The sum is the zero Fubini–Study potential in the class of $H$.

## 13.3 Local convexity and relative nefness

This section gives a local criterion for plurisubharmonicity on a normal integral projective scheme (later, a compactification of $Y$): for an ample rational line bundle $L$, a locally $L$-convex function in the sense defined below is $[L]$-psh (Theorem 13.3.1). The proof represents such a function on a test configuration and checks the degree of the marked line bundle on every curve in the special fibre.

On an analytic open subset of an algebraic analytification, call a finite function *locally convex PL* if around each point it is a maximum of finitely many functions of the form

$$b + \sum_i a_i \log |f_i|, \qquad a_i, b \in \mathbb{Q},$$

with the $f_i$ regular units on a Zariski open set containing the support of the point. Such functions are continuous. They are stable under finite maxima, sums, positive rational multiplication and addition of a rational multiple of the log norm of a regular unit. A finite maximum of positive rational multiples of log norms of regular functions plus constants is locally convex PL wherever it is finite. Near any such point, the terms vanishing at its support can be discarded: one remaining term is finite there, and the discarded terms tend to $-\infty$.

Let $P$ be normal integral and projective and $L$ an ample rational line bundle. If $mL$ is globally generated and $s$ is a section of it, put $P_s = \{s \neq 0\}$. A finite function $\phi$ on $P^{\mathrm{an}}$ is *locally $L$-convex* if near every point one can choose such a section nonzero at its support with $\phi - m^{-1} \log |s|$ locally convex PL there. Changing the section adds a rational multiple of the log norm of an invertible ratio, after taking common powers. A constant rational function is locally $L$-convex. Indeed, if the sections $s_j$ generate $mL$, then on $P_s^{\mathrm{an}}$

$$-m^{-1} \log |s| = m^{-1} \max_j \log |s_j/s|.$$

The maximum of two locally $L$-convex functions is locally $L$-convex by using the same local reference. Also $m^{-1} \log |s|$ plus a locally convex PL function is locally $L$-convex on $P_s^{\mathrm{an}}$.

**Lemma 13.3.1** *A locally $L$-convex function on $P^{\mathrm{an}}$ belongs to* $\mathrm{PL}_{\mathbb{Q}}(P)$.

***Proof*** First, the log norm of a regular unit on a Zariski neighborhood of the support of a point agrees near that point with a global rational PL function. Write the unit as a rational function $f$ on $P$. Its coherent denominator ideal consists locally of $a$ with $af$ regular. The ideal is trivial at the support in question. Twist it by a sufficiently high power of an ample line bundle and choose a global section $a$ nonzero at the support. Then $a$ and $fa$ are global sections of the same globally generated line bundle, neither vanishing at the support. Their log norms, truncated below at a sufficiently negative rational constant, are global rational Fubini–Study functions in that line bundle. To obtain the truncation, add generators with that constant as their common weight. The difference of the two truncations equals $\log |f|$ near the given point. The same truncation argument handles the local reference $m^{-1} \log |s|$.

Consequently $\phi$ agrees locally with global rational PL functions. Choose finitely many compact sets $K_i$ whose interiors cover $P^{\mathrm{an}}$, such that $\phi = G_i$ near $K_i$, with $G_i$ global rational PL. Choose rational PL cutoffs $0 \leq \beta_i \leq 1$ equal to one on $K_i$ and supported in neighborhoods $V_i$ where $\phi = G_i$. More explicitly, choose continuous cutoffs with those supports, approximate within $1/8$ by rational PL functions $\eta_i$, and take $\beta_i = \max(0, \min(1, 2\eta_i - 1/2))$. This uses the lattice and density in Theorem 5.4.1. If $M > 0$ is a rational bound for the absolute values of $\phi$ and all $G_i$, then

$$\phi = \min_i \{G_i + 2M(1 - \beta_i)\}.$$

For every point a term from a $K_i$ attains equality. All other terms dominate $\phi$, by equality on $V_i$ or by the size of the penalty where $\beta_i = 0$. □

**Lemma 13.3.2** *Let $\mathcal{Z}$ be an integral scheme of finite type over $k$, with function field $F$, and let $E$ be a prime divisor at whose generic point $\mathcal{Z}$ is normal. Put $\nu_1 = \mathrm{ord}_E$. Let $\nu_2$ be a valuation of $k(E)$ whose centre on $E$ is $y$, and let $t \in F^\times$ have $\nu_1(t) > 0$. In an affine neighborhood of $y$ where $t$ is regular, one can approximate the normalized values $\nu_1/\nu_1(t)$ on any finite set of nonzero rational functions by a positive multiple $w$ of a divisor order, with $w(t) = 1$ and centre in that neighborhood. These divisorial valuations can be required to have $w(g) < 0$ for any prescribed $g \in F^\times$ with $\nu_1(g) = 0$ and $\nu_2(\bar{g}) < 0$.*

***Proof*** The composition of the valuation of $E$ with the residue valuation $\nu_2$ has valuation ring

$$R = \{0\} \cup \{a : \nu_1(a) > 0\} \cup \{a : \nu_1(a) = 0,\ \nu_2(\bar{a}) \geq 0\}.$$

It contains the coordinate ring $A$ of the specified affine neighborhood, since $\nu_2$ is centred at $y$. If the additional pole inequality is prescribed, then $g^{-1}$ lies in the maximal ideal of $R$. For each of the finitely many test functions $h \in F^\times$, enclose $\nu_1(h)/\nu_1(t)$ in a rational interval $(p/q, p'/q')$, with $q, q' > 0$. Then $t^{-p}h^q$ and $t^{p'}h^{-q'}$ have strictly positive $\nu_1$-value, so lie in the maximal ideal of $R$.

Adjoin these elements and, when prescribed, $g^{-1}$ to $A$, giving a finite type $k$-algebra $B \subset R$ with fraction field $F$. Put $\mathfrak{p} = \mathfrak{m}_R \cap B$. This prime contains $t$. On the normalized blowup of $\mathfrak{p}$ choose a component of the exceptional Cartier divisor dominating $V(\mathfrak{p})$. Such a component exists by properness and surjectivity, since the preimage of $V(\mathfrak{p})$ is the support of that Cartier divisor. Its order is centred at $\mathfrak{p}$ on $\mathrm{Spec}\, B$. Normalize it by its positive value on $t$. It has all prescribed strict inequalities and has centre on $\mathrm{Spec}\, A$. Taking small rational intervals proves the claim. □

**Theorem 13.3.1** *Let $P$ be normal integral and projective over a trivially valued field and let $L$ be an ample rational line bundle. Every locally $L$-convex function belongs to* $\mathrm{PSH}(P, [L])$.

***Proof*** By Lemma 13.3.1 and Theorem 5.4.1, write the function as $\phi = \varphi_D$ on a normal test configuration $\mathcal{P}$ dominating the product. Put $\mathcal{L} = L + D$ in additive

notation, where $L$ is pulled back from $P$. We show that $\mathcal{L}$ has nonnegative degree on every integral curve $C$ in the special fibre. Blow up $C$ in $\mathcal{P}$ and normalize, obtaining $\mathcal{Z} \to \mathcal{P}$ with a prime divisor $E$ dominating $C$. It exists by the same Cartier-support argument as in Lemma 13.3.2. Let $w_0 = \mathrm{ord}_E / \mathrm{ord}_E(t)$ and let $x$ be its restriction to $k(P)$. All these valuations have generic support.

For any real valuation $w$ of $k(P)(t)$ centred on $\mathcal{Z}$ with $w(t) = 1$, the value of the pullback of $D$ is $\phi(x_w)$, where $x_w$ is the restriction to $k(P)$. To see this, express $D$, after taking a multiple, as the difference of the Cartier divisors of two fractional flag ideals as in Theorem 5.4.1. On a common dominating model the ideals are generated locally by the labelled monomials in $t$. The value of each ideal is the minimum of the values of those generators. It is therefore the same for $w$ and for the Gauss extension of $x_w$. Properness gives the centres on the common model.

Choose a local convex presentation for $\phi - m_0^{-1} \log |s|$ near $x$ and one term $\ell = b + \sum_i a_i \log |f_i|$ attaining the maximum at $x$. For $m$ divisible enough to clear all denominators, the rational section

$$G = t^{-mb} s^{m/m_0} \prod_i f_i^{ma_i}$$

of $m\mathcal{L}$ has model log norm

$$\log \|G\|_{m\mathcal{L}}(w) = m\big(m_0^{-1} \log |s|(x_w) + \ell(x_w) - \phi(x_w)\big)$$

at the valuations under consideration. In a local frame for the model bundle this log norm means minus the value of its coefficient. The formula follows by subtracting $mw(D)$ from the log norm in the product bundle. In particular it is zero at $w_0$, and is nonpositive for all $w$ whose restrictions are sufficiently close to $x$.

Thus $G$ gives a nonzero rational section of $m\mathcal{L}|_E$ on reduction at the generic point of $E$. Pull this residual section back to the normalization $E^\nu$. If it has a pole at a prime divisor there, let $\nu_2$ be the divisor order and let $y$ be its centre on $E$. Trivialize the model bundle on an affine neighborhood of $y$ in $\mathcal{Z}$; its coefficient $g$ satisfies $\mathrm{ord}_E(g) = 0$ and $\nu_2(\bar{g}) < 0$. Apply Lemma 13.3.2. Using finitely many evaluations on an affine chart of $P$, require the restriction of $w$ to belong to the neighborhood where the convex presentation holds. This is possible since these evaluations generate the analytic topology there and the support is generic. The condition $w(g) < 0$ then contradicts the nonpositive log norm of $G$. Hence the residual section has no poles at prime divisors of $E^\nu$, and is regular by normality.

Choose a closed point of the generic fibre of $E^\nu \to C$ where the section does not vanish. Its closure is a projective integral curve $\Gamma$ generically finite over $C$. A nonzero section on $\Gamma$ has nonnegative degree. By the projection formula its degree is $m[k(\Gamma) : k(C)] \deg_C \mathcal{L}$. This proves the required nefness.

Finally, add to $\mathcal{L}$ a positive rational multiple $\varepsilon\mathcal{A}$ of a relatively ample linearized line bundle. The result is relatively ample: it is ample on the special fibre by Proposition A.1.1, since there $\mathcal{L}$ is nef by the degree computation and $\varepsilon\mathcal{A}$ is ample, and on the other fibres since $L$ and the marked class of $\mathcal{A}$ are ample. Use fibrewise ampleness [Stacks, Tag 0D2N]. If $\psi$ is the model potential of $\mathcal{A}$, shift it by a rational

constant to make it nonnegative. The functions $\phi+\varepsilon\psi$ after this shift are finite rational Fubini–Study by Theorem 5.5.1. Taking rational $\varepsilon \downarrow 0$ gives the desired decreasing approximation in classes tending to $[L]$. □

## 13.4 From compact neighborhoods to affine approximation

In the compactification of Proposition 13.2.1, the section $s$ measures how a point approaches the boundary. Once the coefficient of $\log|s|$ is strictly larger than $\kappa$, a sufficiently negative truncation level can be crossed in a compact collar inside $Y^{\mathrm{an}}$.

**Lemma 13.4.1** *Let $L$ be rational and ample on a nonempty reduced projective scheme. If an $[L]$-psh function $\Phi$ is the pointwise infimum of a countable family of finite continuous functions $h_i$, it admits a decreasing sequence of finite rational Fubini–Study approximants in class $[L]$.*

***Proof*** Take the decreasing net in Proposition 6.4.1. For each $n$, the open sets where a member of the net is less than $h_i + 2^{-n}$ cover the compact analytification, for each $i \le n$. The open sets increase in the net. One common tail works for all $i \le n$. Choose the $n$-th approximant in that tail and later than the preceding one. Its limit is bounded above by every $h_i$ and below by $\Phi$. □

**Lemma 13.4.2** *Let $H$ be an ample line bundle on a projective scheme $P$, and let $s \in H^0(P,H)$ satisfy $P_s = Y$; for example, $P, H, s$ may come from a compactification $P, H, s, \kappa$ associated with some $g \in \mathrm{FS}_{\mathrm{aff}}(Y)$ by Proposition 13.2.1. If $\Phi$ is a finite rational Fubini–Study function for $cH$, with $c > 0$ rational, then $\Phi - c\log|s|$ on $Y^{\mathrm{an}}$ belongs to $\mathrm{FS}_{\mathrm{aff}}(Y)$.*

***Proof*** Write $\Phi$ as a maximum using sections of $mcH$, where $mc$ is integral. Dividing those sections by $s^{mc}$ expresses $\Phi - c\log|s|$ as a maximum of the required logarithmic terms. To supply a constant and a generating list, choose a rational lower bound $-B$ for $\Phi$. For each generator $z_l$ of $k[Y]$, write $z_l = \tau_l / s^{n_l}$ with $\tau_l \in H^0(P, n_l H)$ and $n_l \ge 1$. Such a section exists by clearing powers of the section defining the complement, on a finite affine cover. Since $\log|\tau_l| \le 0$, the terms

$$-B, \qquad \varepsilon_l \log|z_l| - B, \qquad 0 < \varepsilon_l \le c/n_l$$

with rational $\varepsilon_l$ are dominated by $-B - c\log|s|$. They may all be included in the maximum. □

**Lemma 13.4.3** *Let $P$ be a normal integral projective scheme, $H$ an ample line bundle on $P$, and $s \in H^0(P,H)$ a section with $P_s = Y$. Let $u : Y^{\mathrm{an}} \to [-\infty, \infty)$ be finite somewhere, and suppose every compact subset of $Y^{\mathrm{an}}$ has a neighborhood on which $u$ is a decreasing limit of a sequence of locally convex PL functions. Suppose $u \le$*

*$C_0 - \kappa \log|s|$ on $Y^{\mathrm{an}}$ for real constants $\kappa \geq 0$ and $C_0$. Then for every rational $c > \kappa$ the function*

$$\Phi = u + c \log|s| \quad \text{on } Y^{\mathrm{an}}, \qquad \Phi = -\infty \quad \text{on } P^{\mathrm{an}} \setminus Y^{\mathrm{an}}$$

*is $[cH]$-psh and admits a decreasing sequence of finite rational Fubini–Study approximants in that class. In particular $u \in \mathcal{R}(Y)$.*

***Proof*** The function $\Phi$ is bounded above by $C_0 + (c-\kappa)\log|s|$ on $Y^{\mathrm{an}}$, with $c - \kappa > 0$. Fix a rational level $b$. Choose $a$ small enough that $\Phi < b - 1$ on $\{\log|s| \leq a\}$, and choose locally convex PL functions $u_j$ decreasing to $u$ on a neighborhood of the compact set $\{\log|s| \geq a - 1\}$. On the compact collar $a - 1 \leq \log|s| \leq a$, the functions $u_j + c\log|s|$ are eventually strictly below $b$: the corresponding strict open sets increase and cover the collar. Thus $\max(b, u_j + c\log|s|)$ on $\{\log|s| > a - 1/2\}$ extends by $b$ to a finite locally $cH$-convex function $\Phi_{b,j}$ on $P^{\mathrm{an}}$. Where it is not constant, local convexity follows by using a common power of $s$ as the reference. The collar ensures the same property at the seam.

By Theorem 13.3.1 these functions are $[cH]$-psh. Their decreasing limit is $\max(b, \Phi)$, which is psh by Theorem 6.2.2. As $b \downarrow -\infty$, the new sequence decreases to $\Phi$. At the trivial valuation this limit stays finite, since every potential in the sequence attains its supremum there and $\Phi$ is finite somewhere. Thus $\Phi$ is psh. Taking integer levels $b = -i$, the family of all $\Phi_{-i,j}$ is a countable family of finite continuous majorants whose infimum is $\Phi$. The sequence follows from Lemma 13.4.1. Finally apply Lemma 13.4.2 to the sequence to regularize $u$. □

**Proposition 13.4.1** *Let $u : Y^{\mathrm{an}} \to [-\infty, \infty)$ be finite somewhere, with $u \leq g + C$ for $g \in \mathrm{FS}_{\mathrm{aff}}(Y)$. Suppose every compact subset has a neighborhood on which $u$ is a decreasing limit of a sequence of locally convex PL functions. Then $u \in \mathcal{R}(Y)$. For the compactification $P, H, s, \kappa$ associated with $g$ and any rational $c > \kappa$, the function*

$$\Phi = u + c \log|s| \quad \text{on } Y^{\mathrm{an}}, \qquad \Phi = -\infty \quad \text{on } P^{\mathrm{an}} \setminus Y^{\mathrm{an}}$$

*is $[cH]$-psh and admits a decreasing sequence of finite rational Fubini–Study approximants in that class.*

***Proof*** By Proposition 13.2.1, $P$ is normal integral and projective, $H$ is ample and $P_s = Y$. Since $g + \kappa \log|s|$ extends continuously to the compact space $P^{\mathrm{an}}$, it is bounded above there. Hence $u \leq C' - \kappa \log|s|$ on $Y^{\mathrm{an}}$ for a constant $C'$. Apply Lemma 13.4.3. □

**Corollary 13.4.1** *If $u \in \mathcal{R}(Y)$ and $u \leq g + C$ with $g$ affine Fubini–Study, then for any rational $\lambda > 1$ there exist $u_j \in \mathrm{FS}_{\mathrm{aff}}(Y)$ decreasing to $u$ with $u_j - \lambda g$ bounded for each $j$. A decreasing sequence in $\mathcal{R}(Y)$ with a limit finite somewhere again has limit in $\mathcal{R}(Y)$.*

***Proof*** Affine Fubini–Study functions are locally convex PL. For the first assertion use $c = \lambda\kappa$ in Proposition 13.4.1. Each approximant on $Y^{\mathrm{an}}$ differs from $-c \log |s|$ by its bounded projective approximant, and $\lambda g + c \log |s|$ is bounded too.

For the second assertion, a growth bound on the first potential bounds the decreasing sequence. For a fixed $c > \kappa$, add $c \log |s|$ and extend by $-\infty$ to obtain decreasing psh potentials on the same $P$ in class $[cH]$. Their limit is psh by Theorem 6.2.2, since their values at the trivial point are bounded below by their values at any fixed finite point of the limit. It is the infimum of a countable family of finite continuous majorants, by taking all the majorants for each member of the sequence. Apply Lemmas 13.4.1 and 13.4.2. □

For later use, finite maxima, finite sums, positive rational multiples and real translates preserve $\mathcal{R}(Y)$. Use the same operations on decreasing approximants, approximating a real translation by rational constants from above. The limits are finite at the trivial valuation by Corollary 13.2.1. Moreover a finite maximum of positive rational log terms of regular functions and rational constants belongs to $\mathcal{R}(Y)$ if it is finite somewhere, even without the generating list and constant term in the presentation. For such a maximum $q$, take $\max(q, g_*/j - j)$ with a fixed nonnegative $g_* \in \mathrm{FS}_{\mathrm{aff}}(Y)$. The same auxiliary construction after pullback by a morphism of normal integral affine schemes gives preservation under pullback whenever the limit is finite somewhere.

## 13.5 Principal opens and polynomial hulls

**Proposition 13.5.1** *Let $0 \neq f \in k[Y]$ and $u \in \mathcal{R}(Y_f)$. Suppose $u \leq g + C$ on $Y_f^{\mathrm{an}}$ for some $g \in \mathrm{FS}_{\mathrm{aff}}(Y)$. For rational $c > 0$, the function $u + c \log |f|$, extended by $-\infty$ over $V(f)^{\mathrm{an}}$, belongs to $\mathcal{R}(Y)$.*

***Proof*** Set $\varepsilon = c/3$ and $g_\varepsilon = \max(g, -\varepsilon \log |f|)$ on $Y_f^{\mathrm{an}}$. The presentations of $g$ and $f^{-1}$ supply generators of $k[Y_f]$. Use Corollary 13.4.1 with $g_\varepsilon$ and $\lambda = 2$ to choose $u_j \downarrow u$ with $u_j \leq 2g_\varepsilon + C_j$ for constants $C_j$. The bound after adding $c \log |f|$ becomes

$$u_j + c \log |f| \leq h + C_j, \qquad h = \max\{2g + c \log |f|, (c - 2\varepsilon) \log |f|\}.$$

Extend $h$ by $-\infty$ over $V(f)^{\mathrm{an}}$. It is upper semicontinuous there because both coefficients of the log norm are strictly positive and $g$ is locally bounded. The extension by $-j$ of $\max(u_j + c \log |f|, -j)$ is locally convex PL on $Y^{\mathrm{an}}$. Indeed it is constant on $\{h + C_j < -j\}$, an open neighborhood of the zero set, and is locally convex PL on $Y_f^{\mathrm{an}}$. This neighborhood may depend on $j$.

Each truncation has logarithmic growth, by the polynomial bound for $\log |f|$ in terms of $g$. Thus it is regularizable by Proposition 13.4.1, using its constant sequence on compact neighborhoods. The truncations decrease to the asserted function. Its value at the trivial valuation of $k(Y)$, where $|f| = 1$, equals that of $u$, which is finite

by Corollary 13.2.1. The second assertion of Corollary 13.4.1 therefore places the limit in $\mathcal{R}(Y)$. □

For a nonempty compact $K \subset Y^{\mathrm{an}}$, define its *polynomial hull* by

$$\widehat{K} = \{x : |a(x)| \le \sup_K |a| \text{ for all } a \in k[Y]\}.$$

It is compact, since the inequalities on generators place it in a compact closed polydisc and the other inequalities are closed. It is polynomially convex, meaning $\widehat{\widehat{K}} = \widehat{K}$, because the polynomial suprema on $K$ and $\widehat{K}$ are the same. Set $\widehat{\varnothing} = \varnothing$. An open set $D$ is *Runge* if $\widehat{K} \subset D$ for every compact $K \subset D$.

Every affine Fubini–Study function satisfies $u(x) \le \sup_K u$ for $x \in \widehat{K}$, by applying the hull inequalities to each logarithmic term. So does every regularizable function. Indeed for a decreasing sequence of continuous approximants the suprema decrease to the supremum of the limit on the compact set, by the finite subcover argument for strict upper bounds. In particular its strict sublevel sets are Runge: upper semicontinuity gives a strict upper bound on any compact set therein. Also a directed increasing union of Runge opens is Runge, since any compact subset lies in one member.

**Corollary 13.5.1** *Under the hypotheses of Proposition 13.5.1,*

$$u(x) \le \sup_{K \cap Y_f^{\mathrm{an}}} u \qquad (x \in \widehat{K} \cap Y_f^{\mathrm{an}})$$

*for any compact $K \subset Y^{\mathrm{an}}$.*

***Proof*** The intersection $K \cap Y_f^{\mathrm{an}}$ is nonempty: otherwise the polynomial $f$ vanishes on the hull. Choose $M \ge 0$ with $\log|f| \le M$ on $K$ and $\log|f(x)| \ge -M$. Apply the hull inequality to the extended potential for rational $c > 0$. It gives

$$u(x) - cM \le \sup_{K \cap Y_f^{\mathrm{an}}} u + cM.$$

The supremum is bounded above by the growth hypothesis on the compact $K$. If it is finite let $c$ decrease to zero. If it is $-\infty$ the same inequality forces that value at $x$. □

## 13.6 Strict polynomial neighborhoods

**Lemma 13.6.1** *If $H_0$ is a nonempty polynomially convex compact set and $U$ is an open neighborhood of $H_0$, there is an affine Fubini–Study function $g \ge 0$, equal to zero on a neighborhood of $H_0$, with $\{g \le 1\} \subset U$. If $k$ is countable, a nonempty Runge open is an increasing union of strict sublevel sets $\{g_j < 1\}$, with $\{g_j \le 1\} \subset \{g_{j+1} < 1\}$.*

***Proof*** Choose algebra generators $z_i$ and rational $R$ with $\log|z_i| < R$ on $H_0$. The part of the closed polydisc $\{\max_i \log|z_i| \leq R+1\}$ outside $U$ is compact. Every point $x$ there fails a polynomial hull inequality, so a power of a separating polynomial $a_x$ gives a rational $b_x$ with

$$\sup_{H_0} \log|a_x| < b_x < \log|a_x(x)| - 1.$$

The strict superlevel sets on the right give a finite cover of the compact part outside $U$. Use the maximum of 0, all $\log|z_i| - R$, and the finitely many $\log|a_x| - b_x$ as $g$. Near $H_0$ all the nonconstant terms are negative. Outside $U$ at least one term is strictly larger than one.

If $k$ is countable, the topology on $Y^{\mathrm{an}}$ has a countable base of inequalities on polynomials with rational positive thresholds. It is locally compact by the compact polydisc bounds. A Runge open $U$ therefore admits an increasing compact exhaustion $K_j$ whose interiors cover it, by taking finite unions of closures of a countable relatively compact open cover. Inductively separate the hull of $K_j \cup \{g_{j-1} \leq 1\}$ from the complement of $U$, starting with $K_1$. This hull lies in $U$ by the Runge property. The constructed strict sublevel contains the hull and its closed sublevel stays in $U$. These containments give the assertion. □

The strict inequalities matter. Even for $Y = \mathbb{A}^1$, the open set $\{|z-1| < 1\}$ is contained in $\{|z| \leq 1\}$ and meets $\{|z| = 1\}$. The interior of a closed polynomial sublevel can contain points of equality.

## 13.7 Hartogs radii

Write $p : Y \times \mathbb{A}^1 \to Y$ and let $w$ be the coordinate of the second factor. For an open set $D$ in the analytification containing the zero section, its fibre radius $d_D(y)$ is the supremum of the radii $r > 0$ for which the whole closed disc $\{|w| \leq r\}$ in the fibre over $y$ lies in $D$. This disc is compact. Indeed the values of each polynomial in $w$ are bounded by the maximum of the values of its coefficients at $y$ times the corresponding powers of $r$. The closed conditions in the product of the resulting compact intervals give compactness, just as for affine polydiscs. The bound on a polynomial is attained at the Gauss point of radius $r$ over $y$, using the Gauss norm over the completed residue field $\mathcal{H}(y)$. A small enough positive disc is contained in $D$, by testing an open neighborhood of the origin on finitely many polynomial evaluations.

**Proposition 13.7.1** *Suppose $k$ is countable, $D$ is Runge and contains the zero section. If $-\log d_D$ has logarithmic growth and is finite somewhere, then $-\log d_D \in \mathcal{R}(Y)$. In particular, for an upper semicontinuous function $u$ of logarithmic growth, finite somewhere, the set*

$$D_u = \{u \circ p + \log|w| < 0\}$$

*is Runge if and only if* $u \in \mathcal{R}(Y)$.

***Proof*** Use Lemma 13.6.1 on the product, and put $W_j = \{g_j < 1\} \cap \{|w| < e^j\}$. These opens increase to $D$; let $\Omega_j$ be the open set of base points whose zero belongs to $W_j$. On a fibre over $y \in \Omega_j$, the defining inequalities for $W_j$ are

$$\log|P_\alpha(w)| < b_\alpha, \qquad \log|w| < j, \qquad P_\alpha = \sum_{l\geq 0} a_{\alpha l} w^l,$$

with $a_{\alpha l} \in k[Y]$ and rational $b_\alpha$. The constant terms satisfy their strict inequalities by definition of $\Omega_j$. Applying the Gauss bound to all the other coefficients gives

$$-\log d_{W_j}(y) = \max\left\{-j,\ \max_{\alpha,\, l\geq 1} \frac{\log|a_{\alpha l}(y)| - b_\alpha}{l}\right\}.$$

In fact a radius strictly below the exponential of the negative maximum satisfies all inequalities throughout its disc. A radius strictly above it fails one inequality at its Gauss point, or exceeds the explicit bound on $|w|$. The convention for zero coefficients allows the inner maximum to be $-\infty$. The outer constant makes the function locally convex PL, including where coefficients vanish.

For any $y$ and $r < d_D(y)$ the compact closed disc of radius $r$ is eventually contained in $W_j$. Thus $d_{W_j}(y) \uparrow d_D(y)$. Every compact subset of $Y^{\mathrm{an}}$ also lies in some $\Omega_j$, so the decreasing approximants for $-\log d_D$ are defined on a neighborhood of that compact set. Apply Proposition 13.4.1.

For $D_u$, its radius is $e^{-u}$, with radius $+\infty$ when $u = -\infty$. The set is open by upper semicontinuity and contains the zero section. If $u \in \mathcal{R}(Y)$, then $u \circ p + \log|w|$ is regularizable by affine pullback and addition of a logarithmic term. It is finite at the trivial valuation of the product. Its strict sublevel is Runge by the hull inequality. The other direction follows from the radius calculation. □

*Example 13.7.1* For $u = \frac{1}{2}\log|z|$ on the affine line, $D_u$ is the strict polynomial open set $\{|zw^2| < 1\}$. Over a point with $|z| > 0$, the closed disc of radius $r$ in the fibre lies in $D_u$ precisely when $r < |z|^{-1/2}$, so $d_{D_u} = e^{-u}$. Over $z = 0$ the fibre radius is infinite. The function is regularizable on the full line, with a pole at $z = 0$.

## 13.8 The boundary of a polynomial hull

The following local maximum principle for polynomial hulls lets us use a principal-open estimate at a point of the hull, even when the original compact set is elsewhere.

**Proposition 13.8.1** *Let $K \subset Y^{\mathrm{an}}$ be compact, $x \in \widehat{K}$, and $V$ a relatively compact open neighborhood of $x$ disjoint from $K$. Then $x$ belongs to the polynomial hull of $\partial V \cap \widehat{K}$.*

***Proof*** Otherwise choose a polynomial $f$ and a rational $b$ with

$$\sup_{\partial V \cap \widehat{K}} \log|f| < b < \log|f(x)|.$$

This includes the case of an empty intersection, with supremum $-\infty$. The closed subset of $\partial V$ where $\log|f| \geq b$ is compact and disjoint from $\widehat{K}$. Apply Lemma 13.6.1 with the complement of that closed subset as the neighborhood: take $g \geq 0$ affine Fubini–Study, zero near $\widehat{K}$, and $g > 1$ on that closed subset. For a sufficiently large rational $a > 0$, $ag > \log|f| - b$ on $\partial V$. It holds on an open neighborhood of the boundary as well. Therefore $\max(ag, \log|f| - b)$ on $V$ and $ag$ off $V$ paste to a finite locally convex PL function of logarithmic growth. By Proposition 13.4.1 it is regularizable. It is zero on $K$ and strictly positive at $x$, contradicting the hull inequality.□

## 13.9 Strict supporting functions

We need to make an upper semicontinuous maximum drop off a chosen neighborhood. Countably many polynomial evaluations give a single function with this property throughout $Y^{\mathrm{an}}$.

**Lemma 13.9.1** *Suppose $k$ is countable. There is a nonnegative continuous $\rho \in \mathcal{R}(Y)$ with compact sublevel sets such that for every $x \in Y^{\mathrm{an}}$ and open neighborhood $V$ of $x$, there are $\rho' \in \mathcal{R}(Y)$ and $\delta > 0$ with*

$$\rho' \leq \rho, \qquad \rho'(x) = \rho(x), \qquad \rho' \leq \rho - \delta \quad \textit{outside } V.$$

***Proof*** Put $\lambda(t) = \log(1+e^t)$ for real $t$ and $\lambda(-\infty) = 0$. For $e \in \mathbb{R}$ let $\ell_e$ be the tangent line of $\lambda$ at $e$, extended by $-\infty$ at $-\infty$. For $e = -\infty$, put $\ell_e = 0$. Then $\lambda - \ell_e \geq 0$, with equality at $e$. For finite $e$ the difference tends to $+\infty$ at either end of the real line, since the tangent slope lies strictly between zero and one. It is strictly positive away from $e$ by strict convexity. For $e = -\infty$ the difference is $\lambda$, strictly increasing on the real line. Thus for all $e$ there is a uniform positive drop off every neighborhood of $e$.

For a nonzero polynomial $f$, the function $\lambda(\log|f|)$ is regularizable. Apply the real binomial formula to $s = |f|$. The largest of the $m+1$ terms of $(1+s)^m$ lies between $(1+s)^m/(m+1)$ and $(1+s)^m$. Taking logarithms and dividing by $m$ gives a maximum of terms $(j/m)\log|f| + (1/m)\log\binom{m}{j}$ with uniform error at most $\log(m+1)/m$. The binomial coefficients only contribute real weights. Round them up after adding the error bound, and take $m$ large to obtain rational logarithmic maxima $q_n$ satisfying

$$\lambda(\log|f|) \leq q_n \leq \lambda(\log|f|) + 2^{-n-2}.$$

The functions $q_n + 2^{-n}$ decrease to $\lambda(\log|f|)$, and each belongs to $\mathcal{R}(Y)$. Use decreasing closure in Corollary 13.4.1. The functions $\ell_e(\log|f|)$ are also regularizable. For finite $e$, approximate the tangent slope from both sides by a shrinking interval of positive rationals, take the maximum of the two resulting affine functions with their

common intercept approximated by a decreasing rational sequence, and include an auxiliary term $g_*/j - j$ as above. The maximum decreases for either sign of $\log|f|$, and gives the claimed limit. For $e = -\infty$ the function is zero.

List the nonzero polynomials as $f_j$, using repetitions if the set is finite. Write $h_j = \lambda(\log|f_j|)$. Choose $b_* \geq 1$ affine Fubini–Study and integers $C_j \geq 1$ such that $0 \leq h_j \leq C_j b_*$, by the polynomial growth estimate. Set $\gamma_j = 2^{-j}/C_j$ and $\rho = \sum_j \gamma_j h_j$. The sum converges uniformly on compact subsets and is at most $b_*$. The regularizable functions $\sum_{j\leq n} \gamma_j h_j + 2^{-n} b_*$ decrease to it, so $\rho$ is regularizable. It has compact sublevel sets because a finite generating list occurs among the $f_j$ and each of its log norms is bounded above on a sublevel.

Given $x, V$, find a finite set $J$ of indices and neighborhoods $U_j$ of $e_j = \log|f_j|(x)$ whose simultaneous inverse image lies in $V$. For $j \in J$, replace the summand $h_j$ by

$$\ell_{e_j}(\log|f_j|).$$

Call the resulting sum $\rho'$. The same finite partial sums with the upper tail $2^{-n} b_*$, for $n \geq \max J$, prove regularizability, since the limit is finite at $x$. The replacement does not change its value at $x$. Outside $V$ at least one of the replaced terms drops by a positive constant, and the minimum over the finite set $J$, with the weights $\gamma_j$, gives $\delta > 0$. If $V = Y^{\mathrm{an}}$, take $\rho' = \rho$. □

## 13.10 Gluing on affine opens

The boundary maximum of Proposition 13.8.1 and the strict supporting function of Lemma 13.9.1 combine to glue regularizability over an affine open cover.

**Proposition 13.10.1** *Let $k$ be countable and $Z \subsetneq Y$ closed. Let $u$ be upper semicontinuous of logarithmic growth and regularizable on an affine Zariski open cover of $Y \setminus Z$. Let $\rho$ be as in Lemma 13.9.1. For rational $c > 0$,*

$$(u + c\rho)(x) \leq \sup_K (u + c\rho) \qquad (x \in \widehat{K} \setminus Z^{\mathrm{an}})$$

*for every compact $K \subset Y^{\mathrm{an}}$. If $Z$ is empty, $u \in \mathcal{R}(Y)$.*

***Proof*** Put $u_c = u + c\rho$. Suppose the hull inequality fails at $x$. Choose generators $a_i$ of the ideal of $Z$ and put $v = \max_i \log|a_i|$, using $v = 0$ if $Z$ is empty. It is regularizable and its pole set is $Z^{\mathrm{an}}$. For a sufficiently small positive rational $\varepsilon$, $u_c + \varepsilon v$ still violates the inequality at $x$, since $v(x)$ is finite and $v$ is bounded above on $K$. Take a point $y$ where this perturbation attains its maximum on $\widehat{K}$. The value is finite and strictly above the supremum on $K$, so $y \notin Z^{\mathrm{an}} \cup K$.

Choose a principal open $Y_f$ in an affine member of the cover with $y \in Y_f^{\mathrm{an}}$. By restriction $u$ is regularizable there. Take a relatively compact open neighborhood $V$ of $y$, with closure in $Y_f^{\mathrm{an}} \setminus K$. Using $\rho'$ from Lemma 13.9.1, put $U = u + c\rho' + \varepsilon v$. Since $\rho' \leq \rho$, all three summands have an upper bound of logarithmic growth on

$Y^{\mathrm{an}}$. The function $U$ is finite at $y$ and regularizable on $Y_f^{\mathrm{an}}$. On $B = \widehat{K} \cap \partial V$ it is bounded above by $U(y) - c\delta$, by the strict drop and the maximizing property of $y$. But $y \in \widehat{B}$ by Proposition 13.8.1, so the principal-open inequality Corollary 13.5.1 gives $U(y) \leq \sup_B U$, a contradiction.

If $Z$ is empty, apply the just-proved part on $Y \times \mathbb{A}^1$ to $u \circ p + \log|w|$. Its affine growth and local regularizability on products of the affine covering sets follow from pullback and addition of a log norm. For each rational $c > 0$, the strict sublevel at zero of the function plus $c\rho$ for that product is Runge by the proved hull inequalities. As $c \downarrow 0$ the opens increase to $D_u$ of Proposition 13.7.1. That proposition now gives $u \in \mathcal{R}(Y)$. □

## 13.11 Norms for finite flat affine maps

Let $p : Y \to S$ be finite flat and surjective between normal integral affine schemes of finite type over $k$. Write $d$ for its rank. At an analytic point $x$ of $S$, write the finite $\mathcal{H}(x)$-algebra of the fibre as $\prod_{y \mapsto x} B_y$, and put $d_y = \dim_{\mathcal{H}(x)} B_y$. Each local Artinian factor gives one analytic point by the unique extension of the absolute value to its residue field, using Lemma 2.3.1. The weights sum to $d$ and retain the nilpotent multiplicities.

**Proposition 13.11.1** *For $u \in \mathcal{R}(Y)$, the function*

$$(N_p u)(x) = \sum_{y \mapsto x} d_y u(y)$$

*is in $\mathcal{R}(S)$. The fibrewise formula commutes with complete extensions of the ground field.*

***Proof*** For an affine Fubini–Study function put $u = m^{-1} \max_j(\log|b_j| + \lambda_j)$, with $b_j \in k[Y]$ and rational weights, by clearing denominators. One $b_j$ is 1. On the finite locally free algebra $p_*\mathcal{O}_Y$ take the determinant of multiplication by $\sum_j T_j b_j$. It is a homogeneous polynomial $\sum_\alpha c_\alpha T^\alpha$ of degree $d$ with $c_\alpha \in k[S]$. Its value under the Gauss norm of radii $e^{\lambda_j}$, after specializing at $x$, is

$$\max_\alpha |c_\alpha(x)| e^{\sum_j \alpha_j \lambda_j}.$$

On each local factor $B_y$, the determinant is the field norm of the residue linear polynomial, raised to the sum of the dimensions of the graded pieces of its maximal-ideal filtration over that field. This follows by taking successive quotients of that filtration. After a finite splitting extension the norm is a product of linear polynomials, with inseparable embedding multiplicities. For the factor at $y$, their total multiplicity is $d_y$, and all the factors have Gauss norm $\max_j |b_j(y)| e^{\lambda_j}$ by uniqueness of the valued extension. Multiplicativity gives

$$N_p u = \frac{1}{m} \max_{\alpha} \bigl( \log |c_\alpha| + \textstyle\sum_j \alpha_j \lambda_j \bigr).$$

It is regularizable: it is a finite rational maximum with a constant term from the determinant monomial corresponding to $b_j = 1$.

Apply the formula to decreasing Fubini–Study approximants for a general $u$. The weighted sums decrease to $N_p u$ because the fibres are finite. At the trivial valuation of $k(S)$ the sole point in its fibre is the trivial valuation of $k(Y)$. Thus the limit is finite there, and the decreasing closure in Corollary 13.4.1 proves regularizability.

For compatibility, the algebra at a point over $x$ after complete base extension is the extension of the finite algebra at $x$ to the new completed residue field. The factors descending from one $B_y$ have total dimension $d_y$, by invariance of dimension under field extension. They restrict to $y$ by uniqueness of the absolute value on its finite residue field. Summing their identical values proves compatibility, even on a fibre with nilpotents. □

## 13.12 Specialization of the finite norm

A finite projection of a normal variety need only be flat over a dense open set. At a point of the complement, the dimension of the scheme-theoretic fibre algebra can jump. To extend the norm of a function, we count conjugates of the generic fibre with their repetitions after specialization. This also gives a direct comparison with the flat norm of Proposition 13.11.1.

**Proposition 13.12.1** *Let $p : Y \to S$ be a finite surjective morphism of normal integral affine schemes of finite type over a trivially valued field. Put $d = [k(Y) : k(S)]$. There are positive integers $d_y(p)$, indexed by $y \in Y^{\mathrm{an}}$, with the following properties.*

(1) *At each $x \in S^{\mathrm{an}}$ the $d_y(p)$ for $y \mapsto x$ sum to $d$. Where $p$ is flat they are the dimensions of the local factors of the analytic fibre algebra.*
(2) *If $u \in \mathcal{R}(Y)$, then $N_p u(x) = \sum_{y \mapsto x} d_y(p) u(y)$ belongs to $\mathcal{R}(S)$.*
(3) *For any upper semicontinuous $u : Y^{\mathrm{an}} \to [-\infty, \infty)$, the same weighted sum is upper semicontinuous.*
(4) *There are a normal integral affine scheme $T$, a finite surjective morphism $\pi : T \to S$, finitely many finite morphisms $p_\sigma : T \to Y$ over $S$ and an integer $e \geq 1$ such that, for every function $u : Y^{\mathrm{an}} \to [-\infty, \infty)$, $N_p u \circ \pi^{\mathrm{an}} = e \sum_\sigma u \circ p_\sigma^{\mathrm{an}}$ on $T^{\mathrm{an}}$, where $N_p u(x) = \sum_{y \mapsto x} d_y(p) u(y)$.*

***Proof*** Write $A = k[S]$, $B = k[Y]$, $K = k(S)$, and $E = k(Y)$. Choose a finite normal extension $M/K$ containing the images of all $K$-embeddings of $E$ into an algebraic closure, and let $D$ be the integral closure of $A$ in $M$. The normalization $\operatorname{Spec} D \to S$ is finite. Indeed, clear denominators in generators of the finite extension $M/K$ to obtain an integral finite $A$-algebra with fraction field $M$, then normalize it [Stacks, Tag 0BXR]. Each embedding $\sigma : E \to M$ sends $B$ into $D$. It therefore defines a finite map $p_\sigma : \operatorname{Spec} D \to Y$ over $S$. Indeed, any finite $A$-module generating set for

$D$ also generates it over $\sigma(B)$. These maps are surjective by integrality. Denote the inseparable degree of $E/K$ by $e$.

For an analytic lift $z$ of $x$ to $\operatorname{Spec} D$, count each $p_\sigma^{\mathrm{an}}(z)$ with weight $e$. There is such a lift by the finite extension of absolute values. These counts are independent of the lift and of $M$. To see this, choose a finite list $b_1, \dots, b_l$ generating $B$ as an $A$-algebra and form the homogeneous polynomial

$$Q(U,T) = \operatorname{Norm}_{E(U,T)/K(U,T)}\Big(U - \sum_j T_j b_j\Big) = \prod_\sigma \Big(U - \sum_j T_j \sigma(b_j)\Big)^e .$$

The product expression follows by passing to the local Artinian factors over the splitting field $M$ and taking successive quotients by powers of their maximal ideals. All coefficients are integral over $A$ and lie in $K$. Thus $Q \in A[U,T]$, since $A$ is normal.

Specialize $Q$ at $x$, and consider its linear factors over an algebraic closure of $\mathcal{H}(x)$, with the unique absolute value on each finite subextension. For any lift $z$, the factorization displayed above specializes to such a factorization via an embedding of $\mathcal{H}(z)$ into the algebraic closure. Each coefficient tuple determines the corresponding multiplicative seminorm on $B$, by evaluation in the valued field, because the $b_j$ generate $B$ over $A$. The factors are monic in $U$, so unique factorization shows that the multiset of seminorms with their multiplicities is independent of the lift. This also proves independence of the normal extension. Let $d_y(p)$ be the multiplicity of $y$ in this multiset. Every $y$ occurs: choose $M$ to contain $E$, and lift $y$ along the finite surjective map for the inclusion $E \hookrightarrow M$. The sum of the weights is $d$ by the degree of $Q$.

Over the flat locus $Q$ agrees with the characteristic polynomial of multiplication by the indicated linear combination on $p_*\mathcal{O}_Y$, since they agree generically and the base is integral. The Artinian filtration argument, now applied to the specialized characteristic polynomial, counts the factors at $y$ with total multiplicity equal to the dimension of the local fibre algebra at $y$. This proves the flat-locus claim.

For an affine Fubini–Study function $u = m^{-1}\max_j(\log|a_j| + \lambda_j)$, take one $a_j = 1$. The polynomial $\operatorname{Norm}_{E(T)/K(T)}(\sum_j T_j a_j) = \sum_\alpha c_\alpha T^\alpha$ likewise has coefficients in $A$. Evaluate it in $\mathcal{H}(z)$ for an analytic lift $z$ of $x$. The Gauss norm of its linear factor for $\sigma$, at radii $e^{\lambda_j}$, is $\max_j e^{\lambda_j}|a_j(p_\sigma^{\mathrm{an}}(z))|$. Multiplying and using the definition of the multiplicities gives

$$N_p u = \frac{1}{m}\max_\alpha\Big(\log|c_\alpha| + \sum_j \alpha_j \lambda_j\Big).$$

The pure monomial corresponding to $a_j = 1$ has coefficient 1. The result is a finite rational maximum and belongs to $\mathcal{R}(S)$ by adjoining the auxiliary Fubini–Study terms as in Section 13.4. For a decreasing sequence approximating $u \in \mathcal{R}(Y)$, the weighted finite sums decrease to $N_p u$. The limit at the generic trivial valuation is finite because the sole point of its fibre is the trivial valuation of $E$, where $u$ is finite. Decreasing closure of $\mathcal{R}(S)$, from Corollary 13.4.1, proves the second assertion.

Put $T = \operatorname{Spec} D$, which is normal and integral, and let $\pi : T \to S$ be the finite map above; it is surjective by integrality. By the definition of the multiplicities, for

every function $u$ the pullback $N_p u \circ \pi^{\mathrm{an}}$ is the sum of the $e\, u \circ p_\sigma^{\mathrm{an}}$. This proves the last assertion. For upper semicontinuous $u$ this pullback is upper semicontinuous, since finite sums preserve upper semicontinuity for these values. The map to $S^{\mathrm{an}}$ is closed and surjective, hence is a quotient map. Closedness follows from properness, or directly by bounding the norms of generators of $D$ using monic equations over any compact subset of $S^{\mathrm{an}}$. Therefore $N_p u$ is upper semicontinuous. □

*Example 13.12.1* Suppose $\operatorname{char} k \neq 2$. The simultaneous sign change on $B = k[a, b]$ has invariant ring $A = k[a^2, ab, b^2]$. The ring $A$ is normal: an element in its fraction field integral over $A$ lies in $B$ by normality of $B$ and is fixed by the sign change. The field extension has degree two, since $k(a, b) = k(a^2, ab, b^2)(a)$ and the sign change acts nontrivially. Over the origin the fibre algebra $k[a, b]/(a^2, ab, b^2)$ has dimension three. There is only one analytic point above the origin, and its specialized norm weight is two.

## 13.13 Local potentials of semiample classes

A function with values in $[-\infty, \infty)$ on an analytic open set is *locally psh* here if each point has an open neighborhood on which it is a decreasing limit of a sequence of locally convex PL functions. In particular such a function is upper semicontinuous. Locally convex PL functions and their local decreasing limits are preserved under algebraic analytic pullback, since a regular unit remains invertible on the inverse image of its domain. The same assertion holds for restriction of scalars along an extension of complete valued fields. For a separated scheme $X$ of finite type over $k$, a function on an open subset of $X^{\mathrm{an}}$ is *weakly psh* if its pullback to a proper birational modification of $X$ is locally psh on the inverse image of the open subset. We call the modification a *witness*. The next lemma shows that weakly psh functions are upper semicontinuous.

**Lemma 13.13.1** *Let $X$ be separated and of finite type over $k$, and let $\mu : T \to X$ be proper and surjective. Then $\mu^{\mathrm{an}}$ is closed and surjective. Consequently, if $U \subset X^{\mathrm{an}}$ is open and $u : U \to [-\infty, \infty)$ has upper semicontinuous pullback to $(\mu^{\mathrm{an}})^{-1}(U)$, then $u$ is upper semicontinuous. In particular, every weakly psh function on an open subset of $X^{\mathrm{an}}$ is upper semicontinuous.*

***Proof*** The scheme $T$ is separated and of finite type over $k$ [Stacks, Tags 01KU and 01T3]. For surjectivity let $x = (\eta, v) \in X^{\mathrm{an}}$. The base change of $\mu$ along $\operatorname{Spec} \mathcal{H}(v) \to \operatorname{Spec} \kappa(\eta) \to X$ is a nonempty scheme proper over $\mathcal{H}(v)$ [Stacks, Tags 01S1 and 01W4]. A maximal ideal of the coordinate ring of a nonempty affine open subset gives a point $q$ of it with $\kappa(q)$ finite over $\mathcal{H}(v)$ [Stacks, Tag 00FV]. By Lemma 2.3.1 the absolute value of $\mathcal{H}(v)$ extends to $\kappa(q)$. Let $\xi \in T$ be the image of $q$. Then $\mu(\xi) = \eta$, and the composite $\kappa(\eta) \to \kappa(\xi) \to \kappa(q)$ is $\kappa(\eta) \to \mathcal{H}(v) \to \kappa(q)$. Restricting the extended absolute value to $\kappa(\xi)$ therefore gives a point of $T^{\mathrm{an}}$ over $x$.

If $f : A \to B$ is a closed map and $B' \subset B$, then $f^{-1}(B') \to B'$ is closed, since $f(C \cap f^{-1}(B')) = f(C) \cap B'$ for $C \subset A$. By Nagata's compactification theorem [Stacks, Tags 0ATT and 0F41], $X$ is an open subscheme of a scheme $\overline{X}$ proper over $k$. The morphism $T \to \overline{X}$ is separated and of finite type, since $T$ and $\overline{X}$ are separated and of finite type over $k$ [Stacks, Tags 01KV, 01T8 and 03GI]. The same theorem, applied over $\overline{X}$, gives an open immersion of $T$ into a scheme $\overline{T}$ and a proper morphism $\bar{\mu} : \overline{T} \to \overline{X}$ extending $\mu$. Put $T_X = \bar{\mu}^{-1}(X)$. It is proper over $X$ [Stacks, Tag 01W4]. Since $T$ is proper over $X$, the open subset $T$ of $T_X$ is also closed [Stacks, Tag 01W6]. The analytification of a Zariski open subset consists of the points with support in it. Hence $T^{\mathrm{an}}$ and $(T_X \setminus T)^{\mathrm{an}}$ are complementary open subsets of $T_X^{\mathrm{an}} = (\bar{\mu}^{\mathrm{an}})^{-1}(X^{\mathrm{an}})$. The scheme $\overline{T}$ is proper over $k$ [Stacks, Tag 01W3]. By Theorem 1.3.1, $\bar{\mu}^{\mathrm{an}}$ is a continuous map from a compact space to a Hausdorff space, hence closed. By the preceding remark, its restriction $T_X^{\mathrm{an}} \to X^{\mathrm{an}}$ is closed, and so is the further restriction $\mu^{\mathrm{an}}$ to the closed subset $T^{\mathrm{an}}$.

For the consequence, the restriction $(\mu^{\mathrm{an}})^{-1}(U) \to U$ is closed by the same remark and surjective. For real $c$, the set $\{u \geq c\}$ is the image of the closed set $\{u \circ \mu^{\mathrm{an}} \geq c\}$, and is therefore closed in $U$. Finally, the pullback of a weakly psh function to its witness is locally psh, hence upper semicontinuous. The witness is proper and surjective, since its image is closed and dense. □

**Lemma 13.13.2** *Over a countable $k$, let $P$ be nonempty, reduced and projective, and let $L$ be semiample and rational. Suppose $mL$ is globally generated and $s \in H^0(P, mL)$. If $\phi \in \mathrm{PSH}(P, [L])$, then $\phi - m^{-1} \log |s|$ is locally psh on $P_s^{\mathrm{an}}$. More precisely, for a globally generated ample $H$, and $a \in H^0(P, H)$, it has a decreasing sequence of locally convex PL approximants on $P_s^{\mathrm{an}} \cap P_a^{\mathrm{an}}$.*

***Proof*** The analytification of a finite type $k$-scheme is second countable, using finitely many affine charts and countably many polynomial evaluations with rational thresholds. Use Proposition 6.2.1 to choose a decreasing net of finite rational Fubini–Study functions $\phi_i$ in classes tending to $[L]$. For every rational $b$, the opens $\{\phi_i < b\}$ cover $\{\phi < b\}$, so a countable subfamily still covers it. Take the union of the resulting index sets over all rational $b$, and choose successively later indices that eventually dominate that countable list. At the $j$-th step also require the index to be late enough that $L + 2^{-j}H - L_i$ is ample, where $L_i$ is the rational line bundle in the chosen presentation. This uses convergence of numerical classes and openness of the ample cone. The selected sequence decreases to $\phi$. By (6.3) its $j$-th term is a finite rational Fubini–Study function for $L + 2^{-j}H$.

Subtract the local reference $m^{-1} \log |s| + 2^{-j} \log |a|$ on the intersection. After common powers, the result is a maximum of positive multiples of log norms of regular functions plus rational constants, with a finite value everywhere on the intersection. It is locally convex PL. Since $\log |a| \leq 0$, the correction $-2^{-j} \log |a|$ decreases to zero. Thus the resulting functions decrease to $\phi - m^{-1} \log |s|$. Sections $a$ of $H$ that are nonzero at the support cover the analytic points under consideration. □

**Proposition 13.13.1** *Let $k$ be countable and let $\mu : T \to X$ be a projective birational morphism of normal integral projective schemes. Let $L$ be a rational ample line bundle on $X$, and let $\phi : X^{\mathrm{an}} \to [-\infty, \infty)$ be a function whose pullback is $[\mu^* L]$-psh. For a nonzero section $s$ of a very ample power $mL$, put $Y = X_s$. Then $\phi - m^{-1} \log |s|$ is weakly psh on $Y^{\mathrm{an}}$ and has logarithmic growth. For each Zariski open $W \subset X$ over which $\mu$ is an isomorphism, it is regularizable on an affine open cover of $Y \cap W$.*

***Proof*** The function $\phi$ is upper semicontinuous by Lemma 13.13.1, since $\mu$ is proper and surjective. The pullback of $\phi$ is bounded above by compactness. Choose a basis $(s_j)$ of the very ample linear system containing $s$. On $Y^{\mathrm{an}}$ we have

$$-m^{-1} \log |s| = m^{-1} \max_j \log |s_j / s|.$$

The ratios generate $k[Y]$, and one of them is 1. The displayed function is thus affine Fubini–Study, which proves the growth bound.

Apply Lemma 13.13.2 to $\mu^* L$ and $\mu^* s$, using a globally generated ample line bundle on $T$. This proves the weak psh assertion. Over the isomorphism locus choose affine principal opens of $Y$ contained in the images of the charts $T_a$ supplied by that lemma. On each of these affine opens, the same decreasing sequence of locally convex PL functions is available on the whole analytification. There is also a logarithmic growth bound, by pulling back the bound from $Y$ and using Proposition 13.1.1. Apply Proposition 13.4.1. The function is finite at the trivial point since $\phi \circ \mu^{\mathrm{an}}$ is psh and the section is nonzero. This gives the stated regularizability. □

## 13.14 Reduction to a countable field

The rational obstacle calculation descends along a field extension, because it uses the section spaces of one marked line bundle. This allows a countability hypothesis in the local proofs to be discharged at the level of projective envelopes.

**Proposition 13.14.1** *Suppose for every countable algebraically closed trivially valued field, normal integral projective schemes have the envelope property in rational ample classes. Then the same holds over any algebraically closed trivially valued field.*

***Proof*** Let $k$ be algebraically closed, $X/k$ normal and integral projective, and $L$ rational ample. By Proposition 12.1.1 it suffices to use a rational PL obstacle. After scaling write it as $\varphi_D$ for a vertical Cartier divisor on a normal test configuration $\mathcal{X}$ of $X$ dominating the product. The marked polarization is $\mathcal{L} = L + D$. All the finite presentation data are defined over a finitely generated subfield of $k$. Take its algebraic closure $k_0$ inside $k$, a countable algebraically closed field. This yields $X_0$, $L_0$, $D_0$, $\mathcal{X}_0$, $\mathcal{L}_0$ whose scalar extensions are the chosen data. Normality and integrality descend along the field extension, as do the marking and flatness. Ampleness descends by faithful flatness, or by descending an embedding given by a power of $L$.

Write $r : X^{\mathrm{an}} \to X_0^{\mathrm{an}}$ for restriction of scalars. For every $n$, the base ideal of $n\mathcal{L}_0$ pulls back to the base ideal of $n\mathcal{L}$. Indeed global sections commute with flat field extension, by the finite affine equalizer for sections, and the images of the evaluation maps commute with that extension. Gauss valuation also commutes with restriction of scalars. Thus the model potential of each base ideal upstairs is its potential downstairs composed with $r$. The same holds for the obstacle itself. The pointwise formula of Proposition 12.4.1 gives

$$P_{[L]}(\varphi_D) = P_{[L_0]}(\varphi_{D_0}) \circ r.$$

The downstairs envelope is psh by the assumed countable-field theorem. Pull its fixed-class Fubini–Study approximation back to $X$. This gives a decreasing Fubini–Study approximation in class $[L]$ of the right-hand side, finite at the trivial point. Hence $P_{[L]}(\varphi_D)$ is psh, as required. □

# Chapter 14
# Extension across an exceptional set

On the affine line, raising the zero function at the origin preserves upper semicontinuity and leaves the approximation problem off the origin unchanged. The Gauss point of any positive-radius disc centred at the origin has value zero, while its polynomial hull contains the origin. To extend regularizability across a closed subset we need a hypothesis that rules out such a change in value.

We work with weakly psh functions on normal varieties over a countable algebraically closed trivially valued field. For projective envelopes the countability hypothesis is removed by Proposition 13.14.1. The main result, Theorem 14.3.1, extends regularizability of a weakly psh function of logarithmic growth on a normal affine variety across a proper closed subset. The proof uses a disc over a complete extension of the ground field to identify each missing value of a finite norm (Propositions 14.1.1 and 14.2.1). The hull inequality for this norm then gives regularization on the original variety. We apply the result to a function whose pullback to a projective birational modification is psh (Corollary 14.3.1). The alteration argument of Chapter 16 uses this descent.

## 14.1 The value at a rigid point of a curve

In this section $F$ is a complete algebraically closed nontrivially valued extension of the ground field. The definitions of locally convex PL and locally psh apply to the analytification of an algebraic $F$-scheme, using algebraic units on that scheme. We only need a maximum principle in a disc around an $F$-point.

**Lemma 14.1.1** *Let $C/F$ be a smooth algebraic curve and $p \in C(F)$. The point $p$ has arbitrarily small closed analytic disc neighborhoods with $p$ at the origin. The restrictions of regular functions on the affine chart used to construct such a disc are convergent power series on the disc.*

***Proof*** Use a smooth affine chart at $p$ with a local parameter $z$ and equations $G(z, U) = 0$ whose Jacobian with respect to the remaining coordinates $U$ is in-

vertible at $p$. This is the local Jacobian presentation of a smooth curve. Translate $p$ to the origin and multiply the equations by a constant inverse matrix to write them as $U + H(z, U) = 0$, where $H(0, 0) = 0$ and the terms linear in $U$ at $z = 0$ vanish. Choose a sufficiently small radius $s$ for $U$ in $|F^\times|$. Then choose a sufficiently small radius $r$ for $z$ in $|F^\times|$. On these bounds the polynomial map $U \mapsto -H(z, U)$ sends the closed polydisc of $U$-radius $s$ to itself and has Lipschitz constant strictly below one. This follows term by term: the terms containing $U$ either have degree at least two in $U$ or have a positive power of $z$. The terms independent of $U$ tend uniformly to zero with $r$.

Iteration from zero in the Banach algebra of the closed $z$-disc gives a tuple of convergent power series $U(z)$. At every point of that disc, specialization gives the unique solution with the specified bound in its completed residue field. Conversely, a point of the chart on the specified bounds satisfies the equations in its own completed residue field, so its coordinates $U$ are the evaluations of the same power series. This identifies the two sets of points homeomorphically by the $z$ coordinate and gives the claimed restriction of functions. Any elements inverted in the affine presentation have nonzero value at the origin. On small enough bounds each is a nonzero constant plus a term of smaller norm, so its inverse is also represented by a convergent series. Make both bounds as small as needed to lie in a prescribed neighborhood and to keep the origin in the interior. □

**Proposition 14.1.1** *Let $u$ be locally psh on an open neighborhood of $p \in C(F)$, with $C/F$ smooth. Then*

$$u(p) = \limsup_{\substack{x \to p \\ x \neq p}} u(x).$$

*In particular two locally psh functions on $C^{\mathrm{an}}$ that agree off a finite set of rigid points agree everywhere.*

***Proof*** First we prove a boundary maximum for a finite locally convex PL function $v$ on one of the closed discs $D$ of Lemma 14.1.1, with a neighborhood of $D$ in its domain. Write $R$ for its radius and $\eta_{a,r}$ for the Gauss point of the $z$-disc centred at $a$ of radius $r > 0$. Put

$$\operatorname{diam}(x) = \inf_{a \in F} |z - a|(x).$$

This is upper semicontinuous. Choose a point $x$ of maximum of $v$ on the compact disc at which $\operatorname{diam}(x)$ is maximal among the maximizers.

Around $x$, pick a touching summand $\ell = b + \sum_i b_i \log |f_i|$ from a local presentation of $v$. Thus $\ell(x) = v(x)$ and $\ell \leq v$ near $x$. The functions $f_i$ are units near the support of $x$. Write each as a quotient of two regular functions on the affine curve chart, with numerator and denominator nonzero at the support of $x$. Such a presentation comes from the local ring at that support. Their restrictions to the disc are convergent series. Approximate them by polynomials with uniform error less than half their respective absolute values at $x$. The ultrametric inequality implies their norms agree with those of these polynomials on a neighborhood of $x$. As $F$ is algebraically closed, after

factoring those finitely many polynomials we can write $\ell = b' + \sum_j c_j \log|z - a_j|$ near $x$, with the displayed log norms finite there.

Write $r = \operatorname{diam}(x)$. If the infimum is attained at $a$, then for every $c \in F$,

$$|z - c|(x) = \max(r, |a - c|).$$

For unequal terms this follows from the ultrametric inequality, and in the equal case the definition of $r$ supplies the reverse inequality. By factoring polynomials, $x$ is therefore $\eta_{a,r}$ if $r > 0$, and the rigid point $a$ if $r = 0$.

If the infimum is not attained, choose $a_n$ with $r_n = |z - a_n|(x)$ decreasing strictly to $r$. The centres and radii can be chosen to satisfy $|a_n| \le R, r_n \le R$, since $|z|(x) \le R$. For each $c$, eventually $r_n < |z - c|(x)$, so $|a_n - c| = |z - c|(x)$. Hence $\eta_{a_n,r_n} \to x$ and the finitely many log norms in $\ell$ eventually have exactly their values at $x$. Thus $v$ attains its maximum at one of these points of larger diameter, a contradiction. For a rigid $x = a$, all the displayed roots have positive distance from $a$. Along $\eta_{a,t}$ for small $t > 0$ the same log norms are constant. This gives the same contradiction.

It remains to consider $x = \eta_{a,r}$ with $0 < r < R$. Increase the radius along $\eta_{a,re^t}$ for small $t \ge 0$. The outgoing slope of $\log|z - a_j|$ is 1 if $|a_j - a| \le r$, and 0 otherwise. Partition the roots $|a_j - a| \le r$ into classes by the relation $|a_j - a_i| < r$. For each class choose a representative $c$. Along the path $\eta_{c,re^{-t}}$ its outgoing slope is $-1$ on the roots in that class and zero on the others. All these paths start at $x$. Thus the sum of the outgoing slopes of $\ell$, on the outward path and these finitely many inward paths, is zero. Because $\ell$ touches the maximum at $x$, each slope is nonpositive. The outward slope is therefore zero, and a short outward segment has the same value of $\ell$, hence of $v$, again contradicting maximality of the diameter. The only remaining possibility is $r = R$, which forces $x = \eta_{0,R}$. This proves the boundary maximum.

Now choose a closed disc about $p$ in one open neighborhood on which $u$ is the limit of a decreasing sequence of locally convex PL functions. The boundary maximum applied to each approximant gives $u(p) \le u(\eta_{0,R})$ for every sufficiently small $R$. The Gauss points tend to $p$ as $R \downarrow 0$, and upper semicontinuity gives the reverse inequality for the upper limit over the punctured neighborhood. This proves the formula. Removing finitely many other rigid points from a neighborhood of each missing point gives the uniqueness claim. □

## 14.2 The norm across a closed subset

We apply the rigid-point calculation on curves over extensions of completed residue fields. This identifies the norm on a nonflat fibre while retaining the same multiplicities as in Proposition 13.12.1.

**Proposition 14.2.1** *Let $k$ be countable and algebraically closed. Let $p : Y \to S$ be a finite surjective morphism from a normal integral affine variety to a nonempty principal open subset of $\mathbb{A}^n_k$, with $n \ge 1$. Assume $u$ on $Y^{\mathrm{an}}$ is weakly psh and has logarithmic growth. If $Z \subsetneq Y$ is closed and $Y \setminus Z$ has an affine open cover on which*

*$u$ is regularizable, then*

$$x \longmapsto N_p u(x) = \sum_{y \mapsto x} d_y(p) u(y),$$

*with the multiplicities $d_y(p)$ of Proposition 13.12.1, belongs to $\mathcal{R}(S)$.*

***Proof*** Choose $0 \neq h \in k[S]$ so that $S_h$ avoids $p(Z)$ and $p$ is flat over $S_h$. Such an open exists because $p$ is finite, the image of $Z$ has smaller dimension than $S$, and the finite module $p_* \mathcal{O}_Y$ is free after inverting a suitable nonzero element. Write $Y_h = p^{-1}(S_h)$. Intersect the affine cover with $Y_h$; the intersections are affine, and regularizability is preserved under restriction by the pullback property of $\mathcal{R}$. The restriction of $u$ has logarithmic growth by Proposition 13.1.1, and $u$ is upper semicontinuous by Lemma 13.13.1. By Proposition 13.10.1 with the empty exceptional set, $u|_{Y_h^{\mathrm{an}}}$ is regularizable. Its norm $q_0$ is regularizable on $S_h$ by Proposition 13.11.1, and agrees there with $N_p u$ by part (1) of Proposition 13.12.1. The growth comparison for finite maps shows $q_0 \leq g + B$ on $S_h^{\mathrm{an}}$, for an affine Fubini–Study function $g$ on $S$ and a constant $B$, after enlarging the rational coefficient of $g$ if necessary.

First extend $q_0$ as a regularizable function on $S$. Write $S = (\mathbb{A}^n)_f$ with $f \neq 0$ of degree $e$. In $P = \mathbb{P}^n$ put $H = \mathcal{O}_P(e+1)$, and let $s$ be the product of $X_0$ with the homogenization of $f$, so $P_s = S$. Every regular function on $S$ is $\tau / s^b$ for a section $\tau \in H^0(P, bH)$ for some integer $b \geq 0$. It follows that $g$ and every positive rational multiple of $\max(0, \log|h|)$ are bounded above by a constant plus a multiple of $-\log|s|$.

Choose positive rational $c_i \downarrow 0$. By Proposition 13.5.1, the extension $w_i$ of $q_0 + c_i \log|h|$ by $-\infty$ belongs to $\mathcal{R}(S)$. The preceding bounds give $w_i \leq -a_0 \log|s| + B_0$ with $a_0, B_0$ independent of $i$, since $c_i \leq c_1$. Choose a positive rational $a > a_0$. Apply Lemma 13.4.3 to $w_i$, with $P, H, s, \kappa = a_0$ and $c = a$; the local approximation on compacts in $S^{\mathrm{an}}$ holds since $w_i \in \mathcal{R}(S)$. It gives $[aH]$-psh functions $\phi_i$ equal to $w_i + a \log|s|$ on $S^{\mathrm{an}}$ and to $-\infty$ off $S^{\mathrm{an}}$. They are bounded above by $B_0$.

By Corollary 12.5.1 and Theorem 12.6.1, the tail envelope

$$\psi_j = \left(\sup_{i \geq j} \phi_i\right)^*$$

is $[aH]$-psh. On $S_h^{\mathrm{an}}$ the supremum before upper regularization equals the upper semicontinuous function

$$q_0 + a \log|s| + \max(c_j \log|h|, 0).$$

Thus $\psi_j$ decreases to $\psi$, whose restriction to that open set is $q_0 + a \log|s|$. The limit is finite at the trivial point by Corollary 13.2.1, so Theorem 6.2.2 makes $\psi$ psh. By Lemma 13.13.2, with powers of $s$ as the two reference sections, the function $q = \psi - a \log|s|$ on $S^{\mathrm{an}}$ is a decreasing limit of locally convex PL functions. It has logarithmic growth: $\psi \leq B_0$, and $-\log|s|$ is the maximum of the log norms of ratios of the generating sections of $H$ with $s$, including 1. Use Proposition 13.4.1 to get $q \in \mathcal{R}(S)$.

We now identify $q = N_p u$ at each $x \in S^{\mathrm{an}}$ outside $S_h^{\mathrm{an}}$. Choose a complete algebraically closed nontrivially valued extension $F$ of $\mathcal{H}(x)$. To obtain one, adjoin a variable with a nontrivial Gauss norm and complete if the original norm is trivial, then complete an algebraic closure. The latter completion is algebraically closed: approximate a given monic polynomial by monic polynomials of the same degree in the algebraic closure. Their roots are uniformly bounded by their coefficients. If $b$ is a root of one approximant of degree $m$, the ultrametric coefficient bound makes the next approximant small at $b$. Its linear factorization gives a root at distance at most the $m$-th root of that small value. Choose successive roots this way. They are Cauchy, and their limit is a root of the given polynomial.

The point $x$ induces an $F$-point $x_F \in S_F$. Let $T$, $\pi$, the $p_\sigma$ and $e$ be as in part (4) of Proposition 13.12.1, so that

$$N_p u \circ \pi^{\mathrm{an}} = e \sum_\sigma u \circ p_\sigma^{\mathrm{an}} \qquad \text{on } T^{\mathrm{an}}. \tag{14.1}$$

Choose $z \in T_F(F)$ over $x_F$. Take an affine line through $x_F$ meeting $(S_h)_F$, and let $C$ be its intersection with $S_F$. Such a line exists by taking a line between $x_F$ and a point of $(S_h)_F(F)$. Each component of $T_F \times_{S_F} C$ is a curve dominating $C$. Indeed $T_F$ is integral of dimension $n$, since $T$ is integral over an algebraically closed field. The inverse image is cut out by $n-1$ affine-linear equations. The height bound gives dimension at least one for each component, and finiteness over $C$ gives dimension at most one. Choose the normalization $C_1$ of a reduced component through $z$ and a point $z_1$ above $z$. This is a smooth algebraic curve over $F$.

Let $\mu : Y' \to Y$ be a proper birational witness to weak plurisubharmonicity of $u$. Each $p_\sigma$ gives a morphism $C_1 \to Y_F$. Their pullbacks of $Y'_F$ along $C_1$ have a proper surjective fibre product over $C_1$. In its generic fibre choose a closed point. The normalization $C_2$ of its reduced closure is a smooth curve finite and surjective over $C_1$: it is proper, integral of dimension one and dominant, so its fibres have dimension zero. All the conjugate maps from $C_2$ lift to $Y'_F$. Choose $z_2 \in C_2(F)$ over $z_1$.

Pull $q$ back to $C_2^{\mathrm{an}}$ by the map to $S_F$ and restriction to $S^{\mathrm{an}}$. Do the same for $N_p u$. The first pullback is locally psh by the affine approximation of $q$. The second is locally psh by (14.1), since all the $u \circ p_\sigma$ pull back through the witness on $C_2$. Away from the finitely many rigid points over $C \setminus (S_h)_F$, these two functions agree because $q = q_0$ there. By Proposition 14.1.1 they agree on $C_2^{\mathrm{an}}$. At $z_2$ this is precisely $q(x) = N_p u(x)$. The identification over $S_h$ was already proved. □

## 14.3 Regularization across the exceptional set

We reduce the extension to finite projections for which the exceptional set is a closed subscheme of the target. A polynomial partition then separates that set from the other points of its inverse image.

**Lemma 14.3.1** *Let $Y$ be integral and affine of dimension $n \geq 1$ over an algebraically closed field. Let $Z \subsetneq Y$ be a reduced closed subscheme and $z$ a smooth closed point of $Z$. There is an affine open neighborhood $U$ of $z$ with a finite surjective map onto a principal open of $\mathbb{A}^n$ that restricts to a closed immersion on $Z \cap U$.*

***Proof*** Embed $Y$ as a closed subscheme of $\mathbb{A}^N$. In the hyperplane at infinity of $\mathbb{P}^N$, choose a projection centre of dimension $N - n - 1$. Require that it miss the intersection of the closure of $Y$ with infinity, the closure of the directions from $z$ towards other points of $Z$, and the directions of the embedded tangent space of $Z$ at $z$. These three closed sets have dimension at most $n - 1$: the first is a hyperplane section, the second is the closure of the image of $Z \setminus \{z\}$ under projection from $z$, and the third is linear of dimension $\dim_z Z - 1$. One can choose $n$ linear equations cutting them out of the centre by successive general hyperplanes. At each step avoid containing any component of the current intersection, including isolated points. If $N = n$ the centre is empty.

Projection extends to a morphism from the projective closure of $Y$ to $\mathbb{P}^n$. The hyperplane line bundle on the target pulls back to the hyperplane bundle on the closure. A positive-dimensional projective fibre would therefore carry a line bundle that is both ample and trivial, which is impossible by restricting to an integral curve of the fibre. Thus the map is proper with finite fibres and is finite. Its image is all of $\mathbb{P}^n$ by dimension, and the inverse image of the affine chart is $Y$. Write $p : Y \to \mathbb{A}^n$ for the affine projection.

The finite morphism on $Z$ has a single point over $p(z)$, by avoidance of the secant directions. At that point it is injective on tangent spaces by the condition on the tangent directions. Hence its fibre at $p(z)$ is $\operatorname{Spec} k$: the cotangent space of that local Artinian fibre vanishes, and Nakayama's lemma kills its maximal ideal. The cokernel of the map from the target coordinate ring to the finite algebra of $Z$ is a finite module and vanishes after tensoring with the residue field of $p(z)$. It therefore vanishes on a principal neighborhood of $p(z)$. Take $U$ to be its inverse image. The restricted morphism on $Z \cap U$ is then a closed immersion. □

**Proposition 14.3.1** *Assume $k$ is countable and algebraically closed. Let $p : Y \to S$ satisfy the hypotheses of Proposition 14.2.1, and let $Z \subsetneq Y$ be reduced and closed. Suppose that $p|_Z$ is a closed immersion and $Z' = p^{-1}(p(Z)) \setminus Z$ is closed in $Y$. If $u$ is weakly psh of logarithmic growth and regularizable on an affine open cover of $Y \setminus Z$, then $u \in \mathcal{R}(Y)$.*

***Proof*** The function $u$ is upper semicontinuous by Lemma 13.13.1. Let $\rho$ be the supporting function from Lemma 13.9.1. First we prove the hull inequality for $u_c = u + c\rho$, with $c > 0$ rational, on each compact $K \subset Y^{\mathrm{an}}$. If it fails, choose a maximum point $x$ of $u_c$ on the compact hull $\widehat{K}$. After subtracting a constant from $u$, suppose $u_c(x) = 0$ and choose a rational $\delta > 0$ with $\sup_K u_c \leq -\delta$. Then $x \notin K$, and Proposition 13.10.1 gives $x \in Z^{\mathrm{an}}$.

Choose $f_1, f_2 \in k[Y]$ with $f_1 + f_2 = 1$, $f_1|_{Z'} = 0$ and $f_2|_Z = 0$. The Zariski closed sets are disjoint, so their ideals are comaximal. The opens

$$O_1 = \{\log|f_2| < -1\}, \qquad O_2 = \{\log|f_1| < -1\}$$

are disjoint; $|f_1| = 1$ on $O_1$ and $|f_2| = 1$ on $O_2$. The analytic fibre at $p(x)$ meets $Z^{\mathrm{an}}$ only at $x$. Cover the rest of this finite fibre by a relatively compact open $A \subset O_2$. Take a rational $M \geq 0$ with $u_c \leq M$ on $A$. Take an open neighborhood $V_x \subset O_1$ of $x$ on which $u_c < \varepsilon = \delta/(2d)$, where $d$ is the degree of $p$. Since $p$ is finite, hence proper [Stacks, Tag 01WN], and surjective, $p^{\mathrm{an}}$ is closed by Lemma 13.13.1. So there is an open neighborhood $V$ of $p(x)$ with $p^{-1}(V) \subset V_x \cup A$. Choose a relatively compact open neighborhood $B$ of $x$ whose closure is contained in $V_x \cap p^{-1}(V) \setminus K$. Obtain $\rho' \leq \rho$ touching at $x$ and dropping by at least $\delta' > 0$ outside $B$, by Lemma 13.9.1.

Put $u' = u + c\rho'$ and

$$\eta = \max(\delta \log|f_2|, -\delta), \qquad u^\sharp = \max\bigl(u' + (M+1)\log|f_1|, \eta\bigr).$$

The function $u^\sharp$ has logarithmic growth and is weakly psh with the witness of $u$. Indeed the regularizable functions are locally psh, the log norm of a regular function is a decreasing limit of its finite lower truncations, and finite maxima and sums commute with the decreasing limits used here. Pullback of locally psh functions preserves these properties. By the same finite maximum and sum rules for regularizable functions, $u^\sharp$ is regularizable on the same affine cover off $Z$. On $V_x$ it equals $\max(u', -\delta) < \varepsilon$. On $A$ it is zero. At $x$ it is zero as well because $u'(x) = u_c(x)$. By Proposition 14.2.1, $q = N_p u^\sharp$ is regularizable on $S$ and $q(p(x)) = 0$.

By Proposition 13.8.1, $x$ belongs to the hull of $D = \partial B \cap \widehat{K}$. Pullback of polynomials gives $p(x) \in \widehat{p(D)}$. If $y \in D \setminus Z^{\mathrm{an}}$, then $u'(y) \leq u_c(y) \leq -\delta$ by the off-exceptional hull inequality, and $u^\sharp(y) = -\delta$. All other fibre values are bounded by $\varepsilon$, and the weight of $y$ is at least one. It follows that $q(p(y)) \leq -\delta + (d-1)\varepsilon < 0$. If $y \in D \cap Z^{\mathrm{an}}$, all the other points above $p(y)$ lie in $Z'^{\mathrm{an}}$, hence in $A$ because they lie in $V_x \cup A$ and $Z'^{\mathrm{an}}$ misses $V_x$. The contribution at $y$ is strictly negative: $u'(y) \leq -c\delta'$ since $y \notin B$ and $u_c(y) \leq 0$. Thus $q < 0$ on the compact set $p(D)$. Upper semicontinuity gives $\sup_{p(D)} q < 0$, contradicting $q(p(x)) = 0$ by the hull inequality.

Apply this argument also to $Y \times \mathbb{A}^1$, $p \times \mathrm{id}$, and $Z \times \mathbb{A}^1$, with function $u \circ \mathrm{pr}_Y + \log|w|$. The weak witness is the product of the original one, and the growth and off-exceptional approximation properties follow from pullback and addition of the log norm. The target is again principal open in an affine space. For $c > 0$ rational its sum with $c\rho$ for the product has Runge strict sublevels by the hull inequality just proved. As $c \downarrow 0$, their strict sublevels at zero increase to the Hartogs open of $u$. That open is Runge. The function $u$ is finite at the trivial valuation of $k(Y)$, since it is regularizable on a nonempty affine open subset of $Y \setminus Z$ (Corollary 13.2.1). Hence Proposition 13.7.1 gives $u \in \mathcal{R}(Y)$. □

**Theorem 14.3.1** *Over a countable algebraically closed trivially valued field, let $Y$ be normal integral and affine. A weakly psh function $u$ on $Y^{\mathrm{an}}$ of logarithmic growth that is regularizable on an affine open cover of the complement of a proper closed subset is regularizable on $Y$.*

***Proof*** The function $u$ is upper semicontinuous by Lemma 13.13.1. Let $W$ be the reduced closed complement of the union of all affine Zariski opens on which $u$ is regularizable. If $W$ is empty, Proposition 13.10.1 gives the conclusion. Otherwise choose a smooth closed point $z$ of a component of $W$, away from the other components. After passing to a principal open about $z$, suppose $W$ is smooth and irreducible. Apply Lemma 14.3.1 to $W$ on this affine open, obtaining $p : U \to S$ finite with $W_U = W \cap U$ a closed subscheme of $S$.

The closure $B'$ of $p^{-1}(p(W_U)) \setminus W_U$ has dimension at most $\dim W_U$, by finiteness. No component of that closure lies in $W_U$, since the subset being closed up is dense and relatively open in its closure. Thus $B'$ cannot contain the irreducible $W_U$. The closed subset $p(B' \cap W_U)$ is proper in $p(W_U)$ since $p|_{W_U}$ is an isomorphism onto its image. Choose a principal open of $S$ disjoint from $p(B' \cap W_U)$ and meeting $p(W_U)$. Its inverse image $U_1$ still meets $W$. Now $p^{-1}(p(W \cap U_1)) \setminus (W \cap U_1)$ is closed in $U_1$. The growth condition restricts by Proposition 13.1.1, the weak witness restricts, and the original regularizable opens intersect $U_1$ in an affine open cover of $U_1 \setminus W$. Apply Proposition 14.3.1. It shows that $u$ is regularizable on $U_1$, contradicting the definition of $W$. □

**Corollary 14.3.1** *Let $k$ be countable and algebraically closed, and let $\mu : T \to X$ be a projective birational morphism of normal integral projective schemes. For a rational ample line bundle $L$ on $X$, a function $\phi$ on $X^{\mathrm{an}}$ is $[L]$-psh if its pullback is $[\mu^* L]$-psh.*

***Proof*** Scale so that $L$ is very ample. Choose generating sections $s_i$ of $L$, and put $Y_i = X_{s_i}$. By Proposition 13.13.1, $u_i = \phi - \log |s_i|$ on $Y_i^{\mathrm{an}}$ is weakly psh of logarithmic growth and regularizable on an affine open cover off the nonisomorphism locus. Thus $u_i \in \mathcal{R}(Y_i)$ by Theorem 14.3.1. Since $\phi$ is bounded above, $u_i \leq C - \log |s_i|$. For rational $a > 1$, apply Lemma 13.4.3 with $P = X$, $H = L$, $s = s_i$, $\kappa = 1$ and $c = a$ to $u_i$. The local approximation is provided by $u_i \in \mathcal{R}(Y_i)$. This gives an $[aL]$-psh potential equal to $\phi + (a-1) \log |s_i|$ on $Y_i^{\mathrm{an}}$ and to $-\infty$ off it.

The maximum over $i$ equals $\phi$, since the generating sections satisfy $\max_i \log |s_i| = 0$ and at least one section attaining the maximum is nonzero at the support. Consequently $\phi$ is psh in $[aL]$ for every rational $a > 1$. It is finite at the trivial valuation because its pullback is psh. Let $a \downarrow 1$ and use Theorem 6.2.2. □

# Chapter 15
# Finite descent of the envelope property

When a variety is replaced by a finite cover, a valuation may have several extensions. To descend a potential, we need an operation that fixes pullbacks and preserves plurisubharmonicity. Over a normal base we construct this operation using the norm of a linear combination of sections (Theorem 15.1.1). Its value at a divisorial point is an average over the extensions, with weights given by the dimensions of the factors of the completed generic fibre. In positive characteristic a finite universal homeomorphism admits a second construction, by raising sections to a power of the characteristic, that also works over a nonnormal base (Theorem 15.2.1). In any characteristic, lifting jets through the conductor handles a finite birational universal homeomorphism that is an isomorphism away from finitely many closed points (Theorem 15.3.1). In characteristic zero, two consecutive corrected powers descend potentials along every finite universal homeomorphism (Theorem 15.4.1). For purely inseparable extensions of the ground field, section descent also works over an infinite extension (Proposition 15.5.1). Consequently, on a unibranch projective variety over an algebraically closed field, the envelope property descends from the normalization (Corollaries 15.2.1 and 15.4.1).

In the first two sections $h : T \to X$ is a finite surjective morphism of integral projective schemes over a trivially valued field $k$. We denote the map of analytifications by $h^{\mathrm{an}}$, and set $K = k(X)$, $E = k(T)$, and $d = [E : K]$.

## 15.1 A norm of section families

First we arrange for the section families approximating a potential on $T$ to have line bundles from $X$.

**Lemma 15.1.1** *Let $\theta \in N^1(X)_{\mathbb{R}}$. Every $u \in \mathrm{PSH}(T, h^*\theta)$ is the decreasing limit of $u_i \in \mathcal{H}^{\mathrm{gf}}(h^*L_i)$, where $L_i$ are rational line bundles on $X$ with $[L_i] \to \theta$.*

***Proof*** Start with $u_i \in \mathcal{H}^{\mathrm{gf}}(M_i)$ in the defining net. Choose an ample line bundle $H$ on $X$; its finite pullback is ample [Stacks, Tag 0B5V]. Fix norms on the two numerical

spaces. Reindex by the product with the positive integers, writing $u_{i,j} = u_i$, and put

$$\varepsilon_{i,j} = \max\left\{\left\|[M_i] - h^*\theta\right\|^{1/2}, j^{-1}\right\}.$$

Choose a rational line bundle $L_{i,j}$ on $X$ whose class is within $\varepsilon_{i,j}^2$ of $\theta+\varepsilon_{i,j}[H]$. The numerical class of $h^*L_{i,j} - M_i$, after division by $\varepsilon_{i,j}$, tends to $h^*[H]$, by linearity of pullback and the chosen bounds. On a tail, $h^*L_{i,j} - M_i$ is ample. By (6.3), $u_{i,j}$ is also in $\mathcal{H}^{\mathrm{gf}}(h^*L_{i,j})$. Its values and order were unchanged by this choice of presentation.□

Let $v \in \mathcal{D}(X)$. It has generic support, and its completed residue field $\mathcal{H}(v)$ is the completion of $K$ for its norm. Decompose the finite-dimensional algebra

$$E \otimes_K \mathcal{H}(v) = \prod_w B_w$$

into local Artinian factors with residue fields $F_w$. Each $F_w$ has the unique norm extending that of $\mathcal{H}(v)$ by Lemma 2.3.1. Its restriction to $E$ gives a point $w$ above $v$. Conversely, an extension $w$ of $v$ to $E$ gives a homomorphism from the displayed algebra to $\mathcal{H}(w)$, factoring through one of the residue fields. The image of that field is complete and contains the dense image of $E$, so it is all of $\mathcal{H}(w)$. Also, the image of $E$ in any $F_w$ is dense, since $E$ spans $F_w$ over $\mathcal{H}(v)$ and $K$ is dense there. Thus the two constructions are inverse, with the homomorphism to $\mathcal{H}(w)$ determined by its restrictions to $E$ and $\mathcal{H}(v)$. Write

$$d_w = \dim_{\mathcal{H}(v)} B_w, \qquad \sum_{w|v} d_w = d.$$

The dimensions include the lengths of the local factors. In particular, a factor that is a field extension contributes its field degree.

All points $w$ just constructed are in $\mathcal{D}(T)$. If $v$ is trivial, the extended valuation is trivial on $E$. For nontrivial $v$, the residue field $\kappa_w$ of the valuation ring of $w$ contains $\kappa_v$, which has transcendence degree $\dim X - 1$ over $k$ by Theorem 3.1.1. The Abhyankar inequality and $\dim T = \dim X$ give equality, and properness supplies a centre on $T$, so Theorem 3.1.1 applies. The scaling is rational: in an algebraic relation for an element of $E^\times$, two terms have the same minimal value, so a nonzero integer multiple of its value lies in the value group of $v$. Applying this to a uniformizer of the divisor representing $w$ proves the claim. The same algebraic relation argument gives the assertion for the trivial valuation.

**Proposition 15.1.1** *Assume $X$ is normal and $L$ is a rational line bundle on $X$. For each $u \in \mathcal{H}^{\mathrm{gf}}(h^*L)$ there is a unique $N_h u \in \mathcal{H}^{\mathrm{gf}}(L)$ such that*

$$N_h u(v) = \frac{1}{d}\sum_{w|v} d_w u(w) \qquad (v \in \mathcal{D}(X)). \tag{15.1}$$

***Proof*** Represent $u$ by sections $s_j \in H^0(T, h^*(mL))$ and weights $\lambda_j$, as in (6.1). On an affine open $U$ with frame $e$ of $mL$, write $s_j = f_j h^* e$. The functions $f_j$ are integral

over $R = \mathcal{O}_X(U)$, by finiteness. In independent variables $Z_j$, form the polynomial

$$N_{E/K}\Big(\sum_j Z_j f_j\Big) = \sum_{|I|=d} c_{I,e} Z^I,$$

where the norm is the determinant of multiplication on the free $K[Z]$-module $E \otimes_K K[Z]$. Choose a finite extension $K'/K$ containing the images of all $K$-embeddings of $E$ into an algebraic closure. Decompose $E \otimes_K K'$ into local Artinian factors with residue field $K'$ and adjoin the variables. The norm polynomial is a product of the linear forms from the embeddings of $E$, with multiplicities. On each local factor, multiplication preserves the filtration by powers of its maximal ideal and acts on the graded pieces by its residue. All the conjugates of the $f_j$ are integral over $R$, so the coefficients are integral over $R$. They belong to $K$, hence are in $R$ by normality. Replacing $e$ by $ge$ replaces each coefficient by $g^{-d} c_{I,e}$. The coefficients therefore define sections $c_I \in H^0(X, mdL)$. At least one is generically nonzero since the norm of a nonzero element of $E(Z)$ over $K(Z)$ is nonzero. Use these sections to set

$$N_h u = \frac{1}{md} \max_{|I|=d} \Big( \log |c_I| + \sum_j I_j \lambda_j \Big).$$

To evaluate it at $v \in \mathcal{D}(X)$, use a frame near the centre of $v$. In its pullback frame at any extension $w$, $\log |s_j|(w) = \log |f_j|(w)$. Extend the scalars of the norm polynomial to $\mathcal{H}(v)$ and then to a finite extension splitting the residue fields of all factors $B_w$. By the same determinant calculation there is a linear factor for each resulting residue embedding, repeated according to the dimension of its local factor over this last field. All embeddings above a fixed $w$ give the same norms of the $f_j$, by the uniqueness of the extension of the absolute value. The corresponding linear forms thus all have Gauss norm $\max_j \big( |f_j|(w) \exp(\lambda_j) \big)$ at polyradius $(\exp(\lambda_j))_j$. The sum of their multiplicities is $d_w$, since dimension is unchanged by scalar extension. Multiplicativity of the Gauss norm, from Proposition 2.1.1, now gives (15.1). Uniqueness and independence of presentation follow from Theorem 6.3.1. □

**Theorem 15.1.1** *Assume $X$ is normal. For every real numerical class $\theta$ on $X$, the operation in Proposition 15.1.1 extends to an order-preserving map*

$$N_h : \mathrm{PSH}(T, h^*\theta) \longrightarrow \mathrm{PSH}(X, \theta)$$

*satisfying* (15.1) *and fixing pullbacks from* $\mathrm{PSH}(X, \theta)$. *If $h^*\theta$ has the envelope property, so does $\theta$.*

***Proof*** Use the decreasing approximants $u_i$ in Lemma 15.1.1. The functions $N_h u_i$ decrease, since they do so on $\mathcal{D}(X)$ by the average formula, and this order implies global order by Theorem 6.3.1. Their classes tend to $\theta$. The limit is finite at the trivial point of $X$, because there it equals the value of $u$ at the trivial point of $T$. By Definition 6.2.1 the limit is $\theta$-psh. The formula passes to the limit since the sum is finite and $u$ is finite at divisorial points. It therefore characterizes $N_h u$ and proves

order preservation. Pullbacks of $\theta$-psh potentials are $h^*\theta$-psh by Theorem 6.2.1, since $h^{\mathrm{an}}$ sends the trivial point of $T$ to that of $X$, where such potentials are finite. For such a pullback, each term of the average at $v$ has the same value, proving the pullback assertion.

Let $u_i \in \mathrm{PSH}(X, \theta)$ be a bounded-above increasing net. The pullback net has a psh regularized supremum $U$ on $T$ under the envelope hypothesis. At each extension $w$ of $v \in \mathcal{D}(X)$, Theorem 12.2.1 gives $U(w) = \sup_i u_i(v)$. Thus $N_h U$ is a psh limit of $u_i$ on $\mathcal{D}(X)$. By Corollary 12.2.1, this suffices for the envelope property. □

## 15.2 Universal homeomorphisms and Frobenius

**Theorem 15.2.1** *Suppose* $\operatorname{char} k = p > 0$ *and $h$ is a universal homeomorphism. Then $h^{\mathrm{an}}$ is a homeomorphism, and pullback along it gives a bijection*

$$\mathrm{PSH}(X, \theta) \longrightarrow \mathrm{PSH}(T, h^*\theta)$$

*for every real numerical class $\theta$. In particular, the envelope property is equivalent for these two classes.*

***Proof*** The map on underlying algebraic points is bijective and the residue field extensions are purely inseparable. If an element $b$ of such an extension satisfies $b^q \in \kappa(x)$, where $q$ is a power of $p$, its norm must be $|b^q|^{1/q}$. The formula defines a multiplicative ultrametric norm by using a common power for any finite set of elements. This proves bijectivity of the map of analytifications. Compactness and Hausdorffness, from Chapter 1, then give a homeomorphism.

On any affine open subset of $X$ write the finite extension of rings as $R \subset S$. By [Stacks, Tag 0CNF], for each $b \in S$ some $p$-power of $b$ is in $R$. Taking the maximum of the exponents for a finite list of algebra generators works for all of $S$, since taking that power is a ring homomorphism. A finite affine cover gives one exponent $q$ valid on the whole cover, and the property persists after localization. If $s \in H^0(T, h^*(mL))$, the $q$-th powers of its local coefficients in pulled-back frames come from coefficients on $X$. They glue to a section of $qmL$, since the transitions for $mL$ are raised to their $q$-th powers and pullback is injective on the functions. Thus a function of $\mathcal{H}^{\mathrm{gf}}(h^*L)$ descends to one of $\mathcal{H}^{\mathrm{gf}}(L)$ by replacing all its sections by their $q$-th powers on $X$ and scaling both its weights and its denominator by $q$. The pointwise identity follows by evaluation of the powers at the corresponding analytic points.

Now use Lemma 15.1.1. The descended approximants decrease and their classes tend to $\theta$. Their limit is finite at the trivial point, so it belongs to $\mathrm{PSH}(X, \theta)$. Together with psh pullback, from Theorem 6.2.1, this proves the bijection. Upper semicontinuous regularization and suprema commute with the homeomorphism, giving the last assertion. □

**Corollary 15.2.1** *Let $X$ be geometrically unibranch and let $k$ have positive characteristic. Then a real class on $X$ has the envelope property if and only if its pullback to the normalization does. Over an algebraically closed field, unibranchness suffices.*

***Proof*** The normalization is finite and projective, by finiteness of normalization over a field [Stacks, Tags 035B and 035S]. The stalk of its pushforward algebra at a point is the integral closure of the local ring. Geometric unibranchness says that this closure has a single maximal ideal and the residue field extension is purely inseparable [Stacks, Tags 0BQ2, 0C37 and 0C3B]. Thus the map is finite, surjective and radicial, hence a universal homeomorphism [Stacks, Tags 01S4 and 04DF].

If $k$ is algebraically closed, unibranchness gives bijectivity for the normalization $T \to X$. Its diagonal is a closed immersion in $T \times_X T$. A nonempty complement to the image would have a closed $k$-point, giving two distinct $k$-points over the same point of $X$, which is impossible. The surjectivity of the diagonal gives radiciality by the same criterion. In both cases apply Theorem 15.2.1. □

*Example 15.2.1* Let $C \subset \mathbb{P}^2_k$ be the cuspidal cubic $y^2 z = x^3$ over a perfect field of positive characteristic. Its normalization $\mathbb{P}^1_k \to C$ is finite and is an isomorphism away from the cusp, over which it has one point with residue field $k$. Indeed on $z = 1$ the map is given by $k[t^2, t^3] \subset k[t]$, and on $y = 1$ it is an isomorphism. Thus it is a universal homeomorphism. The pullback of a nef class is nef. By Theorems 12.5.1 and 15.2.1, every nef class on $C$ has the envelope property. Locally at the cusp, any power $q$ of $p$ with $q \geq 2$ suffices for the section construction, since $t^q \in k[t^2, t^3]$.

## 15.3 Conductor descent at closed points

On the cuspidal curve a local coefficient on the normalization descends precisely when its linear term at the cusp vanishes. For a large power of a section, that term can be cancelled by multiplying by a prescribed first jet. To keep the same log norm, we lift the jet in a line bundle of fixed degree and use several lifts that generate that bundle everywhere.

**Theorem 15.3.1** *Let $h : T \to X$ be a finite birational universal homeomorphism of integral projective schemes over a trivially valued field $k$. Suppose it is an isomorphism away from finitely many closed points of $X$. For any real numerical class $\theta$ on $X$, pullback identifies* $\mathrm{PSH}(X, \theta)$ *with* $\mathrm{PSH}(T, h^*\theta)$*, and the envelope property is equivalent for the two classes.*

***Proof*** The map of analytifications is a homeomorphism. Indeed all scheme points lift uniquely, and their residue extensions are purely inseparable. The norm extends uniquely through such an extension, by taking the root of the norm of a power in the smaller field. In characteristic zero the residue extensions are trivial. The resulting continuous bijection is a homeomorphism by compactness and Hausdorffness.

Let $J$ denote the conductor, the annihilator of $h_*\mathcal{O}_T/\mathcal{O}_X$. It is a coherent ideal, and is also an ideal of $h_*\mathcal{O}_T$: locally it is the set of $b$ with $bh_*\mathcal{O}_T \subset \mathcal{O}_X$. Write $C \subset T$ for the conductor subscheme defined by $J\mathcal{O}_T$. It is zero-dimensional, since the map is an isomorphism away from the stated points. If $C$ is empty the theorem is immediate. Otherwise choose $N \geq 1$ such that the $N$-th power of the maximal ideal of each Artinian local ring of $C$ is zero. Write $A_T = h^*A$, where $A$ is a sufficiently large power of a very ample line bundle on $X$. Since finite pullback is ample, Serre vanishing and generation give

$$H^1(T, J\mathcal{O}_T \otimes A_T) = 0, \qquad J\mathcal{O}_T \otimes A_T \text{ globally generated.}$$

They apply to this fixed conductor ideal, so $A$ will be independent of all sections and powers used below. In particular $H^0(T, A_T) \to H^0(C, A_T|_C)$ is surjective.

Let $M$ be a line bundle on $X$, $s$ a section of $h^*M$ and $q \geq N$ an integer. At each point $x$ under $C$, choose local frames of $M$ and $A$ on $X$, and write $b$ for the coefficient of $s$ in the pulled-back frame of $M$. Let $\bar{b}$ be its image in the Artinian local ring of $C$ at the unique point over $x$. Prescribe a jet of $A_T$ at that point with coefficient $\bar{b}^{-q}$ if $\bar{b}$ is invertible, and with coefficient 1 otherwise. In the second case $\bar{b}^q = 0$, by the choice of $N$. These prescriptions give a section on $C$, which lifts to a section $r_0$ of $A_T$. Together with a finite generating family $r_1, \dots, r_\ell$ of $J\mathcal{O}_T \otimes A_T$, it generates $A_T$. The first section is a frame at each point of $C$, and the others generate off $C$.

For each $j$, the coefficient of $s^q r_j$ in the pulled-back frame of $qM+A$ is congruent to 0 or 1 modulo the conductor at $x$. Since the conductor lies in $\mathcal{O}_X$, the coefficient belongs to $\mathcal{O}_X$ there. At all other points the map is an isomorphism, so there is a unique descended section $S_j \in H^0(X, qM + A)$. To check this coefficient criterion on stalks, note that a finite algebra over $\mathcal{O}_{X,x}$ with a unique point above its maximal ideal is local. It is therefore the local ring at the point of $T$ above $x$.

At each analytic point $v$ of $X$, let $w$ be its lift. With the log norms given by frames at the centres,

$$\max_j \log|S_j|(v) = q \log|s|(w) + \max_j \log|r_j|(w) = q \log|s|(w).$$

The last maximum is zero because the $r_j$ generate $A_T$. The identity also holds at a pole of $\log|s|$.

Now let $u \in \mathrm{PSH}(T, h^*\theta)$ and let $u_i$ be a decreasing Fubini–Study approximation of $u$. By Lemma 15.1.1 the section families of the $u_i$ may be taken in $h^*(m_iL_i)$, for $L_i$ rational with $[L_i] \to \theta$ and $m_i \geq 1$. Apply the construction to every section of a given family with $M = m_iL_i$ and the same integer $q$. Multiply the weights and the denominator of that Fubini–Study formula by $q$. The displayed identity represents exactly the same function on $X$ in the class of $L_i + A/(qm_i)$. Repeat indices over the product with the integers $q \geq N$, ordered by increasing $q$. These functions still decrease, and the classes converge to $\theta$. Their limit is finite at the trivial generic point, so it is $\theta$-psh by definition. Pullback gives the converse, by Theorem 6.2.1, and the homeomorphism preserves suprema and upper semicontinuous regularization. □

**Corollary 15.3.1** *On a unibranch integral projective curve over an algebraically closed trivially valued field, every nef real class has the envelope property.*

***Proof*** The normalization $\nu : T \to X$ of the curve $X$ is finite [Stacks, Tags 035B and 035S]. The map $\mathcal{O}_X \to \nu_*\mathcal{O}_T$ is injective, and its cokernel is coherent and vanishes at the generic point. Its support is therefore a proper closed subset of $X$, hence a finite set of closed points, and $\nu$ is an isomorphism over the complement. Unibranchness over an algebraically closed field makes it a universal homeomorphism, by the argument of Corollary 15.2.1. The normal projective curve is smooth, since the field is perfect and the local rings of a normal curve are regular. Its pullback classes are nef. Apply Theorem 12.5.1 or Theorem 12.6.1, according to the characteristic, and then Theorem 15.3.1. □

*Example 15.3.1* At the ordinary cusp take $k[t^2, t^3] \subset k[t]$. The conductor is $t^2k[t]$, and the quotient on the normalization is $k[t]/(t^2)$. A local coefficient with unit jet $b_0 + b_1 t$ has $q$-th power inverted by the jet $b_0^{-q}(1 - qb_1 t/b_0)$. If $b_0 = 0$, its $q$-th power already belongs to the conductor for $q \geq 2$. Thus the two cases of the jet prescription can be read directly on the cusp.

## 15.4 Subintegral descent in characteristic zero

A finite universal homeomorphism in characteristic zero has identical residue fields. A section upstairs need not have any power that descends, even when the residue fields agree. We first factor the morphism into extensions with one extra generator. Two consecutive corrected powers then give lower and upper valuation bounds. These bounds let us descend decreasing families in varying classes while keeping their order at points where the functions have poles.

A finite extension of rings is called *subintegral* if its spectrum is a universal homeomorphism and all its residue field maps are isomorphisms. We use the elementary criterion [Stacks, Tag 0CND]: in a subintegral extension $A \subset B$ each finite set of elements of $B$ is contained in a tower obtained by adjoining elements whose squares and cubes belong to the preceding ring. Conversely adjoining such an element gives a subintegral extension. A finite morphism $h : T \to X$ of integral schemes is *subintegral* if it is a universal homeomorphism and induces isomorphisms of residue fields. It is then surjective, so $\mathcal{O}_X(U) \subset \mathcal{O}_T(h^{-1}U)$ is a subintegral extension for every affine open $U \subset X$.

**Lemma 15.4.1** *A finite subintegral morphism $h : T \to X$ between integral projective schemes factors into finitely many such morphisms that are locally given by $A \subset A[b]$ with $b^2, b^3 \in A$.*

***Proof*** Let $\mathcal{B} = h_*\mathcal{O}_T$ and start with $\mathcal{A} = \mathcal{O}_X \subset \mathcal{B}$. Fix a finite affine cover by opens $U_\alpha = (\tau_\alpha \neq 0)$, where the $\tau_\alpha$ are sections of one very ample line bundle $H$. Over each $U_\alpha$, the criterion and finiteness give a finite tower of adjunctions generating

$\mathcal{B}(U_\alpha)$ over $\mathcal{O}_X(U_\alpha)$. Process those towers in order, enlarging $\mathcal{A}$ globally at each adjunction.

For an element $b$ at the next step on $U_\alpha$, the preceding adjunctions ensure $b^2, b^3 \in \mathcal{A}(U_\alpha)$. Multiply $b$ by $\tau_\alpha^r$ to extend it to a global section $\widetilde{b}$ of $\mathcal{B}(rH)$ for $r$ sufficiently large. This uses the elementary clearing of denominators for a coherent sheaf on a principal open defined by a section of an invertible sheaf. On a finite affine cover trivializing the line bundle, clear the denominators and then use a common higher power to kill the differences on the overlaps. The images of $\widetilde{b}^2, \widetilde{b}^3$ in the appropriate twists of $\mathcal{B}/\mathcal{A}$ vanish on $U_\alpha$. Hence further multiplication of $\widetilde{b}$ by a fixed power of $\tau_\alpha$ makes both images zero. Indeed the union of the kernels of multiplication by powers of $\tau_\alpha$ stabilizes on this coherent quotient, as seen on the same finite affine cover.

Now enlarge $\mathcal{A}$ by the image of $\mathcal{A}\otimes\mathcal{O}_X(-rH)$ under multiplication by $\widetilde{b}$, using the final value of $r$. The sum is an algebra, since in a frame the square of the new element belongs to $\mathcal{A}$. Its relative spectrum is elementary subintegral over $\operatorname{Spec}_X \mathcal{A}$, since the cube also belongs to $\mathcal{A}$. After the finitely many steps the resulting algebra equals $\mathcal{B}$ on every $U_\alpha$. All intermediate schemes are finite over $X$ and integral, since the algebras lie in the same function field. They are projective because a finite pullback of an ample bundle is ample. □

**Lemma 15.4.2** *Suppose $h : T \to X$ is a finite subintegral morphism of integral projective schemes over a trivially valued characteristic zero field, locally of the elementary form above. Given a line bundle $M$ on $X$ and a section $s$ of $h^*M$, there is an ample line bundle $A$ on $X$ such that for every integer $n \geq 1$ there is a finite family of sections $(S_{n,j})_j$ of $(n+2)M + A$ satisfying*

$$\log|s|(w) \;\leq\; \frac{1}{n+2}\max_j \log|S_{n,j}|(v) \;\leq\; \frac{n}{n+2}\log|s|(w).$$

*Here $w$ is the unique point of $T^{\mathrm{an}}$ over $v$. It exists and is unique because $h$ is bijective with isomorphic residue fields, so the absolute value of $v$ extends uniquely. The same $A$ can be chosen for a finite family of sections of $h^*M$.*

***Proof*** Take a finite cover $U_\alpha = (\tau_\alpha \neq 0)$ trivializing $M$ with frame $e_\alpha$. Use one common ample bundle $H$ for the $\tau_\alpha$, and choose the cover fine enough that the extension is elementary on each member. Such opens form a basis on a projective scheme, by taking sections of powers of a very ample line bundle and passing to a common power. Write the coefficient $b_\alpha$ of $s$ as $a_\alpha + d_\alpha$, where $a_\alpha$ is in the base ring and $d_\alpha$ is a multiple of the elementary generator by a base element. All powers of $d_\alpha$ of order at least two are in the base ring. For every $q \geq 1$ this gives

$$c_{\alpha,q} = b_\alpha^q - q d_\alpha b_\alpha^{q-1} \in \mathcal{O}_X(U_\alpha).$$

In fact modulo the base ring the only term of the binomial expansion of $b_\alpha^q$ that can survive is $q a_\alpha^{q-1} d_\alpha$. It is cancelled by the second summand. The two consecutive expressions satisfy

$$(q+1)b_\alpha c_{\alpha,q} - q c_{\alpha,q+1} = b_\alpha^{q+1}.$$

Fix $r$ large enough that $\widetilde{d}_\alpha = \tau_\alpha^r d_\alpha e_\alpha$ extends to a global section of $h^*(M+rH)$ for every $\alpha$. Choose it also so that multiplication by $\tau_\alpha^r$ kills sections of any line-bundle twist of $h_*\mathcal{O}_T/\mathcal{O}_X$ that vanish on $U_\alpha$. This is the stabilization of kernels used in the preceding lemma, and is independent of the twist. Finally require global generation of $rH$ and $M+rH$, by increasing $r$ if needed. The section $\tau_\alpha^r s^q - q\widetilde{d}_\alpha s^{q-1}$ of $h^*(qM+rH)$ belongs to the base ring on $U_\alpha$, in the chosen frames. Multiplying by another $\tau_\alpha^r$ therefore gives a section descending to $qM+2rH$ on $X$. For $q = n+1, n+2$, multiply each descended section by a generating family of $(n+2-q)M+rH$. The resulting family $(S_{n,j})_j$ has $A = 3rH$.

Every member of the family pulls back to a multiple of $s^n$ by a regular section. Thus $\min_j v(S_{n,j}) \geq nw(s)$, in frames at the centres. Choose $\alpha$ containing the centre of $v$. The factor $\tau_\alpha$ has valuation zero there, and some section of each of the two generating families is a local frame. The resulting minimum is at most

$$\min_{q=n+1,n+2} w(c_{\alpha,q}).$$

The consecutive-power identity and the ultrametric inequality bound this minimum by $(n+2)w(s)$: multiplication by $b_\alpha$ cannot lower the value, and the rational coefficients are trivially valued. This proves the two inequalities, also at a pole by the common factor $s^n$. For finitely many sections take $r$ large enough to work for all the local coefficients at once. □

**Theorem 15.4.1** *Let $h : T \to X$ be a finite universal homeomorphism of integral projective schemes over a trivially valued field of characteristic zero. Then $h^{\mathrm{an}}$ is a homeomorphism, and for every real numerical class $\theta$ on $X$, pullback identifies* $\mathrm{PSH}(X,\theta)$ *with* $\mathrm{PSH}(T, h^*\theta)$. *In particular the envelope properties in these classes are equivalent.*

***Proof*** The residue field extensions are purely inseparable and hence trivial, so $h$ is subintegral. There is consequently a unique lift of every analytic point through $h$, and compactness and Hausdorffness give the homeomorphism. Apply Lemma 15.4.1; it suffices to descend through one elementary step.

Let $u$ be $h^*\theta$-psh and choose a decreasing Fubini–Study approximation $u_i$ in bundles pulled back from $X$, by Lemma 15.1.1. After restricting to a tail and subtracting a common constant, assume $u_i \leq 0$. In the defining finite maximum in $h^*(m_i L_i)$, remove zero sections. Every remaining weight is nonpositive: at the trivial generic point the log norms of the retained sections vanish. Apply Lemma 15.4.2 to the finite family defining $u_i$, using a common $A_i$. Put $\gamma_n = n/(n+2)$. For each original term $(\log|s|+\lambda)/m_i$, use the descended logarithmic maximum divided by $m_i(n+2)$, with weight $\gamma_n\lambda/m_i$ added. Taking the maximum over the original terms gives a Fubini–Study function $f_{i,n}$ in the rational line bundle $L_i + A_i/(m_i(n+2))$, with

$$u_i \leq f_{i,n} \leq \gamma_n u_i.$$

The lower inequality uses $\lambda \le 0$, and the upper one scales both the log norm and the weight by $\gamma_n$. The inequalities preserve poles with value $-\infty$.

Choose a rational $0 < \rho < 1$ and put $g_{i,n,\rho} = \rho f_{i,n}$, which is Fubini–Study in the correspondingly scaled class. Fix a norm on the numerical space and write $e(i,n,\rho) = (1-\rho) + \|[A_i]\|/(m_i(n+2))$. Put $(i,n,\rho) \preceq (i',n',\rho')$ if $i \le i'$, $g_{i',n',\rho'} \le g_{i,n,\rho}$ pointwise on $X^{\mathrm{an}}$ and $e(i',n',\rho') \le e(i,n,\rho)$. This is a preorder. To dominate two triples, choose $i$ above their indices, $\rho$ close enough to one and then $n$ large so that $\rho\gamma_n$ exceeds their two $\rho$-values and the error $e$ is as small as desired. Indeed $g_{i,n,\rho} \le \rho\gamma_n u_i \le \rho' u_{i'} \le g_{i',n',\rho'}$ for either old triple, since $u_i \le u_{i'} \le 0$. This gives a directed decreasing net. The classes converge to $\theta$, since $[L_i] \to \theta$ and the error tends to zero. The infimum of the net is $u$ by the pointwise bounds and by letting $\rho, \gamma_n$ tend to one at any fixed $i$. It is finite at the trivial point, so this is the required psh approximation on $X$. Restoring the common constant and using psh pullback (Theorem 6.2.1) proves that pullback identifies the two cones. The analytic homeomorphism commutes with the regularized supremum. □

**Corollary 15.4.1** *In characteristic zero, the normalization of a geometrically unibranch integral projective scheme reflects the envelope property in every real numerical class. Over an algebraically closed field, unibranchness suffices.*

***Proof*** The algebraic argument in Corollary 15.2.1 shows that the normalization is finite and a universal homeomorphism. Apply Theorem 15.4.1. □

## 15.5 Descent from the perfection of the field

The Frobenius calculation for a smooth scheme used a perfect ground field to choose the monomial basis in the local expansion. We now remove that condition by descending Fubini–Study approximants. Each approximant uses only finitely many coefficients, even though the perfection may have infinite degree over the given field.

**Proposition 15.5.1** *Let $k'/k$ be a purely inseparable algebraic extension of trivially valued fields of characteristic $p > 0$. Let $X$ be a geometrically reduced integral projective scheme over $k$ and put $X' = X_{k'}$. Then $X'$ is integral. The projection $h : X' \to X$ induces a homeomorphism of analytifications and an isomorphism on real numerical spaces. For every real numerical class $\theta$ on $X$, pullback identifies* $\mathrm{PSH}(X,\theta)$ *with* $\mathrm{PSH}(X', h^*\theta)$. *It also identifies their envelope properties.*

***Proof*** The projection is a universal homeomorphism, since $\operatorname{Spec} k' \to \operatorname{Spec} k$ is integral, surjective and radicial. Hence $X'$ is irreducible, and it is reduced by assumption. At every scheme point, a valuation extends uniquely to the purely inseparable algebraic extension of its residue field by the formula $|b| = |b^q|^{1/q}$ for suitable $p$-powers $q$, as in Theorem 15.2.1. Both analytifications are compact Hausdorff, so the map on analytic points is a homeomorphism.

For any line bundle $M$ on $X'$, choose finitely many affine opens $U_i \subset X$ whose inverse images trivialize $M$. This is possible because $h$ is a homeomorphism on Zariski spaces. The intersections $U_i \cap U_j$ are affine by separatedness. The transition functions and their inverses in the chosen frames are finite sums with coefficients in $k'$ in the coordinate rings of $h^{-1}(U_i \cap U_j)$. If a finite family of sections of $M$ is given, include their local coefficients in the same finite list. Choose a common $p$-power $q$ sending the scalar coefficients of all these functions into $k$. The $q$-th powers of the transition functions descend to units on the overlaps of $X$. The cocycle identity descends by faithful flatness. The resulting line bundle $N$ on $X$ pulls back to $qM$, and the $q$-th powers of the prescribed sections descend to $N$.

By the second assertion of Proposition A.1.1, base change to an algebraic closure preserves and detects numerical equivalence of real classes. We may use the same algebraic closure for both fields. Hence pullback from $X$ to $X'$ is well-defined and injective. Each integral class on $X'$ has a positive multiple that descends by the preceding construction, proving surjectivity of $h^*$ on real numerical spaces. These spaces are finite-dimensional, so the isomorphism is a homeomorphism.

Now let $u_i$ be a decreasing net of generically finite Fubini–Study potentials on $X'$ in rational line bundles $M_i$ whose classes tend to $h^*\theta$. At a given index let $m_i M_i$ be the bundle of its sections. Apply the section construction to this bundle and its finite family. Replace each section by its descended $q_i$-th power and multiply all weights and the denominator by $q_i$. This gives exactly the same function under the analytic identification. Its rational line bundle on $X$ has numerical class $(h^*)^{-1}[M_i]$, so the classes tend to $\theta$. The functions remain decreasing and the limit is finite at the trivial point if the original limit is. Thus Definition 6.2.1 gives descent of psh potentials. Pullback follows from the section presentation. The homeomorphism identifies suprema and their regularizations. □

**Corollary 15.5.1** *Let $X$ be smooth integral and projective over a trivially valued field of positive characteristic. Every nef real class on $X$ has the envelope property.*

***Proof*** Let $k'$ be the perfection of $k$. Smoothness gives geometric reducedness and is preserved by this base change. Pullback sends ample classes to ample classes and is continuous on numerical spaces. It therefore sends nef to nef. Apply Theorem 12.5.1 on $X'$ and descend by Proposition 15.5.1. □

# Chapter 16
# Proper fibres and projective envelopes

On an analytic disc a locally convex PL function can increase towards the boundary. If we fill in the rest of the projective line, every annular direction has another side. A submean inequality on a disc with finitely many holes then propagates the largest value through the line.

We take locally psh functions in the sense of Section 13.13. The first sections work over complete fields that can carry nontrivial norms, since such fields occur as completed residue fields of points over the trivially valued ground field. Locally psh functions are constant on the projective line (Theorem 16.1.1), on smooth projective curves by finite traces (Theorem 16.3.1), and on geometrically connected projective schemes by joining algebraic points with curves (Theorem 16.4.1). We apply this constancy to the connected fibres of the Stein factorization of a smooth alteration. Over an algebraically closed field this gives the envelope property in every real numerical class on normal projective varieties (Theorem 16.5.1) and on unibranch ones (Corollary 16.5.1). The envelope property then gives decreasing closure when the numerical class varies (Theorem 16.6.1). Results of the next part that assume the envelope property therefore apply to unibranch projective varieties over an algebraically closed field.

## 16.1 A maximum principle on the projective line

Let $F$ be a complete algebraically closed nontrivially valued field. We use locally convex PL functions on algebraic analytifications over $F$, as in Section 14.1. Write $z$ for a coordinate on $\mathbb{A}^1_F$. A disc with finitely many holes is a compact set

$$D = \{|z - a| \leq R\} \setminus \bigcup_{j=1}^{l} \{|z - a_j| < r_j\},$$

where $|a_j - a| \leq R$, $0 < r_j < R$, and the removed discs are disjoint. Its boundary points in the following lemma are the outer Gauss point $\eta_{a,R}$ and the inner Gauss points $\eta_{a_j,r_j}$ that belong to $D$.

**Lemma 16.1.1** *A finite locally convex PL function on an open neighborhood of $D$ attains its maximum on $D$ at one of these boundary points. The same conclusion holds after subtracting a real linear combination of the log norms of rational functions that are regular units near each point of $D$.*

***Proof*** Use the diameter $\inf_b |z - b|(x)$ as in the proof of Proposition 14.1.1, and choose a point of maximum with largest diameter. Any point of $D$ other than the listed Gauss points lies in its interior. Indeed, if $|z - b|(x) = s$ in one of its defining inequalities and $x \neq \eta_{b,s}$, then the diameter is strictly less than $s$. Choose $c$ with $|z - c|(x) < s$, so that $|b - c| = s$. On the open set $\{|z - c| < s\}$ the original equality holds identically. Strict inequalities already hold on a neighborhood.

At an interior maximizer choose a touching summand in the local presentation. Factoring its rational functions over $F$, the summand is a constant plus a finite real linear combination of $\log |z - b|$. This also holds for each touching summand after the subtraction in the statement. If the diameter is attained and positive, the outward slope of each log norm is cancelled by the inward slope in the direction of its root when the root lies in the disc. If the root is outside the disc, all the initial slopes vanish. As in the disc calculation of Proposition 14.1.1, the sum of all the slopes of the touching summand is zero. All are nonpositive at a maximum. The outward slope is therefore zero, so a short outward segment consists of maximizers of larger diameter. At a rigid point the same outward segment has constant value for each of the log norms, since they are finite there. If the diameter is not attained, the Gauss points of discs with radii decreasing to the diameter, constructed in that proof, converge to the point and give eventually exactly the same values to the finitely many log norms. They too yield maximizers of larger diameter. In each case the points used are in the interior for sufficiently short segments or for sufficiently late indices. The contradiction proves the lemma. □

**Theorem 16.1.1** *Every locally psh function on $(\mathbb{P}^1_F)^{\mathrm{an}}$ is constant.*

***Proof*** We first prove an inequality at a Gauss point $x = \eta_{a,r}$, where $r > 0$. Fix an open neighborhood $U$ of $x$ in the affine chart on which the function $u$ is a decreasing limit of locally convex PL functions $u_i$. In the disc construction above, take $R > r$, $|a_j - a| \leq r$, and $r_j < r$, with the $a_j$ in distinct classes for the relation $|a_j - a_k| < r$. Such a disc $D$ can be chosen inside $U$. Indeed a smaller neighborhood is given by inequalities on finitely many $|z-b|$, since these determine all polynomial evaluations. For the $b$ outside the disc of radius $r$ take $R < |b - a|$. Group the remaining $b$'s by their classes, and in each class put a hole with radius close to $r$, strictly greater than the distances from its centre to all the listed $b$'s in that class. On $D$, the corresponding $|z-b|$ lie between this inner radius and $R$, as close to $r$ as required. We can also leave an outer or inner boundary point at any sufficiently nearby prescribed point along an

outward or inward radial segment from $x$. In the inward case, use its centre for the representative of its class, adding that class if necessary.

Let $x_0 = \eta_{a,R}$, $x_j = \eta_{a_j,r_j}$ for $j > 0$, and let $L_0 = \log(R/r)$, $L_j = \log(r/r_j)$. Given finite target values $m_0, \dots, m_l$ on these boundary points, a function $h = b + \sum_{j=1}^{l} c_j \log|z - a_j|$ takes the prescribed values when

$$h(x) = \frac{\sum_{j=0}^{l} m_j/L_j}{\sum_{j=0}^{l} 1/L_j}, \qquad c_j = \frac{h(x) - m_j}{L_j}.$$

Indeed the $j$-th inward segment changes only the $j$-th log norm, by $-L_j$, and the outward segment changes each one by $L_0$. Choose $m_j = u_i(x_j)$. The function $u_i - h$ has boundary values zero, so Lemma 16.1.1 gives $u_i(x) \le h(x)$. Passing to the decreasing limit gives the same weighted inequality for $u$, with positive weights at all the boundary points. If $u$ attains its global finite maximum at $x$, each boundary value is that maximum. As the boundary points can be chosen along every inward and outward radial segment as above, the maximum propagates along an initial interval of each such segment.

The upper semicontinuous function $u$ attains a maximum on the compact projective line. If the maximum is $-\infty$ the assertion holds. We can find a maximizer that is a Gauss point of positive radius. For a rigid maximizer in a chosen affine chart, take a small closed disc in its approximation neighborhood. For a maximizer whose diameter is not attained, use one of the discs of radii decreasing to its diameter. The latter discs eventually lie in any fixed neighborhood of the point because the finitely many $|z-b|$ used to specify the neighborhood take the same values throughout the disc as at the point. In both cases Lemma 16.1.1, applied to the decreasing approximants, puts the maximum at the disc's Gauss point. A change to the reciprocal coordinate handles a rigid maximizer at infinity. Inversion takes the Gauss point of a disc avoiding zero to the Gauss point of its image disc. It takes the Gauss point of a disc containing zero to the Gauss point with reciprocal radius. Both formulas follow by evaluating linear polynomials.

Any two Gauss points of positive radius in the $z$ chart are joined by the path of radial segments through the smallest disc containing the two discs. Each radial segment $t \mapsto \eta_{a,t}$ is a continuous injection of an interval, since every $|z - b|$ is continuous in $t$. Along such a path the maximum set is closed by upper semicontinuity and open by the propagation just proved. Hence all these Gauss points are maximizers. They are dense, by the disc description of rigid points and of points with nonattained diameter. At infinity use the reciprocal chart and the inversion formulas above, which send $\eta_{0,R}$ to the Gauss point of radius $1/R$ about zero in the reciprocal coordinate. Upper semicontinuity gives the same value at every point of the projective line. □

## 16.2 Finite traces on a proper curve

Continue with $F$ complete, algebraically closed and nontrivially valued. Let $p : C \to \mathbb{P}^1_F$ be a nonconstant morphism from a smooth integral projective curve. It is finite, since it is proper with finite fibres, and flat since its local direct image modules are torsion-free over the regular base curve. For a point $x$ of the base analytification, write $B_x$ for the finite analytic fibre algebra over $\mathcal{H}(x)$. Its local Artinian factors are indexed by $y$ over $x$, and $d_y$ denotes the $\mathcal{H}(x)$-dimension of the factor at $y$. The determinant computation of Proposition 13.11.1 applies over $F$: after extension to a splitting field, the determinant of multiplication by a polynomial is the product of the polynomials from all residue embeddings. The factors at $y$ have total multiplicity $d_y$ and the same Gauss norm by uniqueness of the extending absolute value. The Artinian filtration repeats the residue field norm by the length of the local factor, and the field norm repeats each residue embedding by the inseparable degree. Hence the determinant's Gauss norm is the product of the Gauss norms at $y$, raised to the powers $d_y$.

**Lemma 16.2.1** *Let $p^{-1}(x) = \{y_1, \dots, y_l\}$, and choose pairwise disjoint analytic neighborhoods $U_j$ of the $y_j$. After shrinking to a neighborhood $V$ of $x$, all the fibres over $V$ lie in $\bigcup_j U_j$ and*

$$\sum_{y \in p^{-1}(x') \cap U_j} d_y = d_{y_j} \qquad (x' \in V).$$

*If $v$ is finite and locally convex PL on $\bigcup_j U_j$, then $\sum_{p(y)=x'} d_y v(y)$ is locally convex PL near $x$.*

***Proof*** Shrink the $U_j$'s as needed and work over an affine open neighborhood Spec $A$ of the support of $x$, with $p^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$. The algebra $B$ is finite free over $A$ after further algebraic shrinking. The map on analytifications is closed by compactness. Thus when we shrink each $U_j$, we may shrink $V$ so that every fibre still lies in their union.

Fix $j$ and $\varepsilon > 0$. There is an element $b$ of $B$ on a possibly smaller algebraic base neighborhood with

$$|b(y_j) - 1| < \varepsilon, \qquad |b(y_i)| < \varepsilon \quad (i \neq j).$$

Start with the idempotent of the $j$-th factor of $B_x$. If $x$ is rigid, $B_x = B \otimes_A F$ at the corresponding $F$-point and we choose any element of $B$ mapping to the fibre idempotent. Otherwise $\mathcal{H}(x)$ is the completion of $F(\mathbb{P}^1)$. Approximate the coordinates of the idempotent in a finite free basis of $B$ by elements of $F(\mathbb{P}^1)$. The finitely many seminorms at the $y_i$'s extend to the Artinian factors via their residue fields, so sufficiently close coordinates give the displayed bounds. Invert the finitely many denominators in $A$. This does not remove $x$, which has generic support in this case. The strict bounds hold in smaller neighborhoods of the $y_i$'s.

For a fixed $\gamma \in F^\times$ with $|\gamma| \neq 1$, choose $\varepsilon$ so small that $1 + b(\gamma - 1)$ has absolute value $|\gamma|$ on $U_j$ and 1 on the other shrunk neighborhoods. Indeed its differences from $\gamma$ on the first group and from 1 on the others are strictly smaller than those absolute values. The determinant of multiplication by $1 + b(\gamma - 1)$ is a regular base function. The log of its absolute value near $x$ is $\log|\gamma|$ times the sum of $d_y$ on the $j$-th group. The determinant is nonvanishing on a neighborhood of $x$, so the sum is continuous there. It is integer-valued and equals $d_{y_j}$ at $x$. Repeat for the finitely many groups to obtain the first claim.

For the second claim, present $v$ near $y_j$ as

$$v = \frac{1}{m} \max_t \{\lambda_t + \log|h_t|\},$$

with $m > 0$ integral, $\lambda_t$ rational and $h_t$ rational functions that are units near its support. Such a presentation is obtained by clearing the denominators of the coefficients in the local log sums. If $x$ is nonrigid, the $h_t$ are regular at every support above $x$, since these supports are generic. Remove their poles from the inverse image by shrinking the base algebraically. If $x$ is rigid, each $y_j$ is an $F$-point. Replace each $h_t$ by its nonzero value at $y_j$, which has the same absolute value as $h_t$ on a neighborhood of $y_j$. In either case we can use $h_t \in B$ and the same representation on a smaller $U_j$.

Take $Q(T) = \sum_t T_t h_t$, with the Gauss radii $e^{\lambda_t}$. The Gauss norms of $Q - 1$ near all the $y_i$'s are bounded, and the Gauss norm of $Q$ near $y_j$ is bounded away from zero. Choose the selector $b$ above with $\varepsilon$ small enough relative to these bounds. On the $j$-th group, $1 + b(Q - 1)$ has the same Gauss norm as $Q$. On all other groups it has Gauss norm 1. Now take the determinant of multiplication by $1 + b(Q - 1)$ on $B[T]$. It is a polynomial $\sum_I f_I T^I$ with coefficients regular on the base. The Gauss determinant identity gives

$$\sum_{y \in p^{-1}(x') \cap U_j} d_y v(y) = \frac{1}{m} \max_I \left( \log|f_I(x')| + \sum_t I_t \lambda_t \right)$$

on the shrunk neighborhood. The maximum is finite. Terms whose coefficient vanishes at the support of $x$ may be discarded after shrinking analytically, while the other coefficients are units near that support. This is a locally convex PL presentation for the group near $x$. Sum over the groups to prove the claim. □

**Proposition 16.2.1** *If $u$ is locally psh on $C^{\mathrm{an}}$, then*

$$p_* u(x) = \sum_{p(y)=x} d_y u(y)$$

*is locally psh on $(\mathbb{P}^1_F)^{\mathrm{an}}$.*

***Proof*** At a fixed fibre take disjoint $U_j$'s on which $u$ has decreasing locally convex PL approximants. Take $V$ so that its whole inverse image lies in their union. For each

index, use the approximant of that index on each $U_j$. Its weighted sum is locally convex PL at every point of $V$: apply Lemma 16.2.1 at that point, with one neighborhood for each point of its fibre. The summed approximants decrease to $p_*u$, since each fibre is finite and the weights are positive. □

## 16.3 Connected curves and constant potentials

We will need that a smooth integral projective curve over $F$ has connected analytification. The finite map to the projective line allows us to prove this using just two algebraic coordinate charts and Laurent series.

**Lemma 16.3.1** *Let $A$ be a strictly $F$-affinoid algebra. Each subset of $\mathcal{M}(A)$ that is both open and closed is given by an idempotent of $A$, which evaluates to one there and to zero on the complement.*

***Proof*** First consider the split by $|f| \leq 1$ and $|f| \geq 1$ for $f \in A$. Put $P = A\langle T\rangle$, $Q = A\langle T^{-1}\rangle$, and $W = A\langle T, T^{-1}\rangle$. The two algebras in this split and the algebra of the intersection are

$$P/(T-f), \qquad Q/(1-fT^{-1}), \qquad W/(T-f),$$

respectively. Their spectra are the indicated subsets by the presentation of affinoid domains. The algebra $W$ consists of Laurent series with coefficients tending to zero at both ends. This description follows by reducing powers of two Tate variables using $TU = 1$, as in Section 2.6. The ideals in the displayed quotients are closed by Theorem 2.1.1.

By Lemma 2.3.2, the norm induced on a submodule of a finite affinoid module is equivalent to any of its quotient module norms.

In particular, if $C$ is an integral affinoid and $a \neq 0$ in $C$, compare the induced norm on $aC$ with the quotient norm for the one generator $a$. The latter norm of $ab$ is $\|b\|$ since multiplication by $a$ is injective. We obtain $\|ab\| \geq c\|b\|$ for a fixed $c > 0$. Iteration and Theorem 2.2.1 give a spectral point at which $a$ is nonzero.

The map $T - f : W \to W$ is injective. For an integral affinoid algebra, detect a nonzero coefficient of a nonzero Laurent series at a spectral point, as just proved. Evaluation there gives a nonzero Laurent series over a complete valued field, where multiplication by $T - f(x)$ is injective by the multiplicative Gauss norm. For general $A$, use a finite prime filtration of the finite $A$-module $A$. Such a filtration is obtained by choosing an element with maximal annihilator in each successive nonzero quotient, and terminates by Noetherianity. Laurent series preserve short exact sequences of finite modules because the submodule norm agrees up to equivalence with the induced norm and lifts of quotient coefficients can be chosen within a fixed factor of their quotient norms. These are the finite module norm comparisons of Lemma 2.3.2. Injectivity for the Laurent series with coefficients in each prime quotient therefore gives injectivity on $W$.

It follows that $A$ is the equalizer of the two restriction maps to the overlap. Indeed if $p \in P$ and $q \in Q$ agree there, then $p - q = (T - f)w$ with $w \in W$. Split $w = u - T^{-1}v$ into its nonnegative and negative powers, with $u \in P$ and $v \in Q$. Then $p - (T - f)u = q - (1 - fT^{-1})v$ is a common constant. If a constant has zero restrictions, write it as both $(T - f)u$ and $(1 - fT^{-1})v$. Injectivity on $W$ gives $u = T^{-1}v$, and disjoint exponent supports show that both vanish. This proves the equalizer assertion.

Let $U$ and $V$ be complementary open and closed subsets of $\mathcal{M}(A)$. They are compact. For every pair in $U \times V$, some $|f|$ has distinct values at its two coordinates because the points are distinct seminorms on $A$. Rescale $f$ so that the smaller value is strictly below one and the larger strictly above one. This is possible since $|F^\times|$ is dense in $\mathbb{R}_{>0}$. Compactness gives finitely many such tests that separate every pair. Successively split by the two inequalities for each test, using its image in each intermediate affinoid algebra. Every final chart lies in $U$ or lies in $V$. A pair with one point in each set would be separated strictly by a test despite lying on the same side of its split. Assign constants 1 and 0 on these charts, respectively, and glue back up the tree of splits. Idempotents that agree at all spectral points of an overlap are equal. Indeed the two products $e(1 - e')$ and $e'(1 - e)$ would otherwise give nonzero idempotents vanishing at every point. A nonzero idempotent cuts out a nonzero affinoid quotient, with a spectral point by Theorem 2.2.1. This also handles empty charts. The equalizer yields a unique element at each step. It is idempotent by injectivity of the two restrictions and has the prescribed values. The result at the root gives the required idempotent of $A$. □

**Lemma 16.3.2** *If $C$ is smooth, integral and projective over $F$, then $C^{\mathrm{an}}$ is connected.*

***Proof*** Choose a nonconstant rational function on $C$. Normality of the curve gives a morphism $p : C \to \mathbb{P}^1_F$, by using its inverse at poles. It is finite flat as in Section 16.2. Let $B_0$ and $B_\infty$ be the finite free algebras over $F[z]$ and $F[z^{-1}]$ on the two affine charts of the target. They are free because they are finite and torsion-free over these principal ideal domains. The inverse images of the two closed unit discs are the spectra of

$$B_0 \otimes_{F[z]} F\langle z\rangle, \qquad B_\infty \otimes_{F[z^{-1}]} F\langle z^{-1}\rangle.$$

Indeed on either chart, evaluation at a point where the base coordinate has absolute value at most one extends to its Tate algebra. With respect to a basis of the algebraic module, the finitely many values of the basis elements bound the evaluation on the finite tensor algebra. Conversely a seminorm on the finite tensor algebra restricts to such an algebraic point and is determined by restriction, by density. The two spectra are strict affinoid spectra by Proposition 2.3.1, choosing the bounding radius in the value group of $F$ in that proof. Compactness and the Hausdorff property identify the topologies. On the overlap use $F\langle z, z^{-1}\rangle$, the Laurent series algebra of the unit circle.

Let $U \subset C^{\mathrm{an}}$ be open and closed. Applying Lemma 16.3.1 gives idempotents on the two disc algebras. They agree on the overlap because their evaluations coincide there, by the same idempotent argument as in that lemma. In the free basis of $B_0$ the first idempotent has coefficients in $F\langle z\rangle$. The second has coefficients in $F\langle z^{-1}\rangle$

in the basis of $B_\infty$. The transition matrix and its inverse have entries in $F[z, z^{-1}]$, since the bases are algebraic and trivialize the modules on the affine charts. Multiply the second coefficient vector by the transition matrix to express it in the first basis. The resulting Laurent series have exponents bounded above. Thus the first vector has only finitely many powers of $z$. In the other basis the first vector likewise shows that the second vector has exponents bounded below. Consequently both idempotents are algebraic and glue to a global idempotent on the integral curve $C$. The only such idempotents are zero and one. Hence $U$ is empty or is all of $C^{\mathrm{an}}$. □

**Theorem 16.3.1** *Let $C$ be a smooth integral projective curve over $F$. Every locally psh function on $C^{\mathrm{an}}$ is constant.*

***Proof*** Fix a finite flat morphism $p$ to the projective line as above. For $N > 0$ integral, put $u_N = \max(u, -N)$. This function and all $\max(u_N, c)$, for $c$ rational, are locally psh by taking the same maxima on decreasing approximants. Their traces under $p$ are locally psh by Proposition 16.2.1, and so are constant by Theorem 16.1.1.

In particular the weighted multiset of values of $u_N$ on a fibre is independent of the fibre. To see this, note that the constant values of $p_* \max(u_N, c)$ determine the convex piecewise affine function of $c$ on each fibre, first on the rationals and hence everywhere. The jumps of its slope record the values of $u_N$ and their multiplicities. Thus $u_N$ has finite range. Near each point $y$ of a fixed fibre, upper semicontinuity and finite range give $u_N \leq u_N(y)$. Take disjoint neighborhoods with these bounds. The local sums of the degrees in the neighborhoods are the original degrees, by Lemma 16.2.1. Since $p_* u_N$ is constant, the sum of the bounds is attained on every nearby fibre. Every term equals its bound since all degrees are positive. Thus $u_N$ is locally constant. Connectedness from Lemma 16.3.2 makes it constant, and letting $N$ increase gives the result. □

## 16.4 Constancy on a proper fibre

We now allow a complete ground field $K$ with any ultrametric absolute value. The extension to a larger complete field can turn specified analytic points into rational points, without changing the local log presentations of an approximating sequence.

**Theorem 16.4.1** *Let $V$ be a geometrically connected projective scheme over $K$. Every locally psh function on $V^{\mathrm{an}}$ is constant.*

***Proof*** Choose two analytic points $x, y$ of $V$. Embed $\mathcal{H}(x)$ isometrically into a complete algebraically closed nontrivially valued field $K_1$. Such an extension is obtained by using a Gauss extension first if necessary, then completing an algebraic closure, as in the proof of Proposition 14.2.1. The point $x$ lifts to a $K_1$-rational point of $V_{K_1}$. The point $y$ has an analytic lift $y_1$ on $V_{K_1}$, by scalar extension of an affine box around it. The proof of the box description in Proposition 2.2.1 uses only the bounds on the coordinates and the evaluated polynomial series. It therefore works over $K$. Then

Proposition 2.4.1 gives the lift. Enlarge $K_1$ to a complete algebraically closed field $F$ containing $\mathcal{H}(y_1)$. Both $x$ and $y$ now lift to $F$-rational points. Pullback of the local log descriptions shows that the lifted function is locally psh. The scheme $V_F$ is connected by geometric connectedness.

The rational points of a connected projective $F$-scheme can be joined by chains of integral projective curves. We may take its reduction. The graph of irreducible components joined along nonempty intersections is connected. Each intersection has an $F$-point. It therefore suffices to join two distinct points on one irreducible component. In dimension one the component is the desired curve. In dimension at least two, blow up the two reduced points and embed the resulting integral projective scheme by a very ample line bundle. The inverse images of the points are nonempty effective Cartier divisors. Cut by successive general hyperplanes to obtain an integral curve, taking reductions after each section. The very ample system has no base points, and two general hyperplanes have a codimension-two intersection with a component that a third hyperplane does not contain. Hence Bertini gives irreducibility of each general section [Stacks, Tag 0G4F]. Choose the hyperplanes also to cut the two exceptional divisors properly. The complete intersection meets each of them by projective dimension, and contains neither as a curve component. Its image is an integral projective curve through the two points.

Normalize the curves of this chain. Their normalizations are smooth integral projective curves over $F$, and the rational points where the curves meet lift to rational points on the normalizations. The pullback of the lifted function to every such curve is locally psh and therefore constant by Theorem 16.3.1. The constants agree at the meeting points, so the values at the lifts of $x, y$ agree. This proves the theorem. □

**Corollary 16.4.1** *Let $k$ be countable and trivially valued. Let $p : Y \to T$ be a projective surjective morphism of integral projective schemes with geometrically connected fibres, and let $M$ be a semiample rational line bundle on $T$. Every $p^*[M]$-psh function on $Y^{\mathrm{an}}$ is constant on each fibre of $p^{\mathrm{an}}$.*

***Proof*** Let $\phi$ be a $p^*[M]$-psh function on $Y^{\mathrm{an}}$, and let $t \in T^{\mathrm{an}}$, with completed residue field $K = \mathcal{H}(t)$. The analytic fibre over $t$ is the analytification of $V = Y \times_T \operatorname{Spec} K$. On an affine chart this identification is restriction from the base field $K$. In the reverse direction the seminorm at a point over $t$ extends the given norm on the residue field at its support, and its completed residue field contains $K$ isometrically. Evaluation there yields the required point of $V^{\mathrm{an}}$. The two constructions are inverse by density of the uncompleted residue field in $K$. They respect restriction on algebraic charts.

Choose a section $s$ of a generating power $mM$ that is nonzero at the support of $t$. By Lemma 13.13.2, $\phi - m^{-1} \log |p^* s|$ is locally psh near the fibre. Its restriction to $V^{\mathrm{an}}$ is locally psh by restriction of scalars and algebraic pullback. The reference $m^{-1} \log |p^* s|$ is constant on that fibre, since the centre-based norm of a pulled-back section is its norm downstairs. The scheme $V$ is projective and geometrically connected over $K$, so Theorem 16.4.1 gives the conclusion. □

## 16.5 From smooth alterations to envelopes

A smooth projective alteration factors through a finite cover of the given variety. We first descend the envelope property to that cover along the part of the factorization with connected fibres. The finite descent theorem then gives the property on the original variety.

**Theorem 16.5.1** *Let $X$ be normal, integral and projective over an algebraically closed trivially valued field. Every real numerical class on $X$ has the envelope property.*

***Proof*** First suppose the ground field $k$ is countable and let $L$ be a rational ample line bundle on $X$. By Theorem C.7.1 there is a smooth projective alteration $g : Y \to X$. Factor it as $Y \xrightarrow{\beta} T \xrightarrow{h} X$ by Stein factorization [Stacks, Tag 03H0]. Thus $h$ is finite and $\beta$ has geometrically connected fibres. Over an affine open of $X$, the coordinate ring of $T$ is the ring of sections on the inverse image in $Y$. This is an intersection of integrally closed local rings in $k(Y)$, so $T$ is integral and normal. The generic fibre of $g$ is the spectrum of $k(Y)$ since $g$ is generically finite and $Y$ is integral. So $k(T) = k(Y)$ and $\beta$ is birational. It is projective since its graph is closed in the product of $Y$ with $T$ over $k$, and $Y$ is projective. Set $M = h^*L$, which is ample by finiteness of $h$.

For a bounded-above increasing net $u_i$ of $[M]$-psh potentials on $T$, the pullbacks to $Y$ are psh in the nef class $[\beta^*M]$. By Theorems 12.5.1 and 12.6.1, their regularized supremum is a $[\beta^*M]$-psh function $\Phi$ on $Y$. It has the pointwise supremum of the pullback values on $\mathcal{D}(Y)$ by Theorem 12.2.1. Since $\beta$ is proper and dominant, it is surjective, and $\beta^{\mathrm{an}}$ is surjective by Lemma 13.13.1. By Corollary 16.4.1, $\Phi = \phi \circ \beta^{\mathrm{an}}$ for a function $\phi$ on $T^{\mathrm{an}}$. Then Corollary 14.3.1 gives $\phi \in \mathrm{PSH}(T, [M])$. The map $\beta$ identifies the function fields and, by properness, gives centres for the same divisorial valuations. Thus $\phi = \sup_i u_i$ on $\mathcal{D}(T)$. The increasing-net criterion in Corollary 12.2.1 gives the envelope property for $[M]$ on $T$. Apply Theorem 15.1.1 to $h$ to obtain the property for $[L]$ on $X$.

By Proposition 13.14.1, the rational ample case holds over every algebraically closed trivially valued field.

For a real ample class $\omega$ choose rational ample classes $\omega_j \to \omega$ with $\omega_j - \omega$ ample. For any nonempty bounded-above family of $\omega$-psh potentials, its regularized supremum is $\omega_j$-psh for each $j$, by the rational ample case and positivity of the differences. The supremum is finite at the trivial valuation. Regard the function as a constant decreasing net in these classes and apply Theorem 6.2.2. This gives the envelope property in every real ample class. In particular an increasing net of homogeneous psh potentials in an ample class has a psh regularized supremum, since such functions are nonpositive. Rescaling is by homeomorphisms on the analytification, so this regularized supremum is homogeneous. It agrees with the pointwise supremum on the divisorial set by Theorem 12.2.1. The hypothesis of the homogeneous transfer Corollary 7.6.1 is therefore satisfied. Apply that transfer and the increasing-net criterion to obtain the property in every real numerical class. □

**Corollary 16.5.1** *Let $X$ be unibranch, integral and projective over an algebraically closed trivially valued field. Every real numerical class on $X$ has the envelope property.*

***Proof*** Apply Theorem 16.5.1 to the normalization, and descend the resulting property by Corollary 15.2.1 in positive characteristic and Corollary 15.4.1 in characteristic zero. □

Together with Proposition 12.3.1, this shows that an integral projective variety over an algebraically closed trivially valued field is unibranch if and only if one of its ample classes has the envelope property. For the implication from the envelope property to unibranchness, let $x \in X$. Since the normalization is finite, its points over $x$ correspond to the maximal ideals of the stalk at $x$ of its pushforward algebra. This stalk is the integral closure of $\mathcal{O}_{X,x}$ [Stacks, Tag 0C3B]. The number of its maximal ideals is the number of branches of $\mathcal{O}_{X,x}$, which is one exactly when $\mathcal{O}_{X,x}$ is unibranch [Stacks, Tag 0C37]. Thus a normalization that is a homeomorphism makes $X$ unibranch.

## 16.6 Decreasing limits in varying classes

The envelope property in the limiting class supplies an upper bound for all the homogeneous slices of a decreasing limit. This bound is enough to return from the slices to the original functions, even at their poles.

**Theorem 16.6.1** *Let $X$ be integral and projective. Suppose that the envelope property holds in $\theta \in N^1(X)_{\mathbb{R}}$. If $\theta_i \to \theta$ and a net of $u_i \in \mathrm{PSH}(X, \theta_i)$ decreases pointwise to $u \not\equiv -\infty$, then $u \in \mathrm{PSH}(X, \theta)$. In particular this holds in every real numerical class on an integral unibranch projective variety over an algebraically closed field.*

***Proof*** The inequalities $u_i(v) \le u_i(v_{X,0})$ pass to the limit, so $b = u(v_{X,0}) = \sup u$ is finite. Subtract $b$ from every function. For each $\lambda < 0$, $\widehat{(u_i)}_\lambda$ is homogeneous $\theta_i$-psh by Theorem 7.1.1. These slices decrease with $i$. Their limits $q_\lambda$ belong to $\mathrm{PSH}^{\mathrm{hom}}(X, \theta)$ by Theorem 7.2.1. Commuting the infima in $i$ and in $t > 0$ gives

$$q_\lambda(v) = \inf_{t>0} \{ tu(t^{-1}v) - t\lambda \}.$$

For any fixed $v$, the function $t \mapsto tu(t^{-1}v)$ is the decreasing pointwise limit of the convex perspectives of the $u_i$, hence convex with the extended-value convention of Theorem 7.1.2. It is bounded above by zero. The supporting-line argument of that theorem therefore applies to $u$ and gives $u = \sup_{\lambda<0}(q_\lambda + \lambda)$.

The sequence $q_{-n}$ increases and is bounded above by zero. By the envelope property its upper regularized supremum $Q$ is $\theta$-psh. Its pointwise supremum before regularization is homogeneous, and regularization commutes with scaling since scaling

by each positive real number is a homeomorphism. Thus $Q$ is homogeneous and dominates every $q_\lambda$. Apply Lemma 7.6.2. The last assertion uses Corollary 16.5.1.□

# Part IV
# Global energy theory

Intersections on test configurations give atomic measures on a proper analytification. Energy allows us to pass from these measures to potentials that may have poles. Differentiation with respect to an obstacle leads to the contact-set identity, and duality gives a variational formulation of the Monge–Ampère equation. The preceding part proves the envelope property for integral unibranch projective varieties over an algebraically closed field. On an integral regular projective curve we compare the intersection measure to the local curvature current by calculating the contribution of each valuation ray.

# Chapter 17
# Mixed measures and energy

Blow up a rational point on the special fibre of $\mathbb{P}^1_k \times \mathbb{A}^1_k$. A polarized model on this blowup places mass at the trivial valuation and at the order of the chosen point, and its two masses are the degrees on the two components of the special fibre; Example 17.3.1 computes them. A further blowup can add components without changing the potential. The projection formula explains which of the new degrees vanish and why the measure depends only on the original potential.

Throughout this chapter, $X$ is a reduced projective scheme of pure dimension $n$ over a trivially valued field $k$. Measures and energy use intersection degrees over $k$, and we keep the total mass in the formulas, including when it is zero. The first construction is multilinear on model data: an intersection pairing and its mixed atomic measures (Proposition 17.2.1). A real model pair $(\theta, \varphi)$ is semipositive exactly when $\varphi$ is $\theta$-psh (Theorem 17.4.1). Positivity for relatively nef classes then gives the estimates that extend mixed measures to continuous potentials (Theorem 17.5.1) and to finite-energy potentials, first in ample classes (Theorem 17.8.1) and then in nef classes (Proposition 17.9.1). The last sections prove locality at a strict maximum (Theorem 17.10.1), compare two energy differences (Section 17.11), establish mixed energy estimates (Theorem 17.12.1), and test mixed measures against finite-energy potentials (Theorem 17.13.1). Chapters 18 and 19 rely on these tools.

## 17.1 The intersection pairing on models

Put $\mathrm{PL}_{\mathbb{R}}(X) = \operatorname{span}_{\mathbb{R}} \mathrm{PL}_{\mathbb{Q}}(X)$. Choose a test configuration dominating the product $X \times \mathbb{A}^1$, integrally closed along the puncture. Write $\overline{\mathcal{X}}$ for its compactification over $\mathbb{P}^1$, and $\rho : \overline{\mathcal{X}} \to X$ for the projection through the product. For a real numerical class $\theta$ on $X$ and a function $\varphi \in \mathrm{PL}_{\mathbb{R}}(X)$, the pair $P = (\theta, \varphi)$ is represented on a sufficiently large model by the class

$$P_{\overline{\mathcal{X}}} = \rho^*\theta + D, \qquad \varphi = \varphi_D.$$

Here $D$ is a finite real linear combination of vertical Cartier divisors. To represent finitely many pairs, choose a common model by Section 5.2.

A real vertical divisor on a fixed model integrally closed along the puncture is determined by its values at the special-fibre valuations. Indeed, choose a finite list of rational Cartier divisors spanning all the expressions in question. Their orders at the generic points of special-fibre components form a matrix with rational coefficients. Its real kernel is spanned by its rational kernel. For a rational linear combination the assertion follows from Proposition 5.2.1, applied with both signs. Hence it holds for every real combination. Consequently, two representatives of a pair give the same class after pullback to a common model, so the following intersection is well defined.

**Definition 17.1.1** The intersection pairing of $n+1$ pairs $P_0, \dots, P_n$ is

$$\langle P_0, \dots, P_n \rangle = \big( (P_0)_{\overline{\mathcal{X}}} \cdots (P_n)_{\overline{\mathcal{X}}} \cdot [\overline{\mathcal{X}}] \big).$$

The pairing is symmetric and multilinear. It is independent of the chosen model by the projection formula of Section 3.5, since the fundamental cycle of a dominating model pushes forward to the fundamental cycle by birationality on each component. Replacing a numerical representative for $\theta$ has no effect, since the projection formula pairs the change with a curve class on $X$.

## 17.2 The mixed atomic measure

For pairs $P_i = (\theta_i, \varphi_i)$, $1 \le i \le n$, write $M_i = (P_i)_{\overline{\mathcal{X}}}$ on a common model. Write $E$ for a component of its special fibre, $b_E = \operatorname{ord}_E(t)$, and $v_E$ for the corresponding valuation on $X$ from Proposition 5.3.1. Define

$$\mathrm{MA}(P_1, \dots, P_n) = \sum_E b_E (M_1 \cdots M_n \cdot [E])\, \delta_{v_E}. \tag{17.1}$$

Each component $E$ has dimension $n$. At its generic point it lies in a single component of the total space by the DVR criterion, and $t$ gives a principal codimension-one condition there. Each component of the total space is the closure of the punctured product of one component of $X$.

**Proposition 17.2.1** *The measure in* (17.1) *is independent of the common model. For* $h \in \mathrm{PL}_{\mathbb{R}}(X)$,

$$\int_{X^{\mathrm{an}}} h \, \mathrm{d}\, \mathrm{MA}(P_1, \dots, P_n) = \langle (0, h), P_1, \dots, P_n \rangle.$$

*Its total mass is* $(\theta_1 \cdots \theta_n \cdot [X])$. *It is positive if the classes* $M_i$ *are nef on the curves in the special fibre.*

***Proof*** First use a common model on which $h = \varphi_D$. The vertical divisor $D$ has Weil coefficient $b_E h(v_E)$ at $E$. Intersecting its cycle with the classes $M_i$ gives the

displayed formula. For the same formula on a model representing just the $P_i$, use a dominating model also representing $h$. A component upstairs mapping onto a component downstairs has the same normalized valuation as its image. The two DVRs lie in the same function field and one dominates the other, so they are equal, since a valuation ring is maximal for domination [Stacks, Tag 00I9]. Hence their orders and multiplicities $b_E$ agree. Any component mapping to smaller dimension contributes zero by the projection formula. The pushforward of the upstairs special-fibre cycle is the downstairs special-fibre cycle, again by the projection formula for the Cartier fibre. Thus the integral is unchanged. The identity and the uniform density of rational model functions prove independence.

The special-fibre cycle on the compactification is rationally equivalent to the fibre at infinity because they are pullbacks of two rationally equivalent points of $\mathbb{P}^1$. The model at infinity is the product, where the divisor corrections vanish. This gives the stated mixed intersection degree for the mass. For positivity, the classes restricted to each projective component $E$ are nef. By Proposition A.1.1 they are limits of ample classes, with nonnegative mixed intersection degrees. □

On a reduced projective scheme with components of different dimensions the same construction applies to each component in its own dimension. Restrict a model to a component by closing its marked punctured product in the model, then taking the modification integrally closed along the puncture. The pullback of the marked line bundles gives the restriction of the potentials. Push the component measures forward to $X^{\mathrm{an}}$, and sum them with specified nonnegative weights. The rest of this chapter uses the pure-dimensional pairing.

A pair is *semipositive on the model* if its representative is nef on the special-fibre curves. Pullback to a dominating model preserves this property by the degree formula. Conversely, for a curve on the original special fibre, close a closed point of its generic fibre upstairs to obtain a curve mapping onto it. The pushforward degree is positive, so nefness upstairs implies nefness downstairs. The condition can therefore be checked on any common dominating model.

## 17.3 Integration by parts on models

For $n \geq 1$, real model functions $f, g$, and fixed pairs $P_2, \ldots, P_n$, symmetry gives

$$\int f \, \mathrm{d}\,\mathrm{MA}((0,g), P_2, \ldots, P_n) = \int g \, \mathrm{d}\,\mathrm{MA}((0,f), P_2, \ldots, P_n). \tag{17.2}$$

A pair with zero numerical class is allowed in this identity, and its mixed measure may be signed. When all $P_1, \ldots, P_n$ are semipositive, the integral formula implies

$$\left|\langle (0,h), P_1, \ldots, P_n \rangle\right| \leq \|h\|_\infty (\theta_1 \cdots \theta_n \cdot [X]).$$

Thus the pairing is monotone in the potential of a slot whose numerical class is fixed, provided the other slots are semipositive. Adding a constant to the potential in that slot adds the constant times the stated mass.

*Example 17.3.1* Let $X = \mathbb{P}^1$ and let $p$ be a rational point of $X$. On the blowup of the point $(p, 0)$ of the special fibre of the product $\mathbb{P}^1 \times \mathbb{A}^1$, take $L = \mathcal{O}(d)$ for a positive integer $d$ and $D = -aE$, where $E$ is exceptional and $0 \le a \le d$ is real. The degrees on the strict transform and the exceptional component are $d - a$ and $a$, since the exceptional curve has self-intersection $-1$ and meets the strict transform once. Both special-fibre multiplicities equal one. The measure is $(d-a)\delta_{v_{X,0}} + a\delta_{\mathrm{ord}_p}$. Its mass is $d$, and the potential changes slope by $a$ at $\mathrm{ord}_p$. This recovers the slope calculation of Section 1.5.

## 17.4 Semipositivity and psh potentials

A semipositive model pair has nef base class. Indeed, for a closed integral curve $C \subset X$, take the closure of $C$ times the punctured base in the compactified model, with its reduced structure. This gives an integral surface. Its fibre at zero is an effective one-cycle, all of whose components are curves in the original special fibre. Its degree against the model class equals the degree of the fibre at infinity, namely $(\theta \cdot C)$, by rational equivalence of the fibres. Each summand at zero is nonnegative.

**Theorem 17.4.1** *A real model pair $(\theta, \varphi)$ is semipositive if and only if $\varphi \in \mathrm{PSH}(X, \theta)$. If it is semipositive, there exist finite rational Fubini–Study approximants $\varphi_j$ decreasing uniformly to $\varphi$, with classes tending to $\theta$.*

***Proof*** Suppose first that the pair is semipositive. On its model choose a relatively ample linearized line bundle $\mathcal{A}$. Its marking on the product defines a line bundle $A$ on $X$ and a vertical Cartier correction. For $\varepsilon > 0$, the restriction of the real class $\rho^*\theta + D + \varepsilon\mathcal{A}$ to the special fibre is ample by Proposition A.1.1, as is the base class $\theta + \varepsilon[A]$. Approximate its two coefficients $(\theta + \varepsilon[A], D + \varepsilon(\mathcal{A} - \rho^*A))$ in finite-dimensional real spaces spanned by rational classes and rational Cartier divisors. Openness of the ample cone, applied separately to the special fibre and to $X$, ensures that sufficiently close rational approximations are ample on both. Every fibre away from zero is the scalar extension of the marked fibre. By openness of fibrewise ampleness for a projective morphism [Stacks, Tag 0D2N], the rational line bundles thus obtained are relatively ample. Their functions are finite rational Fubini–Study by Theorem 5.5.1. For a finite real span of vertical divisors, convergence of coefficients gives uniform convergence of the functions by continuity and compactness. Choose a sequence with uniform errors at most $2^{-j-2}$ and class limit $\theta$, then add the rational constants $2^{-j}$. The result decreases, because the lower bound $\varphi + (3/4)2^{-j}$ is greater than the upper bound $\varphi + (5/4)2^{-j-1}$ for the next term. This proves $\varphi \in \mathrm{PSH}(X, \theta)$.

Conversely, let $P = (\theta, \varphi)$ be a model pair with $\varphi \in \mathrm{PSH}(X, \theta)$. Since $\varphi$ is finite and bounded, $\theta$ is nef by the last paragraph of Section 6.2.1. Choose finite rational

Fubini–Study approximants $P_j$ with classes tending to $\theta$ and functions decreasing to $\varphi$, by Proposition 6.2.1. They are semipositive on their models by Theorem 5.5.1. Fix a model integrally closed along the puncture and determining $P$.

We first reduce the degree test on this special fibre to invariant integral curves. For a curve $C$ there, take the closure of the family $s \cdot C$ for $s \in \mathbb{G}_m$, in the product of the special fibre with $\mathbb{P}^1_s$. This integral surface dominates the parameter line. The action of the torus on the special fibre together with multiplication on $s$ preserves the family. Its fibre at $s = 0$ is therefore an effective invariant one-cycle. Each of its components is invariant. Indeed, multiplying an integral component by the torus gives an irreducible image containing the original component and contained in the invariant support of this one-cycle, a finite union of curves; its closure is therefore that component. The degree of $P$ on $C$, the fibre at $s = 1$, equals the degree on this one-cycle by rational equivalence. Thus it is enough to test every invariant integral curve.

If $n = 0$, there are no curves and the condition is automatic. Otherwise fix such a curve $C$, blow it up, and take the relative integral closure along the puncture. Write $\mu : \mathcal{Y} \to \mathcal{X}$ for the composite and $F$ for the effective Cartier divisor given by the pullback of the ideal of $C$. This is a test configuration. Indeed the centre is invariant, the blowup is an isomorphism on the punctured product, and its rings embed in their localizations away from zero, so it is flat over the base. The line bundle $-F$ is ample over $\mathcal{X}$ by the blowup construction and finite pullback. Tensoring with a sufficiently high power of the pullback of the auxiliary bundle on $\mathcal{X}$ gives a relatively ample bundle over $\mathbb{A}^1$ [Stacks, Tag 0C4K].

Let $Q$ denote the pair corresponding to this ample bundle on $\mathcal{Y}$. The pushforward of $[F] \cdot Q_{\mathcal{Y}}^{n-1}$ is a positive multiple $d[C]$. Every component of $F$ maps into $C$, and at least one dominates it because $\mu$ surjects and $F$ is the inverse image of $C$ as a set. The coefficient is the sum of the top degrees of the ample restriction over the generic point of $C$, with the fibre multiplicities. Equivalently cut the components by $n - 1$ general very ample sections of a fixed power of the bundle and push forward the resulting effective curve cycle. Its intersection over the generic point of $C$ is nonempty by ampleness on that fibre. A finite field extension for general sections only multiplies the pushforward degrees by its degree.

The projection formula now yields

$$d(P_{\overline{\mathcal{X}}} \cdot C) = \langle (0, \varphi_F), P, Q, \dots, Q \rangle,$$

where $Q$ occurs $n - 1$ times. With $P_j$ in that slot the right-hand side is nonnegative, because $F$ is effective and the mixed measure is positive. The pairings with $P_j$ converge to that with $P$. For the numerical part this uses its convergence on the common fixed model for $\varphi_F$ and $Q$. For the difference of the two functions in the $P_j$ and $P$ slots, use symmetry to integrate it against the fixed signed atomic measure $\mathrm{MA}((0, \varphi_F), Q, \dots, Q)$. Pointwise convergence suffices on its finite support. Thus $P$ has nonnegative degree on $C$, completing the degree test. □

## 17.5 Continuous mixed measures

By Propositions 4.1.1 and 6.2.1, every continuous $\theta$-psh function admits uniform approximation by semipositive model pairs. To pass mixed measures to the limit, we first need a bound for each model test function.

Every $h \in \mathrm{PL}_{\mathbb{Q}}(X)$ can be written as $(0,h) = Q^+ - Q^-$, where $Q^+$ and $Q^-$ are semipositive model pairs with the same ample base class. Indeed, write $h = m^{-1}(\varphi_{\mathcal{I}} - \varphi_{\mathcal{J}})$ using Theorem 5.4.1. Choose a sufficiently large ample bundle $A$ so that twisting each of the finitely many coefficient ideals of the two flags by $A$ makes it globally generated. The product bundle tensored with either flag is generated by sections of the form $st^j$ for sections of a twisted coefficient ideal. On the blowup the invertible pullbacks are relatively generated. Dividing their model pairs by $m$ gives the decomposition with base class $[A]/m$.

**Theorem 17.5.1** *For $i = 1, \ldots, n$, let $u_i$ be continuous and $\theta_i$-psh. There is a positive mixed Radon measure $\mathrm{MA}((\theta_i, u_i)_{i=1}^n)$ of mass $(\theta_1 \cdots \theta_n \cdot [X])$. For semipositive model pairs it agrees with (17.1). It is continuous for uniform convergence of the potentials accompanied by convergence of their numerical classes.*

***Proof*** We first work with semipositive model pairs in bounded sets of numerical classes. Fix a rational model function $h$ and compare two model measures by telescoping their integrals in the $n$ slots. For one difference of pairs, write

$$(\theta, f) - (\eta, g) = (\theta - \eta, 0) + (0, f - g).$$

For the latter term, use symmetry to integrate $f - g$ against the mixed measure with one slot $(0,h)$. Use $Q^+ - Q^-$ for this slot and bound each integral by $\|f-g\|_\infty$ times the mass of the corresponding positive measure. These masses are bounded since all other numerical classes lie in bounded sets.

For the numerical difference, fix a basis of rational classes $D_l$ and express each as a difference of two ample classes by adding an ample multiple. Express $(\theta - \eta, 0)$ as a linear combination of the resulting differences of pairs with zero function. The absolute value of each paired term is bounded by $\|h\|_\infty$ times the sum of two mixed masses. The coefficient sum is bounded by a fixed constant times $\|\theta - \eta\|$. Hence for the chosen $h$ the change of integral is bounded by a constant times $\|f-g\|_\infty + \|\theta-\eta\|$, uniformly for all the other model pairs with bounded classes.

Take uniform model approximation for each $u_i$, using the product directed set, with the classes converging to $\theta_i$. The estimates give a Cauchy net for each rational model test, with the same limit for any two choices of approximation. Since the masses are uniformly bounded, density gives the same conclusion for every continuous test function. The limiting functional is positive and its value on one is the limiting mass. Theorem 4.3.1 gives the desired Radon measure. The model estimate passes to the limit for every rational model test, giving continuity there, and the mass bound and density give weak continuity on all continuous tests. □

For a nef class $\theta$ and continuous $\theta$-psh potential $u$, define the unnormalized energy by

$$E_\theta(u) = \frac{1}{n+1} \sum_{j=0}^{n} \int u \, \mathrm{d}\,\mathrm{MA}\big((\theta, u)^j, (\theta, 0)^{n-j}\big). \tag{17.3}$$

Here exponents indicate repeated pairs. In the model case the sum telescopes to

$$\frac{1}{n+1}\big(\langle (\theta, u)^{n+1} \rangle - \langle (\theta, 0)^{n+1} \rangle\big).$$

The second term vanishes by projection to $X$. Thus (17.3) extends the top intersection energy, even when the volume is zero.

For two continuous potentials $u, v$ in the same class, the difference identity is

$$E_\theta(u) - E_\theta(v) = \frac{1}{n+1} \sum_{j=0}^{n} \int (u - v) \, \mathrm{d}\,\mathrm{MA}\big((\theta, u)^j, (\theta, v)^{n-j}\big).$$

For models this is the difference-of-powers identity and Proposition 17.2.1. To pass to continuous potentials, use uniform approximation of $u, v$ with varying rational classes. Promote both at each index to one rational class using small ample differences, as in Theorem 6.2.1. The functions do not change under this promotion. Apply the identity to the common rational classes and pass to the limit by Theorem 17.5.1 and uniform convergence of the test functions. In particular energy is monotone and

$$E_\theta(u + c) = E_\theta(u) + c(\theta^n \cdot [X]).$$

These identities retain their meaning when the mass vanishes. In positive mass one can divide them by $(\theta^n \cdot [X])$ to work with probability measures.

## 17.6 The vertical intersection inequality

**Proposition 17.6.1** *For $n \geq 1$, a real model function $h$, and semipositive model pairs $Q_1, \ldots, Q_{n-1}$,*

$$\langle (0, h), (0, h), Q_1, \ldots, Q_{n-1} \rangle \leq 0.$$

***Proof*** On a common model write $h = \varphi_D$. Approximate the $Q_i$ on this fixed model by rational relatively ample pairs, as in the first part of the proof of Theorem 17.4.1. The intersection is continuous in these coefficients. For relatively ample rational bundles, add sufficiently large multiples of the pullback of a point of $\mathbb{P}^1$ to make their extensions ample over $k$ [Stacks, Tag 0C4K]. This does not change the intersection, since the fibre line bundle is trivial on each component of the support of $D$.

Take sufficiently divisible very ample powers and cut successively by general sections. The resulting two-dimensional cycle is effective, and each component meets the fibre at zero properly. To arrange this, at each cut choose a section that is nonzero generically on every component of the current cycle and of its intersection with the fibre at zero. Such a choice exists over an infinite coefficient field, and over a finite field can be made after a finite extension. Pushforward of the base change multiplies

the full degree by the field degree, so the sign is unchanged. The degrees for the powers differ by positive factors from the desired degrees. Thus it suffices to show that the square of the restriction of $D$ has nonpositive degree on each integral surface in the cut. A surface mapping into a point away from zero has trivial restriction.

On a surface dominating $\mathbb{P}^1$, pull back to a proper regular resolution, which exists for integral surfaces of finite type over a field [Stacks, Tags 0ADX and 0BGP]. The projection formula preserves the degrees. On this regular surface write the fibre as $F = \sum_i b_i E_i$ and the pullback of $D$ as $\sum_i b_i a_i E_i$. Prime divisors are Cartier on a regular surface, and $F \cdot E_i = 0$ since the fibre line bundle is trivial on each $E_i$. Thus expanding the square and eliminating the diagonal terms gives

$$(D^2) = -\sum_{i<j} b_i b_j (a_i - a_j)^2 (E_i \cdot E_j).$$

Distinct prime curves intersect with nonnegative degree. This proves the inequality on every surface and hence on the original model. □

In particular $E_\theta$ is concave on the space of continuous $\theta$-psh potentials. For semipositive model pairs in a fixed class, the second derivative of $E_\theta((1-s)u + sv)$ is

$$n\langle (0, v-u), (0, v-u), (\theta, (1-s)u + sv)^{n-1} \rangle \le 0.$$

This follows by differentiating the intersection polynomial. The assertion passes to continuous potentials by simultaneous approximation in common varying rational classes. For $n = 0$ the energy is affine. The first derivative on a segment of continuous potentials is

$$\frac{\mathrm{d}}{\mathrm{d}s} E_\theta((1-s)u + sv) = \int (v-u)\, \mathrm{d}\,\mathrm{MA}((\theta, (1-s)u + sv)^n).$$

Indeed the difference identity from Section 17.5 shows that each of its $n+1$ mixed measures converges weakly to the measure on the right as the increment in $s$ goes to zero. At the ends use the one-sided derivative.

## 17.7 Approximation in a fixed ample class

Let $\theta$ be ample and real. The continuous approximants in Proposition 6.4.2 are real model functions, since each is a real multiple of a rational Fubini–Study potential plus a constant. Their model pairs are semipositive by Theorem 17.4.1. Thus every $\theta$-psh function is a decreasing limit of semipositive model potentials in exactly the class $\theta$.

For an arbitrary $\theta$-psh $u$ in this ample class we may therefore set

$$E_\theta(u) = \inf\{E_\theta(f) : f \text{ continuous and } \theta\text{-psh},\ f \ge u\}.$$

It extends the continuous energy by monotonicity. For every decreasing continuous psh approximation $f_j \downarrow u$, the infimum equals $\lim_j E_\theta(f_j)$. Indeed if $g$ is a continuous psh majorant of $u$, for every $\delta > 0$ compactness gives $f_j < g + \delta$ eventually. Energy monotonicity and the constant identity then give the required upper bound after letting $\delta$ decrease to zero. The opposite bound holds by the definition. The *finite-energy class* in this ample class consists of the potentials with $E_\theta(u) > -\infty$.

The same compact comparison proves decreasing continuity for psh nets in a fixed ample class with a psh limit. In detail, if $u_j \downarrow u$ and $g \geq u$ is a continuous psh majorant, the open sets $\{u_j < g + \delta\}$ increase to cover the analytification. Hence $g+\delta$ is an eligible majorant eventually. This gives $\inf_j E_\theta(u_j) \leq E_\theta(g)+\delta(\theta^n \cdot [X])$. Take the infimum over $g$ and then let $\delta \to 0$, also if the infimum is $-\infty$. The reverse inequality follows from monotonicity of the infimum definition.

## 17.8 Mixed measures of finite-energy potentials

Let $\omega_0, \ldots, \omega_n$ be ample real classes and $u_l \in \mathrm{PSH}(X, \omega_l)$. Define their *extended mixed pairing* as the infimum of the model pairings of semipositive model majorants in these same numerical classes. Such majorants exist by Section 17.7. The infimum is the limit along any simultaneous decreasing model approximations. Indeed, given any finite continuous model majorants $g_l \geq u_l$ and $\delta > 0$, compactness gives that each approximant to $u_l$ is at most $g_l + \delta$ eventually. The bound for adding a constant in each slot is $\delta$ times the corresponding mixed degree on $X$, independent of the potentials. Comparison and letting $\delta$ decrease prove the equality with the infimum. The same comparison with $g_l + \delta$ for decreasing nets of psh potentials proves joint continuity along decreasing nets, allowing value $-\infty$. In particular the diagonal pairing is $(n + 1)E_{\omega_0}(u_0)$ when all the slots are equal.

We next show that the extended pairing is finite when each $u_l$ has finite energy in $\omega_l$. First suppose the potentials are nonpositive, which can be arranged by translation. Use a tail of model approximants whose functions are nonpositive as well, after translating by a common larger constant if needed. A pairing of nonpositive semipositive model potentials is nonpositive. Compare it with the pairing of the same classes with all functions zero, one variable at a time. The latter pairing is zero because all factors pull back from $X$. Moreover, increasing any of the numerical classes by a nef class can only decrease the pairing. The difference has one slot of the form $(\eta, 0)$ with $\eta$ nef, and all slots still have nonpositive semipositive potentials, so its sign is nonpositive. These comparisons pass to the extended pairings in ample classes by choosing the same approximants after each increase of class.

Now let each $u_l$ have finite energy in $\omega_l$. Choose an ample $\Omega$ with $\Omega - \omega_l$ nef for every $l$. The potentials have finite energy in $\Omega$. Indeed, choose $C \geq 1$ with $C\omega_l - \Omega$ nef, and apply the preceding comparison on the diagonal, followed by homogeneity of the model pairing and monotonicity in the potential:

$$E_\Omega(u_l) \geq E_{C\omega_l}(u_l) = C^{n+1}E_{\omega_l}(u_l/C) \geq C^{n+1}E_{\omega_l}(u_l).$$

These identities and inequalities follow first on models and then by decreasing approximation. Thus it suffices to bound the mixed pairing from below in the common class $\Omega$.

Use decreasing model approximants $f_l$ in that class and form $g = (f_0 + \cdots + f_n)/(n+1)$. When all the functions are nonpositive, every term in the expansion of $(n+1)E_\Omega(g)$ is nonpositive. The coefficient of the pairing with each $f_l$ in one slot is $(n+1)!/(n+1)^{n+1}$. It follows that this mixed pairing is bounded below by

$$\frac{(n+1)^{n+1}}{(n+1)!}\,(n+1)E_\Omega(g).$$

Concavity on models bounds $E_\Omega(g)$ from below by the average of the energies of the $f_l$, which are bounded below by the finite energies of the $u_l$. The lower bound passes to the infimum. This proves finiteness. Adding constants changes the pairing by finite mixed degrees, so the assertion applies without the nonpositivity assumption.

**Theorem 17.8.1** *For ample classes $\omega_1, \ldots, \omega_n$ and finite-energy $\omega_i$-psh potentials $u_i$, the mixed measure is the weak limit of the model measures along any simultaneous decreasing model approximations in the respective classes. It has mass $(\omega_1 \cdots \omega_n \cdot [X])$, and it is continuous along decreasing nets of finite-energy potentials with finite-energy limits in those classes.*

***Proof*** For rational PL $h$, express $(0, h) = Q^+ - Q^-$ as in Section 17.5. Each of the two pairings against the approximants to $u_1, \ldots, u_n$ has a finite limit by the bound just proved. Their difference gives the limit of the integrals of $h$ against the model measures, independently of choices. Those measures are positive with a fixed total mass. Uniform density and the compact-space representation theorem give the limit measure and its mass. Joint decreasing continuity of the extended pairing gives convergence of the same rational PL tests for decreasing finite-energy tuples, and the mass bound gives weak convergence on all continuous tests. □

If $u_0$ has finite energy in another ample class $\omega_0$, it is integrable against the mixed measure. Write $P_i = (\omega_i, u_i)$ for $1 \le i \le n$. For a continuous model majorant $f_0$ in that class, the identity for models, followed by decreasing approximation in the other slots, gives

$$\int f_0 \,\mathrm{d}\,\mathrm{MA}(P_1, \ldots, P_n) = \langle (\omega_0, f_0), P_1, \ldots, P_n \rangle - \langle (\omega_0, 0), P_1, \ldots, P_n \rangle.$$

For this step the real model test $f_0$ is a finite linear combination of rational PL tests. Let $f_0$ decrease to $u_0$, and apply monotone convergence for Radon measures from Theorem 4.2.1 on the left. The right-hand side converges to a difference of finite extended pairings. This proves integrability and the identity with $u_0$.

## 17.9 Changing numerical classes

A potential that has finite energy in one ample class has finite energy in every ample class in which it is psh. Indeed one can first enlarge the original ample class to dominate both classes, using the finiteness estimate of Section 17.8. After subtracting a constant to make the function nonpositive, enlarging the second class only decreases its energy, which gives the assertion for the second class. Write $\mathcal{E}^1(X)$ for the functions having finite energy in some ample class.

For $u \in \mathcal{E}^1(X)$ and any real class $\theta$, choose an ample class $\Omega$ large enough that $u$ is $\Omega$-psh and $\Omega - \theta$ is ample. Use the decomposition

$$(\theta, u) = (\Omega, u) - (\Omega - \theta, 0)$$

to define mixed measures with this pair by expansion. The same procedure defines the extended pairing in $n + 1$ slots. The expansions are independent of $\Omega$. For two choices $\Omega, \Omega'$, compare both with their sum. For instance in the first expansion the class can be enlarged from $\Omega$ to $\Omega + \Omega'$ by adding the pair $(\Omega', 0)$ to both terms. At the level of finite-energy pairings this is a multilinear identity: approximate $u$ from above by models in $\Omega$, use the same models in the larger class, and pass to the finite limits. The other comparison is the same with $\Omega$ and $\Omega'$ exchanged. Applying these identities one slot at a time and testing by rational model functions proves the asserted independence of the mixed measure. Its total mass is the mixed degree of the $\theta_i$, by the same expansion. Every finite-energy test potential is integrable against it, since it is a finite linear combination of positive ample mixed measures with that property.

The extended pairing and mixed measure are additive in a slot that is a sum of pairs. For ample classes, choose simultaneous decreasing model approximants to the two potentials in their fixed classes. Their sums give a decreasing semipositive approximation in the sum class. The sum has finite energy there by concavity and positive scaling after placing the potentials in a common ample class. Model additivity passes to the limit for the pairing and for the weak limits of the measures. For general classes, decompose both pairs into differences as above and use the sum of the two ample ambient classes in the decomposition of their sum. This reduces the identity to the ample case.

**Proposition 17.9.1** *If $\theta_i$ are nef and $u_i \in \mathcal{E}^1(X) \cap \mathrm{PSH}(X, \theta_i)$, the expanded mixed measure is positive. It is the weak limit of the positive rational model measures for any simultaneous decreasing finite Fubini–Study approximation with classes tending to $\theta_i$.*

***Proof*** Let $f_{i,j}$ be such approximants in classes $\eta_{i,j}$. They are semipositive on models, and their measures are positive. Choose a common $\Omega$ ample and large enough that $\Omega - \eta_{i,j}$ is ample for every $i$ on a tail, with $\Omega - \theta_i$ ample as well. This is possible because the numerical classes converge and the ample cone contains a neighborhood of each ample class. Also choose $\Omega$ to dominate the ample classes in which the limits

have finite energy. Then $f_{i,j}$ is $\Omega$-psh on the tail, decreases to $u_i$, and all the limits have finite energy in $\Omega$.

Expand each model pair using $(\Omega, f_{i,j})$ and $(\Omega - \eta_{i,j}, 0)$. To pass to the limit in the second kind of slot, express the numerical class in a fixed basis, each basis element being a difference of two ample classes. The coefficients converge. In each term the nonzero potentials all decrease in fixed class $\Omega$, so Theorem 17.8.1 gives convergence of its measures, including the slots of constant zero potential in fixed ample classes. The expanded limits are exactly the prescribed measure for $(\theta_i, u_i)$. A weak limit of positive measures is positive, giving the conclusion. The existence of the required approximants in nef classes follows from Proposition 6.2.1. □

For fixed nef classes the measures are continuous along decreasing nets of finite-energy potentials with finite-energy limits. Choose an ample class $\Omega$ with $\Omega - \theta_i$ ample for every $i$. Every potential in the nets and every limit is $\Omega$-psh by Theorem 6.2.1, and so has finite energy in $\Omega$ by the first paragraph of this section. In the expansion above the constant slots then have fixed classes, and the varying slots decrease in that one ample class. Theorem 17.8.1 then applies directly.

## 17.10 Locality at a strict maximum

For a nef class $\theta$ and $u \in \mathcal{E}^1(X) \cap \mathrm{PSH}(X, \theta)$, write $\mu_u = \mathrm{MA}((\theta, u)^n)$.

**Theorem 17.10.1** *If $u, v$ are two such potentials and $w = \max\{u, v\}$, then*

$$\mathbf{1}_{\{u>v\}}\mu_w = \mathbf{1}_{\{u>v\}}\mu_u.$$

***Proof*** First let $u, v$ be rational model functions. Use a common denominator and a common subtraction to write

$$u = m^{-1}(\varphi_{\mathcal{I}} - \varphi_{\mathcal{J}}), \qquad v = m^{-1}(\varphi_{\mathcal{K}} - \varphi_{\mathcal{J}}).$$

This follows from Theorem 5.4.1: multiply the two denominators to one, then replace two subtracted ideals by their product and multiply the numerator ideals accordingly. The maximum uses $\mathcal{I} + \mathcal{K}$ in the numerator. Take a common model integrally closed along the puncture on which all four ideals are invertible.

If $u(v_E) > v(v_E)$, the integral invertible ideal $\mathcal{K}(\mathcal{I} + \mathcal{K})^{-1}$ on the model has strictly positive order at $E$. It vanishes along all of $E$, since the zero set is closed and contains the generic point. There is an equality of integral ideals

$$\mathcal{I}(\mathcal{I} + \mathcal{K})^{-1} + \mathcal{K}(\mathcal{I} + \mathcal{K})^{-1} = \mathcal{O}.$$

At every point of $E$, the second summand is contained in the maximal ideal, so the first is the unit ideal locally. Thus $\mathcal{I} = \mathcal{I} + \mathcal{K}$ on an open neighborhood of $E$. The marked model bundles of $u$ and $w$ have the same restriction to $E$, so their contributions to the atom at $v_E$ coincide. This argument applies to each component contribut-

ing to the strict region, even if several components give the same valuation. It proves the model assertion by (17.1).

Choose decreasing rational Fubini–Study approximants $u_j, v_j$ in rational classes tending to $\theta$. Promote each pair to a common rational ample class tending to $\theta$ by adding small ample differences. Put $w_j = \max(u_j, v_j)$. The three nets decrease to $u, v, w$, respectively, with classes tending to $\theta$. All limits have finite energy in some common ample class, since $w$ is psh and dominates $u$. For both choices $\nu_j = \mu_{w_j}$ and $\nu_j = \mu_{u_j}$, use the varying class at index $j$ in the measure. The measures converge weakly to $\nu = \mu_w$ or $\mu_u$ by Proposition 17.9.1. Moreover, for $a_j = w_j$ or $a_j = v_j$ and respective limit $a = w$ or $a = v$,

$$\int a_j \, \mathrm{d}\nu_j \longrightarrow \int a \, \mathrm{d}\nu.$$

Indeed each integral is a difference of the extended pairings by the finite-energy integration formula of Section 17.8. Expand each varying class from a common fixed ample class and then in the fixed basis of ample differences, as in the proof of Proposition 17.9.1. Convergence of the coefficients and decreasing continuity of the ample pairings prove the displayed convergence.

For any continuous $h$ with $0 \le h \le 1$, we also have $\int h a_j \, \mathrm{d}\nu_j \to \int h a \, \mathrm{d}\nu$. To verify this, give $a_j$ a common upper bound by working on a tail, and note that $h a_j$ is upper semicontinuous, with value zero where $h = 0$. For any fixed index $j_0$, eventually $h a_j \le h a_{j_0}$. The upper Portmanteau bound Proposition 4.3.1 then gives

$$\limsup_j \int h a_j \, \mathrm{d}\nu_j \le \int h a_{j_0} \, \mathrm{d}\nu.$$

Take the infimum over $j_0$ and use Theorem 4.2.1. The reverse lower bound follows by applying the same upper estimate to $1 - h$ and using convergence of the full integrals. Both limiting integrals are finite by the preceding finite-energy integrability result.

Model locality implies $\int h(w_j - v_j) \, \mathrm{d}\mu_{w_j} = \int h(w_j - v_j) \, \mathrm{d}\mu_{u_j}$. Pass to the weighted limits just established, then use the uniqueness of Radon measures to obtain $(w - v)\mu_w = (w - v)\mu_u$. The difference $w - v$ can be given any value on the common null set where either function is infinite, since both functions are integrable against both measures. On the strict region away from this null set, it is finite and strictly positive. Dividing there proves the assertion. □

**Corollary 17.10.1** *For the same nef class and finite-energy potentials,*

$$\mu_v\{u < v\} \le \mu_u\{u < v\}.$$

*For $j > 0$, the measure of $\max\{u, -j\}$ agrees with $\mu_u$ on $\{u > -j\}$. The mass of $\mu_u$ on $\{u \le -j\}$ tends to zero as $j \to \infty$.*

***Proof*** For $\varepsilon > 0$, apply Theorem 17.10.1 on both strict regions for $\max\{u, v - \varepsilon\}$. The mass on $\{u < v - \varepsilon\}$ equals that of $\mu_v$, and the mass on $\{u > v - \varepsilon\}$ equals that

of $\mu_u$. Since all three have the same total mass, this gives $\mu_v\{u < v - \varepsilon\} \le \mu_u\{u \le v - \varepsilon\}$. Let $\varepsilon \downarrow 0$, discarding the common pole set, which is null by integrability. The truncation identity is the theorem with the constant potential $-j$. The discarded mass is bounded by $j^{-1}\int(-u)_+ \,\mathrm{d}\mu_u$, which tends to zero. □

If $u = v$ on an open subset $G$ of $X^{\mathrm{an}}$, then $\mu_u = \mu_v$ on $G$. Indeed, on the subset where the potentials are finite, $u + \varepsilon > v$. Theorem 17.10.1 identifies $\mu_u$ there with the measure of $\max(u + \varepsilon, v)$, and both measures give zero mass to the poles in $G$. Let $\varepsilon \downarrow 0$ and test with continuous functions supported inside $G$, using decreasing continuity for the measures of the maxima. This identifies $\mu_u$ with the measure of $\max(u, v)$ on $G$. Interchanging the potentials proves the assertion. Polarization gives locality for two tuples of mixed measures whose corresponding slots have the same nef class and agree in potential on $G$. For each subset of the slots, sum the pairs in that subset and apply the diagonal result to the two sums. They have finite energy in some ample class by concavity and positive scaling of energy there, and are psh in the sum of the original nef classes. The alternating sum of the diagonal measures is $n!$ times the mixed measure.

## 17.11 Two energy differences

For a nef class $\theta$ and $u, v \in \mathcal{E}^1(X) \cap \mathrm{PSH}(X, \theta)$, put $P = (\theta, u)$ and $Q = (\theta, v)$. Use the extended pairing to define $E_\theta(u) = (n+1)^{-1}\langle P^{n+1}\rangle$. This agrees with the energies defined earlier in the chapter wherever both are defined. The difference-of-powers formula gives

$$E_\theta(u) - E_\theta(v) = \frac{1}{n+1}\sum_{j=0}^{n} a_j, \qquad a_j = \int (u - v)\,\mathrm{d}\,\mathrm{MA}(P^j, Q^{n-j}).$$

All these integrals exist by the finite-energy integration formula and the expansion into ample classes.

The sequence $a_j$ is decreasing in $j$. For model potentials this follows from the difference identity

$$a_{j+1} - a_j = \langle (0, u - v), (0, u - v), P^j, Q^{n-1-j}\rangle \le 0,$$

by Proposition 17.6.1. For finite-energy potentials use simultaneous decreasing finite approximants in varying rational classes. Promote the two classes at each index to a common rational class as before, leaving the functions fixed. In a sufficiently ample fixed class each sequence of approximating functions is psh and decreases to its limit. To compare extended pairings, expand the differences from that fixed ample class in a fixed basis of numerical classes, as in Proposition 17.9.1. All the coefficients converge, and all the remaining pairings converge by decreasing continuity in the fixed ample classes. Thus the inequality survives the limit.

Define

$$I_\theta(u,v) = \int (u-v)\,\mathrm{d}\big(\mathrm{MA}(Q^n) - \mathrm{MA}(P^n)\big),$$
$$J_{\theta,v}(u) = E_\theta(v) - E_\theta(u) + \int (u-v)\,\mathrm{d}\,\mathrm{MA}(Q^n).$$

We have

$$I_\theta(u,v) = a_0 - a_n \geq 0, \qquad J_{\theta,v}(u) = \frac{1}{n+1}\sum_{j=0}^{n}(a_0 - a_j).$$

In particular

$$\frac{1}{n+1} I_\theta(u,v) \leq J_{\theta,v}(u) \leq \frac{n}{n+1} I_\theta(u,v) \qquad (n \geq 1).$$

In dimension zero both differences vanish. The inequalities compare two finite-energy potentials without requiring bounded differences or a normalization of the mixed mass. When the top degree is positive, later chapters use the same quantities divided by that degree.

The same vertical intersection inequality gives a weak measure estimate in terms of $I_\theta$. For a rational model function $h$, choose a difference $(0,h) = R^+ - R^-$ of semipositive model pairs as in Section 17.5, with common ample base class $\beta$. For $n \geq 1$ one has

$$\left|\int h\,\mathrm{d}(\mu_u - \mu_v)\right|^2 \leq 2n\|h\|_\infty(\beta\cdot\theta^{n-1}\cdot[X])I_\theta(u,v).$$

First prove this on models. Telescope the measure difference and apply Cauchy–Schwarz to

$$(D_1,D_2) \longmapsto -\langle (0,D_1),(0,D_2),P^j,Q^{n-1-j}\rangle, \qquad 0 \leq j < n.$$

This form is positive semidefinite by Proposition 17.6.1. In this form the square of $u-v$ equals $a_j - a_{j+1}$. The square of $h$ is at most $2\|h\|_\infty(\beta\cdot\theta^{n-1}\cdot[X])$, by replacing one $(0,h)$ slot by $R^+ - R^-$ and using the positive mass formula on each term. Cauchy–Schwarz in the remaining finite sum gives the displayed bound. Pass to finite-energy potentials by the same simultaneous approximations used for the $a_j$. In particular, if $I_\theta(u_i,u) \to 0$, then $\mu_{u_i}$ converges weakly to $\mu_u$, using uniform density of the rational PL tests and the fixed total mass.

## 17.12 Mixed energy estimates

Suppose $n \geq 1$. A *semipositive finite-energy pair* is a pair $(\eta,w)$ with $\eta$ nef and $w \in \mathcal{E}^1(X) \cap \mathrm{PSH}(X,\eta)$. For semipositive finite-energy pairs $R_1,\dots,R_{n-1}$, put

$$\|h\|^2_{R_1,\dots,R_{n-1}} = -\langle (0,h),(0,h),R_1,\dots,R_{n-1}\rangle,$$

where $h$ is a formal linear combination of finite-energy potentials. Products involving their differences are evaluated by multilinear expansion. In particular their meaning survives when two potentials take the value $-\infty$ at the same point. The function $h \mapsto \|h\|_{R_1,\dots,R_{n-1}}$ is a seminorm. It is increasing in each of the $R_i$ under addition of a semipositive finite-energy pair. Both assertions hold on models by Proposition 17.6.1. To pass to finite-energy potentials in nef classes, approximate each pair by semipositive model pairs in rational ample classes tending to its numerical class, with decreasing potentials. Form the same linear combinations of approximating pairs when a slot is prescribed as a sum or difference. For function slots keep base class zero at every stage. As in Section 17.11, expand each pairing in a fixed ample ambient class with fixed ample classes spanning the numerical space for the corrections. The coefficients converge and each pairing in the expansion converges by decreasing continuity. This proves the assertions, including the Cauchy–Schwarz inequality for the associated bilinear form.

Fix a nef class $\theta$, and abbreviate $P_z = (\theta, z)$, $I = I_\theta$ and $J_v = J_{\theta,v}$. Finite-energy $\theta$-psh potentials form a convex set. Indeed, place them in a common ample class and use concavity of energy on simultaneous decreasing model approximants. This gives finite energy of their convex combinations in that ample class. Concavity of $E_\theta$ on this convex set follows by the same approximation argument as for the square above, using equal approximating classes for all potentials in the combination. In particular, $J_v$ is convex. We also have

$$I(u,v) = \sum_{j=0}^{n-1} \|u-v\|^2_{P_u^j, P_v^{n-1-j}}.$$

**Theorem 17.12.1** *Let $u, v, w$ and $w_1, \dots, w_{n-1}$ be finite-energy $\theta$-psh potentials. With $\alpha = 2^{1-n}$, one has*

$$\begin{aligned} \|u-v\|^2_{P_w^{n-1}} &\le C_n I(u,v)^\alpha \max\{I(u,w), I(v,w)\}^{1-\alpha}, \\ I(u,v) &\le C_n \max\{I(u,w), I(v,w)\}, \\ \|u-v\|^2_{P_{w_1},\dots,P_{w_{n-1}}} &\le C_n I(u,v)^\alpha M^{1-\alpha}, \end{aligned}$$

*where $C_n$ is a constant depending only on $n$ and $M = \max_{z\in\{u,v,w_1,\dots,w_{n-1}\}} I(z,0)$. A factor with exponent zero is interpreted as one.*

***Proof*** For $n = 1$ the first and third estimates are identities, and the second follows from the triangle inequality for the seminorm with empty subscript. Assume $n \ge 2$. Write $A = I(u,v)$, $B = \max\{I(u,w), I(v,w)\}$, and $\tau = (u+v)/2$. The nonnegative numbers

$$b_j = \|u-v\|^2_{P_w^j, P_\tau^{n-1-j}} \qquad (0 \le j \le n-1)$$

satisfy $2^{1-n}A \le b_0 \le A$ by the binomial expansion. The triangle inequality for the last seminorm gives $b_{n-1} \le 4B$. Convexity and the comparison between $I$ and $J$ give

$$I(\tau, w) \le (n+1)J_w(\tau) \le \tfrac{n+1}{2}\big(J_w(u) + J_w(v)\big) \le (n+1)B.$$

For $j < n-1$, set $S = (P_w^j, P_\tau^{n-2-j})$. By replacing one $P_\tau$ with $P_w$, we obtain

$$b_{j+1} - b_j = -\langle (0, u-v), (0, w-\tau), P_u - P_v, S\rangle.$$

For each of the two terms, Cauchy–Schwarz bounds its absolute value by the product of the seminorms of $u - v$ and $w - \tau$ with remaining slots $(P_u, S)$ or $(P_v, S)$. Each squared seminorm is at most twice its counterpart with $(P_\tau, S)$, since $2P_\tau = P_u + P_v$. The latter squared seminorms are $b_j$ and one of the summands of $I(\tau, w)$. Consequently

$$b_{j+1} \le b_j + 4\sqrt{(n+1)Bb_j}.$$

If $0 < A \le B$, induction gives $b_j \le C_n A^{2^{-j}} B^{1-2^{-j}}$. The endpoint bound gives the first estimate if $A > B$ or $B = 0$. If $A = 0$, the recursion gives $b_j = 0$ for every $j$.

Next, the triangle inequality in the seminorm defining $b_0$ gives

$$A \le 2^n\big(\|u - w\|^2_{P_\tau^{n-1}} + \|v - w\|^2_{P_\tau^{n-1}}\big).$$

Apply the first estimate with third potential $\tau$. We have $I(\tau, w) \le (n+1)B$ from above and $I(\tau, u) \le (n+1)J_u(\tau) \le (n+1)A/2$, with the same bound for $v$. Hence $A \le C_n B^\alpha \max\{A, B\}^{1-\alpha}$. If $A > B$, divide by $A^{1-\alpha}$. This gives the second estimate with an enlarged constant.

Finally, put $\bar w = (w_1 + \cdots + w_{n-1})/(n-1)$. Expansion of $P_{\bar w}^{n-1}$ and nonnegativity of each summand bound the mixed square in the third estimate by $(n-1)^{n-1}\|u-v\|^2_{P_{\bar w}^{n-1}}$. Two applications of the comparison between $I$ and $J$, with convexity in between, give $I(\bar w, 0) \le (n+1)M$. The second estimate now bounds $I(u, \bar w)$ and $I(v, \bar w)$ by $C_n M$. The first estimate finishes the proof. □

## 17.13 Testing mixed measures with finite energy

In dimension zero the mixed measure is a finite sum of atoms at the points of $X^{\mathrm{an}}$, and the pairing is linear in the single potential. The testing and convergence assertions below then hold directly. We assume $n \ge 1$ in the estimates.

For an ample class $\Omega$, write $I_\Omega(z) = I_\Omega(z, 0)$. For numerical classes write $\eta \le \Omega$ when $\Omega - \eta$ is nef. Let $R_i = (\eta_i, w_i)$ for nef $\eta_i \le \Omega$ and finite-energy $\eta_i$-psh potentials $w_i$. Let $u, v$ have finite energy in $\Omega$. Set $M = \max_{z \in \{u, v, w_1, \ldots, w_n\}} I_\Omega(z)$. Then

$$\left|\int (u-v)\,\mathrm{d}\big(\mathrm{MA}(R_1, \ldots, R_n) - \mathrm{MA}(\eta_1, \ldots, \eta_n)\big)\right| \le C_n I_\Omega(u, v)^{2^{-n}} M^{1-2^{-n}}. \quad (17.4)$$

Here $C_n$ depends only on $n$, and the numerical classes alone designate pairs with zero potential. To see this, telescope by replacing the potentials $w_i$ by zero in

turn. Each difference has two function slots, $(0, u - v)$ and $(0, w_i)$. Apply Cauchy–Schwarz to each term. Increasing the remaining slots to class $\Omega$ can only increase both squares. The third inequality in Theorem 17.12.1 bounds the first square by $C_n I_\Omega(u,v)^{2^{1-n}} M^{1-2^{1-n}}$ and the second by $C_n M$. This proves the estimate, also when $M = 0$. The subtracted measure is given by the atomic formula on the product model. In particular it has support among the trivial valuations of the irreducible components.

For an ample class $\omega \le \Omega$, choose $C \ge 1$ such that $\Omega \le C\omega$. If $u, v$ have finite energy in $\omega$, they also do in $\Omega$ and $C\omega$. To compare the energy differences, first bound the squares in $I_\Omega(u,v)$ by the corresponding squares for $C\omega$. In each remaining slot write $(C\omega, z) = (\omega, z) + (C-1)(\omega, 0)$, for $z = u, v$. Expansion and Theorem 17.12.1 give

$$I_\Omega(u,v) \le C_n C^{n-1} I_\omega(u,v)^\alpha \max\{I_\omega(u,0), I_\omega(v,0)\}^{1-\alpha}, \qquad \alpha = 2^{1-n}, \quad (17.5)$$

where $C_n$ depends only on $n$. In particular a bound on $I_\omega(u,0)$ gives a bound on $I_\Omega(u,0)$, with a constant depending only on $n, C$.

**Theorem 17.13.1** *Let $\mu$ be a mixed measure of finite-energy potentials in ample classes. Fix an ample class $\omega$, and suppose $\psi_j, \psi$ are finite-energy $\omega$-psh potentials with $I_\omega(\psi_j, 0)$ bounded. If $\psi_j \to \psi$ on $\mathcal{D}(X)$, then $\int \psi_j \, \mathrm{d}\mu \to \int \psi \, \mathrm{d}\mu$.*

***Proof*** Write $\mu = \mathrm{MA}((\eta_1, w_1), \ldots, (\eta_n, w_n))$ and let $w_{i,l}$ be decreasing semipositive model approximants in the fixed ample classes $\eta_i$. Write $\mu_l$ for their mixed measures. Decreasing continuity of the pairing gives $I_{\eta_i}(w_{i,l}, w_i) \to 0$ and bounds on $I_{\eta_i}(w_{i,l}, 0)$ on a tail. The differences $w_{i,l} - w_i$ tend to zero at the trivial valuations of the components. Choose $\Omega$ ample dominating all the classes involved. By (17.5) the corresponding bounds and limits hold in $\Omega$.

For any finite-energy $\omega$-psh test $\xi$ with $I_\omega(\xi, 0)$ bounded by a fixed constant, telescope $\int \xi \, \mathrm{d}(\mu_l - \mu)$. Each resulting term, by symmetry of the pairing, is the integral of $w_{i,l} - w_i$ against a mixed measure with one slot $(\omega, \xi)$ minus the same integral with that slot replaced by $(\omega, 0)$. Estimate both integrals by (17.4) after subtracting the same product-model measure. Their difference tends to zero uniformly in $\xi$, by the bound and convergence in $\Omega$. Thus the integrals against $\mu_l$ converge uniformly on each such set of tests. For fixed $l$ the desired convergence holds because $\mu_l$ is a finite sum of atoms in $\mathcal{D}(X)$. Combine these two assertions to conclude. □

**Corollary 17.13.1** *Suppose $u_{i,j}$ increase on $\mathcal{D}(X)$ to $u_i$ and all these potentials have finite energy in their respective ample classes $\omega_i$. Then their mixed pairings converge to the pairing of the $u_i$, with the option of holding any slots fixed. The mixed measures converge weakly, and $I_{\omega_i}(u_{i,j}, u_i) \to 0$ for each $i$.*

***Proof*** Order on $\mathcal{D}(X)$ determines order everywhere by Theorem 6.3.1. Monotonicity of energy and boundedness of the potentials at the trivial valuations give a bound on $J_{\omega_i,0}(u_{i,j})$ on a tail, and thus on $I_{\omega_i}(u_{i,j}, 0)$. With all other slots fixed, convergence of a variable slot now follows from Theorem 17.13.1 and the integration formula for the pairing. The pairing is monotone in each slot. To pass to simultaneous

limits, take the supremum over independent indices and use the single-slot result repeatedly. The diagonal is cofinal in this product of directed sets. Apply the result to the pairings in the definition of $I$ to obtain the last assertion. For weak convergence test the measures against a rational PL function, expressed as a difference of semipositive pairs in an ample class, and then use uniform density and the mass formula. □

# Chapter 18
# Extended energy and orthogonality

For the truncated ray of Example 12.1.1, the envelope has slope $-d$ up to the truncation point and slope zero afterward. Up to the constant $d - a$, it is the potential of the divisor $D = -dE$ in Example 17.3.1. The parameter of that example is therefore $d$, not the depth $a$ of the obstacle, and its two component degrees give the measure $d\delta_{v_1}$. Thus the mass lies where the envelope reaches the obstacle. In an ample class with the envelope property, the same statement for arbitrary continuous obstacles follows from the first variation of extended energy.

Throughout, $X$ is a reduced projective scheme over a trivially valued field $k$. We use the envelopes from Chapter 12 and the intersection normalization of Chapter 17, and the measure results are stated for pure-dimensional schemes. Section 18.1 treats a birational weakening of the envelope property, under which a bounded obstacle and the upper regularization of its envelope have the same energy (Theorem 18.1.1). Section 18.2 compares relative section counts on models with energy differences. In an ample class with the envelope property, Section 18.3 then proves the orthogonality identity for continuous obstacles (Theorem 18.3.1), shows that it is equivalent to differentiability of the extended energy (Proposition 18.3.2), and extends it to lower semicontinuous obstacles (Corollary 18.3.1). The first-variation estimates of Proposition 18.3.1 return in Chapter 19, and the section lattices in Chapter 34.

## 18.1 Energy under the weak envelope property

In this section $X$ has pure dimension $n$. For an ample class $\omega$, the *(extended) energy* of a bounded obstacle $f$ is

$$E_\omega(f) = \sup_{u \le f,\ u \in \mathrm{PSH}(X,\omega)} E_\omega(u).$$

Constants bound this supremum from both sides. The functions $f$ and $P_\omega(f)$ have the same psh competitors below them, so $E_\omega(f) = E_\omega(P_\omega(f))$.

**Definition 18.1.1** Let $\omega$ be an ample class on $X$. The pair $(X, \omega)$ has the *weak envelope property* if there are a reduced projective scheme $Y$, a projective birational morphism $\pi : Y \to X$ and an ample class $\Omega$ on $Y$ with the envelope property such that $\Omega - \pi^*\omega$ is nef.

**Theorem 18.1.1** *Let $X$ be integral and projective. If $(X, \omega)$ has the weak envelope property, then every bounded obstacle $f$ satisfies*

$$E_\omega(f) = E_\omega(P_\omega(f)^*).$$

***Proof*** Write $\tau = P_\omega(f)$. Take the directed family of competitors $u_i \le f$ bounded below by $\inf f$, ordered by pointwise comparison. Finite maxima stay in the family, and adjoining the constant $\inf f$ as a maximum with any competitor shows that the family increases to $\tau$. Let $\pi : Y \to X$ and $\Omega$ witness the weak envelope property. The function $b = (\sup_i \pi^* u_i)^*$ is $\Omega$-psh by that property. It is bounded, and $\pi^* u_i$ increases to $b$ on $\mathcal{D}(Y)$ by divisorial agreement.

For any bounded $\omega$-psh $u$ we have $E_{\pi^*\omega}(\pi^* u) = E_\omega(u)$. On model pairs this is the projection formula on a model dominating the pullback of the given model to $Y$. In fact pushforward carries the fundamental cycle to the fundamental cycle by birationality, and the classes on the new model are pullbacks. For bounded $u$, take semipositive model majorants decreasing in the fixed class $\omega$, pull back, then apply the decreasing pairing identities of Section 17.9. The potentials upstairs are bounded and psh in the ample class $\Omega$, which gives finite energy there. Also, by Corollary 17.13.1, applied in ample classes and then to the expansion with fixed constant slots, we have

$$\lim_i E_{\pi^*\omega}(\pi^* u_i) = E_{\pi^*\omega}(b).$$

Here the right-hand side is the expanded pairing in the numerical class $\pi^*\omega$. The expansion is defined because $b$ has finite energy in $\Omega$. In particular the right-hand side is at most $E_\omega(f)$.

For a competitor $u \le \tau^*$, we may replace $u$ by $\max(u, \inf f)$ to bound its energy from above by that of a bounded competitor. Then $u \le \tau$ on $\mathcal{D}(X)$ by divisorial agreement. Since $X$ is integral and $\pi$ is birational, the reduced scheme $Y$ is integral with the same function field. Both schemes are proper, so $\pi^{\mathrm{an}}$ identifies $\mathcal{D}(Y)$ with $\mathcal{D}(X)$, and hence $\pi^* u \le b$ on $\mathcal{D}(Y)$. The order holds everywhere by divisorial determination. To compare their energies in class $\pi^*\omega$, use the difference of powers as in Section 17.11. The mixed measures in this formula are positive, by the following increasing approximation.

For each $0 \le j \le n$ the mixed measures with $j$ copies of $(\pi^*\omega, \pi^* u_i)$ and $n - j$ copies of $(\pi^*\omega, \pi^* u)$ are positive. Their weak limits are the expanded mixed measures with $(\pi^*\omega, b)$ in the first $j$ slots. To see the convergence, expand using $(\Omega, \pi^* u_i)$ and $(\Omega - \pi^*\omega, 0)$, and express the latter fixed numerical difference using ample classes. Apply the increasing convergence theorem to each resulting measure. Thus the limit measures are positive. Now the difference of powers integrates the

nonnegative function $b - \pi^* u$ against the sum of these measures, divided by $n + 1$. The integration formula holds by linearity for the expansion. Consequently

$$E_\omega(u) = E_{\pi^*\omega}(\pi^* u) \leq E_{\pi^*\omega}(b) \leq E_\omega(f).$$

Take the supremum over $u \leq \tau^*$. The reverse inequality follows since $\tau \leq \tau^*$. □

**Lemma 18.1.1** *The extended energy $E_\omega(f)$ of a bounded obstacle is locally Lipschitz in the ample class $\omega$, uniformly for obstacles of bounded uniform norm. In the obstacle variable it is Lipschitz with constant $(\omega^n \cdot [X])$.*

***Proof*** Translation by a constant changes the energy of every competitor by that constant times $(\omega^n \cdot [X])$. Monotonicity therefore gives the estimate in the obstacle variable.

For the class variable take nearby ample classes $\omega, \eta$ in a fixed compact neighborhood in the ample cone. We can arrange $(1-\varepsilon)\omega \leq \eta \leq (1+\varepsilon)\omega$ with $\varepsilon \leq C\|\omega - \eta\|$ and $\varepsilon < 1/2$, by openness of the cone. The same bound holds with the classes interchanged and with a larger constant. Suppose $\|f\|_\infty \leq B$. Taking the maximum with $\inf f$ allows us to use only competitors with $|u| \leq B$. For a competitor $u$ in class $\omega$, the function $v = (1 - \varepsilon)u - \varepsilon B$ is a competitor in class $\eta$, since $v \leq u$ and it is psh in $\omega' = (1 - \varepsilon)\omega$. Homogeneity and the translation formula give

$$E_{\omega'}(v) = (1 - \varepsilon)^{n+1} E_\omega(u) - \varepsilon B((\omega')^n \cdot [X]).$$

The difference from $E_\omega(u)$ is at most a constant times $B\varepsilon$, because monotonicity bounds $|E_\omega(u)|$ by $B(\omega^n \cdot [X])$.

To pass from $E_{\omega'}(v)$ to $E_\eta(v)$, telescope the difference of the diagonal pairings. Every term has a slot $(\eta - \omega', 0)$ and the other slots are pairs $(\eta, v)$ or $(\omega', v)$. Replace the potentials of those slots by zero in turn. The all-zero term vanishes, and each remaining term is an integral of $v$ against a mixed measure with one slot $(\eta - \omega', 0)$. It is positive as a measure because all these slots are semipositive. Its mass is bounded by a constant times $\varepsilon$ by multilinearity of numerical degrees and the bound $\|\eta - \omega'\| = O(\varepsilon)$. Since $|v| \leq B$, we have $|E_\eta(v) - E_{\omega'}(v)| \leq CB\varepsilon$. For $n = 0$ the all-zero term is the entire class difference and the same estimate holds. Take the supremum over $u$, and repeat with $\omega, \eta$ interchanged, to obtain the Lipschitz bound. □

## 18.2 Relative section counts on a model

Let $X$ have pure dimension $n$. On a common model $\mathcal{X}$ integrally closed along the puncture take relatively ample marked line bundles $\mathcal{L}$, $\mathcal{M}_1$, and $\mathcal{M}_2$. The last two have the same marked restriction $M$ on $X$. Write $D = \mathcal{M}_1 - \mathcal{M}_2$ for the resulting vertical Cartier divisor in additive notation and $h = \varphi_D$. The generic bundle of $\mathcal{L}$ is written $L$.

For two finite $k[t]$-modules $A, B$ in the same free $k[t, t^{-1}]$-module which become that module after inverting $t$, define the *signed length*

$$\ell(A, B) = \dim_k A/(A \cap B) - \dim_k B/(A \cap B).$$

Both quotients are finite-dimensional because they are finitely generated and killed by a power of $t$. Using a common intersection for three modules shows $\ell(A, C) = \ell(A, B) + \ell(B, C)$. In particular $\ell(B, A) = -\ell(A, B)$. Global sections of $m\mathcal{L} + jD$ give such modules inside $H^0(X, mL)[t, t^{-1}]$ by the marking and properness. Denote the module by $A_{m,j}$ and put

$$S_m = \frac{n!}{m^{n+1}} \ell(A_{m,m}, A_{m,0}).$$

**Proposition 18.2.1** *Let $\mu$ be the model measure for $\mathcal{L} + \mathcal{M}_1$. If $h \geq 0$, then $\limsup_m S_m \leq \int h \, \mathrm{d}\mu$. If $h \leq 0$, then $\liminf_m S_m \geq \int h \, \mathrm{d}\mu$.*

***Proof*** If $h \geq 0$, then $D$ is effective by the vertical pole criterion. For $1 \leq j \leq m$, the restriction exact sequence gives an injection

$$A_{m,j}/A_{m,j-1} \hookrightarrow H^0(D, (m\mathcal{L} + jD)|_D).$$

Choose $b$ so that $q\mathcal{M}_i$ is generated over $\mathbb{A}^1$ for every integer $q \geq b$ and both $i$. This is possible because $\mathcal{M}_i$ is ample on the model, the base being affine, and high powers of an ample line bundle are globally generated [Stacks, Tags 01VK and 01Q3]. For each $m, j$ the restrictions of $(m - j + b)\mathcal{M}_1$ and $(j + b)\mathcal{M}_2$ to $D$ have sections nonvanishing at every associated point of $D$. Global generation makes the forbidden choices a finite union of proper linear subspaces. If $k$ is finite, choose the sections after a finite extension of $k$. The dimension inequality obtained there holds over $k$ since cohomology commutes with the field extension. Multiplication by the product of these sections gives a sheaf injection on $D$, since it is a nonzerodivisor in every local frame. This bounds each of the dimensions on the right by

$$h^0\big(D, (m(\mathcal{L} + \mathcal{M}_1) + b(\mathcal{M}_1 + \mathcal{M}_2))|_D\big).$$

The restriction of $\mathcal{L}+\mathcal{M}_1$ is ample on $D$, which is projective over a finite thickening of the point zero. The leading term of this section count is $m^n((\mathcal{L}+\mathcal{M}_1)^n \cdot [D])/n!$. This is the Hilbert polynomial formula, with the coefficient computed on the fundamental cycle using generic lengths, as in the section count (7.1). Summing the $m$ increments and using the factor $n!/m^{n+1}$ yields the first bound. The coefficient is $\int h \, \mathrm{d}\mu$ by the model integration formula.

If $h \leq 0$, apply the restriction sequence on the effective Cartier divisor $B = -D$ to the inclusions $A_{m,j+1} \subseteq A_{m,j}$ for $0 \leq j < m$. The quotient injects into $H^0(B, (m\mathcal{L} - jB)|_B)$. The same tensor correction $(m - j + b)\mathcal{M}_1 + (j + b)\mathcal{M}_2$ makes the bound independent of $j$. Summing now gives $\limsup_m(-S_m) \leq \int(-h) \, \mathrm{d}\mu$. □

The relative section count agrees with the energy difference when the two endpoints are themselves relatively ample model bundles. The next lemma assumes no decomposition of their difference.

**Lemma 18.2.1** *Let $\mathcal{L}_0, \mathcal{L}_1$ be relatively ample marked line bundles on the same model with generic bundle $L$. Write $u_0, u_1$ for their potentials and $A_m(\mathcal{L}_i)$ for the module of sections of $m\mathcal{L}_i$. Then*

$$\lim_m \frac{n!}{m^{n+1}} \ell(A_m(\mathcal{L}_1), A_m(\mathcal{L}_0)) = E_{[L]}(u_1) - E_{[L]}(u_0).$$

***Proof*** Compactify the two line bundles by the marked product toward infinity. Let $F_\infty$ be the fibre divisor over infinity and choose a positive integer $c$ for which both $\overline{\mathcal{L}}_i + cF_\infty$ are ample, by the composition property for relative ampleness [Stacks, Tag 0C4K]. Increase $c$ so that $c \geq \max_i \|u_i\|_\infty$.

The module $A_m(\mathcal{L}_i)$ is graded by Laurent powers of $t$. For a term $st^j$, with $s \in H^0(X, mL)$, the pole criterion tests membership in $A_m(\mathcal{L}_i)$ by

$$\log|s|(v_E) - j \leq m u_i(v_E)$$

at every special-fibre valuation. Tests for a sum of terms are equivalent to those for each term by the Gauss formula. Since $\log|s| \leq 0$, both lattices contain every coefficient space in exponents $j \geq cm$. At infinity the condition for $m(\overline{\mathcal{L}}_i + cF_\infty)$ is $j \leq cm$, so its space of global sections is the sum of the weight spaces of $A_m(\mathcal{L}_i)$ up to that exponent. The signed length $\ell(A_m(\mathcal{L}_1), A_m(\mathcal{L}_0))$ is the difference of the dimensions of these two section spaces, since the higher weight spaces are identical.

The leading term of the difference of their Hilbert polynomials gives

$$\lim_m \frac{n!}{m^{n+1}} \ell(A_m(\mathcal{L}_1), A_m(\mathcal{L}_0)) = \frac{\big((\overline{\mathcal{L}}_1 + cF_\infty)^{n+1} - (\overline{\mathcal{L}}_0 + cF_\infty)^{n+1}\big)}{n+1}.$$

In the expansion, terms with two fibre factors vanish and the terms with one fibre factor cancel, since the bundles have the same degree on $X$. The remaining term is the stated energy difference by the model pairing. □

### 18.2.1 Section lattices of bounded obstacles

For a rational ample line bundle $L$ and a bounded function $f$, define a *section lattice* by

$$A_m(f) = \bigoplus_{j \in \mathbb{Z}} \big\{ s \in H^0(X, mL) : \log|s| \leq mf + j \big\} t^j \qquad (mL \text{ integral}).$$

In each exponent the coefficients form a vector space, by the ultrametric inequality. The spaces are nested in $j$ and are zero for $j < -m\|f\|_\infty$, since a nonzero section on a

reduced scheme is nonzero at some generic point. They are the full space $H^0(X, mL)$ for $j \geq m\|f\|_\infty$ because $\log|s| \leq 0$. Thus $A_m(f)$ is a finite free $k[t]$-module of rank $h^0(X, mL)$ that becomes $H^0(X, mL)[t, t^{-1}]$ after inverting $t$. For a model potential and sufficiently divisible $m$, it is the module of model sections by the vertical pole criterion and the Gauss formula. For the inequality at all points in the lattice definition, use nonnegative values of regular local representatives at the centres on the model.

Put

$$S_m(f, g) = \frac{n!}{m^{n+1}}\ell(A_m(f), A_m(g)), \qquad V_m = \frac{n!}{m^n}h^0(X, mL).$$

The lengths are additive in the middle obstacle. The inequality $f \leq g$ gives lattice inclusion. For an integer $a$ we have $A_m(f + a/m) = t^{-a}A_m(f)$, and each step by $1/m$ has signed length $h^0(X, mL)$. By sandwiching the two lattices between such shifts,

$$|S_m(f, g)| \leq V_m\big(\|f - g\|_\infty + m^{-1}\big). \tag{18.1}$$

The Hilbert polynomial gives $V_m \to ([L]^n \cdot [X])$. We take limits of section counts over the positive integers ordered by divisibility. A limiting statement in this order permits us to restrict to multiples of any fixed integer.

**Lemma 18.2.2** *For any bounded obstacle $f$ and all sufficiently divisible $m$, one has $A_m(f) = A_m(P_{[L]}(f))$. If $u, v$ are continuous $[L]$-psh potentials, then in divisibility order*

$$S_m(u, v) \longrightarrow E_{[L]}(u) - E_{[L]}(v).$$

***Proof*** Suppose $\log|s| \leq mf + j$ and choose a rational constant $a \leq \inf f$. For large divisible $m$ choose generating sections of $mL$. They express the constant $a$ as a finite Fubini–Study function in degree $m$, by taking their maximum with weight $ma$. Adjoin $s$ with weight $-j$. The resulting potential $\max\{m^{-1}(\log|s| - j), a\}$ is a competitor below $f$. Hence $\log|s| \leq mP_{[L]}(f) + j$. This proves one inclusion, and the other follows from $P_{[L]}(f) \leq f$.

For the limit, first suppose $u$ and $v$ are potentials of relatively ample rational models of $L$ on one common model. Clear denominators and use Lemma 18.2.1, keeping the factor $m^{n+1}$ in the original degree. General continuous psh potentials can be approximated uniformly by such potentials in the fixed rational line bundle. Indeed fixed-class approximation gives semipositive rational model potentials. On any common dominating model, an auxiliary relatively ample marked bundle gives a Cartier correction ample over the product. A sufficiently small positive rational multiple of that correction added to the pullback of $L$ is relatively ample over $\mathbb{A}^1$, by the composition property [Stacks, Tag 0C4K]. Interpolate with this relatively ample model of $L$ with arbitrarily small rational weight. This preserves the marked line bundle $L$ on $X$ and gives the required uniform approximation. The general limit now follows from (18.1), additivity of the lengths, and uniform continuity of energy on continuous psh potentials. □

**Lemma 18.2.3** *Let $\lambda = [L]$ be ample and rational, $u$ continuous and $\lambda$-psh, and $h$ rational PL. Represent $h = g_1 - g_2$ by relatively ample rational models of the same marked rational ample line bundle of class $\eta$. For rational $s > 0$ put $C(s) = ((\lambda + s\eta)^n - \lambda^n) \cdot [X]$ and $\mu_s = \mathrm{MA}((\lambda + s\eta, u + sg_1)^n)$. In divisibility order one has*

$$\limsup_m S_m(u + sh, u) \le s \int h \, \mathrm{d}\mu_s - sC(s) \inf h,$$
$$\liminf_m S_m(u + sh, u) \ge s \int h \, \mathrm{d}\mu_s - sC(s) \sup h.$$

***Proof*** To obtain the representation of $h$, choose a model carrying its vertical rational Cartier divisor. Start with a relatively ample marked bundle on the model and take a sufficiently large common multiple, so that adding the divisor leaves it relatively ample. These are the pairs for $g_2$ and $g_1$.

When $u$ is the potential of a relatively ample rational model of $L$ on this model, first suppose $h \ge 0$ or $h \le 0$ and apply Proposition 18.2.1 with the auxiliary bundles scaled by $s$. Clear denominators in all the input bundles at once and use the normalized length in the original degree. For arbitrary $h$, apply the first inequality to $h - c$ with rational $c \le \inf h$, and the second with rational $c \ge \sup h$. This amounts to shifting the first auxiliary model potential by a constant, which leaves its measure unchanged. Each shift moves the section-count limit bounds by $sc(\lambda^n \cdot [X])$. The integral of the constant shifts by $sc((\lambda + s\eta)^n \cdot [X])$. Taking the two rational constants to $\inf h$ and $\sup h$ gives the stated correction terms.

To pass to continuous $u$, approximate it uniformly by finite rational model potentials in class $\lambda$. On each common model carrying an approximant and the auxiliary bundles, the pullbacks are relatively nef. Perturb the approximant towards a relatively ample model of $L$, and add the same arbitrarily small rational vertical correction to both auxiliary bundles so that they are relatively ample in their original class. Such a correction can be chosen ample relative to the original model of the auxiliary bundles. Add a small positive rational multiple of it to the pullbacks of those relatively ample bundles and use the composition property for relative ampleness [Stacks, Tag 0C4K]. This keeps their difference equal to $h$. The perturbed potentials can be made uniformly close to the original ones. The length estimates are stable by (18.1), and the measures converge weakly by Theorem 17.5.1. The two limits give the claim. □

## 18.3 Contact and differentiability

Assume the envelope property in an ample class $\omega$. For a continuous obstacle $f$, put $p = P_\omega(f)$ and $\mu_f = \mathrm{MA}((\omega, p)^n)$. We call $\int (f - p) \, \mathrm{d}\mu_f = 0$ the orthogonality identity. Since the gap $f - p$ is continuous and nonnegative, the identity says that $\mu_f$ is supported on $\{p = f\}$.

**Proposition 18.3.1 (First variation at a continuous potential)** *Let $X$ be a reduced projective scheme of pure dimension $n$, let $\omega$ be a real ample class with the envelope property, and let $u$ be continuous and $\omega$-psh. Let $h$ be rational PL, written $h = g_1 - g_2$ with auxiliary class $\eta$ as in Lemma 18.2.3. For real $s \geq 0$ put*

$$\nu_s = \mathrm{MA}((\omega + s\eta, u + sg_1)^n), \qquad C_\omega(s) = ((\omega + s\eta)^n - \omega^n) \cdot [X].$$

*Then for every real $s > 0$,*

$$\begin{aligned} s \int h \, \mathrm{d}\nu_s - sC_\omega(s) \sup h \leq E_\omega(u + sh) - E_\omega(u) \\ \leq s \int h \, \mathrm{d}\nu_s - sC_\omega(s) \inf h. \end{aligned} \tag{18.2}$$

*Consequently, for every real $s$ with $|s| \leq 1$,*

$$\left| E_\omega(u + sh) - E_\omega(u) - s \int h \, \mathrm{d}\,\mathrm{MA}((\omega, u)^n) \right| \leq 2\|h\|_\infty C_\omega(1)\, s^2. \tag{18.3}$$

*Finally, for every continuous $h$,*

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} E_\omega(u + sh) = \int h \, \mathrm{d}\,\mathrm{MA}((\omega, u)^n).$$

***Proof*** First let $s > 0$ be rational. Put $f = u + sh$ and $p = P_\omega(f)$, which is continuous by the envelope property and Proposition 12.1.1. Choose rational ample classes $\lambda_j \to \omega$ and numbers $\varepsilon_j \downarrow 0$ such that $(1 - \varepsilon_j)\omega \leq \lambda_j \leq (1 + \varepsilon_j)\omega$. Fix $B$ bounding the norms of $u, f, p$. The functions

$$u_j = (1 - \varepsilon_j)u - \varepsilon_j B, \qquad p_j = (1 - \varepsilon_j)p - \varepsilon_j B$$

are continuous $\lambda_j$-psh and converge uniformly to $u, p$. In particular $p_j$ is a competitor below $f$. In the other direction,

$$P_{\lambda_j}(f) \leq (1 + \varepsilon_j) P_\omega\big(f/(1 + \varepsilon_j)\big),$$

by monotonicity in the class and positive scaling of psh potentials. The Lipschitz estimate for $P_\omega$ bounds the right-hand side by $p + 2\varepsilon_j B$. Together these bounds give $\|P_{\lambda_j}(f) - p_j\|_\infty \leq 4\varepsilon_j B$, even if the envelope in class $\lambda_j$ is discontinuous.

In each rational class choose a representing rational line bundle and form the section counts for $u_j + sh, u_j$. Replacing $u_j + sh$ by $f$ changes the counts by at most $V_m(2\varepsilon_j B + m^{-1})$ by (18.1). By Lemma 18.2.2 the lattice for $f$ is the lattice for $P_{\lambda_j}(f)$, which is within the bound just obtained from $p_j$. Thus both the upper and lower asymptotic section counts differ from $E_{\lambda_j}(p_j) - E_{\lambda_j}(u_j)$ by at most a constant times $\varepsilon_j B$, using the continuous psh limit in the same lemma. This energy difference tends to $E_\omega(f) - E_\omega(u)$. Indeed, $\|p_j - p\|_\infty$ and $\|u_j - u\|_\infty$ are at most $2\varepsilon_j B$. By the obstacle part of Lemma 18.1.1, $|E_{\lambda_j}(p_j) - E_{\lambda_j}(p)|$ and $|E_{\lambda_j}(u_j) - E_{\lambda_j}(u)|$

are therefore at most $2\varepsilon_j B(\lambda_j^n \cdot [X])$, where $E_{\lambda_j}(p)$ and $E_{\lambda_j}(u)$ are the energies of the obstacles $p$ and $u$. By the class part of the same lemma, $E_{\lambda_j}(p) \to E_\omega(p)$ and $E_{\lambda_j}(u) \to E_\omega(u)$. Finally $E_\omega(f) = E_\omega(p)$, since $p$ is itself a competitor and dominates all the others.

Apply Lemma 18.2.3 in each $\lambda_j$ and pass to the limit in $j$. Continuity of the mixed measures under uniform convergence and converging classes gives (18.2) for rational $s$. By multilinearity, $\int h\,\mathrm{d}\nu_s$ and $C_\omega(s)$ are polynomials in $s$, and the extended energy is Lipschitz in the obstacle by Lemma 18.1.1. All terms of (18.2) are therefore continuous in $s$, and the bounds hold for real $s > 0$.

Multilinearity also gives

$$\nu_s - \mathrm{MA}((\omega,u)^n) = \sum_{k=1}^{n} \binom{n}{k} s^k\, \mathrm{MA}((\omega,u)^{n-k}, (\eta,g_1)^k),$$

a positive measure of mass $C_\omega(s)$. As a polynomial in $s$ with nonnegative coefficients and zero constant term, $C_\omega(s)$ is at most $sC_\omega(1)$ for $0 < s \le 1$. Hence both bounds in (18.2) differ from $s\int h\,\mathrm{d}\,\mathrm{MA}((\omega,u)^n)$ by at most $sC_\omega(s)(\sup h - \inf h) \le 2\|h\|_\infty C_\omega(1)s^2$. This proves (18.3) for $0 < s \le 1$. For $-1 \le s < 0$ apply it to $-h = g_2 - g_1$, which has the same auxiliary class. In particular the derivative formula holds for rational PL $h$. For continuous $h$ and rational PL $h'$, the same Lipschitz estimate shows that the difference quotients of $E_\omega(u+sh)$ and $E_\omega(u+sh')$ differ by at most $(\omega^n \cdot [X])\|h - h'\|_\infty$ for every $s \neq 0$. The integrals of $h$ and $h'$ against $\mathrm{MA}((\omega,u)^n)$ differ by at most the same amount. Uniform density of rational PL functions (Theorem 5.4.1) therefore gives the derivative formula for continuous $h$.□

**Theorem 18.3.1** *For a reduced projective scheme of pure dimension and a real ample class with the envelope property, every continuous obstacle satisfies the orthogonality identity.*

***Proof*** Let $f$ be a continuous obstacle. Its envelope $p = P_\omega(f)$ is continuous and $\omega$-psh by the envelope property and Proposition 12.1.1. Put $h = f - p$. For $0 \le s \le 1$ the obstacles $p + sh$ lie between $p$ and $f$, so their envelope is $p$ and their energy is $E_\omega(p)$. The derivative of $E_\omega(p+sh)$ at $s = 0$, which exists by Proposition 18.3.1, is therefore zero, and that proposition identifies it with $\int (f-p)\,\mathrm{d}\mu_f$. □

**Proposition 18.3.2** *Under the envelope property, orthogonality for all continuous obstacles is equivalent to differentiability of the extended energy at every continuous obstacle $f$ along every continuous direction $h$, with*

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} E_\omega(f+sh) = \int h\,\mathrm{d}\mu_f.$$

*In either condition it suffices to test orthogonality on rational PL obstacles or differentiability with this derivative on rational PL directions at every continuous obstacle.*

***Proof*** Let $p_s = P_\omega(f + sh)$. The energy difference bounds of Section 17.11 give

$$\int (p_s - p)\,\mathrm{d}\mu_{f+sh} \le E_\omega(p_s) - E_\omega(p) \le \int (p_s - p)\,\mathrm{d}\mu_f.$$

If both orthogonality identities hold, the left integral is at least $s\int h\,\mathrm{d}\mu_{f+sh}$, since $p \le f$ and $p_s = f + sh$ almost everywhere for that measure. The right integral is at most $s\int h\,\mathrm{d}\mu_f$, by the other orthogonality identity. The potentials $p_s$ converge uniformly to $p$ by the envelope Lipschitz bound, so their measures converge weakly. Dividing by $s$ with the appropriate inequality direction proves the derivative formula.

Conversely, take the direction $h = f - p$ at the obstacle $p$. For $0 \le s \le 1$, the envelopes of $p + sh$ equal $p$ since these obstacles lie between $p$ and $f$. The derivative formula at $p$ therefore gives $\int (f - p)\,\mathrm{d}\mu_f = 0$.

The reduction to rational PL obstacles in the orthogonality identities follows from uniform density of rational PL functions, the Lipschitz bound for envelopes and weak convergence of their measures. For directions, the extended energy is Lipschitz in the uniform norm with constant $(\omega^n \cdot [X])$. This follows from monotonicity and the change in energy under addition of a constant. Thus the error in a difference quotient when the direction changes from $h$ to $h'$ is bounded by $(\omega^n \cdot [X])\|h - h'\|_\infty$, uniformly in the nonzero increment. The same bound holds for the integrals against $\mu_f$. Density therefore extends the derivative formula from rational PL to continuous directions. □

**Corollary 18.3.1** *Let $X$ be reduced and projective of pure dimension, and let $\omega$ be ample with the envelope property. Suppose $f$ is lower semicontinuous and bounded below, with values allowed to be $+\infty$. If $P_\omega(f)$ is bounded above, put $p = P_\omega(f)^*$ and $\mu = \mathrm{MA}((\omega, p)^n)$. Then $f = P_\omega(f) = p$ almost everywhere for $\mu$.*

***Proof*** Let $f_i$ range over continuous minorants of $f$ bounded below by a fixed constant $a \le \inf f$. They form an upward-directed family under maximum and have supremum $f$ by lower semicontinuity and compact Hausdorff separation. Put $p_i = P_\omega(f_i)$. The $p_i$ are continuous and increase to $P_\omega(f)$. Indeed, for a psh competitor below $f$, the compactness argument of Proposition 12.1.1 produces a continuous psh minorant arbitrarily close to it from above after subtracting a small constant. This also works when $f$ takes the value $+\infty$. Taking a maximum with $a$ gives a minorant of the required kind, which can itself serve as one of the $f_i$.

The envelope property makes $p$ bounded and psh, and $p_i$ increase to $p$ on $\mathcal{D}(X)$ by divisorial agreement. By Theorem 17.13.1, $\int p_i\,\mathrm{d}\mu \to \int p\,\mathrm{d}\mu$. The energy bounds needed there follow from uniform boundedness. The monotone convergence theorem for lower semicontinuous nets also shows that the integrals increase to $\int P_\omega(f)\,\mathrm{d}\mu$. Since $P_\omega(f) \le p$, they are equal $\mu$-almost everywhere.

Write $\mu_j = \mathrm{MA}((\omega, p_j)^n)$. For a fixed $i$ and $j \ge i$, continuous orthogonality shows $\int f_i\,\mathrm{d}\mu_j \le \int f_j\,\mathrm{d}\mu_j = \int p_j\,\mathrm{d}\mu_j$. Use Corollary 17.13.1 to pass to the limit. It gives weak convergence on the left and convergence of the mixed pairings of the right-hand integral. Thus $\int f_i\,\mathrm{d}\mu \le \int p\,\mathrm{d}\mu$. Increasing convergence for $f_i$ gives the same inequality with $f$ in place of $f_i$. As $f \ge P_\omega(f) = p$ almost everywhere, this is the asserted identity. □

# Chapter 19
# Dual energy and the Monge–Ampère equation

On the product test configuration of $X$, the zero potential has normalized Monge–Ampère measure $\delta_{v_0}$, the Dirac mass at the trivial valuation. Every potential $u$ with $u(v_0) = \sup u = 0$ has nonpositive energy, so the zero potential maximizes the energy of a potential minus its integral against $\delta_{v_0}$ (Example 19.7.1). For a probability measure $\mu$ on $X^{\mathrm{an}}$, this suggests a variational approach to finding a potential whose normalized Monge–Ampère measure is $\mu$: maximize the energy of a potential minus its integral against $\mu$. The finiteness of the supremum imposes a condition on $\mu$ itself. This condition does not suffice: on an integral projective nodal curve over an algebraically closed field, some measures satisfying it are not normalized Monge–Ampère measures (Example 19.7.2). We first identify the condition and examine the maximizing problem before addressing existence of a potential.

Throughout this chapter, $X$ is an integral projective scheme of dimension $n$ over a trivially valued field $k$, and $\omega$ is an ample class. Write $V = (\omega^n \cdot [X]) > 0$, $e(u) = V^{-1}E_\omega(u)$, and $\mu_u = V^{-1}\,\mathrm{MA}((\omega, u)^n)$. Thus $\mu_u$ is a probability measure. The same normalization is used for the energy of bounded obstacles. Let $\mathcal{E}^1_\omega = \mathcal{E}^1(X) \cap \mathrm{PSH}(X, \omega)$. We use the normalization $u(v_0) = \sup u = 0$ when needed, where $v_0$ is the trivial valuation of the function field.

Theorem 19.1.1 characterizes the measures of finite energy as those that integrate every potential in $\mathcal{E}^1_\omega$. Proposition 19.2.1 expresses the energy of a continuous obstacle or of a finite-energy potential $u$ as a minimum over these measures; for $u$ the minimum is attained at $\mu_u$. Theorem 19.4.1 describes the strong topology of potentials by a quasi-distance. Under the envelope property, Theorem 19.6.1 solves $\mu_u = \mu$ uniquely up to constants for every measure of finite energy. By Corollary 16.5.1 this hypothesis holds in every real ample class on a unibranch integral projective scheme over an algebraically closed field. Conversely, by Corollary 19.7.1, the envelope property is equivalent to surjectivity of $u \mapsto \mu_u$ onto these measures and to completeness of the normalized finite-energy potentials. For the strong topologies, $u \mapsto \mu_u$ is always a homeomorphism from the normalized finite-energy potentials onto its image (Corollary 19.7.2).

Later chapters use these results as follows. Chapter 31 computes $e^\vee$ and the solutions on curves with Proposition 19.2.1 and Theorem 19.4.1. Chapter 32 uses the lat-

ter theorem for uniqueness, and Chapter 33 uses both results and the strong topology of Section 19.4. Chapter 34 uses Theorem 19.6.1 and Corollaries 19.4.1 and 19.7.1.

## 19.1 Energy of a probability measure

Write $\mathcal{M}$ for the Radon probability measures on $X^{\mathrm{an}}$. For $\mu \in \mathcal{M}$, set

$$e^\vee(\mu) = \sup_{u \in \mathcal{E}^1_\omega} \left( e(u) - \int u \, \mathrm{d}\mu \right).$$

The integral of a psh potential may be $-\infty$, in which case this test gives $+\infty$. The expression is unchanged when a constant is added to $u$. The test $u = 0$ gives $e^\vee(\mu) \geq 0$. We call $\mu$ a measure of finite energy if this supremum is finite, and write $\mathcal{M}^1_\omega$ for these measures.

**Proposition 19.1.1** *Semipositive real model potentials in the fixed class $\omega$ suffice to compute $e^\vee$. Moreover,*

$$e^\vee(\mu) = \sup_{f \in C(X^{\mathrm{an}})} \left( e(f) - \int f \, \mathrm{d}\mu \right),$$

*where $e(f)$ is the normalized energy of an obstacle. The sublevel sets of $e^\vee$ are convex and compact for weak convergence of measures.*

***Proof*** For $u \in \mathcal{E}^1_\omega$, choose decreasing semipositive model approximants in class $\omega$. Their energies decrease to $e(u)$ and their integrals against $\mu$ decrease to $\int u \, \mathrm{d}\mu$ by Theorem 4.2.1. This also gives the required limit of the differences if the integral is $-\infty$. The expression on continuous obstacles therefore gives at least $e^\vee(\mu)$. Conversely, for each competitor $u \leq f$ of finite energy, $e(u) - \int f \, \mathrm{d}\mu \leq e^\vee(\mu)$ by positivity of $\mu$. Candidates of finite energy suffice for $e(f)$, since a constant is a competitor. Taking the supremum gives the reverse inequality.

The tests by continuous functions are weakly continuous affine functions of the measure, so their supremum is lower semicontinuous and convex. Hence its sublevel sets are closed and convex in $\mathcal{M} = \mathcal{P}(X^{\mathrm{an}})$, which is weakly compact by Corollary 4.3.1. This proves the claim about sublevel sets. □

**Theorem 19.1.1** *A probability measure has finite energy if and only if it integrates every potential in $\mathcal{E}^1_\omega$. This condition is independent of the ample class $\omega$.*

***Proof*** The forward implication follows from the definition. For the converse suppose all these potentials are integrable. Consider those with $\sup u = 0$ and $-e(u) \leq 1$. The numbers $-\int u \, \mathrm{d}\mu$ are bounded on this set. Otherwise choose $u_j$ on the set with $-\int u_j \, \mathrm{d}\mu \geq 2^j$. The sums $v_N = \sum_{j=1}^N 2^{-j} u_j$ decrease, since the $u_j$ are nonpositive. The missing weight can be assigned to the zero potential, so concavity of energy

gives $e(v_N) \geq -1$. Their limit has value zero at $v_0$ and is psh by Theorem 6.2.2. It has finite energy by decreasing continuity. But its negative integral is infinite by monotone convergence, a contradiction. Let $C$ be the bound on the set.

For a general normalized $u$, put $K = -e(u) \geq 0$. For $0 \leq t \leq 1$ one has

$$-e(tu) \leq C_n t^2 K. \tag{19.1}$$

Indeed the difference of powers with the zero potential expresses $-e(tu)$ as a sum of mixed integrals of $-tu$. Expand each pair as

$$(\omega, tu) = t(\omega, u) + (1-t)(\omega, 0).$$

The summand with all zero potentials in the measure integrates to zero because the measure is supported at $v_0$. The other summands have coefficients bounded by $2^n t$ before the factor from the integrand. They are nonnegative integrals of $-u$, and their sum, up to a constant depending on $n$, is bounded by $K$ by the difference of powers with $t = 1$. This proves (19.1). For $n = 0$ all normalized potentials vanish and the assertion is direct.

Take $t = (1 + C_n K)^{-1/2}$. The potential $tu$ now belongs to the bounded test set, so $-\int u \, \mathrm{d}\mu \leq C(1 + C_n K)^{1/2}$. Subtracting $K$ gives a uniform upper bound on $e(u) - \int u \, \mathrm{d}\mu$, proving finite energy.

If $w$ has finite energy in another ample class $\eta$, choose $a > 0$ so small that $\omega - a\eta$ is nef. The potential $aw$ is psh in $\omega$, and has finite energy there by homogeneity and Section 17.9. Integrating $aw$ is equivalent to integrating $w$. The integrability criterion therefore gives independence of the class. □

## 19.2 Supporting measures and maximizers

**Proposition 19.2.1** *For a continuous obstacle $f$, one has*

$$e(f) = \min_{\mu \in \mathcal{M}^1_\omega} \left( e^\vee(\mu) + \int f \, \mathrm{d}\mu \right).$$

*For a finite-energy psh potential $u$, the same formula holds with $u$ in place of $f$, and its normalized Monge–Ampère measure attains the minimum. In particular $u$ maximizes $e(v) - \int v \, \mathrm{d}\mu_u$ over finite-energy psh potentials $v$.*

***Proof*** The bound from above by every term of the minimum is the definition of $e^\vee$ and the equivalent obstacle formula. For continuous obstacles $e$ is a concave Lipschitz functional on $C(X^{\mathrm{an}})$. Lipschitz continuity follows by monotonicity and translation. Concavity follows by taking convex combinations of two competitors and applying concavity of psh energy. The right directional derivative of $e$ at $f$ exists in each direction. It is positively homogeneous and superadditive by concavity, and bounded by the uniform norm. Hahn–Banach applied to its negative gives a continuous linear

functional $\Lambda$ majorizing the derivative. Concavity then gives $e(f+h)-e(f) \le \Lambda(h)$ for every continuous $h$. Monotonicity implies $\Lambda(h) \ge 0$ for $h \ge 0$. Testing both constant directions gives $\Lambda(1) = 1$. Thus $\Lambda$ is integration against a Radon probability measure. The supporting inequality gives $e^\vee(\mu) = e(f) - \int f \, \mathrm{d}\mu$ by Proposition 19.1.1. This gives attainment and shows $\mu$ has finite energy.

For $u \in \mathcal{E}^1_\omega$, the energy difference bound of Section 17.11 says $e(v) - e(u) \le \int (v - u) \, \mathrm{d}\mu_u$. This shows $e^\vee(\mu_u) = e(u) - \int u \, \mathrm{d}\mu_u$ and proves the remaining assertions. The difference between this supremum and the value at $v$ is exactly $V^{-1} J_{\omega,u}(v)$. □

## 19.3 The midpoint estimate

For a finite-energy measure $\mu$ and a finite-energy potential $u$, define the *deficit*

$$\Delta_\mu(u) = e^\vee(\mu) - e(u) + \int u \, \mathrm{d}\mu.$$

This is nonnegative and vanishes exactly at the maximizers. A net $u_i \in \mathcal{E}^1_\omega$ is *maximizing* for $\mu$ if $\Delta_\mu(u_i) \to 0$. Write $\iota = V^{-1} I_\omega$ for the normalized energy difference.

**Lemma 19.3.1** *There is $c_n > 0$ such that for $u, v \in \mathcal{E}^1_\omega$,*

$$e((u+v)/2) - \tfrac{1}{2}(e(u) + e(v)) \ge c_n \iota(u, v).$$

*In particular, for every $\mu \in \mathcal{M}^1_\omega$, $\iota(u, v) \le C_n(\Delta_\mu(u) + \Delta_\mu(v))$.*

***Proof*** For $n = 0$ the energy is affine and $\iota = 0$. For $n \ge 1$, put $P_t = (\omega, (1-t)u + tv)$. The energy along this segment is a polynomial by multilinearity of the finite-energy pairing. Its second derivative is $-nV^{-1}\|u - v\|^2_{P_t^{n-1}}$. For $1/4 \le t \le 3/4$, the binomial expansion and positivity of every square give $\|u - v\|^2_{P_t^{n-1}} \ge 4^{1-n} I_\omega(u, v)$. The midpoint gap of a twice differentiable function $g$ on $[0, 1]$ equals the integral of $-g''(t)$ against $\frac{1}{2}\min(t, 1-t)$, by integration by parts. Restrict the integral to $[1/4, 3/4]$ to get the first inequality. For the second, subtract twice the deficit of the midpoint from the sum of the two endpoint deficits. The linear integrals cancel, leaving twice the midpoint gap. The midpoint deficit is nonnegative. □

## 19.4 Strong convergence of potentials

The strong topology on $\mathcal{E}^1_\omega$ is the weak topology of divisorial evaluation refined by continuity of $e$. We first show it is detected by the energy difference. The key is to compare ordered potentials and to use model measures that charge a given divisorial point.

**Lemma 19.4.1** *For each $w \in \mathcal{D}(X)$ there is a continuous $\omega$-psh model potential $q$ with $\mu_q\{w\} > 0$.*

***Proof*** Choose a model with a special-fibre component $F$ representing $w$, as in Proposition 5.4.1. Choose a relatively ample marked bundle on that model, with base class $\eta$ and potential $g$. Its mixed atomic formula places mass at least $b_F(\mathcal{A}^n \cdot [F]) > 0$ at $w$, where $\mathcal{A}$ is the chosen bundle. Take $a > 0$ with $\omega - a\eta$ ample and put $q = ag$. The pair $(\omega, q) = a(\eta, g) + (\omega - a\eta, 0)$ is semipositive. Its measure contains the original measure multiplied by $a^n/V$, by multilinearity and positivity of all the other terms. □

**Lemma 19.4.2** *For $u, v \in \mathcal{E}^1_\omega$ and $b = \max(u, v)$, one has*

$$\iota(u, v) = \iota(b, u) + \iota(b, v).$$

*If $n \geq 1$ and both potentials have supremum zero, then at every $w \in \mathcal{D}(X)$,*

$$|u(w) - v(w)| \leq C_w \iota(u, v)^{2^{-n}} \max\{1, \iota(u, 0), \iota(v, 0)\}^{1-2^{-n}}.$$

***Proof*** The maximum has finite energy by monotonicity. In the sum $\iota(b, u) + \iota(b, v)$, consider the regions $u > v$, $u < v$, and $u = v$. On the first region $b = u$ and $\mu_b = \mu_u$ by the strict locality of Theorem 17.10.1. Its contribution is $(u - v)(\mu_v - \mu_u)$ there. The second region gives the same integrand by interchanging the variables. On the equality set all differences vanish wherever the potentials are finite. Their poles have zero mass for the three measures by finite-energy integrability. This proves the identity.

Now assume the normalizations. The maximum has the same normalization and $\iota(b, 0) \leq (n+1)\iota(u, 0)$, using the comparison with $J_{\omega,0}$ and monotonicity of energy. Fix $q$ as in Lemma 19.4.1. Apply (17.4) with the measure $\mu_q$ to both nonnegative differences $b - u$ and $b - v$. The constant-potential measure in that estimate is supported at $v_0$, where the differences are zero. Each integral thus satisfies the stated bound with $\iota(b, u)$ or $\iota(b, v)$ in place of $\iota(u, v)$. Absorb the fixed $\iota(q, 0)$ and normalization factors in the constant. Since $\mu_q$ charges $w$, the sum of the integrals is at least $\mu_q\{w\}|u(w) - v(w)|$. The identity just proved gives the result. □

**Theorem 19.4.1** *The function*

$$d_\omega(u, v) = \iota(u, v) + |u(v_0) - v(v_0)|$$

*is a quasi-distance defining the strong topology. The equation $\iota(u, v) = 0$ holds exactly when $u - v$ is constant, that is, when $u - u(v_0) = v - v(v_0)$.*

***Proof*** In dimension zero the integral scheme has a single analytic point, so the assertion reduces to the usual distance on constants. Otherwise the pointwise estimate and divisorial determination show definiteness. The quasi-triangle inequality follows from Theorem 17.12.1 and the triangle inequality on the constants.

If $d_\omega(u_i, u) \to 0$, normalize the potentials by subtracting their values at $v_0$. This leaves $\iota$ unchanged. The normalized energies stay bounded by the quasi-triangle

inequality and the comparison with $J_{\omega,0}$. The pointwise estimate then gives weak convergence. The same holds after restoring the constants, which converge. Theorem 17.13.1 supplies $\int u_i \, d\mu_u \to \int u \, d\mu_u$. Since $V^{-1} J_{\omega,u}(u_i) \le \iota(u_i, u)$, the energy difference formula gives convergence of the energies.

Conversely suppose the potentials converge weakly and the energies converge. Their values at $v_0$ converge, so the comparison of $I_\omega$ with $J_{\omega,0}$ gives bounded energy differences from zero. The same integration argument gives $J_{\omega,u}(u_i) \to 0$, hence $\iota(u_i, u) \to 0$. □

**Corollary 19.4.1** *If $u \in \mathcal{E}^1_\omega$, $v$ is $\omega$-psh, and $v \le u$ almost everywhere for $\mu_u$, then $v \le u$ everywhere.*

***Proof*** Replace $v$ by $w = \max(u, v)$, which has finite energy by monotonicity. Then $w - u \ge 0$ and it is zero almost everywhere for $\mu_u$. Hence $0 \le \iota(w, u) = -\int (w - u) \, d\mu_w \le 0$. Theorem 19.4.1 shows that $w - u$ is constant. Since $w = u$ almost everywhere for the probability measure $\mu_u$, this constant is zero. □

## 19.5 Approximating measures by maximizing nets

**Theorem 19.5.1** *Assume the envelope property in $\omega$ and let $\mu \in \mathcal{M}^1_\omega$. For every net $u_i \in \mathcal{E}^1_\omega$ with $\Delta_\mu(u_i) \to 0$, one has $\mu_{u_i} \to \mu$ weakly. For every $C > 0$ the integrals against $\mu_{u_i}$ converge to the integrals against $\mu$ uniformly on $\{a \in \mathcal{E}^1_\omega : \iota(a, 0) \le C\}$.*

***Proof*** First extend the obstacle energy formula to obstacles $f = u + th$ with $u \in \mathcal{E}^1_\omega$ and $h$ continuous. Use the same supremum over psh minorants. The minorant $u - |t| \|h\|_\infty$ bounds the energy from below. Such an obstacle is upper semicontinuous, so the envelope property makes $P_\omega(f)$ psh, finite-energy, and a maximal minorant. For $\psi_k$ decreasing to $u$ through continuous psh potentials, $P_\omega(\psi_k + th)$ decreases to $P_\omega(u + th)$. Indeed Theorem 6.2.2 applies because of the fixed minorant just given, and the limit is below $u + th$. By decreasing continuity of the pairing the energies converge. The measures $\mu_{\psi_k}$ converge weakly to $\mu_u$.

For a rational PL direction $h$ and $|t| \le 1$, the estimate (18.3) of Proposition 18.3.1, divided by $V$, gives

$$\left| e(u + th) - e(u) - t \int h \, d\mu_u \right| \le C_h t^2$$

for continuous $u$. The constant $C_h$ depends only on $\|h\|_\infty$, $\omega$ and the auxiliary class of a model difference representing $h$, not on $u$. The decreasing limits of the preceding paragraph extend the estimate to all finite-energy $u$.

For any finite-energy competitor $v \le u_i + th$, $e(v) - \int (u_i + th) \, d\mu \le e^\vee(\mu)$. It follows that $e(u_i + th) - e(u_i) \le \Delta_\mu(u_i) + t \int h \, d\mu$. The estimate and its counterpart for the direction $-h$ now give, for $0 < t \le 1$,

$$\left| \int h \, \mathrm{d}(\mu_{u_i} - \mu) \right| \leq C_h t + t^{-1} \Delta_\mu(u_i).$$

Take the net limit and then let $t$ tend to zero. Rational PL density proves weak convergence.

For the uniform assertion take $u = u_i$, $v = u_j$. On a tail $\iota(u, 0)$ and $\iota(v, 0)$ are bounded by Lemma 19.3.1, using zero as the other endpoint and $\Delta_\mu(0) = e^\vee(\mu)$. Telescope the difference of the measures in the integral of a test $a$. In each of the $n$ terms there are two function slots, $(0, a)$ and $(0, u - v)$, and $n - 1$ slots filled by $(\omega, u)$ or $(\omega, v)$. Cauchy–Schwarz bounds the term by the two vertical seminorms. The first seminorm is bounded uniformly for $\iota(a, 0) \leq C$, by the mixed estimate of Theorem 17.12.1. The sum of the squares of the second seminorms is $I_\omega(u, v)$. Consequently

$$\left| \int a \, \mathrm{d}(\mu_{u_i} - \mu_{u_j}) \right| \leq C' \big( \Delta_\mu(u_i) + \Delta_\mu(u_j) \big)^{1/2}$$

with $C'$ depending on the test energy bound and the bound on the tail of the net. For continuous tests pass to the limit in $j$ by weak convergence. For a general test $a$ approximate it from above by continuous psh potentials in the fixed class. Their energy differences from zero are bounded on a tail by $C + 1$, by decreasing continuity. Their integrals converge against both $\mu_{u_i}$ and $\mu$ by monotone convergence and finite-energy integrability. Thus the bound persists, with a constant uniform on the given energy ball. □

## 19.6 Existence and uniqueness

**Theorem 19.6.1** *Assume the envelope property in $\omega$. For every $\mu \in \mathcal{M}^1_\omega$ there exists a unique $u \in \mathcal{E}^1_\omega$ with $\sup u = 0$ and $\mu_u = \mu$. Its translates by constants are exactly the maximizers of $e(v) - \int v \, \mathrm{d}\mu$ among finite-energy potentials $v$.*

***Proof*** Choose a maximizing net and subtract its suprema to normalize. By Corollary 12.2.1, pass to a subnet converging pointwise on $\mathcal{D}(X)$ to a normalized psh $u$. Its energy is at least the limit superior of the approximating energies. To see this, form the regularized suprema of the tails of the subnet. They are psh and decrease to $u$, by divisorial agreement, Theorem 6.2.2, and divisorial determination. By monotonicity, each tail supremum has energy at least the supremum of the energies on that tail. Now use decreasing continuity. The midpoint gap bounds $\iota(u_i, 0)$ on a tail, and $-e(u_i) = V^{-1} J_{\omega,0}(u_i)$ is bounded there. Hence $u$ has finite energy.

For every $\nu \in \mathcal{M}^1_\omega$, and in particular for $\nu = \mu$, integration against $\nu$ along this subnet converges to $\int u \, \mathrm{d}\nu$. Indeed choose a maximizing net $z_j$ for $\nu$. By Theorem 19.5.1, the integrals against $\mu_{z_j}$ approximate the integrals against $\nu$ uniformly on the energy ball containing the subnet and $u$. For fixed $j$ the integrals against $\mu_{z_j}$ converge by Theorem 17.13.1. Combining these two assertions gives the convergence.

The energy upper bound at the limit now shows $\Delta_\mu(u) = 0$. Apply Theorem 19.5.1 to the constant net with value $u$, obtaining $\mu_u = \mu$. Every solution is a maximizer by Proposition 19.2.1, and the constant-net argument proves the converse for every maximizer.

If $\mu_u = \mu_v$, then $\iota(u, v) = 0$ by its definition, so the normalized potentials agree by Theorem 19.4.1. □

## 19.7 Completeness and surjectivity

A net is Cauchy for a quasi-distance if the quasi-distance between two terms tends to zero as both indices tend to infinity.

**Theorem 19.7.1** *The envelope property in an ample class $\omega$ is equivalent to completeness of the normalized finite-energy potentials for $\iota$ and to completeness of all finite-energy potentials for $d_\omega$.*

***Proof*** Assume the envelope property. A Cauchy net $u_i$ of normalized potentials has bounded energy differences from zero on a tail, by the quasi-triangle inequality. By weak compactness from Corollary 12.2.1, a subnet converges to a normalized psh $u$. The tail-supremum argument in the proof of Theorem 19.6.1 gives upper semicontinuity of $e$ there, so $u$ has finite energy. For each fixed index $i$, integration of the subnet against $\mu_{u_i}$ converges by Theorem 17.13.1. Thus

$$J_{\omega,u_i}(u) \le \liminf_j J_{\omega,u_i}(u_j),$$

where $j$ ranges over the subnet. The comparison of $J$ and $I$ and the Cauchy condition give $\iota(u, u_i) \to 0$.

Conversely, assume completeness on normalized potentials and take an increasing psh net $u_i$ uniformly bounded above and below. Its energies increase to a finite limit. For $i \le j$, the difference-of-powers formula and positivity give

$$\iota(u_i, u_j) \le \int (u_j - u_i)\, \mathrm{d}\mu_{u_i} \le (n+1)\big(e(u_j) - e(u_i)\big).$$

For incomparable indices choose a common later index and use the quasi-triangle inequality. The normalized net is Cauchy, and its normalizing constants increase to a finite limit. Restoring that limit gives a psh limit on the divisorial set by Theorem 19.4.1. For an increasing net only bounded above, apply the preceding step to $\max\{u_i, -m\}$ for each positive integer $m$. The limiting psh potentials decrease with $m$ by divisorial determination. At each divisorial point they decrease to $\sup_i u_i$. In particular the limit at $v_0$ is finite. Theorem 6.2.2 gives a psh limit on $X^{\mathrm{an}}$, so the increasing-net criterion of Corollary 12.2.1 gives the envelope property.

Completeness with and without normalization is equivalent because $u \mapsto (u - u(v_0), u(v_0))$ identifies $d_\omega$ with the sum of the normalized quasi-distance and the usual real distance. □

### 19.7.1 Uniform testing of measures

The strong topology on $\mathcal{M}^1_\omega$ refines weak convergence by continuity of $e^\vee$. We also use the distance on $\mathcal{M}^1_\omega$ given by

$$d_{\text{test}}(\mu,\nu) = \sup_{\substack{u\in\mathcal{E}^1_\omega,\ u(v_0)=0\\ -e(u)\le 1}} \left|\int u\,\mathrm{d}(\mu-\nu)\right|.$$

**Proposition 19.7.1** *This distance is a finite complete metric. Metric convergence implies strong convergence. If the normalized Monge–Ampère operator is onto $\mathcal{M}^1_\omega$, this metric defines the strong topology.*

***Proof*** Let $B_\mu$ be the supremum of $-\int u\,\mathrm{d}\mu$ on the normalized energy unit ball. It is at most $1 + e^\vee(\mu)$, so the distance is finite. The triangle inequality is immediate. A rational PL function is a difference of continuous psh potentials in an ample class by Section 17.5. Scale both to enter $\omega$, normalize, and scale again to enter the unit ball. Since both measures are probabilities, this proves separation and weak convergence for metric-convergent nets by PL density.

For $K = -e(u)$ and $\sup u = 0$, the small-slope estimate (19.1), with $t = (1 + C_nK)^{-1/2}$, gives

$$-\int u\,\mathrm{d}\mu \le B_\mu(1 + C_nK)^{1/2}.$$

If $B_\mu$ is bounded, the tests computing $e^\vee(\mu)$ can therefore all be taken from one fixed ball $-e(u) \le C$. Outside that ball the objective is negative, whereas zero is a test. Differences of integrals on this larger ball are bounded by a constant times $d_{\text{test}}$, by the same scaling. Metric convergence bounds $B_{\mu_i}$, so the dual energies converge by taking suprema on the larger ball.

If $\mu_i$ is metric-Cauchy then $B_{\mu_i}$, and hence the dual energies, are bounded on a tail. A subnet converges weakly to $\mu \in \mathcal{M}^1_\omega$ by Proposition 19.1.1. Given $\varepsilon > 0$, the Cauchy bound gives $\left|\int u\,\mathrm{d}(\mu_i - \mu)\right| \le \varepsilon$ for late $i$ and continuous tests in the unit ball, by passing to the weak limit in one index. For an arbitrary test, take decreasing continuous psh majorants and subtract their values at $v_0$. Their energies converge to $e(u)$. Scaling each by at most one if necessary makes it a unit-ball test, with the multipliers tending to one by concavity. Their integrals converge against each finite-energy measure by monotone convergence before subtracting the constants. Thus the estimate holds for the entire unit ball and proves completeness.

For the last assertion put

$$D(\mu,\nu) = \inf_{u\in\mathcal{E}^1_\omega} \big(\Delta_\mu(u) + \Delta_\nu(u)\big).$$

For any $a, b \in \mathcal{E}^1_\omega$, the identity $\Delta_{\mu_a}(u) = V^{-1}J_{\omega,a}(u)$, the comparison of $J$ and $I$, and the quasi-triangle inequality give

$$c_n\iota(a,b) \le D(\mu_a,\mu_b) \le C_n\iota(a,b).$$

For the upper bound take $u = a$. Along a strongly convergent net $\mu_i \to \mu$, choose a fixed continuous psh test whose deficit for $\mu$ is arbitrarily small. Its deficit for $\mu_i$ tends to that for $\mu$, so $D(\mu_i, \mu) \to 0$. Under surjectivity choose potentials $a_i, a$ for the measures. Then $\iota(a_i, a) \to 0$, so the energies of the normalized potentials stay bounded. The mixed-measure telescope in the proof of Theorem 19.5.1 gives $d_{\mathrm{test}}(\mu_{a_i}, \mu_a) \le C\iota(a_i, a)^{1/2}$ on this energy ball. The telescope uses only the vertical Cauchy–Schwarz inequality and mixed energy estimates. This proves the asserted metric convergence. □

**Corollary 19.7.1** *For an integral projective scheme over any trivially valued field and any real ample class, the envelope property, surjectivity onto $\mathcal{M}^1_\omega$, and completeness of the normalized finite-energy potentials for $\iota$ are equivalent.*

***Proof*** Existence under the envelope property is Theorem 19.6.1. Suppose instead surjectivity and take a Cauchy net $a_i$ of normalized potentials. Its energies are bounded on a tail. The same mixed estimate makes the measures Cauchy for $d_{\mathrm{test}}$, hence strongly convergent. By surjectivity the limit is the measure of a normalized finite-energy potential $a$. The common-deficit comparison gives $\iota(a_i, a) \to 0$. The implication from completeness to the envelope property is Theorem 19.7.1. □

**Corollary 19.7.2** *Give the normalized finite-energy potentials the strong topology and $\mathcal{M}^1_\omega$ its strong topology. The map $u \mapsto \mu_u$ is continuous and is a homeomorphism onto its image. Its image is $\mathcal{M}^1_\omega$ if and only if $\omega$ has the envelope property.*

***Proof*** Let $a_i, a$ be normalized finite-energy potentials, so that $d_\omega(a_i, a) = \iota(a_i, a)$. Suppose first that $\iota(a_i, a) \to 0$. As in the last paragraph of the proof of Proposition 19.7.1, the energies of the $a_i$ are bounded on a tail, and the mixed-measure telescope gives $d_{\mathrm{test}}(\mu_{a_i}, \mu_a) \le C\iota(a_i, a)^{1/2}$ there. Metric convergence implies strong convergence by Proposition 19.7.1, so $\mu_{a_i} \to \mu_a$ strongly. Conversely, if $\mu_{a_i} \to \mu_a$ strongly, the same proof gives $D(\mu_{a_i}, \mu_a) \to 0$, and the inequality $c_n\iota(a_i, a) \le D(\mu_{a_i}, \mu_a)$ proved there gives $\iota(a_i, a) \to 0$. If $\mu_a = \mu_b$, then $\iota(a, b) = 0$ by the definition of $\iota$, so $a = b$ by Theorem 19.4.1 and the normalization. Thus the map is injective, and the two implications show that it is a homeomorphism onto its image. The last assertion is Corollary 19.7.1. □

*Example 19.7.1 (The measure of the product model)* Keep the standing hypotheses on $X$ and $\omega$. On the product test configuration the measure of the zero potential is $\mu_0 = \delta_{v_0}$, by the mixed atomic formula. A normalized psh potential $u$ has $u(v_0) = 0$ and $e(u) \le 0$. Thus $e^\vee(\delta_{v_0}) = 0$. The deficit for any finite-energy potential is $u(v_0) - e(u)$. Its vanishing implies $I_\omega(u, 0) = 0$ by the comparison with $J_{\omega,0}$, and the divisorial estimate shows that the normalized maximizer is zero.

*Example 19.7.2 (The energy quasi-distance on a nodal curve)* Let an integral projective nodal curve over an algebraically closed field have a node with two branches. Its normalization separates the branches, so no ample class has the envelope property by Proposition 12.3.1. By Corollary 19.7.1 the normalized finite-energy space is incomplete in every ample class, and its Monge–Ampère map has proper image in the finite-energy measures.

# Chapter 20
# Local and global measures on curves

On the ray of a closed point $p$, a log norm is affine in the multiple of $\mathrm{ord}_p$. The second derivative of a truncated log norm is a point mass at its bend. At the common origin there is a second contribution, from the outgoing slope of the ray. Its weight involves the degree of $p$ over the ground field. We first calculate this weight from the local real forms, then recover it by intersecting a divisor on a projective model.

Throughout, $C$ is an integral regular projective curve over a trivially valued field $k$. We use the ray notation of Section 1.4 and Theorem 1.4.1: $v_0$ is the trivial valuation of $k(C)$, and the ray of a closed point $p$ consists of the points $x_{p,t}$ for $0 < t \leq \infty$, with endpoint $x_{p,\infty}$ supported at $p$. Write $d_p = [\kappa(p) : k]$ and, for a function $h$, write $h_p(t) = h(x_{p,t})$. The valuation parameter is normalized so that a uniformizer at $p$ has value $t$ at $x_{p,t}$.

Proposition 20.1.1 computes integrals of real forms ray by ray, with the weight $d_p$ on the ray of $p$. Theorem 20.3.1 identifies the intersection measure of a model potential with the curvature current $\deg \theta\, \delta_{v_0} + \Delta u$. Theorem 20.4.1 characterizes the psh functions that are finite at $v_0$ by finite, convex, decreasing ray profiles whose endpoint values are the limits along the rays, subject to an initial-slope budget. This gives the envelope property for every real numerical class (Corollary 20.4.1). The comparison of measures then extends to continuous psh potentials (Theorem 20.5.1) and to finite-energy potentials in ample classes (Corollary 20.5.1). Chapter 31 builds on these formulas.

## 20.1 Real forms on the rays

Let $U \subset C^{\mathrm{an}}$ be open. A *moment* on $U$ is a map $F = (-\log|f_1|, \dots, -\log|f_m|)$ from $U$ to $\mathbb{R}^m$, given by finitely many invertible analytic functions $f_j$ on $U$. A function on $U$ is *smooth* if each point of $U$ has an open neighborhood $W$ on which it equals $H \circ F$ for a moment $F$ on $W$ and a smooth function $H$ on an open subset of $\mathbb{R}^m$ containing $F(W)$. A *smooth real form* on $U$ is locally the pullback $F^*\alpha$ of a smooth form $\alpha$ of Section 10.1 by a moment $F$. On a finite open segment of the $p$-ray the log norms

in a moment are integral multiples of $t$. At the origin they come from a finite list of rational functions. Their directions are the vectors of orders at the closed points. Near a closed endpoint the norms of every such list are constant.

To prove the assertions about local presentations, use an algebraic affine chart at the chosen point. A neighborhood in it contains a neighborhood of the point defined by finitely many closed inequalities $|g| \le r$ and $|g'| \ge r'$ on regular functions $g, g'$, with $r' > 0$. To obtain it, take closed upper and positive lower bounds on the values of regular functions, inside the original open bounds. Add strict upper bounds on algebra generators, then use the affinoid algebra of this closed box. Each extra upper bound is a rational inequality with denominator 1. A positive lower bound on $|g'|$ is the rational inequality on $1/g'$. These finite rational constructions use the presentations of Proposition 2.2.1 and (2.3). Polynomials in the algebraic generators and the adjoined inverses are dense by the series description. For a unit $\phi$ of an analytic presentation at $x$, choose one such algebraic rational approximant $f$ with $|f - \phi|(x) < |\phi(x)|$. On a smaller neighborhood the strict inequality persists, so $|f| = |\phi|$. Repeat for the finite list. The functions $f$ can be regarded as units on a Zariski open neighborhood of the support of $x$. The ray formulas for rational functions now give the assertions. At a closed endpoint the rational functions used there have order zero.

A smooth function $\chi$ on $C^{\mathrm{an}}$ is thus smooth in $t$ on each ray, constant on an end interval of each ray and constant on all but finitely many rays. Its derivatives at $t = 0$ exist. They satisfy

$$\sum_p d_p \chi_p'(0) = 0. \tag{20.1}$$

To see this, near $v_0$ write $\chi$ as a smooth function $H$ of the moment of rational units $f_1, \dots, f_m$. Differentiating the moment on the $p$-ray gives

$$\chi_p'(0) = \sum_j (\partial_j H)(0) \operatorname{ord}_p(f_j).$$

Each principal divisor $\operatorname{div}(f_j)$ has degree zero, as shown in the proof of Theorem 8.7.1, so summing with the weights $d_p$ gives (20.1).

Smooth functions on $C^{\mathrm{an}}$ serve as *smooth tests*. A smooth function of $t$ with compact support in a finite open interval of a ray is one: a uniformizer gives the coordinate there, and the function extends by zero. The same construction allows functions constant for large $t$ on a single ray and vanishing near $v_0$. In particular these tests together with the constants separate points. Smooth functions form a unital real algebra and are uniformly dense in $C(C^{\mathrm{an}})$.

Since $v_0$ and the closed endpoints carry no one-dimensional cell, we define the integral of a compactly supported smooth real $(1,1)$-form $\gamma$ on $C^{\mathrm{an}}$ by (20.2) below.

**Proposition 20.1.1** *For a smooth real $(1,1)$-form $\gamma$ of compact support, write its restriction to the finite part of the $p$-ray as $\gamma_p(t)\, d't \wedge d''t$. Over a closed subinterval of a finite open segment of the $p$-ray, the calibration (10.1) of Section 10.5, for the projection of a moment to any coordinate that is nonconstant in norm there, integrates $\gamma$ as $d_p \gamma_p(t)\, \mathrm{d}t$. These local integrals glue to the integral*

$$\int \gamma = \sum_p d_p \int_0^\infty \gamma_p(t)\,\mathrm{d}t. \tag{20.2}$$

*In particular integration of $d'd''\chi$ is zero for every global smooth function $\chi$.*

***Proof*** We calculate the degree attached to an open segment of a ray. For a rational function $z$ of positive order $e$ at $p$, the rational map $C \to \mathbb{P}^1_z$ is finite, as in Lemma 8.7.2. The inverse image of a closed subannulus $0 < r \le |z| \le R < 1$ has algebra

$$k((z)) \otimes_{k[z]} D, \qquad D = \mathcal{O}(C \times_{\mathbb{P}^1_z} \mathbb{A}^1_z).$$

Here the annulus algebra is $k((z))$ with the two-radius norm of (2.5). The inverse-image presentation follows by adjoining finite algebra generators of $D$ and their relations to that annulus algebra. Since $D$ is finite over $k[z]$, a finite module presentation gives the quotient module topology of Proposition 2.3.1. Equivalently one can bound the generators by their monic equations, use the analytic box with these bounds, and impose the annulus inequalities there. The finite algebra has one field factor $\widehat{k(C)}_p$ per point over $z = 0$, by completing the finite $k[z]$-module at $(z)$ and then inverting $z$. This is the decomposition used in the proof of Lemma 8.7.2. By Theorem 1.4.1 the factor for $p$ has spectrum the indicated closed subinterval of that ray. Taking $z$ of order one gives arbitrary closed subintervals with positive finite endpoints.

The same degree calculation works for any analytic moment projected to a single coordinate $g$ that is nonconstant in norm along this interval. Use its inverse if its order is negative. On a closed subinterval the function $g$ belongs to $\widehat{k(C)}_p$, and its order there is an integer $e > 0$. Its map from the annulus algebra $k((Z))$ sends $Z$ to $g$. The valuation ring of $\widehat{k(C)}_p$ is a free $k[[g]]$-module of rank $ed_p$. Indeed, its quotient by $g$ has $k$-dimension $ed_p$ by the successive powers of the maximal ideal. Lifting a basis and expanding successively in $g$ gives a basis over $k[[g]]$. For uniqueness divide a relation by the smallest power of $g$ and reduce. This argument uses completeness and works for inseparable residue extensions. Inverting $g$ gives degree $ed_p$. For this finite map the quotient module norm is equivalent to the maximum of the two endpoint field norms. Indeed a fixed basis compares the two norms with coordinate norms at each endpoint separately. Thus this projection of the moment is finite and flat of that degree along the open segment.

The calibration for an interval is computed by integrating in such a projected coordinate with its finite flat degree. Here the coordinate is $y = et$ with degree $ed_p$. If the form in that coordinate is $c(y)\,d'y \wedge d''y$, its coefficient in $t$ is $e^2c(et)$. The weighted integral in $y$ is

$$ed_p \int c(y)\,\mathrm{d}y = d_p \int e^2 c(et)\,\mathrm{d}t.$$

Reversing the coordinate of negative slope gives the same formula, since the two differentials give the square of the slope and the integration length is positive. Thus the calibration is independent of the projection and of the presentation on each branch. For an algebraic moment the degrees of the branches over the same tropical ray add.

In a primitive coordinate their sum is the weight of Lemma 8.7.2 for the pushforward cycle, that is, with $l_i$ replaced by the degree of $C$ over its image in the torus. A moment with constant norm on an interval contributes no $(1,1)$-form there. The common origin and the closed endpoints contribute no one-dimensional cell. On a compact support finitely many local presentations suffice, and each uses only finitely many nonconstant rays. This proves the gluing to (20.2). Finally the integral of $d'd''\chi$ is $\sum_p d_p \int_0^\infty \chi_p''(t)\,\mathrm{d}t = -\sum_p d_p \chi_p'(0)$, which vanishes by (20.1). □

Thus (20.2) is the sum of the segment calibrations, and it agrees with the calibration of every such projection on an open ray segment.

## 20.2 Curvature as a current

For a continuous function $u$ define the current $\Delta u = d'd''[u]$ by

$$\langle \Delta u, \chi \rangle = \sum_p d_p \int_0^\infty u_p(t)\chi_p''(t)\,\mathrm{d}t. \tag{20.3}$$

We use the same formula when $u$ is integrable against compactly supported smooth top forms. Tests for which the support lies in an open set give the restriction to that set. The distributional definition agrees with the second derivative of a smooth function by integration by parts. At $v_0$ the boundary terms for two smooth functions are their values times the sums in (20.1).

For $f \in k(C)^\times$, set $m_p = \operatorname{ord}_p(f)$. The function $\log|f|$ is integrable against smooth top forms. Their coefficients vanish near the closed endpoints, and on a finite ray the log norm equals $-m_p t$. Integration by parts gives

$$\int_0^\infty (-m_p t)\chi_p''(t)\,\mathrm{d}t = m_p\big(\chi(x_{p,\infty}) - \chi(v_0)\big).$$

The $v_0$-terms sum to zero, by the degree of a principal divisor. This is the curve Poincaré–Lelong formula

$$\Delta \log|f| = \sum_p d_p m_p \delta_{x_{p,\infty}}. \tag{20.4}$$

Let $L$ be a line bundle with its trivial model metric $\|\cdot\|_0$, defined by declaring an algebraic frame at the centre to have norm one. For a rational nonzero section $\sigma$, the function $-\log\|\sigma\|_0$ is $t\,\operatorname{ord}_p(\sigma)$ on the finite part of the $p$-ray. If $\|\sigma\| = e^{-u}\|\sigma\|_0$, set

$$c_1(L, \|\cdot\|) = \Delta(-\log\|\sigma\|) + \sum_p d_p \operatorname{ord}_p(\sigma)\delta_{x_{p,\infty}}.$$

It is independent of $\sigma$ by (20.4) and restricts to $\Delta(-\log\|\ell\|)$ in a local frame $\ell$. The preceding integration by parts gives

$$c_1(L, \|\cdot\|) = \deg(L)\delta_{v_0} + \Delta u. \tag{20.5}$$

For a real numerical class $\theta$ replace $\deg(L)$ by $\deg\theta$. Degree determines the numerical class on a curve.

## 20.3 Model potentials and intersection degrees

A real model function is constant on all but finitely many rays, with the constant equal to its value at $v_0$. On each remaining ray it is continuous and piecewise affine with finitely many rational breakpoints, and is constant for large $t$. To see this, apply (5.2) and Theorem 5.4.1. Nonzero coefficient ideals of a flag are invertible on a regular curve. Their orders are nonzero at only finitely many closed points. On the $p$-ray each flag function is a maximum of finitely many affine functions in $t$ with integer slopes and intercepts, including a constant term. Conversely, for rational $s > 0$ let $h_{p,s}$ have value $-\min(t, s)$ on the $p$-ray and zero elsewhere. If $s = a/b$ with $a, b > 0$, it is $1/b$ times the function of the flag $(\mathcal{I}_p^b, \tau^a)$, where $\mathcal{I}_p \subset \mathcal{O}_C$ is the ideal sheaf of $p$ and $\tau$ is the model parameter. A continuous profile with the indicated properties is a linear combination of these functions and a constant. Its coefficient at $h_{p,s}$ is the jump of its derivative at $s$, as is seen by reading the slopes back from the constant end interval. Hence all such profiles are real model functions.

**Lemma 20.3.1** *For every closed $p$ and rational $s > 0$,*

$$\mathrm{MA}(0, h_{p,s}) = d_p(\delta_{x_{p,s}} - \delta_{v_0}) = \Delta h_{p,s}.$$

***Proof*** For the current identity, $h_{p,s}(v_0) = 0$, its initial derivative is $-1$, and its derivative jumps by 1 at $s$. Substitution in (20.3) and integration by parts give the formula.

For intersections take $s = a/b$ in lowest terms and blow up $(\mathcal{I}_p^b, \tau^a)$. Write $E$ for the effective Cartier divisor of the pulled-back ideal. At $p$, choose a uniformizer $z$. The two Rees charts have equations $\tau^a = z^b w$ and $z^b = \tau^a w'$. Their exceptional reduced curve is $\mathbb{P}^1_{\kappa(p)}$ with $w' = 1/w$ on the overlap, and $\mathcal{O}(-E)$ restricts there to $\mathcal{O}(1)$. Its degree over $k$ is $d_p$. In the first chart the exceptional curve is generically given by $z = \tau = 0$ with $w$ invertible. Modulo $\tau$, the local ring at its generic point is $\mathcal{O}_{C,p}[w]_{(z)}/(z^b)$. Since $\mathcal{O}_{C,p}[w]_{(z)}$ is a discrete valuation ring with uniformizer $z$ and residue field $\kappa(p)(w)$, this ring has length $b$. Thus the exceptional curve has multiplicity $b$ in the special-fibre cycle. The strict transform at $w = 0$ is isomorphic to $C$, has multiplicity one, and the ideal restricts there to $\mathcal{I}_p^b$. The degree of $\mathcal{O}(-E)$ on it is $-bd_p$.

On normalizing, an exceptional component has $w$ a unit at its generic point, so $\tau^a = z^b w$ gives

$$\frac{\mathrm{ord}(z)}{\mathrm{ord}(\tau)} = \frac{a}{b}.$$

Its valuation on $C$ is therefore $x_{p,s}$. Over the generic point of the strict transform the valuation is $v_0$. For the normalization $\nu$, the pushforward of its special-fibre cycle

is the special-fibre cycle of the blowup. By the projection formula, the components over the exceptional curve contribute $bd_p$ in total, all at $x_{p,s}$, and those over the strict transform contribute $-bd_p$ at $v_0$. Consequently $O(-E)$ contributes $bd_p(\delta_{x_{p,s}} - \delta_{v_0})$. Divide by $b$. □

**Theorem 20.3.1** *For a real model function $u$ and a real numerical class $\theta$ on $C$,*

$$\mathrm{MA}(\theta, u) = \deg\theta\, \delta_{v_0} + \Delta u. \tag{20.6}$$

*The pair is semipositive exactly when the measure on the right is positive. In slope terms, each ray profile must be convex and*

$$\deg\theta + \sum_p d_p u_p'(0) \geq 0.$$

***Proof*** The product model for $(\theta, 0)$ gives $\deg\theta\, \delta_{v_0}$. Use linearity of model intersections in dimension one, the decomposition into the functions $h_{p,s}$, and Lemma 20.3.1 to obtain (20.6). On an integral normal model dominating the product, the special-fibre components have distinct valuations on $C$. Indeed their valuations on $k(C)(\tau)$, normalized by the order of $\tau$, are the Gauss extensions of the corresponding valuations on $C$, by Proposition 5.3.1. Distinct prime divisors of a normal model define distinct valuations because the centre on this separated model is unique. The mass at a component valuation is therefore its degree times a positive multiplicity. Nonnegative degrees on the components are equivalent to relative nefness, since the special fibre is a curve. The slope description follows from the current formula for $h_{p,s}$ and the derivative jumps. □

*Example 20.3.1* Let $p$ be a closed point of $\mathbb{P}^1_k$ of degree $d_p$, and let $s > 0$ be rational. In the numerical class of $O(1)$, the function $d_p^{-1} h_{p,s}$ has model measure $\delta_{x_{p,s}}$. The local slope jump is $d_p^{-1}$, so the mass at the bend is one after calibration. On the model the same factor compensates for the degree of the exceptional curve over $k$.

## 20.4 Psh profiles and envelopes

**Theorem 20.4.1** *Let $u : C^{\mathrm{an}} \to [-\infty, +\infty)$ have $a = u(v_0) \in \mathbb{R}$, and let $\theta$ have degree $V$. Then $u$ is $\theta$-psh exactly when the following hold.*

(1) *Each profile $u_p$, with $u_p(0) = a$, is finite, convex and decreasing on $[0, \infty)$. One has $u(x_{p,\infty}) = \lim_{t\to\infty} u_p(t)$.*
(2) *For all finite sets $F$ of closed points and all $t_p > 0$ for $p \in F$,*

$$\sum_{p\in F} d_p \frac{a - u_p(t_p)}{t_p} \leq V. \tag{20.7}$$

***Proof*** If $u$ is psh, then $\theta$ is pseudoeffective and has nonnegative degree, so it is nef on $C$. Use the decreasing finite rational Fubini–Study approximation in nef limiting classes of Proposition 6.2.1. The approximants are semipositive model functions of degrees $V_i \to V$ by Theorem 5.5.1. They satisfy (20.7) with $V_i$, since each secant from the origin is at least the initial derivative and all slopes are nonpositive. The one-ray bound in this inequality gives finiteness on the finite rays in the limit. Convexity, monotonicity and (20.7) then pass to the limit. Monotonicity gives an upper bound for the endpoint value by the limit along its ray. Upper semicontinuity gives the reverse inequality.

Conversely, the condition with $F$ empty gives $V \geq 0$. Choose a finite $F$ and for each $p \in F$ a finite partition of $[0, R_p]$ with rational breakpoints and positive endpoint $R_p$. Interpolate $u_p$ linearly on these partitions, extend by constants after $R_p$, and put $a$ on the other rays. The resulting profiles are convex. Their first secants satisfy the initial derivative bound of Theorem 20.3.1 by (20.7). Thus they are semipositive models in the fixed class $\theta$. They majorize $u$, by convexity before $R_p$ and monotonicity after $R_p$. They decrease when one adds rays, extends the partitions or inserts breakpoints. These choices form a directed family. The profiles are continuous on the finite ray, including at zero by the one-ray estimate in (20.7). It follows by refining the partitions that the limit is $u$ there. At a closed endpoint the limit is $u$ by the condition on the profile at infinity. Use Theorems 6.2.2 and 17.4.1 to conclude.□

**Corollary 20.4.1** *Every real numerical class on an integral regular projective curve over a trivially valued field has the envelope property.*

***Proof*** For a nonempty uniformly bounded-above family of $\theta$-psh functions, use finite maxima to make the family increasing, by Theorem 6.2.1. On the finite rays the pointwise supremum preserves convexity, monotonicity and finiteness. It preserves (20.7) by taking the increasing limit in each of the finitely many evaluations, including at $v_0$. Consequently it is continuous on every finite open ray. It is upper semicontinuous at $v_0$, since all values are at most the value at $v_0$. Upper regularization therefore affects only the closed endpoints. At an endpoint it gives the limit along the ray, since the profile decreases and its original value there is at most this limit. Apply Theorem 20.4.1. □

The upper regularization in this proof can change values at a closed endpoint. On $\mathbb{P}^1$ the potentials $-\min(t/j, 1)$ on one rational ray and zero elsewhere increase in the class of $O(1)$. Their supremum is zero at all finite radii and $-1$ at the endpoint. The upper regularization is zero also at the endpoint.

## 20.5 Continuous and finite-energy potentials

**Theorem 20.5.1** *Let $u$ be continuous on $C^{\mathrm{an}}$ and $\theta$ a real numerical class of degree $V$. Then $u$ is $\theta$-psh if and only if the current $V\delta_{v_0} + \Delta u$ is nonnegative on nonnegative smooth tests. In this case the current is a positive Radon measure, and it equals* $\mathrm{MA}(\theta, u)$.

***Proof*** If $u$ is $\theta$-psh, the semipositive model approximation of Theorem 20.4.1 converges uniformly, by Proposition 4.1.1. For every smooth test, (20.3) passes to the uniform limit, since the second derivative of the test is integrable on the finitely many contributing rays. Use Theorem 20.3.1 and Theorem 17.5.1. The two measures agree by density of smooth tests.

Conversely let $T = V\delta_{v_0} + \Delta u$ be nonnegative on nonnegative tests. It satisfies $T(1) = V$ and $|T(\chi)| \leq V\|\chi\|_\infty$. Density extends it to a positive Radon measure $\nu$ of mass $V$. In particular $V \geq 0$. On each finite open ray its restriction is $d_p u_p''$ in the sense of distributions, by tests supported there. Thus $u_p$ is convex. To see this, convolve with nonnegative smooth approximate identities on compact subintervals. The smoothed functions are convex and tend uniformly to $u_p$ there. The convex function $u_p$ is bounded towards infinity, so its slopes are nonpositive and tend to zero at infinity.

There is no endpoint mass. Fix a smooth cutoff $\zeta$ on the real line with $0 \leq \zeta \leq 1$, equal to zero up to 1 and one beyond 2. Test with $\zeta(t/R)$ on the $p$-ray, extended by zero on the other rays. Its second derivative has $L^1$ norm $R^{-1}\|\zeta''\|_1$. By boundedness of $u$, its pairing with $T$ tends to zero. Dominated convergence for $\nu$ identifies the limit with $\nu(\{x_{p,\infty}\})$. The total mass of $d_p u_p''$ on the open ray is $-d_p u_p'(0+)$, by the one-dimensional derivative formula and the zero slope at infinity. The bound by $V$ on the sums of the ray masses gives (20.7), since each secant is at least the initial slope. Continuity of $u$ at the endpoints gives the remaining condition of Theorem 20.4.1. Thus $u$ is psh, and the first part of the proof gives equality of the measures. □

For a continuous $\theta$-psh function $u$ the mass remaining at the common origin is

$$\mathrm{MA}(\theta, u)(\{v_0\}) = V + \sum_p d_p u_p'(0+).$$

Indeed, the proof of Theorem 20.5.1 gives mass $-d_p u_p'(0+)$ on the $p$-ray and zero mass at its endpoint. Every compact subset of the complement of $v_0$ meets only finitely many rays, by Theorem 1.4.1. Inner regularity therefore gives the mass of that complement as the supremum of the finite sums of the ray masses. These sums are bounded by $V$, so the series of initial slopes is absolutely convergent, and subtraction from the total mass proves the formula.

**Corollary 20.5.1** *For a finite-energy potential $u$ in an ample real class $\omega$, the current* $\deg\omega\,\delta_{v_0} + \Delta u$ *equals the global finite-energy measure* $\mathrm{MA}(\omega, u)$. *In particular a closed endpoint has zero mass for this measure.*

***Proof*** Use the decreasing semipositive model approximations $u_i$ in exactly the class $\omega$ from Section 17.7, and set $V = \deg\omega$. The slope criterion in Theorem 20.3.1 bounds each initial slope below by $-V/d_p$. Convexity and the constant tails give this

bound on every slope of every approximant, with upper bound zero. Thus on $[0, S]$ on any ray, for $S < \infty$, the $u_{i,p}$ are $V/d_p$-Lipschitz. Their pointwise limit is finite there, since $u_i(v_0) \downarrow u(v_0)$ is finite, and has the same Lipschitz bound. Hence the convergence is uniform on that interval by Proposition 4.1.1. This gives local integrability of $u$ against the test forms and passage to the limit in (20.3), since each test has second derivatives supported on finitely many such intervals. Global decreasing continuity (Theorem 17.8.1) and density of the tests prove equality.

At the endpoint of a ray put $\ell = \lim_{t\to\infty} u_p'(t+) \le 0$. Testing with the scaled cutoffs from the proof of Theorem 20.5.1 gives endpoint mass $-d_p\ell$, by integrating by parts once and letting $R \to \infty$. If $\ell < 0$, $u_p(t) \le u(v_0) + \ell t$, so $u(x_{p,\infty}) = -\infty$ by decreasing approximation and monotonicity on the ray. The measure $\mathrm{MA}(\omega, u)$ integrates $u$ by the finite-energy pairing estimates, which excludes that mass. Thus $\ell = 0$. □

# Part V
# Transcendental theory

On a complex space, coherent ideal sheaves give coordinates on a space of semivaluations, which includes the centred semivaluations of the analytic local ring at each point. On a separated scheme of finite type over a trivially valued field the same construction recovers the points of the analytification that have a centre, which form the whole analytification when the scheme is proper. Flag ideals on the product with the projective line give model functions. Intersections on their blowups define mixed measures and make it possible to compare the analytic and algebraic models term by term. A vertical intersection identity controls energy on these models and gives mixed measures on the uniform closure of semipositive model functions and on its decreasing limits of finite energy. The construction also applies relative to a specified positive part of a big class.

A test curve on a compact Kähler manifold assigns a multiplier-ideal type to each slope. Taking compatible curves under Kähler perturbations handles pseudoeffective classes. Evaluating their divisorial types gives functions on the valuation space. The Taylor coefficient ideals extend these functions to the whole space of ideal semivaluations. In nef classes the perturbation also gives an obstacle energy and its first variation.

The chapters proceed by class. After the space of semivaluations and the energy of cohomological models, two chapters treat a Kähler class: the first proves continuity of envelopes, orthogonality and the variational equation, and the second compares positivity on smooth models and identifies maximal rays. Test curves then reach pseudoeffective classes. The last five chapters return to the equation in nef classes of positive volume and in big classes. They construct energies and mixed measures from Kähler perturbations, solve the equation in a nef class, treat obstacles and envelopes in a big class, solve the equation there relative to a minimal singularity type, and finally transport the equation along divisorial parts and modifications, identify the image of the operator in every big class and prove that the normalized operator is a homeomorphism onto it.

# Chapter 21
# Ideal semivaluations on complex spaces

On a compact connected Riemann surface, every nonzero coherent ideal is the ideal of an effective divisor. A semivaluation of ideals can give positive value to the ideal of at most one point, since the ideals of two different points sum to the unit ideal. Thus we obtain one ray for every point, as on a regular projective curve over a trivially valued field.

On a higher-dimensional complex space, we use coherent ideal sheaves as the test objects. This gives a compact Hausdorff space before any choice of a modification or of a Kähler class. On an algebraic scheme the same construction recovers analytic points that admit a centre. At an individual complex point we recover the centred semivaluations of the analytic local ring.

The first three sections work with locally ringed spaces whose structure sheaf is coherent, chiefly locally Noetherian schemes and complex spaces. From Section 21.4 on we work on complex spaces, and flag models are formed over a normal irreducible compact complex space. The model measures of Section 21.7 are defined on a connected compact complex manifold. Each later section states its own hypotheses on $X$.

Proposition 21.2.1 identifies $X^{\beth}$ with the centred part of the analytification of a separated scheme of finite type over a trivially valued field, and Proposition 21.3.1 describes the ideal semivaluations that are positive on the ideal of a point. The Gauss extension of Theorem 21.4.1 identifies $X^{\beth}$ with the $\mathbb{C}^{\times}$-invariant ideal semivaluations on $X \times \mathbb{P}^1$ with value one on the ideal of the central fibre. Invariant ideals containing a power of the ideal of that fibre, the flag ideals, then define model functions. They are uniformly dense (Proposition 21.5.1), and each rational model function attains its maximum at the valuation of a central component of a flag model (Proposition 21.6.1). Intersections on flag models define the pairing and mixed measures of Theorem 21.7.1. Proposition 21.9.1 computes them for the blowup of a point, a calculation that also applies on a nonprojective torus. The pairing and the measures agree with the algebraic constructions on a smooth projective variety (Corollary 21.8.1) and are compatible with modifications of the base (Theorem 21.10.1).

Chapter 22 extends the energy and the mixed measures in a nef class to the uniform closure of the model functions semipositive on smooth models, using the den-

sity of model functions and the divisor formula (21.2), and Section 23.3 extends them to decreasing limits of finite energy. The pullback along modifications returns in Section 24.2, the Gauss extension in Section 25.11, and the point blowup in Sections 24.2 and 25.6.

## 21.1 The space of ideal semivaluations

We use locally ringed spaces $X$ with a coherent structure sheaf. The examples here are locally Noetherian schemes, by [Stacks, Tag 01XZ], and complex spaces, whose structure sheaf is locally the quotient of the structure sheaf of a domain in some $\mathbb{C}^N$ by a coherent ideal and is therefore coherent by Oka's theorem [Xia26a, Theorem A.1.5 and Lemma A.1.4]. Write $\mathcal{J}(X)$ for the coherent ideals of $\mathcal{O}_X$. They are closed under sum and product. A morphism $f : Y \to X$ of locally ringed spaces among these examples extends a coherent ideal $I$ on $X$ to its image $I\mathcal{O}_Y$ in $\mathcal{O}_Y$. This extension is coherent, since $I$ is locally of finite type and $\mathcal{O}_Y$ is coherent.

**Definition 21.1.1** An *ideal semivaluation* on $X$ is a map

$$v : \mathcal{J}(X) \longrightarrow [0, +\infty]$$

satisfying

$$\begin{aligned} v(0) = +\infty, \qquad v(\mathcal{O}_X) = 0, \\ v(I + J) = \min(v(I), v(J)), \\ v(IJ) = v(I) + v(J). \end{aligned}$$

We write $X^{\sqsupset}$ for the space of these functions with the topology of pointwise convergence.

Addition extends continuously to $[0, +\infty]$, with $+\infty$ absorbing. Hence the defining equations cut out a closed subset of the compact Hausdorff product $[0, +\infty]^{\mathcal{J}(X)}$, and $X^{\sqsupset}$ is compact Hausdorff. If $I \subset J$, then $I + J = J$, and so $v(I) \geq v(J)$. Positive real scaling gives an action on $X^{\sqsupset}$, and every morphism $f : Y \to X$ as above induces the continuous map $f^{\sqsupset} : Y^{\sqsupset} \to X^{\sqsupset}$ by

$$f^{\sqsupset}(v)(I) = v(I\mathcal{O}_Y).$$

This construction respects composition.

Let $i : Y \hookrightarrow X$ be a closed embedding defined by a coherent ideal $J$. Then $i^{\sqsupset}$ identifies $Y^{\sqsupset}$ with the closed locus $v(J) = +\infty$. Indeed an ideal on $Y$ is the image of a coherent ideal on $X$ containing $J$: take its inverse image in the structure sheaf, using its finite generators locally along $Y$ and the unit ideal off $Y$. Two ideals on $X$ have the same extension to $Y$ exactly when their sums with $J$ agree. When $v(J) = +\infty$, adding $J$ preserves each ideal value. Hence $v$ descends uniquely, and its values on

the quotient ideals satisfy the defining equations. Compactness gives the claimed topological identification. If $X$ is quasi-compact and its coherent nilradical is locally nilpotent, one power of the nilradical vanishes globally by a finite cover. This forces its value to be infinite for each ideal semivaluation. Thus, in the quasi-compact cases below, passing to the reduction does not change $X^{\sqsupset}$.

## 21.2 An algebraic comparison

On a Noetherian affine scheme $U = \operatorname{Spec} R$, an ideal semivaluation gives a nonnegative ring semivaluation by $f \mapsto v((f))$. Conversely, if the semivaluation is nonnegative on $R$, the value of an ideal generated by $f_1, \ldots, f_s$ is the minimum of their values. Every element of the ideal has value at least that minimum by the ultrametric inequality and nonnegativity. These constructions are inverse and continuous. Over a trivially valued field this identifies $U^{\sqsupset}$ with the points of $U^{\mathrm{an}}$ bounded by one on $R$.

**Proposition 21.2.1** *Let $X$ be a separated scheme of finite type over a trivially valued field. The space $X^{\sqsupset}$ identifies with the subspace of $X^{\mathrm{an}}$ consisting of points with a centre on $X$. If $X$ is proper, the subspace is all of $X^{\mathrm{an}}$.*

***Proof*** A point with a centre assigns to each coherent ideal the minimum of the values of generators in the local ring at the centre. The same minimum argument as in the affine case proves the ideal semivaluation rules. Conversely, given an ideal semivaluation $v$, the ideals with strictly positive value are closed under finite sum and have a largest member $I_c$, by Noetherianity of $X$. This ideal is radical: for a coherent radical $\sqrt{I_c}$ some power is contained in $I_c$, so its value is strictly positive and it is contained in $I_c$. The closed subset cut out by $I_c$ is nonempty because $v(\mathcal{O}_X) = 0$. It is also irreducible: otherwise write it as a union of two proper closed subsets. Their radical ideals properly contain $I_c$, but their product is contained in $I_c$, forcing one of them to have strictly positive value. Let $c$ be the generic point of this irreducible subset.

Choose an affine neighborhood $U = \operatorname{Spec} R$ of $c$ and write $J$ for the reduced ideal of $X \setminus U$. Then $v(J) = 0$, since $c$ is outside that closed subset. If two coherent ideals $I, I'$ agree on $U$, the coherent module $(I + I')/I'$ is supported in $X \setminus U$. A power of $J$ annihilates it, by finite generation and the Noetherian property, so $J^N I \subset I'$ for some $N$. It follows that $v(I) \geq v(I')$. Interchanging the ideals gives equality. Every ideal of $R$ extends to a coherent ideal on $X$ by taking the scheme-theoretic closure of its zero scheme on $U$. Applying $v$ to an extension therefore gives a well-defined ideal semivaluation on $U$. Its nonnegative ring semivaluation has centre $c$. In fact an ideal of $R$ has strictly positive value exactly when it vanishes at $c$. Indeed, the image of the extended ideal in the structure sheaf of the reduced irreducible centre is an ideal sheaf with generic stalk zero. It therefore vanishes on the whole centre, where the structure sheaf is integral. This is equivalent to containment in $I_c$. Recovery of each global ideal from its restriction shows that the constructions are inverse.

For the topology, cover $X$ by finitely many affine opens $U = \operatorname{Spec} R$. The points of $U^{\mathrm{an}}$ that are nonnegative on $R$ form the generator-bounded subset with all bounds one, as in the proof of Theorem 1.3.1, and this subset is compact by Lemma 1.2.1. On it all ideal values are continuous by finite generation. These finitely many compact subsets cover the centred locus, which is Hausdorff by Proposition 1.2.1. They are therefore closed in that locus, so continuity on each gives continuity on their union. Thus the map from that locus into $X^{\sqsupset}$ is a continuous bijection from a compact space to a Hausdorff one. Properness supplies all centres by Theorem 1.3.1. □

## 21.3 Semivaluations at a complex point

For a complex point $x$ let $\mathfrak{m}_x$ denote its coherent ideal sheaf on $X$, and let $\mathfrak{m}$ be the maximal ideal of the analytic local ring $R = \mathcal{O}_{X,x}$. The next argument also works on a locally Noetherian scheme at a closed point.

**Proposition 21.3.1** *Evaluation on stalks gives a homeomorphism from nonnegative ring semivaluations $w$ of $R$ satisfying $w(\mathfrak{m}) > 0$ onto the ideal semivaluations $v$ with $v(\mathfrak{m}_x) > 0$.*

***Proof*** The forward map takes $v(I) = w(I_x)$, using the minimum over local generators. To recover $w$ from $v$, write $I_k(f)$ for the coherent ideal on $X$ given by $(f) + \mathfrak{m}^k$ at $x$ and by the unit ideal away from $x$. To construct it, take a representative of $f$ on an open neighborhood, add $\mathfrak{m}_x^k$, and glue with the unit ideal. Two representatives of the germ give the same global ideal. Put

$$w(f) = \lim_{k \to \infty} v(I_k(f)).$$

The values increase as the ideals decrease, and the limit at $f = 0$ is $+\infty$ since $v(\mathfrak{m}_x) > 0$. The value at 1 is zero. The inclusion $I_k(f+g) \subset I_k(f) + I_k(g)$ gives the ultrametric inequality. To check multiplicativity use the two inclusions

$$I_{2k}(fg) \subset I_k(f) I_k(g) \subset I_k(fg).$$

For the first, both $fg$ and $\mathfrak{m}^{2k}$ lie in the product. For the second, all the cross-terms belong to $\mathfrak{m}^k$. Evaluation reverses the inclusions, so letting $k$ increase gives $w(fg) = w(f) + w(g)$.

For a coherent ideal $I$ with local generators $f_j$, the ideal $I + \mathfrak{m}_x^k$ is the sum of the ideals $I_k(f_j)$. Its value tends to $v(I)$, since that value is $\min(v(I), kv(\mathfrak{m}_x))$. The same limit equals $\min_j w(f_j)$, as the minimum is finite over the index set, with values allowed to be infinite. Thus the recovered semivaluation returns $v$ on all coherent ideals, and $w(\mathfrak{m}) = v(\mathfrak{m}_x) > 0$. For the other composition, starting with $w$, the truncated value is $\min(w(f), kw(\mathfrak{m}))$, which converges back to $w(f)$.

For the topology, evaluation of a fixed coherent ideal is a finite minimum on stalk generators, hence continuous in the ring semivaluation. The set of nonnegative ring

semivaluations is compact by the same product argument as for ideal semivaluations. For each $\varepsilon > 0$, the closed subset with $w(\mathfrak{m}) \geq \varepsilon$ therefore maps homeomorphically onto the corresponding closed subset with $v(\mathfrak{m}_x) \geq \varepsilon$. The relative interiors of these subsets cover the locus of strictly positive values. This proves continuity of the inverse there. □

*Example 21.3.1* Let $X$ be a compact connected smooth complex curve. Its local rings are discrete valuation rings, and every nonzero coherent ideal has a discrete zero locus, necessarily finite by compactness. Thus every such ideal is a finite product of the $\mathfrak{m}_x$ with nonnegative integral powers. Since $\mathfrak{m}_x + \mathfrak{m}_y = O_X$ for $x \neq y$, at most one of their values can be positive. The ideal semivaluation with all these values zero is the trivial point. For each $x$, assigning any value $r \in (0, +\infty]$ to $\mathfrak{m}_x$ and zero to the other point ideals gives a point of $X^\beth$. The topology is the pointwise topology of these ray coordinates, just as for a proper regular algebraic curve in Theorem 1.4.1.

## 21.4 Gauss extension on a complex space

Let $X$ be a complex space and put $Y = X \times \mathbb{P}^1$. Write $p : Y \to X$ for projection and $\mathfrak{t}$ for the ideal of $X \times \{0\}$. Choose a coordinate $t$ at 0 in which the action of $\mathbb{C}^\times$ on the second factor is multiplication. We will extend an ideal semivaluation on $X$ to an invariant ideal semivaluation on $Y$ with value one on $\mathfrak{t}$.

For a coherent ideal $I$ on $Y$ and a positive integer $m$, reduce $I$ modulo $\mathfrak{t}^m$ and regard its image as an $O_X$-submodule of $\bigoplus_{0 \leq j < m} O_X t^j$. The identification comes from Taylor expansion in the second factor. For $0 \leq j < m$, project that submodule onto its $j$-th coefficient. The image is a coherent ideal on $X$, and it is independent of $m > j$ because truncation carries the image of $I$ onto its image at each lower order. Denote it by $C_j(I)$. Thus $C_j(I)$ is the ideal of $j$-th Taylor coefficients of local sections of $I$, and these sections need not be homogeneous in $t$.

**Theorem 21.4.1** *For $v \in X^\beth$, the formula*

$$\sigma(v)(I) = \inf_{j \geq 0} \big( v(C_j(I)) + j \big)$$

*defines an ideal semivaluation on $Y$. The resulting map $\sigma : X^\beth \to Y^\beth$ is a homeomorphism onto the subset of $\mathbb{C}^\times$-invariant ideal semivaluations $w$ with $w(\mathfrak{t}) = 1$. Its inverse takes $w$ to the semivaluation $A \mapsto w(AO_Y)$ on $X$.*

***Proof*** The value is zero on the unit ideal and infinite on the zero ideal. For sums, $C_j(I + J) = C_j(I) + C_j(J)$, which gives the minimum rule. For products, the coefficient formula gives

$$C_n(IJ) \subseteq \sum_{j+l=n} C_j(I) C_l(J).$$

The inclusion implies $\sigma(v)(IJ) \geq \sigma(v)(I) + \sigma(v)(J)$. In particular equality holds if either term on the right is infinite.

Suppose both terms are finite. Since $v(C_j(I)) \geq 0$, the infimum for $I$ is attained. Let $a$ be the least index attaining it and take the corresponding least index $b$ for $J$. Write $n = a + b$. Products of local sections with coefficients in degrees $a, b$ give the reverse inclusion up to the other terms,

$$C_a(I)C_b(J) \subseteq C_n(IJ) + \sum_{\substack{j+l=n \\ (j,l)\neq(a,b)}} C_j(I)C_l(J).$$

Indeed, expand the coefficient in degree $n$ of the product and move the other terms to the right. Every other pair has $j < a$ or $l < b$. Its sum of weighted values is strictly larger than $\sigma(v)(I) + \sigma(v)(J)$ by minimality of $a, b$, so its unweighted value is strictly larger than $v(C_a(I)C_b(J))$. Applying $v$ to the inclusion forces $v(C_n(IJ))$ to be at most this last value. This proves the product rule.

Scaling the second coordinate multiplies each coefficient in a fixed degree by a nonzero scalar, so $\sigma(v)$ is invariant. The coefficients of $\mathfrak{t}$ vanish in degree zero and generate the unit ideal in degree one, giving $\sigma(v)(\mathfrak{t}) = 1$. An ideal extended from $X$ has that ideal in every degree, so restriction of $\sigma(v)$ to $X$ gives $v$. For continuity, for each $I$ and $m > 0$ integral,

$$\min\big(m, \sigma(v)(I)\big) = \min\big(m, \min_{j<m}(v(C_j(I)) + j)\big).$$

These finite minima are continuous in $v$, and the truncations for all $m$ determine the topology of the extended interval.

For the converse let $w$ be invariant and satisfy $w(\mathfrak{t}) = 1$. In any fixed truncation of order $m$, choose $m$ distinct nonzero scalars in $\mathbb{C}$. The sum of the corresponding translates of the image of $I$ is its homogeneous saturation $\bigoplus_{j<m} C_j(I)t^j$. To see this, each translated tuple has its $j$-th entry multiplied by the $j$-th power of the scalar. The Vandermonde matrix is invertible, so every coefficient of every local section occurs separately in the sum. Conversely each translate has its coefficients in these same ideals. Adding $\mathfrak{t}^m$ on $Y$ gives the equality of sheaves between the sum of translated ideals with $\mathfrak{t}^m$ and the ideal $\mathfrak{t}^m + \sum_{j<m}(C_j(I)\mathcal{O}_Y)\mathfrak{t}^j$. By invariance, taking this finite sum of translates does not change the value after truncation. Therefore

$$\min(m, w(I)) = \min\big(m, \min_{j<m}(w(C_j(I)\mathcal{O}_Y) + j)\big).$$

Letting $m$ increase gives the desired formula for $w$. Its inverse is continuous by evaluation on the extended ideals. □

In particular, by the equality of sheaves in the proof, an invariant coherent ideal $I$ on $Y$ containing $\mathfrak{t}^m$ is a finite sum of extended coefficient ideals times powers of $\mathfrak{t}$, together with that pure power:

$$I = \mathfrak{t}^m + \sum_{0\leq j<m} \big(C_j(I)\mathcal{O}_Y\big)\mathfrak{t}^j. \tag{21.1}$$

Its function $-\sigma(v)(I)$ is a finite maximum of shifted functions $v \mapsto -v(A)$ with the constant $-m$ among the terms. Conversely a finite maximum of functions $v \mapsto -v(A)$ shifted by nonpositive integers, with one term the constant $-m$, comes from such an ideal by summing the corresponding powers of $\mathfrak{t}$ times the extended coefficient ideals.

## 21.5 Flag ideals and model functions

Let $X$ be a normal irreducible compact complex space. Keep the product $Y = X \times \mathbb{P}^1$ and the ideal $\mathfrak{t}$ of its fibre over zero. The space $Y$ is again normal and irreducible: a product with a complex manifold preserves normality, as in the proof of Lemma D.1.1, and a connected normal complex space is irreducible. A *flag ideal* is an invariant coherent ideal $I$ on $Y$ containing $\mathfrak{t}^m$ for some $m$. It defines the function $\varphi_I(v) = -\sigma(v)(I)$. This is continuous and takes values in $[-m, 0]$. Write $\mathrm{PL}_{\mathbb{Q}}(X^{\beth})$ for the rational linear span of these functions and $\mathrm{PL}_{\mathbb{R}}(X^{\beth})$ for their real linear span. Their elements are the rational and real *model functions*, respectively.

**Proposition 21.5.1** *The rational space of model functions is a lattice containing the rational constants. It is uniformly dense in the real continuous functions on* $X^{\beth}$.

***Proof*** The semivaluation rules give $\varphi_{IJ} = \varphi_I + \varphi_J$ and $\varphi_{I+J} = \max(\varphi_I, \varphi_J)$. Also $\varphi_{\mathfrak{t}} = -1$. Any two rational model functions can be written $(\varphi_{I_1} - \varphi_{J_1})/m$ and $(\varphi_{I_2} - \varphi_{J_2})/m$ with the same positive integer $m$. Their maximum is the difference of the potentials of $I_1J_2 + I_2J_1$ and $J_1J_2$, divided by $m$. Negation gives closure under minima. For any coherent ideal $A$ on $X$, the flag ideal $AO_Y + \mathfrak{t}^m$ has function $-\min(v(A), m)$. These functions separate points since the ideal evaluations separate points. The uniform closure of the rational lattice is a real vector subspace containing all constants, since rational scalars and constants approximate real ones. It is still a lattice. Apply Lemma 4.4.1. □

Blow up a flag ideal and normalize the result. We call this a *flag model*, with the product itself included by taking the unit ideal. The construction and its uniqueness are given in Lemma D.1.1. A flag model $q : M \to Y$ is compact and normal, and it is isomorphic to $Y$ over the punctured parameter line $\mathbb{P}^1 \setminus \{0\}$. By its finite homogeneous decomposition (21.1), the flag ideal is preserved by the $\mathbb{C}^\times$-action over $\mathbb{C}^\times \times Y$, so the action lifts to $M$. The composite $\pi : M \to \mathbb{P}^1$ is flat, since a local ring of $M$ is a domain and has no torsion over the local discrete valuation ring of the parameter. Normalized blowups of products of flag ideals dominate those of their factors. We can therefore pass to one common flag model for any finite collection of models or ideals. If $IO_M$ is invertible, denote its associated Cartier divisor by $D_I$, with the sign convention $IO_M = O_M(D_I)$. Thus $D_I$ is the negative of the effective zero divisor of the ideal.

Write

$$M_0 = \sum_E b_E E$$

for the Cartier fibre viewed as a cycle. For a coherent ideal $I$ on $Y$, put $w_E(I) = b_E^{-1}\,\mathrm{ord}_E(IO_M)$. The order along $E$ is computed on a dense open of its smooth locus in the smooth part of $M$. In coordinates there, let $z$ be a local equation of $E$, and write each local generator of $IO_M$ as $z^a h$ with $h$ not divisible by $z$. If a generator $z^a h$ attains the smallest such exponent $a$, then the ideal equals $z^a O_M$ near each point of $E$ where $h$ does not vanish. Such points are dense in $E$ within the chart, and this $a$ is the order of the ideal along $E$. These orders agree on overlaps by the identity principle on the irreducible $E$. The local minimum and product rules then show that $w_E$ is an ideal semivaluation. The connected group $\mathbb{C}^\times$ preserves every $E$, since its orbit on the finite set of irreducible fibre components is constant. It follows that $w_E$ is invariant and $w_E(\mathfrak{t}) = 1$. By Theorem 21.4.1, $w_E = \sigma(v_E)$ for its restriction $v_E \in X^\sqsupset$. Consequently

$$\mathrm{ord}_E(D_I) = b_E \varphi_I(v_E). \tag{21.2}$$

**Lemma 21.5.1** *A finite collection of real model functions is represented on one flag model by real linear combinations of the divisors $D_I$. On any such model the divisor representing each function is unique among these linear combinations.*

***Proof*** Use a common model on which all the ideals in the given expressions are invertible. If two expressions have the same function, their difference has order zero along every central component by (21.2). The difference is a real linear combination of finitely many Cartier divisors with integer Weil coefficients, supported on the central fibre. The kernel of the matrix of these coefficients is spanned by rational vectors. A rational combination with all coefficients zero is zero as a Cartier divisor: clear denominators and use that a meromorphic function with neither zeros nor poles in codimension one on a normal space is a unit. This assertion is local, by the height-one intersection description of a Noetherian normal local domain. It proves the real assertion as well. □

## 21.6 Testing inequalities on central components

An inequality between Cartier divisors on a flag model implies an inequality for all ideal semivaluations on the base. Lemma 21.6.1 proves this descent for ideals, through a power identity to which an arbitrary ideal semivaluation can be applied. Proposition 21.6.1 deduces that rational model functions attain their maxima at central component valuations, and that these valuations are dense. In this section all ideals considered on a compact normal irreducible complex space are coherent.

**Lemma 21.6.1** *Let $I, J$ be nonzero ideals on a compact normal irreducible complex space $Y$. Suppose $q : W \to Y$ is a proper normal modification, $JO_W$ is invertible and $IO_W \subset JO_W$. There is an integer $N > 0$ such that*

$$(I + J)^N = J(I + J)^{N-1}.$$

*Consequently $w(I) \geq w(J)$ for every $w \in Y^\sqsupset$.*

***Proof*** Set $L = I + J$. Its pullback to $W$ is the invertible ideal $JO_W$. Thus $W$ maps onto the blowup $B = \mathrm{Bl}_L Y$. The factorization follows from the graph construction of Lemma D.1.1, before normalization. The map is proper and its image contains the dense punctured graph, hence is all of $B$. On $B$ the map from the extended $J$ to the invertible ideal $LO_B$ generates it. Indeed, if the coefficients in a local frame of $LO_B$ all vanish at a point of $B$, they still vanish at every point of $W$ above it. That contradicts $LO_W = JO_W$.

Fix $y \in Y$ and put $R = O_{Y,y}$, with maximal ideal $\mathfrak{m}$. The points of the fibre $B_y$ are the complex points of the projective scheme of the graded algebra

$$T = \bigoplus_{r \geq 0} L_y^r / \mathfrak{m} L_y^r .$$

Here this projective scheme is regarded as a closed subscheme of the projective space of a fixed generating tuple for $L_y$. To check this description, use the local graph charts. For a generator $f_i$, the algebraic chart of the blowup of $L_y$ over $\operatorname{Spec} R$ is given by the kernel of localization at $f_i$ in the algebra with the equations $f_j = f_i u_j$. At any chart point over $y$ with coordinates $c_1, \ldots, c_s$, the local map from $R[u_1, \ldots, u_s]_{(\mathfrak{m}, u_1 - c_1, \ldots, u_s - c_s)}$ to the analytic local ring of the product chart is flat. Indeed their completions both equal the power series ring over $\widehat{R}$ in the differences $u_j - c_j$. The shared completion is faithfully flat over either Noetherian local ring. Flatness preserves the localization kernel, which cuts out the analytic graph closure on the chart. Taking the fibre at $y$ gives the displayed projective description.

The image of $J_y$ in $T_1$ generates the restriction of the tautological line bundle on this fibre, as shown above. Thus the projective spectrum of $T/(J_y T)$, where $J_y$ acts in degree one, is empty. To see that all sufficiently high graded pieces vanish, use that the algebra is generated in degree one by finitely many elements. The standard affine opens where each such element is invertible are empty after quotienting. Hence each generator is nilpotent in the quotient, and all sufficiently long homogeneous monomials vanish. This gives

$$L_y^N = J_y L_y^{N-1} + \mathfrak{m} L_y^N$$

for $N$ sufficiently large. Nakayama's lemma removes the last term. The equality holds on a neighborhood of $y$, by coherence. Multiplying by powers of $L$ gives the equalities for all larger $N$. Compactness permits one common $N$ for $Y$.

Now $w(L) = \min(w(I), w(J))$. If this value is finite, applying $w$ to the power identity forces $w(L) = w(J)$. If it is infinite, then both $w(I)$ and $w(J)$ are infinite. This proves the inequality in both cases. □

**Proposition 21.6.1** *For a normal irreducible compact complex space $X$, every rational model function on $X^{\sqsupset}$ attains its maximum at a normalized central component valuation of a flag model. These valuations are dense in $X^{\sqsupset}$.*

***Proof*** By the product rule and clearing denominators, it suffices to maximize $\varphi_J - \varphi_I$ for two flag ideals on $Y = X \times \mathbb{P}^1$. Use the normalized blowup of $I$ and let $C$ be the

maximum of the difference on the valuations of its central components. It is rational because the normalized orders have integral numerators and denominators. Write $c = dC \in \mathbb{Z}$ with $d > 0$ integral. On the chosen model one has the ideal containment

$$J^d \mathfrak{t}^{\max(c,0)} \mathcal{O}_W \subset I^d \mathfrak{t}^{\max(-c,0)} \mathcal{O}_W .$$

In fact the right side is invertible, and the orders of the left minus the right along a central component are $d\big(C - (\varphi_J - \varphi_I)(v_E)\big) b_E \geq 0$. Away from the central fibre both ideals are the unit ideal. Dividing by a local generator of the right side therefore gives holomorphic functions, by normality and the height-one criterion. Apply Lemma 21.6.1 to the two ideals on $Y$ and then to the semivaluation $\sigma(v)$. The resulting inequality on all $v$ is $\varphi_J(v) - \varphi_I(v) \leq C$.

For density, let $U$ be a nonempty open of $X^{\beth}$. Compact Hausdorff regularity gives a continuous function positive at a point of $U$ and zero on its complement. By Proposition 21.5.1, approximate it uniformly by a rational model function, then subtract a small rational constant so that the function is still positive somewhere and is negative off $U$. It has a maximum at one of the stated valuations, which must be in $U$. □

## 21.7 The cohomological model measure

In this section $X$ is a connected compact complex manifold of dimension $n$. Let $\alpha \in H^2_{\mathrm{dR}}(X, \mathbb{R})$ be represented by a closed real $(1,1)$-form. A real Bott–Chern class of type $(1,1)$ can be used through its image in de Rham cohomology. For a pair $P = (\alpha, \varphi)$, with $\varphi \in \mathrm{PL}_{\mathbb{R}}(X^{\beth})$, choose a flag model $q$ representing $\varphi$ by $D$. On $M$ put $P_M = \rho^* \alpha + c_1(D)$, with $\rho = p \circ q$. Use pullback of closed forms for the first summand and the curvatures of smooth line-bundle metrics for the second, extending linearly for real divisors. The intersection conventions for a singular model are in Appendix D.2.

**Theorem 21.7.1** *For $P_0, \dots, P_n$ choose a common flag model $M$. The pairing*

$$\langle P_0, \dots, P_n \rangle_X = \int_{[M]} (P_0)_M \cdots (P_n)_M$$

*and the mixed signed measure*

$$\mathrm{MA}_X(P_1, \dots, P_n) = \sum_E b_E \left( \int_{[E]} (P_1)_M \cdots (P_n)_M \right) \delta_{v_E}$$

*are independent of $M$. The pairing is symmetric and multilinear. The mass of the measure is $\int_X \alpha_1 \cdots \alpha_n$, and for a real model function $h$ one has*

$$\int_{X^{\beth}} h \, \mathrm{d}\, \mathrm{MA}_X(P_1, \dots, P_n) = \langle (0, h), P_1, \dots, P_n \rangle_X .$$

***Proof*** On a common dominating model the divisors for the same function coincide by Lemma 21.5.1. Pullback preserves each model class. Integration of a top-degree pullback along a proper modification equals the integral downstairs, by Lemma D.2.1, so the pairing is independent of the model. Symmetry and multilinearity follow from the formula.

Consider a domination $M' \to M$, and use the classes pulled back from $M$. Over a dense open of each old component $E$ the domination is an isomorphism, by Lemma D.1.2. Its strict transform has the same $b_E$ and gives the same $v_E$, which can be computed there for each ideal. The map of the strict transform to $E$ preserves the integral of the restricted product by Lemma D.2.1. Each other component maps to dimension less than $n$, so the pullback of a form of degree $2n$ has zero integral on that component. This proves independence of the measure.

Apply Lemma D.2.2 to the fibre divisor. Its first Chern class is the same as that of the fibre at infinity. Along that fibre the divisor corrections are trivial and $\rho$ identifies it with $X$. This computes the mass. For the last formula use a model also representing $h$ by a real divisor $H$. Its coefficient along $E$ is $b_E h(v_E)$ by (21.2). Applying Lemma D.2.2 to $H$ gives the integral of $h$. □

Suppose the chosen model admits a Kähler form $\Omega$ and each of the classes $(P_i)_M$ has, for every $\varepsilon > 0$, a smooth representative with lower bound $-\varepsilon\Omega$ on the smooth loci of all central components. Then the measure is positive. To see this, replace the restriction of each representative to $E$ by its sum with $\varepsilon\Omega|_E$. The product is a non-negative density. Its integral depends polynomially on $\varepsilon$ by closedness and Stokes, and its constant term is the weight in the measure divided by $b_E$. This positivity in particular applies to Kähler model classes. When $X$ is Kähler and all $\alpha_i = \alpha$ are the same Kähler class, division by $\int_X \alpha^n$ gives a probability measure for such data. The conclusions of the theorem itself do not require a positivity hypothesis.

## 21.8 Comparison on projective varieties

Let $X$ be a smooth projective variety over $\mathbb{C}$ and denote its complex analytification by $X_{\mathbb{C}}$. Coherent ideal sheaves on $X_{\mathbb{C}}$ are exactly the analytifications of coherent ideals on $X$, with sums, products and pullback to the projective models preserved by Lemma D.3.1. This gives $X_{\mathbb{C}}^{\beth} \simeq X^{\beth} \simeq X^{\mathrm{an}}$, where $X^{\mathrm{an}}$ on the right is over the trivial valuation. Both identifications are homeomorphisms, by pointwise evaluation and Proposition 21.2.1. The analytic flag ideal decomposition uses finitely many coefficients on $X_{\mathbb{C}}$. Algebraize those ideals to see that each analytic flag ideal comes from an algebraic flag ideal. Conversely algebraic flag ideals give analytic ones after extension by the unit ideal near infinity. The two formulas for the Gauss value are identical because the coefficients of the flag agree.

**Corollary 21.8.1** *On a smooth projective $X/\mathbb{C}$ the analytic and algebraic rational model functions coincide under this homeomorphism. For pairs represented by real combinations of line bundles and vertical divisors, the cohomological pairing and*

*mixed measure of Theorem 21.7.1 agree with the algebraic pairing and measure of Definition 17.1.1 and Proposition 17.2.1.*

***Proof*** The equality of the rational model functions follows from the flag formula in Theorem 5.4.1. Use one normalized flag blowup for a finite collection of representatives. The algebraic and analytic blowups and their normalizations agree by Lemma D.3.1. The component multiplicities agree, since analytification preserves the orders of the parameter at a smooth point of each component where no other component meets it. The normalized orders on ideal sheaves agree by the same local calculation. Thus the points carrying the measures are the same. Every irreducible component of the analytification $E_{\mathbb{C}}$ of an algebraic component $E$ contains such smooth points, so it has multiplicity $b_E$ and valuation $v_E$. Hence the analytic weights over $E$ add up to $b_E$ times an integral over the cycle $[E_{\mathbb{C}}]$. Finally the algebraic intersection degrees on the model and its components equal the Chern class integrals on the corresponding analytic cycles, by Lemma D.2.3. This identifies the individual weights and the pairings. □

## 21.9 A model supported at a point

Let $X$ be a connected compact Kähler manifold of dimension $n > 0$, let $\alpha$ be a Kähler class, and fix $p \in X$. Write $V = \int_X \alpha^n$. Blow up $(p, 0)$ in $X \times \mathbb{P}^1$, with exceptional divisor $F$. This is the flag blowup of $\mathfrak{m}_p \mathcal{O}_{X\times\mathbb{P}^1} + \mathfrak{t}$. For $c > 0$ its real model function is

$$\varphi_c(v) = -c \min(v(\mathfrak{m}_p), 1).$$

The correction divisor on the model is $-cF$.

**Proposition 21.9.1** *For all sufficiently small $c > 0$, the pair $P_c = (\alpha, \varphi_c)$ has a positive model measure. Writing $v_0$ for the trivial ideal semivaluation and $w_p$ for the order at $p$ on coherent ideals, one has*

$$\mathrm{MA}_X(P_c, \ldots, P_c) = (V - c^n)\delta_{v_0} + c^n \delta_{w_p}, \qquad \frac{\langle P_c, \ldots, P_c\rangle_X}{n+1} = -\frac{c^{n+1}}{n+1}.$$

*The pairing in the second formula has $n + 1$ entries.*

***Proof*** The model is smooth. Its central fibre has two components with multiplicity one. In the blowup charts the strict transform of $X \times \{0\}$ is $\mathrm{Bl}_p X$, and the exceptional component is $F = \mathbb{P}^n$. The strict transform has valuation $v_0$, because the pullback of every nonzero base ideal has order zero at its general point. In local coordinates at $p$, the exceptional order of a function on $X$ is the least total degree of its nonzero Taylor terms. This gives $w_p$ after restricting to ideals on $X$.

On $F$, the pullback of $\alpha$ vanishes and $\mathcal{O}(-F)|_F = \mathcal{O}_{\mathbb{P}^n}(1)$. Hence the top intersection on $F$ is $c^n$. The mass formula of Theorem 21.7.1 gives $V - c^n$ for the other component. These weights are nonnegative for small $c$. In fact the model class is

Kähler after adding a pullback from the parameter line. To see this, choose a smooth metric on $O(-F)$ that near $F$ is the pullback of the Fubini–Study metric in the local projective-product description of the blowup. Extend the metric with a smooth cutoff in the logarithms of the metric ratios. Near $F$, the sum of the pullback of a Kähler form on $X \times \mathbb{P}^1$ and $c$ times this curvature is positive, since those are the two factors of a local product embedding. On the compact complement of that neighborhood the pullback is strictly positive, so it absorbs the bounded curvature term for small $c$. The added parameter class has zero restriction on each central component.

For the energy expand the $(n+1)$-fold product of $\rho^*\alpha - c\,c_1(O(F))$. Its term with no divisor vanishes by dimension. Every mixed term vanishes because $\rho^*\alpha|_F = 0$, using Lemma D.2.2. The last term is

$$(-c)^{n+1} \int_{[F]} c_1(O(F)|_F)^n = (-c)^{n+1}(-1)^n = -c^{n+1}.$$

This proves the formula. □

The same calculation applies on a nonprojective complex torus. Consider complex structures $J$ on $\mathbb{R}^4$ inducing one fixed orientation, and the quotients by $\mathbb{Z}^4$. Every such $J$ gives a Kähler torus by using a constant positive Hermitian form. An integral cohomology class in degree two corresponds to an integral alternating form $q$ on the lattice. Its invariant real representative is of type $(1,1)$ exactly when $q(Ju, Jv) = q(u, v)$. Indeed averaging differential forms over translations preserves type and leaves their cohomology classes unchanged, since every translation is homotopic to the identity.

For any nonzero $q$, these equations cut out a proper real analytic subset of the connected space of orientation-compatible complex structures. Choose independent $u, v$ with $q(u, v) \neq 0$ and complete them by $w, z$ to a basis with the required orientation in the order $u, w, v, z$. Rescale $w$ by a positive constant so that $q(w, z) \neq q(u, v)$. The complex structure sending $u$ to $w$ and $v$ to $z$ then fails the equations. The space of these complex structures is the connected homogeneous space $\mathrm{GL}^+(4, \mathbb{R})/\mathrm{GL}(2, \mathbb{C})$. Baire's theorem gives a complex structure outside the countable union of the proper subsets for the nonzero integral $q$. Its torus has no nonzero integral class of type $(1,1)$, hence no ample line bundle. The model measure above still uses any Kähler class of that torus, and neither its class nor its construction requires an algebraic polarization.

## 21.10 Maps and modifications of the base

A map of complex spaces induces a map on their spaces of ideal semivaluations by extending ideals. We first lift central component valuations along a surjection, which proves surjectivity in Proposition 21.10.1. Theorem 21.10.1 then compares pairings and measures along a modification.

**Proposition 21.10.1** *If $f : Y \to X$ is a surjective holomorphic map of normal irreducible compact complex spaces, then $f^{\beth} : Y^{\beth} \to X^{\beth}$ is surjective.*

***Proof*** Fix a flag model $M_X$ obtained by normalizing a blowup over $X \times \mathbb{P}^1$. Let $M_Y$ be the normalized blowup of its extended flag ideal over $Y \times \mathbb{P}^1$. The graph construction in Lemma D.1.1 gives a lift to the blowup of $X \times \mathbb{P}^1$: the model map makes the extended ideal invertible and its image off the zero set determines the projective coordinates on a dense open. The lift is dominant, so each meromorphic function defining the normalization pulls back meromorphically. Its integral equation pulls back to one with holomorphic coefficients. Normality of $M_Y$ makes the function holomorphic, giving a lift $M_Y \to M_X$ to the normalized blowup. This map is proper and its image contains the punctured model by surjectivity of $f$. It is therefore surjective.

Fix a component $E$ of the central fibre of $M_X$. Some component $G$ of the central fibre of $M_Y$ maps onto $E$. Indeed there are finitely many components, with closed analytic images, and no finite union of proper analytic subsets covers $E$. On a dense open of the smooth part of $E$ in the smooth locus of $M_X$ take a local equation $z$ of $E$ where no other central component meets it. Write $e = \operatorname{ord}_G(z)$ for the order of its pullback near a general point. This is positive, and the parameter order upstairs is $eb_E$. For any nonzero ideal on $X$, its pullback to $M_X$ has order $a$ along $E$. On a dense open of $E$ it is generated by $z^a$, because at least one coefficient after division is nonzero there. Since $G$ dominates $E$, its order upstairs is $ea$. Thus its normalized order upstairs is $a/b_E$. The zero ideal also has the same infinite value. This shows $f^{\beth}(v_G) = v_E$ on all coherent ideals.

The image of $f^{\beth}$ contains all such central valuations and is closed by compactness. By Proposition 21.6.1 it is all of $X^{\beth}$. □

For the measure comparison let $f : Y \to X$ be a proper modification of connected compact complex manifolds of dimension $n$. Write $f^{\beth}_* \mu$ for the pushforward of a measure $\mu$ on $Y^{\beth}$ under $f^{\beth}$.

**Theorem 21.10.1** *If $P_i = (\alpha_i, \varphi_i)$ are pairs on $X$ as in Section 21.7 and $Q_i = (f^*\alpha_i, \varphi_i \circ f^{\beth})$, then the $Q_i$ are such pairs on $Y$, and*

$$\langle Q_0, \dots, Q_n \rangle_Y = \langle P_0, \dots, P_n \rangle_X,$$
$$f^{\beth}_* \operatorname{MA}_Y(Q_1, \dots, Q_n) = \operatorname{MA}_X(P_1, \dots, P_n).$$

***Proof*** Consider first a flag ideal $I$ on $X \times \mathbb{P}^1$, written using the finite decomposition (21.1). Its extension to $Y \times \mathbb{P}^1$ is given by extending each coefficient ideal and keeping the same powers of the ideal of the parameter. Taking the finite minimum in the Gauss formula therefore composes $\varphi_I$ with $f^{\beth}$. Linearity gives the statement for all real model functions.

Use a common model $M_X$ over $X \times \mathbb{P}^1$ obtained by normalizing the blowup of a flag ideal $J$. Let $M_Y$ be the normalized blowup of the extension of $J$ to $Y \times \mathbb{P}^1$. Its map to $X \times \mathbb{P}^1$ makes $J$ invertible, so it factors through $M_X$. The graph factorization and the normality argument in Lemma D.1.1 apply since the map is bimeromorphic.

The divisors representing the $\varphi_i$ pull back to divisors representing the $\varphi_i \circ f^{\sqsupset}$. If a further flag ideal was used in any presentation of the functions, include it among the factors of $J$.

The map $M_Y \to M_X$ is a proper modification. The integrations of the pulled-back products over the two total spaces therefore agree. For each $E$ of the central fibre of $M_X$, its strict transform in $M_Y$ has the same multiplicity and maps bimeromorphically to $E$ by Lemma D.1.2. The normalized order of any ideal from $X$ computed along this strict transform is the normalized order computed along $E$. Thus $f^{\sqsupset}$ sends the atom of that component to the old atom. The integrals on the strict transform and on $E$ are equal. All the other components map to smaller dimension and have zero top-degree integrals for the pulled-back forms. Summing proves the measure identity. □

# Chapter 22
# Energy of cohomological models

Intersecting two distinct components of a smooth central fibre against semipositive classes gives a nonnegative degree. Intersecting with the full fibre gives zero along each component. Hence a vertical self-intersection is minus a sum of squares (Proposition 22.2.1), and this sign controls the energy and its mixed measures.

Throughout this chapter $X$ is a connected compact Kähler manifold of dimension $n > 0$. We use real classes of closed smooth $(1,1)$-forms and the model pairings of Theorem 21.7.1. On a compact Kähler manifold two smooth representatives of the same class differ by $dd^c$ of a smooth function, by the $dd^c$-lemma [Xia26a, Theorem 14.1.7]. From Section 22.3 on, the background classes are nef. In Section 22.5 $X$ is a smooth projective variety, and in Section 22.7 the class is Kähler.

For a nef class $\alpha$, the mixed measures and the energy extend to the uniform closure $C_\alpha$ of the model functions semipositive on a smooth model, by Theorems 22.3.1 and 22.4.1. The energy is concave, and its derivative along a segment is the integral of the slope against the Monge–Ampère measure. On a smooth projective variety, when $\alpha$ is obtained from a real ample class $\theta$ as in Section 22.5, Theorem 22.5.1 identifies $C_\alpha$ with the continuous algebraic $\theta$-psh functions, with the same measures and energies. For a decreasing limit $u$ of functions in $C_\alpha$ and a model function $h$, Theorem 22.6.1 shows that the supremum of $u - h$ is attained at a central component of a smooth model representing $h$. In a Kähler class, a nonpositive subgeodesic ray with a logarithmic lower bound defines such a limit, whose divisorial values are the normalized generic Lelong numbers of the ray with the opposite sign (Proposition 22.7.1). Every nonpositive semipositive model potential is approximated in this way, by Proposition 22.7.2.

Chapter 23 uses these results to construct continuous envelopes and to solve the Monge–Ampère equation in a Kähler class. Chapter 24 builds maximal rays from the same ray transform. The projective comparison and the divisorial tests also enter the study of test curves in Sections 25.7 and 25.12.

## 22.1 Semipositivity on smooth refinements

Use the smooth refinements of Lemma D.4.2. Let $q : W \to X \times \mathbb{P}^1$ be such a refinement dominating a flag model representing $P = (\alpha, u)$, and write $P_W = \rho^*\alpha + c_1(D_u)$, where $\rho$ is the projection to $X$ and $D_u$ is the pullback of its vertical divisor. Write $\pi : W \to \mathbb{P}^1$. We use the reduced components of the central fibre with their smooth analytic structures.

**Definition 22.1.1** The pair $P$ is *semipositive on a smooth model* if it has such a refinement $W$ on which the following holds. For a Kähler form $\Omega_W$ on $W$ and every $\varepsilon > 0$, the class $P_W + \varepsilon[\Omega_W]$ has a smooth closed representative whose restriction to each central component is semipositive.

This condition is independent of the choice of $\Omega_W$ on a fixed $W$, since two Kähler forms are uniformly comparable. It is preserved on further smooth refinements. Indeed, every central component upstairs maps into some central component downstairs. Pull back the representative and then replace the pulled-back Kähler form by a sufficiently large multiple of the new one. Since the required property holds for arbitrarily small positive $\varepsilon$, this gives the same property upstairs. Passing to common refinements, such pairs therefore form a convex cone under addition and nonnegative scaling. The pair $(\alpha, 0)$ belongs to this cone when $\alpha$ is nef, meaning that it is a limit of Kähler classes. Indeed, after adding a small positive multiple of a fixed Kähler class, the resulting class is Kähler by the openness and convexity of the Kähler cone. Pull back such representatives from $X$. Adding a constant to a model function adds a multiple of $W_0$ to its divisor. The corresponding class is pulled back from $\mathbb{P}^1$ and has a representative vanishing on each central component.

The intersection and measure identities may be computed on a smooth refinement, even if it is not required to carry an action. For every central component of a flag model, its strict transform has the same multiplicity and gives the same valuation, and its top intersection integrals of pulled-back classes agree. Every other component has image of smaller dimension and gives zero in these integrals. This follows from Lemmas D.1.2 and D.2.1, as in the proof of Theorem 21.7.1, which uses only that the dominating space is normal and irreducible. In particular, mixed measures of pairs semipositive on a smooth model are positive. On a common refinement, add $\varepsilon$ times a Kähler form to all the pulled-back classes. The resulting mixed degrees on the smooth components are nonnegative. Passing to the constant term of these intersection polynomials gives the assertion.

We will need an order bound for divisors on a smooth refinement. For any real model function $h$, let $D_h$ be its representative. If $a \le h \le b$ then

$$aW_0 \le D_h \le bW_0. \tag{22.1}$$

First check the inequalities on a flag model representing $h$. The coefficient of $D_h$ at $E$ is $b_E h(v_E)$ by (21.2). The resulting Weil inequalities pull back to the smooth refinement. For this last assertion with real coefficients, approximate a nonnegative real vertical Cartier divisor by rational combinations in the finite span of the Cartier

divisors in its expression, allowing also the fibre divisor. Add a small positive multiple of the fibre before approximating. Its coefficients are then strictly positive on the finitely many central components, so the rational approximants may be chosen effective. They pull back to effective rational Cartier divisors by normality, and letting the added multiple tend to zero proves the bound.

## 22.2 A vertical intersection identity

The following inequality is the source of the concavity of the energy in Section 22.4.

**Proposition 22.2.1** *Let $P_2, \dots, P_n$ be semipositive on smooth models and let $h$ be a real model function. Then*

$$\langle (0,h), (0,h), P_2, \dots, P_n \rangle_X \le 0.$$

*For $n = 1$ there are no background pairs in this formula.*

***Proof*** Choose a common smooth refinement $W$. Put $A = \prod_{i=2}^{n} (P_i)_W$, using $A = 1$ for $n = 1$. Write $W_0 = \sum_E b_E E$ and $D_h = \sum_E b_E x_E E$. For distinct components $E, G$, the intersection $E \cap G$ is smooth of the required dimension, possibly empty. Thus

$$B_{EG} = \int_{[W]} c_1(E) c_1(G) A = \int_{[E \cap G]} A \ge 0.$$

The equality uses the divisor integration formula twice, and positivity follows from the smooth semipositive approximants used above. Define $B_{EE}$ by the same intersection on $W$. Since $[W_0]$ is pulled back from the parameter line, its restriction to $E$ vanishes in cohomology. Hence $\sum_G b_G B_{EG} = 0$ for every $E$. Expanding the square and using these row identities gives

$$-\int_{[W]} c_1(D_h)^2 A = \sum_{E<G} b_E b_G B_{EG} (x_E - x_G)^2.$$

This proves the inequality. It also gives Cauchy–Schwarz for the associated nonnegative bilinear form on vertical model functions by polarization. □

## 22.3 Uniform limits of mixed measures

For a nef class $\alpha$, let $C_\alpha$ be the uniform closure in $C(X^{\beth})$ of the model functions for which $(\alpha, u)$ is semipositive on a smooth model. It is a convex set stable under translations by constants. We first control the integrals of model tests.

**Lemma 22.3.1** *Fix nef classes $\alpha_1, \dots, \alpha_n$ and a real model function $h$. For two tuples of functions $u_i, v_i$ semipositive on smooth models in those classes,*

$$\left| \int h \, \mathrm{d}\,\mathrm{MA}_X((\alpha_i, u_i)_{i=1}^n) - \int h \, \mathrm{d}\,\mathrm{MA}_X((\alpha_i, v_i)_{i=1}^n) \right| \leq C_h \sum_{i=1}^n \|u_i - v_i\|_\infty,$$

*where $C_h$ depends only on $h$ and the fixed classes.*

***Proof*** Represent $h$ on a fixed smooth refinement $W_h$. Choose a smooth closed representative $\gamma_h$ of $c_1(D_h)$ there, and a Kähler form $\Omega_h$. There is a $C$ with $-C\Omega_h \leq \gamma_h \leq C\Omega_h$ on $W_h$. Compute on a common smooth refinement $W$ dominating $W_h$ and the models for all the pairs. The two bounds pull back to $W$. Telescope in the $n$ slots and consider one term in the $i$-th slot. The difference there has divisor $D_{u_i - v_i}$ and zero background class. By symmetry of the pairing and divisor integration the term is a sum on central components $E$ of $W$, with coefficient $\mathrm{ord}_E(D_{u_i - v_i})$ multiplying

$$\int_{[E]} \gamma_h \prod_{j \neq i} A_j.$$

Here each $A_j$ is the pulled-back class from one of the semipositive pairs. The bound for $\gamma_h$ and smooth semipositive approximation along $E$ give an absolute bound by $C \int_{[E]} \Omega_h \prod_{j \neq i} A_j$. By (22.1) the absolute value of the coefficient is at most $b_E \|u_i - v_i\|_\infty$. With the weights $b_E$, the sum of the new integrals is the intersection with the full central fibre, so equals the intersection over the fibre at infinity. That fibre is identified with $X$ on every refinement used. If $\omega_h$ is the restriction of $\Omega_h$ there, this sum equals $\int_X \omega_h \prod_{j \neq i} \alpha_j$, independent of the approximants. Summing the estimates proves the lemma. □

**Theorem 22.3.1** *For fixed nef classes $\alpha_i$, the mixed model measure extends to the tuples $u_i \in C_{\alpha_i}$. The extension is positive with mass $\int_X \alpha_1 \cdots \alpha_n$, is symmetric and respects convex combinations in each slot. It is weakly continuous for uniform convergence of the $u_i$.*

***Proof*** Take a uniform model approximation in each slot. Lemma 22.3.1 makes the model integrals Cauchy for every real model test and independent of the approximating sequences. Their positive measures have fixed mixed mass. Use the density of rational model functions from Proposition 21.5.1 to get the same conclusions for arbitrary continuous tests. The positive limiting functional represents a unique Radon measure by Theorem 4.3.1. The fixed mass and the model estimates pass to the limit, giving continuity first on model tests and then on continuous ones. Symmetry and convex linearity pass to the limit from the model identities on common refinements. □

## 22.4 Energy and its variation

Fix a nef class $\alpha$, put $V = \int_X \alpha^n$, and write $P_u = (\alpha, u)$. For model functions in $C_\alpha$ that are semipositive on a smooth model, use the unnormalized energy $E_\alpha(u) = \langle P_u^{n+1} \rangle_X / (n+1)$. The repeated pairing has $n+1$ entries.

**Theorem 22.4.1** *The energy extends to $C_\alpha$, is monotone and concave, and satisfies*

$$|E_\alpha(u) - E_\alpha(v)| \le V \|u - v\|_\infty, \qquad E_\alpha(u + c) = E_\alpha(u) + cV.$$

*For $u, v \in C_\alpha$, along the segment $u_s = (1-s)u + sv$ it has derivative*

$$\frac{\mathrm{d}}{\mathrm{d}s} E_\alpha(u_s) = \int_{X^\beth} (v - u)\, \mathrm{d}\,\mathrm{MA}_X(P^n_{u_s}).$$

*At the endpoints take the one-sided derivatives.*

***Proof*** On models, multilinearity gives

$$E_\alpha(u) - E_\alpha(v) = \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^\beth} (u - v)\, \mathrm{d}\,\mathrm{MA}_X(P_u^j, P_v^{n-j}).$$

Positivity and the common mass $V$ give monotonicity and the Lipschitz and translation identities. The second derivative along a segment of model functions is $n\langle (0, v-u), (0, v-u), P_{u_s}^{n-1} \rangle_X$, which is nonpositive by Proposition 22.2.1. Concavity thus passes to the uniform closure by the Lipschitz estimate. The difference identity also passes to this closure, using weak convergence of the mixed measures and uniform convergence of the test functions. Apply it to $u_{s+t}$ and $u_s$ and divide by $t$. All the mixed measures in the quotient tend to the same measure by Theorem 22.3.1, giving the asserted derivative. □

## 22.5 The continuous projective comparison

Let $X$ be a smooth projective variety over $\mathbb{C}$. Choose a real ample numerical class $\theta$, and write it as a positive real combination of ample line bundle classes. The same combination of their analytic Chern classes defines a Kähler class $\alpha$. Use Corollary 21.8.1 to identify the valuation spaces and their real model functions.

**Theorem 22.5.1** *A real model function is semipositive on a smooth analytic model in class $\alpha$ if and only if it is an algebraic $\theta$-psh function. Under the identification of valuation spaces, $C_\alpha$ consists of all the continuous algebraic $\theta$-psh functions. Their mixed measures and energies in the corresponding ample classes agree, with unnormalized masses on both sides.*

***Proof*** Every smooth refinement from Lemma D.4.2 is the analytification of a smooth projective variety over $X \times \mathbb{P}^1$. In fact each smooth centre on a projective stage of the sequence is algebraic by the coherent-ideal comparison in Lemma D.3.1. The blowup is its algebraic blowup by the graph charts. Its algebraic smoothness follows from the analytic smoothness and the comparison of completed local rings. Induction gives the assertion. When the refinement dominates a normalized algebraic flag model, the subsequent lift is algebraic. The extension of the flag ideal is invertible, since analytic invertibility of the coherent ideal descends by faithful flatness, and the algebraic blowup and normalization have the same lifting property.

If the analytic pair is semipositive on a smooth refinement, its algebraic class there has nonnegative degree on every central curve. Test the approximate smooth representatives on the normalization of the curve, then let the perturbation decrease to zero. Given a central curve on the original flag model, take an irreducible component of its inverse image dominating it on the refinement. Intersect with sufficiently many general very ample divisors to get a curve dominating the given one. The degree on that curve is the degree downstairs multiplied by the generic degree of the map. This proves relative nefness on the flag model, and Theorem 17.4.1 yields the algebraic psh property.

Conversely suppose the algebraic pair is psh. Its class is nef on the central fibre of a flag model by the same theorem and remains nef on a smooth refinement $W$. Choose a very ample line bundle on $W$ with analytic Kähler curvature $\Omega$. Adding $\varepsilon[\Omega]$ to the model class gives an ample real class on the central fibre, by Proposition A.1.1. It is ample on the other fibres as well, where the original class is ample. To pass to relative ample line bundles, work in the finite rational span of the base line bundles, the vertical divisors and this one very ample line bundle. The very ample bundle restricts to the same numerical class on the product fibres away from zero, since its Chern class restricts there along homotopic inclusions. Openness of the ample cone on the two fibres therefore allows one to express the perturbed class in this span as a positive convex combination of rational line bundle data ample on all fibres. For each rational summand take an integral multiple and use fibrewise ampleness [Stacks, Tag 0D2N]. After adding a large multiple of the pullback of an ample class on $\mathbb{P}^1$, the summand is ample on $W$ [Stacks, Tag 0892]. Take the corresponding combination of Kähler curvatures and subtract the added pullback form. This gives a representative of the perturbed class nonnegative along every central component, as required.

For the continuous statement use the uniform fixed-class approximation by semipositive models from Section 17.7 and Proposition 6.4.2. Conversely, a uniform sequence of algebraic psh approximants has a subsequence with errors at most $2^{-j-2}$. Adding $2^{-j}$ to that subsequence gives decreasing psh approximants with the same limit. Theorem 6.2.2 gives the converse. The mixed measures agree on models by Corollary 21.8.1, so agree on the continuous cones by weak continuity in both constructions. Energy equality follows by the common difference formula and the zero energy at the zero potential. □

## 22.6 Divisorial tests for semipositive potentials

Fix a nef class $\alpha$ and let $\mathcal{D}_\alpha$ consist of the pointwise decreasing limits of nets in $C_\alpha$, excluding the identically $-\infty$ function. Each such limit is upper semicontinuous. We use the normalized base restrictions $v_E$ of the orders of all central components $E$ of smooth refinements $W$, and call them the *divisorial tests*. They are ideal semivaluations by the order construction in Section 21.5. Their set is dense in $X^\beth$, since it contains the central component valuations of flag models by strict transform and Proposition 21.6.1.

Put $b_E = \operatorname{ord}_E(t)$. For an ideal $I$ on the product defining a flag model, the following formula for $\operatorname{ord}_E(I)$ holds on any smooth refinement. Write $I$ as the sum of its homogeneous terms $A_k \mathfrak{t}^k$, including a pure power of $\mathfrak{t}$, using (21.1). Then

$$b_E^{-1} \operatorname{ord}_E(I) = \min_k \big(v_E(A_k) + k\big) = \sigma(v_E)(I).$$

The first equality uses the minimum rule for a sum of ideals. Thus (21.2) holds for a real model function on a smooth refinement where all the flag ideals of its expression are invertible.

**Theorem 22.6.1** *Let $h$ be a real model function and choose a smooth refinement $W_h$ representing it. For $u$ in $\mathcal{D}_\alpha$,*

$$\sup_{X^\beth}(u - h) = \max_{E \subset (W_h)_0} (u - h)(v_E), \tag{22.2}$$

*where the maximum runs over the irreducible components. For any continuous $f$, $u \le f$ on $X^\beth$ if $u(v_E) \le f(v_E)$ at all the divisorial tests.*

***Proof*** First take $u$ semipositive on a smooth model. A constant shift of $h$ allows us to assume $u(v_E) \le h(v_E)$ for the components of $(W_h)_0$. Use a common smooth refinement $p : W' \to W_h$ representing both functions, obtained by smooth blowups over zero by Lemma D.4.2. The divisor $\Delta = D_u - p^* D_h$ on $W'$ has nonpositive coefficient along the strict transform of every component of $(W_h)_0$. It has nonnegative degree on every curve contracted by $p$. Such a curve is contained in a central component. The classes from $X$ and $W_h$ have degree zero on the curve. Integrating the semipositive representatives of Definition 22.1.1 on its normalization and letting the error tend to zero gives the assertion about $\Delta$.

Choose a Kähler form $\Omega$ on $W_h$. The second assertion of Lemma D.4.1 gives a Kähler class $p^*[\Omega] - c_1(F)$, where $F$ is an effective exceptional divisor with positive coefficient on all exceptional components. At each blowup one pulls back the preceding effective divisor and adds a small positive multiple of the new exceptional divisor. This makes $F \cdot C < 0$ on each contracted curve $C$.

If an exceptional coefficient of $\Delta$ is positive, let $s > 0$ be the maximum of its exceptional coefficient ratios to $F$. The effective divisor $sF - \Delta$ omits a component attaining this ratio. Choose a general point of that component outside the effective divisor and a curve $C$ in its fibre over $W_h$ through the point. There is such a curve since

the component has positive-dimensional general fibres and a composition of projective blowups has projective fibres. One may cut a fibre component by hyperplanes through the chosen point. Thus $(sF - \Delta) \cdot C \geq 0$, which gives $\Delta \cdot C \leq sF \cdot C < 0$, a contradiction.

We conclude $\Delta \leq 0$. On further common smooth refinements this inequality pulls back by the effective divisor argument of (22.1). Computing at strict transforms gives $u \leq h$ at all divisorial tests and hence everywhere, by continuity and density. Applying the argument to a shift of $h$ by the right side of (22.2) gives that equality for model $u$.

The equality passes to a uniform limit in $C_\alpha$. For a decreasing net from this space the maxima over the finite set on the right decrease to the maximum of the limits. Each earlier maximum bounds $u-h$, so the limiting maximum still bounds it, proving the equality for $\mathcal{D}_\alpha$. Finally, for $\varepsilon > 0$, approximate $f$ uniformly within $\varepsilon$ by a rational model function $h$ using Proposition 21.5.1. The hypothesis gives $u - h \leq \varepsilon$ at the divisorial tests, so (22.2) gives $u \leq f + 2\varepsilon$ everywhere. Let $\varepsilon$ decrease to zero.□

## 22.7 Subgeodesic rays and model potentials

From here through Section 23.2, fix a Kähler form $\omega$ in $\alpha$. Write $p_X : X \times \mathbb{D} \to X$ for the projection from the product with the unit disk. A subgeodesic ray in this section is a circle-invariant $p_X^*\omega$-psh function on $X \times \mathbb{D}$ that is bounded above and nontrivial. Its domain includes the central fibre $X \times \{0\}$. Let $U$ be any $p_X^*\omega$-psh function on $X \times \mathbb{D}$ that is bounded above and nontrivial, not necessarily circle-invariant. For a central component $E$ of a smooth refinement $q : W \to X \times \mathbb{P}^1$, which we call a *test component*, set

$$T_{W,E}(U) = -b_E^{-1} \nu_E(U \circ q). \tag{22.3}$$

We normalize $\nu_0(\log |z|^2) = 1$. The generic Lelong number $\nu_E$ is the infimum of the point Lelong numbers along $E$. Adding a smooth local potential of the background form leaves these numbers unchanged. The point numbers are upper semicontinuous [Xia26a, Proposition 12.1.4], so their infimum is the same on any dense open part of $E$. In particular the generic number is unchanged on the strict transform under a further smooth refinement. We retain the pair $W, E$ in the notation until we have identified its slope with a function of $\nu_E$.

**Lemma 22.7.1** *For any nonempty family $U_i \leq 0$ of nontrivial $p_X^*\omega$-psh functions, its upper regularized supremum $U = (\sup_i U_i)^*$ is $p_X^*\omega$-psh, and*

$$T_{W,E}(U) = \sup_i T_{W,E}(U_i) \tag{22.4}$$

*for every test component. If the family is stable under circle rotations, $U$ is circle-invariant.*

***Proof*** The local upper envelope theorem [Xia26a, Theorem 8.3.2], applied after adding a common local potential, gives the psh property. Stability under rotations

gives the last assertion. For a locally uniformly upper bounded family of psh functions $V_i$, the point Lelong number of $(\sup_i V_i)^*$ at $x$ is $\inf_i \nu_x(V_i)$ by [Xia26a, Proposition 12.1.4].

On $W$, the pullback $U \circ q$ and the upper regularized supremum of the $U_i \circ q$ agree off the central fibre, where $q$ is a biholomorphism. They agree everywhere, since after adding smooth local potentials they are psh and these functions are determined by their almost-everywhere values via shrinking ball means. Apply the point formula on $W$ and take the infimum over $E$. Interchanging the two infima gives (22.4). □

**Proposition 22.7.1** *Suppose a subgeodesic ray satisfies $L \log|t|^2 \le U \le 0$ for a real $L \ge 0$. There is a decreasing sequence of rational model functions semipositive on smooth models in class $\alpha$ whose limit $u \in \mathcal{D}_\alpha$ satisfies $u(v_E) = T_{W,E}(U)$ at every divisorial test. This limit is unique in $\mathcal{D}_\alpha$.*

***Proof*** First compactify the ray near the central fibre. Choose $0 < r_1 < r_2 < 1$. For $A > L$ sufficiently large, one can choose $B$ so that $A \log|t|^2 + B$ lies below $L \log|t|^2$ for $|t| \le r_1$ and is positive near $|t| = r_2$. Indeed the conditions on $B$ are the strict inequalities $-A \log r_2^2 < B < (L - A) \log r_1^2$, whose interval is nonempty for large $A$. Extend this radial function smoothly off zero to $\mathbb{P}^1$ by a cutoff beyond $r_2$. Its curvature there is bounded below by $-c\omega_{\mathrm{FS}}$ for some $c > 0$, since the logarithm is harmonic near zero off the origin. Inside $|t| < r_2$ take its maximum with $U$. The two prescriptions agree near $|t| = r_2$, and give a circle-invariant $\theta$-psh function $\widetilde{U}$ on the compact product for $\theta = p_X^*\omega + c p_{\mathbb{P}^1}^* \omega_{\mathrm{FS}}$. This function equals $U$ near zero and is locally bounded off the central fibre.

For $m_j = 2^j$, put $I_j = \mathcal{I}(m_j \widetilde{U})$ with the integrability convention $|g|^2 e^{-m_j \widetilde{U}}$. The ray bound implies $\mathfrak{t}^{\lceil m_j L \rceil} \subset I_j$: the local generator $t^{\lceil m_j L \rceil}$ satisfies the integrability condition. The ideals are coherent [Xia26a, Theorem 12.2.7], applied locally after adding a smooth local potential, and are the unit ideal off zero. Circle invariance yields their finite homogeneous decomposition modulo this pure power, by taking finitely many distinct circle scalars in the Vandermonde argument of Theorem 21.4.1. Thus the $I_j$ are flag ideals.

Apply [Xia26a, Theorem 13.4.5] with reference form $p_X^*\omega + p_{\mathbb{P}^1}^* \omega_{\mathrm{FS}}$. It gives quasi-psh approximants with gentle analytic singularities of type $(m_j^{-1}, I_j)$ and curvature loss bounded by $c'/m_j$ times this reference form. On a smooth refinement principalizing $I_j$, write $I_j \mathcal{O}_W = \mathcal{O}_W(-F_j)$. Subtracting the logarithmic part $m_j^{-1} \log \|s_{F_j}\|^2$ from the pulled-back approximation leaves a smooth function by [Xia26a, Definition 13.1.11]. The smooth part of the pulled-back approximating current after the curvature correction is positive semidefinite off the fibre and hence everywhere.

Choose a rational $C > c'$ and make the correction using $C/m_j$ times the reference form. Modulo a class pulled back from the parameter line, the resulting smooth semipositive class is $(1 + C/m_j)\rho^*\alpha - c_1(F_j)/m_j$. Divide by $1 + C/m_j$. This produces the rational model function

$$u_j(v) = -\frac{\sigma(v)(I_j)}{m_j(1 + C/m_j)}$$

semipositive on a smooth model in class $\alpha$. A class pulled back from the parameter line has zero restriction on each central component, so the semipositivity condition is unchanged by removing it.

Subadditivity gives $I_{j+1} \subset I_j^2$ [Xia26b, Theorem 1.4.2], applied locally after adding a smooth local potential. Before division by $1+C/m_j$, the functions therefore decrease and are nonpositive. The reciprocals of these factors increase to one, so the divided functions still decrease. They are bounded below by $-L-1$ by containment of the pure powers of $\mathfrak{t}$. The divisorial order formula [Xia26a, Corollary 12.6.8], applied near a point of $E$ after adding a smooth local potential, gives $m_j^{-1}\operatorname{ord}_E(I_j) \to \nu_E(U\circ q)$ on each test component. Here the generic Lelong number along the resulting open part of $E$ is $\nu_E(U\circ q)$, since the generic value is attained on a dense subset of $E$ [Xia26a, Proposition 12.3.5]. This proves $u(\nu_E) = T_{W,E}(U)$. Uniqueness follows from the last assertion of Theorem 22.6.1. Let $u' \in \mathcal{D}_\alpha$ have the same divisorial values, and let $u'_k$ be a net in $C_\alpha$ decreasing to $u'$. Then $u(\nu_E) = u'(\nu_E) \le u'_k(\nu_E)$ at every divisorial test, so $u \le u'_k$ for each $k$, and hence $u \le u'$. By symmetry $u = u'$.□

**Proposition 22.7.2** *Let $h_D \le 0$ be a model function semipositive on a smooth model in class $\alpha$. For every $\eta > 0$ there is a nonpositive subgeodesic ray $U_\eta$ with*

$$h_D(\nu_E) - \eta \le T_{W,E}(U_\eta) \le h_D(\nu_E)$$

*at every divisorial test.*

***Proof*** Choose a smooth refinement representing $h_D$ on which the semipositivity condition holds. The divisor $D$ is nonpositive by (22.1). The second assertion of Lemma D.4.1 gives a Kähler form $\Omega_W$ with class

$$[\Omega_W] = \rho^*\alpha + c\pi^*[\omega_{\mathrm{FS}}] - c_1(F)$$

for some $c > 0$ and an effective exceptional divisor $F$ over zero. Start with the product Kähler class and subtract small positive multiples of the new exceptional classes on successive blowups.

For $\varepsilon > 0$ the class $\rho^*\alpha + c_1(D) + \varepsilon[\Omega_W]$ restricts to a Kähler class on each central component. Use the semipositive representatives with error $\varepsilon/2$ and add $(\varepsilon/2)\Omega_W$. By Lemma D.5.1, the class has one smooth representative Kähler near the whole fibre. After dividing by $1+\varepsilon$, the class is

$$\rho^*\alpha + c_1(D_\varepsilon) + c_\varepsilon\pi^*[\omega_{\mathrm{FS}}], \qquad D_\varepsilon = \frac{D-\varepsilon F}{1+\varepsilon}, \quad c_\varepsilon = \frac{\varepsilon c}{1+\varepsilon}.$$

Write $D_\varepsilon = \sum_i a_i E_i$, with $a_i \le 0$, choose smooth Hermitian norms of the defining sections, and put $L_\varepsilon = \sum_i a_i \log\|s_i\|^2$. If $\Theta_\varepsilon$ is the corresponding curvature form then $dd^c L_\varepsilon = [D_\varepsilon] - \Theta_\varepsilon$ as currents. The smooth positive representative just constructed is $\rho^*\omega + \Theta_\varepsilon + c_\varepsilon\pi^*\omega_{\mathrm{FS}} + dd^c\psi$ near the fibre for a smooth $\psi$. For sufficiently small $r > 0$, this neighborhood contains $\pi^{-1}(\{|t| \le r\})$ by properness. On the inverse image of the punctured disk of radius $r$, the function $\psi - L_\varepsilon + c_\varepsilon\chi\circ\pi$ descends to the product, where $dd^c\chi = \omega_{\mathrm{FS}}$ on the disk. Its $p_X^*\omega$-curvature pulls

back to the positive representative. It is bounded above because $D_\varepsilon \le 0$, and the removable singularity theorem [Xia26a, Theorem 8.4.1] extends it across zero. After restriction to a smaller disk and rescaling to the unit disk, subtract a constant to make it nonpositive.

On any common smooth refinement the normalized logarithmic slopes are the normalized coefficients of $D_\varepsilon$. Indeed the other terms are smooth on the model, and pullbacks of the logarithms have the stated generic orders; passing to strict transforms preserves the generic Lelong numbers. Since $0 \le F \le CW_0$ for some $C$, these slopes differ from $h_D(v_E)$ by at most $\varepsilon(\|h_D\|_\infty + C)$. The estimate also holds under rescaling and rotation: for $a_\lambda(x,t) = (x,\lambda t)$, use the marking $a_\lambda^{-1} \circ q$ for the transformed function. The flag ideals defining $h_D$ and the fibre are unchanged, and the bound on $F$ holds for the new marking. All test components can be compared on further common refinements.

Take the upper regularized supremum of the circle rotations. It is a subgeodesic ray with the same slope bounds by Lemma 22.7.1. Adding $\varepsilon(\|h_D\|_\infty + C)\log|t|^2$ lowers its transform by that constant, since the generic Lelong number of $\log|t|^2$ is $b_E$. Choose $\varepsilon$ so that twice this constant is less than $\eta$. This gives the required ray.□

# Chapter 23
# The variational theory in a Kähler class

Let $p$ be a point of the compact Kähler manifold $X$, let $\alpha$ be a Kähler class and $c > 0$, and take the obstacle $f(v) = c\min(v(\mathfrak{m}_p), 1)$ on $X^{\beth}$, where $\mathfrak{m}_p$ is the ideal sheaf of $p$. It vanishes at the trivial semivaluation $v_0$. On the product model, Theorem 22.6.1 gives $\sup u = u(v_0) \le 0$ for every $u \in \mathcal{D}_\alpha$ below $f$. Since zero is itself such a potential, the envelope of $f$ is zero. Its Monge–Ampère measure is $(\int_X \alpha^n)\delta_{v_0}$, which is carried by the contact set because $f(v_0) = 0$. This calculation is carried out in an example in Section 23.1. In this chapter we prove the contact property for every continuous obstacle and use it to solve the Monge–Ampère equation in a Kähler class by the variational method.

We keep the conventions of Chapter 22. Thus $X$ is a connected compact Kähler manifold of dimension $n > 0$, $C_\alpha$ is the uniform closure of the potentials semipositive on smooth models, $\mathcal{D}_\alpha$ consists of its decreasing limits, and $v_E$ are the divisorial tests of Section 22.6. Except at the start of Section 23.3, which allows a nef class of positive volume, $\alpha$ is a Kähler class. Through Section 23.2 we keep the Kähler form $\omega$ in $\alpha$ and the subgeodesic rays of Section 22.7. In Section 23.7 the class varies.

Theorem 23.1.1 shows that the envelope $P_\alpha(f)$, the supremum of the potentials in $\mathcal{D}_\alpha$ below a continuous obstacle $f$, lies in $C_\alpha$. For a model obstacle, Theorem 23.2.1 computes the gap $f - P_\alpha(f)$ at the central components by generic Lelong numbers of a class on the model. Measures and mixed measures extend to decreasing limits of finite energy by Theorems 23.3.1 and 23.3.2. Theorem 23.4.1 proves the contact property and expresses the energy of a model envelope as a volume on the model. It gives the first variation of the obstacle energy in Corollary 23.4.1. With the weak compactness of Theorem 23.5.1, this yields Theorem 23.6.1: every probability measure of finite dual energy is the normalized Monge–Ampère measure of a unique normalized finite-energy potential. The solution map is a homeomorphism for the strong topologies (Corollary 23.6.2), and Theorem 23.7.1 extends this continuity to a varying Kähler class.

Chapter 24 uses the solution of the equation and weak compactness to construct maximal finite-energy rays. For nef backgrounds, the volume formula (23.5), its derivative in Corollary 23.4.2 and the ordered estimate of Lemma 23.3.1 give the

obstacle energy in Section 26.2. Weak compactness and the estimates along maximizing nets return in Sections 27.3 and 27.4.

## 23.1 Continuous envelopes in a Kähler class

We first check that the maximum of two semipositive rational model potentials is again semipositive.

**Lemma 23.1.1** *In the fixed Kähler class $\alpha$, the maximum of two rational model potentials semipositive on smooth models is again a rational model potential semipositive on a smooth model.*

***Proof*** Represent the potentials as $(\varphi_{I_i} - \varphi_{J_i})/m$ for $i = 1, 2$, with a common positive denominator $m$. As in Proposition 21.5.1, their maximum is represented by $I_1J_2+I_2J_1$ and $J_1J_2$. Choose a smooth refinement where all the ideals are invertible and both original pairs are semipositive. If $D_\vee$ and $D_i$ are the corresponding divisors, then $G_i = D_\vee - D_i$ are effective. Their supports have empty intersection: in a frame for the pullback of $I_1J_2 + I_2J_1$, the two pulled-back summands generate the unit ideal, so at each point one of their generators is a unit.

Fix $\varepsilon > 0$. Each class $\rho^*\alpha + c_1(D_i) + \varepsilon[\Omega_W]$ is Kähler on every central component by the semipositivity condition with $\varepsilon/2$. Use Lemma D.5.1 to choose representatives positive near the whole fibre. Write these representatives as $\rho^*\omega + \Theta_i + \varepsilon\Omega_W + dd^c\psi_i$, using curvature forms $\Theta_i$ of smooth norms on the Cartier multiples $mD_i$, divided by $m$. Choose a norm for $mD_\vee$ as well, and use the quotient norms on $mG_i$. Away from $G_i$, the potential $\psi_i + m^{-1}\log\|s_{mG_i}\|^2$ then gives the same positive form with background $\rho^*\omega + \Theta_\vee + \varepsilon\Omega_W$. Take the regularized maximum of the two potentials as in Lemma D.5.1. It is smooth even near the supports of $G_i$, since the potential with a pole there is smaller than the other by more than the smoothing width locally. This yields a positive representative near the fibre of the required perturbed class for $D_\vee$. Extend its potential with a cutoff. Since $\varepsilon$ was arbitrary, the maximum is semipositive on a smooth model. □

For a real continuous function $f$ on $X^\beth$ define its envelope by

$$P_\alpha(f) = \sup\{u \in \mathcal{D}_\alpha : u \le f\}.$$

Since constants belong to $\mathcal{D}_\alpha$, this function takes values between $\inf f$ and $f$.

**Theorem 23.1.1** *The envelope $P_\alpha(f)$ belongs to $C_\alpha$. It is a uniform increasing limit of rational model potentials semipositive on smooth models and lying below $f$. The envelope operator is nonexpansive for the uniform norm.*

***Proof*** For $u \in \mathcal{D}_\alpha$ with $u \le f$, take a decreasing net $u_i \in C_\alpha$ with limit $u$. For $\varepsilon > 0$, the open sets $\{u_i < f + \varepsilon\}$ exhaust the compact space $X^\beth$ in the directed order, so one $u_i$ is everywhere below $f + \varepsilon$. Choose a semipositive model function $h$ within $\varepsilon$ of $u_i$.

Then $u - 3\varepsilon \le h - 2\varepsilon \le f$. Thus the supremum defining the envelope is unchanged when taken over semipositive model candidates, and it is lower semicontinuous.

Translating by a constant reduces to $f \le 0$. Take the upper regularized supremum $U_f$ of all nonpositive subgeodesic rays with $T_{W,E}(U) \le f(v_E)$ at every test component. The constant-slope ray $L \log |t|^2$ is such a candidate for $L \ge -\inf f$. Thus $L \log |t|^2 \le U_f \le 0$, and Lemma 22.7.1 gives the same divisorial obstacle inequalities for $U_f$. By Proposition 22.7.1 its transform extends to $u_f \in \mathcal{D}_\alpha$. The order test in Theorem 22.6.1 gives $u_f \le f$.

For every semipositive model function $h \le f$, Proposition 22.7.2 yields rays satisfying the obstacle inequalities and having slopes arbitrarily close to $h$ at all test components. Therefore $h \le u_f$ on the dense set of tests. The set where $u_f \ge h$ is closed because $u_f$ is upper semicontinuous and $h$ is continuous. It follows that $u_f = P_\alpha(f)$, and the latter is continuous.

The rational models decreasing to $u_f$ in Proposition 22.7.1 converge uniformly by Proposition 4.1.1. Select a subsequence whose distance to the limit at index $j$ is at most $2^{-j}$. Subtract $2^{-j}$ at that index to give rational model potentials below $u_f$ and still approaching it uniformly. Their successive finite maxima remain semipositive by Lemma 23.1.1. They give the claimed increasing approximation. For the translation made at the beginning one may use a rational constant, so rationality also holds for the original obstacle. Monotonicity and translation by constants in the defining supremum give the uniform norm bound. □

The next proposition uses the approximation in Theorem 23.1.1 to bound the envelope measure from below at points of contact with semipositive models. If a rational semipositive model potential $h_E \le f$ agrees with $f$ at $v_E$, the atom of the envelope measure at $v_E$ is at least $b_E \int_E (\rho^*\alpha + c_1(D_{h_E}))^n$. For model obstacles, Corollary 23.4.3 computes these atoms exactly.

**Proposition 23.1.1** *Let $f$ be a real continuous function on $X^\beth$. Let $S$ be a nonempty set of central components on a smooth refinement $W$. For each $E \in S$, suppose there is a rational semipositive model potential $h_E$ represented on $W$, with $h_E \le f$ and $h_E(v_E) = f(v_E)$. Then for $u = P_\alpha(f)$,*

$$\mathrm{MA}_X((\alpha, u)^n) \ge \sum_{E \in S} b_E \left( \int_E (\rho^*\alpha + c_1(D_{h_E}))^n \right) \delta_{v_E}.$$

***Proof*** Take the increasing rational model approximants $r_j$ from Theorem 23.1.1. By Lemma 23.1.1, $s_j = \max(r_j, \max_{E \in S} h_E)$ are semipositive models with $h_E \le s_j \le f$ and $s_j(v_E) = h_E(v_E)$ for each $E \in S$. Since each $h_E$ is a candidate in the envelope, $h_E \le P_\alpha(f) = u$. Hence $r_j \le s_j \le u$, so the $s_j$ converge uniformly to $u$. For each $j$, choose a common smooth refinement $W'_j$ over $W$ on which $s_j$ and the finite family of $h_E$ are semipositive. For each $E$, $D_{s_j} - D_{h_E}$ is effective by (22.1) and has zero coefficient at the strict transform $\widetilde{E}$. Its restriction to $\widetilde{E}$ is effective. Expand the difference of the two top intersection powers on $\widetilde{E}$ as this divisor class times a sum of mixed powers. Each summand has nonnegative integral. Use the smooth semipositive representatives with arbitrarily small errors and integrate over the effective

restriction, as in Section 22.1. The contribution of $\widetilde{E}$ to the model measure of $s_j$ is at least the stated degree, by the projection formula for $h_E$ and preservation of $b_E$. These contributions can be added even when some $v_E$ agree. Other contributions are nonnegative.

The measures converge weakly by Theorem 22.3.1. The lower bound passes to the limit on every nonnegative continuous test. □

*Example 23.1.1 (An obstacle from a point blowup)* Let $p \in X$ and take $f(v) = c\min(v(\mathfrak{m}_p), 1)$ for $c > 0$. This is $-c$ times the model function of $\mathfrak{m}_p O_{X\times\mathbb{P}^1} + \mathfrak{t}$. The obstacle is nonnegative and equals zero at the trivial semivaluation $v_0$. For $h = 0$, the product model has only one central component, with valuation $v_0$. Theorem 22.6.1 gives $\sup u = u(v_0)$ for every candidate $u \le f$. Thus $P_\alpha(f) = 0$, since zero is itself a candidate. The product model gives $\mathrm{MA}_X((\alpha, 0)^n) = (\int_X \alpha^n)\delta_{v_0}$ by Theorem 21.7.1. In particular the envelope measure is carried by the contact set.

**Corollary 23.1.1** *In the smooth projective setting of Theorem 22.5.1, $\mathcal{D}_\alpha$ agrees with the algebraic $\theta$-psh cone. The two envelope operators agree on continuous obstacles.*

***Proof*** Theorem 22.5.1 identifies the continuous cones, and Proposition 6.4.2 supplies decreasing continuous majorants for each algebraic $\theta$-psh potential. Conversely a decreasing limit from the common continuous cone which is not identically $-\infty$ is finite at the trivial valuation by (6.4). It is therefore algebraic $\theta$-psh by Theorem 6.2.2. The suprema defining the envelopes are over the same functions, and the common envelope is continuous. □

## 23.2 The envelope gap and divisorial Lelong numbers

Take a real model obstacle $f = h_D$ on a smooth refinement $q : W \to X \times \mathbb{P}^1$ representing $D$. Write $W_0 = \sum_E b_E E$ and let $d_E$ be the coefficient of $D$ along $E$. For a real class $\beta$ containing positive currents, write $\nu_E(\beta)$ for the infimum of $\nu_E(T)$ over closed positive currents $T$ in $\beta$.

**Theorem 23.2.1** *For all sufficiently large real $\lambda$, the class $\beta_\lambda = \rho^*\alpha + c_1(D) + \lambda c_1(W_0)$ contains positive currents, and*

$$f(v_E) - P_\alpha(f)(v_E) = b_E^{-1}\nu_E(\beta_\lambda) \tag{23.1}$$

*for every central component $E$ of $W$.*

***Proof*** Translate $f$ by a constant to arrange $f \le 0$. This translates its envelope by the same constant and reparametrizes $\lambda$, since a constant model function has divisor a multiple of $W_0$. By (22.1), $D \le 0$. Choose $L \ge -\inf f$.

Use the ray $U_f$ in the proof of Theorem 23.1.1. It satisfies $L \log |t|^2 \leq U_f \leq 0$ and has slope $P_\alpha(f)(v_E)$ along every test component. The compactification in the proof of Proposition 22.7.1 extends it near zero to a potential $\widetilde{U}_f$ on the product, positive relative to $p_X^*\omega + \lambda p_{\mathbb{P}^1}^*\omega_{\mathrm{FS}}$ for all sufficiently large $\lambda$. Normalize $\omega_{\mathrm{FS}}$ to have integral one. Pull back this positive current using its local potentials and call the pullback $S$. Along $E$ it has $\nu_E(S) = -b_E P_\alpha(f)(v_E) \geq -d_E$. By divisorial decomposition [Xia26a, Theorem 14.3.7], $S + [D]$ is positive and its generic Lelong number along $E$ is $d_E - b_E P_\alpha(f)(v_E)$. It lies in $\beta_\lambda$, giving one inequality.

For the reverse inequality take any positive current $T$ in $\beta_\lambda$. Write $\ell_D$ for the sum of the logarithms of squared section norms of the central components with coefficients $d_E$, so $dd^c\ell_D = [D] - \gamma_D$ for a smooth representative $\gamma_D$ of $c_1(D)$. By the $dd^c$-lemma for currents [Xia26a, Theorem 14.1.7], there is a quasi-psh function $\varphi$ on $W$ with

$$T = \rho^*\omega + \gamma_D + \lambda\pi^*\omega_{\mathrm{FS}} + dd^c\varphi.$$

Choose a smooth potential $a$ for $\omega_{\mathrm{FS}}$ on the affine chart containing the closed unit disk. On the inverse image of the disk the function $\varphi - \ell_D + \lambda a \circ \pi$ is $\rho^*\omega$-psh, since its current is $T - [D] \geq 0$. It is bounded above over the disk, since $-D$ is effective and $a$ is bounded there. Off zero it therefore gives a potential on the product, and this extends across zero by the bounded-above psh extension theorem [Xia26a, Theorem 8.4.1]. Denote the extended function by $U$. Its pullback is the given function on $W$, since the two agree off the fibre and are determined by their values there. In particular

$$\nu_E(U \circ q) = \nu_E(T) - d_E.$$

We check that enforcing circle invariance preserves the obstacle. Locally on the product, take homogeneous generators of each of the finitely many flag ideals representing $D$. Write $IO_W = O_W(-F_I)$ for each such ideal. In a frame, factoring out a local generator leaves generators of the unit ideal. Thus the logarithm of the sum of squared norms of the generators differs from $\log \|s_{F_I}\|^2$ by a locally bounded function. The model divisor of $I$ is $-F_I$, so the real linear combination representing $D$ expresses $\ell_D$ up to a locally bounded term, using logarithms on the product that are invariant under circle rotations. Thus, on each relatively compact product chart over a smaller disk, the upper bound $U \circ q \leq -\ell_D + C$ off zero gives the same bound for the pulled-back rotations of $U$, with a common larger constant. Here the bounds may be taken on a smaller chart by compactness of its closed inverse image in $W$.

Subtract a constant from $U$ to make it nonpositive. This leaves its slopes unchanged. Take the upper regularized supremum of its rotations and $L \log |t|^2$. The latter has slope $-L \leq f(v_E)$. The rotational bounds give slopes at most $f(v_E)$ on each component of $W_0$, by comparison of generic Lelong numbers. By Lemma 22.7.1 the result is a ray with these obstacle bounds and with slope at least $b_E^{-1}(d_E - \nu_E(T))$ along $E$. Its transform from Proposition 22.7.1 lies below $f$ by Theorem 22.6.1. Consequently $b_E(f(v_E) - P_\alpha(f)(v_E)) \leq \nu_E(T)$ for every $T$. This finishes the proof. □

## 23.3 Decreasing limits of finite energy

Fix a nef class $\alpha$ with $V = \int_X \alpha^n > 0$. In this section the functions under consideration are pointwise decreasing limits of nets in $C_\alpha$. For such a limit $u$ set

$$E_\alpha(u) = \inf\{E_\alpha(w) : w \in C_\alpha,\ w \geq u\}.$$

If $u_i \downarrow u$ is one such net, then $E_\alpha(u) = \inf_i E_\alpha(u_i)$. Indeed for any continuous majorant $w$ in the infimum and any $\varepsilon > 0$, the open sets $\{u_i < w + \varepsilon\}$ cover the compact space $X^{\beth}$ and increase in the directed order. Thus $u_i < w+\varepsilon$ eventually, and the monotonicity and translation identities from Theorem 22.4.1 give the required upper bound. The other bound follows since each $u_i$ is a majorant. We say $u$ has finite energy when this infimum is finite. In particular a bounded limit has finite energy, by comparison with constant potentials.

Write $\mu_w = \mathrm{MA}_X((\alpha, w)^n)$ for continuous $w \in C_\alpha$. The next estimate uses an energy difference to control a measure difference.

**Lemma 23.3.1** *For a real model test function $h$, there is a constant $K_h$ such that whenever $u, v \in C_\alpha$ and $u \leq v$,*

$$\left| \int h \, \mathrm{d}(\mu_v - \mu_u) \right|^2 \leq K_h \big(E_\alpha(v) - E_\alpha(u)\big).$$

***Proof*** First let $u, v$ be model functions semipositive on smooth models. Put $\delta = v - u$ and for $0 \leq j < n$ use the notation $\Gamma_j = P_v^j P_u^{n-1-j}$, with $P_w = (\alpha, w)$. For a model function $f$ define $\|f\|^2_{\Gamma_j} = -\langle (0, f), (0, f), \Gamma_j \rangle_X$. By the vertical intersection identity of Proposition 22.2.1, these quadratic forms are nonnegative, so they satisfy the Cauchy–Schwarz inequality. One has

$$\int h \, \mathrm{d}(\mu_v - \mu_u) = \sum_{j=0}^{n-1} \langle (0, h), (0, \delta), \Gamma_j \rangle_X.$$

To bound $\|h\|^2_{\Gamma_j}$, use a fixed smooth refinement representing $h$ and the curvature bound $-C\Omega_h \leq \gamma_h \leq C\Omega_h$ of Lemma 22.3.1. In the divisor integration formula the coefficient of $D_h$ along $E$ is bounded in absolute value by $b_E \|h\|_\infty$ on every common refinement by (22.1). Thus the norm squared is bounded by $C\|h\|_\infty \int_X \omega_h \alpha^{n-1}$, where $\omega_h$ is the restriction of $\Omega_h$ at infinity. The computation again uses the full fibre, so this bound is independent of $u, v, j$.

The difference of powers also gives

$$\sum_{j=0}^{n-1} \|\delta\|^2_{\Gamma_j} = \int \delta \, \mathrm{d}(\mu_u - \mu_v) \leq \int \delta \, \mathrm{d}\mu_u \leq (n+1)\big(E_\alpha(v) - E_\alpha(u)\big).$$

The first inequality uses $\delta \geq 0$ and positivity of $\mu_v$. The last follows by keeping the $\mu_u$ term in the mixed energy difference. Apply Cauchy–Schwarz in each term and then to the finite sum. This proves the claimed estimate on models.

For continuous $u \leq v$, take uniform model approximations to both functions and add a nonnegative constant tending to zero to the upper approximant so as to preserve the inequality. Uniform continuity of the measures and energy gives the result in the limit. □

**Theorem 23.3.1** *Let $u_i \downarrow u$ in $C_\alpha$ with $E_\alpha(u) > -\infty$. Then $\mu_{u_i}$ converges weakly to a positive Radon measure $\mu_u$ of mass $V$, independently of the net. This measure integrates $u$.*

*For $n$ possibly different finite-energy decreasing limits in this same class, their mixed measures are the weak limits along simultaneous decreasing approximations. They are positive with mass $V$, and every finite-energy decreasing limit in the class is integrable against them.*

***Proof*** Lemma 23.3.1 shows that the integrals against each real model function are Cauchy, since the decreasing energies tend to the finite number $E_\alpha(u)$. For two incomparable indices apply the estimate to each with a common later index. Positivity and the common mass extend Cauchy convergence to all continuous tests by Proposition 21.5.1. The limit is a Radon measure by Theorem 4.3.1.

If $v_j \downarrow u$ is another net, fix $j$ and $\varepsilon > 0$. Then $u_i \leq v_j + \varepsilon$ eventually by the compact comparison used for energy above. The measure of $v_j + \varepsilon$ equals that of $v_j$: on models a constant has divisor a multiple of the fibre, whose class restricts to zero on every central component. This passes to continuous functions by uniform approximation. Apply the test estimate to $u_i$ and $v_j + \varepsilon$, then let $i$ tend to the limit. The resulting squared bound is at most

$$K_h\big(E_\alpha(v_j) + \varepsilon V - E_\alpha(u)\big).$$

Sending $j$ to the limit and $\varepsilon$ to zero proves independence on all model tests, hence on all continuous ones. The same argument with translation shows that adding a constant to $u$ preserves $\mu_u$.

For integrability discard an initial segment and subtract a constant so that every approximant on the remaining tail is nonpositive. The mixed energy difference from zero gives

$$\int u_i \, \mathrm{d}\mu_{u_i} \geq (n+1)E_\alpha(u_i) \geq (n+1)E_\alpha(u).$$

All terms in that mixed sum are nonpositive. For any fixed earlier index $j$ on the tail, replace $u_i$ on the left by $u_j$ for $i \geq j$. Weak convergence gives the same lower bound for $\int u_j \, \mathrm{d}\mu_u$. Decreasing convergence on continuous tests under a Radon measure (Theorem 4.2.1, applied to their negatives) gives a finite lower bound for $\int u \, \mathrm{d}\mu_u$.

For the mixed statement, each average of a nonempty subset of the $n$ limits is a decreasing limit of continuous functions in $C_\alpha$, and has finite energy by concavity. The same assertion holds for a simultaneous decreasing approximation on the product

of their directed sets. Polarization expresses a mixed measure of the approximants as a linear combination of the diagonal measures of these averages, with constant coefficients. Indeed, by the alternating sum over all subsets, every mixed term with a missing index cancels, and the term with all $n$ indices occurs $n!$ times. A sum over a nonempty subset is the cardinality of the subset times the model pair of its average. This identity holds on continuous data by uniform convergence from models. Thus weak convergence and independence for each diagonal term give them for the mixed measures. Positivity and mass pass from the approximants.

Finally normalize all $n+1$ given limits $u_0, \dots, u_n$ by constants to be nonpositive and take their average $w = (u_0 + \cdots + u_n)/(n+1)$. By concavity of the continuous approximants it has finite energy. The diagonal measure of $w$ dominates $n!/(n+1)^n$ times the mixed measure of $u_1, \dots, u_n$. This follows by the positive expansion on the continuous approximants, then weak convergence of the two sides. Since $0 \geq u_0 \geq (n+1)w$ and $w$ is integrable against its diagonal measure, $u_0$ is integrable against the mixed measure. □

**Theorem 23.3.2** *Let $\alpha_1, \dots, \alpha_n$ be Kähler classes and let $u_i$ be finite-energy decreasing limits in the respective classes. The mixed measure along any simultaneous decreasing continuous approximations has a weak limit independent of the approximations. It is positive and has mass $\int_X \alpha_1 \cdots \alpha_n$. Every finite-energy decreasing limit in any Kähler class is integrable against it.*

***Proof*** We first compare energies under increasing the background class. If $\beta - \alpha$ is nef and $v \in C_\alpha$ is nonpositive, then $v \in C_\beta$ and $E_\beta(v) \leq E_\alpha(v)$. For a model $v$, the semipositivity follows by adding the pair $(\beta - \alpha, 0)$. Expand the energy difference by telescoping in the $n+1$ slots of the pairing. Each term has a slot $(\beta - \alpha, 0)$, and all other slots have nonpositive potentials. Such a pairing is nonpositive. Replacing the potentials by zero increases the pairing one slot at a time, by the divisor integration formula and positivity of the other factors. With all potentials zero, the pairing vanishes since all $n+1$ classes are pulled back from $X$. This proves the comparison for model $v$. Uniform approximation, with a downward constant correction to keep the approximants nonpositive, gives the comparison on $C_\alpha$. The model scaling identity $E_{a\alpha}(av) = a^{n+1} E_\alpha(v)$ for $a > 0$ also passes to the continuous cones.

Consider any nonempty finite family of the pairs $(\alpha_i, u_i)$ and put $\beta = \sum_i \alpha_i$ for that family. Choose a Kähler class $\Omega$ such that $\Omega - \alpha_i$ is nef for each $i$. Choose $C_i \geq 1$ with $C_i \alpha_i - \Omega$ nef. These choices exist by openness of the Kähler cone. Normalize the approximants by constants on a tail so they are all nonpositive. Adding constants shifts the energy by a fixed finite amount and preserves every continuous mixed measure because the fibre class restricts to zero along the central fibre. For each of the resulting continuous approximants $v_i$ to $u_i$, the comparison, scaling and monotonicity give

$$E_\Omega(v_i) \geq E_{C_i \alpha_i}(v_i) = C_i^{n+1} E_{\alpha_i}(v_i / C_i) \geq C_i^{n+1} E_{\alpha_i}(v_i) \geq C_i^{n+1} E_{\alpha_i}(u_i).$$

If $k$ is the size of the family, its sum $\sum_i v_i$ belongs to both $C_\beta$ and $C_{k\Omega}$. Since $k\Omega - \beta$ is nef, the energy of the sum in $\beta$ is at least its energy in $k\Omega$. This last energy

is $k^{n+1}$ times the energy of the average in $\Omega$. Concavity bounds it from below by $k^n \sum_i E_\Omega(v_i)$, uniformly. Thus $\sum_i u_i$ has finite energy in $\beta$.

For $n$ inputs, expand the alternating sum of the diagonal measures of the sums over their nonempty subsets. On continuous approximants the terms missing an input cancel, leaving $n!$ times the mixed measure. This identity follows first on models by multilinearity and then on continuous potentials by uniform approximation. For every nonempty subset, the diagonal term converges by Theorem 23.3.1 in the sum of its classes. This proves existence and independence. Positivity and mixed mass pass from the approximants.

Finally, include one further finite-energy input $u_0$ in an arbitrary Kähler class. After the same normalization all inputs are nonpositive. Their sum $w = \sum_{i=0}^n u_i$ has finite energy in $\sum_{i=0}^n \alpha_i$ and satisfies $w \le u_0 \le 0$. The diagonal measure of $w$ dominates $n!$ times the mixed measure of the last $n$ inputs by positive expansion on the approximants and weak convergence. Since $w$ is integrable against its diagonal measure by Theorem 23.3.1, $u_0$ is integrable against the mixed measure. □

**Corollary 23.3.1** *In the setting of Theorem 22.5.1, the extended algebraic and cohomological energies agree, and the finite-energy classes coincide. The finite-energy mixed measures agree for any tuple of real ample numerical classes and their corresponding Kähler classes.*

***Proof*** Corollary 23.1.1 identifies the decreasing cone with the algebraic psh cone. In the definitions of extended energy the continuous majorants are the same, and their energies agree by Theorem 22.5.1. This gives equality of their infima, including the value $-\infty$. For the mixed measures, choose the simultaneous decreasing model approximants in the respective ample classes from Section 17.7. They are semipositive cohomological model potentials. Their model measures agree by Corollary 21.8.1. The weak limits given by Theorems 17.8.1 and 23.3.2 therefore agree. □

## 23.4 Obstacle volume and orthogonality

For a rational model obstacle we compute the energy of the envelope as the volume of a class on a smooth model. Comparing this volume with the top degrees of semipositive models below the obstacle gives the contact property, which then passes to continuous obstacles by uniform approximation.

Fix a Kähler class $\alpha$ on $X$, represented by $\omega$, and put $N = n + 1$ and $V = \int_X \alpha^n$. We write top intersection numbers as scalars. We use the volume of a big class on a compact Kähler manifold, defined as the non-pluripolar top mass of a current with minimal singularities. This is the largest non-pluripolar top mass in the class. Volume is increasing under addition of a pseudoeffective class, is continuous under decreasing Kähler perturbations, and equals the top intersection for nef classes [Xia26a, Definition 15.8.1 and Proposition 15.8.2].

All modifications used to compute a volume in this section are isomorphisms off the central fibre, so the following lemma applies to them.

**Lemma 23.4.1** *Let $p : W' \to W$ be a modification of connected compact Kähler manifolds that restricts to a biholomorphism $W' \setminus p^{-1}(Z) \to W \setminus Z$ for a proper analytic subset $Z \subset W$. If $\beta$ is a big class on $W$, then $p^*\beta$ is big and* $\mathrm{vol}(p^*\beta) = \mathrm{vol}(\beta)$.

***Proof*** The pullback is big by [Xia26a, Proposition 14.4.7]. Fix a smooth representative of $\beta$. A positive current in $p^*\beta$ has a global potential for the pulled-back smooth background. On $W \setminus Z$ this gives a potential downstairs bounded above locally near $Z$, since the potential upstairs is bounded above on the compact manifold $W'$. It extends across $Z$ by the bounded-above extension theorem [Xia26a, Theorem 8.4.1], applied after adding a smooth local potential of the background. The pullback of the extension equals the original potential off the proper analytic subset $p^{-1}(Z)$, hence everywhere by uniqueness of the psh representative. The non-pluripolar top masses of the two currents agree off these analytic sets, which they do not charge. Conversely the pullback of a positive current in $\beta$ is a positive current in $p^*\beta$ with the same non-pluripolar top mass. Since the volume of a big class is its largest non-pluripolar top mass, the two volumes agree. □

**Lemma 23.4.2** *If $h$ is a rational model function semipositive on a smooth refinement and $h \geq a > 0$, its class $\rho^*\alpha + c_1(D_h)$ is nef there.*

***Proof*** Set $D = D_h$, so $D \geq aW_0$ by (22.1). Choose a smooth curvature form $\gamma$ for $D$ and the logarithmic section norm $\ell_D$ with $dd^c\ell_D = [D] - \gamma$. Use a Cartier multiple and divide its norm weight by the multiple. Then $\ell_D$ tends uniformly to $-\infty$ near $W_0$, by the coefficient inequality and the local smooth norm of the fibre.

For every $\varepsilon > 0$ the class with background $\rho^*\omega + \gamma + \varepsilon\Omega$, with $\Omega$ a Kähler form on the refinement, has Kähler restrictions to the central components by Definition 22.1.1 with perturbation $\varepsilon/2$. By Lemma D.5.1 it has a smooth potential $\psi$ giving positivity near the whole fibre. Off the fibre $\ell_D$ gives the positive form $\rho^*\omega + \varepsilon\Omega$. Take a small tube $\pi^{-1}(\{|t| < r\})$ in the positive neighborhood of $\psi$. Subtract a constant from $\psi$ so that it is strictly below $\ell_D$ on an inner collar of the boundary of that tube. In the tube take their smooth regularized maximum with sufficiently small smoothing width. It agrees with $\psi$ minus the constant near the fibre and with $\ell_D$ on the collar. Gluing it to $\ell_D$ outside gives a smooth potential for a Kähler form in the perturbed class. Let $\varepsilon$ decrease to zero. □

**Lemma 23.4.3** *Let $f$ be a rational model function with $f \geq 1$ and let $D$ represent it on a fixed smooth refinement $W$. The class $\beta = \rho^*\alpha + c_1(D)$ on $W$ is big. There are smooth refinements $p_j : W_j \to W$ and rational model functions $g_j$ represented on $W_j$ with the following properties. Writing $D_j$ for their divisors, $\rho_j = \rho \circ p_j$ and $F_j = p_j^*D - D_j$, one has*

$$0 \leq F_j \leq C(W_j)_0, \qquad A_j = \rho_j^*\alpha + c_1(D_j), \qquad \lim_j A_j^N = \mathrm{vol}(\beta), \tag{23.2}$$

*where $C$ is fixed and the classes $A_j$ admit globally smooth semipositive representatives.*

***Proof*** By the second assertion of Lemma D.4.1, there is an effective rational exceptional divisor $F$ on $W$ such that $\rho^*\alpha + c_1(W_0) - c_1(F)$ is Kähler. Start from the product Kähler class with coefficient one on the parameter class and subtract successively small positive rational multiples of the exceptional divisors. Since $D \geq W_0$, adding the effective divisor $D - W_0 + F$ to this Kähler form gives a Kähler current $T_0$ in $\beta$, smooth off $W_0$. Let $\theta$ be a smooth representative of $\beta$, and take a potential $\psi_0 \leq 0$ for $T_0$ in that background. Thus $V_\theta \geq \psi_0$, where $V_\theta$ is the extremal $\theta$-psh potential of minimal singularities. The function $\psi_0$ is a logarithmic section norm of the effective divisor plus a smooth term. In a neighborhood of $W_0$ it is bounded below by $L \log |t \circ \pi|^2 - K$ for fixed $L, K$, since $D - W_0 + F \leq LW_0$. Here $t$ denotes the parameter near zero.

For $0 < \delta < 1$, the potential $\psi_\delta = (1-\delta)V_\theta + \delta\psi_0$ has a strict curvature lower bound. It is locally bounded off $W_0$ and is at least $\psi_0$. Regularize it as in [Xia26a, Theorem 13.4.5], in a background reduced by a fixed part of its strict lower bound. The regularized potentials $\psi_{\delta,k}$ decrease to $\psi_\delta$ and for large $k$ their currents in $\beta$ dominate $c_\delta\Omega$ for a fixed Kähler form $\Omega$ and $c_\delta > 0$. Their ideals are the unit ideal off $W_0$ by local boundedness of $\psi_\delta$. Resolve each such ideal by a smooth tower over $W_0$, principalizing its product with the fibre ideal so the reduced fibre has strict normal crossings. This uses the first assertion of Lemma D.4.1 and the strictification in the proof of Lemma D.4.2, starting on $W$. On this resolution the pulled-back current is the sum of a smooth form and an effective rational divisor by the smooth remainder in [Xia26a, Definition 13.1.11]. The smooth form dominates the pullback of $c_\delta\Omega$, first off the fibre and then everywhere by continuity. The pole divisor is bounded above by $L$ times the full fibre upstairs. Indeed $\psi_{\delta,k} \geq \psi_0$ and the lower logarithmic bound on $\psi_0$ bounds the generic Lelong number along each component by $L$ times its fibre multiplicity. The smooth residual form has zero generic Lelong numbers, so these numbers are precisely the coefficients of the pole divisor.

Subtract from the residual class a small positive rational exceptional correction to make it Kähler, by the second assertion of Lemma D.4.1 applied on $W$ to $c_\delta\Omega$. On the fixed resolution the correction can be arbitrarily small, by scaling an exceptional correction and the corresponding smooth form. It can be chosen bounded above by the full fibre and with arbitrarily small change in top degree. Thus the resulting Kähler classes have the form $\rho'^*\alpha + c_1(B)$ for rational divisors $B$ with $0 \leq p^*D - B \leq (L+1)W_0'$. They can be chosen with top degrees arbitrarily close to $\mathrm{vol}(\beta)$. In fact on each relatively compact open set off $W_0$ the regularizations decrease and are locally uniformly bounded. Their masses there converge on compactly supported continuous tests by the bounded Bedford–Taylor convergence theorem [Xia26a, Theorem 9.2.5]. Positivity of mixed products shows that the limiting mass dominates $(1-\delta)^N$ times the mass of the minimal current there. This latter current has full mass $\mathrm{vol}(\beta)$ off $W_0$, since its non-pluripolar product does not charge the fibre. Exhaust the complement, then let $\delta$ tend to zero and let the exceptional correction errors tend to zero. The upper degree bound follows from volume monotonicity and Lemma 23.4.1, since the difference from $p^*\beta$ is effective.

Apply Lemma D.6.1 to obtain $B = cW_0' - m^{-1}F_I$ with $F_I$ the effective divisor of a coherent ideal $I$ on the product containing $\mathfrak{t}^a$ with $a \geq 1$. For $|\zeta| = 1$ write $R_\zeta(x,t) = (x, \zeta t)$. Choose $a$ distinct such $\zeta$, including $\zeta = 1$, and let $J$ be the sum of the translates $R_\zeta^* I$. Modulo $\mathfrak{t}^a$, the matrix of these rotations on the first $a$ powers of $t$ is invertible. Thus the sum contains every homogeneous term of each germ of $I$ modulo $\mathfrak{t}^a$, and is generated by these terms and the pure power. In particular $J$ is a flag ideal. This is the finite Taylor argument of Theorem 21.4.1.

Start on the tower carrying $B$ and take a common smooth refinement over $W$ that dominates all the rotated blowup towers and principalizes $J$. This is possible by Lemma D.4.2. On it set $B^\vee = cW_0' - m^{-1}F_J$, with $W_0'$ now denoting the fibre on the common refinement. For each chosen $\zeta$ let $B_\zeta = cW_0' - m^{-1}F_{R_\zeta^* I}$ be the divisor $B$ transported by $R_\zeta$ to the rotated tower and pulled back; thus $B_1$ is the pullback of $B$. Since $D$ comes from flag data, it is unchanged by rotating the ideals that represent it. The coefficient of $B^\vee$ on each prime is the largest coefficient of the $B_\zeta$, by the minimum rule for orders of an ideal sum. Hence the same bounds $0 \leq D - B^\vee \leq (L+1)W_0'$ hold after pullback.

The class for $B^\vee$ admits a smooth semipositive representative on the entire common refinement. To see this, use a common smooth background for that class and write the pulled-back Kähler forms of the rotated classes with smooth potentials. Add the log section norms of the effective differences $B^\vee - B_\zeta$ to these potentials with quotient metrics, as in Lemma 23.1.1. Away from their poles the resulting positive currents are given by the pulled-back smooth semipositive forms. The pole sets have empty common intersection, since the summands generate the unit ideal after dividing by a local generator of $J$. The smooth regularized maximum of this finite family is therefore smooth and gives a semipositive representative, by the same local argument as for two summands. Concretely convolution of the finite maximum with a smooth nonnegative product kernel of small compact support gives a convex increasing operation, commuting with addition of a common constant. It ignores an entry strictly below the largest by more than twice the width. Thus at every pole it uses only the entries smooth nearby. Since $B^\vee \geq B$ after pulling back $B$, its top degree is at least that before rotation. This follows by expanding the difference of powers against the effective divisor and the smooth semipositive forms. It is at most $\mathrm{vol}(\beta)$ by volume monotonicity and Lemma 23.4.1. Taking these flag-model functions for a sequence of approximations proves (23.2). □

**Lemma 23.4.4** *Let $M$ be a connected compact Kähler manifold of dimension $N$. Let $a, \kappa$ be smooth closed semipositive $(1,1)$-forms, let $G$ be an effective real divisor, and choose a smooth representative $\gamma$ of $c_1(G)$ with $-\kappa \leq \gamma \leq \kappa$. Fix $0 < s \leq 1$ such that $[a] - sc_1(G)$ is big and contains a positive current with locally bounded potentials off a proper analytic set containing $G$. Then*

$$\begin{aligned} \mathrm{vol}([a] - sc_1(G)) \;\geq\; & \int_M a^N - Ns \int_{[G]} (a + s\kappa)^{N-1} \\ & - 2^N s^2 \int_M (a+\kappa)^N . \end{aligned} \tag{23.3}$$

***Proof*** We can assume $a$ Kähler by adding a small Kähler form and then using decreasing perturbation continuity of the volume and polynomial continuity of the intersections. Let $g \leq 0$ satisfy $dd^c g + \gamma = [G]$, using the log section norms and the $dd^c$-lemma. It is smooth off $G$. Take $g_r = \chi_r(g)$, with $\chi_r$ a smooth convex increasing rounding of $\max(x, -r)$, equal to a constant well below $-r$ and equal to $x$ well above $-r$. For instance convolve $\max(x, -r)$ with a symmetric smooth probability density on $[-1, 1]$. Then $g_r$ is smooth and decreases to $g$, and

$$S_r = dd^c g_r + \chi_r'(g)\gamma \quad \text{is smooth semipositive.}$$

Off $G$ this follows by differentiating, since $dd^c g = -\gamma$ there and the remaining term is the nonnegative form $\chi_r''(g)\mathrm{d}g \wedge d^c g$. Near $G$, $g_r$ is locally constant. Also $S_r \leq dd^c g_r + \kappa$ and $S_r$ converges as a current to $[G]$, by local integrability of $g$ and dominated convergence for $\chi_r'(g)$.

Let $v_r = P_a(sg_r)$ be the complex envelope in the background $a$. It is bounded and decreases to a nontrivial $a$-psh function $v$. Indeed $sg + w$ is a common competitor, where $w \leq 0$ is a potential of the positive current in the hypothesis in the background $a - s\gamma$. This competitor is locally bounded off the indicated analytic set. Thus $v_r$ are locally uniformly bounded there and their measures converge on continuous tests with support there by bounded Bedford–Taylor convergence [Xia26a, Theorem 9.2.5]. Moreover $v \leq sg$, so the current $a + dd^c v$ has at least the coefficients of $s[G]$ as generic Lelong numbers. By divisorial decomposition [Xia26a, Theorem 14.3.7], subtracting $s[G]$ leaves a positive current in $[a] - sc_1(G)$. Its mass off the analytic set agrees with the mass of $a + dd^c v$.

The measure of $v_r$ has total mass $\int a^N$ and is carried by its contact set with $sg_r$. On that set it has the top-power density of $a + sdd^c g_r$. These assertions follow from the Kähler envelope regularity, concentration and contact formulas [Xia26a, Theorems 16.2.24 and 16.2.5 and Proposition 16.2.9]. The last of these also shows that the bounded form $a + dd^c v_r$ is semipositive almost everywhere. By [Xia26a, Proposition 9.12.3], applied in local charts, the complex Hessians of $v_r$ and $sg_r$ agree almost everywhere on the contact set. Hence there $a + sdd^c g_r$ coincides almost everywhere with the semipositive form $a + dd^c v_r$. Since $-\chi_r'(g)\gamma \leq \kappa$, that density is bounded by $(a + s\kappa + sS_r)^N$. For any continuous cutoff $0 \leq \eta \leq 1$ with compact support off the analytic set, we therefore bound the mass lost by multiplying by $\eta$ by

$$\int_M (1 - \eta)(a + s\kappa)^N + Ns \int_M (a + s\kappa)^{N-1} \wedge S_r + 2^N s^2 \int_M (a + \kappa)^N.$$

For the terms of degree at least two in $S_r$, use $0 \leq S_r \leq dd^c g_r + \kappa$, expand the positive products and use Stokes for the larger closed form. Each mixed integral is then at most $\int_M (a + \kappa)^N$, since $s \leq 1$, and the sum of binomial coefficients is at most $2^N$. Let $r$ tend to infinity. Local convergence gives the cutoff mass of $a + dd^c v$, which is at most $\mathrm{vol}([a] - sc_1(G))$. The first-order term converges to the integral on $G$. Finally exhaust the complement of the analytic set with the cutoffs. The zeroth term tends to zero, since the smooth measure gives that set zero mass. This proves (23.3). □

**Theorem 23.4.1** *For every real continuous obstacle $f$ on $X^{\sqsupset}$, the measure of $u = P_\alpha(f)$ is carried by the contact set $\{u = f\}$. In particular*

$$\int_{X^{\sqsupset}} (f - u)\, \mathrm{d}\,\mathrm{MA}_X((\alpha, u)^n) = 0. \tag{23.4}$$

*If $f \geq 1$ is a rational model function represented by $D$ on $W$, then*

$$(n+1)E_\alpha(u) = \mathrm{vol}(\rho^*\alpha + c_1(D)). \tag{23.5}$$

***Proof*** Start with a rational model obstacle, shifted by a rational constant to arrange $f \geq 1$. A constant shift translates the envelope and preserves the measure because the fibre class vanishes on central components. Thus it preserves the integral in (23.4).

Choose a fixed smooth refinement $W$ representing $f$ by $D$, and let $\theta$ represent $\beta = \rho^*\alpha + c_1(D)$. Use Lemma 23.4.3 with smooth semipositive forms $\omega_j$ in $A_j$, and let $\eta_j = \pi_j^*\omega_{\mathrm{FS}}$, with integral one on the parameter line. Write $M_j$ for the full central fibre $(W_j)_0$, and put $G_j = CM_j - F_j \geq 0$. The class $c_1(G_j)$ is represented by $\gamma_j = C\eta_j - p_j^*\theta + \omega_j$. Fix a Kähler form $\Omega$ on $W$. For a fixed large $C_0$, $\kappa_j = \omega_j + C_0 p_j^*\Omega$ bounds both $\gamma_j$ and $-\gamma_j$. For $0 < s \leq 1$ apply Lemma 23.4.4 with $a_j = \omega_j + sC\eta_j$, $\kappa_j$ and $G_j$. The class $[a_j] - sc_1(G_j) = A_j + sc_1(F_j)$ is big, since it dominates the pullback of the Kähler class before the rotation step in the proof of Lemma 23.4.3. A pullback of a Kähler class by a smooth blowup tower contains a Kähler current by the exceptional correction. The current $\omega_j + s[F_j]$ is smooth off $M_j$.

We record why the quadratic errors are uniform. A fixed large Kähler class $H$ on $W$ can be chosen with $H - \beta$ Kähler. Then $p_j^*H - A_j = p_j^*(H - \beta) + c_1(F_j)$ is pseudoeffective. Taking $H$ larger if needed, all the nef factors $[\omega_j]$, $[a_j]$ and $[\kappa_j]$ are bounded above in this order by a fixed multiple of $p_j^*H$. Their products have uniformly bounded top intersections. Indeed replacing a nef factor by a larger nef class increases the intersection, by positivity against the pseudoeffective difference. This uses approximation by smooth Kähler forms and integration against a positive representative of the difference. For an integral on $G_j$ of a product of these smooth semipositive forms, replace $G_j$ by $CM_j$ using effectivity. The full fibre class has smooth semipositive representative $\eta_j$, so the same bound applies. Expanding powers thus gives

$$\int_{[G_j]} (a_j + s\kappa_j)^{N-1} \leq \int_{[G_j]} \omega_j^{N-1} + C_1 s$$

for a fixed $C_1$. Since $\eta_j^2 = 0$, and $\int_{[M_j]} \omega_j^{N-1} = V$ by comparison with the noncentral fibre, the linear term in (23.3) simplifies to

$$\int_{W_j} a_j^N - Ns \int_{[G_j]} \omega_j^{N-1} = A_j^N + Ns \int_{[F_j]} \omega_j^{N-1}.$$

Here the equality of fibre classes is used with all their multiplicities. The volume on the left of (23.3) is at most $\mathrm{vol}(\beta)$ by volume monotonicity and Lemma 23.4.1, since its class differs from $p_j^*\beta$ by the subtraction of $(1 - s)c_1(F_j)$. Hence

$$Ns \int_{[F_j]} \omega_j^{N-1} \leq \mathrm{vol}(\beta) - A_j^N + C_2 s^2$$

with $C_2$ fixed. First let $j$ tend to infinity for fixed $s$, using (23.2), and then let $s$ decrease to zero. The intersection on $F_j$ tends to zero. By the model measure formula and the coefficient formula on smooth refinements, it is precisely $\int (f - g_j)\, \mathrm{d}\mu_{g_j}$, where $\mu_h = \mathrm{MA}_X((\alpha, h)^n)$.

Now take the uniform rational approximants $r_j \leq u$ from Theorem 23.1.1. Since $u \geq 1$, we may assume $r_j \geq 1/2$. Since $F_j \geq 0$, the coefficient formula on smooth refinements (Section 22.6) gives $g_j(v_E) \leq f(v_E)$ for every central component $E$ of $W_j$, where $f$ is represented by $p_j^* D$. As $g_j$ is semipositive on $W_j$, Theorem 22.6.1 gives $g_j \leq f$ on $X^{\sqsupset}$, and so $g_j \leq P_\alpha(f) = u$. Put $h_j = \max(r_j, g_j)$. They are rational semipositive model functions and converge uniformly to $u$, since $r_j \leq h_j \leq u$. Since $h_j \geq 1/2$, their classes are globally nef by Lemma 23.4.2. On a common smooth refinement their differences from the pullback of $\beta$ are the negatives of effective divisors by $h_j \leq f$. Consequently

$$A_j^N = NE_\alpha(g_j) \leq NE_\alpha(h_j) \leq \mathrm{vol}(\beta).$$

The first equality uses intersection invariance. The first inequality is energy monotonicity and the last uses the nef volume formula, volume monotonicity and Lemma 23.4.1. Uniform energy convergence gives (23.5).

The difference $E_\alpha(h_j) - E_\alpha(g_j)$ tends to zero. Apply Lemma 23.3.1 to this ordered pair and use the density of model tests and the fixed mass $V$. This shows that $\mu_{g_j} - \mu_{h_j}$ tends weakly to zero, while $\mu_{h_j} \to \mu_u$ by uniform convergence. The continuous function $f - u$ satisfies $0 \leq f - u \leq f - g_j$. Testing with it gives (23.4) in this case.

For continuous $f$, approximate it uniformly by rational model obstacles. The envelopes converge uniformly by Theorem 23.1.1 and their measures converge weakly by Theorem 22.3.1. The corresponding gaps converge uniformly, so the integrals converge by the fixed mass bound. A nonnegative continuous function with zero integral vanishes almost everywhere for a positive measure, which proves the contact assertion. □

**Corollary 23.4.1** *For $f, h \in C(X^{\sqsupset}, \mathbb{R})$, the derivative of the obstacle energy in the direction $h$ is*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E_\alpha(P_\alpha(f + th)) = \int_{X^{\sqsupset}} h\, \mathrm{d}\,\mathrm{MA}_X((\alpha, P_\alpha(f))^n). \tag{23.6}$$

***Proof*** Write $u_t = P_\alpha(f + th)$. Concavity and the derivative along a segment in Theorem 22.4.1 give

$$\int (u_t - u_0)\, \mathrm{d}\mu_{u_t} \leq E_\alpha(u_t) - E_\alpha(u_0) \leq \int (u_t - u_0)\, \mathrm{d}\mu_{u_0}.$$

On the support of $\mu_{u_t}$ the equality $u_t = f + th$ and $u_0 \le f$ give a lower bound by $t\int h\,\mathrm{d}\mu_{u_t}$. On the support of $\mu_{u_0}$ the equality $u_0 = f$ and $u_t \le f + th$ give an upper bound by $t\int h\,\mathrm{d}\mu_{u_0}$. The potentials $u_t$ converge uniformly to $u_0$, so their measures converge weakly. Divide the bounds by $t$, reversing them when $t < 0$, and take the limit. □

**Corollary 23.4.2** *Let $f$, $h$ be real model functions with* $\inf f > 1$. *Represent them by $D, G$ on a common smooth refinement $W$ and put $\beta = \rho^*\alpha + c_1(D)$. Then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{vol}\big(\beta + tc_1(G)\big) = (n+1)\int_{X^{\sqsupset}} h\,\mathrm{d}\,\mathrm{MA}_X((\alpha, P_\alpha(f))^n). \tag{23.7}$$

*In particular the volume in the fibre direction $G = W_0$ is affine near zero, with slope $(n+1)V$.*

***Proof*** On the fixed model every class in question is big for small $t$, by the effective-divisor construction in Lemma 23.4.3, which uses only $D + tG \ge W_0$. Volume is continuous in the big cone. For the needed continuity, let $\beta'$ be a big class and choose $a > 0$ so that $\beta' - a[\Omega]$ is pseudoeffective for a fixed Kähler form $\Omega$. Such an $a$ exists by subtracting from a Kähler current. A class $\gamma$ sufficiently close to $\beta'$ lies between $\beta' - \varepsilon[\Omega]$ and $\beta' + \varepsilon[\Omega]$ in the pseudoeffective order, by bounding the smooth representatives of coefficients in a finite-dimensional basis. These two bounds lie respectively above $(1 - \varepsilon/a)\beta'$ and below $(1 + \varepsilon/a)\beta'$ in the same order. Volume monotonicity and homogeneity give continuity.

Approximate the coefficients of the finite flag expressions of $D + tG$ by rational ones on $W$. Their model functions converge uniformly to $f + th$, and they are eventually at least one. Both sides of (23.5) pass to the limit by volume continuity and uniform continuity of the envelope and energy. Apply Corollary 23.4.1 to obtain (23.7). For $G = W_0$, the model function is the constant one. Translation of the envelope and the energy identity of Theorem 22.4.1 give the affine formula on the whole small interval. □

**Corollary 23.4.3** *For a real model obstacle $f$ represented on a smooth refinement $W$, the envelope measure is supported on the finite set of normalized base restrictions $v_E$ of the central components of $W$. At any one of these points $v$, its mass can be computed by (23.7), using a real model function $h$ that is one at $v$ and zero at the other points of the set, after shifting $f$ by a constant to satisfy* $\inf f > 1$.

***Proof*** Let $S$ be the finite set in question. For every nonnegative continuous $k$ that vanishes on $S$, the competitors $w \le f + k$ satisfy $w(v_E) \le f(v_E)$. Since $f$ is a model function, the finite test (22.2) gives $w \le f$ on $X^{\sqsupset}$. Hence $P_\alpha(f + k) = P_\alpha(f) = u$. Applying (23.4) to both obstacles gives $\int k\,\mathrm{d}\mu_u = 0$. Every compact subset of the complement of $S$ admits such a test equal to one on it, by normality of the compact Hausdorff space. Inner regularity proves that $\mu_u$ is supported on $S$.

To construct $h$, start with continuous functions whose value vectors on $S$ are the standard basis vectors. Approximate each uniformly by real model functions using

Proposition 21.5.1. The matrix of the resulting value vectors is arbitrarily close to the identity, so is invertible for sufficiently good approximations. Linear combinations give all the desired value vectors. Use a common refinement representing $f$ and $h$. Integration of $h$ against $\mu_u$ is exactly $\mu_u\{v\}$, and shifting $f$ by a constant preserves $\mu_u$. The mass formula follows. □

## 23.5 Weak compactness in a Kähler class

Fix a Kähler class $\alpha$. We normalize $u \in \mathcal{D}_\alpha$ by $u(v_0) = 0$, where $v_0$ is the trivial valuation. By Theorem 22.6.1 on the product model, $u(v_0) = \sup u$. We use the weak topology of pointwise convergence at the divisorial tests $v_E$ of Section 22.6. These tests determine the order on $\mathcal{D}_\alpha$. Indeed, if the test values of $u$ are bounded above by those of $v$, each continuous majorant of $v$ bounds $u$ by Theorem 22.6.1. Taking the infimum of the continuous semipositive majorants of $v$ proves $u \le v$.

**Lemma 23.5.1** *At each fixed test $v_E$ there is a lower bound for the values of all normalized potentials in $\mathcal{D}_\alpha$.*

***Proof*** First let $u$ be a normalized semipositive model potential. Fix a smooth refinement $W$ containing the chosen component, and a Kähler form $\Omega$ on $W$. Take a common smooth refinement $p : W' \to W$ where $u$ is represented by $D_u$ and is semipositive. For any central component $E$ on $W$, the pulled-back divisor $p^*E$ is effective and supported on the central fibre. Its mixed degree with $\rho'^*\alpha + c_1(D_u)$ and $p^*[\Omega]^{n-1}$ is nonnegative by semipositivity along central components. In pushing the degree down to $W$, the divisor $D_u$ contributes only the coefficients on strict transforms of the central components: the other components map to sets of dimension $< n$, where the top-degree pullbacks have zero integral. Write $u_E = u(v_E)$, $a_E = \int_E \rho^*\alpha\,[\Omega]^{n-1}$, and $m_{EH} = \int_{E\cap H} \Omega^{n-1}$ for $H \neq E$. With $W_0 = \sum_H b_H H$, the row identity from Proposition 22.2.1 gives

$$\sum_{H\neq E} b_H m_{EH}(u_E - u_H) \le a_E.$$

Here $m_{EH} > 0$ precisely when the two components meet, by positivity on their smooth intersections.

The graph whose vertices are the central components of $W$, with an edge between two components that meet, is connected. The initial fibre $X \times \{0\}$ is connected. Each smooth blowup is proper with connected fibres, so the preimage of a compact connected set is connected: a separation would give disjoint closed images covering the set because each fibre is connected. This proves connectedness of $W_0$, hence of its finite component graph. The strict transform of the initial fibre has value $u_{E_0} = u(v_0) = 0$. If $u_E \ge -L$ at a component and $H$ meets it, then every other difference $u_E - u_{H'}$ is at least $-L$, since $u_{H'} \le 0$. Consequently

$$u_H \geq -L - \frac{a_E + L \sum_{H' \neq E,H} b_{H'} m_{EH'}}{b_H m_{EH}}.$$

Following paths from $E_0$ gives all the claimed bounds without dependence on $u$. Uniform approximation, with normalization at $v_0$, gives the bounds in $C_\alpha$. Decreasing continuous semipositive approximation and convergence of the values at $v_0$ give them on $\mathcal{D}_\alpha$. □

**Theorem 23.5.1** *The normalized cone in $\mathcal{D}_\alpha$ is weakly compact and its normalized rational semipositive models are weakly dense. If a net of normalized potentials converges weakly to $u$, then*

$$E_\alpha(u) \geq \limsup_i E_\alpha(u_i).$$

***Proof*** We first construct a potential with prescribed upper values on the tests. Let $k$ be a positive integer and $h_i$ a nonempty family of rational semipositive models between $-k$ and zero. Use all their approximate rays from Proposition 22.7.2 and include the ray $k \log |t|^2$. Their upper regularized supremum has test slopes $\sup_i h_i(v_E)$ by Lemma 22.7.1. It is bounded below by $k \log |t|^2$, so the transform in Proposition 22.7.1 gives a potential in $\mathcal{D}_\alpha$ with these values.

We also use closure under nested decreasing limits. If $w_\lambda$ decreases in $\mathcal{D}_\alpha$ to $w$ with $w(v_0) > -\infty$, consider all $g \in C_\alpha$ majorizing some $w_\lambda$. Their infimum is $w$ by the construction of the cone. For any two such majorants, their minimum majorizes a common later $w_\lambda$. The continuous envelope of that minimum is again in the family by Theorem 23.1.1, and lies below both. The family is therefore directed downwards and gives $w \in \mathcal{D}_\alpha$. Moreover the energies decrease to $E_\alpha(w)$. If $g \in C_\alpha$ majorizes $w$, the increasing open sets $\{w_\lambda < g + \varepsilon\}$ cover the compact space $X^\beth$. Thus eventually $w_\lambda \leq g + \varepsilon$. Monotonicity and translation of the energy, followed by the infimum over such $g$, give the upper bound for the decreasing energy limit. The lower bound is monotonicity.

By Lemma 23.5.1, the normalized values on tests belong to a product of compact intervals. Suppose a net of normalized rational models $u_i$ has a pointwise limit in that product. For each $i, k$ take the potential $w_{i,k}$ constructed above from the clipped models $\max(u_j, -k)$ for $j \geq i$, which are rational semipositive by Lemma 23.1.1. On the tests its values are $\sup_{j \geq i} \max(u_j(v_E), -k)$. By order determination these potentials decrease with $(i, k)$ and majorize each $u_j$ on the indicated tail. Their values at $v_0$ are zero, so their decreasing limit is in the normalized cone. On each test its value is the original pointwise limit. This proves that the closure of the normalized rational models in the compact product is contained in the cone. Conversely every normalized potential is weakly approximated by normalizing its continuous decreasing majorants and then using uniform rational approximation of each continuous majorant from Theorem 23.1.1. The normalization of a rational model is rational, by the coefficient formula on the product component. This proves compactness and density.

For the energy assertion, $E_\alpha(w_{i,k})$ is at least $\sup_{j \geq i} E_\alpha(u_j)$. Decreasing energy continuity for the limit of $w_{i,k}$ gives the upper semicontinuity along model nets. For

a net of general normalized potentials $u_i$, index approximating models by $i$, finite sets of tests, and positive errors $\varepsilon$ tending to zero. For each index choose a normalized rational model whose values on the specified tests differ by at most $\varepsilon$ from those of $u_i$, and whose energy is at least $E_\alpha(u_i) - \varepsilon$. Decreasing continuous approximation gives such finite accuracy and an energy lower bound by monotonicity before normalization. The normalization constant can be made arbitrarily small as the continuous values at $v_0$ converge to zero. Rational uniform approximation gives the same accuracy with a rational model. The resulting models converge weakly to the limit of $u_i$. Their energy upper semicontinuity gives the assertion for the original net. □

## 23.6 The variational equation in a Kähler class

Continue with the fixed Kähler class $\alpha$, and write $\mathcal{E}^1_\alpha$ for the finite-energy members of $\mathcal{D}_\alpha$. Set $V = \int_X \alpha^n$, $e = E_\alpha/V$, and $\nu_u = \mathrm{MA}_X((\alpha, u)^n)/V$. Thus $\nu_u$ is a probability measure and $\nu_0 = \delta_{v_0}$ by the product-model formula. For a Radon probability measure $\nu$ put

$$e^\vee(\nu) = \sup_{u \in \mathcal{E}^1_\alpha} \left( e(u) - \int u \, \mathrm{d}\nu \right). \tag{23.8}$$

A measure has *finite dual energy* when this supremum is finite. It then integrates every finite-energy potential, since such potentials are bounded above and an integral of $-\infty$ would give $+\infty$ in (23.8).

The supremum may be taken over normalized rational semipositive models. The expression is invariant under constants by the energy translation rule and the probability normalization. Decreasing continuous approximation gives convergence of energy and integrals, using Theorem 4.2.1, with the convention $e(u) - (-\infty) = +\infty$. Uniform rational approximation of continuous semipositive potentials then gives the claim. In particular every test point $v_E$ gives a finite-dual-energy Dirac mass: normalized potentials have $e \le 0$ and their values at $v_E$ are bounded below by Lemma 23.5.1. Finite mixtures of these Dirac masses also have finite dual energy by (23.8).

**Proposition 23.6.1** *For $u \in \mathcal{E}^1_\alpha$ and any Radon probability measure $\nu$, one has $\nu = \nu_u$ if and only if $\nu$ has finite dual energy and $u$ attains the supremum in* (23.8).

***Proof*** First extend the supporting inequality from continuous potentials to $\mathcal{E}^1_\alpha$. Take a decreasing continuous approximation $u_i$ to $u$. For $j \ge i$ the mixed energy difference gives

$$0 \le \int (u_i - u_j) \, \mathrm{d}\nu_{u_j} \le (n+1)(e(u_i) - e(u_j)).$$

Here the upper bound retains the term with all $n$ slots equal to $u_j$. With $i$ fixed, the integrals of $u_i$ against $\nu_{u_j}$ converge to its integral against $\nu_u$ by Theorem 23.3.1. Consequently the upper and lower limits of $\int u_j \, \mathrm{d}\nu_{u_j}$ are bounded above by $\int u_i \, \mathrm{d}\nu_u$ and below by that quantity minus $(n+1)(e(u_i) - e(u))$, respectively. Letting $i$ tend to its

limit proves convergence to $\int u\,\mathrm{d}\nu_u$. For continuous semipositive $\nu$, the supporting inequality at $u_j$ follows from concavity and Theorem 22.4.1. Take its limit by the integral convergence just proved. Decreasing continuous approximation of $\nu \in \mathcal{E}^1_\alpha$ now gives

$$e(\nu) - e(u) \le \int (\nu - u)\,\mathrm{d}\nu_u. \tag{23.9}$$

All tests are integrable by Theorem 23.3.1. This proves the forward implication and gives $e^\vee(\nu_u) = e(u) - \int u\,\mathrm{d}\nu_u$.

For the reverse implication we need a first variation at $u$. For a real model test $h$ take $u_i \downarrow u$ as before and put $u_i(t) = P_\alpha(u_i + th)$. These continuous potentials decrease for fixed $t$ and have limits $u(t) \in \mathcal{E}^1_\alpha$, since

$$u_i - \|h\|_\infty |t| \le u_i(t) \le u_i + \|h\|_\infty |t|.$$

Indeed $P_\alpha(u_i) = u_i$ and the envelope is nonexpansive. The same estimate shows $u(0) = u$. Use Lemma 23.3.1 for the two comparisons with the common upper potential $u_i + \|h\|_\infty |t|$. The ordered energy gaps are bounded by $2V\|h\|_\infty |t|$ by the energy Lipschitz estimate. The measure of the common upper potential equals $\nu_{u_i}$ by translation invariance. Hence

$$\left| \int h\,\mathrm{d}(\nu_{u_i(t)} - \nu_{u_i}) \right| \le C_h |t|^{1/2}.$$

The constant is independent of $i$ and $u_i$. The continuous first variation of Corollary 23.4.1, applied at each parameter $t$, has continuous derivative by uniform continuity of the envelope and measures. Integrating it and using the preceding estimate yields

$$\left| e(u_i(t)) - e(u_i) - t \int h\,\mathrm{d}\nu_{u_i} \right| \le C'_h |t|^{3/2}. \tag{23.10}$$

Decreasing continuity of the energies and weak convergence of $\nu_{u_i}$ give the same formula for $u(t)$ and $u$. If $u$ maximizes (23.8), the obstacle bound gives

$$e(u_i(t)) \le e^\vee(\nu) + \int (u_i + th)\,\mathrm{d}\nu.$$

Let $i$ tend to its limit, and use the maximality identity for $u$. The two signs of $t$ in (23.10) give $\int h\,\mathrm{d}\nu_u = \int h\,\mathrm{d}\nu$. Model test density proves $\nu_u = \nu$. □

### 23.6.1 Testing along maximizing nets

Fix $\nu$ with finite dual energy and write

$$\Delta_\nu(u) = e^\vee(\nu) - e(u) + \int u\,\mathrm{d}\nu.$$

We call $\Delta_\nu(u) \geq 0$ the *deficit* of $u \in \mathcal{E}^1_\alpha$, and call a net $u_i$ in $\mathcal{E}^1_\alpha$ *maximizing* for $\nu$ when $\Delta_\nu(u_i) \to 0$. For normalized potentials write $j(u) = -e(u) \geq 0$. On rational semipositive models set $P_u = (\alpha, u)$. For a tuple $\Gamma$ of $n-1$ semipositive background pairs define $\|h\|^2_\Gamma = -V^{-1}\langle(0,h),(0,h),\Gamma\rangle_X$ for a real model function $h$. These squares are nonnegative by the vertical intersection identity and their bilinear forms satisfy Cauchy–Schwarz. Define

$$I(u,v) = \int (u-v)\,\mathrm{d}(\nu_v - \nu_u) = \sum_{\ell=0}^{n-1} \|u-v\|^2_{P_u^\ell P_v^{n-1-\ell}}$$

on models, by the difference of powers.

**Lemma 23.6.1** *A maximizing net of normalized rational semipositive models has bounded $j$ on a tail. For any two normalized rational semipositive model potentials $u, v$ in the class,*

$$I(u,v) \leq C_n(\Delta_\nu(u) + \Delta_\nu(v)).$$

*For every normalized rational semipositive model $a$,*

$$\left|\int a\,\mathrm{d}(\nu_u - \nu_v)\right| \leq C_n\big(j(a) + j(u) + j(v)\big)^{1/2} I(u,v)^{1/2}.$$

***Proof*** For $0 \leq t \leq 1$, expand the mixed energy difference of $tu$ from zero by $P_{tu} = (1-t)P_0 + tP_u$. The term integrated against the product measure $\nu_0$ vanishes since $u(v_0) = 0$. Each other term has at least two factors $t$: one from the integrand $tu$ and one from the occurrence of $P_u$. All the integrals of $u$ are nonpositive. The sum of their absolute values, before the scalar coefficients, is bounded by a dimension multiple of $j(u)$: the mixed difference from zero at $t = 1$ contains every mixed term with $P_u$ and $P_0$, and the expansion has only a bounded number of repetitions. Thus $e(tu) \geq -C_n t^2 j(u)$. Using $tu$ as a dual test with $t = (1 + j(u))^{-1/2}$ gives

$$-\int u\,\mathrm{d}\nu \leq (e^\vee(\nu) + C_n)(1 + j(u))^{1/2}.$$

Hence $e(u) - \int u\,\mathrm{d}\nu$ tends to $-\infty$ as $j(u) \to \infty$, proving the tail bound.

For the second assertion put $g(t) = e((1-t)u + tv)$. The vertical second derivative identity in the proof of Theorem 22.4.1 writes $-g''(t)$ as the sum

$$n\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (1-t)^\ell t^{n-1-\ell} \|u-v\|^2_{P_u^\ell P_v^{n-1-\ell}}.$$

The midpoint gap $g(1/2) - (g(0)+g(1))/2$ is $\frac{1}{2}\int_0^1 \min(t, 1-t)(-g''(t))\,\mathrm{d}t$, by integration by parts on the two half intervals. Each of the finitely many basis polynomials has strictly positive integral against the kernel. The gap is therefore at least $c_n I(u,v)$ for a positive dimension constant $c_n$. The integral against $\nu$ is affine on the segment, so the midpoint gap is at most $(\Delta_\nu(u) + \Delta_\nu(v))/2$. This yields the second assertion.

For the third assertion, telescope the measure difference into pairings of $(0, a)$ and $(0, u - v)$ with backgrounds $\Gamma = P_u^{\ell} P_v^{n-1-\ell}$. By Cauchy–Schwarz it suffices to bound each $\|a\|_{\Gamma}^2$. Its definition and the integration formula give

$$\|a\|_{\Gamma}^2 = V^{-1} \int (-a)\, \mathrm{d}\,\mathrm{MA}_X(P_a, \Gamma) - V^{-1} \int (-a)\, \mathrm{d}\,\mathrm{MA}_X(P_0, \Gamma).$$

The second integral is nonnegative. Put $w = (a + u + v)/3$, so $-a \le -3w$. Expansion gives $\nu_w \ge 3^{-n} V^{-1}\, \mathrm{MA}_X(P_a, \Gamma)$. The mixed energy difference from zero gives $\int (-w)\, \mathrm{d}\nu_w \le (n+1) j(w)$. Finally concavity bounds $j(w)$ by the average of $j(a), j(u), j(v)$. This bounds the first square. Sum the Cauchy–Schwarz estimates, using the formula for $I(u, v)$ to bound the sum of squares for the second slot. □

**Corollary 23.6.1** *A Radon probability measure has finite dual energy if and only if it integrates every member of* $\mathcal{E}_{\alpha}^1$.

***Proof*** The forward implication follows from the definition. For the reverse implication the integrals $\int (-u)\, \mathrm{d}\nu$ are bounded on the normalized $j \le 1$ ball. Otherwise choose $u_k$ on that ball with integrals at least $2^k$. The finite sums $w_m = \sum_{k=1}^{m} 2^{-k} u_k$ are normalized and have $j(w_m) \le 1$ by concavity, assigning the leftover coefficient to zero. Convexity and concavity on the finite-energy cone follow by simultaneous decreasing continuous approximation and the continuous energy inequality. The sums decrease because $u_k \le 0$. Their limit belongs to $\mathcal{E}_{\alpha}^1$ by the nested decreasing-limit argument in the proof of Theorem 23.5.1, with energy bounded below. Its negative integral is infinite by monotone convergence, a contradiction.

Let $A$ bound the negative integrals on the ball. The scaling inequality $j(tu) \le C_n t^2 j(u)$ in the proof of Lemma 23.6.1 applies to normalized finite-energy potentials by approximation, as the energy of each scaled continuous approximant converges by decreasing continuity before normalization. Take $t = (1 + C_n j(u))^{-1/2}$ to get $\int (-u)\, \mathrm{d}\nu \le A\sqrt{1 + C_n j(u)}$. Subtracting $j(u)$ gives an upper bound uniform in $u$, which proves finite dual energy. □

**Proposition 23.6.2** *For a maximizing net* $u_i \in \mathcal{E}_{\alpha}^1$ *for a measure* $\nu$ *of finite dual energy, the measures* $\nu_{u_i}$ *converge weakly to* $\nu$. *For* $B, R, S \ge 0$ *there is a constant* $C_{B,R,S}$ *such that if* $e^{\vee}(\nu) \le B$, *then*

$$\left| \int a\, \mathrm{d}(\nu_q - \nu) \right| \le C_{B,R,S} \Delta_{\nu}(q)^{1/2} \tag{23.11}$$

*for normalized* $a, q \in \mathcal{E}_{\alpha}^1$ *with* $j(a) \le R$ *and* $j(q) \le S$.

***Proof*** Apply (23.10) to each base potential $u_i$, using its decreasing-limit version when the base is singular. The variation lies below $u_i + th$ by its construction. Its dual bound therefore gives, for $t > 0$ and either test $h$ or $-h$,

$$\left| \int h\, \mathrm{d}(\nu_{u_i} - \nu) \right| \le C_h' t^{1/2} + t^{-1} \Delta_{\nu}(u_i).$$

First take the net limit, then $t \downarrow 0$. Model density and the probability normalization give weak convergence.

For (23.11) first take $a, q$ to be normalized rational models and choose a maximizing net of normalized rational models $u_i$ for $\nu$. Its tail satisfies $j(u_i) \le T_B$ for a constant depending only on $B$ and $n$. Indeed the proof of Lemma 23.6.1 gives an upper bound $-j + (B + C_n)\sqrt{1 + j}$ for the objective, which is at least $-1$ on a tail since $e^\vee(\nu) \ge 0$. The two model comparison inequalities of that lemma give

$$\left| \int a \, \mathrm{d}(\nu_q - \nu_{u_i}) \right| \le C\big(\Delta_\nu(q) + \Delta_\nu(u_i)\big)^{1/2},$$

where $C$ depends only on $B, R, S, n$. The weak limit just proved gives the desired bound.

Continuous normalized semipositive tests are uniform limits of normalized rational models. Energy and measure continuity give the bound for continuous $a, q$, increasing the energy radii by one to include approximants. Next take $q \in \mathcal{E}^1_\alpha$ by decreasing continuous approximation, for continuous $a$. Normalization at $v_0$ changes the constants by amounts tending to zero. Energies and integrals against $\nu$ converge, and the measures of the approximants converge by Theorem 23.3.1. Finally take $a \in \mathcal{E}^1_\alpha$ by the same normalized approximation. Its integrals against $\nu$ and $\nu_q$ converge by monotone integration before normalization, since both measures have finite dual energy. The energies of each normalized approximating net converge, so a fixed enlarged radius suffices throughout. This proves (23.11). □

### 23.6.2 Existence and uniqueness

**Theorem 23.6.1** *For each Radon probability measure $\nu$ of finite dual energy there is a unique normalized $u \in \mathcal{E}^1_\alpha$ with $\nu_u = \nu$.*

***Proof*** Choose a maximizing net of normalized rational models and pass to a weakly convergent subnet by Theorem 23.5.1. The tail energy bound of Lemma 23.6.1 and the upper semicontinuity along models give a finite-energy limit $u$. We show its integral against $\nu$ is the limit of the subnet integrals. Each fixed $\nu_{u_\ell}$ in the original net is supported on finitely many tests, by the model measure formula. Hence the subnet integrals against it converge to the integral of $u$ by pointwise convergence, with finite values by the fixed-test bound. By Proposition 23.6.2, the bound (23.11) with $q = u_\ell$ approximates integrals against $\nu$ uniformly for the subnet and its limit as $\ell$ advances, since they belong to a common energy ball on a tail. This proves convergence against $\nu$. Upper semicontinuity of the energy now gives maximality of $u$. Apply Proposition 23.6.1.

For uniqueness let $u, v$ be two normalized solutions. By the variational characterization both maximize the scalar objective. Approximate each by normalized rational models $u_i, v_i$ on a common directed set, with convergence of energies and integrals against all measures of finite dual energy. Such nets result from decreasing continu-

ous approximation followed by rational uniform approximation with errors tending to zero, and normalization. The continuous nets converge in energy and in the integrals by decreasing continuity and monotone integration. Uniform errors preserve these limits, as do the normalization constants tending to zero. The normalized models therefore form two maximizing nets. In particular $I(u_i, v_i) \to 0$ by Lemma 23.6.1.

Fix a normalized rational semipositive model $q$. Set $w = q/2$, and put $\tau_i = (u_i + v_i)/2$. For $0 \le r \le n-1$ set

$$b_{i,r} = \|u_i - v_i\|^2_{P_w^r P_{\tau_i}^{n-1-r}}.$$

By binomial expansion $b_{i,0} \le I(u_i, v_i)$. For $r < n-1$, put $S = P_w^r P_{\tau_i}^{n-2-r}$. The difference $b_{i,r+1} - b_{i,r}$ is the normalized negative pairing with slots $(0, u_i - v_i)$, $(0, w - \tau_i)$, $P_{u_i} - P_{v_i}$, $S$. Indeed replacing $P_{\tau_i}$ by $P_w$ in the square gives a zero-class slot, which interchanges with a slot $(0, u_i - v_i)$. In each term of the resulting difference, Cauchy–Schwarz applies with $P_{u_i}, S$ or $P_{v_i}, S$ as background. Each squared norm there is at most twice the one with $P_{\tau_i}, S$, by nonnegativity and linearity in that background slot. Thus

$$b_{i,r+1} \le b_{i,r} + 4\sqrt{b_{i,r} I(\tau_i, w)}.$$

The last $I$ is bounded: for normalized nonpositive potentials its defining integral is at most the sum of the two integrals $\int (-\tau_i)\, d\nu_{\tau_i}$ and $\int (-w)\, d\nu_w$, because the crossed integrals are subtracted. Each is bounded by $(n+1)$ times the corresponding $j$, and $j(\tau_i)$ is bounded by concavity and the maximizing tail bounds. Induction gives $b_{i,n-1} \to 0$, including $n = 1$ where only $b_{i,0}$ occurs.

Telescope $\int (u_i - v_i)\, d(\nu_q - \nu_0)$ using the backgrounds $P_q^\ell P_0^{n-1-\ell}$. In each term the squared seminorm of $u_i - v_i$ is at most $2^{n-1} b_{i,n-1}$, by expanding $P_w = (P_q + P_0)/2$ and using positivity. The other seminorm, for $q$, is fixed. Thus the integral tends to zero by Cauchy–Schwarz. The integral of $u_i - v_i$ against $\nu_0$ is zero by normalization. Passing to the limit against $\nu_q$, which has finite dual energy by Proposition 23.6.1, gives $\int (u - v)\, d\nu_q = 0$ for every such $q$.

For a fixed test $v_E$, choose a maximizing model net for the finite-dual-energy measure $\delta_{v_E}$. By Proposition 23.6.2 its measures test $u$ and $v$ in the limit as evaluation at $v_E$, by (23.11). The equality just proved for every normalized model measure therefore gives $u(v_E) = v(v_E)$. Order determination on $\mathcal{D}_\alpha$ now gives $u = v$. □

### 23.6.3 Strong convergence

Give the normalized finite-energy cone the topology of weak convergence of potentials together with convergence of $e$. Give the finite-dual-energy probabilities the topology of weak convergence of measures together with convergence of $e^\vee$. These are the respective strong topologies, in the same convention as in Chapter 19.

**Corollary 23.6.2** *The normalized Monge–Ampère map of Theorem 23.6.1 is a homeomorphism for the strong topologies.*

***Proof*** For any measure $\nu$ of finite dual energy, integration against $\nu$ is weakly continuous on each normalized energy ball of potentials. Indeed by (23.11) a maximizing model net approximates that integral uniformly on the ball. The approximating integrals depend on finitely many test values, so are weakly continuous there.

Suppose $u_i$ converges strongly to $u$ and write $\nu = \nu_u$. The energies $j(u_i)$ are bounded on a tail. By the continuity just proved and Proposition 23.6.1, $\Delta_\nu(u_i) \to 0$. Thus $\nu_{u_i} \to \nu$ weakly by Proposition 23.6.2. The estimate (23.11) with $q = u_i$ and $a = u_i$ also gives $\int u_i \, \mathrm{d}(\nu_{u_i} - \nu) \to 0$. Together with convergence of $e(u_i)$ and of their integrals against $\nu$, the identity $e^\vee(\nu_{u_i}) = e(u_i) - \int u_i \, \mathrm{d}\nu_{u_i}$ proves strong convergence of the measures.

Conversely, suppose $\nu_i$ converges strongly to $\nu$, and let $u_i, u$ be their normalized solutions. Fix $B$ bounding $e^\vee(\nu_i)$ and $e^\vee(\nu)$ on a tail. The scalar estimate in the proof of Lemma 23.6.1 extends to normalized finite-energy potentials by decreasing continuous approximation and rational uniform approximation, with normalization at $v_0$. For the maximizers $u_i$ it gives a uniform bound $j(u_i) \le S$, since their objectives equal $e^\vee(\nu_i) \ge 0$. We claim that the integrals of $\nu_i - \nu$ tend uniformly to zero on the normalized $j \le R$ ball for each $R$. Choose a normalized rational model $q$ in a maximizing net for $\nu$, with $\Delta_\nu(q)$ arbitrarily small and $j(q) \le T_B$ on its tail as in the proof of Proposition 23.6.2. For this fixed $q$ the continuity of $q$ gives $\Delta_{\nu_i}(q) \to \Delta_\nu(q)$. Apply (23.11) to both measures using the same $q$, with constants independent of its choice on that tail. This proves the claim by the triangle inequality.

In particular the deficits satisfy

$$\Delta_\nu(u_i) = e^\vee(\nu) - e^\vee(\nu_i) + \int u_i \, \mathrm{d}(\nu - \nu_i) \longrightarrow 0.$$

Every weak cluster point of $u_i$ is finite energy by weak compactness and energy upper semicontinuity. Continuity of integration against $\nu$ on the bounded-energy ball gives maximality at that point, so the point equals $u$ by uniqueness. The compact Hausdorff weak topology then gives $u_i \to u$ weakly. Convergence of their integrals against $\nu$ and of their deficits gives $e(u_i) \to e(u)$. This proves strong convergence. □

## 23.7 Continuity with respect to the Kähler class

We compare classes in the nef order, in which $\beta \le \gamma$ means that $\gamma - \beta$ is nef. For every Kähler class $\alpha$ and every $0 < \varepsilon < 1$, all classes $\beta$ sufficiently close to $\alpha$ satisfy $(1-\varepsilon)\alpha \le \beta \le (1+\varepsilon)\alpha$. Indeed the two differences tend to the Kähler class $\varepsilon\alpha$ as $\beta \to \alpha$.

**Proposition 23.7.1** *Suppose $\beta_i \to \alpha$ are Kähler classes and $f_i \to f$ uniformly are continuous obstacles. Then $P_{\beta_i}(f_i) \to P_\alpha(f)$ uniformly. Their energies converge in the respective classes, and their Monge–Ampère measures converge weakly.*

***Proof*** Write $a_i = 1 - \varepsilon_i$ and $b_i = 1 + \varepsilon_i$ with $a_i\alpha \le \beta_i \le b_i\alpha$ and $\varepsilon_i \to 0$. Such choices follow from the openness of the Kähler cone in the finite-dimensional space of classes. Adding the pair with nef class and zero potential gives inclusions of the continuous semipositive cones. Scaling on models and taking uniform limits then gives

$$a_i P_\alpha(f_i/a_i) \le P_{\beta_i}(f_i) \le b_i P_\alpha(f_i/b_i).$$

The fixed-class envelope is nonexpansive, so both outside terms tend uniformly to $P_\alpha(f)$.

After discarding an initial segment, subtract a common constant so the obstacles are nonpositive. This does not affect the measures, and the energy shifts converge since the top volumes converge. Write $w_i = a_i P_\alpha(f_i/a_i)$ and $u_i = P_{\beta_i}(f_i)$. The class-order and scaling identities in the proof of Theorem 23.3.2 give

$$b_i^{n+1} E_\alpha(w_i/b_i) \le E_{\beta_i}(w_i) \le E_{\beta_i}(u_i).$$

Conversely $u_i/b_i \in C_\alpha$ and $a_i u_i/b_i \in C_{\beta_i}$. The latter lies above $u_i$ because $u_i \le 0$. Thus $E_{\beta_i}(u_i) \le E_{\beta_i}(a_i u_i/b_i) \le a_i^{n+1} E_\alpha(u_i/b_i)$. Energy continuity in class $\alpha$ bounds the limit above and below by $E_\alpha(P_\alpha(f))$.

For $h$ continuous, consider the energy of the envelope of $f_i + th$ as a function of $t$. It is concave: the convex combination of two envelopes lies below the envelope of the combined obstacle, and energy is monotone and concave. For every $t$ the energies converge to the corresponding energy of $P_\alpha(f + th)$. By Corollary 23.4.1, the derivatives at zero are the integrals of $h$ against the envelope measures. The left and right secant slopes at $\pm t$, for $t > 0$, bound each derivative at zero. Advance $i$ at fixed $t$ and then let $t \downarrow 0$. Differentiability of the limiting energy gives convergence of the integrals. □

### 23.7.1 The inverse equation

For a Kähler class $\gamma$ write $V_\gamma = \int_X \gamma^n$, $e_\gamma = E_\gamma/V_\gamma$, and $j_\gamma = -e_\gamma$ on potentials normalized at $v_0$. Let $\nu_y^\gamma$ denote the probability measure of a potential $y$ in class $\gamma$. The dual energy $e_\gamma^\vee$ and the $\gamma$-deficit are those of Section 23.6 in class $\gamma$.

**Proposition 23.7.2** *The set of finite-dual-energy probabilities is independent of the Kähler class. For such a probability $\nu$, if $(1-\varepsilon)\alpha \le \beta \le (1+\varepsilon)\alpha$ with $0 < \varepsilon \le 1/2$, then*

$$|e^\vee_\beta(\nu) - e^\vee_\alpha(\nu)| \le C_n \varepsilon\, e^\vee_\alpha(\nu). \tag{23.12}$$

*For the normalized solution $y$ of $\nu$ in class $\beta$, the potential $z = y/(1+\varepsilon)$ lies in $\mathcal{E}^1_\alpha$ and satisfies $j_\alpha(z) \le C_n e^\vee_\alpha(\nu)$. Its $\alpha$-deficit is at most $2C_n\varepsilon\, e^\vee_\alpha(\nu)$.*

***Proof*** Put $a = 1-\varepsilon$ and $b = 1+\varepsilon$. If $\alpha, \beta$ are arbitrary Kähler classes, one has $a\alpha \le \beta \le b\alpha$ for some $0 < a < b$. For continuous semipositive potentials $y$ in $\beta$ and $u$ in $\alpha$, both normalized, set $z = y/b$. Then $z \in C_\alpha$ and $au \in C_\beta$ by adding the pair with a nef class and zero potential and by scaling. In both cases the estimates are

$$\frac{a^{n+1}V_\alpha}{V_\beta}\, j_\alpha(z) \le j_\beta(y) \le \frac{b^{n+1}V_\alpha}{V_\beta}\, j_\alpha(z), \tag{23.13}$$

with the pair $(u, au)$ in place of $(z, y)$ in the second case. For the first, $E_{b\alpha}(y) \le E_\beta(y)$ and $E_\beta(y) \le E_\beta(az) \le E_{a\alpha}(az)$, using $az \ge y$ and the class order on nonpositive potentials. For the second, $E_\beta(au) \le E_{a\alpha}(au)$ and

$$E_\beta(au) \ge E_{b\alpha}(au) = b^{n+1}E_\alpha((a/b)u) \ge b^{n+1}E_\alpha(u).$$

The last step uses monotonicity since $u \le 0$. The inclusions pass to decreasing limits. Apply the energy comparisons termwise on continuous approximants normalized at $v_0$. Before normalization the approximants decrease, and their normalization constants tend to zero. Thus by energy translation and decreasing continuity the same comparisons hold for normalized limits. For $y$ of finite energy the lower bound for $j_\beta(y)$ bounds $j_\alpha(y/b)$ from above. For $u$ of finite energy the upper bound in the comparison bounds $j_\beta(au)$. This proves the finite-energy inclusions in both directions. By Corollary 23.6.1, the scaling inclusions prove independence of the class. Mixed positivity gives $a^n V_\alpha \le V_\beta \le b^n V_\alpha$.

Return to $a = 1-\varepsilon$, $b = 1+\varepsilon$ as in the statement. Write $L(q) = \int(-q)\,\mathrm{d}\nu$ and $F_{\gamma,\nu}(q) = e_\gamma(q) + L(q)$. Take the normalized solution $u$ for $\nu$ in $\alpha$ and $y$ for $\nu$ in $\beta$. These exist by Theorem 23.6.1. Their objectives are nonnegative. Hence $j_\alpha(u) \le L(u)$ and $j_\beta(y) \le L(y)$. For $z = y/b$, the lower energy comparison in (23.13) gives $j_\alpha(z) \le C_n L(z)$, since $L(y) = bL(z)$ and the volume ratio is bounded by $a^n, b^n$. By the quadratic scaling bound in the proof of Corollary 23.6.1, for either $q = u$ or $q = z$ and $0 < t \le 1$ we have

$$e^\vee_\alpha(\nu) \ge F_{\alpha,\nu}(tq) \ge tL(q) - C_n t^2 j_\alpha(q).$$

Fix $t > 0$ small enough depending only on $n$. Since $j_\alpha(q) \le C_n L(q)$, the right side is at least $tL(q)/2$. This bounds both $L(q)$ and $j_\alpha(q)$ by $C_n e^\vee_\alpha(\nu)$.

The volume bounds show that the coefficients in (23.13) differ from one by $O_n(\varepsilon)$. The integrals scale by $a$ and $b$, respectively. Thus both

$$|F_{\beta,\nu}(au) - F_{\alpha,\nu}(u)|, \qquad |F_{\beta,\nu}(y) - F_{\alpha,\nu}(z)|$$

are at most $C_n \varepsilon e^{\vee}_{\alpha}(\nu)$. By maximizing in the target class of each scaled potential, we get the two directions of (23.12). The second objective difference and that inequality bound the $\alpha$-deficit of $z$. □

**Theorem 23.7.1** *Let $\beta_i \to \alpha$ be Kähler classes, and let $y_i, y$ be the normalized solutions for finite-dual-energy probabilities $\nu_i, \nu$ in classes $\beta_i, \alpha$, respectively. The following are equivalent.*

(1) *The values of $y_i$ converge to those of $y$ at each divisorial test and $e_{\beta_i}(y_i) \to e_{\alpha}(y)$.*
(2) *The measures $\nu_i$ converge weakly to $\nu$ and $e^{\vee}_{\beta_i}(\nu_i) \to e^{\vee}_{\alpha}(\nu)$.*

*In the second condition one may equivalently use $e^{\vee}_{\alpha}(\nu_i)$ in the convergence.*

***Proof*** Choose $a_i, b_i$ as in the proof of Proposition 23.7.1, with $\varepsilon_i \le 1/4$ on a tail. Write $z_i = y_i / b_i$. By (23.13) the first condition is exactly strong convergence of $z_i$ to $y$ in the fixed class $\alpha$. The sandwiches also give $(1 - 2\varepsilon_i)\beta_i \le \alpha \le (1 + 2\varepsilon_i)\beta_i$. Consequently a uniform upper bound on either of $e^{\vee}_{\beta_i}(\nu_i)$ and $e^{\vee}_{\alpha}(\nu_i)$ yields one for the other by (23.12), and the two energies then have difference tending to zero. If the first condition holds, $j_{\beta_i}(y_i)$ is bounded. The ordered energy formula, keeping the diagonal measure term, gives

$$e^{\vee}_{\beta_i}(\nu_i) = e_{\beta_i}(y_i) - \int y_i \, \mathrm{d}\nu_i \le n j_{\beta_i}(y_i).$$

In particular either condition implies that $e^{\vee}_{\alpha}(\nu_i)$ is bounded on a tail.

Under this bound Proposition 23.7.2 gives bounded $j_{\alpha}(z_i)$ and deficits $\Delta_{\nu_i}(z_i) \to 0$ in class $\alpha$. For a fixed real model test $h$, apply the uniform first variation (23.10) at $z_i$ with both signs of $h$. The obstacle bound and the dual definition give for $t > 0$

$$\left| \int h \, \mathrm{d}(\nu^{\alpha}_{z_i} - \nu_i) \right| \le C'_h t^{1/2} + t^{-1} \Delta_{\nu_i}(z_i).$$

Advance $i$ and then let $t \downarrow 0$. Density and the probability mass bound give convergence of the signed differences to zero against every continuous test. By (23.11) with $q = z_i$ and test $z_i$, we have

$$|e^{\vee}_{\alpha}(\nu_i) - e^{\vee}_{\alpha}(\nu^{\alpha}_{z_i})| \le \Delta_{\nu_i}(z_i) + C \Delta_{\nu_i}(z_i)^{1/2}.$$

In fact the $\nu_i$-objective at $z_i$ differs from $e^{\vee}_{\alpha}(\nu_i)$ by the deficit, and the two objectives at $z_i$ differ by the integral of $z_i$ against the signed difference. It follows that $\nu_i$ and $\nu^{\alpha}_{z_i}$ have the same strong convergence in the fixed class. Apply Corollary 23.6.2 to $z_i$ and $\nu^{\alpha}_{z_i}$. □

**Corollary 23.7.1** *The strong topology of finite-dual-energy probabilities is independent of the reference Kähler class. Consequently the measure condition of Theorem 23.7.1 can use the strong topology in any fixed Kähler class.*

***Proof*** Suppose $\nu_i \to \nu$ strongly in $\alpha$. The proof of Corollary 23.6.2 gives convergence of their integrals uniformly on each normalized $j_\alpha$-ball. Fix a Kähler class $\gamma$ and $0 < a < b$ with $a\alpha \le \gamma \le b\alpha$. Scaling a normalized $j_\gamma$-ball by $1/b$ puts it in a normalized $j_\alpha$-ball by (23.13), whose energy comparison holds for any such $a, b$. Thus the integrals also converge uniformly on every normalized $j_\gamma$-ball.

There is one $j_\gamma$-ball on which the $\gamma$-dual suprema of $\nu_i$ on a tail and of $\nu$ can all be computed. In fact, for $w$ normalized of finite energy in $\gamma$, the term $z = w/b$ belongs to $\mathcal{E}^1_\alpha$. The scalar bound in class $\alpha$ bounds $\int(-w)\,\mathrm{d}\nu_i$ by $b(B + C_n)\sqrt{1 + j_\alpha(z)}$ for a common bound $B$ on their $\alpha$-dual energies and that of $\nu$. By (23.13) this grows at most as a fixed multiple of $\sqrt{1 + j_\gamma(w)}$. The $\gamma$-objectives are all bounded above by $-j_\gamma(w)$ plus that multiple. Their suprema are nonnegative, so all their candidates with objective at least $-1$ lie in one $j_\gamma$-ball. Uniform comparison of the integrals there proves $e^\vee_\gamma(\nu_i) \to e^\vee_\gamma(\nu)$. Weak measure convergence is the same in both classes. Exchanging $\alpha$ and $\gamma$ proves equality of topologies. □

# Chapter 24
# Positivity on models and maximal rays

On $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(x, t)$, blow up the origin and then the point $x/t = 1$ of the exceptional curve. The strict transform of the first exceptional curve and the second exceptional curve both have multiplicity one in the central fibre, and both restrict to the order of vanishing at $x = 0$. Nevertheless, the exceptional correction divisor with which the second assertion of Lemma D.4.1 produces a Kähler class on this model has different coefficients along them (Example 24.1.1). Perturbations of model classes are therefore compared on the full model over the parameter line, and not through the valuations of the central components alone.

Throughout this chapter we keep the conventions of Chapter 22: $X$ is a connected compact Kähler manifold of dimension $n > 0$, and $C_\alpha \subset \mathcal{D}_\alpha$ are the continuous semipositive cone and its decreasing limits. The class $\alpha$ is Kähler, and $\mathcal{E}^1_\alpha$ denotes the finite-energy potentials of Section 23.6. In Section 24.2, $\theta$ is a big class with a fixed decomposition $f^*\theta = \beta + [F]$ on a modification $f : Y \to X$ from a compact Kähler manifold, where $\beta$ is Kähler, or nef with positive volume, and $F$ is effective. In Sections 24.3 and 24.4 we fix a Kähler form $\omega$ in $\alpha$ and compare the cohomological energy with the complex energy $\mathcal{E}_\omega$ of the slices of rays. Section 24.5 uses Kähler forms $\omega_i$ in several classes $\alpha_i$.

Proposition 24.1.1 identifies semipositivity on a smooth model with relative nefness of the model class. Section 24.2 shows that pullback along a modification maps the continuous semipositive cone of a nef class into that of its pullback and preserves the pushed-forward mixed measures and the energies. It uses this pullback to define measures and energy relative to a specified positive part of a big class, and Example 24.2.1 shows that the probability measures depend on that choice. For a ray with model singularities, Lemma 24.3.1 computes the slope of the circular mean of its slice energies as a top intersection on the model. Proposition 24.3.1 attaches to every nonpositive subgeodesic ray a potential in $\mathcal{D}_\alpha$ whose energy bounds the energy slope of the ray from above. Conversely, by Theorem 24.4.1, every normalized $u \in \mathcal{E}^1_\alpha$ has a maximal ray, a weak geodesic ray from zero with transform $u$ whose slice energy at time $s$ is $sE_\alpha(u)$. Theorem 24.4.2 allows any initial potential of finite energy, and Theorem 24.5.1 identifies the mixed energy slopes of these rays with the cohomological mixed pairings.

Chapter 25 uses the ray transform and the maximal rays to identify finite-energy test curves with normalized finite-energy potentials in Section 25.10. The mixed pairing of Section 24.5 is used at the nef boundary in Sections 26.4 and 26.8, and the pullback of Section 24.2 returns in Section 30.2.

## 24.1 Relative positivity on smooth models

Fix a smooth refinement $q : W \to X \times \mathbb{P}^1$, with projections $\rho : W \to X$ and $\pi : W \to \mathbb{P}^1$. In the space of real $(1,1)$-classes on $W$ modulo the fibre class, the relatively Kähler cone is the image of the Kähler cone, and the relatively nef cone is its closure. A relatively Kähler class therefore has a Kähler lift after adding a real multiple of $\pi^*[\omega_{\mathrm{FS}}]$, for a positive form $\omega_{\mathrm{FS}}$ on the parameter line.

**Proposition 24.1.1** *Let $\alpha$ be Kähler and $D$ be a real vertical divisor on $W$. The class $\gamma = \rho^*\alpha + c_1(D)$ is relatively Kähler if and only if its restriction to each central component is Kähler. It is relatively nef if and only if each of those restrictions is nef. In particular, for a divisor representing a model function, the latter condition is the semipositivity test on $W$ of Definition 22.1.1.*

***Proof*** Restriction to any central component kills the fibre class, so it descends to the quotient. It sends Kähler classes to Kähler classes and is continuous. This proves both necessity statements.

Choose a Kähler form $\omega$ in $\alpha$ and a Kähler form $\Omega_W$ on $W$. Fix $\varepsilon \geq 0$ such that $\gamma + \varepsilon[\Omega_W]$ has Kähler restriction to every central component. By Lemma D.5.1, this class has a smooth representative $\eta$ Kähler near the fibre. By properness, there is a small base disk whose inverse image lies in that neighborhood. Choose a smooth form $\kappa$ in $c_1(D)$ supported over a smaller base disk. For an integral component of $D$, take a metric on its line bundle for which the defining section has constant norm outside the smaller disk. On an annulus away from the fibre this metric is obtained by interpolating the logarithms of the norms. Its curvature vanishes where the norm is constant. Real linear combinations of these curvature forms give the form $\kappa$ for $D$.

Put $\eta_0 = \rho^*\omega + \varepsilon\Omega_W + \kappa$ and write $\eta = \eta_0 + dd^c\psi$ with $\psi$ smooth, by the $dd^c$-lemma. Let $\chi$ be a smooth cutoff on the base with values in $[0,1]$ that equals one on the smaller disk and equals zero outside the larger disk, with the transition outside the support of $\kappa$. The representative $\eta_0 + dd^c((\chi \circ \pi)\psi)$ agrees with $\eta$ near the central fibre. On the transition annulus, all terms involving derivatives of $\chi \circ \pi$ vanish on vertical tangent vectors. The restriction to the vertical tangent bundle is the convex combination of the restrictions of $\eta_0$ and $\eta$, which are positive there. Outside the larger disk it equals $\eta_0$, with positive restriction to the vertical tangent bundle since the model is the product away from zero.

On the compact complement of a smaller neighborhood of the central fibre, $\pi$ is a submersion and the positive vertical block has a uniform lower bound. Adding $C\pi^*\omega_{\mathrm{FS}}$ with $C$ large makes the form Kähler by the Schur complement. Near the

central fibre the original form was already Kähler and the added form is semipositive. This proves relative Kählerness for the perturbed class. Take $\varepsilon = 0$ when the restrictions are Kähler. For nef restrictions the argument applies for every $\varepsilon > 0$, giving relative nefness in the limit.

For semipositivity on $W$, the representatives in Definition 22.1.1, with error $\varepsilon/2$, become Kähler on each component after adding $(\varepsilon/2)\Omega_W$, which gives nef restrictions in the limit. Conversely, the global Kähler lift constructed for each positive $\varepsilon$, after subtracting its base form, has positive restrictions to all components. □

Choose $[\Omega_W] = \rho^*\alpha + c\pi^*[\omega_{\mathrm{FS}}] - c_1(F)$ as in Proposition 22.7.2. If $D$ is relatively nef after adding the background $\rho^*\alpha$, the proof above and division by $1+\varepsilon$ show that $\rho^*\alpha + c_1(D_\varepsilon)$ is relatively Kähler for

$$D_\varepsilon = \frac{D - \varepsilon F}{1+\varepsilon}.$$

This keeps the class on the product fibres fixed. The coefficients of the pullbacks of $D_\varepsilon - D$ to all further smooth refinements are bounded in absolute value by a fixed multiple of $\varepsilon$ times the fibre multiplicities. Indeed, on $W$ one has $-AW_0 \le D + F \le AW_0$ for some $A$, and these divisor inequalities pull back. This comparison takes place on the full model over the parameter line: for a nonequivariant refinement, two components can have the same normalized restriction to $X$ but different normalized coefficients in a correction divisor, as the following example shows.

*Example 24.1.1* Let $X = \mathbb{P}^1$ and use a coordinate $x$ near zero. Blow up $(x,t) = (0,0)$ and denote the exceptional divisor by $E$. Blow up the point $x/t = 1$ of $E$, with new exceptional divisor $G$ and blowup map $p$. Denote the strict transform of $E$ by $\widetilde{E}$. At the generic point of $E$ the expression $x = t(x/t)$ gives both $x$ and $t$ order one. Near the second centre, $t$ and $x/t - 1$ are coordinates and $x = t(1 + (x/t - 1))$. On the new exceptional divisor $x$ and $t$ again have order one. Every nonzero meromorphic germ at zero on $X$ is a power of $x$ times a unit. Thus $\widetilde{E}$ and $G$ both give $\mathrm{ord}_{x=0}$ on the base after division by their parameter multiplicities, which are both one. Their full orders of $x - t$ are one and two, respectively, since $x - t = t(x/t - 1)$.

By the second assertion of Lemma D.4.1, a correction divisor for a Kähler form on this model is $F = ap^*E + bG$ with successively small $a, b > 0$. Since $p^*E = \widetilde{E} + G$, its two coefficients are $a$ and $a + b$. Even when $D = 0$, the correction $D_\varepsilon$ assigns different coefficients to the two components giving the same base valuation.

## 24.2 A specified positive part of a big class

We first extend the modification formula from models to the uniform closures. Let $g : Y' \to Y$ be a proper modification of connected compact Kähler manifolds. If a pair on $Y$ is semipositive on a smooth model, its pullback in Theorem 21.10.1 is semipositive on a smooth model over $Y'$. Indeed, take the sequence of smooth blowups defining the semipositive refinement over $Y$. Starting with $Y' \times \mathbb{P}^1$, principalize the pullbacks of

their centre ideals successively by the first assertion of Lemma D.4.1 and lift through each blowup, as in Lemma D.4.2. Include the flag ideals representing the pulled-back functions and achieve strict normal crossings of the central fibre in the final step. On the resulting common refinement the semipositivity follows by pullback on the central components and comparison of Kähler forms. In particular, if $\beta$ is nef then pullback gives

$$g^{\sqsupset *} : C_\beta \longrightarrow C_{g^*\beta}.$$

The pullback class on the base is nef by smooth semipositive approximation. The map on potentials preserves the uniform norm because $g^{\sqsupset}$ is surjective by Proposition 21.10.1. By weak continuity and the energy Lipschitz bound, the two identities in Theorem 21.10.1 give the same pushed-forward mixed measures and unnormalized energies for the continuous data.

Now let $X$ be compact Kähler, let $\theta$ be a big real $(1,1)$-class, and fix a modification $f : Y \to X$ from a compact Kähler manifold together with a decomposition

$$f^*\theta = \beta + [F]$$

where $F$ is an effective real Cartier divisor and $\beta$ is Kähler. One can also use a nef $\beta$ with $\int_Y \beta^n > 0$. On a flag model over $Y$ the class used for a vertical correction $D$ is $\rho^* f^*\theta - [\rho^* F] + [D] = \rho^*\beta + [D]$, where $\rho$ denotes the projection to $Y$. For a continuous function $u$ on $X^{\sqsupset}$ with $u \circ f^{\sqsupset} \in C_\beta$, define

$$\mu_{\theta;f,F}(u) = f_*^{\sqsupset}\,\mathrm{MA}_Y\big((\beta, u \circ f^{\sqsupset})^n\big),$$
$$E_{\theta;f,F}(u) = E_\beta(u \circ f^{\sqsupset}).$$

The mixed measures in the specified class are defined by the same rule with several potentials, and their mass is $V_\beta = \int_Y \beta^n$. On the convex domain of such $u$ the energy is concave, and its derivative in the direction of a segment is the integral of the slope against $\mu_{\theta;f,F}$. This follows by applying Theorem 22.4.1 on $Y$ and changing variables by $f^{\sqsupset}$. Dividing by $V_\beta$ gives probability measures. The same measure and energy definitions apply when $u \circ f^{\sqsupset}$ is a decreasing limit of finite energy in class $\beta$. The measures are then the weak limits of the pushed-forward measures, taken with Theorem 23.3.1, and their mass is still $V_\beta$.

These formulas are unchanged for the existing data if the decomposition is pulled back to a further modification. More generally, let $(f_i : Y_i \to X, F_i)$ be two specified decompositions admitting a common compact Kähler refinement $g_i : Z \to Y_i$ with $f_1 g_1 = f_2 g_2$ and $g_1^*\beta_1 = g_2^*\beta_2$. For a continuous potential in both domains, the pullbacks to $Z^{\sqsupset}$ coincide in the uniform cone of the common class. Applying the modification formula from the beginning of this section twice shows that both the pushed-forward mixed measures and the energies agree. The same compatibility holds for the finite-energy decreasing limits. The continuous pullback formula preserves the energies of their approximants, so the pulled-back limits still have finite energy. Theorem 23.3.1 then gives equality of the weak limits after pushforward.

*Example 24.2.1 (Changing the positive part on a surface)* Let $\tau : X \to S$ be the blowup of a point of a connected compact Kähler surface, with exceptional curve $C$. Let $\omega_S$ be a Kähler form on $S$, and fix $a > 0$. Set $\theta = \tau^*[\omega_S] + a[C]$. For sufficiently small $b > 0$, $\beta_b = \tau^*[\omega_S] - b[C]$ is Kähler. Indeed, the second assertion of Lemma D.4.1 makes $\tau^*[\omega_S] - \varepsilon_1[C]$ Kähler for some rational $\varepsilon_1 > 0$. For $0 < b \le \varepsilon_1$, the class $\beta_b$ is $(b/\varepsilon_1)(\tau^*[\omega_S] - \varepsilon_1[C]) + (1 - b/\varepsilon_1)\tau^*[\omega_S]$, represented by a Kähler form plus a semipositive form, hence Kähler. Thus $\theta = \beta_b + (a+b)[C]$ is big, as it has a Kähler current given by the smooth positive part and the effective divisor $(a+b)C$ of the decomposition. Its degree on $C$ is $-a$. Writing $V = \int_S \omega_S^2$, the specified mass is $V_b = V - b^2$, since $\tau^*[\omega_S]|_C = 0$ and $C^2 = -1$.

Fix a point $p \notin C$ and use the model function $\varphi_c = -c\min(v(\mathfrak{m}_p), 1)$ from Proposition 21.9.1. Choose two different small values of $b$ and then a common $c > 0$ small enough for both model pairs. Up to a parameter class, the positive representative furnished by that proposition is a valid smooth-model semipositivity test. With $f$ the identity, the probability measure assigns mass $c^2/(V - b^2)$ to $w_p$. Thus the choice of positive part affects the measure even after probability normalization. The same calculation works when $S$ is the nonprojective torus from Section 21.9.

## 24.3 The energy slope of a ray

Fix a Kähler form $\omega$ in $\alpha$, put $s = -\log|t|^2$, and write $U_s$ for the slice of a circle-invariant subgeodesic ray at time $s$. Use the unnormalized complex energy

$$\mathcal{E}_\omega(u) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X u\,(\omega + dd^c u)^j \wedge \omega^{n-j}$$

for bounded $\omega$-psh potentials, and extend it to unbounded potentials as the decreasing limit along truncations. It is monotone and continuous under decreasing limits, and adding $c$ changes it by $c\int_X \omega^n$ [Xia26a, Propositions 17.2.6–17.2.7].

We first allow a function $V$ on a punctured product disk with no invariance assumption. On a smooth refinement $q : W \to X \times \mathbb{P}^1$, let $D$ be a real vertical divisor. For smooth Hermitian norms of the sections defining its components, write $L_D = \sum_E \operatorname{ord}_E(D)\log\|s_E\|^2$. Thus $dd^c L_D = [D] - \Theta_D$, where $\Theta_D$ is the sum of the curvature forms with the same coefficients.

**Lemma 24.3.1** *Suppose $V \circ q + L_D$ extends smoothly near $W_0$ and $V(\cdot, t)$ is $\omega$-psh for sufficiently small $0 < |t|$. Then*

$$\lim_{s\to\infty}\frac{1}{s}\int_0^{2\pi}\mathcal{E}_\omega(V(\cdot, e^{-s/2+\mathrm{i}\vartheta}))\frac{\mathrm{d}\vartheta}{2\pi} = \frac{1}{n+1}\int_W \big(\rho^*\alpha + c_1(D)\big)^{n+1}. \tag{24.1}$$

***Proof*** Use a cutoff pulled back from the base to extend $V \circ q + L_D$ to a smooth function $\psi$ on $W$. The function $\psi - L_D$ descends off the central fibre to a smooth

function $\widetilde{V}$ on the punctured product with $\mathbb{P}^1$. Its curvature there pulls back to the smooth form $\Theta = \rho^*\omega + \Theta_D + dd^c\psi$. On the punctured parameter line use the smooth fibre energy $F(t) = \mathcal{E}_\omega(\widetilde{V}(\cdot, t))$, given by the same mixed integral even outside the region of positivity. Differentiating this fibre integral gives

$$dd^c F = \frac{1}{n+1}\pi_*(\Theta^{n+1}) \quad \text{off } 0.$$

Indeed, the mixed forms in the integrand are closed, and applying $dd^c$ to the potential produces the difference of the two full forms. The mixed sum telescopes to the difference of their powers, with the $(n+1)$-st power of the background from $X$ equal to zero.

Let $A(s)$ be the circular average of $F$ at $r = e^{-s/2}$. With the convention $dd^c \log|t|^2 = [0]$, polar coordinates give $A'(s) = -\int_{|t|=r} d^c F$ on the positively oriented circle. Stokes' theorem on its compact exterior therefore gives

$$A'(s) = \frac{1}{n+1}\int_{\pi^{-1}(\{|t|\geq r\})} \Theta^{n+1},$$

where the exterior includes infinity. The omitted fibre has measure zero for the smooth top form. The derivative tends to its full integral divided by $n+1$, so integrating in $s$ gives (24.1). □

**Proposition 24.3.1** *Each nonpositive subgeodesic ray $U$ has a unique potential $T(U) \in \mathcal{D}_\alpha$ with values $T_{W,E}(U)$ on all divisorial tests. The same formula holds for a vertical prime divisor $E$ on a smooth local modification of an open subset of $X \times \mathbb{D}$, using its normalized order restricted to ideals extended from $X$. The ray satisfies*

$$\limsup_{s\to\infty} \frac{\mathcal{E}_\omega(U_s)}{s} \leq E_\alpha(T(U)). \tag{24.2}$$

***Proof*** First suppose $-Ls \leq U_s \leq 0$ for some $L$. Use the compactification and dyadic regularization in the proof of Proposition 22.7.1. Write $\Phi_j$ for the gentle approximants of $\widetilde{U}$ and $\varepsilon_j = C/2^j$ for their chosen curvature correction. The approximants dominate $\widetilde{U}$ [Xia26a, Theorem 13.4.5]. On a smaller disk choose a bounded smooth potential $b_j(t)$ of the base term of the corrected form. Then

$$V_j = \frac{\Phi_j + b_j}{1+\varepsilon_j}$$

has positive semidefinite curvature with respect to the background pulled back from $\omega$. Near zero, $\widetilde{U} = U \leq 0$, so $V_j \geq U - O_j(1)$. This bound is uniform in the angle of $t$. The functions $V_j$ are smooth off the central fibre and satisfy the compatibility of Lemma 24.3.1 with $D_j = -F_j/(2^j(1+\varepsilon_j))$ on a principalization of the multiplier ideal. This divisor represents the model potential $u_j$ in the proof of the fixed-class approximation. By monotonicity and translation of the complex energy, the circular mean energy of $V_j$ bounds $\mathcal{E}_\omega(U_s)$ from above up to a constant depending on $j$. By

(24.1) its slope is $E_\alpha(u_j)$, by the definition of the model energy. Let first $s \to \infty$ and then $j \to \infty$. Decreasing continuity gives (24.2) in this case.

For the local test statement, an order along a prime divisor on a smooth local modification is evaluated at a general point, where the pullback of a nonzero ideal has a well-defined order. The product and minimum rules give an ideal semivaluation on $X$ by restriction to its extended ideals and division by $b_E = \mathrm{ord}_E(t) > 0$. For each flag ideal, the minimum over its homogeneous terms still gives its normalized order on that divisor, by the same coefficient formula as on a global model. On this test $u_j$ therefore converges to $T_{W,E}(U)$ by the local divisorial multiplier-ideal formula [Xia26a, Corollary 12.6.8]. The local formula thus holds for the bounded ray.

In general take $U^{(k)} = \max(U, k \log|t|^2)$ for integers $k > 0$. The potentials of these rays decrease by the finite divisorial order test and have values $\max(T_{W,E}(U), -k)$, by (22.4). The limiting value at $v_0$ is finite since a nontrivial local psh function has finite generic Lelong number. The nested decreasing-limit argument in Theorem 23.5.1 gives $T(U)$ in $\mathcal{D}_\alpha$. The test formulas pass to this limit on both the global and local tests. Uniqueness follows from the divisorial order test. By monotonicity, $\mathcal{E}_\omega(U_s) \le \mathcal{E}_\omega(U_s^{(k)})$ for each $s, k$. Apply the bounded-ray inequality and let $k \to \infty$ using decreasing continuity of $E_\alpha$. □

## 24.4 Maximal finite-energy rays

We first consider nonpositive rays with initial value zero. A ray is maximal for a potential $u \in \mathcal{D}_\alpha$ if it is the greatest nonpositive subgeodesic ray with transform bounded above by $u$. On the cohomological side, normalize potentials at the trivial valuation $v_0$, so they are nonpositive.

For $0 \le s_0 < s_1$, a subgeodesic on $[s_0, s_1]$ is a circle-invariant $p_X^*\omega$-psh function on the annulus $e^{-s_1/2} < |t| < e^{-s_0/2}$, with slices $U_s$ as above. For $\omega$-psh potentials $a_0, a_1$, the upper regularized envelope of the subgeodesics on $[s_0, s_1]$ whose slices have $L^1$ limits at most $a_0$ and $a_1$ at the two ends is the Perron segment. When the $L^1$ limits of its own slices at the two ends are $a_0$ and $a_1$, it is the *weak geodesic* between them [Xia26a, Definition 17.6.7]; see also [Xia26b, Definition 4.2.1 and Proposition 4.2.1]. These references use the time $-\log|t| = s/2$, and the affine change of time does not affect these notions. A subgeodesic ray $U$ is a *weak geodesic ray from* an $\omega$-psh potential $a$ if $U_s \to a$ in $L^1$ as $s \downarrow 0$ and, with $U_0 = a$, its restriction to every interval $[s_0, s_1]$ with $0 \le s_0 < s_1$ is the weak geodesic between $U_{s_0}$ and $U_{s_1}$.

**Proposition 24.4.1** *For $u \in C_\alpha$ normalized, the upper regularized supremum $U^u$ of all nonpositive subgeodesic rays with $T(U) \le u$ is the maximal ray for $u$. It is a weak geodesic ray from zero and has transform $u$. For $s > 0$,*

$$\mathcal{E}_\omega(U_s^u) = sE_\alpha(u). \tag{24.3}$$

*For normalized $u, w \in C_\alpha$, $\|U_s^u - U_s^w\|_\infty \le s\|u - w\|_\infty$.*

***Proof*** Put $M = \|u\|_\infty$. The ray $-Ms$ is admissible, and Lemma 22.7.1 shows that the supremum is admissible and lies between $-Ms$ and zero. In particular its initial value is zero uniformly on $X$. For every $\delta > 0$ take a rational semipositive model $h \le u$ with $\|h - u\|_\infty < \delta$, by uniform approximation and subtraction of a small rational constant. Its approximate rays of Proposition 22.7.2 lie in the defining family. Their test values tend to $h$. Letting $\delta$ decrease proves $T(U^u) = u$ on the divisorial tests, hence everywhere.

On a positive compact time interval take the weak geodesic between the two bounded endpoint slices [Xia26b, Proposition 4.2.1]. The old ray is a candidate. The geodesic is nonpositive since it is bounded above by the linear interpolant of the endpoints. The upper boundary inequalities also follow from that interpolant and upper semicontinuity of the endpoints. Replace the ray on the interval by this geodesic. The local psh gluing rule for maxima gives a nonpositive subgeodesic with the same vertical slopes, since the central germ is unchanged. Maximality forces equality with the old ray. The same argument applies to an interval starting at zero, where the original ray has uniform boundary value zero. Thus it is a weak geodesic ray from zero.

To prove the energy identity, fix $h = h_D \le u$ as above on a smooth refinement and use the proof of Proposition 22.7.2. Before taking the supremum over rotations, that proof constructs a smooth compatible potential $V_\varepsilon$ with divisor $D_\varepsilon = (D - \varepsilon F)/(1 + \varepsilon)$ on the marked model. It is nonpositive after a constant subtraction. Adding $\varepsilon C_h \log |t|^2$, with $C_h$ the slope bound from that proof, makes the slopes at most $h$ for every rotation. Their upper regularized supremum is thus admissible, and $U^u_s$ bounds each rotated slice from above. Apply monotonicity of $\mathcal{E}_\omega$, take the circular mean, and use (24.1). It follows that the lower energy slope of $U^u$ is at least the model intersection of $\rho^*\alpha + c_1(D_\varepsilon - \varepsilon C_h W_0)$ divided by $n + 1$. Letting $\varepsilon \downarrow 0$ on the fixed refinement gives $E_\alpha(h)$. Let $h$ converge uniformly to $u$. The opposite bound is (24.2), and complex energy is affine along the weak geodesic [Xia26a, Theorem 17.7.3]. This proves (24.3).

For the last estimate, put $\delta = \|u - w\|_\infty$. The ray $U^u - \delta s$ has transform $u - \delta$ and is admissible for $w$. Interchanging the two potentials gives the other inequality. □

**Lemma 24.4.1** *If $w, u \in \mathcal{E}^1_\alpha$ satisfy $w \le u$ and $E_\alpha(w) = E_\alpha(u)$, then $w = u$.*

***Proof*** Take simultaneous decreasing continuous semipositive approximants $w_i \le u_i$, using for $w_i$ the envelope of the minimum of its original approximant and $u_i$. The order and the limits follow since the minimum bounds $w$ from above. As in Section 23.6, write $V = \int_X \alpha^n$, $e_\alpha = E_\alpha/V$ and $\nu_\varphi = \mathrm{MA}_X((\alpha, \varphi)^n)/V$, and use the dual energy $e^\vee$ and the deficits $\Delta_\nu$ defined there. The mixed energy difference gives

$$\int (u_i - w_i)\, \mathrm{d}\nu_{w_i} \le (n + 1)(e_\alpha(u_i) - e_\alpha(w_i)).$$

The energies $e_\alpha(w_i)$ decrease to $e_\alpha(w)$, and $\int w_i\, \mathrm{d}\nu_w \to \int w\, \mathrm{d}\nu_w$ by monotone integration, since $\nu_w$ integrates $w$. As $e^\vee(\nu_w) = e_\alpha(w) - \int w\, \mathrm{d}\nu_w$ by Proposition 23.6.1, it follows that $\Delta_{\nu_w}(w_i) \to 0$. Deficits and measures are unchanged by adding constants, and the normalized approximants $u_i, w_i$ lie in a fixed normalized

energy ball on a tail, since the constants at $v_0$ converge. Hence (23.11), applied with $\nu = \nu_w$, with $q$ the normalized $w_i$, and with $a$ the normalized $u_i$ or $w_i$, allows $\nu_{w_i}$ to be replaced by $\nu_w$ with error tending to zero: the normalizing constants cancel in integration against the difference of two probabilities. Apply monotone integration to the two decreasing nets separately against $\nu_w$, then subtract the finite limiting integrals to obtain $\int (u - w)\, \mathrm{d}\nu_w = 0$. As the energies agree, $u$ and $w$ give the same scalar objective for $\nu_w$, maximized at $w$. By Proposition 23.6.1 and Theorem 23.6.1, their normalized potentials coincide. The equality of energies makes the constant difference zero. □

**Theorem 24.4.1** *For each normalized $u \in \mathcal{E}^1_\alpha$ there is a maximal ray $U^u$ for $u$. It is a weak geodesic ray from zero, its transform is $u$, and for every $s > 0$,*

$$\mathcal{E}_\omega(U^u_s) = sE_\alpha(u). \tag{24.4}$$

*It is the decreasing limit of the rays of any normalized decreasing continuous semipositive approximation to $u$.*

***Proof*** Take continuous decreasing majorants $u_i$ and replace them by $P_\alpha(\min(u_i, 0))$. These are normalized majorants and still decrease to $u$. The rays $U^{u_i}$ decrease by maximality, and (24.3) bounds their slice energies below by $sE_\alpha(u)$. All potentials are nonpositive, so the term against $\omega^n$ in the complex energy formula gives

$$\int_X (-U^{u_i}_s)\, \omega^n \leq -(n+1)sE_\alpha(u).$$

For the decreasing ray limit one can take a sequence of indices with the same pointwise limit on the punctured product. Indeed the increasing open sets $\{U^{u_i} < c\}$, for rational $c$, have the same union after selecting countably many indices by second countability. Take a sequence of successive common upper bounds of the selected indices. The integral bound shows that the limit has nontrivial slices and is nontrivial on the product. Decreasing psh convergence gives a subgeodesic $U$, extending across the central fibre by the common upper bound. Decreasing continuity of complex energy gives $\mathcal{E}_\omega(U_s) = sE_\alpha(u)$, and the same integral bound gives $U_s \to 0$ in $L^1$ at the initial time.

The original decreasing net consists of weak geodesics. On each finite interval every subgeodesic with endpoints below those of $U$ is a candidate for every one of those weak geodesics, so it lies below their limit. The limit itself has the required endpoint values in $L^1$. At positive times this follows from pointwise convexity and local uniform $L^1$ bounds, or from continuity of subgeodesic slices [Xia26b, Corollary 4.1.2 and Proposition 4.1.4]. At zero it was proved in the preceding paragraph. Thus $U$ is the weak geodesic on each such interval.

Monotonicity of the ray transform gives $T(U) \leq u$. By (24.2) and the complex energy identity, $E_\alpha(T(U)) \geq E_\alpha(u)$. In particular $T(U)$ has finite energy. Monotonicity gives equality of the energies, and Lemma 24.4.1 gives $T(U) = u$.

Finally every nonpositive ray with transform at most $u$ lies below each $U^{u_i}$, hence below $U$. This proves maximality and the claimed independence. □

### 24.4.1 Changing the analytic initial potential

For a subgeodesic ray bounded above, define its transform by subtracting an upper bound before applying Proposition 24.3.1. This is independent of the subtraction because generic Lelong numbers are invariant under constants.

**Theorem 24.4.2** *Let $u \in \mathcal{E}^1_\alpha$ be normalized and let $a$ be an $\omega$-psh potential with finite complex energy. Among bounded-above subgeodesic rays with transform at most $u$ and $L^1$ initial limit at most $a$, there is a greatest ray $U^{u,a}$. It is a weak geodesic ray from $a$, its transform is $u$, and*

$$\mathcal{E}_\omega(U^{u,a}_s) = \mathcal{E}_\omega(a) + sE_\alpha(u) \qquad (s > 0). \tag{24.5}$$

*For normalized $u_i \in C_\alpha$ decreasing to $u$ and bounded $\omega$-psh $a_j$ decreasing to $a$, the rays $U^{u_i,a_j}$ decrease to $U^{u,a}$ on the product directed set.*

***Proof*** Shifting the initial potential translates the admissible rays by the same constant. It changes neither their transforms nor the energy difference from the initial potential. Thus suppose $a \le 0$. Each candidate $V$ satisfies $V_s \le a$. Indeed the radial functions $s \mapsto V_s(x)$ are convex and bounded above, so they are decreasing. Take a subsequence of the $L^1$ initial convergence which converges almost everywhere. The resulting inequality $V_s \le a$ almost everywhere holds everywhere for the psh representatives. In particular all candidates are nonpositive.

First let $u$ be continuous and $a$ bounded. The candidate $a - s\|u\|_\infty$ has constant transform $-\|u\|_\infty$, since bounded potentials have zero Lelong numbers. The upper regularized supremum of all candidates therefore lies between $a - s\|u\|_\infty$ and $a$. It has the prescribed initial limit. Its transform satisfies the bound by Lemma 22.7.1, so it is a candidate. If $-C \le a \le 0$, comparison of the admissible families gives

$$U^u_s - C \le U^{u,a}_s \le U^u_s.$$

The two bounds have transform $u$. On a finite time interval the weak geodesic between the bounded endpoint slices can replace $U^{u,a}$ by the same argument as in Proposition 24.4.1. At an interval starting at 0 its initial value is $a$, so the replacement is still admissible. This proves geodesicity. The two translates bound the energy slope by the same value $E_\alpha(u)$, and affinity of the complex energy gives (24.5).

For general $a, u$, take normalized $u_i \in C_\alpha$ decreasing to $u$ and bounded $\omega$-psh $a_j$ decreasing to $a$, for instance $a_j = \max(a, -j)$. Passing to a tail of the $a_j$ and translating $a$ and all $a_j$ by a common constant, we may assume $a_j \le 0$. On the product directed set the rays $U^{u_i,a_j}$ decrease by comparison of the admissible families. Their complex energies at each $s > 0$ are bounded below by $\mathcal{E}_\omega(a) + sE_\alpha(u)$. The countable selection by strict sublevel sets and the integral estimate in the proof of Theorem 24.4.1 give a nontrivial decreasing subgeodesic limit $U$. In that estimate the lower energy bound is now $\mathcal{E}_\omega(a) + sE_\alpha(u)$. Decreasing continuity gives (24.5) at each $s > 0$.

The slices satisfy $U_s \le a$ and increase as $s \downarrow 0$. Let $a'$ be their upper regularized increasing limit, so $a' \le a$. Increasing energy continuity [Xia26a, Proposition 17.3.1]

gives $\mathcal{E}_\omega(a') = \mathcal{E}_\omega(a)$. The ordered energy formula [Xia26a, Proposition 17.2.12], retaining the term against the Monge–Ampère measure of $a'$, gives

$$0 \le \int_X (a - a') \, \langle (\omega + dd^c a')^n \rangle \le (n+1)\big(\mathcal{E}_\omega(a) - \mathcal{E}_\omega(a')\big) = 0.$$

Both potentials have full mass by finite energy [Xia26a, Theorem 17.2.8]. The domination principle [Xia26a, Theorem 16.4.6] gives $a' \ge a$, hence $a' = a$. The pointwise increasing limit coincides with its regularization almost everywhere by the psh upper envelope theorem. Monotone integration, using any fixed positive-time slice as an integrable lower bound for the earlier slices, gives the initial $L^1$ limit $a$. On each finite interval the endpoint comparison with the approximating weak geodesics, as in Theorem 24.4.1, shows that $U$ is itself the weak geodesic.

We have $T(U) \le u$. By (24.2) and (24.5) its cohomological energy is at least $E_\alpha(u)$. Monotonicity and Lemma 24.4.1 give $T(U) = u$. Every admissible ray for $a, u$ is admissible for each $a_j, u_i$, so it lies below the limit. Hence $U$ is the asserted greatest ray $U^{u,a}$, for every choice of the approximations. This finishes the proof. □

## 24.5 Mixed energy slopes

This section computes the mixed energy slopes of the rays of Theorem 24.4.2 (Theorem 24.5.1). Its proof approximates the data and needs an energy estimate on the complex slices that is uniform in the ray parameter. For a Kähler form $\eta$, write $\eta_w = \eta + dd^c w$.

**Lemma 24.5.1** *Let $b \le a \le 0$ and $w_1, \dots, w_n \le 0$ be bounded $\eta$-psh potentials. Put $\Delta = \mathcal{E}_\eta(a) - \mathcal{E}_\eta(b)$, and let $B$ bound their negative energies. With $\gamma = 2^{-n}$ one has*

$$\int_X (a - b)\, \eta_{w_1} \wedge \cdots \wedge \eta_{w_n} \le C_n \Delta^\gamma B^{1-\gamma}. \tag{24.6}$$

***Proof*** Let $d = a - b$. For $0 \le k \le n$ let $A_k$ be the maximum of the integrals of $d$ against $n$ factors, at most $k$ of which are selected from the currents of the $w_j$, and all others from $\eta_a, \eta_b$. Repetition is allowed. The bounded energy difference formula [Xia26a, Proposition 17.2.3] gives $A_0 \le (n+1)\Delta$. In a term with $k$ of the $w_j$ factors, write the product as $\eta_w \wedge R$, where $R$ is the product of the other $n - 1$ factors. Replacing $\eta_w$ by $\eta_b$ leaves a term at most $A_{k-1}$.

On bounded differences the symmetric form

$$Q_R(f, g) = -\int_X f \, dd^c g \wedge R$$

is positive semidefinite by integration by parts and the Dirichlet inequality [Xia26a, Theorem 17.1.3 and Lemma 17.1.4]. For the latter, a linear combination of bounded

potentials is a bounded difference in a common multiple of the background. The error of the replacement is $-Q_R(d, w-b)$, and Cauchy–Schwarz gives

$$|Q_R(d, w-b)| \le \big(Q_R(d,d)\, Q_R(w-b, w-b)\big)^{1/2}.$$

Since $dd^c d = \eta_a - \eta_b$, positivity and $d \ge 0$ give

$$Q_R(d,d) \le \int_X d\,\eta_b \wedge R \le A_{k-1}.$$

For $q = w$ or $q = b$,

$$Q_R(q,q) \le \int_X (-q)\,\eta_q \wedge R.$$

Take the average $z$ of the $n{+}1$ potentials given by the weight $q$ and the $n$ current factors on the right. Each current factor is at most $(n{+}1)\eta_z$, and $-q \le (n{+}1)(-z)$. Therefore the integral is at most $(n{+}1)^{n+1}\int_X (-z)\eta_z^n$. The energy formula and concavity bound this by $(n+1)^{n+2}B$. The bound follows from the formula since all its integrals are nonpositive, and concavity bounds $-\mathcal{E}_\eta(z)$ by the average of the negative energies. Consequently $Q_R(w-b, w-b) \le C_n' B$ with $C_n' = 4(n{+}1)^{n+2}$, by Cauchy–Schwarz. We obtain

$$A_k \le A_{k-1} + \big(C_n' B A_{k-1}\big)^{1/2}.$$

If $\Delta = 0$, this recurrence gives $A_n = 0$, including the case $B = 0$. Otherwise divide by $B > 0$ and use $0 < \Delta/B \le 1$ at each step. Induction gives $A_k \le C_{n,k} B (\Delta/B)^{2^{-k}}$, which proves (24.6). □

**Lemma 24.5.2** *Let $\omega_i$, $1 \le i \le m$, be Kähler forms, and $b_i \le a_i \le 0$ be finite-energy $\omega_i$-psh potentials. Write $\Omega = \sum_i \omega_i$ and $\Delta_i = \mathcal{E}_{\omega_i}(a_i) - \mathcal{E}_{\omega_i}(b_i)$. The two sums have finite energy in $\Omega$. If $B$ bounds the negative individual energies, then*

$$0 \le \mathcal{E}_\Omega\Big(\sum_i a_i\Big) - \mathcal{E}_\Omega\Big(\sum_i b_i\Big) \le C B^{1-\gamma} \sum_i \Delta_i^\gamma, \qquad \gamma = 2^{-n}, \tag{24.7}$$

*where $C$ depends only on the forms, $m$, and $n$.*

***Proof*** First take all potentials bounded. Telescope the difference by replacing one $b_i$ with $a_i$ at a time. In the mixed energy formula for this step the weight is $a_i - b_i$ and the $n$ current factors are sums of the currents of the individual potentials. Expand those sums. Choose $C_0 \ge 1$ such that $C_0\omega_i \ge \omega_j$ for all $i, j$. If $p \le 0$ is one of the other entries in class $\omega_j$, the current bound

$$\omega_j + dd^c p \le C_0(\omega_i + dd^c(p/C_0))$$

permits replacement by the right side in each positive integral. The scaled potential is $\omega_i$-psh. Its energy satisfies

$$\mathcal{E}_{\omega_i}(p/C_0) \ge \mathcal{E}_{C_0\omega_j}(p/C_0) = C_0^{n+1}\mathcal{E}_{\omega_j}(p/C_0^2) \ge C_0^{n-1}\mathcal{E}_{\omega_j}(p).$$

In the energy formula an increase of the background by a semipositive form adds nonnegative terms to the products against the nonpositive weight, giving the first inequality. The last inequality is concavity of the energy on the segment from 0 to $p$. Apply Lemma 24.5.1 in background $\omega_i$ to all the expanded terms, with common bound $C_0^{n-1}B$. The number of terms and the current-bound constants depend only on the given forms and $m, n$, giving (24.7).

For unbounded data truncate each entry from below by the same constant $-k$. The order is preserved. To see finiteness of the limiting sums, let $p_i$ denote any such bounded truncated entries. Since $\Omega \le D_i\omega_i$ for some $D_i \ge 1$, the same background, scaling, and concavity comparisons give $\mathcal{E}_\Omega(p_i) \ge D_i^n \mathcal{E}_{\omega_i}(p_i)$. In the sum class use

$$\mathcal{E}_\Omega\Big(\sum_i p_i\Big) \ge m^{n+1}\mathcal{E}_\Omega\Big(\frac{1}{m}\sum_i p_i\Big) \ge m^n \sum_i \mathcal{E}_\Omega(p_i).$$

The first inequality compares $\Omega$ with $m\Omega$, and the second is concavity. Thus the decreasing sums have a uniform lower energy bound. Their pointwise limits are psh because each original entry is finite almost everywhere. Decreasing energy continuity for the individual entries and the sums now passes (24.7) to the limit. This uses only limits of scalar energies. □

For $n+1$ pairs $P_i = (\alpha_i, u_i)$ in Kähler classes with finite energy, define the mixed pairing by

$$\langle P_0, \dots, P_n\rangle_X = \frac{1}{n!}\sum_{\varnothing \ne I \subseteq \{0,\dots,n\}} (-1)^{n+1-|I|} E_{\sum_{i\in I}\alpha_i}\Big(\sum_{i\in I} u_i\Big).$$

The sums have finite energy by the proof of Theorem 23.3.2. On models this is the intersection pairing of Theorem 21.7.1. In the expansion of the alternating sum, terms missing an input cancel and the full mixed term occurs $(n+1)!$ times before the energy factor $1/(n+1)$. For finite complex energies in Kähler forms $\omega_i$, use the same formula with $\mathcal{E}_{\sum_{i\in I}\omega_i}$ for the complex mixed pairing $\langle(\omega_0, w_0), \dots, (\omega_n, w_n)\rangle_{\mathbb{C}}$. Finiteness for sums follows from Lemma 24.5.2 after constant translations, which preserve finite energy.

**Theorem 24.5.1** *Let $\alpha_i = [\omega_i]$ be Kähler classes, $u_i$ normalized cohomological finite-energy potentials, and $a_i$ finite-energy $\omega_i$-psh potentials, for $0 \le i \le n$. For $U^i = U^{u_i, a_i}$ one has*

$$\lim_{s\to\infty} \frac{\langle(\omega_0, U_s^0), \dots, (\omega_n, U_s^n)\rangle_{\mathbb{C}}}{s} = \langle(\alpha_0, u_0), \dots, (\alpha_n, u_n)\rangle_X. \tag{24.8}$$

***Proof*** It suffices to prove convergence for each energy of a sum, by the two pairing formulas. For a nonempty subset of the inputs, write $U = \sum_i U^i$, $u = \sum_i u_i$, $\beta = \sum_i \alpha_i$ and $\omega = \sum_i \omega_i$, restricting every sum to that subset. We may shift each initial potential by a constant to be nonpositive. This translates the corresponding ray and changes the energy of its sum by only a fixed constant. The ray $U$ has transform $u$ by additivity of generic Lelong numbers [Xia26a, Propositions 12.1.4 and 12.3.5]

(as in the proof of [Xia26a, Lemma 16.6.2]) and the divisorial order test. The upper estimate (24.2) thus bounds its upper energy slope by $E_\beta(u)$.

For the lower estimate, first let the $u_i$ be continuous and $a_i = 0$. Approximate each $u_i$ uniformly from below by rational semipositive models $h_i$. Use the compatible competitors in the proof of Proposition 24.4.1 on one fixed smooth refinement of all the models of the $h_i$. At each $\varepsilon$ choose the same smaller disk and rescaling for the finitely many competitors. The rescaling changes the marking by one base automorphism and leaves $\rho$ unchanged. This gives unaveraged punctured product functions $V^i_\varepsilon$ whose rotated slices are all dominated by $U^i_s$. Their sum at the same $t$ is smoothly compatible with the sum of the corrected divisors on the marked refinement. Its circular mean energy is bounded above by $\mathcal{E}_\omega(U_s)$. Letting $s$ tend to infinity, $\varepsilon$ tend to zero on the fixed refinements, and then $h_i$ tend uniformly to $u_i$, we obtain the lower slope $E_\beta(u)$ from (24.1). For bounded $a_i$, the corresponding sums lie between the zero-based sum and its translate down by a fixed constant, by the bounded comparison of Theorem 24.4.2. They have the same slope.

Finally choose simultaneous normalized continuous majorants $u_{i,j} \downarrow u_i$ and bounded truncations $a_{i,j} \downarrow a_i$. The resulting rays $U^{i,j}$ decrease to $U^i$, by the last assertion of Theorem 24.4.2 applied along the diagonal, which is cofinal in the product directed set. By (24.5), for $s \geq 1$ their individual negative energies and those of the limit are bounded by $C_1 s$ for a fixed $C_1 > 0$. The individual energy gaps are at most $s\delta_{i,j}$, where

$$\delta_{i,j} = \mathcal{E}_{\omega_i}(a_{i,j}) - \mathcal{E}_{\omega_i}(a_i) + E_{\alpha_i}(u_{i,j}) - E_{\alpha_i}(u_i) \longrightarrow 0.$$

Apply (24.7) at each $s$, and divide by $s$. The error in the slopes is bounded by $C_2 \sum_i \delta^\gamma_{i,j}$ uniformly for $s \geq 1$. The slopes for the approximating sums equal $E_\beta(\sum_i u_{i,j})$ by the bounded-initial, continuous case. The sums decrease and have finite energy in $\beta$, so these energies tend to $E_\beta(u)$. This gives the lower bound and proves convergence for every subset. Inclusion–exclusion proves (24.8). □

# Chapter 25
# Test curves in pseudoeffective classes

On the projective line, imposing vanishing of order $a$ at a point uses $a$ units of degree. A test curve records the same constraint on a scale of slopes. The curve can be analytified by reading off its divisorial singularities and then taking a Legendre transform. For the curve of Example 25.3.1 this produces the truncation of an ordinary valuation ray.

Throughout this chapter $X$ is a connected compact Kähler manifold of dimension $n$. In Sections 25.6 and 25.7 it is moreover projective. Test curves are first defined in classes where they have positive mass. A pseudoeffective class is then treated by adding arbitrarily small Kähler forms: a non-Archimedean potential is a compatible family of positive-mass test curves in the perturbed classes, and the transition toward a larger class retains the multiplier ideals at each slope. A sequence of perturbations determines the family (Proposition 25.2.1). Divisorial evaluation is an order embedding, and it recovers the divisorial Lelong numbers of every slice (Proposition 25.4.1). The mass distribution of a test curve is its Duistermaat–Heckman measure (Proposition 25.5.1).

On a projective manifold, a finite weighted system of sections gives a compatible family whose divisorial evaluation is the corresponding Fubini–Study function (Proposition 25.6.1). Divisorial evaluation then extends to an order isomorphism between the compatible families and the algebraic psh functions, in each pseudoeffective real class spanned by line bundle classes (Theorem 25.7.1). For a test curve in a big class, the first moment of its mass distribution gives the energy of its relative geodesic ray (Theorem 25.8.1). A family of positive volume comes from an ordinary test curve (Proposition 25.9.1), which gives an intrinsic finite-energy criterion in a big class (Corollary 25.9.1). In a Kähler class $\alpha$, the ray transform and the maximal rays of Chapter 24 identify the normalized finite-energy test curves of left limit zero with the normalized potentials in $\mathcal{E}^1_\alpha$ (Theorem 25.10.1). The coefficient ideals of the Legendre ray extend evaluation to the full valuation space $X^{\sqsupset}$ (Theorem 25.11.1). In a Kähler class this jet extension identifies the compatible families with the whole decreasing semipositive cone $\mathcal{D}_\alpha$ (Theorem 25.12.1), and on a projective manifold it agrees with the projective comparison (Corollary 25.12.1).

Chapter 26 uses the jet extension to identify the perturbative cone of a nef class with the compatible families, and Chapter 28 realizes the obstacle energy of a big class by test curves. The test-curve energy of Corollary 25.9.1 and the jet extension return in Chapters 29 and 30.

## 25.1 Singularities and test curves

Let $\alpha$ be a smooth real closed $(1,1)$-form on $X$. Write $\mathrm{PSH}_{\mathbb{C}}(X,\alpha)$ for the quasi-plurisubharmonic functions $u$ with $\alpha + \mathrm{dd}^c u \geq 0$. Use the normalization in which $\mathrm{dd}^c \log|z|^2 = [0]$ in one variable. For a quasi-psh potential $u$, its multiplier ideal $\mathcal{I}(u)$ consists of germs of holomorphic functions $g$ for which $|g|^2\exp(-u)$ is locally integrable. Write $u \sim_{\mathcal{I}} v$ if $\mathcal{I}(mu) = \mathcal{I}(mv)$ for every positive integer $m$. The same condition at all real positive indices is equivalent. The basic facts about multiplier ideals and their description by all divisorial Lelong numbers are recalled in [Xia26b, §§1.4 and 3.2.1].

The $\mathcal{I}$-envelope of a potential in the class of $\alpha$ is

$$P_\alpha[u]_{\mathcal{I}} = \big(\sup\{w \in \mathrm{PSH}_{\mathbb{C}}(X,\alpha) : w \leq 0,\ w \sim_{\mathcal{I}} u\}\big)^*.$$

It is $\alpha$-psh and $\mathcal{I}$-equivalent to $u$, so the supremum is attained. A potential equal to its $\mathcal{I}$-envelope is called *$\mathcal{I}$-model*. We use its envelope, concavity and mass properties from [Xia26b, §3.2]. For an ordinary $P$-model potential, the defining envelope is the upper regularized supremum of nonpositive potentials of the same singularity type modulo bounded functions. A positive-mass $\mathcal{I}$-model potential is $P$-model, since equality modulo bounded functions implies $\mathcal{I}$-equivalence.

The mass of a potential in a big class is the total mass of the non-pluripolar product $\langle(\alpha + \mathrm{dd}^c u)^n\rangle$. A positive-mass $\mathcal{I}$-model test curve in the class of $\alpha$ consists of a finite number $M$ and a concave decreasing family $\Gamma_\tau$, $-\infty < \tau < M$, of $\mathcal{I}$-model potentials such that the left limit $\Gamma_{-\infty} = (\sup_{\tau<M}\Gamma_\tau)^*$ has positive mass. This is the test-curve convention of [Xia26b, §§9.1 and 9.3]. Every interior slice has positive mass. Indeed, by concavity that slice is bounded below by a positive combination of an earlier positive-mass slice and a later slice, and positivity of the mixed products and mass monotonicity give the assertion. Such an earlier positive-mass slice exists by increasing continuity of mass for model potentials with positive-mass limit. We will always take suprema over $\tau < M$, so the right endpoint need not be assigned a value in the definition.

For a smooth semipositive real closed form $\eta$ there is a transition map

$$J_{\alpha+\eta,\alpha}(\Gamma)_\tau = P_{\alpha+\eta}[\Gamma_\tau]_{\mathcal{I}} \qquad (\tau < M).$$

Indeed $\Gamma_\tau$ is $(\alpha+\eta)$-psh. The monotonicity and concavity of the envelope give a decreasing concave family. Its levels are $\mathcal{I}$-model and its left limit is $P_{\alpha+\eta}[\Gamma_{-\infty}]_{\mathcal{I}}$, with positive mass, by the increasing-limit and mass properties in [Xia26b, §§3.2 and 9.3]. The cutoff remains $M$. Two successive transitions preserve the original $\mathcal{I}$-

type and yield model potentials in the final class. They therefore equal the direct transition.

## 25.2 Taking the perturbation limit

Now let $\theta$ represent an arbitrary pseudoeffective class. For every Kähler form $\omega$, the class of $\theta + \omega$ is big. A non-Archimedean potential in the test-curve construction is a family

$$\Gamma = (\Gamma^{\omega})_{\omega}, \qquad J_{\theta+\omega,\theta+\omega'}(\Gamma^{\omega'}) = \Gamma^{\omega} \quad \text{if } \omega \geq \omega',$$

where each coordinate is a positive-mass $\mathcal{I}$-model test curve in $\theta + \omega$. The inequality of forms means semipositivity of $\omega - \omega'$ at every point. Denote this set by $\mathrm{PSH}_{\mathrm{tc}}(X, \theta)$. Any two perturbations admit a common smaller Kähler form, and the transition maps preserve the cutoff. Thus the coordinates have one common cutoff $\Gamma_{\max}$. This is the projective-limit construction of [Xia26b, §13.1].

**Proposition 25.2.1** *Fix a Kähler form $\omega_0$. The coordinates at $\omega_j = 2^{-j}\omega_0$ for $j \geq 1$ determine the entire compatible family. Any compatible family over these perturbations extends uniquely to all perturbations.*

***Proof*** For any $\omega$, compactness and positive definiteness give $c > 0$ with $\omega \geq c\omega_0$. Choose $j$ with $2^{-j} \leq c$ and set $\Gamma^{\omega}$ equal to the image of $\Gamma^{\omega_j}$. Two choices give the same image by comparison with a later index. If $\omega \geq \omega'$, use one index below both forms to check the claimed transition. Any extension must have these coordinates.□

An ordinary positive-mass $\mathcal{I}$-model test curve in $\theta$ gives a compatible family by enlargement to each $\theta + \omega$. If two positive-mass curves have the same image at one perturbation, they have the same cutoff and the same $\mathcal{I}$-type at each interior slope. They coincide since both original curves are $\mathcal{I}$-model in $\theta$. More generally, equality of two families can be checked at one perturbation. To see this, descend to a common smaller perturbation of the given perturbation and any target one. At the smaller perturbation the two model curves have the same types, since both transition maps to the given perturbation preserve those types. They coincide in the smaller class, so their images agree in the target.

This construction depends only on the real Bott–Chern class. For $\theta' = \theta + \mathrm{dd}^c g$ with $g$ smooth, take the $\mathcal{I}$-envelope of $\Gamma^{\omega}_{\tau} - g$ in the class represented by $\theta' + \omega$. This change of representative preserves the cutoff, concavity and the mass condition [Xia26b, §9.3]. Smooth changes preserve multiplier ideals. Hence this map commutes with the transition maps, and changing back by $-g$ gives the identity. The same argument for a sum of changes gives the cocycle identity.

## 25.3 Divisorial evaluation

Let $v = r\,\mathrm{ord}_F$, where $r > 0$ and $F$ is a prime divisor on a modification of $X$. For a quasi-psh potential $u$ write $v(u) = r\nu_F(u)$, where $\nu_F(u)$ is the generic Lelong number of the pullback along $F$. Equivalently this is

$$v(u) = \lim_{m\to\infty} \frac{1}{m} v(\mathcal{I}(mu)).$$

The divisorial multiplier-ideal estimate gives this identity with the normalization $\mathrm{dd}^c \log|z|^2 = [0]$ fixed above [Xia26b, §§1.4 and 13.4.2]. The trivial valuation has value zero on every such potential. These values depend only on the $\mathcal{I}$-type.

For a compatible family $\Gamma$, its *divisorial evaluation* is

$$\Gamma^{\mathrm{an}}(v) = \sup_{\tau<\Gamma_{\max}} \{\tau - v(\Gamma^{\omega}_{\tau})\}. \tag{25.1}$$

The transition maps preserve the $\mathcal{I}$-type at each slope, so the expression is independent of $\omega$. It is at most $\Gamma_{\max}$ since the Lelong numbers are nonnegative, and it is finite at each fixed divisorial valuation by evaluating one interior term. At the trivial valuation it equals $\Gamma_{\max}$. On a divisorial ray it is convex and decreasing as a function of $r$, because it is a supremum of affine functions with nonpositive slopes. This agrees with the raywise behavior in Section 6.3.

For two slope families with the same positive-mass slices up to a translation of the slope parameter by $c$, the divisorial functions differ by the same additive constant $c$. For the change of slopes $\Gamma'_\tau = \Gamma_{\tau/a}$ with $a > 0$, the change is $\Gamma'^{\mathrm{an}}(v) = a\Gamma^{\mathrm{an}}(v/a)$. Both identities follow by changing the variable in (25.1).

*Example 25.3.1* Let $X = \mathbb{P}^1_{\mathbb{C}}$, take $\theta$ to be the curvature form of the Fubini–Study metric on $O(1)$, and fix a point $p$. Choose $g_p \le 0$ with $\theta + \mathrm{dd}^c g_p = [p]$, obtained by taking the logarithm of the squared norm of a section vanishing at $p$. For $0 < b < 1$, set

$$a(\tau) = \max(0, \tau + b), \qquad \Gamma^{\omega}_{\tau} = P_{\theta+\omega}[a(\tau)g_p]_{\mathcal{I}} \quad (\tau < 0).$$

Here we use the convention $0g_p = 0$. The original family $a(\tau)g_p$ is decreasing and concave since $a$ is increasing and convex and $g_p \le 0$. Since $\Gamma^{\omega}_{\tau} \ge a(\tau)g_p$, its only possible pole is $p$. Its Lelong number there equals $a(\tau)$ by $\mathcal{I}$-equivalence. If $a(\tau) > 0$, the point is a pole and the curvature measure has an atom of mass $a(\tau)$ there. The non-pluripolar mass is therefore at most the total mass minus this atom. It is at least the mass $1 + \int_X \omega - a(\tau)$ of $a(\tau)g_p$, by mass monotonicity [Xia26b, Theorem 2.4.3], since $\Gamma^{\omega}_{\tau} \ge a(\tau)g_p$. If $a(\tau) = 0$, the envelope is zero. Thus the mass is $1 + \int_X \omega - a(\tau) > 0$ for all $\tau < 0$. The envelope properties give the required test curves, with left limit zero, and the transition identity gives compatibility. At $v = r\,\mathrm{ord}_p$ one obtains

$$\Gamma^{\mathrm{an}}(v) = \sup_{\tau<0}\{\tau - ra(\tau)\} = -b\min(r, 1).$$

Indeed on $\tau \le -b$ the supremum is $-b$, and on $-b < \tau < 0$ the expression has slope $1 - r$. At valuations based at other points the value is zero. For $b = \ell/m$ rational, this is the algebraic Fubini–Study potential given by a basis of $H^0(X, \mathcal{O}(m))$ with weight $-mb$, together with a basis of the subspace of sections divisible by the $\ell$-th power of a linear section at $p$, with weight zero. The latter system vanishes to order $\ell$ at $p$ and has no common zero elsewhere. The Fubini–Study potential gives the continuous extension of the computed divisorial values. Its value at the endpoint of the ray at $p$ is $-b$, and it is zero on every other ray, including the endpoints.

## 25.4 Order and recovery from divisorial rays

For two compatible families $\Gamma, \Lambda$ in the same class, say $\Gamma \le \Lambda$ if $\Gamma_{\max} \le \Lambda_{\max}$ and $\Gamma^{\omega}_{\tau} \le \Lambda^{\omega}_{\tau}$ for $\tau < \Gamma_{\max}$ in each perturbation. The order can be checked in one perturbation. Indeed, for two $\mathcal{I}$-model potentials $u, w$ in a common class, pointwise order $u \le w$ holds exactly when $v(u) \ge v(w)$ for every divisorial valuation. One implication is monotonicity of Lelong numbers. For the other, $\max(u, w)$ has generic Lelong number equal to the smaller of the two at every divisor, so it is $\mathcal{I}$-equivalent to $w$. It is nonpositive and psh in the same class. The maximality of the $\mathcal{I}$-model representative gives $\max(u, w) = w$. Since transition maps preserve all divisorial values of the slices, this test gives the assertion about perturbations.

**Proposition 25.4.1** *The evaluation of* (25.1) *is an order embedding on* $\mathrm{PSH}_{\mathrm{tc}}(X, \theta)$. *More precisely, for a divisor order $v$, any perturbation $\omega$ and any $\tau < \Gamma_{\max}$,*

$$v(\Gamma^{\omega}_{\tau}) = \sup_{r>0} \frac{\tau - \Gamma^{\mathrm{an}}(rv)}{r}.$$

*The trivial valuation determines the cutoff.*

***Proof*** Put $M = \Gamma_{\max}$ and $a(\sigma) = v(\Gamma^{\omega}_{\sigma})$. The function $a$ is finite, nonnegative and increasing for $\sigma < M$. It is convex by concavity of the test curve and the additivity of generic Lelong numbers. For $s > 0$ put

$$B(s) = \sup_{\sigma < M} \{s\sigma - a(\sigma)\}.$$

Then $B(s) = s\Gamma^{\mathrm{an}}(v/s)$, and $s\tau - B(s) \le a(\tau)$. If $a$ has a strictly positive supporting slope $s$ at $\tau$, equality holds for that $s$. Otherwise zero is a supporting slope, because $a$ is finite, convex and increasing and $\tau$ is interior. Then $a(\tau) = \inf_{\sigma<M} a(\sigma)$. As $s \downarrow 0$, $B(s) \to -\inf_{\sigma<M} a(\sigma)$: the upper bound uses $\sigma < M$, and for the lower bound one first fixes $\sigma$, then takes the supremum over $\sigma$. Thus the supremum of $s\tau - B(s)$ over $s > 0$ equals $a(\tau)$ in this case as well. Substitute $s = 1/r$.

If $\Gamma \le \Lambda$, the Lelong order reverses and the slope domain of $\Gamma$ is contained in that of $\Lambda$. Hence $\Gamma^{\mathrm{an}} \le \Lambda^{\mathrm{an}}$. Conversely this inequality at the trivial valuation gives the cutoff comparison, and the formula just proved reverses the inequality for the values

of the slices at each divisor. The model test for the pointwise order completes the proof. □

Let $\lambda > 0$. Multiplication by $\lambda$ takes $\mathrm{PSH}_{\mathrm{tc}}(X,\theta)$ to $\mathrm{PSH}_{\mathrm{tc}}(X,\lambda\theta)$ by

$$(\lambda\Gamma)^{\omega}_{\tau} = \lambda\Gamma^{\omega/\lambda}_{\tau/\lambda}, \qquad \tau < \lambda\Gamma_{\max}.$$

Multiplication of potentials by $\lambda$ preserves $\mathcal{I}$-equivalence in both directions, since one can test the multiplier ideals at all positive real indices. Thus it commutes with the envelope when the form is scaled as well. This proves compatibility with the transitions. Concavity and monotonicity persist, and the left mass is multiplied by $\lambda^n$, by homogeneity of the non-pluripolar product. Divisorial evaluation transforms by $(\lambda\Gamma)^{\mathrm{an}} = \lambda\Gamma^{\mathrm{an}}$, since $v(\lambda u) = \lambda v(u)$. Here both the class and the slope parameter have been scaled.

## 25.5 The mass distribution of a test curve

For a positive-mass $\mathcal{I}$-model test curve in a big class $\alpha$, write $M = \Gamma_{\max}$ and put

$$m(\tau) = \int_X \langle(\alpha + \mathrm{dd}^c\Gamma_\tau)^n\rangle \qquad (\tau < M),$$
$$m_- = \int_X \langle(\alpha + \mathrm{dd}^c\Gamma_{-\infty})^n\rangle.$$

Mass monotonicity, log concavity and continuity at the left endpoint give a decreasing continuous function on $(-\infty, M)$ with limit $m_-$ at $-\infty$ [Xia26b, §9.1]. Extend the function by zero at and beyond $M$. There is a unique finite positive Radon measure $\mu_\Gamma$ with

$$\mu_\Gamma((\tau,\infty)) = m(\tau) \quad (\tau < M), \qquad \mu_\Gamma((M,\infty)) = 0.$$

Indeed the extended function is decreasing and right continuous, so its negative distributional derivative is a positive measure. Its total mass is $m_-$ and its mass at $M$ is $\lim_{\tau\uparrow M} m(\tau)$. All interior levels are continuity levels.

**Proposition 25.5.1** *The measure $\mu_\Gamma$ is the Duistermaat–Heckman measure of $\Gamma$. It is $n!$ times the measure of its Okounkov test curve. The first moment, with $-\infty$ allowed, is*

$$\int_{\mathbb{R}} x\,\mathrm{d}\mu_\Gamma(x) = Mm_- + \int_{-\infty}^{M} (m(\tau) - m_-)\,\mathrm{d}\tau.$$

***Proof*** For a smooth flag use the nested convex bodies $\Delta_\tau$ of the Okounkov test curve in the body $\Delta$ of the left endpoint, which is $\mathcal{I}$-model by [Xia26b, Proposition 9.3.1]. Their volumes are $m(\tau)/n!$ and $m_-/n!$, respectively [Xia26b, Ch. 10 and §13.3]. The slope function is $G(a) = \sup\{\tau < M : a \in \Delta_\tau\}$. For $\tau < M$, the set $\{G > \tau\}$ is the increasing union of $\Delta_{\tau_j}$ for any sequence $\tau_j \downarrow \tau$ with $\tau_j > \tau$. Its volume is $m(\tau)/n!$

by continuity. Letting $\tau \to -\infty$ shows that $G$ is finite almost everywhere on $\Delta$. The pushforward of $n!$ times the volume on $\Delta$ therefore has the stated tails. Independence of the flag follows. For the construction via a proper modification admitting a flag, the pullback is again an $\mathcal{I}$-model test curve [Xia26b, Proposition 9.3.3]. The non-pluripolar masses are unchanged, since the morphism is an isomorphism off analytic sets, which these measures do not charge.

For the moment identity use $M - x = \int_{-\infty}^{M} \mathbf{1}_{\{x \le \tau\}}\, \mathrm{d}\tau$ for $x \le M$, and integrate by Tonelli. Since $\mu_\Gamma(\{x : x \le \tau\}) = m_- - m(\tau)$ at $\tau < M$, the stated identity follows, including the case of divergent negative moment. □

## 25.6 Test curves of weighted sections

Let $X$ be connected, smooth and projective over $\mathbb{C}$, and let $\theta = c_1(L, h)$ for a rational line bundle $L$ with a smooth Hermitian metric. A finite weighted system of sections gives both a Fubini–Study function on the algebraic valuation space and a compatible family of complex test curves. This section compares the two for a single system; limits of such data are taken in Section 25.7.

**Proposition 25.6.1** *Let $s_1, \dots, s_N$ be nonzero sections of $mL$ for some positive integer $m$ for which $mL$ is a line bundle. Give them rational weights $\lambda_1, \dots, \lambda_N$. There is a compatible family of $\mathcal{I}$-model test curves $\Gamma$ in class $\theta$, with cutoff $M = \max_i \lambda_i/m$, whose divisorial evaluation is*

$$\Gamma^{\mathrm{an}}(v) = \frac{1}{m} \max_i \{\lambda_i - v(s_i)\}.$$

*The sections may have a common zero set.*

***Proof*** Group the sections of the same weight and let $\tau_1, \dots, \tau_r$ be the distinct weights divided by $m$. Set

$$A_j = \frac{1}{m} \max_{\lambda_i/m = \tau_j} \log |s_i|^2_{h^m}.$$

Each $A_j$ is $\theta$-psh, by the local psh property of logarithms of holomorphic sections and of their finite maximum with a common background. Subtract a common constant so that every $A_j \le 0$. This changes none of their divisorial orders.

Let $K_\tau$ be the compact polytope

$$\{(a_1, \dots, a_r) : a_j \ge 0,\ \textstyle\sum_j a_j = 1,\ \sum_j a_j \tau_j \ge \tau\} \quad (\tau < M).$$

Define $B_\tau = \max_{a \in K_\tau} \sum_j a_j A_j$. A term with zero coefficient is omitted. At each point of $X$ the set of feasible $a$ giving a finite sum is a face obtained by vanishing of the coefficients for the infinite negative terms. If that face is nonempty, a maximum occurs at a vertex of the face and hence at a vertex of $K_\tau$. The formula is therefore

a finite maximum of $\theta$-psh convex combinations. The shrinking feasible sets give monotonicity in $\tau$, and taking a convex combination of two maximizers gives concavity. For $\tau \le \min_j \tau_j$ the polytope is the full simplex and $B_\tau = \max_j A_j$.

For each Kähler form $\omega$ put $\Gamma^\omega_\tau = P_{\theta+\omega}[B_\tau]_{\mathcal{I}}$. Monotonicity of the envelope gives decreasing slices. For concavity, take a convex combination of two model slices. Its divisorial Lelong numbers are those of the same combination of the corresponding $B_\tau$'s by additivity. The latter is at most $B$ at the interpolated slope. Order of the $\mathcal{I}$-types and the envelope property then bound the combination by the model slice there. The envelope preserves the $\mathcal{I}$-type, so the transitions defining the compatible families give the required compatibility. On the left tail the model slices have positive mass: $B_\tau$ has analytic singularities and $\theta + \omega + \mathrm{dd}^c B_\tau \ge \omega$. On the complement of the common zero set the non-pluripolar product has at least the mass of $\omega^n$, and the envelope can only increase the mass.

At a divisorial valuation $v$, put $x_j = v(A_j)$. The finite maximum formula gives

$$v(B_\tau) = \min_{a \in K_\tau} \sum_j a_j x_j .$$

Indeed, near a general point of the divisor on a smooth model every $A_j$ is a logarithmic multiple of a local defining equation plus a bounded function. The finite maximum keeps the smallest coefficient. The envelope preserves this value. Thus for any feasible $a$,

$$\tau - \sum_j a_j x_j \le \sum_j a_j(\tau_j - x_j) \le \max_j(\tau_j - x_j).$$

Taking the single vertex $a_j = 1$ and letting $\tau$ increase to $\tau_j$ gives the opposite bound in the supremum over $\tau < M$. Finally $x_j = m^{-1} \min_{\lambda_i/m = \tau_j} v(s_i)$, which proves the formula. □

*Example 25.6.1 (A point blowup)* Assume $\dim X > 0$, let $L$ be ample and fix $p \in X$. Choose $m$ sufficiently large that both $mL$ and $\mathfrak{m}_p \otimes mL$ are generated by global sections. Give a generating system of $mL$ weight $-1$ and a generating system of $\mathfrak{m}_p \otimes mL$ weight 0. The resulting valuative potential is

$$u(v) = \frac{1}{m} \max\{-1, -v(\mathfrak{m}_p)\}.$$

The section calculation produces a test curve with this divisorial evaluation. Under the projective comparison of Corollary 21.8.1, its continuous valuative function is the cohomological point-blowup function of Proposition 21.9.1 with $c = 1/m$. Increasing $m$ if necessary makes the model semipositive. This identifies the point measure computed there on these common data.

## 25.7 The projective comparison

The comparison involves two different limits. A decreasing family of algebraic approximants gives a decreasing family of test curves in each fixed perturbation. In the other direction, a single test curve yields algebraic potentials in smaller and smaller perturbations of the numerical class.

**Lemma 25.7.1** *Let $(C_i)_i$ be a decreasing net of ordinary positive-mass $\mathcal{I}$-model test curves in one big class $\beta$. Write $M = \inf_i (C_i)_{\max}$. Assume that $M > -\infty$ and that the masses of $C_{i,\tau_0}$ are bounded below by a positive constant for some $\tau_0 < M$. Then the curve $C_\tau = \inf_i C_{i,\tau}$ for $\tau < M$ is an ordinary positive-mass $\mathcal{I}$-model test curve of cutoff $M$, and for every $\tau < M$ the masses of $C_{i,\tau}$ converge to the mass of $C_\tau$. At a divisorial or trivial valuation,*

$$C^{\mathrm{an}}(v) = \inf_i C_i^{\mathrm{an}}(v).$$

*For an ordinary curve, the superscript denotes the evaluation of its compatible family.*

***Proof*** The analytic decreasing-limit theorem gives the curve. At any fixed $\tau < M$ the masses of the decreasing model slices converge to the mass of the limit slice, which is positive. Thus the slices converge for $d_S$, the distance on singularity types. Divisorial Lelong numbers are continuous for this convergence, also on a modification carrying the divisor. These facts are [Xia26b, Proposition 9.4.9, Corollary 6.2.6, Theorem 6.2.4, and the slice argument in Proposition 13.4.5]. In particular $v(C_\tau) = \sup_i v(C_{i,\tau})$.

The cutoff calculation proves the claim at the trivial valuation. At a divisorial valuation put $M_i = (C_i)_{\max}$ and $f_i(\tau) = \tau - v(C_{i,\tau})$ on $\tau < M_i$. The function $f_i$ is continuous there by the convexity of the slice orders in Proposition 25.4.1. Extend it to $M_i$ by its left limit, which exists in $[-\infty, M_i]$ by monotonicity of the slice orders. Set it equal to $-\infty$ above $M_i$. The extended functions are upper semicontinuous, satisfy $f_i(\tau) \le \tau$, and decrease in $i$. For the last assertion at an endpoint, take left limits of the order on interior levels if the cutoffs coincide, or use continuity in the larger domain if they differ. Their suprema are unchanged by the endpoint extension.

Put $s = \inf_i \sup_\tau f_i(\tau)$. The inequality $C^{\mathrm{an}}(v) \le s$ follows by order. Take a real $a < s$ and restrict to a tail with $M_i \le M_{i_0}$. The closed sets

$$\{\tau \in [a, M_{i_0}] : f_i(\tau) \ge a\}$$

are nonempty and have the finite intersection property since the net is decreasing. Choose a common point $\tau_*$. Then $\tau_* \le M$. If $\tau_* < M$, slice convergence gives $C^{\mathrm{an}}(v) \ge \inf_i f_i(\tau_*) \ge a$. If $\tau_* = M$, then for every $\tau < M$,

$$f_i(\tau) \ge f_i(M) + \tau - M \ge a + \tau - M$$

by monotonicity of the slice orders. First take the infimum in $i$, then let $\tau$ increase to $M$, to get the same bound. This proves the formula. □

**Lemma 25.7.2** *Let $X$ be smooth and projective over $\mathbb{C}$. Let $\beta = c_1(B, h)$ for a big rational line bundle $B$. The divisorial evaluation of any ordinary positive-mass $\mathcal{I}$-model test curve in $\beta$ extends to a unique $[B]$-psh function on $X^{\mathrm{an}}$.*

***Proof*** By the positive scaling rule, clear denominators in $B$. Translate the cutoff to zero. The Legendre dual of the curve $C$ is the relative ray $U_s = \sup_{\tau<0}(C_\tau + s\tau)$ for $s > 0$, based at the positive-mass model potential $C_{-\infty}$. On the punctured disk its complexification is $\Phi(x, \zeta) = U_{-\log|\zeta|^2}(x)$. We use the positive-mass argument in [Xia26b, Propositions 13.4.3 and 13.4.4 and Lemma 13.4.6]. With the normalization of (25.1), the Gauss extension $\sigma(v)$ has value one on the parameter. For $v$ a positive rational multiple of a divisor order the ray formula gives $C^{\mathrm{an}}(v) = -\sigma(v)(\Phi)$.

The algebraic approximation in that argument enters the definition of $[B]$-psh as follows. Over the punctured product the multiplier ideal $\mathcal{I}(m\Phi)$ equals the pullback of $\mathcal{I}(mC_{-\infty})$, since every slice of the ray has full mass relative to $C_{-\infty}$. Gluing with this pullback over $X \times \mathbb{C}^\times$ extends it to a circle-invariant coherent ideal on $X\times\mathbb{C}$, which is algebraic [Xia26b, Lemmas 13.4.4 and 13.4.5 and Proposition 13.4.4]. Its homogeneous expansion has coefficient ideals $\mathfrak{a}_{m,j}$ on $X$, for $0 \le j \le N_m$, with the last coefficient repeated in all higher degrees. Keeping that coefficient as an ideal allows horizontal singularities. For one fixed ample twist $T$, the sheaves $\mathfrak{a}_{m,j} \otimes (mB + T)$ are globally generated. Their generators with weights $-j$ give the function

$$h_m(v) = -\frac{1}{m} \min_{0\le j\le N_m} \{v(\mathfrak{a}_{m,j}) + j\} \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(m^{-1}T + B).$$

Indeed the Gauss formula uses the minimum of exactly these terms. Subadditivity gives $\mathcal{I}(2m\Phi) \subseteq \mathcal{I}(m\Phi)^2$. The orders of the coefficient ideals therefore give $h_{2m} \le h_m$ at every semivaluation, including at poles. At a rational divisorial valuation the multiplier-ideal estimate gives $h_{2^k}(v) \downarrow -\sigma(v)(\Phi)$. It also gives convergence at the trivial base valuation, since its Gauss extension is the order of the central fibre of the product. The limit there is zero. The classes of the $h_{2^k}$ tend to $[B]$, so their pointwise limit is $[B]$-psh. On the ray $rv$ of a fixed divisor order $v$, the displayed formula for $h_m$ is convex in $r > 0$. Their decreasing limit is convex and finite at all positive rational $r$, so it is finite and continuous throughout $r > 0$. Indeed, if the limit were $-\infty$ at some $r_0 > 0$, convexity on the segment from $r_0$ to a positive rational point would force the value $-\infty$ at the rational points strictly between them, contradicting finiteness at positive rationals. The evaluation of $C$ is also finite convex in $r$ by (25.1). The equality therefore extends to real scalings. Uniqueness follows from Theorem 6.3.1. Undo the translation and scaling to recover the original data. □

**Theorem 25.7.1** *Let $X$ be smooth, connected and projective over $\mathbb{C}$. Let $\alpha$ be a pseudoeffective form that is a real linear combination of smooth curvature forms of line bundles, and let $\theta$ be the same combination of their numerical classes. Divisorial evaluation extends to an order isomorphism*

$$\mathrm{PSH}_{\mathrm{tc}}(X, \alpha) \longrightarrow \mathrm{PSH}(X, \theta).$$

*It commutes with adding constants and with positive scaling of both class and potential. If the given combination is a positive combination of ample line bundles, so that* $[\alpha]$ *is the Kähler class of Section 22.5, then, under the identification of valuation spaces, the continuous functions in the image of this isomorphism are exactly the functions of* $C_{[\alpha]}$ *in Theorem 22.5.1.*

***Proof*** We first choose rational perturbations of the given class. Let $\Omega$ be the curvature form of an ample line bundle. Approximate the real coefficients defining $\alpha + 2^{-j}\Omega$ by rationals with error small enough that the resulting curvature form is $\alpha + \eta_j$ with

$$\tfrac{3}{4}2^{-j}\Omega \le \eta_j \le \tfrac{5}{4}2^{-j}\Omega.$$

Write $B_j$ for the resulting rational line bundle and $A$ for the ample bundle with curvature $\Omega$. For a small rational $\delta > 0$, the bundle $B_j - \delta A$ has a psh Hermitian weight, since $\alpha$ is pseudoeffective and $\eta_j - \delta\Omega$ is positive. The fixed-twist generation in the proof of [Xia26b, Lemma 13.4.2] gives a nonzero section of $m(B_j - \delta A) + T$ for divisible $m$ and a fixed twist $T$. The section is algebraic by full faithfulness of the coherent comparison in the proof of Lemma D.3.1. For large $m$, $m\delta A - T$ is ample. This expresses $mB_j$ as the sum of an effective divisor and an ample bundle, so $B_j$ is big. The forms $\eta_j$ decrease and are coinitial among Kähler forms, and $[B_j] \to \theta$.

If $\Gamma$ is a compatible family, apply Lemma 25.7.2 to $\Gamma^{\eta_j}$. The resulting $[B_j]$-psh functions have the same divisorial values. They are the same pointwise by Theorem 6.3.1, which can compare psh functions of different classes. Their common value at the trivial point is the cutoff of $\Gamma$. By Theorem 16.6.1, this constant sequence in varying classes has a $\theta$-psh limit. This defines the comparison map. Its order and injectivity follow from Proposition 25.4.1 and Theorem 6.3.1.

For surjectivity take $u \in \mathrm{PSH}(X, \theta)$. By Proposition 6.2.1, choose a decreasing rational Fubini–Study net $f_i \downarrow u$ for line bundles $L_i$ with $[L_i] \to \theta$. For each fixed $j$, on a tail in $i$, $B_j - L_i$ is ample. Indeed $[B_j] - \theta$ is ample, by making the rational coefficient errors in the preceding choice small enough in $N^1(X)_{\mathbb{R}}$, and ampleness is open. By (6.3) the same $f_i$ is a rational weighted section function for $B_j$. Let $\Gamma^{i,j}$ be the family in $\alpha + \eta_j$ given by Proposition 25.6.1.

Fix a Kähler form $\omega$ and choose a $j$ so large that $\eta_j < \omega/2$. For all $i$ on the tail just considered, let $C_i^{\omega}$ be the coordinate of $\Gamma^{i,j}$ at the perturbation $\omega - \eta_j$. This is an ordinary test curve in $\alpha + \omega$ with divisorial evaluation $f_i$. Its cutoff is $f_i(v_{X,0})$, so the limiting cutoff is finite. The curves decrease by Proposition 25.4.1, using their images in the compatible families for $\alpha + \omega$ and the order test on their original model slices. For each interior level common to the curves, the coordinate at $\omega/2 - \eta_j$ is $(\alpha + \omega/2)$-psh. In the class $\alpha + \omega$ its current has the added positive form $\omega/2$. The non-pluripolar mass in that class is at least $\int_X (\omega/2)^n$, by expanding the product with this smooth form. Taking the $\mathcal{I}$-envelope in $\alpha + \omega$ preserves this lower bound by mass monotonicity. This envelope gives the corresponding slice of $C_i^{\omega}$. Thus Lemma 25.7.1 gives an ordinary limit curve $C^{\omega}$ of evaluation $\inf_i f_i = u$ on divisorial and trivial valuations. It is independent of $j$ and of the chosen tail by the recovery formula in the common class $\alpha + \omega$.

For $\omega' \le \omega$, the transition applied to $C^{\omega'}$ has the same divisorial evaluation as $C^{\omega}$, since it preserves slice types and the cutoff. By the same recovery formula the curves coincide. Consequently the $C^{\omega}$ form a compatible family. Its image is $u$ by Theorem 6.3.1. The assertions about constants and scaling follow by changing variables in (25.1). The continuous comparison is Theorem 22.5.1 under the identification of the two valuation spaces. □

*Example 25.7.1 (A pseudoeffective class on the boundary)* Let $X$ be the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at a point and let $E$ be its exceptional curve. The class of $E$ is effective and satisfies $E^2 = -1$. The pullback of a line is nef and has zero intersection with $E$, so $[E]$ is on the boundary of the pseudoeffective cone. Choose a curvature form $\alpha$ on $\mathcal{O}(E)$ and a normalized log norm $g_E$ of its canonical section. Then $\alpha + \mathrm{dd}^c g_E = [E]$ as currents. At each perturbation take the constant slices

$$C^{\omega}_{\tau} = P_{\alpha+\omega}[g_E]_{\mathcal{I}}, \qquad \tau < 0.$$

They have positive mass since the curvature of $g_E$ off $E$ is $\omega$ in the perturbed class. The transitions preserve their $\mathcal{I}$-type. Evaluation therefore gives $-v(E)$ on every divisorial ray. The projective comparison identifies the whole function with the log norm of the canonical section, including the poles on $E^{\mathrm{an}}$.

## 25.8 Relative rays and the energy of a test curve

Let $X$ be a connected compact Kähler manifold and let $\alpha$ represent a big class. Fix an $\mathcal{I}$-model potential $\varphi$ of positive mass in $\alpha$. Write $V_{\varphi} = \int_X \langle(\alpha + \mathrm{dd}^c\varphi)^n\rangle$. The relative weak geodesic from $u_0$ to $u_1$ is the upper regularized envelope on the strip $0 < \mathrm{Re}\, z < 1$ of subgeodesics whose $L^1$ endpoint limits are bounded above by the prescribed potentials. Here a subgeodesic is independent of $\mathrm{Im}\, z$ and psh with respect to the background pulled back from $X$. A weak geodesic ray based at $\varphi$ starts at $\varphi$ and restricts to the weak geodesic on each finite interval, after reparametrization. We take the relative full-mass rays of [Xia26b, Definitions 4.1.1, 4.2.1 and 4.2.2]. The reference condition requires that the slices be more singular than $\varphi$ up to constants and have mass $V_{\varphi}$.

Write $E^{\varphi}_{\alpha}$ for relative energy, normalized to vanish at $\varphi$. For bounded differences from $\varphi$, its formula is

$$E^{\varphi}_{\alpha}(u) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X (u-\varphi)\langle(\alpha + \mathrm{dd}^c u)^j(\alpha + \mathrm{dd}^c\varphi)^{n-j}\rangle.$$

For general relative potentials the energy [Xia26b, Definition 3.1.6] equals the decreasing limit of the energies of the truncations $\max\{u, \varphi - b\}$ as $b \to \infty$ [Xia26b, Proposition 3.1.16]. A ray has finite energy if its positive-time slices have finite rel-

ative energy. The chosen normalization leaves the mass in the energy: $E^{\varphi}_{\alpha}(\varphi + c) = cV_{\varphi}$.

**Theorem 25.8.1** *A positive-mass $\mathcal{I}$-model test curve $\Gamma$ with $\Gamma_{-\infty} = \varphi$ gives a weak geodesic ray*

$$U_t = \sup_{\tau < \Gamma_{\max}} \{\Gamma_\tau + t\tau\} \quad (t > 0), \qquad U_0 = \varphi.$$

*It has $\mathcal{I}$-model inverse slices $\Gamma_\tau = \inf_{t>0}(U_t - t\tau)$ for $\tau < \Gamma_{\max}$. The correspondence is a bijection with the relative weak geodesic rays whose inverse slices are $\mathcal{I}$-model. The ray has finite energy exactly when $\int_{\mathbb{R}} x \, \mathrm{d}\mu_\Gamma(x) > -\infty$. In this case, for $t > 0$,*

$$E^{\varphi}_{\alpha}(U_t) = t \int_{\mathbb{R}} x \, \mathrm{d}\mu_\Gamma(x).$$

***Proof*** We use the complex analytic construction of geodesics from ordinary model slices and of ordinary model slices from geodesics [Xia26b, Lemmas 9.2.1, 9.2.3 and 9.2.4]. These results apply since every positive-mass $\mathcal{I}$-model slice is ordinary model and the left limit is the prescribed positive-mass model potential. We first verify the inversion pointwise. For each interior slope $s$, the decreasing limit of the slices as $\tau \uparrow s$ equals $\Gamma_s$ off the polar set of a later slice, by concavity. Both sides are psh, so the equality holds everywhere by uniqueness of the psh representative in $L^1$. Extend to the cutoff by the decreasing limit and set the slices above the cutoff equal to $-\infty$. For each $x$, the negative of the resulting slope function is closed and convex. If it is identically $+\infty$, inversion is immediate. Otherwise its transform with the sign convention in the theorem is $+\infty$ for $t < 0$, by monotonicity of the slices, and finite for $t > 0$, since $\Gamma_\tau \leq 0$ and the cutoff is finite. The value at $t = 0$ is the limit from $t > 0$, by testing each fixed $\tau$ and using $t\tau \leq t\Gamma_{\max}$. Thus Theorem 4.5.1 gives inversion using $t > 0$ alone. Conversely, off a null set the time slices of a geodesic are finite and convex as functions of $t > 0$. The same convex duality recovers them from their inverse slices. Slopes beyond the cutoff of the inverse curve give the value $-\infty$, and a finite value at the cutoff is approached from below by concavity and monotonicity. For each $t > 0$, the resulting psh potentials agree everywhere by uniqueness of their representatives. Restricting to the rays with $\mathcal{I}$-model inverse slices gives the asserted bijection.

We prove the energy formula at each time, including the possibility that its value is $-\infty$. Translating slopes by the cutoff subtracts $t\Gamma_{\max}$ from $U_t$ and subtracts $t\Gamma_{\max}V_\varphi$ from the energy, by the formula for bounded differences and truncation. We may therefore set $\Gamma_{\max} = 0$. First suppose that $\Gamma_\tau = \varphi$ for $\tau \leq -A$ with $A > 0$. Fix $b \in (-A, 0)$ and subdivide $[-A, b]$ into equal intervals of length $h$, with endpoints $s_0, \ldots, s_N$. Put

$$W_j = \max_{0 \leq i \leq j} (\Gamma_{s_i} + t s_i).$$

These potentials lie between $\varphi - tA$ and $\varphi$. On the set where $W_{j+1} > W_j$, concavity of the slope function gives $W_j = \Gamma_{s_j} + t s_j$ and $W_{j+1} = \Gamma_{s_{j+1}} + t s_{j+1}$: the last grid value exceeds the preceding one, so all earlier increments are nonnegative. The energy difference is bounded below by integrating $W_{j+1} - W_j$ against the top measure

of $W_{j+1}$, and above by integrating against that of $W_j$ [Xia26b, Proposition 3.1.19]. The difference vanishes off the set of strict inequality. On that set, away from the common polar locus, locality identifies the top measures with those of the slices at $s_{j+1}$ and $s_j$. We use locality in the plurifine topology [Xia26b, Corollary 1.3.4 and Proposition 2.2.1]. The top measure of an ordinary model slice is carried by its zero set [Xia26b, Theorem 3.1.1]. The zero set at $s_{j+1}$ lies in the strict inequality set, since both slices vanish there and $th > 0$. On it the difference is $th$. Everywhere away from the polar locus the difference is at most $th$, since the slices decrease. Hence

$$th\, m(s_{j+1}) \le E^{\varphi}_{\alpha}(W_{j+1}) - E^{\varphi}_{\alpha}(W_j) \le th\, m(s_j).$$

Both Riemann sums converge to $t\int_{-A}^{b} m(s)\,\mathrm{d}s$. Along successive dyadic subdivisions, the maxima increase to the transform restricted to $s \le b$, by continuity of the finite slope functions in the interior. This transform is psh by the analytic construction applied to the curve cut off at $b$. Then let $b \uparrow 0$, so the restricted transforms increase to $U_t$. All differences from $\varphi$ are bounded by $tA$ in absolute value. Monotone continuity of the relative energy [Xia26b, Proposition 3.1.20] and $E^{\varphi}_{\alpha}(W_0) = -tAV_{\varphi}$ give

$$E^{\varphi}_{\alpha}(U_t) = t\int_{-\infty}^{0} (m(s) - V_{\varphi})\,\mathrm{d}s.$$

For general $\Gamma$, let $\varepsilon > 0$. Put $a(s) = \max\{0, 1 + \varepsilon s\}$ for $s < 0$ and apply the ordinary model envelope to $(1 - a(s))\varphi + a(s)\Gamma_s$, with the slice interpreted as $\varphi$ when $a(s) = 0$. Call the result $\Gamma^{\varepsilon}_s$. Before taking the envelope the family is decreasing and concave. Indeed, outside a common null set the function $\Gamma_s - \varphi$ is concave, nonpositive and decreasing. Multiplication by the nonnegative increasing convex function $a$ preserves concavity in this case. For two slopes the value at their convex combination is bounded below by the product of the two weighted means, by the sign of $\Gamma_s - \varphi$. This product is at least the weighted mean of the products because the two factors are oppositely ordered. The envelope preserves order and concavity, so $\Gamma^{\varepsilon}$ is an ordinary test curve equal to $\varphi$ on $s \le -1/\varepsilon$. It is again $\mathcal{I}$-model. Indeed, positive-mass $\mathcal{I}$-model potentials are $\mathcal{I}$-good, and $\mathcal{I}$-goodness is preserved by positive multiples and sums [Xia26b, Example 7.1.2 and Proposition 7.2.1]. So the slices before enveloping are $\mathcal{I}$-good, and their ordinary model envelopes equal their $\mathcal{I}$-envelopes [Xia26b, Theorem 7.1.1]. The bounded case therefore applies to $\Gamma^{\varepsilon}$. At each fixed $s < 0$, these envelopes decrease as $\varepsilon \downarrow 0$ to $\Gamma_s$, by continuity of the ordinary envelope along convex combinations [Xia26b, Corollaries 6.2.6 and 6.2.10]. The masses decrease to $m(s)$ by [Xia26b, Proposition 3.1.12], applied in the fixed background class.

For fixed $t > 0$ the transforms decrease to $U_t$. To see this at $x$ outside a common null set, use a fixed finite slice as a lower bound and $\Gamma^{\varepsilon}_s \le 0$ as an upper bound to restrict nearly maximizing slopes to a compact interval ending at zero. Along a subsequence the slopes tend to some $s_* \le 0$. For any $r < s_*$, monotonicity bounds the limiting upper value by $\Gamma_r(x) + ts_*$. Letting $r \uparrow s_*$ bounds it by $U_t(x)$, also when $s_* = 0$. The reverse inequality follows from $\Gamma^{\varepsilon} \ge \Gamma$. The decreasing limit and $U_t$ are psh, so they agree everywhere. Relative energy is continuous from above in this

passage, even at $-\infty$. For a fixed truncation level $c > 0$, the maxima with $\varphi - c$ decrease and have bounded differences from $\varphi$. Apply monotone continuity at that level and interchange the two infima over $c$ and $\varepsilon$. Monotone convergence of the nonnegative deficits $V_\varphi - m^\varepsilon(s)$ now gives the displayed integral for $\Gamma$. Restoring the cutoff and using $V_\varphi = m_-$, Proposition 25.5.1 identifies the integral with the first moment. This proves the finite-energy criterion and the energy identity. □

For the version based at minimal singularities in a big class, let $\varphi_{\min} = (\sup\{u \in \mathrm{PSH}_{\mathbb{C}}(X,\alpha) : u \le 0\})^*$. An ordinary $\mathcal{I}$-model test curve with $\Gamma_{-\infty} = \varphi_{\min}$ embeds in the compatible family space. The theorem gives its relative weak geodesic ray based at $\varphi_{\min}$, whose inverse slices are $\mathcal{I}$-model (a maximal ray in the sense of [Xia26b, Definition 9.3.3]), with finite energy exactly when the first moment is finite. In the ambient mass convention put $V = \int_X \langle(\alpha + \mathrm{dd}^c\varphi_{\min})^n\rangle$ in place of $V_\varphi$ in the integral criterion. For an ordinary curve with $m_- < V$ that integral is $-\infty$, since the mass deficit at every sufficiently small slope is bounded below by a positive constant. If $m_- = V$, then $\Gamma_{-\infty} = \varphi_{\min}$ by full-mass characterization of the ordinary model envelope [Xia26b, Theorem 3.1.2 and Proposition 9.3.1]. Thus the ambient finite-energy and relative finite-energy conditions coincide for the ordinary curves based at minimal singularities.

*Example 25.8.1 (The point blowup in a Kähler class)* Let $\alpha$ now be a Kähler form, let $n > 0$, and fix $p \in X$. Let $\pi : Y \to X$ be the blowup of $p$ and write $E$ for the effective divisor of the extended ideal of $p$. In dimension one this is the point divisor on $X = Y$. Choose $c > 0$ small enough that $\pi^*\alpha - c[E]$ is Kähler. For instance, one can use the restriction to the strict transform of the central fibre of the Kähler model class in Proposition 21.9.1. A smooth positive representative of $\pi^*\alpha - c[E]$ has the expression $\pi^*\alpha - cR_E + \mathrm{dd}^c h$, where $R_E$ is the curvature of a smooth metric on $\mathcal{O}(E)$. Adding $c[E]$ to this form gives the current with potential $c\log|s_E|^2 + h$ above $\pi^*\alpha$. Pushing it down gives an $\alpha$-psh potential $g$ after choosing an additive constant. Near $p$ it is $c\log\sum_j |z_j|^2 + O(1)$, as one sees by pulling the coordinate ideal back to its invertible ideal on $Y$. Off $p$ it is smooth. Normalize $g \le 0$.

For $\tau < 0$ put $a(\tau) = \max\{0, c + \tau\}$ and

$$\Gamma_\tau = P_\alpha[(a(\tau)/c)g]_{\mathcal{I}}.$$

Here we use the convention $0g = 0$. Convexity and monotonicity of $a$ and $g \le 0$ give decreasing, concave slices before taking the envelope, and these properties persist under the envelope. The curve is ordinary and its left limit is zero. For $a = a(\tau)$, the non-pluripolar mass before taking the envelope is

$$\int_Y (\pi^*\alpha - a[E])^n = V - a^n, \qquad V = \int_X \alpha^n.$$

Indeed the non-pluripolar product off the pole agrees with the smooth product on $Y \setminus E$. The mixed terms vanish since $\pi^*\alpha|_E = 0$, and $\int_Y [E]^n = (-1)^{n-1}$, by the normal bundle of the exceptional divisor. For rational $a$, the slice has analytic singularities and the envelope preserves its mass [Xia26b, Theorem 7.1.1 and Example 7.1.1].

For $a > 0$ real, take rational $a_j \uparrow a$. The envelopes at $a_j$ dominate the envelope at $a$, since $g \le 0$, so its mass is at most $\inf_j (V - a_j^n) = V - a^n$. Its mass is also at least that of $(a/c)g$, by monotonicity. At $a = 0$ the envelope is zero and its mass is $V$.

For a divisorial valuation $v$,

$$\sup_{\tau<0}\{\tau - a(\tau)v(\mathfrak{m}_p)\} = -c\min(v(\mathfrak{m}_p), 1).$$

This agrees with the point model in Proposition 21.9.1. The test-curve measure has density $n(c+\tau)^{n-1}$ on $(-c, 0)$ and an atom of mass $V - c^n$ at zero. Its first moment is

$$-\int_{-c}^{0} (c+\tau)^n \, \mathrm{d}\tau = -\frac{c^{n+1}}{n+1}.$$

The radial energy and the cohomological model energy therefore agree for this example on every compact Kähler manifold.

*Example 25.8.2 (Full left mass and infinite first moment)* In the point construction, take $a(\tau) = c(1-\tau)^{-1/n}$ for $\tau < 0$. This is increasing and convex with limit zero as $\tau \to -\infty$. The same envelope construction gives a test curve. Its left limit is zero, since the unprojected potentials converge to zero off $p$. The mass deficit at level $\tau$ is $c^n/(1-\tau)$. Its integral from $-\infty$ to zero diverges, so the first moment is $-\infty$ even though the left mass equals $V$.

## 25.9 Positive-volume compatible test curves

Fix a pseudoeffective representative $\theta$ and a compatible family $\Gamma$ as in Proposition 25.2.1. Write $M$ for its cutoff. For a Kähler perturbation $\omega$ and $\sigma \in \{-\infty\} \cup (-\infty, M)$ put

$$m_\omega(\sigma) = \int_X \langle (\theta + \omega + \mathrm{dd}^c \Gamma_\sigma^\omega)^n \rangle, \qquad m(\sigma) = \inf_\omega m_\omega(\sigma).$$

The mass is increasing in the perturbation with respect to the order of forms. Indeed, the larger perturbation's slice majorizes the smaller perturbation's slice, including at the regularized left limit. In the larger background the mass increases under this order, by mixed mass monotonicity [Xia26b, Theorem 2.4.3]. Expanding after adding the nonnegative difference of backgrounds shows that the smaller slice has at least its original mass in the larger background. This proves the asserted monotonicity. Thus the infima can be taken over every decreasing coinitial sequence $\varepsilon_j \omega_0$, with $\varepsilon_j \downarrow 0$ and $\omega_0$ Kähler. We call $m(-\infty)$ the *volume* of the family.

**Proposition 25.9.1** *A compatible family is obtained by enlargement of an ordinary positive-mass $\mathcal{I}$-model test curve in $\theta$ if and only if its volume is positive. The ordinary representative is unique. Its slice masses, including the left mass, are the limits $m(\sigma)$.*

***Proof*** Write $m_i$ for the masses in $\theta + \varepsilon_i\omega_0$. For $i \le j$ we claim that

$$0 \le m_j(-\infty) - m_j(\tau) \le m_i(-\infty) - m_i(\tau) \qquad (\tau < M). \tag{25.2}$$

For a strictly larger perturbation the $\mathcal{I}$-envelope transition of an interior slice is also its ordinary $P$-envelope. Indeed a positive-mass $\mathcal{I}$-model potential is $\mathcal{I}$-good, and this property persists under enlargement [Xia26b, Example 7.1.2 and Lemma 7.1.1]. The same transition holds at the left limit by increasing continuity of both envelopes [Xia26b, Propositions 3.1.13 and 3.2.13]. Ordinary enveloping preserves mass in the larger class [Xia26b, Proposition 3.1.3].

Compute $m_i(\sigma)$, for $\sigma = -\infty$ and $\sigma = \tau$, in that larger background using the $j$-th slice. Put $T_{j,\sigma} = \theta + \varepsilon_j\omega_0 + \mathrm{dd}^c\Gamma_\sigma^{\varepsilon_j\omega_0}$. Expand the top products of $T_{j,\sigma} + (\varepsilon_i - \varepsilon_j)\omega_0$ by multilinearity of non-pluripolar products [Xia26b, Proposition 2.2.1(6)]. Subtract the expansion for the interior slice from that for the left slice. Every difference of mixed masses is nonnegative by mass monotonicity, since the left slice majorizes the interior slice. The term with no added background is $m_j(-\infty) - m_j(\tau)$. This gives (25.2), also when the two perturbations coincide.

Suppose $m(-\infty) > 0$. Choose a slope $\tau_0 < M$ sufficiently far to the left that $m_1(-\infty) - m_1(\tau_0) < m(-\infty)/2$, using increasing continuity of the left mass in the first perturbation. The gap bound gives $m_j(\tau_0) \ge m(-\infty)/2$ for all $j$. For $\tau \le \tau_0$, use monotonicity of the slices. For $\tau_0 < \tau < M$, take a slope $\tau' > \tau$ before $M$ and express $\tau$ as a convex combination with weight $a > 0$ on $\tau_0$. Concavity, positivity and mass monotonicity give $m_j(\tau) \ge a^n m_j(\tau_0)$ uniformly in $j$. This establishes positive limiting mass at every interior slice.

At each such slope the decreasing limit $\Gamma_\tau^0 = \inf_j \Gamma_\tau^{\varepsilon_j\omega_0}$ is an ordinary model potential. Its mass is $m(\tau) > 0$, and its divisorial Lelong numbers are the limits of the corresponding numbers, by the decreasing model theorem with moving backgrounds [Xia26b, Proposition 3.1.12]. In particular $\theta$ is big by the positive volume criterion [Xia26a, Proposition 15.8.2 and Theorem 17.4.20]. The divisorial Lelong numbers were constant in $j$, so the limit slice has the same $\mathcal{I}$-type as each earlier slice [Xia26b, Proposition 3.2.1(5)]. Its $\theta$-$\mathcal{I}$-envelope, considered in the larger class, is a candidate for the $\mathcal{I}$-envelope of each approximant. Thus it is at most their infimum. Since it also majorizes the limit slice, the latter is $\mathcal{I}$-model. Concavity and monotonicity pass to the limits, and the left limit has positive mass since it majorizes an interior slice. The enlarged curve agrees with the compatible family by equality of the types at each slope.

Let $j \to \infty$ in (25.2). This gives $0 \le m(-\infty) - m(\tau) \le m_1(-\infty) - m_1(\tau)$, which tends to zero as $\tau \to -\infty$. Increasing continuity of the ordinary left mass therefore identifies it with $m(-\infty)$. Conversely an ordinary positive-mass curve has enlargements with left mass at least its original positive left mass, by the same order and added-background mass comparison. Its image has positive volume, so the preceding construction applies. Uniqueness follows from the common multiplier types of the ordinary slices and their model property. □

### 25.9.1 Energy in a big class

Suppose the class of $\theta$ is big, with volume $V_\theta > 0$. We use $V_\theta$ here for the number $\mathrm{vol}([\theta])$ and write $\varphi_{\min}$ for the envelope of all nonpositive $\theta$-psh potentials. Thus $\varphi_{\min}$ has minimal singularities and mass $V_\theta$. Define the energy of a compatible family, relative to the full class volume, by

$$\mathcal{E}_{\mathrm{tc}}(\Gamma) = MV_\theta + \int_{-\infty}^{M} \bigl(m(\tau) - V_\theta\bigr)\,\mathrm{d}\tau. \tag{25.3}$$

The slice masses in each perturbation are bounded by its class volume, and the class volumes decrease to $V_\theta$ [Xia26a, Proposition 15.8.2]. The integrand is therefore nonpositive, and it is measurable by monotonicity in the slope. This defines a number in $[-\infty, \infty)$ independent of coinitial direction. A smooth change of representative also preserves it: at each perturbation the mass of an $\mathcal{I}$-model potential is its volume for the prescribed multiplier type, invariant under the smooth change [Xia26b, Propositions 3.2.4 and 3.2.7].

**Corollary 25.9.1** *In a big class the compatible families of finite energy in* (25.3) *are exactly the enlargements of ordinary $\mathcal{I}$-model test curves with left limit $\varphi_{\min}$ and finite first moment for their mass distribution. For such a family the energy in* (25.3) *equals that first moment.*

***Proof*** If the integral is finite, $m(\tau)$ tends to $V_\theta$ as $\tau \to -\infty$. Otherwise the deficit from $V_\theta$ would be bounded below by a positive constant on an infinite left interval. It follows that $m(-\infty) = V_\theta > 0$, since $m(\tau) \le m(-\infty) \le V_\theta$. By Proposition 25.9.1 the ordinary representative exists and its left limit is full mass. It is an $\mathcal{I}$-model potential by increasing continuity of the $\mathcal{I}$-envelope, hence also an ordinary model potential. The full-mass characterization [Xia26b, Proposition 3.1.14] makes it $\varphi_{\min}$. Now $m(\tau)$ is the tail mass of its ordinary distribution $\mu_\Gamma$. Since $\mu_\Gamma$ has mass $V_\theta$ and is carried by $(-\infty, M]$, layer integration gives

$$\int_{\mathbb{R}} (M - x)\,\mathrm{d}\mu_\Gamma(x) = \int_{-\infty}^{M} (V_\theta - m(\tau))\,\mathrm{d}\tau.$$

This proves the equality of energies and the first moment criterion. In the opposite direction the same identity and the mass equality in Proposition 25.9.1 give finite energy for the enlarged curve. □

## 25.10 Comparison in finite Kähler energy

In this section $\omega$ is a Kähler form and $\alpha = [\omega]$. Consider ordinary $\mathcal{I}$-model test curves in $\omega$ with left limit zero, cutoff zero, and finite first moment for the mass distribution. Their left mass is $V = \int_X \omega^n$. They give compatible families by enlargement to every perturbation of $\omega$. The cutoff normalization corresponds to $u(v_0) = 0$ on the cohomological side.

Let $\Gamma$ be such a curve and $U^\Gamma$ its weak geodesic ray from Theorem 25.8.1. Its nonpositive complexification on the product disk is

$$U^\Gamma(x,t) = \sup_{\tau<0}\bigl\{\Gamma_\tau(x) - \tau \log|t|^2\bigr\}.$$

On the punctured disk this is the pointwise Legendre formula. It extends as a psh family after adding local potentials across $t = 0$ by the common upper bound.

As in Proposition 24.3.1, a *full vertical divisor* is either a central component $E$ of a smooth refinement $W$ of $X \times \mathbb{P}^1$ (the global case) or a vertical prime divisor $E$ on a smooth local modification $W$ of an open subset of $X \times \mathbb{D}$ (the local case). Put $b_E = \operatorname{ord}_E(t) > 0$. The *full-model test* of $E$ is the normalized order $v_E = b_E^{-1}\operatorname{ord}_E$ restricted to ideals extended from $X$. In the global case it is the divisorial test $v_E$.

**Lemma 25.10.1** *For a full vertical divisor $E$ on $W$, global or local, one has*

$$T_{W,E}(U^\Gamma) = \sup_{\tau<0}\Bigl\{\tau - \lim_{m\to\infty} m^{-1} v_E\bigl(\mathcal{I}(m\Gamma_\tau)\bigr)\Bigr\}. \tag{25.4}$$

*In particular, if $v_E = r\operatorname{ord}_F$ for a divisor order on $X$, the right side is $\Gamma^{\mathrm{an}}(r\operatorname{ord}_F)$.*

***Proof*** Apply Lemma 22.7.1 to the summands in the Legendre formula. The argument on a local model is the same: compare the pullbacks and the regularized supremum on the open set where the modification is biholomorphic and $t \neq 0$. It has full measure, and the two psh expressions, after adding common local potentials, are determined by their almost-everywhere values. The $\tau$-summand has generic Lelong number $v_E(\Gamma_\tau \circ \rho) - b_E\tau$. The product formula for multiplier ideals [Xia26a, Theorem 12.4.1] identifies the multiplier ideal of each $m\Gamma_\tau$ pulled back to the product with the ideal extended from $X$. Thus the divisorial multiplier-ideal limit on the full model, divided by $b_E$, is the limit in (25.4). For $v_E = r\operatorname{ord}_F$ it is $v_E(\Gamma_\tau)$ by the same divisorial limit on $X$. □

To test every rational multiple $r = a/b > 0$ of a divisor order on $X$, take a smooth modification of $X$ carrying $F$. Make its strict transform smooth by a log resolution [Xia26a, Theorem A.5.5]. On the product with the parameter disk, use a local equation $z = 0$ for the strict transform. Successive blowups along the intersections of the two divisors in the crossing $z = t = 0$ and its transforms realize the order with values $a, b$ on $z, t$ when $a, b$ are relatively prime. Indeed, if $a > b$, use the chart $(z/t, t)$ and replace $(a, b)$ by $(a - b, b)$. If $b > a$ use $(z, t/z)$ and replace $(a, b)$ by $(a, b - a)$. At $(1, 1)$ take the exceptional divisor of the next blowup. Undoing these substitutions

gives the required orders. Each crossing centre is a smooth stratum of the two indicated components, closed in the product model. The normalized restriction of the resulting order on ideals from $X$ is $r\,\mathrm{ord}_F$: each local generator is a power of $z$ times a function nonzero at a general point of $F$. One may use the composed local modification of the product even when it has exceptional divisors over the punctured disk. The local statement of Proposition 24.3.1 applies.

**Theorem 25.10.1** *The map $\Gamma \mapsto T(U^\Gamma)$ is an order isomorphism from the normalized finite-energy $\mathcal{I}$-model test curves in $\omega$ with left limit zero to the normalized $\mathcal{E}^1_\alpha$. Writing $u = T(U^\Gamma)$, the ray $U^\Gamma$ is the maximal ray $U^u$ of Theorem 24.4.1, and*

$$E_\alpha(u) = \int_{\mathbb{R}} x \, \mathrm{d}\mu_\Gamma(x). \tag{25.5}$$

*On divisor orders the map is the evaluation* (25.1).

***Proof*** Let $u \in \mathcal{E}^1_\alpha$ be normalized and take its maximal ray $U^u$. Its inverse Legendre slices form an ordinary model test curve with left limit zero [Xia26b, Lemma 9.2.4 and Theorem 9.2.1]. The cutoff is at most zero since $U^u \le 0$. A strictly negative cutoff would give an upper bound by that cutoff times $s$ in the Legendre formula, and hence a strictly negative transform at $v_0$, a contradiction. Thus the cutoff is zero.

Replace each inverse slice by its $\mathcal{I}$-envelope. The resulting family $\Gamma$ is an $\mathcal{I}$-model test curve with the same cutoff and left limit $P_\omega[0]_{\mathcal{I}} = 0$ [Xia26b, Proposition 9.3.4]. Its Legendre dual dominates $U^u$. The formula of Lemma 25.10.1 applies to the original slices as well: its proof uses only the Legendre formula and nonpositivity. The multiplier ideals at each slope are unchanged by the replacement, so the full ray slopes are unchanged. The dual ray of $\Gamma$ is admissible for $u$, and maximality of $U^u$ gives equality. Its complex energy is finite, so Theorem 25.8.1 gives finiteness of the first moment of $\Gamma$. Together with (24.4), the energy formula there gives (25.5).

Conversely let $\Gamma$ be a normalized test curve with finite first moment and left limit zero. By (24.2) and Theorem 25.8.1, $T(U^\Gamma)$ has finite energy. Its value at $v_0$ is zero by the formula (25.4), since the trivial valuation vanishes on every nonzero multiplier ideal. Thus its maximal ray exists and gives a normalized $\mathcal{I}$-model test curve $\Lambda$ by the first part of the proof. The two curves evaluate equally at every positive rational multiple of each base divisor order: their rays have the same cohomological transform, and those multiples are local divisorial tests by the blowup construction preceding the theorem. For each divisor order $v$, the function

$$s \longmapsto s\,\Gamma^{\mathrm{an}}(v/s) = \sup_{\tau<0}\bigl\{s\tau - v(\Gamma_\tau)\bigr\}$$

is finite and convex for $s > 0$, and therefore continuous there. The same holds for $\Lambda$. Thus their evaluations agree also at real multiples. The recovery formula of Proposition 25.4.1 determines the multiplier-ideal types of all interior slices. The $\mathcal{I}$-model property then gives $\Lambda = \Gamma$, hence $U^\Gamma$ is maximal.

Finally, if $\Gamma \le \Lambda$, then their Legendre rays are ordered and so are their cohomological transforms. Conversely if $u \le w$, the maximal rays satisfy $U^u \le U^w$ by inclusion

of their admissible families. Taking inverse Legendre slices preserves the order. This proves the order isomorphism. Formula (25.4) gives divisorial evaluation for rational scalings. For the actual values of $u$ at real scalings, use the bounded clippings in the proof of Proposition 24.3.1. Every dyadic model in a clipping has values at $rv$ given by a positive multiple of $-\min_j\{rv(A_j)+j\}$, where $A_j$ are the coefficient ideals of a flag ideal. This is convex in $r$. Decreasing limits first in the dyadic approximants and then in the clippings give convexity of $r \mapsto u(rv)$. It is finite on the positive rationals by (25.4). If it took the value $-\infty$ at some interior point, convexity with a rational point to its right would make it $-\infty$ at intermediate rational points. Thus it is finite and continuous on $r > 0$. The evaluation of $\Gamma$ is also continuous there by the displayed convex formula in $s = 1/r$. Equality therefore holds for all real scalings.□

In the class of a Kähler form the minimal potential $\varphi_{\min}$ is zero. By Corollary 25.9.1, every compatible family of finite energy in (25.3) has the ordinary representative required in Theorem 25.10.1 after making the cutoff zero. More explicitly, for cutoff $M$ replace the slices by $\Gamma'_\tau = \Gamma_{\tau+M}$ for $\tau < 0$. The energy changes by $-M\int_X \omega^n$ in (25.3), and the divisorial evaluation changes by $-M$ in (25.1). Adding back the constant $M$ to the cohomological potential restores the evaluation and its energy by the translation law. For ordered curves with possibly different cutoffs, first subtract a common upper bound for their cutoffs from the slope parameters. The corresponding Legendre rays are nonpositive and ordered. Their transforms are ordered by the divisorial test, and adding back the common constant preserves that order. Equivalently, a shift of slopes adds the corresponding multiple of $s$ to the ray, which shifts all its divisorial slopes by the same amount. Conversely the order of cohomological potentials implies the order of their divisorial evaluations, which recovers the order of the compatible families by Proposition 25.4.1. This gives the order and energy comparison for all finite-energy compatible families in a Kähler class.

## 25.11 Evaluation on the full valuation space

Let $X$ be a connected compact Kähler manifold and use the product disk with parameter $t$. For a coherent ideal $I$ defined near $X \times \{0\}$, denote by $C_j(I)$ the Taylor coefficient ideal in degree $j$. As in Section 21.4, it is the image under the coefficient projection of the reduction of $I$ modulo $t^L$ for any integer $L > j$. This uses only the germ of $I$ along the fibre. For $v \in X^{\beth}$ put

$$S(I)(v) = \inf_{j\geq 0}\big(v(C_j(I)) + j\big). \tag{25.6}$$

The proof of the sum and product rules in Theorem 21.4.1 uses only the finite coefficient projections, so both rules hold here. In particular a finite infimum is attained, and for the product the least minimizing degrees give its value by considering the coefficient in their sum degree. If a factor has infinite value, the coefficient inclusion already gives infinite product value. Inclusion of ideals reverses this order.

For any $L$ the function

$$\min(L, S(I)) = \min\bigl(L, \min_{0\le j<L}(v(C_j(I)) + j)\bigr)$$

is finite and continuous. Consequently $-S(I)$ is upper semicontinuous, as an infimum of the negatives of these continuous functions. For a nonpositive circle-invariant quasi-psh potential $\Phi$ on the product disk, set

$$J_\Phi(v) = \lim_{m=2^k\to\infty} -\frac{1}{m} S(\mathcal{I}(m\Phi))(v). \tag{25.7}$$

This is a decreasing limit by multiplier subadditivity [Xia26b, Theorem 1.4.2] and the product rule for $S$, so it is upper semicontinuous.

**Lemma 25.11.1** *For every full vertical divisor $E$ on a smooth local or global model of the product, one has*

$$J_\Phi(v_E) = -b_E^{-1} v_E(\Phi), \qquad b_E = \operatorname{ord}_E(t) > 0.$$

*Here $v_E(\Phi)$ is the generic Lelong number of the pullback.*

***Proof*** The ideal $I = \mathcal{I}(m\Phi)$ is circle-invariant. In the quotient modulo $t^L$, the sum of the translates by $L$ distinct circle elements equals the direct sum of the coefficient ideals in each degree. Indeed, the Vandermonde matrix isolates every Taylor coefficient. By invariance, the sum of the translates is the original image of $I$. Thus $I+(t^L)$ is generated by the $C_j(I)t^j$ with $j < L$ and by $t^L$. The normalized order of this ideal along $E$ is $\min(L, S(I)(v_E))$. At the same time it is $\min(L, b_E^{-1}\operatorname{ord}_E I)$. Let $L\to\infty$. The divisorial multiplier order estimate [Xia26a, Corollary 12.6.8] now proves the equality. □

**Lemma 25.11.2** *With $s = -\log|t|^2$ and $c \le 0$, one has $J_{\Phi+cs} = J_\Phi + c$.*

***Proof*** For an integer $k \ge 0$ the multiplier ideal of $m\Phi + k\log|t|^2$ is $t^k\mathcal{I}(m\Phi)$. To prove divisibility, local integrability against its weight implies integrability against $|t|^{-2k}$ since $\Phi \le 0$. On almost every base slice, one-variable integrability forces the numerator to vanish to order at least $k$ in $t$. The lower Taylor coefficients vanish identically by holomorphicity. Cancelling $t^k$ proves the remaining multiplier condition and also the reverse inclusion. Multiplication by $t^k$ shifts every $S$-value by $k$ by (25.6). For $k = -mc$ real, sandwich between its integer floor and ceiling using monotonicity of multiplier ideals. Divide the orders by $m$ and take the dyadic limit.□

**Theorem 25.11.1** *Let $\theta$ be a pseudoeffective form and $\Gamma \in \mathrm{PSH}_{\mathrm{tc}}(X,\theta)$. Translate the cutoff of a perturbed coordinate $\Gamma^{\omega}$ to zero, let $\Phi$ be the potential on the product disk of its relative Legendre ray, and put $F_\Gamma = J_\Phi + \Gamma_{\max}$, the* jet extension *of $\Gamma$. Then $F_\Gamma$ does not depend on $\omega$, and the assignment $\Gamma \mapsto F_\Gamma$ is an order embedding of $\mathrm{PSH}_{\mathrm{tc}}(X,\theta)$ into the upper semicontinuous functions on $X^{\beth}$. It extends the divisorial evaluation (25.1) at all real scalings of divisor orders and at the trivial valuation. It commutes with cutoff translations and with smooth changes of representative.*

***Proof*** First normalize the cutoff to zero. The relative ray of each positive-mass coordinate from Theorem 25.8.1 is nonpositive. Its product potential extends over the central fibre by removable singularities for functions with a common upper bound [Xia26a, Theorem 8.4.1]. Let $V_s$ be its time slices, and write $a$ for their left model reference from that coordinate. The ray is decreasing in $s$, since the Legendre formula uses nonpositive slopes.

We show that the ideals of this space-time potential are independent of the coordinate. At a prime divisor $E$ on a smooth model of a local product chart with $\mathrm{ord}_E(t) > 0$, the Lelong number of the ray is the infimum over $\tau < 0$ of

$$\nu_E(\Gamma_\tau^{\omega} \circ p_X) - b_E\tau.$$

The proof is that of Lemma 25.10.1, which uses the common local upper bound and the Legendre formula. The first term is computed by the product multiplier formula [Xia26a, Theorem 12.4.1] and divisorial asymptotic order of the slice ideals extended from $X$. Those ideals agree for all perturbations by compatibility.

For the other prime divisors a general point maps to the punctured disk. Work in an annulus with $s_1 < s < s_2$, where the spatial potentials $V_{s_2}$ and $V_{s_1}$ bound the ray below and above. Both are full mass relative to $a$ by the relative ray construction [Xia26b, Corollary 9.2.1]. Their ordinary model envelopes equal $a$, so they have its multiplier-ideal type [Xia26b, Propositions 3.1.14 and 3.2.9]. The product multiplier formula then bounds the space-time ideal between two copies of the ideal of $a$ extended from $X$. The type of the left model is preserved under transition. Thus horizontal generic Lelong numbers, like vertical ones, agree for all coordinates. Apply the local divisorial criterion [Xia26b, Theorem 1.4.3], adding local smooth potentials of the backgrounds. This proves independence of the space-time multiplier ideals near the central fibre. The same reasoning compares an ordinary positive-mass curve and its enlargements. Smooth changes of representative preserve all slice types, so the same argument proves invariance under them.

By Lemma 25.11.1 and the Legendre formula, the resulting $J_\Phi$ equals $\Gamma^{\mathrm{an}}$ on the trivial valuation and the positive rational multiples of base divisor orders. To realize these multiples, use the local toroidal blowups of Section 25.10. For a fixed base divisor order $v$, the terms in (25.7) are convex functions of $r$ at $rv$, since $-S$ there is the supremum of the affine functions $-rv(C_j(I)) - j$, with identically infinite ideal orders omitted. The decreasing limit is convex and finite on the positive rationals. If it were $-\infty$ at an interior point it would be $-\infty$ at intermediate rational points on a segment to a finite value. Thus it is finite and continuous for $r > 0$. The ordinary evaluation is also finite convex in $r$ by (25.1). Equality extends to real scalings.

The cutoff translation used in the statement gives the same function when a common larger cutoff bound is subtracted, by Lemma 25.11.2. For ordered families make this common subtraction so their Legendre rays in any one perturbation are nonpositive and ordered. Multiplier ideals and then the negative of their $S$-values are ordered in the same direction. Conversely, the order of the functions implies the order on the divisor rays and at the trivial valuation, which recovers the family order by Proposition 25.4.1. This proves the order embedding. □

## 25.12 The whole cone in a Kähler class

Fix a Kähler form $\omega$ and let $\alpha = [\omega]$. Write $\mathcal{D}_\alpha$ for the decreasing semipositive cone of Section 22.6.

**Lemma 25.12.1** *For an ordinary positive-mass $\mathcal{I}$-model test curve in $\omega$ with cutoff zero, its jet extension is the cohomological transform of its relative Legendre ray.*

***Proof*** The jet extension can be computed from that ordinary ray $U$, by the perturbation comparison in Theorem 25.11.1. Put $u = T(U) \in \mathcal{D}_\alpha$, using Proposition 24.3.1. At each full-model test, the order bound [Xia26a, Corollary 12.6.8] and the capping computation of Lemma 25.11.1 give

$$u(v_E) \le -\frac{1}{m} S(\mathcal{I}(mU))(v_E).$$

Replace $S$ on the right by $\min(L, S)$ to obtain a continuous majorant on the test set. The divisorial order test of Theorem 22.6.1 gives the same inequality everywhere. Let $L$ tend to infinity and take the dyadic limit. This proves $u \le J_U$.

Conversely take $h \in C_\alpha$ normalized at $v_0$ with $h \ge u$. Its maximal ray $U^h$ dominates $U$, since $T(U) \le h$, by the definition in Proposition 24.4.1. Then $\mathcal{I}(mU) \subseteq \mathcal{I}(mU^h)$ for every $m$ and $J_U \le J_{U^h}$. The bounded-logarithm construction in Proposition 22.7.1 gives the latter limit as $h$. Indeed its semipositive approximants differ from the raw jet potentials by the factor $1/(1+C/m)$. The raw potentials are bounded uniformly since $U^h \ge -\|h\|_\infty s$, so that its ideals contain $t^{\lceil m\|h\|_\infty \rceil}$. The compactification there leaves the ideals unchanged near the fibre. Thus $J_U \le h$. In this situation $u(v_0) = 0$ by the Legendre formula, so one can choose normalized decreasing continuous majorants of $u$: take the envelopes of the minima of the original majorants with zero. Taking their infimum gives $J_U \le u$. □

**Theorem 25.12.1** *The jet extension of Theorem 25.11.1 gives an order isomorphism*

$$\mathrm{PSH}_{\mathrm{tc}}(X,\omega) \longrightarrow \mathcal{D}_\alpha.$$

*It identifies the extended test-curve energy (25.3) with $E_\alpha$, including the value $-\infty$.*

***Proof*** We first normalize the cutoff to zero. Apply Lemma 25.12.1 to the ordinary coordinates in $(1+\varepsilon_j)\omega$ for a decreasing sequence $\varepsilon_j \to 0$. Perturbation independence puts the same function $F_\Gamma$ in every $\mathcal{D}_{(1+\varepsilon_j)\alpha}$. For $\lambda > 0$, a pair $(\alpha, u)$ is semipositive on a model exactly when $(\lambda\alpha, \lambda u)$ is, so the semipositive cones are homogeneous: $\mathcal{D}_{\lambda\alpha} = \lambda\mathcal{D}_\alpha$. Hence the rescaled functions $F_\Gamma/(1+\varepsilon_j)$ belong to $\mathcal{D}_\alpha$. They decrease to $F_\Gamma \le 0$ and vanish at $v_0$. The closure of $\mathcal{D}_\alpha$ under nested decreasing limits, established in the proof of Theorem 23.5.1, gives $F_\Gamma \in \mathcal{D}_\alpha$.

For surjectivity take $u \in \mathcal{D}_\alpha$ with $u(v_0) = 0$. The truncations $u_j = \max(u, -j)$ decrease to $u$ and have finite energy. Indeed $C_\alpha$ is stable under finite maxima by Lemma 23.1.1 and uniform approximation, so the truncations are bounded decreasing limits in the class. The finite-energy comparison Theorem 25.10.1 gives ordinary test curves $\Gamma_j$ with left limit and cutoff zero. They decrease in order.

For each fixed Kähler perturbation $\eta$, enlarge these curves to $\omega + \eta$. At every interior slope the masses of the enlarged slices are bounded below by $\int_X \eta^n > 0$. To see this, add $\eta$ to the smaller-class current and keep the pure $\eta$ term of the mixed expansion, which has its full mass since the smooth measure does not charge poles. The enlarged envelope can only increase mass in the larger class. By Lemma 25.7.1 the decreasing curves in $\omega+\eta$ have an ordinary positive-mass limit $\Gamma^\eta$. Its evaluation is $u$ on the divisor rays. For two ordered perturbations the transition preserves that evaluation, and the order recovery in Proposition 25.4.1 identifies the transition with the larger limit curve. This gives a compatible family $\Gamma$.

To compare its jet extension with $u$ on every full-model test, fix $\eta$ and an interior slope. Write $\psi_j$ for its decreasing slices in the enlarged class, and $\psi$ for the limiting $\mathcal{I}$-model slice. On $X \times \mathbb{P}^1$ their pullbacks are $\mathcal{I}$-model in the sum of the pulled-back background $\omega + \eta$ and a Fubini–Study form from $\mathbb{P}^1$. Indeed, the $\mathcal{I}$-envelope of a pulled-back slice is invariant under the transitive compact rotation group of $\mathbb{P}^1$, since the group preserves the background and the type. It is therefore a pulled-back potential. Its base potential has the type of the original slice by the product formula for multiplier ideals and restriction to a product stalk. Since the original slice is $\mathcal{I}$-model, the pullback already equals its envelope.

The top mass of each product slice equals the base mass multiplied by $n+1$ and by the integral of the Fubini–Study form. This is the term with one base-line factor in the non-pluripolar expansion, computed on the complement of the poles by truncation. The other terms vanish by dimension. In particular the limiting mass is positive. The decreasing Lelong convergence of [Xia26b, Proposition 3.1.12], with fixed product background, applies to these ordinary model potentials. At each prime divisor $E$ over the product the generic Lelong numbers of the pullbacks thus increase to the limiting number.

Now fix a divisorial test $v_E$, given by a central component $E$ of a smooth refinement of $X \times \mathbb{P}^1$. Repeat the compact slope-interval argument of Lemma 25.7.1 with

the functions $\tau - b_E^{-1} \nu_E(p_X^* \psi_{j,\tau})$ in place of the base-divisor functions. They have the same bound by $\tau$ and the same monotonicity and concavity properties, since generic Lelong numbers are additive on sums and reverse the order of potentials. The slice convergence just proved gives the limiting function on all interior slopes. The full-ray formula from Lemmas 25.10.1 and 25.11.1 identifies their suprema with $u_j(v_E)$ and the limiting supremum with $F_\Gamma(v_E)$. Thus $F_\Gamma(v_E) = u(v_E)$. Both functions belong to $\mathcal{D}_\alpha$, so the divisorial order test gives equality on $X^{\exists}$. In particular $F_\Gamma$ and $u$ also agree at every local full-model test.

General cutoffs and normalizations follow by translation. The finite-energy comparison identifies exactly the finite loci of the two energies under this bijection and gives equality there. Both remaining energies are $-\infty$, which proves the extended identity. □

**Corollary 25.12.1** *On a smooth connected projective manifold, for a pseudoeffective form in the real span of smooth curvatures of line bundles, the jet extension agrees with the projective comparison of Theorem 25.7.1.*

***Proof*** Choose a sufficiently large positive multiple of an ample curvature form so that adding it makes the form Kähler and makes its numerical class ample. The jet extension of the enlarged family is the same function by Theorem 25.11.1. By Theorem 25.12.1 and the ample cohomological comparison Corollary 23.1.1, it is algebraic psh in the enlarged class. The projective extension is likewise psh in the enlarged class by adding this ample background with zero potential. Both extensions agree on the algebraic divisorial tests by their evaluation formulas. The algebraic divisorial order theorem Theorem 6.3.1 now gives equality. □

# Chapter 26
# Nef backgrounds: energy and mixed measures

On $X = \mathbb{P}^1 \times \mathbb{P}^1$, let $\alpha_1$ and $\alpha_2$ be the classes pulled back from $O(1)$ on the two factors. Both are nef with $\alpha_i^2 = 0$, so the diagonal measure of the zero potential in either class has zero mass. The mixed measure of the two zero potentials is nevertheless the unit mass $\delta_{v_0}$, the limit of the mixed intersection degrees of Kähler perturbations (Example 26.4.1). In a nef class we therefore construct energies and mixed measures as limits along Kähler perturbations, without dividing by the volume.

Throughout this chapter $X$ is a connected compact Kähler manifold of dimension $n \geq 1$, $\theta$ is a smooth closed $(1,1)$-form in a nef class $\alpha$, and $V = \int_X \alpha^n$, which may vanish. The perturbative cone $\mathcal{D}_\alpha^{\mathrm{pert}}$ is the intersection of the decreasing cones of the Kähler classes $\alpha + [\eta]$. The jet extension identifies it with the compatible test-curve families of Chapter 25, and the envelope of a continuous obstacle in it is the decreasing limit of the Kähler envelopes (Theorem 26.1.1). The obstacle energy and its measure are limits along arbitrary Kähler approaches, with a first variation formula and contact (Theorem 26.2.1). The same limits exist for bounded potentials, in total variation for the measures (Theorem 26.3.1), and for mixed measures of potentials of finite energy in a Kähler perturbation (Theorem 26.4.1). On a smooth projective variety, for classes spanned by line bundles, they agree with the algebraic nef theory (Corollary 26.4.1).

Finite energy in a Kähler perturbation is too strong a requirement at the nef boundary. On the blowup of $\mathbb{P}^2$ at a point, with $\alpha$ the pullback of the hyperplane class, Example 26.5.1 gives a potential $u$ whose energy is $-\infty$ in every Kähler perturbation, while the infimum $\overline{E}_\alpha^{\mathrm{pert}}(u)$ of the energies of its truncations $\max(u, -j)$ in $\alpha$ is finite. When $V > 0$ this truncation energy is the extended test-curve energy (Theorem 26.5.1). Its finiteness suffices for the diagonal measure (Theorem 26.6.1), and for different nef classes finite intrinsic energy of the sum in the sum class gives mixed measures of full mass (Theorem 26.7.1). Locality at a strict maximum (Theorem 26.8.1) then gives restricted mixed measures for arbitrary tuples (Theorem 26.9.1). On $\mathbb{P}^1 \times \mathbb{P}^1$ a restricted mixed measure can lose all of its mass (Example 26.9.1).

Chapter 27 uses these energies and measures to solve the Monge–Ampère equation when $V > 0$. The big-class constructions of Chapter 28 agree with those of

this chapter when the class is nef, and Section 33.5 uses Theorem 26.5.1 and the projective comparison to compute intrinsic energies on a toric blowup.

## 26.1 The perturbative cone

With $X$, $\theta$ and $\alpha$ as above, write $\mathcal{D}_\alpha$ for the decreasing fixed-class model cone of $\alpha$ and $\mathcal{K}_X$ for the set of Kähler forms on $X$. For every $\eta \in \mathcal{K}_X$ the class $\alpha+[\eta]$ is Kähler by the nef cone description [Xia26a, Proposition 14.3.2]. Define the *perturbative cone*

$$\mathcal{D}_\alpha^{\mathrm{pert}} = \bigcap_{\eta \in \mathcal{K}_X} \mathcal{D}_{\alpha+[\eta]}.$$

Adding a nef background with zero potential preserves semipositivity on models, uniform limits and the decreasing cones, so $\mathcal{D}_\alpha \subset \mathcal{D}_\alpha^{\mathrm{pert}}$. The intersection can be taken along any coinitial sequence of perturbations.

The proofs below change the smooth representative of a class for ordinary positive-mass curves. If two forms differ by $dd^c g$, subtract $g$ from the slices and take the $\mathcal{I}$-envelope in the new form. The resulting curve is concave by concavity of the envelope and has the same types. It keeps positive mass, since the shifted slices have their original masses and enveloping can only increase mass. Increasing continuity of the envelope gives the corresponding change of the left limit. Changing back recovers the original slices by uniqueness of $\mathcal{I}$-model potentials of a given type. For compatible families this is the change of representative in Chapter 25.

**Theorem 26.1.1** *The jet extension is an order isomorphism from* $\mathrm{PSH}_{\mathrm{tc}}(X,\theta)$ *onto* $\mathcal{D}_\alpha^{\mathrm{pert}}$. *For a continuous real obstacle* $f$ *write* $P_\alpha^{\mathrm{pert}}(f)$ *for the greatest member of that cone below* $f$. *It exists and satisfies*

$$P_\alpha^{\mathrm{pert}}(f) = \lim_{\varepsilon \downarrow 0} P_{\alpha+\varepsilon[\omega]}(f) \tag{26.1}$$

*pointwise for any fixed Kähler form* $\omega$.

***Proof*** For a compatible family, each ordinary coordinate belongs to a Kähler class. If necessary change its form to a positive representative in the class. By Lemma 25.12.1 its jet extension is in the decreasing cone of that class. Every coordinate has the same extension by perturbation independence, so the extension lies in the intersection.

Conversely, let $u$ be in the intersection. To construct its ordinary coordinate in $\theta+\eta$, choose a smaller Kähler form $\eta'$ so that $\eta-\eta'$ is Kähler. Choose a Kähler representative of $\alpha+[\eta']$ and use Theorem 25.12.1 there to obtain the compatible family of $u$. Transfer the family to $\theta+\eta'$ by smooth change of representative and take its ordinary coordinate at $\eta-\eta'$. It has jet extension $u$ in $\theta+\eta$. Any smaller choice gives the same ordinary curve because its divisorial evaluation and cutoff are the same. For ordered perturbations the transition preserves the divisorial evaluation and the cutoff, so the resulting ordinary curves are compatible by the same uniqueness. The order comparison now follows from Theorem 25.11.1.

For the last assertion, the Kähler envelopes decrease as $\varepsilon \downarrow 0$ and are bounded between the minimum and maximum of $f$. For any fixed larger perturbation a tail lies in its decreasing cone. The nested-limit closure in a Kähler class therefore puts the pointwise limit in every larger cone. It is bounded above by $f$. Every member of the intersection below $f$ is a candidate in each Kähler envelope, hence lies below the limit. This proves (26.1). □

## 26.2 Energy and measure of a continuous obstacle

For a Kähler class $\beta$ write $G_\beta(f) = E_\beta(P_\beta(f))$, and denote the corresponding Monge–Ampère measure by $\mu_\beta(f)$; its mass is $\int_X \beta^n$. The bounds by constant potentials give $|G_\beta(f)| \le \|f\|_\infty \int_X \beta^n$. The next theorem passes these quantities to the nef class $\alpha$, whose volume $V$ may vanish.

**Theorem 26.2.1** *Let $\beta_i \to \alpha$ through Kähler classes, and let continuous real obstacles $f_i$ on $X^\sqsupset$ converge uniformly to $f$. Then $G_{\beta_i}(f_i)$ and $\mu_{\beta_i}(f_i)$ converge, the latter weakly, to limits $G_\alpha(f)$ and $\mu_\alpha^{\mathrm{obs}}(f)$ independent of the approximations. The measure is positive of mass $V$ and is carried by $\{f = P_\alpha^{\mathrm{pert}}(f)\}$. For every continuous real direction $h$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} G_\alpha(f+th) = \int h \,\mathrm{d}\mu_\alpha^{\mathrm{obs}}(f). \tag{26.2}$$

***Proof*** Put $V_i = \int_X \beta_i^n$. If $V = 0$, the constant bound and the measure mass $V_i \to 0$ give zero limits for every obstacle. All the assertions follow. Suppose $V > 0$, so that $\alpha$ is big by [Xia26a, Proposition 15.8.2 and Theorem 17.4.20].

First take a rational model obstacle $f$ with divisor $D$ on a fixed smooth refinement $W$. Choose a fibre shift $c$ with $D + cW_0 \ge W_0$. Then $\rho^*\alpha + c_1(D) + c\,c_1(W_0)$ is big. Indeed, on $X \times \mathbb{P}^1$ the pullback of $\alpha$ plus the class of the pullback of a Fubini–Study form from the parameter line $\mathbb{P}^1$ has a Kähler current, by adding that form to a pullback of a Kähler current on $X$. The pullback of this current to $W$ lies in $\rho^*\alpha + c_1(W_0)$, since the pulled-back Fubini–Study class on $W$ is $c_1(W_0)$, and it dominates the pullback of a product Kähler form. The second assertion of Lemma D.4.1 gives a Kähler representative of the pulled-back form class after subtracting an effective exceptional divisor. Replace the pulled-back lower form by this representative, then add the exceptional divisor back as a current. This gives a Kähler current in $\rho^*\alpha + c_1(W_0)$, which is therefore big. Adding the effective vertical divisor $D+cW_0-W_0 \ge 0$ preserves bigness.

For each $i$ the Kähler volume identity (23.5) and the constant translation law give

$$G_{\beta_i}(f) = \frac{1}{n+1}\,\mathrm{vol}_W\big(\rho^*\beta_i + c_1(D) + c\,c_1(W_0)\big) - cV_i.$$

The volume converges to that of the class at $\alpha$, by continuity in the big cone as proved in Corollary 23.4.2. Thus the limit for this obstacle is independent of the approximating classes. Since $|G_{\beta_i}(f) - G_{\beta_i}(g)| \le V_i\|f-g\|_\infty$, rational model density gives

the same conclusion for all continuous real $f$, as well as for $f_i \to f$ uniformly. The limit is Lipschitz with constant $V$.

For a fixed real model direction $h$, the ordered energy test of Lemma 23.3.1 has a constant uniform as $\beta_i \to \alpha$. Indeed the proof bounds its constant using the fixed test curvature, $\|h\|_\infty$, and the mixed degree $\int_X \omega_h \beta_i^{n-1}$ for a fixed $\omega_h$ on $X$. For $P_{\beta_i}(f_i + th)$ and $P_{\beta_i}(f_i)$ take the common upper potential $P_{\beta_i}(f_i) + |t|\|h\|_\infty$. By nonexpansiveness and the energy Lipschitz bound the two energy gaps are at most $2V_i|t|\|h\|_\infty$. The measure of the common upper potential is $\mu_{\beta_i}(f_i)$ by translation invariance. The uniform energy test therefore bounds the difference of the two measure tests on $h$ by $C_h|t|^{1/2}$. Integrating the first variation (23.6) on the segment of obstacles gives

$$\left| G_{\beta_i}(f_i + th) - G_{\beta_i}(f_i) - t \int h \, \mathrm{d}\mu_{\beta_i}(f_i) \right| \le C_h' |t|^{3/2} \tag{26.3}$$

with a common constant on a tail.

At each fixed $t \neq 0$ the energies in (26.3) converge. Comparing two late indices and then taking $t$ arbitrarily small shows that the tests $\int h \, \mathrm{d}\mu_{\beta_i}(f_i)$ are Cauchy. The same comparison for any two approximations gives the same limit. Model density, positivity, and the bounded total masses give a weak limit Radon measure of mass $V$ by Theorem 4.3.1. Pass to the limit in (26.3) and divide by $t$ to get (26.2) for real model directions. Approximate a continuous $h$ uniformly by model directions. The error in the energy difference quotient is at most $V$ times the uniform error in the direction. The integral has the same bound. This proves the first variation for all continuous directions.

For contact, take the monotone sequence $\alpha + \varepsilon[\omega]$ and $f$ fixed. For each fixed $\varepsilon_0 > 0$, the gap $f - P_{\alpha+\varepsilon_0[\omega]}(f)$ is continuous and nonnegative. It is at most the gap in the smaller perturbation for $\varepsilon \le \varepsilon_0$, so its integral against the smaller perturbation's measure is zero by Kähler orthogonality. Weak convergence gives zero integral against $\mu_\alpha^{\mathrm{obs}}(f)$ as well. The gaps increase to $f - P_\alpha^{\mathrm{pert}}(f)$, by (26.1). Monotone integration gives contact. □

**Corollary 26.2.1** *Suppose $V > 0$. For a real model obstacle $f$ on a fixed smooth refinement $W$, $\mu_\alpha^{\mathrm{obs}}(f)$ is carried by the finite set $S$ of divisorial tests $v_E$ of the central components $E$ of $W$. For a real model direction $h$, choose a common smooth refinement $W'$, with projection $\rho' : W' \to X$, on which $f$ and $h$ have divisors $D$ and $G$. With a sufficiently large fixed shift $c$, the derivative at $t = 0$ of*

$$\mathrm{vol}_{W'}\left(\rho'^{*}\alpha + c_1(D + tG) + c\, c_1(W_0')\right)$$

*equals $(n+1)\int h \, \mathrm{d}\mu_\alpha^{\mathrm{obs}}(f)$. For each $v \in S$, a real model function $h$ with $h(v) = 1$ and $h = 0$ on $S \setminus \{v\}$, which exists as in Corollary 23.4.3, gives $(n+1)\mu_\alpha^{\mathrm{obs}}(f)(\{v\})$ as this derivative.*

***Proof*** All Kähler measures of the fixed obstacle $f$ are carried by the same finite set by Corollary 23.4.3. The limit is therefore carried by that set. The bigness and volume

identity in the proof of Theorem 26.2.1 hold for every real model obstacle by rational approximation on $W'$. Indeed one can keep the shift fixed when approximating the coefficients of $D + tG$ near $t = 0$, by making $f + c$ strictly greater than one. Uniform convergence of the obstacles and continuity of the big volumes pass the identity to those coefficients. The constant term $-cV$ has zero derivative, so (26.2) gives the asserted formula. Interpolation is by the same argument as for Corollary 23.4.3. □

## 26.3 Bounded potentials and their class limits

This section passes from envelopes of continuous obstacles to arbitrary bounded potentials of the perturbative cone. Write $V_\beta = \int_X \beta^n$ for a nef class $\beta$. The following lemma compares the energies and measures of one bounded potential in two nested nef classes, the smaller of which has positive volume.

**Lemma 26.3.1** *Let $\beta, \gamma$ be nef with $V_\beta > 0$. For bounded $v \in \mathcal{D}_\beta$,*

$$|E_{\beta+\gamma}(v) - E_\beta(v)| \le \|v\|_\infty (V_{\beta+\gamma} - V_\beta). \tag{26.4}$$

*Moreover $\mu_{\beta+\gamma}(v) - \mu_\beta(v)$ is positive and has mass $V_{\beta+\gamma} - V_\beta$.*

***Proof*** Adding $(\gamma, 0)$ preserves the continuous and decreasing cones. For a semipositive model write $P = (\beta, v)$ and $Q = (\gamma, 0)$. Expand $\langle (P+Q)^{n+1} \rangle_X$. For a term with $1 \le k \le n$ factors $Q$, replace the other $n + 1 - k$ factors, one at a time, by $(\beta, 0)$. The term with all zero potentials vanishes by dimension on $X$. Each replacement has absolute value at most $\|v\|_\infty \int_X \beta^{n-k}\gamma^k$, by the divisor integration formula against a positive mixed measure of that mass. Since $\binom{n+1}{k}(n+1-k)/(n+1) = \binom{n}{k}$, division by $n + 1$ gives (26.4). The term with $k = n + 1$ vanishes as well. In the measure expansion all additional terms are positive, so the mass comparison follows from the mixed masses.

Pass to $C_\beta$ by uniform approximation. Then use a decreasing continuous approximation to $v$ in that cone. The approximants also belong to $C_{\beta+\gamma}$. For any $\delta > 0$, compactness bounds their uniform norms on a tail by $\|v\|_\infty + \delta$, since the increasing open sets on which they are less than that constant cover the spectrum. Decreasing continuity of energies and weak convergence of measures in both classes give the claims. □

**Theorem 26.3.1** *For $u \in \mathcal{D}_\alpha^{\mathrm{pert}}$ bounded, the limits*

$$E_\alpha^{\mathrm{pert}}(u) = \lim_{\kappa \to 0} E_{\alpha+\kappa}(u), \qquad \mu_\alpha^{\mathrm{pert}}(u) = \lim_{\kappa \to 0} \mu_{\alpha+\kappa}(u)$$

*exist along Kähler additions $\kappa$, with convergence in total variation for the measures. The energy is real and the measure is positive of mass $V$. For $V > 0$ they agree with the fixed-nef constructions on bounded elements of $\mathcal{D}_\alpha$.*

***Proof*** Fix a Kähler form $\omega$. For each $\delta > 0$, the difference $\delta[\omega] - \kappa$ is Kähler when $\kappa$ is sufficiently small, by openness of the Kähler cone. Compare any two such classes $\alpha + \kappa$ with $\alpha + \delta[\omega]$. By Lemma 26.3.1, the energy distances to that upper class are at most $\|u\|_\infty (V_{\alpha+\delta[\omega]} - V)$. The measure distances are at most $V_{\alpha+\delta[\omega]} - V$, since the measure differences are positive. These estimates tend to zero with $\delta$, proving the Cauchy property without a monotonicity assumption on the approaches.

The limiting measure exists by completeness in total variation. Equivalently, the Cauchy integrals define a positive functional on the continuous functions, and its Radon measure, given by Theorem 4.3.1, satisfies the same Cauchy bound by testing against continuous functions of norm at most one. Its mass is $V$. The energy satisfies $|E^{\mathrm{pert}}_\alpha(u)| \le V\|u\|_\infty$ by the constant bounds in each class. If $u$ is in the fixed cone and $V > 0$, apply Lemma 26.3.1 with $\beta = \alpha$ to obtain the last assertion. □

**Corollary 26.3.1** *For every continuous real obstacle $f$, with $u = P^{\mathrm{pert}}_\alpha(f)$, one has*

$$G_\alpha(f) = E^{\mathrm{pert}}_\alpha(u), \qquad \mu^{\mathrm{obs}}_\alpha(f) = \mu^{\mathrm{pert}}_\alpha(u).$$

***Proof*** Put $\alpha_\varepsilon = \alpha + \varepsilon[\omega]$ for a fixed Kähler form $\omega$, and $U_\varepsilon = P_{\alpha_\varepsilon}(f)$. Both $u$ and $U_\varepsilon$ are bounded in absolute value by $A = \|f\|_\infty$, and $U_\varepsilon \downarrow u$. The lower bound $\liminf_\varepsilon E_{\alpha_\varepsilon}(U_\varepsilon) \ge E^{\mathrm{pert}}_\alpha(u)$ follows from $U_\varepsilon \ge u$. For $0 < \varepsilon \le \delta$, Lemma 26.3.1 gives

$$E_{\alpha_\varepsilon}(U_\varepsilon) \le E_{\alpha_\delta}(U_\varepsilon) + A(V_{\alpha_\delta} - V_{\alpha_\varepsilon}).$$

The potentials on the right decrease to $u$ in the fixed Kähler class $\alpha_\delta$. Take $\varepsilon \downarrow 0$ by decreasing energy continuity, then $\delta \downarrow 0$ by Theorem 26.3.1. This proves the opposite inequality and shows that

$$E_{\alpha_\varepsilon}(U_\varepsilon) - E_{\alpha_\varepsilon}(u) \longrightarrow 0.$$

For each $\varepsilon$ the ordered estimate of Lemma 23.3.1 applies to $u \le U_\varepsilon$. Indeed, if $w_j \in C_{\alpha_\varepsilon}$ decrease to $u$, use the continuous majorants $P_{\alpha_\varepsilon}(\min(w_j, U_\varepsilon))$. These decrease to $u$ since they are between $u$ and $w_j$, and they are at most $U_\varepsilon$. Apply the estimate to the continuous majorants, and take the decreasing limit in the energies and measures. Its constant is uniform as $\varepsilon \downarrow 0$ by the curvature bound in the proof of Theorem 26.2.1. Each model test of the measure difference therefore tends to zero. The masses are bounded and model tests are uniformly dense. Since the measures of $u$ in $\alpha_\varepsilon$ converge in total variation, the weak obstacle limit is its measure in Theorem 26.3.1. □

**Corollary 26.3.2** *Let $V > 0$ and $f$ be continuous. If $u = P^{\mathrm{pert}}_{\alpha}(f)$ belongs to $\mathcal{D}_{\alpha}$, then it is the greatest fixed-nef minorant of $f$, and*

$$G_{\alpha}(f) = E_{\alpha}(u), \qquad \mu^{\mathrm{obs}}_{\alpha}(f) = \mu_{\alpha}(u).$$

***Proof*** The minorant statement follows from $\mathcal{D}_{\alpha} \subset \mathcal{D}^{\mathrm{pert}}_{\alpha}$. The energy and measure equalities follow from Theorem 26.3.1 and Corollary 26.3.1. □

## 26.4 Mixed measures at the nef boundary

For unbounded potentials this section constructs mixed measures under a finite-energy hypothesis. A potential in the perturbative cone is said to have *finite energy in a Kähler perturbation* if its energy is finite for at least one of the classes $\alpha+\kappa$ with $\kappa$ Kähler. The first lemma shows that this does not depend on the auxiliary Kähler class.

**Lemma 26.4.1** *If $u \in \mathcal{D}^{\mathrm{pert}}_{\alpha}$ satisfies $E_{\alpha+\kappa_0}(u) > -\infty$ for one Kähler class $\kappa_0$, then $E_{\alpha+\kappa}(u) > -\infty$ for every Kähler class $\kappa$.*

***Proof*** Subtract a constant so the potential is nonpositive. Suppose $u$ has finite energy in $\beta_0 = \alpha + \kappa_0$, and let $\beta = \alpha + \kappa$ be another perturbation. The class $\Omega = \alpha + \kappa_0 + \kappa$ dominates both in the nef order. Take continuous majorants $w_j$ decreasing to $u$ in $\beta_0$. We may work on a tail on which the majorants are bounded above by a common constant, and translate both them and $u$ further to make them nonpositive. They are also in $C_{\Omega}$. Choose $c \geq 1$ such that $c\beta_0 - \Omega$ is nef. The background comparison and scaling used in Theorem 23.3.2 give

$$E_{\Omega}(w_j) \geq E_{c\beta_0}(w_j) = c^{n+1} E_{\beta_0}(w_j/c) \geq c^{n+1} E_{\beta_0}(u).$$

Thus $u$ has finite energy in $\Omega$. For the bounded truncations $u_j = \max(u, -j)$, the same background comparison gives $E_{\beta}(u_j) \geq E_{\Omega}(u_j)$. It applies to bounded decreasing limits by using their continuous majorants. Decreasing continuity in $\beta$ and $\Omega$ proves finiteness in $\beta$. □

**Theorem 26.4.1** *For $1 \leq i \leq n$ let $\alpha_i$ be nef and let $u_i \in \mathcal{D}^{\mathrm{pert}}_{\alpha_i}$ have finite energy in a Kähler perturbation. The mixed measures*

$$\mathrm{MA}_X((\alpha_i + \kappa_i, u_i)_{i=1}^{n})$$

*converge in total variation as the Kähler additions $\kappa_i$ tend independently to zero. Their limit $\mathrm{MA}^{\mathrm{pert}}_X((\alpha_i, u_i)_{i=1}^{n})$ is positive of mass $\int_X \alpha_1 \cdots \alpha_n$. The diagonal measure extends Theorem 26.3.1. The scalar limit $\lim_{\kappa \to 0} E_{\alpha+\kappa}(u)$ likewise exists and is finite for any such $u$. For both limits one may first take the class limit of the bounded truncations and then pass to the truncation limit, weakly for the measures.*

***Proof*** Put $\beta_i = \alpha_i + \kappa_i$. Choose a fixed Kähler perturbation $\Omega_i$ of $\alpha_i$. For all $\beta_i$ sufficiently close to $\alpha_i$, the difference $\Omega_i - \beta_i$ is Kähler. By Lemma 26.4.1, $u_i$ has finite energy in $\Omega_i$. Use formal differences of pairs as in the intersection formula on models. In the $i$-th slot write

$$(\beta_i, u_i) = (\Omega_i, u_i) - (\Omega_i - \beta_i, 0).$$

Expanding all slots computes the mixed measure. Indeed, take simultaneous decreasing continuous approximations in the $\beta_i$. They are also continuous semipositive approximations in the $\Omega_i$. The signed expansion on models passes to the continuous measures by uniform approximation, and then to the finite-energy measures by Theorem 23.3.2. The same expansion computes the diagonal energy by using $n+1$ slots and dividing by $n+1$. For these pairings, decreasing continuity follows from the polarization in Section 24.5, since each sum of approximants decreases in the fixed sum class and its limit has finite energy there.

Choose a basis of the real Bott–Chern space of type $(1,1)$ whose elements are differences of fixed Kähler classes. Such a basis exists by openness of the Kähler cone. Express each $(\Omega_i - \beta_i, 0)$ in those fixed zero-potential pairs. The expansion in these fixed classes is valid by linearity on models with the functions unchanged. Passing through the same continuous and decreasing approximations proves it for the finite-energy data. Consequently the measure is a finite sum of fixed Radon measures with coefficients polynomial in the $\beta_i$, and the energy is a finite sum of fixed real pairings with polynomial coefficients. Taking $\beta_i \to \alpha_i$ gives total variation and scalar convergence. Positivity and the mass formula pass to the limit. For bounded potentials the diagonal limit is the one already constructed. The same proof applies to any Kähler approaches for which the fixed potentials are psh and of finite energy in the approaching classes, since the $\Omega_i - \beta_i$ are still Kähler near $\alpha_i$.

For the truncations take $u_{i,j} = \max(u_i, -j)$. These are bounded decreasing limits in each perturbed class by stability under maxima there. The coefficients of the expansions above do not depend on the potentials. In all terms with nonconstant potentials the truncations decrease in the fixed Kähler classes $\Omega_i$. For these fixed terms, apply polarization to the sums over nonempty subsets. Each sum decreases in its fixed Kähler class and the limiting energy is finite. For such decreasing limits $v_j \downarrow v$, the energies decrease to $E(v)$. Indeed $v_j < w + \delta$ eventually for any continuous majorant $w$ of $v$ and $\delta > 0$, by upper semicontinuity and compactness. Take the infimum over such majorants. The ordered energy estimate extends to finite-energy $v \le v_j$ as follows. Fix a continuous majorant of $v_j$ in its class, and cut the continuous majorants of $v$ by it using the Kähler envelope of their minimum, as in the proof of Corollary 26.3.1. First pass to the limit in the lower majorants, then in the upper ones. This proves the estimate between $v$ and $v_j$. The energy gap tends to zero, so each model test converges. Density and the constant mass give weak convergence. This proves convergence of each polarized mixed measure. The scalar pairings converge by the energy continuity just used. Thus each fixed term of the expansions has the asserted limit. Evaluation at $\beta_i = \alpha_i$ proves the assertion about truncations. □

*Example 26.4.1* On $X = \mathbb{P}^1 \times \mathbb{P}^1$, let $\alpha_i$ be the class pulled back from $\mathcal{O}(1)$ on the $i$-th factor. Each diagonal measure of the zero potential has zero mass because $\alpha_i^2 = 0$. The mixed measure of the two zero potentials is $\delta_{v_0}$. Indeed, for positive perturbations the product model has the single central component $X$, of multiplicity one, so the mixed measure is its mixed intersection degree times $\delta_{v_0}$. As the perturbations tend to zero this degree tends to $\int_X \alpha_1 \alpha_2 = 1$, computed by intersecting one fibre of each projection.

**Corollary 26.4.1** *Let $X$ be smooth connected projective and let all nef classes be in the real span of curvatures of line bundles. The perturbative cones agree with the algebraic psh cones. The perturbative energies and mixed measures of Theorem 26.4.1 agree with the algebraic nef energies and mixed measures on $\mathcal{E}^1(X)$. The two conditions of finite energy in an ample class correspond, so the comparison includes all bounded potentials.*

***Proof*** The cone identification follows from Theorem 26.1.1 and Corollary 25.12.1. For the finite-energy domain, take a numerical ample perturbation and use Corollary 23.3.1. On the cohomological side Lemma 26.4.1 allows this choice. On the algebraic side a potential of finite energy in one ample class has finite energy in every ample class in which it is psh, by Section 17.9. In the signed expansion of the proof choose numerical ample $\Omega_i$ and choose the basis for the constant backgrounds in the real numerical subspace, using differences of numerical ample classes. Every term agrees with its algebraic counterpart by the ample comparison, also for mixed pairings by their energy polarization. The algebraic nef construction of Section 17.9 is precisely the evaluation of this expansion at the nef classes. This gives the stated identities. □

## 26.5 The extended energy from test curves

A potential in the perturbative cone can have energy $-\infty$ in every Kähler perturbation (Example 26.5.1). To reach such potentials we use bounded truncations. For $u \in \mathcal{D}_\alpha^{\mathrm{pert}}$ define the *truncation energy*

$$\overline{E}_\alpha^{\mathrm{pert}}(u) = \inf_{j \geq 1} E_\alpha^{\mathrm{pert}}(\max(u, -j)). \tag{26.5}$$

Truncations belong to every larger decreasing cone by stability under finite maxima there. Each is bounded above since $u$ is upper semicontinuous on the compact spectrum. The bounded energies decrease in $j$, by monotonicity in the Kähler classes. On the domain of Theorem 26.4.1 this recovers the scalar class limit by its truncation assertion. We also call $\overline{E}_\alpha^{\mathrm{pert}}$ the *intrinsic energy*, and say that $u$ has *finite intrinsic energy* when $\overline{E}_\alpha^{\mathrm{pert}}(u) > -\infty$. The next theorem identifies it with the extended test-curve energy when $V > 0$.

**Theorem 26.5.1** *If $V > 0$, then for the compatible family $\Gamma$ corresponding to $u \in \mathcal{D}_\alpha^{\mathrm{pert}}$, one has*

$$\overline{E}_\alpha^{\mathrm{pert}}(u) = \mathcal{E}_{\mathrm{tc}}(\Gamma)$$

*with the extended energy from (25.3). If $V = 0$, the truncation energy is zero and all the limiting slice masses vanish.*

***Proof*** In a nef class the volume is $V$, by the volume formula for nef classes [Xia26a, Proposition 15.8.2]. When $V = 0$, each bounded energy in (26.5) is zero by Theorem 26.3.1. Every limiting slice mass is at most the limiting class volume, hence is zero.

Suppose $V > 0$. First take $u$ bounded. Fix a Kähler form $\omega$ and use the ordinary coordinates in $\theta + \varepsilon\omega$. Write $m_\varepsilon(\tau)$ for their slice mass and put $V_\varepsilon = \int_X (\alpha + \varepsilon[\omega])^n$. The compatible family of the ordinary $\varepsilon$-coordinate has jet $u$, by Theorem 25.11.1; a change of smooth representative allows the full Kähler comparison in that class. By boundedness and Theorem 25.12.1 and Corollary 25.9.1, the ordinary $\varepsilon$-coordinate has full left mass $V_\varepsilon$ and finite energy. The mass comparison in (25.2) therefore makes the deficits $V_\varepsilon - m_\varepsilon(\tau)$ decrease as $\varepsilon \downarrow 0$. On each tail of the approach an integrable dominant is the deficit of a fixed larger perturbation. Denote the cutoff by $M$. Dominated convergence on $(-\infty, M)$ and $V_\varepsilon \to V$ give

$$\lim_{\varepsilon \downarrow 0} E_{\alpha+\varepsilon[\omega]}(u) = MV - \int_{-\infty}^{M} (V - m(\tau))\, \mathrm{d}\tau.$$

This proves the bounded case.

Now take any $u$ and restrict to $j$ large enough that $-j < M$. The cutoff is the value at the trivial valuation, so $u_j = \max(u, -j)$ and $u$ have cutoff $M$. Write $\Gamma_j$ for the family of $u_j$. By the order isomorphism, the families decrease in $j$. We claim that for each $\tau < M$ the limiting slice masses $m_j(\tau)$ of $\Gamma_j$ decrease to $m(\tau)$.

In the fixed coordinate $\theta + \varepsilon\omega$, every slice of $\Gamma_j^\varepsilon$ is obtained by enlargement from $\theta + (\varepsilon/2)\omega$. Adding $(\varepsilon/2)\omega$ to the smaller current yields a mass at least $(\varepsilon/2)^n \int_X \omega^n > 0$, by retaining the pure smooth term of the non-pluripolar product. Enveloping in the larger class can only increase mass. Thus the decreasing slices have a uniform positive lower bound for their mass. The decreasing-evaluation lemma Lemma 25.7.1 gives an ordinary limit and convergence of its interior slice masses. Its divisorial evaluation is $u$ on divisorial rays by that lemma. The order recovery Proposition 25.4.1 identifies the limit with $\Gamma^\varepsilon$. Thus $m_{\varepsilon,j}(\tau) \downarrow m_\varepsilon(\tau)$. Both class and truncation mass limits are infima, so

$$\inf_j m_j(\tau) = \inf_j \inf_{\varepsilon>0} m_{\varepsilon,j}(\tau) = \inf_{\varepsilon>0} m_\varepsilon(\tau) = m(\tau).$$

This proves the claim. Apply the bounded energy identity to each $u_j$ and then monotone convergence to the nonnegative deficits $V - m_j(\tau)$. The integral may be infinite. Its negative plus $MV$ is the extended energy in (25.3), as required. □

*Example 26.5.1* Let $X$ be the blowup of $\mathbb{P}^2$ at a point and $\alpha$ the pullback of the hyperplane class. There is a potential $u \in \mathcal{D}_\alpha^{\mathrm{pert}}$ with $\overline{E}_\alpha^{\mathrm{pert}}(u) > -\infty$ and $E_{\alpha+\kappa}(u) = -\infty$ for every Kähler addition $\kappa$.

***Proof*** Let $D$ be the exceptional divisor and put $c = 1/2$. The class $\alpha - c[D]$ is Kähler. Indeed, twice this class is the restriction of the ample $O(1,1)$ under the graph presentation of the blowup in $\mathbb{P}^2 \times \mathbb{P}^1$. The second map is the pencil of lines through the blown-up point, of class $\alpha - [D]$. Choose a semipositive representative $\theta$ of $\alpha$ by pullback and a positive smooth representative of $\alpha - c[D]$. By the Poincaré–Lelong formula and a smooth change of potential there is a $\theta$-psh function $g \leq 0$ whose current is this last positive form plus $c[D]$.

For $\tau < 0$ put $a(\tau) = c/(1-\tau)$ and take the $\mathcal{I}$-envelope of $(a(\tau)/c)g$ in $\theta$. The input functions are $\theta$-psh by convex combination with zero. They are decreasing and concave in $\tau$ since $g \leq 0$ and $a$ is increasing and convex. The envelopes retain those properties by monotonicity and concavity of the envelope. Their left limit is zero, since the inputs tend to zero off $D$. To compute the mass let $b = (a(\tau)/c)g$, and let $p$ be its $\mathcal{I}$-envelope in $\theta + \eta$, where $\eta = 0$ or $\eta$ is Kähler. Then $(\theta+\eta) + dd^c b = B_\eta + a(\tau)[D]$, with $B_\eta$ smooth and positive. Since $p$ has the same multiplier type, its generic Lelong number along $D$ is $a(\tau)$ [Xia26b, Proposition 3.2.1(5)]. Siu's decomposition gives $\theta + \eta + dd^c p - a(\tau)[D] \geq 0$ [Xia26b, Lemma 1.7.1]. Thus $dd^c(p-b) \geq -B_\eta$ as currents. The quasi-psh representative of this difference is bounded above on the compact manifold. Together with $p \geq b$, this shows equality of singularity types modulo bounded functions. Their masses agree by mass monotonicity [Xia26b, Theorem 2.4.3]. The mass is $\int_X B_\eta^2$, since off $D$ the input current equals $B_\eta$ and the non-pluripolar product does not charge $D$. In $\theta$ this gives $(\alpha - a(\tau)[D])^2 > 0$. These slices form an ordinary curve of cutoff zero. Enlarge it to all Kähler perturbations and let $u$ be the jet of the resulting family.

The enlarged slice is the envelope of the same type in $\theta+\eta$ for a Kähler representative $\eta$ of $\kappa$. Its mass is $(\alpha+\kappa-a(\tau)[D])^2$ by the same computation. Since $\alpha \cdot D = 0$ and $D^2 = -1$, the deficit from the perturbed full class volume is $a(\tau)^2 + 2a(\tau)(\kappa \cdot D)$. The intersection $D^2$ is the degree of the normal bundle $O_{\mathbb{P}^1}(-1)$, and $\kappa \cdot D > 0$. On $(-\infty, 0)$, the integral of $a^2$ is $c^2$, while that of $a$ is infinite. The extended test-curve energy of the family in $\alpha$ is therefore $-c^2$, which equals the truncation energy by Theorem 26.5.1. Every Kähler perturbation has energy $-\infty$ by the deficit formula and Theorem 25.12.1. □

## 26.6 Measures of intrinsic finite energy

The test-curve energy gives a finite-energy domain in the perturbative cone even when the perturbing Kähler energies are infinite. To define a measure on this domain, we use the ordered energy estimate at the boundary.

**Lemma 26.6.1** *The energy $\overline{E}^{\mathrm{pert}}_{\alpha}$ is increasing and continuous along decreasing nets in the perturbative cone whose limit lies in that cone. For every real model test $h$ there is a constant $C_h$ such that for bounded ordered potentials $u \le v$ in that cone,*

$$\left| \int h \, \mathrm{d}(\mu^{\mathrm{pert}}_{\alpha}(v) - \mu^{\mathrm{pert}}_{\alpha}(u)) \right|^2 \le C_h \big( E^{\mathrm{pert}}_{\alpha}(v) - E^{\mathrm{pert}}_{\alpha}(u) \big).$$

***Proof*** Monotonicity follows from that of the bounded energies and of truncation. Suppose first that bounded $u_i$ decrease to bounded $u$. On a tail the functions have a common bound $A$ in absolute value, using $u$ and a fixed $u_{i_0}$ on that tail. Fix a Kähler form $\omega$ and write $\alpha_\delta = \alpha + \delta[\omega]$. Lemma 26.3.1 for nested perturbations, followed by the class limit of Theorem 26.3.1, gives

$$E^{\mathrm{pert}}_{\alpha}(u_i) \le E_{\alpha_\delta}(u_i) + A(V_{\alpha_\delta} - V).$$

Take $i$ to the limit by decreasing energy continuity in the fixed class. Then take $\delta \downarrow 0$. This bounds the limiting energy above by $E^{\mathrm{pert}}_{\alpha}(u)$; monotonicity gives the reverse. For general decreasing $u_i \downarrow u$, apply the bounded assertion to $\max(u_i, -j) \downarrow \max(u, -j)$ for each $j$. The infima over $i$ and $j$ commute in (26.5), which proves decreasing continuity.

For the estimate use a Kähler perturbation and the proof of Theorem 26.4.1 to pass the ordered test estimate from continuous majorants to $u \le v$. Its constant stays bounded as the classes approach $\alpha$, by the bound with $\int_X \omega_h(\alpha + \delta[\omega])^{n-1}$ in Lemma 23.3.1. The limits for bounded potentials in Theorem 26.3.1 give the claimed inequality. □

**Theorem 26.6.1** *Let $u \in \mathcal{D}^{\mathrm{pert}}_{\alpha}$ with $\overline{E}^{\mathrm{pert}}_{\alpha}(u) > -\infty$. The bounded measures of $\max(u, -j)$ converge weakly to a positive Radon measure $\mu^{\mathrm{pert}}_{\alpha}(u)$ of mass $V$. Every decreasing net of bounded potentials of the cone with limit $u$ gives the same measure. It integrates $u$ and agrees with the perturbative measure when $u$ has finite energy in a Kähler perturbation.*

***Proof*** For any decreasing bounded approximation $u_i \downarrow u$, its energies decrease to the finite limit by Lemma 26.6.1. The ordered test estimate then makes the model tests of the measures Cauchy. For two indices compare each with a common later index. Model density and positivity of fixed mass give convergence on every continuous test, and hence a positive Radon limit. To see independence, interleave two nets $u_i, w_j$ using the upper bound $z_{ij} = \max(u_i, w_j)$ for both. It is a bounded potential of the perturbative cone. The three energies tend to the same limit, since $z_{ij} \downarrow u$. Apply the estimate to each ordered comparison with $z_{ij}$. This gives the same model tests in the two limits. The agreement with the finite-energy class limit follows from the truncation assertion of Theorem 26.4.1.

For integrability translate $u$ to be nonpositive and take $u_j = \max(u, -j)$. In each Kähler perturbation $\beta$,

$$\int u_j \, \mathrm{d}\mu_\beta(u_j) \ge (n+1) E_\beta(u_j).$$

This is the mixed energy difference from zero, keeping the diagonal term as in Theorem 23.3.1. The bounded function $u_j$ is a valid test in the total variation class limit, so the same inequality holds in $\alpha$ for the bounded operator. The right side is bounded below by $(n+1)\overline{E}^{\mathrm{pert}}_{\alpha}(u)$. For a fixed $k$ and $j \geq k$, one may replace $u_j$ on the left by $u_k$. The upper semicontinuous test inequality of Proposition 4.3.1 passes the lower bound for these integrals to the weak limit. Then $k \to \infty$ gives the lower bound for $\int u \, \mathrm{d}\mu^{\mathrm{pert}}_{\alpha}(u)$ by monotone integration. Translation preserves the measures and shifts the energies by $V$ times the constant, first for bounded data by passage from the Kähler classes and then by decreasing bounded approximation. □

**Corollary 26.6.1** *Suppose $V > 0$. For every $u \in \mathcal{D}_{\alpha}$,*

$$\overline{E}^{\mathrm{pert}}_{\alpha}(u) = E_{\alpha}(u),$$

*including the value $-\infty$. If the energy is finite, then the intrinsic and fixed-nef diagonal measures agree.*

***Proof*** Take $w_j \in \mathcal{C}_{\alpha}$ decreasing to $u$. These are bounded members of $\mathcal{D}_{\alpha}$, so their two energies and measures agree by Theorem 26.3.1. The energies decrease to the two stated energies by Lemma 26.6.1 and the definition of fixed-nef energy in Section 23.3. For finite energy the diagonal constructions in Theorems 23.3.1 and 26.6.1 both give the weak limit of the measures of $w_j$. □

## 26.7 Finite energy in the sum class

For potentials in different nef classes, finite intrinsic energy of their sum in the sum class controls the mixed measures. In a single class this condition follows from finite intrinsic energy of the individual potentials, by concavity (Corollary 26.7.1). With different backgrounds it can impose an additional integrability condition.

For finitely many nef classes $\alpha_i$ and $u_i \in \mathcal{D}^{\mathrm{pert}}_{\alpha_i}$, put

$$\alpha_I = \sum_{i \in I} \alpha_i, \qquad u_I = \sum_{i \in I} u_i$$

for each nonempty subset of the indices. Then $u_I \in \mathcal{D}^{\mathrm{pert}}_{\alpha_I}$. Indeed, for a Kähler addition $\kappa$, the individual potentials are in the classes $\alpha_i + \kappa/|I|$. Adding their decreasing continuous approximations puts the sum in the cone of $\alpha_I + \kappa$. The sum is finite at the trivial valuation $v_0$.

If $\gamma$ is nef and $u \leq 0$ is in the perturbative cone of $\alpha$, then

$$\overline{E}^{\mathrm{pert}}_{\alpha+\gamma}(u) \leq \overline{E}^{\mathrm{pert}}_{\alpha}(u). \tag{26.6}$$

For bounded $u$, compare the energies in $\alpha + \kappa$ and $\alpha + \gamma + \kappa$ by the background expansion in the proof of Theorem 23.3.2. The comparison extends from continuous

potentials to bounded decreasing limits by translating the continuous majorants down by constants tending to zero. Such constants are available by compactness on a tail. Take $\kappa \to 0$ using Theorem 26.3.1. For unbounded $u$, take the infimum over the bounded truncations to get (26.6).

As in Theorems 26.3.1 and 26.6.1, $\mu_\beta^{\mathrm{pert}}(w)$ denotes the diagonal measure in a class $\beta$ whenever it is defined.

**Theorem 26.7.1** *Let $u_i \in \mathcal{D}_{\alpha_i}^{\mathrm{pert}}$ for nef $\alpha_i$, $1 \le i \le n$. Suppose the sum $u = \sum_i u_i$ has finite intrinsic energy in $\beta = \sum_i \alpha_i$, that is, $\overline{E}_\beta^{\mathrm{pert}}(u) > -\infty$. The mixed measures of any simultaneous decreasing bounded approximations in the respective cones converge weakly to the same positive Radon measure, of mass $\int_X \alpha_1 \cdots \alpha_n$. It is*

$$\frac{1}{n!} \sum_{\varnothing \ne I \subseteq \{1,\dots,n\}} (-1)^{n-|I|} \mu_{\alpha_I}^{\mathrm{pert}}(u_I). \tag{26.7}$$

*If $u_0 \in \mathcal{D}_{\alpha_0}^{\mathrm{pert}}$ and the sum including $u_0$ has finite intrinsic energy in the enlarged sum class, then $u_0$ is integrable against the mixed measure.*

***Proof*** Translate all the potentials to be nonpositive. Translation preserves finiteness in the sum classes by the energy translation rule and preserves the measures. For any nonempty subset $I$, the sum $u_I$ is at least $u$, and $\beta - \alpha_I$ is nef. By (26.6) and monotonicity of the energy in the potential, its energy in $\alpha_I$ is at least the finite energy of the full sum in the full class. Thus each diagonal measure on the right of (26.7) is defined.

For bounded approximants, the polarization identity holds on models by multilinearity: each term that omits an index cancels, and the mixed term using every index occurs $n!$ times. Uniform and decreasing approximation pass it to bounded potentials in Kähler classes. Taking the Kähler additions to zero gives the identity for bounded potentials in the respective perturbative cones by Theorem 26.4.1. Every subset sum of the bounded approximants decreases to the corresponding $u_I$. The diagonal convergence of Theorem 26.6.1 therefore gives the weak limit in (26.7). Positivity and the mixed mass pass from the bounded measures.

For the integrability assertion let $w = \sum_{i=0}^n u_i$, with all potentials again nonpositive. For bounded approximations of these potentials, the diagonal measure of their sum dominates $n!$ times the mixed measure of the last $n$ entries. This holds by the positive model expansion, then in Kähler classes by approximation, and at the boundary by the class limits. Taking the decreasing limits, the same domination holds for $\mu_{\sum \alpha_i}^{\mathrm{pert}}(w)$. The function $w$ is integrable against that measure by Theorem 26.6.1, and $w \le u_0 \le 0$. □

**Corollary 26.7.1** *If all the $\alpha_i$ equal $\alpha$ and each $u_i$ has finite intrinsic energy, then the sum condition holds and the mixed measure integrates every member of $\mathcal{D}_\alpha^{\mathrm{pert}}$ of finite intrinsic energy. When $V > 0$ and the $u_i$ belong to $\mathcal{D}_\alpha$, the mixed measure agrees with the fixed-nef construction. For distinct or equal nef classes on the smooth projective overlap, the measures of Theorem 26.7.1 are the weak limits of the algebraic nef measures of the bounded approximants.*

***Proof*** The average of finitely many potentials of finite intrinsic energy has finite intrinsic energy. For bounded potentials this follows from concavity in the Kähler perturbations and passage to the class limit. For unbounded potentials apply the bounded result to their truncations and take the decreasing limit using Lemma 26.6.1. For $r > 0$, scaling $(\alpha, u)$ to $(r\alpha, ru)$ scales energy by $r^{n+1}$. The model equality passes to bounded potentials by the Kähler class limits, and to all potentials by decreasing continuity. Thus the sum of $r$ such potentials with $r$ a positive integer has finite intrinsic energy in $r\alpha$, including when a further test potential is added.

For fixed-nef data take simultaneous approximants in $C_\alpha$. The two mixed measures of these bounded approximants agree by polarization of the diagonal comparison in Theorem 26.3.1, using the classes $|I|\alpha$. The fixed-nef and intrinsic constructions are the weak limits of those same measures. For smooth projective data all bounded approximants are algebraic psh, with bounded mixed measures given by Corollary 26.4.1. The identity follows from the weak limit in Theorem 26.7.1. □

## 26.8 Locality at a strict maximum

Bounded mixed measures have a locality property that can be tested on one slot at a time: replacing a potential $u_1$ by $\max(u_1, v)$ does not change the measure on $\{u_1 > v\}$ (Theorem 26.8.1). The strict inequality matters because the potentials can be discontinuous. The proof passes from model inputs to bounded potentials, and the following lemma supplies the required convergence of weighted integrals.

**Lemma 26.8.1** *Fix Kähler classes $\beta_1, \dots, \beta_n, \gamma$. Let $u_{i,j} \downarrow u_i$ in $\mathcal{D}_{\beta_i}$ and $a_j \downarrow a$ in $\mathcal{D}_\gamma$, with all limits and approximants bounded. Write $\nu_j$ and $\nu$ for the mixed measures of the $u_{i,j}$ and the $u_i$, respectively. For every continuous real $h$,*

$$\int h a_j \, \mathrm{d}\nu_j \longrightarrow \int h a \, \mathrm{d}\nu.$$

***Proof*** The mixed pairing of $n + 1$ bounded pairs in Kähler classes is the polarization of the energy in Section 24.5. It is continuous along decreasing nets: the subset sums decrease in their fixed sum classes and decreasing energy continuity applies to each term. The mixed measures are likewise weakly continuous along decreasing nets with bounded limits. For their diagonals, the ordered estimate extends from continuous majorants by taking envelopes of the minima, as in Theorem 26.4.1. The energy gaps tend to zero and the masses are constant. Polarization gives the mixed assertion.

For bounded $w_i \in \mathcal{D}_{\beta_i}$ and $b \in \mathcal{D}_\gamma$, write $P_i = (\beta_i, w_i)$. The integration formula is

$$\int b \, \mathrm{d}\, \mathrm{MA}_X(P_1, \ldots, P_n) = \langle (\gamma, b), P_1, \ldots, P_n \rangle_X - \langle (\gamma, 0), P_1, \ldots, P_n \rangle_X .$$

This is the divisor formula for model inputs, and uniform approximation proves it for continuous ones. For a fixed continuous $b$, let continuous majorants decrease to the $w_i$, using weak convergence on the left and decreasing continuity of the pairings on the right. Next hold the $w_i$ fixed and let continuous majorants decrease to $b$. Monotone Radon integration on the left gives the formula for all the bounded inputs. Applying it to $a_j$ and the $u_{i,j}$ now proves convergence of the unweighted integrals by decreasing continuity of each pairing.

It remains to allow a continuous factor $0 \leq h \leq 1$. For a fixed index $k$, one has $ha_j \leq ha_k$ when $j \geq k$, and $ha_k$ is bounded and upper semicontinuous. Weak convergence and Proposition 4.3.1 give $\limsup_j \int ha_j \, \mathrm{d}\nu_j \leq \int ha_k \, \mathrm{d}\nu$. Let $k$ tend to the limit by monotone integration. Apply the same upper bound to $1 - h$ and use the unweighted convergence to get the lower bound for $h$. Subtracting and scaling continuous nonnegative functions gives the result for real $h$. □

**Theorem 26.8.1** *Take bounded potentials $u_i \in \mathcal{D}^{\mathrm{pert}}_{\alpha_i}$ in nef classes $\alpha_i$, $1 \leq i \leq n$. For bounded $v \in \mathcal{D}^{\mathrm{pert}}_{\alpha_1}$ put $w = \max(u_1, v)$. On $\{u_1 > v\}$, replacing the first potential by $w$ leaves the bounded mixed measure unchanged.*

***Proof*** First take Kähler backgrounds and rational semipositive model inputs. Write $u_1 = (\varphi_I - \varphi_J)/m$ and $v = (\varphi_K - \varphi_J)/m$ using a common positive denominator and the same subtracted ideal. This is possible by multiplying the two separate subtracted ideals and adjusting each numerator. The maximum uses $I + K$ in the numerator, as in Lemma 23.1.1. On a common smooth refinement resolve the three numerators and the denominator, as well as the other pairs. If $u_1(v_E) > v(v_E)$ for a central component $E$, the invertible integral ideal $K(I + K)^{-1}$ has strictly positive order there. It vanishes along $E$. Since the two ideals $I(I+K)^{-1}$ and $K(I+K)^{-1}$ generate the unit ideal, $I(I+K)^{-1}$ is the unit ideal on a neighborhood of $E$. The classes of the first pairs for $u_1$ and $w$ therefore have the same restriction to $E$, so the mixed degrees agree at each such component.

Write $\mu_w, \mu_{u_1}$ for the two mixed measures. The model assertion gives

$$(w - v)\mu_w = (w - v)\mu_{u_1} .$$

For continuous semipositive potentials in Kähler classes, take uniform rational approximations in the respective fixed classes by Theorem 23.1.1. Their maxima converge uniformly to $w$, and the continuous measure theorem passes this equality to the limit. For bounded decreasing potentials in these Kähler classes take simultaneous decreasing continuous approximations. The maxima of the approximants to $u_1, v$ decrease to $w$. Apply Lemma 26.8.1 to their maxima and to the approximants to $v$, with all continuous factors $h$. It passes the weighted equality to the bounded limits. The

bounded Borel factor $w - v$ is positive precisely on $\{u_1 > v\}$, so the two measures agree there.

Finally, for the nef classes keep all the potentials fixed and add a small Kähler class to each background. The Kähler case applies with the same strict region. The mixed measures converge in total variation by Theorem 26.4.1, so restricting to this Borel set preserves the identity in the limit. □

## 26.9 Restricted products of unbounded potentials

Consider arbitrary $u_i \in \mathcal{D}^{\mathrm{pert}}_{\alpha_i}$ for nef classes $\alpha_i$, $1 \le i \le n$. For positive integers $j$ put $u_{i,j} = \max(u_i, -j)$ and define

$$A_j = \bigcap_{i=1}^{n} \{u_i > -j\}, \qquad \nu_j = \mathrm{MA}^{\mathrm{pert}}_X((\alpha_i, u_{i,j})_{i=1}^n).$$

The sets $A_j$ are Borel. Without any finite-energy hypothesis we restrict each mixed measure to the region where every truncation is inactive. The limit of these restrictions can lose all of its mass (Example 26.9.1).

**Theorem 26.9.1** *The measures $\mathbf{1}_{A_j}\nu_j$ increase to a positive Radon measure $\nu$ in total variation. The limit is unchanged if the truncation levels tend independently to infinity in the different slots. It gives zero mass to the union of the pole sets of the inputs and has mass at most $\int_X \alpha_1 \cdots \alpha_n$.*

*If the total sum of the potentials has finite intrinsic energy in the sum class, then $\nu$ is the full-mass measure of Theorem 26.7.1. In that case the unrestricted measures $\nu_j$ also converge in total variation to $\nu$.*

***Proof*** For $k \ge j$ the $j$-th truncation of an input is the maximum of its $k$-th truncation and the constant $-j$. The strict region for this comparison is $\{u_i > -j\}$. Applying Theorem 26.8.1 in each slot gives $\mathbf{1}_{A_j}\nu_j = \mathbf{1}_{A_j}\nu_k$. Consequently the restricted measures increase and have masses bounded by the mixed degree. Their successive differences are positive, so their total variation distances equal the corresponding differences of masses. Thus they converge in total variation. The limit is Radon, for example by Riesz representation of the limits on continuous functions and the uniform bound in the dual norm. Each restriction gives zero mass to the pole sets, and this passes to the total variation limit. The same comparison works for componentwise ordered tuples of levels. A common level above all entries gives cofinality, which proves independence of the approach with separate levels.

For the last part put $\beta = \sum_i \alpha_i$ and choose $B = \sum_i \max(0, \sup u_i)$. The bounded potential $s_j = \sum_i u_{i,j} - B$ is nonpositive, and $s_j \le -j$ on $X \setminus A_j$. The positive mixed expansion, first on models and then in Kähler classes and at the boundary, gives

$$\mu^{\mathrm{pert}}_\beta(s_j) \ge n!\nu_j.$$

For a bounded nonpositive potential the proof of Theorem 26.6.1 gives the diagonal energy inequality

$$\int (-s_j)\,\mathrm{d}\mu_\beta^{\mathrm{pert}}(s_j) \le -(n+1)E_\beta^{\mathrm{pert}}(s_j).$$

The energies on the right are bounded below by the finite energy of $\sum_i u_i - B$ in $\beta$. It follows that

$$\nu_j(X^{\sqsupset} \setminus A_j) \le \frac{(n+1)\bigl(-\overline{E}_\beta^{\mathrm{pert}}(\sum_i u_i - B)\bigr)}{n!\,j}.$$

The distance from $\nu_j$ to its restriction tends to zero. Thus both total variation limits are the same. The full mixed mass and the identity with the intrinsic finite-sum construction follow by the weak convergence of Theorem 26.7.1. □

**Corollary 26.9.1** *Let $v \in \mathcal{D}_{\alpha_1}^{\mathrm{pert}}$ and set $w = \max(u_1, v)$. Replacing the first input by $w$ leaves the restricted mixed measure unchanged on $\{u_1 > v\}$.*

*For $u, v \in \mathcal{D}_\alpha^{\mathrm{pert}}$, write $\nu_u, \nu_v$ for their restricted diagonal measures. If $\nu_u$ has mass $V$, then*

$$\nu_v\{u < v\} \le \nu_u\{u < v\}.$$

***Proof*** On $\{u_1 > v\} \cap A_j$, the truncation of $u_1$ is strictly greater than that of $v$. The truncation of $w$ is their maximum, so the two bounded mixed measures agree there by Theorem 26.8.1. The region $A_j$ for the replaced tuple agrees with the original one on $\{u_1 > v\}$. Taking the restricted limits gives the first assertion.

Subtracting a constant from any input preserves its restricted measure. For bounded inputs this holds on models because the fibre class restricts to zero on central components. Approximation in Kähler classes and then the bounded class limits give it for bounded decreasing inputs. For arbitrary inputs use independent truncation levels, moving the level in the translated slot by the same constant. The strict sets are then unchanged. The cofinality argument for independent levels works for positive real levels as well as for integers.

Fix $t > 0$ and put $w_t = \max(u, v - t)$. Apply the first assertion in each of the $n$ slots. On $\{u < v - t\}$ the diagonal measure of $w_t$ agrees with that of $v - t$, which is $\nu_v$. On $\{u > v - t\}$ it agrees with $\nu_u$. Since the diagonal measure of $w_t$ has mass at most $V = \nu_u(X^{\sqsupset})$, this gives $\nu_v\{u < v - t\} \le \nu_u\{u \le v - t\}$. As $t \downarrow 0$, the left sets increase to $\{u < v\}$. The union of the right sets differs from $\{u < v\}$ only at common poles. The measure $\nu_u$ vanishes on these, which proves the inequality. □

**Corollary 26.9.2** *If $b, w \in \mathcal{D}^{\mathrm{pert}}_{\alpha}$, $w \geq b - C$ for a constant $C$, and the restricted diagonal measure of $b$ has full mass, then so does that of $w$.*

*For an arbitrary nef tuple, if the restricted diagonal measure of its sum in the sum class has full mass, then its restricted mixed measure has full mixed mass. The unrestricted mixed truncations converge to this measure in total variation.*

***Proof*** First take $b \leq q \leq 0$ in one class $\alpha$ with $q$ bounded. We bound the measure of $\{q \leq -j\}$ for $j > 0$. Choose $L > j$ so large that $b_L = \max(b, -L) \leq q$. Put $z = q/2 - j/4$, a bounded potential of the same class by convexity and translation. Then

$$\{q \leq -j\} \subseteq \{b_L < z\} \subseteq \{b_L < -j/4\}.$$

The first inclusion uses $b_L \leq q < q/2 - j/4$ on the stated set, and the second uses $q \leq 0$. The measure of $z$ dominates $2^{-n}$ times the measure of $q$, by expanding the positive mixed measure for the convex combination with zero. Both $z$ and $b_L$ are bounded and have full mass. The comparison in Corollary 26.9.1 gives

$$2^{-n}\mu^{\mathrm{pert}}_{\alpha}(q)\{q \leq -j\} \leq \mu^{\mathrm{pert}}_{\alpha}(b_L)\{b_L < -j/4\}.$$

On $\{b > -j/4\}$, the measure of $b_L$ agrees with that of $b_{j/4} = \max(b, -j/4)$ by bounded strict locality. Let $M_b(r)$ be the mass of the restriction of the measure of $b_r$ to $\{b > -r\}$. Since $L > j$, the last upper bound is at most $V - M_b(j/4)$. This quantity tends to zero if the restricted measure of $b$ has full mass.

For the first assertion, translate $b$ and $w$ so that $b \leq w \leq 0$, using translation invariance. Apply the bound to $q = \max(w, -j)$. On $\{w \leq -j\}$ it gives vanishing discarded mass as $j \to \infty$. The bounded masses are $V$, so the restricted measure of $w$ has mass $V$.

For the second assertion use the sum class $\beta = \sum_i \alpha_i$ and put $b = \sum_i u_i - B$ and $q = \sum_i u_{i,j} - B$ with $B = \sum_i \max(0, \sup u_i)$ as in the proof of Theorem 26.9.1. Then $q \leq -j$ off $A_j$, and its measure in $\beta$ dominates $n!\nu_j$. The bound above shows that $\nu_j(X^{\beth} \setminus A_j) \to 0$ under the full-mass assumption on the sum. The restricted measures increase in total variation, so the unrestricted $\nu_j$ have the same limit in total variation, of full mixed mass. □

*Example 26.9.1 (Loss of all mixed mass on restriction)* On $X = \mathbb{P}^1 \times \mathbb{P}^1$, let $\alpha_i$ be pulled back from the hyperplane class on the $i$-th factor. There are potentials of finite intrinsic energy in the individual classes for which the restricted mixed measure is zero and the unrestricted truncation measures converge weakly to a unit mass at a pole.

***Proof*** Fix $p \in \mathbb{P}^1(\mathbb{C})$ and put $D = \{p\} \times \mathbb{P}^1$. Take $u_1(v) = -v(\mathcal{I}_D)$ and $u_2 = 0$. For $j \geq 1$, $u_{1,j} = -\min(v(\mathcal{I}_D), j)$ is a finite Fubini–Study potential in $\alpha_1$. Indeed, take a section with divisor $D$ with weight zero, together with a generating pair of sections of $\mathcal{O}(D)$ of weight $-j$. The maximum of the last two terms is the constant $-j$. The functions decrease to $u_1$ and are zero at $v_0$. The projective comparison and decreasing closure put $u_1$ in $\mathcal{D}^{\mathrm{pert}}_{\alpha_1}$. Each individual top degree is zero, so both intrinsic energies are zero by Theorem 26.5.1.

The model of $u_{1,j}$ is the pullback of the curve model for the flag ideal $(\mathcal{I}_p, \tau^j)$ by the first projection. It is a product of that curve model with the second $\mathbb{P}^1$. On a central component the mixed degree with $(\alpha_2, 0)$ is the degree of the first model class on the curve component times one, the degree on the second factor. The curve computation in Lemma 20.3.1 and Theorem 20.3.1 places all that mass, with its multiplicity, at the point of radius $j$. Thus the projective comparison for bounded potentials gives $\nu_j = \delta_{j\,\mathrm{ord}_D}$.

At this point $u_1 = -j$, so the restriction to $A_j$ is zero for every $j$. As $j \to \infty$, the evaluation at $j\,\mathrm{ord}_D$ tends to the semivaluation whose support is the generic point of $D$ and which is trivial on its residue field. Indeed, for every coherent ideal $I$ on $X$, the values $j\,\mathrm{ord}_D(I)$ tend to 0 if $\mathrm{ord}_D(I) = 0$ and to $+\infty$ otherwise. These are the values of the limit semivaluation, which is $+\infty$ exactly on the ideals vanishing along $D$. This is convergence in $X^{\sqsupset}$ by the definition of its topology. The unrestricted measures converge weakly to the unit atom at this endpoint, where $u_1 = -\infty$. In particular the sum energy in $\alpha_1 + \alpha_2$ is $-\infty$, since finite sum energy gives full mass to the restricted measure by Theorem 26.9.1. □

# Chapter 27
# The variational equation in nef classes

Let $X$ be the blowup of $\mathbb{P}^2$ at a point and $\alpha$ the pullback of the hyperplane class. The potential of Example 26.5.1 has finite intrinsic energy and energy $-\infty$ in every Kähler perturbation of $\alpha$. Its normalized intrinsic measure has finite dual energy, and the unique normalized solution of the equation for this measure has the same two properties (Theorem 27.4.1 and the paragraph after its proof). The solution thus lies outside the finite-energy class of every Kähler perturbation. We therefore pose the equation with the intrinsic energy of Chapter 26.

We keep the setting of that chapter: $X$ is a connected compact Kähler manifold of dimension $n \geq 1$, $\theta$ is a smooth closed $(1,1)$-form in a nef class $\alpha$, and $V = \int_X \alpha^n$. We use the perturbative cone $\mathcal{D}_\alpha^{\mathrm{pert}}$ with its envelopes $P_\alpha^{\mathrm{pert}}$, the obstacle energy $G_\alpha$ and measure $\mu_\alpha^{\mathrm{obs}}$ of Theorem 26.2.1, the intrinsic energy $\overline{E}_\alpha^{\mathrm{pert}}$ of (26.5), and the measures $\mu_\alpha^{\mathrm{pert}}$ of Theorem 26.6.1. From Section 27.2 on we assume $V > 0$.

The first variation of the obstacle energy extends to bounded upper semicontinuous obstacles, with a remainder uniform in the obstacle (Theorem 27.1.1), although contact can fail at an upward jump (Example 27.1.1). A finite-energy potential solves the equation for a probability $\nu$ exactly when $\nu$ has finite dual energy and the dual supremum is attained at that potential (Proposition 27.2.1). Finite dual energy is equivalent to integrability of every intrinsic finite-energy potential (Corollary 27.3.1), and the measures along a maximizing net converge weakly to $\nu$ (Proposition 27.3.1). For every probability $\nu$ of finite dual energy, a weak cluster point of a maximizing net for $\nu$ in the compact normalized cone solves the equation, and the midpoint estimate of Lemma 27.3.1 makes the normalized solution unique (Theorem 27.4.1). The solution map is a homeomorphism for the strong topologies (Corollary 27.4.1).

The last two sections let the class vary. Along a net of nef classes converging to $\alpha$, obstacle energies and measures converge (Theorem 27.5.1). Convergence of normalized solutions on tests and in energy is equivalent to weak convergence of their measures together with convergence of the dual energies, each computed in its own class (Theorem 27.6.1). Example 27.6.1 shows on $\mathbb{P}^1$ that weak convergence with bounded dual energy does not suffice. When a big class is nef, Chapter 28 recov-

ers the measures of Theorem 27.1.1 for bounded obstacles, and Chapter 30 applies Theorem 27.4.1 and Corollary 27.4.1 to the nef positive part of a big class.

## 27.1 Upper semicontinuous obstacles

For an upper semicontinuous obstacle, decreasing continuous approximation still gives a first variation. Contact can fail at an upward jump of the obstacle.

**Theorem 27.1.1** *A bounded upper semicontinuous obstacle $f$ has a greatest minorant $P_\alpha^{\mathrm{pert}}(f)$ in the perturbative cone. If a net of continuous functions $f_i$ decreases to $f$, then the envelopes $P_\alpha^{\mathrm{pert}}(f_i)$ decrease to $P_\alpha^{\mathrm{pert}}(f)$. Set $G_\alpha(f) = E_\alpha^{\mathrm{pert}}(P_\alpha^{\mathrm{pert}}(f))$ and $\mu_\alpha^{\mathrm{obs}}(f) = \mu_\alpha^{\mathrm{pert}}(P_\alpha^{\mathrm{pert}}(f))$. For continuous $f$ these agree with the limits of Theorem 26.2.1 by Corollary 26.3.1. For a real model function $h$ there is a constant $C_h$, independent of $f$, such that*

$$\left| G_\alpha(f + th) - G_\alpha(f) - t \int h \, \mathrm{d}\mu_\alpha^{\mathrm{obs}}(f) \right| \le C_h |t|^{3/2}. \tag{27.1}$$

*Formula (26.2) holds for every continuous real direction.*

***Proof*** Replace each continuous majorant $g$ of $f$ from Lemma 4.1.1 by $\min(g, \sup f)$. These clipped majorants are still downward directed with infimum $f$, so they form a decreasing net, and they lie between $\inf f$ and $\sup f$. Their envelopes decrease between these constants. The nested decreasing closure of each Kähler cone in Theorem 23.5.1 puts the limit in the perturbative cone. It is at most $f$ and majorizes every perturbative minorant, by comparison with each continuous envelope. The same proof works for any net of continuous majorants decreasing to $f$, using common bounds on a tail.

For continuous $f$, the estimate (26.3) passes through the Kähler perturbations by Theorem 26.2.1, and its limiting energies and measure are those of the perturbative envelopes by Corollary 26.3.1. The proof of Theorem 26.2.1 bounds the constant only by the direction and the backgrounds through the mixed degrees. In particular it gives one $C_h$ for every continuous obstacle in the limit class. Given continuous $f_i \downarrow f$, apply that estimate to $f_i$ and $f_i + th$. The energies decrease to those of the limiting envelopes by Lemma 26.6.1. Their measures converge weakly by Theorem 26.6.1. Thus the estimate survives the decrease to $f$.

Order and constant translation of the envelope and energy give $|G_\alpha(f) - G_\alpha(g)| \le V \|f - g\|_\infty$. Approximation of a continuous direction by real model directions changes the difference quotient by at most $V$ times the uniform approximation error. The integrals against the measure of mass $V$ satisfy the same bound. This extends the directional derivative formula. □

*Example 27.1.1* On the valuation space of $\mathbb{P}^1_{\mathbb{C}}$ in a class of degree one, let $f$ be one at the trivial valuation and zero at every other point. The obstacle is upper semicontinuous. A perturbative psh minorant has value at most zero at the trivial valuation, by approaching it along one finite ray and using the one-ray bound in (20.7). Here the algebraic curve description applies by Corollary 26.4.1. Thus the envelope is zero. Its measure is the unit atom at the trivial valuation, by the product-model formula, and the contact identity fails.

## 27.2 The intrinsic variational equation

For the rest of the chapter suppose $V > 0$. Write $\mathcal{E}^1_{\alpha,\mathrm{pert}}$ for the members of $\mathcal{D}^{\mathrm{pert}}_\alpha$ of finite intrinsic energy, and put

$$e(u) = V^{-1}\overline{E}^{\mathrm{pert}}_\alpha(u), \qquad \nu_u = V^{-1}\mu^{\mathrm{pert}}_\alpha(u).$$

These measures are probabilities by Theorem 26.6.1. In particular $\nu_0 = \delta_{v_0}$ by the product-model formula and the class limit. A potential $u$ is *normalized* if $u(v_0) = 0$. All potentials in the perturbative cone satisfy $u(v_0) = \sup u$, by the order test in any Kähler perturbation.

For a Radon probability measure $\nu$ define the *dual energy*

$$e^\vee(\nu) = \sup_{u \in \mathcal{E}^1_{\alpha,\mathrm{pert}}} \left( e(u) - \int u \, \mathrm{d}\nu \right). \tag{27.2}$$

When $\int u \, \mathrm{d}\nu = -\infty$ the expression is $+\infty$. A measure of finite dual energy therefore integrates every intrinsic finite-energy potential. The expression in the supremum is invariant under translation by constants, because the mass is one and the energy shifts by the same constant.

For bounded potentials the difference-of-powers formula is

$$e(v) - e(u) = \frac{1}{(n+1)V} \sum_{\ell=0}^{n} \int (v-u) \, \mathrm{d}\,\mathrm{MA}^{\mathrm{pert}}_X\big((\alpha,v)^\ell, (\alpha,u)^{n-\ell}\big). \tag{27.3}$$

In a Kähler perturbation, apply the continuous identity from Theorem 22.4.1 to decreasing continuous approximations of $v, u$. The convergence of the two weighted integrals in each term follows from Lemma 26.8.1. Then take the class limit using total variation for the bounded mixed measures and the bounded Borel test $v - u$. The supporting inequality $e(v) - e(u) \le \int (v-u) \, \mathrm{d}\nu_u$ for bounded potentials follows by the same class limit from the Kähler supporting inequality (23.9).

**Proposition 27.2.1** *For $u \in \mathcal{E}^1_{\alpha,\mathrm{pert}}$ and a Radon probability $\nu$, the equality $\nu = \nu_u$ is equivalent to $e^\vee(\nu) < +\infty$ and attainment of (27.2) at $u$.*

***Proof*** Put $u_j = \max(u, -j)$. For $j \geq k$, retaining the term integrated against $\nu_{u_j}$ in (27.3) gives

$$0 \leq \int (u_k - u_j)\, \mathrm{d}\nu_{u_j} \leq (n+1)(e(u_k) - e(u_j)).$$

The same-class sum has finite energy by Corollary 26.7.1. Thus the truncation measures $\nu_{u_j}$ converge in total variation to $\nu_u$ by Theorem 26.9.1. For fixed $k$, pass to the limit in the integral of $u_k$, which is bounded. Let $k \to \infty$ by decreasing integration and energy continuity. The estimate shows that the integrals of $u_j$ against $\nu_{u_j}$ tend to the integral of $u$ against $\nu_u$. The supporting inequality at $u_j$ now passes to $u$ for any bounded $v$, since $\int v\, \mathrm{d}\nu_{u_j}$ also converges. Decreasing truncation of $v$ extends the inequality to $\mathcal{E}^1_{\alpha,\mathrm{pert}}$, using the same-class integrability in Corollary 26.7.1. This proves that

$$e^\vee(\nu_u) = e(u) - \int u\, \mathrm{d}\nu_u. \tag{27.4}$$

For the converse fix a real model function $h$. The bounded envelopes $P^{\mathrm{pert}}_\alpha(u_j + th)$ decrease to a potential $u(t)$. Indeed, with $c = \|h\|_\infty |t|$ they are between $u_j - c$ and $u_j + c$, by the minorant $u_j - c$ and the obstacle bound. The limit is finite at $v_0$, so belongs to the cone by nested decreasing closure in each Kähler class. It has finite energy by $u(t) \geq u - c$. Also $u(t) \leq u + th$ and $u(0) = u$. Pass to the limit in (27.1) for $u_j$ and divide by $V$. This gives

$$\left| e(u(t)) - e(u) - t \int h\, \mathrm{d}\nu_u \right| \leq C'_h |t|^{3/2}. \tag{27.5}$$

If $u$ is a maximizer for $\nu$, the dual bound and $u(t) \leq u + th$ give

$$e(u(t)) \leq e^\vee(\nu) + \int (u + th)\, \mathrm{d}\nu.$$

Subtract the equality at $u$. In (27.5), both signs of $t$ now give $\int h\, \mathrm{d}\nu = \int h\, \mathrm{d}\nu_u$. Real model tests are uniformly dense, so the two probability measures agree. □

## 27.3 Testing along maximizing nets

For normalized potentials write $j(u) = -e(u)$, and for bounded $u, v$ set

$$I(u, v) = \int (u - v)\, \mathrm{d}(\nu_v - \nu_u).$$

The energy difference formula (27.3) and its monotonic bounds give $0 \leq I(u, v) \leq (n+1)(j(u) + j(v))$. For the upper bound, discard the crossed integrals of the non-

positive $u, v$, and use $\int(-u)\,\mathrm{d}\nu_u \le (n+1)j(u)$ and its analogue for $v$. Nonnegativity also follows by adding the two supporting inequalities.

**Lemma 27.3.1** *For bounded normalized potentials $u, v, a$ in the perturbative cone, the following estimates hold with a dimension constant $C_n$.*

$$j(tu) \le C_n t^2 j(u) \qquad (0 \le t \le 1), \tag{27.6}$$

$$I(u,v) \le C_n \left( e((u+v)/2) - \tfrac{1}{2}(e(u)+e(v)) \right), \tag{27.7}$$

$$\left| \int a\,\mathrm{d}(\nu_u - \nu_v) \right| \le C_n \big(j(a)+j(u)+j(v)\big)^{1/2} I(u,v)^{1/2}. \tag{27.8}$$

*If $\Omega$ is a fixed Kähler perturbation of $\alpha$, then every $a \in C_\Omega$ satisfies*

$$\left| \int a\,\mathrm{d}(\nu_u - \nu_v) \right| \le C_\Omega \|a\|_\infty^{1/2} I(u,v)^{1/2}. \tag{27.9}$$

*For bounded normalized $q$ in the perturbative cone, put $J = j(u)+j(v)+j(q)$ and $\eta = 2^{-n}$. Then*

$$\left| \int (u-v)\,\mathrm{d}(\nu_q - \nu_0) \right| \le C_n J^{1-\eta} I(u,v)^{\eta}. \tag{27.10}$$

***Proof*** Work first in a Kähler perturbation $\beta$ and on semipositive models. Normalize energies and measures there by $V_\beta = \int_X \beta^n$. The proofs of the three estimates in Lemma 23.6.1 give (27.6)–(27.8) in $\beta$. For the midpoint estimate use the integral of the negative second derivative over the two halves of the segment, before introducing any dual measure. The constants depend only on $n$.

For (27.9) keep the testing pair $(\Omega, a)$ in its own class. Let $P_z = (\beta, z)$ and let $\Gamma = P_u^{\ell} P_v^{n-1-\ell}$ denote the background in a term of the telescoping difference of measures. Its vertical seminorm satisfies

$$-V_\beta^{-1} \langle (0,a),(0,a),\Gamma \rangle_X \le 2\|a\|_\infty V_\beta^{-1} \int_X \Omega\,\beta^{n-1}.$$

Indeed split one $(0,a)$ slot into $(\Omega,a)-(\Omega,0)$ and use the divisor integration formula. The two positive mixed measures on the right have the same mass $\int_X \Omega\,\beta^{n-1}$. Apply Cauchy–Schwarz to the $n$ telescope terms. The sum of the squares for $u-v$ is $I_\beta(u,v)$, as in Lemma 23.6.1. The constant therefore stays bounded as $\beta \to \alpha$ since $V > 0$.

For (27.10), use the propagation of vertical seminorms in the proof of Theorem 23.6.1. More explicitly, put $w = q/2$ and $p = (u+v)/2$. The squared seminorm $b_r$ of $u - v$ with background $P_w^r P_p^{n-1-r}$ satisfies $b_0 \le I_\beta(u,v)$ and $b_{r+1} \le b_r + 4\sqrt{b_r I_\beta(p,w)}$. The difference of successive squares comes from the slot $(0, w-p)$. Interchange it with one $(0, u-v)$ slot and use $P_u - P_v$ for that slot, as in the proof of Theorem 23.6.1. Each Cauchy–Schwarz square with $P_u$ or $P_v$ is bounded by twice the square with $P_p$. By nonpositivity and concavity, $I_\beta(p,w) \le (n+1)(j_\beta(p)+j_\beta(w)) \le C_n J_\beta$, where $J_\beta$ is the sum of the three energies with a negative sign. Also $I_\beta(u,v) \le C_n J_\beta$. Thus by induction $b_r \le C_n J_\beta^{1-2^{-r}} I_\beta(u,v)^{2^{-r}}$

when $J_\beta > 0$. If $J_\beta = 0$, all the squares vanish. In the telescope against $\nu_q - \nu_0$, the square of $u - v$ is bounded by $2^{n-1} b_{n-1}$, since $P_w = (P_q + P_0)/2$. The sum of the squares of $q$ there is $I_\beta(q, 0) \le (n+1) j_\beta(q)$. Cauchy–Schwarz gives (27.10) on models.

Uniform approximation gives the estimates for continuous semipositive potentials in the indicated Kähler classes. For bounded decreasing potentials in $\beta$, use simultaneous decreasing continuous majorants. The energies of the potentials and their averages converge by decreasing continuity. Each variable integral converges by Lemma 26.8.1. Normalization constants of the majorants tend to the constants of the limits, so subtracting them preserves these convergences. This proves the bounds for bounded decreasing data in $\beta$, keeping the continuous $a$ in $\Omega$ for (27.9). Finally let $\beta \to \alpha$ by Kähler additions. The bounded energies converge and total variation of the bounded measures allows every fixed bounded test in the formulas. This gives the asserted bounds. □

For $\nu$ of finite dual energy, the *deficit* of $u$ is $\Delta_\nu(u) = e^\vee(\nu) - e(u) + \int u \, \mathrm{d}\nu$. A net $u_i$ in $\mathcal{E}^1_{\alpha,\mathrm{pert}}$ is *maximizing* for $\nu$ if $\Delta_\nu(u_i) \to 0$, that is, if $e(u_i) - \int u_i \, \mathrm{d}\nu \to e^\vee(\nu)$.

The supremum defining $e^\vee$ can be taken over bounded potentials by truncation and decreasing integration. In fact it can be taken over the normalized perturbative envelopes $P^{\mathrm{pert}}_\alpha(f)$ of real model obstacles $f$. Indeed, the rational model lattice is uniformly dense by Proposition 21.5.1. Uniform approximation and a small upward rational shift place a rational model function arbitrarily close above a continuous majorant of an upper semicontinuous potential. The rational model majorants of a potential therefore have infimum that potential, by Lemma 4.1.1. They are directed downwards by finite minima. For $u$ in the perturbative cone their envelopes decrease to $u$, since they lie between $u$ and the majorants. Energies and integrals converge by decreasing continuity and integration. For normalized $u$ the normalizing constants of these envelopes tend to zero. After subtracting their constants they are still envelopes of real model obstacles.

By Corollary 26.3.1 the measures of these envelopes are the obstacle measures of their model obstacles, divided by $V$. They are therefore supported on finite sets of tests by Corollary 26.2.1. Every Dirac mass at a test has finite dual energy since normalized test values are bounded below by Lemma 23.5.1 in a fixed Kähler perturbation, and $e(u) \le 0$.

**Corollary 27.3.1** *A Radon probability has finite dual energy if and only if it integrates every intrinsic finite-energy potential.*

***Proof*** The forward assertion is immediate from the definition. For the converse, the negative integrals must be bounded on the normalized ball $j \le 1$. Otherwise take $u_k$ on that ball with negative integrals at least $2^k$. The sums $\sum_{k=1}^m 2^{-k} u_k$ decrease in the perturbative cone, are finite at $v_0$, and have energy at least $-1$ by concavity with the remaining weight assigned to zero. Concavity extends from bounded data by decreasing truncation of both potentials and their convex combinations. The sums' limit belongs to each Kähler cone by nested decreasing closure, has finite energy by decreasing continuity, and its negative integral is infinite by monotone integration. If

$A$ bounds the negative integrals on that ball, take $t = (1+C_n j(u))^{-1/2}$ in (27.6) for $u$. The estimate extends by decreasing truncation, so $j(tu) \le 1$. This bounds $\int(-u)\,\mathrm{d}\nu$ by $A\sqrt{1+C_n j(u)}$ and gives a finite dual supremum. □

**Proposition 27.3.1** *If $u_i$ is a maximizing net for $\nu$ of finite dual energy, then $\nu_{u_i} \to \nu$ weakly. For $B, R, S \ge 0$ there is a constant $C_{B,R,S}$, depending only on n, B, R and S and not on $\alpha$, such that if $e^\vee(\nu) \le B$, then*

$$\left|\int a\,\mathrm{d}(\nu_q - \nu)\right| \le C_{B,R,S}\Delta_\nu(q)^{1/2} \tag{27.11}$$

*for normalized intrinsic finite-energy $a, q$ with $j(a) \le R$ and $j(q) \le S$.*

***Proof*** For any $u$ of finite energy the variation in (27.5) lies below $u + th$. Using the dual bound with each of $h, -h$ gives, for $t > 0$,

$$\left|\int h\,\mathrm{d}(\nu_u - \nu)\right| \le C'_h t^{1/2} + t^{-1}\Delta_\nu(u).$$

On a maximizing net first pass $u$ to the limit, then let $t \downarrow 0$. Real model density proves weak convergence.

To prove (27.11) choose a maximizing net $u_i$ of normalized envelopes of model obstacles. The values $j(u_i)$ are bounded on a tail by a constant $T_B$. Indeed, take $t = (1+j(u_i))^{-1/2}$. Use (27.6) for $u_i$ to bound $j(tu_i)$, and use $tu_i$ in the dual supremum to bound $\int(-u_i)\,\mathrm{d}\nu$. The objective at $u_i$ is at most $-j(u_i)+(B+C_n)(1+j(u_i))^{1/2}$, which tends to $-\infty$ as $j(u_i) \to \infty$. On a tail the objective is at least $e^\vee(\nu)-1 \ge -1$, because the test $u = 0$ gives $e^\vee(\nu) \ge 0$. Hence $j(u_i) \le T_B$ there, for a constant $T_B$ depending only on $B$ and $n$. The same estimate bounds $j$ for all normalized maximizers with dual energy at most $B$. The midpoint estimate and affinity of the integral give $I(u_i, u_k) \le C_n(\Delta_\nu(u_i) + \Delta_\nu(u_k))$.

We first show convergence of the net's integrals against any fixed bounded $a$ in the perturbative cone. For continuous $b \in C_\Omega$ with $\|b\|_\infty \le M$, apply (27.9) to $u_i, u_k$ and let $k$ advance, using weak convergence against $b$. The resulting bound is uniform for these $b$ and tends to zero with $\Delta_\nu(u_i)$. Take decreasing continuous majorants of $a$ in $\Omega$. Their norms are bounded on a tail by $\|a\|_\infty + 1$, by compactness. Decreasing integration against the two fixed measures gives the same comparison for $a$, hence convergence of its integrals.

For bounded normalized $a, q$ with the stated energy bounds, the midpoint bound with $q, u_i$ and (27.8) now give (27.11) by passing to the limit against $a$. To extend to unbounded $q$ with $a$ bounded, truncate $q$. The truncations stay normalized and satisfy the same energy bound. Their measures converge in total variation and their deficits converge by decreasing integration. Then truncate $a$, using integrability against $\nu$ and $\nu_q$, which both have finite dual energy by Proposition 27.2.1. This proves the estimate on the full finite-energy domain. □

## 27.4 The Calabi–Yau theorem in a nef class

We equip the normalized perturbative cone with the weak topology of the common model tests from Section 23.5. Each of the normalized Kähler cones $\mathcal{D}_{\alpha+\kappa}$ is compact in the product of its test values by Theorem 23.5.1. These are closed subsets of the same Hausdorff product, and their intersection is compact. The tests determine the potentials since they do so in any one of the perturbations.

The energy $e$ is weakly upper semicontinuous on this compact cone, allowing the value $-\infty$. Fix $m \geq 1$. Taking $\max(u, -m)$ is continuous on test values, and on the normalized cone these truncations lie between $-m$ and zero. In $\alpha + \delta[\omega]$, for a fixed Kähler form $\omega$, their energies are weakly upper semicontinuous by Theorem 23.5.1. The class estimate of Lemma 26.3.1 bounds the distance to the bounded perturbative energy by $m(V_{\alpha+\delta[\omega]} - V)$, uniformly on these truncations, by comparison with smaller additions and passage to zero. Thus the energy of the truncation in $\alpha$ is a uniform limit of upper semicontinuous functions. Taking the infimum over $m$ gives the assertion.

**Theorem 27.4.1** *For a nef class $\alpha$ of positive volume on $X$, every Radon probability measure $\nu$ with $e^\vee(\nu) < +\infty$, in the notation of (27.2), has a unique normalized solution $u \in \mathcal{E}^1_{\alpha,\mathrm{pert}}$ of $\nu_u = \nu$.*

***Proof*** We first record a testing consequence of (27.11). If $\nu$ has finite dual energy, integration against $\nu$ is weakly continuous on each normalized ball $j \leq R$. To see this, choose a maximizing net of normalized envelopes $q_i$ of model obstacles. Their energies are bounded on a tail by the estimate in the proof of Proposition 27.3.1. Their measures have finite test support, so integration against each is weakly continuous. The uniform dual testing estimate makes the integral against $\nu$ a uniform limit of these integrals on the ball.

Now take a maximizing net of normalized envelopes of model obstacles for $\nu$ and take a weak cluster point $u$. Its energy is finite by the uniform energy bound on a tail and upper semicontinuity. On this energy ball the integral against $\nu$ is weakly continuous by the first paragraph of this proof. Upper semicontinuity of the energy then gives attainment of (27.2) at $u$. Proposition 27.2.1 gives $\nu_u = \nu$.

For uniqueness, let $u, v$ be normalized solutions. Both maximize the same objective. Take normalized bounded envelope approximants $u_i, v_i$ on a common directed set, as in Section 27.3. They converge on the tests and in energy, and their integrals against $\nu$ converge, by decreasing integration before subtracting the constants. Their deficits therefore tend to zero. The midpoint estimate gives $I(u_i, v_i) \to 0$. Fix a normalized envelope $q$ of a model obstacle. Apply (27.10). The energies of $q, u_i, v_i$ are bounded, and the integrals against $\nu_0$ vanish by normalization. Since $\nu_q$ has finite test support, the test limits give $\int (u - v)\, \mathrm{d}\nu_q = 0$.

For a fixed test $\nu_E$, take a maximizing net $q_\ell$ of such envelopes for the finite-dual-energy probability $\delta_{\nu_E}$. By (27.11), their integrals against $u$ and $v$ tend to the corresponding values at $\nu_E$. This proves equality at every test, hence $u = v$. □

The blowup potential of Example 26.5.1 gives a probability to which this theorem applies. Indeed its intrinsic measure has finite dual energy by (27.4). The potential

$u$ of the example is normalized, since its cutoff $u(v_0)$ is zero. By uniqueness it is the normalized solution for $\nu_u$, so this solution has finite intrinsic energy and energy $-\infty$ in every Kähler perturbation.

### 27.4.1 Strong convergence

Use the strong topology on normalized intrinsic finite-energy potentials given by weak convergence on tests together with convergence of $e$. For finite-dual-energy probabilities, use weak convergence of measures together with convergence of $e^\vee$.

**Corollary 27.4.1** *The normalized Monge–Ampère map of Theorem 27.4.1 is a homeomorphism for these strong topologies.*

***Proof*** Suppose $u_i \to u$ strongly and write $\nu = \nu_u$. The potentials belong to one normalized energy ball on a tail, and integration against $\nu$ is continuous there by the proof of Theorem 27.4.1. Hence $\Delta_\nu(u_i) \to 0$. The maximizing-net assertion in Proposition 27.3.1 gives $\nu_{u_i} \to \nu$ weakly. By (27.11), with both potentials there equal to $u_i$, one has $\int u_i \, \mathrm{d}(\nu_{u_i} - \nu) \to 0$. Equation (27.4) therefore gives convergence of the dual energies.

Conversely, suppose $\nu_i \to \nu$ strongly and let $u_i, u$ be their normalized solutions. On a tail $e^\vee(\nu_i)$ and $e^\vee(\nu)$ are bounded by a common $B$. The estimate from (27.6) used in the proof of Proposition 27.3.1 gives a common bound for $j(u_i)$, since these are normalized maximizers. Fix a normalized envelope $q$ of a model obstacle in a maximizing net for $\nu$. Along that net $j(q) \le T_B$ on a tail. The bounded upper semicontinuous $q$ satisfies

$$\limsup_i \Delta_{\nu_i}(q) \le \Delta_\nu(q)$$

by the upper semicontinuous case of Proposition 4.3.1 and convergence of the dual energies. For $R \ge 0$ and normalized intrinsic finite-energy $a$ with $j(a) \le R$, apply (27.11) to both probabilities with this same $q$. The triangle inequality bounds the upper limit of the supremum over these $a$ of the differences of the integrals by a constant times $\Delta_\nu(q)^{1/2}$. The constant is uniform on the maximizing tail of $q$. Let its deficit tend to zero. Thus, for every $R$, the integrals of these $a$ against $\nu_i - \nu$ tend to zero uniformly.

In particular

$$\Delta_\nu(u_i) = e^\vee(\nu) - e^\vee(\nu_i) + \int u_i \, \mathrm{d}(\nu - \nu_i) \longrightarrow 0.$$

By compactness and energy upper semicontinuity, each weak cluster point has finite energy. Integration against $\nu$ is weakly continuous on this energy ball, so each cluster point maximizes the same objective and equals $u$. Compactness gives weak convergence to $u$. The deficits and convergence of integrals give $e(u_i) \to e(u)$. □

## 27.5 Continuity at a positive-volume nef class

In this section let $\alpha_i \to \alpha$ be a net of nef classes, with $V_\alpha > 0$. Put $V_i = \int_X \alpha_i^n$, which is positive on a tail. We use the perturbative cones in all classes, including Kähler ones, and weak convergence on the common model tests. Fix a Kähler form $\omega$ and put $\gamma_\delta = \alpha + \delta[\omega]$ for $\delta > 0$. For each $\delta$, the class $\gamma_\delta - \alpha_i$ is Kähler on a tail by openness, so there every potential in $\mathcal{D}^{\mathrm{pert}}_{\alpha_i}$ lies in the decreasing cone of the fixed Kähler class $\gamma_\delta$. In that cone the values of normalized potentials at each fixed test are bounded below by Lemma 23.5.1, and bounded energies in $\gamma_\delta$ are compared with those in small Kähler perturbations of $\alpha_i$ by Lemma 26.3.1.

**Lemma 27.5.1** *A net of normalized $v_i \in \mathcal{D}^{\mathrm{pert}}_{\alpha_i}$ has weak cluster points. Every such point belongs to $\mathcal{D}^{\mathrm{pert}}_{\alpha}$. If $v_i \to v$ weakly, then*

$$\limsup_i \overline{E}^{\mathrm{pert}}_{\alpha_i}(v_i) \le \overline{E}^{\mathrm{pert}}_{\alpha}(v). \tag{27.12}$$

*The assertions also hold for a net of uniformly bounded potentials without normalization.*

***Proof*** With $\gamma_\delta$ as above, $v_i \in \mathcal{D}_{\gamma_\delta}$ on a tail. For normalized $v_i$, one such $\delta$ gives a common compact set of test values by Theorem 23.5.1. The compact cones for all smaller $\delta$ are closed in the Hausdorff product of these values, and every cluster point belongs to their intersection.

Let $w_i = \max(v_i, -m)$ for $m > 0$. For normalized potentials $-m \le w_i \le 0$. Apply Lemma 26.3.1 to compare $w_i$ in a sufficiently small Kähler perturbation of $\alpha_i$ with its energy in $\gamma_\delta$. First remove the small perturbation at fixed $i$. The larger class differs from the smaller one by a Kähler class for all sufficiently small additions. This gives

$$\overline{E}^{\mathrm{pert}}_{\alpha_i}(v_i) \le E_{\gamma_\delta}(w_i) + m(V_{\gamma_\delta} - V_i).$$

The truncations converge on tests to $w = \max(v, -m)$. Upper semicontinuity in the fixed class gives the same upper bound for the upper limit with $w$ and $V_\alpha$ on the right. Now let $\delta \downarrow 0$ by Theorem 26.3.1. Then let $m \to \infty$ by the definition of intrinsic energy. This proves (27.12). For uniformly bounded potentials, normalize by subtracting their values at $v_0$. These values range in a compact interval. The assertion follows by restoring the constants and using $V_i \to V_\alpha$ in the energy translation formula. □

**Theorem 27.5.1** *If continuous $f_i \to f$ uniformly, then*

$$G_{\alpha_i}(f_i) \longrightarrow G_\alpha(f), \qquad \mu^{\mathrm{obs}}_{\alpha_i}(f_i) \longrightarrow \mu^{\mathrm{obs}}_{\alpha}(f) \quad \textit{weakly}.$$

*The envelopes converge on every model test. For a fixed real model obstacle $f$, the measure convergence for $f_i = f$ holds in total variation.*

***Proof*** For a fixed real model obstacle represented by $D$ on $W$, use the volume expression in the proof of Corollary 26.2.1. A fibre shift $c$ with $D + cW_0 \geq W_0$ gives

$$G_{\alpha_i}(f) = \frac{1}{n+1}\,\mathrm{vol}_W(\rho^*\alpha_i + c_1(D) + c\,c_1(W_0)) - cV_i.$$

The shifted class at $\alpha$ is big, and volume is continuous there by the argument in the proof of Theorem 26.2.1. The real coefficient formula follows on the same refinement by rational approximation and uniform convergence of the obstacles, as in the proof of Corollary 26.2.1. Thus the energies converge for model obstacles. Model density and the Lipschitz bounds with constants $V_i$, bounded on a tail, give convergence for varying continuous obstacles.

For a real model direction $h$, the proof of (26.3) bounds its constant uniformly for Kähler backgrounds near $\alpha$. The bound uses the fixed curvature of $h$, its uniform norm, and bounded mixed class degrees. By perturbing each $\alpha_i$ with Kähler additions and taking their limits as in Theorem 27.1.1, the same bound holds at $\alpha_i$, uniformly on a tail. For fixed $t \neq 0$, apply it to the difference quotient between $f_i + th$ and $f_i$. The difference quotients converge by the energy assertion, and their errors against the measure integrals are at most $C_h|t|^{1/2}$. Now let $t$ tend to zero and use the derivative at $\alpha$. This gives convergence on real model tests, and hence weak convergence by density and $V_i \to V_\alpha$. For fixed $f$ a model obstacle, all the measures are carried by the finite set of divisorial tests of the central components of a refinement representing $f$, by Corollary 26.2.1. Their coefficients converge by testing with continuous functions that interpolate on this finite set. This is total variation convergence.

Put $p_i = P^{\mathrm{pert}}_{\alpha_i}(f_i)$. The envelopes have a uniform bound, so Lemma 27.5.1 applies to their weak cluster points. For every fixed $\gamma_\delta = \alpha + \delta[\omega]$, on a tail we have $p_i \leq P_{\gamma_\delta}(f_i)$. By nonexpansiveness in this Kähler class the latter envelopes converge uniformly to $P_{\gamma_\delta}(f)$. A cluster point $p$ is therefore at most $P_{\gamma_\delta}(f)$ on every test, hence everywhere by order determination in $\gamma_\delta$. Let $\delta \downarrow 0$ in (26.1). It follows that $p \leq P^{\mathrm{pert}}_{\alpha}(f)$. The energy upper bound (27.12), the identities $E^{\mathrm{pert}}_{\alpha_i}(p_i) = G_{\alpha_i}(f_i)$ and $G_\alpha(f) = E^{\mathrm{pert}}_{\alpha}(P^{\mathrm{pert}}_{\alpha}(f))$ of Corollary 26.3.1, and the obstacle energy convergence give

$$E^{\mathrm{pert}}_{\alpha}(p) \geq G_\alpha(f) = E^{\mathrm{pert}}_{\alpha}(P^{\mathrm{pert}}_{\alpha}(f)).$$

By order the energies are equal. The ordered test estimate in Lemma 26.6.1 gives equality of the two measures. Uniqueness in Theorem 27.4.1 gives equality modulo constants, and the energy translation formula gives equality of the potentials since $V_\alpha > 0$. Every cluster point is therefore the stated envelope. Common compactness gives convergence on the tests. □

## 27.6 Strong convergence in varying nef classes

For a positive-volume nef class $\gamma$ write $e_\gamma = \overline{E}^{\mathrm{pert}}_\gamma / V_\gamma$, $j_\gamma = -e_\gamma$ on normalized potentials, and $\nu^\gamma_u = V_\gamma^{-1}\mu^{\mathrm{pert}}_\gamma(u)$. The dual energy $e^\vee_\gamma$ and the deficit $\Delta^\gamma_\nu(q) = e^\vee_\gamma(\nu) - e_\gamma(q) + \int q\,\mathrm{d}\nu$ use the same class $\gamma$. To compare solutions in different cones, we use the common model tests and keep track of their energies in their respective classes.

**Lemma 27.6.1** *For normalized finite-energy $u, v$ in $\gamma$ put*

$$g_\gamma(u,v) = e_\gamma\big((u+v)/2\big) - \tfrac{1}{2}\big(e_\gamma(u) + e_\gamma(v)\big).$$

*For a normalized finite-energy test $a$ in the same class,*

$$\left|\int a\,\mathrm{d}(\nu^\gamma_u - \nu^\gamma_v)\right| \le C_n\big(j_\gamma(a) + j_\gamma(u) + j_\gamma(v)\big)^{1/2} g_\gamma(u,v)^{1/2}. \tag{27.13}$$

*For a fixed real model direction $h$,*

$$\left|\int h\,\mathrm{d}(\nu^\gamma_u - \nu^\gamma_v)\right| \le C_h g_\gamma(u,v)^{1/2}, \tag{27.14}$$

*where $C_h$ is bounded uniformly for nef $\gamma$ near any given nef class of positive volume.*

***Proof*** For bounded potentials (27.13) follows from (27.7) and (27.8). For the model test start with semipositive models in a Kähler class $\beta$. Use the normalized vertical seminorms of the proof of Lemma 23.3.1 with the backgrounds $(\beta,u)^\ell(\beta,v)^{n-1-\ell}$, $0 \le \ell < n$. That proof bounds the squared seminorm of $h$ by

$$C\|h\|_\infty V_\beta^{-1}\int_X \omega_h \beta^{n-1},$$

where $C$ and the Kähler class of $\omega_h$ come from a fixed model for $h$. This bound uses no order between the two potentials. The sum of the squared seminorms of $u - v$ equals $I_\beta(u,v)$ by the difference of powers. The telescope for the measure difference, Cauchy–Schwarz over its terms and (27.7) give (27.14) on models. The bound on the model seminorms is uniform near the given positive-volume class. Uniform model approximation gives the estimate for continuous semipositive potentials. Decreasing continuous approximation and then passage from Kähler perturbations to $\gamma$ give the estimate for bounded perturbative potentials by Theorem 26.3.1.

For finite-energy $u, v$, truncate both at $-m$. Their energies and the energy of their midpoint converge by decreasing continuity. Their measures converge in total variation by Corollary 26.7.1 and Theorem 26.9.1. Thus both bounds pass to the limit for bounded tests. Finally truncate $a$ and use its integrability against both measures, again by Corollary 26.7.1. The energy radius of each truncation is at most $j_\gamma(a)$. This proves the estimates. □

For $\alpha_i \to \alpha$ as in Section 27.5, fix a real model obstacle $f$ and set $p_i = P^{\mathrm{pert}}_{\alpha_i}(f)$, $p = P^{\mathrm{pert}}_\alpha(f)$. By Theorem 27.5.1,

$$p_i \to p \text{ on model tests}, \qquad e_{\alpha_i}(p_i) \to e_\alpha(p),$$
$$\nu_{p_i}^{\alpha_i} \to \nu_p^\alpha \text{ in total variation on a common finite set.}$$

These statements also hold after normalizing both potentials at $v_0$, by the energy translation law and invariance of the measures under addition of constants. For $\nu_i \to \nu$ weakly, comparison with the continuous envelope of $f$ in any fixed larger class $\gamma_\delta = \alpha + \delta[\omega]$ gives

$$\limsup_i \int p_i \, \mathrm{d}\nu_i \le \int p \, \mathrm{d}\nu. \tag{27.15}$$

Indeed $p_i \le P_{\gamma_\delta}(f)$ on a tail, so the left side is at most $\int P_{\gamma_\delta}(f) \, \mathrm{d}\nu$. Decrease $\delta$ to zero and use bounded decreasing integration. Equation (27.15) also holds for the normalized envelopes since the subtracted constants converge.

In particular,

$$\liminf_i e^\vee_{\alpha_i}(\nu_i) \ge e^\vee_\alpha(\nu). \tag{27.16}$$

For this inequality, test the moving dual energy with $p_i$. Use energy convergence and (27.15), then take the supremum over model obstacles. This gives the full supremum: every finite-energy potential is a decreasing limit of bounded model obstacle envelopes as in Section 27.3. The energies and the integrals against $\nu$ converge along this approximation by decreasing continuity and integration, including when the integral is $-\infty$.

**Theorem 27.6.1** *Let $\alpha_i \to \alpha$ be a net of nef classes with $V_\alpha > 0$ and $V_i > 0$. Let $u_i, u$ be normalized intrinsic finite-energy potentials in $\alpha_i, \alpha$ and set $\nu_i = \nu_{u_i}^{\alpha_i}$, $\nu = \nu_u^\alpha$. Then the following are equivalent.*

(1) *The potentials $u_i \to u$ on all model tests and $e_{\alpha_i}(u_i) \to e_\alpha(u)$.*
(2) *The probabilities $\nu_i \to \nu$ weakly and $e^\vee_{\alpha_i}(\nu_i) \to e^\vee_\alpha(\nu)$.*

***Proof*** First assume convergence of the measures and their dual energies. All those energies are bounded by a constant $B$ on a tail. We prove a testing assertion: if normalized finite-energy $a_i$ in $\alpha_i$ satisfy $j_{\alpha_i}(a_i) \le R$ and converge on the tests to $a$, then

$$\int a_i \, \mathrm{d}\nu_i \longrightarrow \int a \, \mathrm{d}\nu. \tag{27.17}$$

By Lemma 27.5.1 and volume convergence, $a$ has finite energy in $\alpha$ with $j_\alpha(a) \le R$. Choose a maximizing net $q^k$ of normalized model obstacle envelopes for $\nu$. As in the proof of Proposition 27.3.1, $j_\alpha(q^k) \le T_B$ on a tail and the constant in (27.11) depends only on $B$, the radii and $n$. For each $k$, let $q_i^k$ be the normalized envelope of the same obstacle in $\alpha_i$. By energy convergence, $j_{\alpha_i}(q_i^k) \le T_B + 1$ eventually for fixed $k$ in the maximizing tail. The integral bound (27.15) and convergence of the dual energies give

$$\limsup_i \Delta_{\nu_i}^{\alpha_i}(q_i^k) \le \Delta_\nu^\alpha(q^k).$$

For fixed $k$, the probabilities of $q_i^k$ and $q^k$ have a common finite support, with converging weights. Since the values of $a_i$ converge at each point of that support,

$$\int a_i \, \mathrm{d}\nu_{q_i^k}^{\alpha_i} \longrightarrow \int a \, \mathrm{d}\nu_{q^k}^{\alpha}.$$

Apply (27.11) to replace the probabilities on both sides by $\nu_i$ and $\nu$, respectively. The upper limit of the error in (27.17) is at most twice the uniform constant times $\Delta_\nu^\alpha (q^k)^{1/2}$. Advance the maximizing net to prove the assertion.

By the uniform objective bound in the proof of Proposition 27.3.1, the maximizers $u_i$ have bounded normalized energies. Every weak cluster point $u'$ has finite energy by Lemma 27.5.1. On a subnet converging to $u'$, apply (27.17) to the solutions. Using the supporting identity (27.4) gives

$$e_\alpha^\vee(\nu) \le e_\alpha(u') - \int u' \, \mathrm{d}\nu.$$

This makes $u'$ a maximizer, hence $u' = u$ by Proposition 27.2.1 and Theorem 27.4.1. Compactness implies full convergence on the model tests. Equation (27.17) now applies on the full net. The supporting identity gives convergence of the normalized energies.

Conversely, assume convergence of the potentials and their normalized energies. Take a decreasing model obstacle envelope approximation of $u$, and call its normalized potentials $q^k$. As above, write $q_i^k$ for the normalized envelopes of the same obstacles in $\alpha_i$. For fixed $k$, the midpoint of $u_i, q_i^k$ converges on tests to the midpoint of $u, q^k$. Upper semicontinuity gives

$$\limsup_i g_{\alpha_i}(u_i, q_i^k) \le g_\alpha(u, q^k).$$

The right side tends to zero by decreasing energy continuity for the envelopes and their midpoints with $u$ before normalization. The gap is unchanged by adding a constant to either potential.

Use (27.14) for $u_i, q_i^k$, with uniform $C_h$. The measures of the envelopes $q_i^k$ converge weakly to $\nu_{q^k}^\alpha$. The latter converges to $\nu_u^\alpha$ by the same test estimate in $\alpha$, or by decreasing continuity before normalization. Let $i$ advance for each fixed $k$, then let $k$ advance. This proves weak convergence of $\nu_i$ on the dense set of real model directions and hence on all continuous tests.

To compute the dual energies, the common finite support and convergence of its weights give

$$\int u_i \, \mathrm{d}\nu_{q_i^k}^{\alpha_i} \longrightarrow \int u \, \mathrm{d}\nu_{q^k}^{\alpha} \quad \text{for fixed } k.$$

Use (27.13) to replace the measures on the left by $\nu_i$ with test $u_i$, and those on the right by $\nu$ with test $u$. The radii of $u_i$ are uniformly bounded by hypothesis. The radii of $q^k$ are bounded since their energies converge, and the same bound with an additional unit holds for $q_i^k$ on a tail at each fixed $k$. The upper limit of the total error is bounded by a fixed constant times $g_\alpha(u, q^k)^{1/2}$. Let $k$ advance. The self-pairings converge, and (27.4) gives the claimed convergence of the dual energies. □

**Corollary 27.6.1** *Suppose $\nu_i \to \nu$ weakly, with all $\nu_i$ carried by a common finite set of model tests. Then their normalized solutions in $\alpha_i$ converge on tests and in normalized energy to the normalized solution of $\nu$ in $\alpha$.*

***Proof*** This finite set contains the support of $\nu$, and all the weights converge. The fixed-test bound in a common larger Kähler class bounds the negative integrals of all normalized potentials on this finite support. In particular, all these probabilities have finite dual energy in the respective classes. Their normalized solutions $u_i$ have bounded normalized energies since $e_{\alpha_i}(u_i) - \int u_i \, \mathrm{d}\nu_i \geq 0$. On any subnet converging on tests to $w$, the energy upper bound gives $w$ finite energy and

$$\limsup_i \bigl(e_{\alpha_i}(u_i) - \int u_i \, \mathrm{d}\nu_i\bigr) \leq e_\alpha(w) - \int w \, \mathrm{d}\nu \leq e^\vee_\alpha(\nu),$$

with the upper limit taken along that subnet. By compactness one can take such a subnet of any subnet realizing the upper limit in the original net. Thus the moving dual energies converge by (27.16). Apply Theorem 27.6.1. □

*Example 27.6.1* Even for a fixed ample class, weak convergence and a uniform bound on dual energy do not imply convergence of the solution energies. On $\mathbb{P}^1_{\mathbb{C}}$ take the degree-one class and a closed point $p$. In the notation of Theorem 31.2.1, for integers $m \geq 2$ let

$$\nu_m = (1 - m^{-1})\delta_{v_0} + m^{-1}\delta_{x_{p,m^2}}.$$

These converge weakly to $\delta_{v_0}$ since the mass away from $v_0$ tends to zero. The tail on the ray of $p$ equals $m^{-1}$ on $[0, m^2)$, and the other tails are zero. Thus $Q(\nu_m) = 1$, giving dual energy $1/2$ and solution energy $-1/2$ for each $m$. Although Theorem 31.2.1 comes later, its proof uses only the algebraic chapters preceding Chapter 21, so this use creates no circularity. The solution for $\delta_{v_0}$ is zero. This calculation applies to the perturbative measures by the projective agreement of Corollary 26.4.1 and the normalized energy comparison there. The curve solutions here are bounded, so they belong to that overlap.

# Chapter 28
# Obstacles and envelopes in big classes

Let $\tau : X \to S$ be the blowup of a point of a compact Kähler manifold of dimension $n \geq 2$, with exceptional divisor $F$, and put $\alpha = \tau^*\omega + a\,c_1(F)$ for a Kähler class $\omega$ on $S$ and $a > 0$. This class is big, of volume $V = \int_S \omega^n$, and has negative degree on the lines in $F$, so it is not nef. Consider the pullback from $S^{\sqsupset}$ of the point model function of depth $c$. For $c \leq a$ its obstacle energy is zero and its measure is $V\delta_{v_0}$, where $v_0$ is the trivial semivaluation. For $c$ slightly larger than $a$ the energy is $-(c-a)^{n+1}/(n+1)$, and a mass $(c-a)^n$ moves to the exceptional order $\mathrm{ord}_F$ (Example 28.2.1). Thus mass reaches the exceptional divisor only once the depth exceeds the coefficient $a$.

Throughout this chapter $X$ is a connected compact Kähler manifold of dimension $n$, $\alpha$ is a big class with $V = \mathrm{vol}_X(\alpha)$, and $N = n+1$. A model obstacle is represented by a real vertical divisor $D$ on a smooth refinement $W$ of $X \times \mathbb{P}^1$, with full central fibre $W_0$ and map $\rho : W \to X$. We write $c_1(H)$ for the class of a divisor $H$ and $[H]$ for its integration current. For a sufficiently large fibre shift $c$, the volume of $\rho^*\alpha + c_1(D) + c\,c_1(W_0)$ is affine in $c$ with slope $NV$ (Lemma 28.1.1). Dividing it by $N$ and subtracting $cV$ defines the obstacle energy $G_\alpha^{\mathrm{big}}$, which agrees with the nef obstacle energy of Chapter 26 when $\alpha$ is nef (Theorem 28.1.1). Its first variation is a positive measure of mass $V$, given on a representing model by positive products against the central components (Theorem 28.2.1).

The test curves of Chapter 25 realize this energy. Every continuous obstacle $f$ has a greatest compatible test-curve family below it. Its energy is $G_\alpha^{\mathrm{big}}(f)$, and the obstacle measure is carried by the contact set (Theorem 28.4.1). Decreasing approximation extends the envelope, its energy and the first variation to upper semicontinuous obstacles of finite relaxed energy (Proposition 28.5.1 and Theorem 28.5.1), although contact can fail at the trivial valuation (Example 28.5.1). This defines the intrinsic Monge–Ampère operator $\mathrm{MA}_\alpha$ on all intrinsic finite-energy potentials (Corollary 28.5.1). On a smooth refinement with strict normal crossings central fibre to which the rotation of the parameter lifts, every distribution of the mass $V$ among the tests of the central components is the measure of a model-obstacle envelope (Theorem 28.6.1).

Chapter 29 solves the equation $\mathrm{MA}_\alpha(u) = \nu$ using these envelopes, the finite-support solutions and the weak convergence of measures along maximizing nets (Proposition 28.5.3). Chapter 30 uses the model rays of Lemma 28.3.2, and Section 33.5 computes $\mathrm{MA}_\alpha$ on a toric blowup.

## 28.1 Obstacle volume in a big class

Let $\alpha$ be a big class on the connected compact Kähler manifold $X$, and write $V = \mathrm{vol}_X(\alpha)$ and $N = n + 1$. A model obstacle $f$ on $X^{\sqsupset}$ is represented by a real vertical divisor $D$ on a smooth refinement $W$. The maps from $W$ to $\mathbb{P}^1$ and $X$ are denoted by $\pi$ and $\rho$, and $W_0$ is the full central fibre. Here $\mathrm{vol}_X$, $\mathrm{vol}_W$ and similar symbols denote the volume of [Xia26a, Definition 15.8.1], the total non-pluripolar mass of a current with minimal singularities; for nef classes it is the top self-intersection [Xia26a, Proposition 15.8.2].

**Lemma 28.1.1 (Translation by the fibre)** *For a real vertical divisor $D \geq aW_0$ with $a > 0$, put $B = \rho^*\alpha + c_1(D)$. The class $B$ is big, and for every $t \geq 0$,*

$$\mathrm{vol}_W(B + t\,c_1(W_0)) = \mathrm{vol}_W(B) + NtV. \tag{28.1}$$

*Moreover, for a positive current $T$ with minimal singularities in $B$ and any smooth representative $\lambda$ of the point class on the parameter line,*

$$\int_W \langle T^n \rangle \wedge \pi^*\lambda = V. \tag{28.2}$$

***Proof*** The product class with background $\alpha$ on $X$ and degree $a$ on $\mathbb{P}^1$ contains a Kähler current obtained from a Kähler current of $\alpha$ and a positive form on the line. Its pullback remains big: subtract a small effective exceptional class from the pullback of a Kähler lower bound by the second assertion of Lemma D.4.1, then add the exceptional divisor back as a current. Bigness of $B$ follows by adding $D - aW_0$.

Choose a smooth representative $\theta$ of $\alpha$ and write $V_\theta$ for its extremal potential of minimal singularities, normalized by supremum zero. Let $\gamma_D$ represent $c_1(D)$ and let $\ell_D$ be a logarithmic norm with $dd^c\ell_D = [D] - \gamma_D$. In the background $\Theta = \rho^*\theta + \gamma_D$ the function $v = \rho^*V_\theta + \ell_D$ is a positive-current potential. For the proof choose a smooth nonnegative $\lambda$ of integral one with support disjoint from a closed disk about zero. Then $\Theta_t = \Theta + t\pi^*\lambda$ represents the shifted class and equals $\Theta$ over the disk.

Let $u_t \leq 0$ be a minimal $\Theta_t$-psh potential. On each noncentral fibre $W_z$, restriction of local psh potentials shows that $u_t|_{W_z} - \ell_D|_{W_z}$ is $\theta$-psh or identically $-\infty$. On the complement of any smaller central disk $\ell_D$ is smooth and bounded below. The supremum normalization of $V_\theta$ therefore gives $u_t \leq v + M$ there with one constant $M$. In the larger disk take $\max(u_t - M, v)$, and use $v$ outside. The two prescriptions agree throughout an inner collar of the boundary of the larger disk. This gives a $\Theta$-psh function and shows, by minimality and the bound outside the smaller disk,

that $u_t$ is at most a minimal $\Theta$-psh potential plus a constant on all of $W$. Conversely, every $\Theta$-psh potential is $\Theta_t$-psh, so the two minimal potentials differ by a bounded amount. Using the same gluing with $t = 0$ gives a minimal current $T_0$ in $B$ equal to the pullback of $\theta + dd^c V_\theta$ over a neighborhood of $\operatorname{supp}\lambda$.

By local truncation on this product neighborhood, the non-pluripolar power $\langle T_0^n \rangle$ is the pullback of the corresponding top measure on $X$. Indeed the bounded local truncations pulled back from $X$ have pulled-back Bedford–Taylor products by the successive distributional definition, and the truncation restrictions on the product are inverse images of the restrictions on $X$. Integration against $\pi^*\lambda$ therefore gives $V \int_{\mathbb{P}^1} \lambda = V$. The value is independent of the choice of minimal current and of the smooth representative, by [Xia26a, Proposition 15.8.2]: for the former use a Kähler representative of the class of a fixed Kähler form plus a small multiple of $\pi^*\lambda$, and subtract the same Kähler form. This compares the two mixed masses against $\pi^*\lambda$. For the latter use closedness of the product and integration against an exact smooth form. This proves (28.2). Finally $T_0 + t\pi^*\lambda$ has minimal singularities in the shifted class. Multilinearity of non-pluripolar products and $(\pi^*\lambda)^2 = 0$ give (28.1). □

**Lemma 28.1.2** *Let $p : W' \to W$ be a modification of compact Kähler manifolds and let $J$ be an effective exceptional real divisor. For a big class $B$ on $W$,*

$$\operatorname{vol}_{W'}(p^*B + c_1(J)) = \operatorname{vol}_W(B).$$

***Proof*** The pullback class is big by [Xia26a, Proposition 14.4.7], and adding $J$ preserves bigness. The pushforward of a positive current in $p^*B + c_1(J)$ belongs to $B$. Indeed the projection formula gives $p_* p^* B = B$, and the image of every exceptional prime has codimension at least two, so its integration current pushes to zero by dimension. Off the exceptional sets the top products of the current and its pushforward agree, and they give no mass to the excluded analytic sets. This bounds the largest mass upstairs by the volume downstairs. The pullback of a minimal current plus $[J]$ is a candidate attaining the lower bound by the same mass calculation. □

**Theorem 28.1.1** *For a real model obstacle $f$ with divisor $D$, the quantity*

$$G_\alpha^{\mathrm{big}}(f) = \frac{\operatorname{vol}_W(\rho^*\alpha + c_1(D) + c\, c_1(W_0))}{N} - cV, \qquad D + cW_0 \geq W_0, \tag{28.3}$$

*is independent of $c$ and the representative on a refinement. It extends uniquely to a concave, increasing functional on $C(X^\beth, \mathbb{R})$ with Lipschitz constant $V$ and $G_\alpha^{\mathrm{big}}(f + a) = G_\alpha^{\mathrm{big}}(f) + aV$ for real constants $a$. For nef and big $\alpha$ it agrees with $G_\alpha$ of Theorem 26.2.1.*

***Proof*** Independence of $c$ is (28.1). Invariance under refinement follows from Lemma 28.1.2 with $J = 0$, since the divisor and the fibre pull back. On a common refinement the divisors of a pair of model functions are ordered whenever the functions are ordered, by (22.1). Monotonicity of volume and the translation formula now give the Lipschitz bound. The volume of $\rho^*\alpha + c\, c_1(W_0)$ for $c > 0$ equals $NcV$:

on the product the minimal potential is the pullback of $V_\theta$, by restriction to each fibre and the product lower candidate. Multilinearity computes its mass, and birational invariance gives the result on $W$. Thus $G_\alpha^{\mathrm{big}}(0) = 0$.

For concavity, apply concavity of the $N$-th root of volume to the classes of two model obstacles and their convex combination on a common refinement, with the same large shift $c$. This concavity follows by applying [Xia26a, Theorem 17.4.23] to minimal currents and using maximality of the volume in the convex combination. For each obstacle the $N$-th root is $(NcV+NG_\alpha^{\mathrm{big}}(f))^{1/N}$. Subtract $(NcV)^{1/N}$, multiply by $(NcV)^{1-1/N}$, and let $c \to \infty$. The resulting quantities are the three corresponding values of $G_\alpha^{\mathrm{big}}$, so the concavity inequality passes to them. Density of real model functions from Proposition 21.5.1 and the Lipschitz bound give all the assertions for continuous obstacles except the nef comparison. There the volume formula in the proof of Theorem 26.2.1 is exactly (28.3) for rational model obstacles. Density gives the equality. □

## 28.2 First variation of big-class obstacle energy

For a big class $B$ on a compact Kähler manifold of dimension $N$, and a smooth closed real $(1,1)$-form $\gamma$, write

$$M_B(\gamma) = \int \langle T^{N-1} \rangle \wedge \gamma,$$

where $T$ has minimal singularities in $B$. This depends only on the classes. Closedness gives independence of the form representative. Add a large Kähler form to $\gamma$ and then subtract it to see independence of $T$ by [Xia26a, Proposition 15.8.2]. We write $M_B(H)$ when the argument is the class of a divisor $H$.

**Lemma 28.2.1** *Under the hypotheses of Lemma 23.4.4, on the compact Kähler manifold $M$ of that lemma and with the effective divisor $H$ in place of $G$, one has*

$$\mathrm{vol}([a] - sc_1(H)) \geq \int_M a^N - Ns \int_{[H]} (a + s\kappa)^{N-1}. \tag{28.4}$$

***Proof*** Use the rounded logarithmic norm $g_r = \chi_r(g)$ in the proof of Lemma 23.4.4. There the form $S_r = dd^c g_r + \chi_r'(g)\gamma$ equals $\chi_r''(g)\mathrm{d}g \wedge d^c g$ off $H$ and vanishes near $H$ for each fixed $r$. Thus $S_r^2 = 0$. The upper density bound in that proof expands as

$$(a + s\kappa + sS_r)^N = (a + s\kappa)^N + Ns(a + s\kappa)^{N-1} \wedge S_r.$$

In the cutoff estimate only these two terms occur. Let $r$ tend to infinity, using $S_r \to [H]$, and exhaust the complement of the analytic set as in that proof. The smooth term outside the cutoff tends to zero. This gives (28.4) for $a$ Kähler. Adding a small Kähler form and taking the limit as in the same proof gives the estimate for semipositive $a$.□

**Lemma 28.2.2** *Fix a smooth model $W_*$ over $X$ and an effective real vertical divisor $H$ on it. Fix $K > 0$ and $c \geq K + 1$. Let $q : W \to W_*$ be a further smooth blowup tower over zero, let $-KW_0 \leq D \leq KW_0$, and set*

$$B = \rho^*\alpha + c_1(D) + c\, c_1(W_0).$$

*There are $s_0 > 0$ and $C$ independent of $q, D, K, c$ such that for $0 \leq s \leq s_0$,*

$$\mathrm{vol}_W(B - sc_1(q^*H)) \geq \mathrm{vol}_W(B) - NsM_B(q^*H) - Cs^2. \tag{28.5}$$

*The constants can also be fixed for all background classes in a sufficiently small neighborhood of $\alpha$.*

***Proof*** On $W_*$ choose a Kähler form $\Omega$, a smooth representative $\gamma$ of $c_1(H)$, and $\kappa = L\Omega$ with $-\kappa \leq \gamma \leq \kappa$. The class $\rho_*^*\alpha + c_1((W_*)_0)$ is big by Lemma 28.1.1, with $\rho_* : W_* \to X$ the projection. Choose a Kähler current $R$ in this class with gentle analytic singularities. Such a current is obtained by regularizing a Kähler current in the background reduced by half its strict lower bound [Xia26a, Theorem 13.4.5]. Write $R \geq 2\delta\Omega$, with $\delta > 0$.

By the second assertion of Lemma D.4.1, choose an effective exceptional divisor $E$ and a Kähler form $\zeta$ in $[q^*\Omega] - c_1(E)$ on $W$. For the identity tower take $E = 0$. Then

$$R_W = q^*(R - \delta\Omega) + \delta\zeta + [\delta E + D + (c-1)W_0]$$

is in $B$ and dominates $\delta q^*\Omega + \delta\zeta$. The bracketed divisor is effective. In particular, for $sL \leq \delta/2$, the current $R_W - sq^*\gamma$ is Kähler, so the subtraction class on $W$ is big. This reference current has smooth remainder after resolving its gentle analytic poles and the divisorial terms. Let $Z$ be their singular set on $W$. Its potential $\psi$ in a fixed smooth background of $B$ can be normalized to have $\psi \leq 0$.

We describe the positive approximations used in the estimate. If $u$ is the extremal potential for $B$, regularize $(1-\varepsilon)u + \varepsilon\psi$ from above, for $0 < \varepsilon < 1$, by [Xia26a, Theorem 13.4.5]. This mixture is at least $\psi$ and its current dominates $\varepsilon\delta\zeta$. Apply the regularization in a background reduced by a fixed fraction of this bound. All sufficiently late currents are then positive in $B$, and their potentials decrease to the mixture for fixed $\varepsilon$. Off $Z$ they are locally bounded, since they are bounded above and dominate $\psi$.

Resolve their ideals and the poles of $R_W$ by smooth blowup towers $p_j : W_j \to W$, using the first assertion of Lemma D.4.1. These auxiliary resolutions may have centres off the central fibre. The smooth-remainder condition for gentle singularities [Xia26a, Definition 13.1.11] gives decompositions

$$p_j^*T_j = a_j + [F_j], \qquad p_j^*R_W = r_j + [F_j^*],$$

where $a_j, r_j$ are smooth semipositive forms and $0 \leq F_j \leq F_j^*$. The divisorial inequality follows from the comparison with $\psi$ and generic Lelong numbers after pulling back. The smooth lower bound $r_j \geq \delta(qp_j)^*\Omega + \delta p_j^*\zeta$ holds off the divisors and extends everywhere. Set $A_j = [a_j]$. Letting $\varepsilon \downarrow 0$ while the regularization is refined,

we can arrange along a diagonal sequence that

$$\int_{W_j} a_j^N \longrightarrow \mathrm{vol}_W(B), \qquad \int_{W_j} a_j^{N-1} \wedge (qp_j)^*\gamma \longrightarrow M_B(q^*\gamma). \tag{28.6}$$

To see this, test the powers of degrees $N$ and $N-1$ against nonnegative smooth cutoffs off $Z$ with respectively the constant 1 and a fixed Kähler form on $W$. For fixed $\varepsilon$, bounded decreasing Bedford–Taylor continuity [Xia26a, Theorem 9.2.5] gives convergence locally off $Z$ to the powers of the mixture. As $\varepsilon \downarrow 0$ these converge locally by multilinearity, using the two locally bounded currents there. Off the exceptional sets the resolved powers are the powers of $T_j$, and neither their pushforwards nor the non-pluripolar products of $T_j$ charge the sets excluded in making this identification. By maximality of the mixed mass in [Xia26a, Proposition 15.8.2], the total trace masses of the resolved products on $W$ are bounded above by those of the minimal current. The latter put no mass on $Z$. Exhausting its complement and choosing the diagonal therefore gives convergence of each trace mass. The same local convergence for signed smooth tests, with domination by the fixed trace form to bound the omitted part, gives (28.6).

Pull back $H, \gamma, \kappa$ to $W_j$, denoting the results by $H_j, \gamma_j, \kappa_j$. For $sL \le \delta/2$, the current

$$r_j - s\gamma_j + [F_j^* - F_j]$$

in $A_j - sc_1(H_j)$ has smooth part at least $\delta p_j^*\zeta$. As in the bigness argument in the proof of Lemma 28.1.1, subtracting a small exceptional correction from this lower bound and adding it back as a divisor gives a Kähler current which is locally bounded off a proper analytic set. Enlarge that set to include $H_j$. We can apply Lemma 28.2.1 to $a_j, \kappa_j, H_j$. The class $p_j^*(B - sc_1(q^*H))$ differs from $A_j - sc_1(H_j)$ by $c_1(F_j)$, so its volume is at least the lower bound of that lemma.

The error in replacing the divisor term by $Ns\int a_j^{N-1}\wedge\gamma_j$ is at most $Ns^2\int_{[H_j]}(a_j+\kappa_j)^{N-1}$. Indeed integrate $\gamma_j$ for the leading divisor term, and expand the remaining powers against $H_j$. There is a constant $C_H$ with $H \le C_H(W_*)_0$ on the fixed model. On pulling back, positivity bounds this last integral by

$$C_H \int_{W_j} \eta_j \wedge (a_j + \kappa_j)^{N-1},$$

where $\eta_j$ is a smooth semipositive pullback of the point class from the parameter line.

Choose a fixed Kähler form $\Sigma$ on $W_*$ whose class is above $\rho_*^*\alpha + [\kappa]$ in the pseudoeffective order. The pullback of $[\Sigma] + (c+K)c_1((W_*)_0)$ is then above $A_j + [\kappa_j]$, using $D \le KW_0$ and $F_j \ge 0$. Both upper and lower classes have smooth semipositive representatives. To compare their $(N-1)$-st powers against $\eta_j$, expand the difference of powers and use a positive representative for the difference in each term. The resulting upper bound is

$$C_H \int_{W_*} \eta_* \wedge (\Sigma + (c+K)\eta_*)^{N-1} = C_H \int_{W_*} \eta_* \wedge \Sigma^{N-1},$$

where $\eta_j = (qp_j)^*\eta_*$ and $\eta_*^2 = 0$. This bound is independent of the obstacle depth and the fibre shift. Now use Lemma 28.1.2 with zero exceptional divisor and pass to the limits in (28.6). Take $s_0 \le 1$ small enough for the curvature bound. For nearby backgrounds, add to $R$ a small smooth change of representative of the base class, chosen in a fixed finite-dimensional space of smooth forms. It keeps the same poles and a common strict lower bound, and the fixed upper Kähler class can be chosen on a smaller neighborhood. This proves the last assertion as well. □

**Proposition 28.2.1** *Let $B = \rho^*\alpha + c_1(D) + c\,c_1(W_0)$ on a smooth model with $D + cW_0 \ge W_0$. For a real vertical divisor $H$, the derivative of $\mathrm{vol}_W(B + tc_1(H))/N$ at zero is $M_B(H)$. For a fixed direction pulled back from $W_*$, the error in this linear approximation is $O(t^2)$, uniform when the divisors $D$ range as in Lemma 28.2.2 and $c$ is sufficiently large. The bound is uniform also for nearby backgrounds.*

***Proof*** Take $L$ so that $H + LW_0$ and $LW_0 - H$ are both effective. Increasing $c$ leaves $M_B(H)$ unchanged. Indeed, in the proof of Lemma 28.1.1 the glued minimal current is a product near the support of $\lambda$. Adding $t\pi^*\lambda$ changes the mixed pairing by a multiple of

$$\int_{X\times U} \langle (\theta + dd^c V_\theta)^{n-1} \rangle \wedge \lambda \wedge \gamma_H.$$

Here the factors from $X$ and the parameter line mean their pullbacks, and $U$ avoids zero. Since $H$ is vertical, $\gamma_H$ is exact on the complement of the central fibre. The current formed by the other two factors is closed and has support away from that fibre, so the integral is zero by Stokes. This works also for $n = 1$, with the zeroth power equal to 1.

On the affine space of real vertical divisors the shifted volume divided by $N$, minus $cV$, is concave by the root-concavity argument of Theorem 28.1.1; that argument uses only the divisorial order bound and the fibre formula. For small $s > 0$, apply Lemma 28.2.2 to subtract $s(H+LW_0)$, and then add $sLW_0$ using (28.1). Do the same for $s(LW_0 - H)$. The shift is admissible at all intermediate divisors if the original $c$ is chosen with extra slack. Since $M_B(W_0) = V$ by (28.2), we obtain the two lower bounds

$$\frac{\mathrm{vol}_W(B \pm sc_1(H))}{N} \ge \frac{\mathrm{vol}_W(B)}{N} \pm sM_B(H) - Cs^2.$$

Concavity on this line gives the opposite bound for each sign, using the lower bound for the other sign. This proves the estimate and the derivative for large shifts. For a smaller admissible $c$, the volume difference quotients near zero agree after a common fibre addition. The perturbed vertical divisors remain at least $\frac{1}{2}W_0$ for small $t$, so Lemma 28.1.1 applies. The mixed pairing comparison above identifies the coefficient at the original shift. □

**Theorem 28.2.1** *For every real continuous obstacle $f$, there is a positive Radon measure $\mu_\alpha^{\mathrm{big}}(f)$ of mass $V$, such that for every real continuous $h$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} G_\alpha^{\mathrm{big}}(f+th) = \int h \,\mathrm{d}\mu_\alpha^{\mathrm{big}}(f). \tag{28.7}$$

*For each fixed real model direction $h$, the expansion in (28.7) has a uniform $O(t^2)$ remainder over all continuous obstacles, with constants locally uniform in the big background. If $f$ is represented on $W$ as above, this measure is*

$$\mu_\alpha^{\mathrm{big}}(f) = \sum_{E \subset W_0} b_E M_B(E)\, \delta_{v_E}, \tag{28.8}$$

*where $b_E$ is the fibre multiplicity and $v_E$ the normalized model test. The scalar energies and measures vary continuously, respectively in the ordinary and weak topologies, as the big class varies and the obstacles converge uniformly. For nef and big $\alpha$ the measure agrees with $\mu_\alpha^{\mathrm{obs}}(f)$.*

***Proof*** For model $f$, the derivative on model directions is given by Proposition 28.2.1 on any common refinement. If $p : W' \to W$ represents such a direction by $H$, split it into $p^*(p_*H)$ and a signed exceptional divisor. On pullbacks the derivative agrees with that on $W$ by birational invariance of volume. For an effective exceptional divisor the right derivative is zero by Lemma 28.1.2; hence its contribution to $M_{p^*B}$ vanishes. Linearity handles signed exceptional divisors. The coefficient of $p_*H$ on each $E$ is $b_E h(v_E)$, by the divisor evaluation formula and invariance of the old test on its strict transform. This gives (28.8) on model tests. Each weight is nonnegative by volume monotonicity in an effective direction on $W$. The sum of the weights is $M_B(W_0) = V$.

Let $f_i$ be model obstacles converging uniformly to a continuous $f$, and temporarily allow big backgrounds $\alpha_i \to \alpha$. For a fixed model obstacle the scalar formula (28.3) varies continuously in the background. The total-space class is big for a fixed admissible shift, and its volume is continuous on the big cone, by the comparison in the proof of Corollary 23.4.2. The bound with Lipschitz constants $V_i = \mathrm{vol}_X(\alpha_i)$, which are bounded on a tail, gives convergence for $f_i$ and also for $f_i + th$ at a fixed real model direction $h$ and fixed $t$. Represent $h$ on $W_*$ and take each common refinement by a smooth blowup tower starting there. The coefficient bound for the model divisors of $f_i$ follows from their uniform norms. Choose a common large shift and use the uniform remainder of Proposition 28.2.1. The difference quotients at fixed $t \neq 0$ thus converge and approximate the measure tests with an error at most $C|t|$ uniformly on a tail. Let $t$ tend to zero. This shows that the model integrals are Cauchy, with limits independent of the approximations by comparison with the same difference quotients. By density and positivity the integrals give a Radon measure of mass $V$, by Theorem 4.3.1. Passing the remainder to the limit gives (28.7) on model directions. On a continuous direction, approximate uniformly by model functions. The Lipschitz bound controls the difference quotients uniformly in $t$, and the mass bound controls the integrals. This proves the formula and uniqueness.

The uniform remainder for model directions is independent of the obstacle by Proposition 28.2.1. It passes to every continuous obstacle by uniform approximation and the construction just given. Apply the fixed difference quotient argument to conclude weak continuity for these obstacles as well. In the nef and big case, Theorem 28.1.1 identifies the energies, so (26.2) identifies their derivative measures. □

*Example 28.2.1 (A fixed exceptional divisor and the onset of mass)* Let $S$ be compact Kähler of dimension $n \geq 2$, with Kähler class $\omega$, and let $\tau : X \to S$ be the blowup of a point $p$, with exceptional divisor $F$. Fix $a > 0$ and put $\alpha = \tau^*\omega + a\, c_1(F)$ and $V = \int_S \omega^n$. The class $\alpha$ is big, has volume $V$, and has negative degree on the lines in $F$. For $c \geq 0$, let $u_c$ be the pullback from $S^{\sqsupset}$ of the point model function $-c \min(v(\mathfrak{m}_p), 1)$. Then for $c \leq a$ and for $c - a > 0$ sufficiently small,

$$G^{\mathrm{big}}_{\alpha}(u_c) = -\frac{(c-a)_+^{n+1}}{n+1},$$
$$\mu^{\mathrm{big}}_{\alpha}(u_c) = \big(V - (c-a)_+^n\big)\delta_{v_0} + (c-a)_+^n \delta_{\mathrm{ord}_F}.$$

Here $v_0$ is the trivial ideal semivaluation of $X$.

Indeed $\tau^*\omega$ is nef and big and Lemma 28.1.2 gives its volume also for $\alpha$. The negative degree follows from $\mathcal{O}(F)|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. The pullback point ideal is $\mathcal{O}_X(-F)$. Blow up $F \times \{0\}$ in $X \times \mathbb{P}^1$; write $R$ for the exceptional component and $F'$ for the strict transform of $F \times \mathbb{P}^1$. The zero divisor of the pulled-back flag ideal is $R$, and the horizontal total transform has divisor $F' + R$. Thus the shifted obstacle class on this model is the pullback of the shifted product class over $S$, plus $a\, c_1(F') + (a-c)\, c_1(R)$. When $c \leq a$, both added divisors are effective and exceptional over $S \times \mathbb{P}^1$. Their addition preserves volume, giving $G^{\mathrm{big}}_{\alpha}(u_c) = 0$.

For $c > a$, put $d = c - a$. The model maps to the blowup of $(p, 0)$ in $S \times \mathbb{P}^1$ since the point flag ideal pulls back to an invertible ideal. Its exceptional divisor pulls back to $R$, and $F'$ is exceptional for this map. In fact its image is the closure of $\{p\} \times (\mathbb{P}^1 \setminus \{0\})$, of dimension one. The obstacle class is therefore the pullback of the shifted class of the point model of depth $d$ on $S$ plus the class $a\, c_1(F')$ of an effective exceptional divisor. For small $d > 0$, volume invariance and Proposition 21.9.1 and Theorem 28.1.1 give energy $-d^{n+1}/(n+1)$.

The model over $X$ has two central components of multiplicity one, with tests $v_0$ and $\mathrm{ord}_F$. For the latter, the exceptional order of the pullback of a base ideal is its order along the smooth divisor $F$, as is seen by blowing up the two generators of the ideal of $F \times \{0\}$. On these two tests, $u_1$ has values 0 and $-1$, respectively. Differentiate the energy formula in $c$, using (28.7) and the support in (28.8). This gives mass $(c-a)_+^n$ at $\mathrm{ord}_F$. The rest of the mass is at $v_0$. At $c = 0$ the same calculation uses the right derivative.

## 28.3 Big-class envelopes from test curves

Fix a smooth representative $\theta$ of the big class $\alpha$. Write $\varphi = V_\theta$ for the nonpositive extremal potential with minimal singularities and $V = \mathrm{vol}_X(\alpha)$. For a compatible test-curve family $\Gamma$, let $F_\Gamma$ be its function on $X^\square$ and $\mathcal{E}_{\mathrm{tc}}(\Gamma)$ its energy, as in Theorem 25.11.1 and (25.3). By Corollary 25.9.1, a finite-energy family is the enlargement to the perturbations of an ordinary curve based at $\varphi$. Write $p_X : X \times \mathbb{D} \to X$ for the projection and use $s = -\log|t|^2$ as the time on the punctured disk. For a nonpositive circle-invariant potential $U$ on the disk product, $J_U$ denotes the jet function of (25.7).

**Lemma 28.3.1** *Fix a condition on circle-invariant potentials $Q$ that depends only on the germs along $X \times \{0\}$ of the multiplier ideals $\mathcal{I}(mQ)$, $m > 0$. Let $U$ be the greatest circle-invariant $p_X^*\theta$-psh potential below $p_X^*\varphi$ satisfying it. Suppose that $U_s \geq \varphi - Ls$ for some finite $L \geq 0$. Then $U$ is a relative weak geodesic ray based at $\varphi$. Its inverse curve is $\mathcal{I}$-model, and $J_U$ is the function of the associated finite-energy compatible family.*

***Proof*** The condition is unchanged by altering a potential away from the central fibre, or by replacing it by one with the same germ of each multiplier ideal. We use these two facts in turn. Replace the restriction of $U$ to any finite time interval by the relative Perron geodesic between the two endpoints [Xia26a, Theorem 17.6.8]. The old restriction is a competitor, by convexity in time, the relative bounds and the endpoint limits. The new segment is bounded above by the linear interpolation of the endpoint potentials, hence by $\varphi$. This upper bound also gives the joint boundary inequalities for the psh gluing rule, using upper semicontinuity of the endpoints. The glued potential agrees with the old one near zero in the disk. If the interval starts at time zero, the initial value is $\varphi$ by the relative bounds. By maximality the replacement agrees with the old ray. This proves geodesicity. The slices have full mass relative to $\varphi$ because their differences from $\varphi$ are bounded.

The inverse curve of this ray is an ordinary model test curve based at $\varphi$, with cutoff $M = \sup_X U_1 \leq 0$, by [Xia26b, Lemma 9.2.4]. Apply the $\mathcal{I}$-envelope to the inverse slices. It gives an $\mathcal{I}$-model test curve with the same cutoff and left limit by [Xia26b, Proposition 9.3.4], since $\varphi$ itself is $\mathcal{I}$-model. Its relative Legendre ray $\widehat{U}$ dominates $U$ and is at most $\varphi$. The product multiplier-ideal formula and the supremum calculation of Lemma 25.10.1 show that the two rays have the same generic Lelong numbers at every vertical prime over the disk. In fact the calculation uses only the relative Legendre formula and the nonpositive cutoff, so it applies to the ordinary inverse curve before $\mathcal{I}$-enveloping. Each slice and its $\mathcal{I}$-envelope have the same ideals at all positive multiples. The pullbacks have equal Lelong numbers by the product formula. Over the punctured disk the two ray potentials differ by a locally bounded amount, by the relative bounds. Thus all their divisorial Lelong numbers, also those of orders not supported at zero, agree. The divisorial criterion [Xia26b, Theorem 1.4.3] gives equality of the space-time multiplier ideals. Thus $\widehat{U}$ is admissible. Maximality gives $U = \widehat{U}$, and Legendre inversion identifies the inverse curve. The relative bounds give finite energy by Theorem 25.8.1 and Corollary 25.9.1. For the actual cutoff $M$, the

translation rule Lemma 25.11.2 and the comparison with ordinary rays in the proof of Theorem 25.11.1 identify $J_U$ with $F_\Gamma$. □

**Lemma 28.3.2** *Let $f \le 0$ be a real model obstacle with divisor $D$ on a smooth refinement $W$ of the product. There is a greatest $p_X^*\theta$-psh potential $U^f \le p_X^*\varphi$ on the disk product subject to*

$$v_E(U^f) \ge -\operatorname{ord}_E(D) \qquad (E \subset W_0). \tag{28.9}$$

*Pullbacks are understood in these generic Lelong numbers. The potential is circle-invariant and is a weak geodesic ray based at $\varphi$, with*

$$\varphi - Ls \le U_s^f \le \varphi \qquad \text{for } L \ge \|f\|_\infty. \tag{28.10}$$

*Its inverse Legendre curve is $\mathcal{I}$-model. The function $J_{U^f}$ is $F_{\Gamma_f}$ for the associated finite-energy family and satisfies $F_{\Gamma_f} \le f$.*

***Proof*** The constraints are equivalent to positivity of $q^*(p_X^*\theta + dd^c U) + [D]$ over the disk, where $q : W \to X \times \mathbb{P}^1$ is the model morphism. This follows from the divisorial decomposition [Xia26a, Theorem 14.3.7], since $D \le 0$. Pulling back this positive current shows that the constraints then hold on every higher model. The coefficient of the pullback of $D$ on any vertical prime $E'$ is $b_{E'} f(v_{E'})$, by the divisor evaluation in Section 22.6. This formula also shows that the constraints are invariant under rotation of the product: the rotated vertical order has the same $b_{E'}$ and the same restriction to base ideals. Apply the formula on a common refinement when the chosen model is nonequivariant.

The potential $p_X^*\varphi + L \log|t|^2$ satisfies the constraints, since its generic Lelong number at $E$ is at least $Lb_E$. The upper regularized supremum of the competitors is a competitor by the upper envelope theorem and the generic Lelong calculation of Lemma 22.7.1. After adding local potentials, that proof works for the background $p_X^*\theta$ as well. Rotation invariance gives a radial potential and the barrier gives (28.10).

By Lemma 28.3.1, $U^f$ has an $\mathcal{I}$-model inverse curve. Indeed the vertical Lelong constraints depend only on the germs of the ideals along the central fibre, by the divisorial multiplier-order estimate [Xia26a, Corollary 12.6.8].

For the full obstacle inequality in the model case, let $I_m = \mathcal{I}(mU^f)$. The divisorial multiplier order estimate [Xia26a, Corollary 12.6.8] at the finitely many components on $W$ gives

$$\operatorname{ord}_E(I_m) \ge -m \operatorname{ord}_E(D) - Kb_E$$

with one constant $K \ge 0$, independent of $m$. The pullback of every local generator of $I_m$ has divisor at least $-mD - KW_0$. The difference is effective, so the inequality pulls back to all dominating models. Passing through a common refinement gives

$$-m^{-1} S(I_m)(v_{E'}) \le f(v_{E'}) + K/m$$

at every full-model test, using the invariant ideal calculation of Lemma 25.11.1. Choose an integer $A > m\|f\|_\infty + K$. The same inequality holds after replacing $S$ by

$\min(A, S)$, which is continuous by (25.6). The tests are dense by Proposition 21.6.1, so the capped inequality holds everywhere. Remove the cap and pass to the dyadic limit. This proves $J_{U^f} \le f$. □

**Proposition 28.3.1** *For a real model obstacle $f \le 0$, $\Gamma_f$ is the greatest compatible family with $F_\Gamma \le f$.*

***Proof*** Fix a Kähler form $\omega$ and $\delta > 0$ small enough that $[\theta - \delta\omega]$ is big. Choose a $(\theta - \delta\omega)$-psh potential $\psi \le 0$. Let $\Phi_\varepsilon$ be the relative ray of the coordinate $\Gamma^{\varepsilon\omega}$ of a competitor $\Gamma$. Its cutoff is at most zero since $F_\Gamma(v_0) \le f(v_0)$. The ray is nonpositive and extends to the disk product. In particular its vertical slopes are $F_\Gamma(v_E)$, by Lemma 25.11.1 and Theorem 25.11.1 and the cutoff translation rule. Set $a = \varepsilon/(\varepsilon + \delta)$ and

$$\Psi_\varepsilon = (1 - a)\Phi_\varepsilon + a\, p_X^*\psi + aL \log |t|^2$$

for $L \ge \|f\|_\infty$. This potential is $p_X^*\theta$-psh because $(1 - a)\varepsilon = a\delta$. It is nonpositive, so by restricting to each fibre it is at most $p_X^*\varphi$. At a component on $W$ its generic Lelong number is at least $(1-a)(-\operatorname{ord}_E D)+aLb_E$. Thus $\Psi_\varepsilon \le U^f$ by Lemma 28.3.2. At any fixed full vertical prime, including primes on local models, divide the resulting inequality of Lelong numbers by its fibre multiplicity. The contribution from $\psi$ is finite at a fixed prime. As $a \to 0$, the first term uses the same slope $F_\Gamma$ for all $\varepsilon$. Hence $F_\Gamma \le F_{\Gamma_f}$ at these tests. They include the trivial valuation and all rational multiples of base divisor orders, by the crossing construction following Lemma 25.10.1. The values on each positive divisor ray are continuous by the convex formula (25.1). The order embedding of Proposition 25.4.1 and Theorem 25.11.1 therefore gives the inequality of the compatible families on the full space. □

## 28.4 Energy and contact for the big-class envelopes

We retain the notation of Section 28.3. The energy of a relative ray is normalized at $\varphi$, and the volume $V$ enters its translation law.

**Proposition 28.4.1** *For a nonpositive real model obstacle $f$ with greatest family $\Gamma_f$, the energy satisfies*

$$\frac{E_\theta^\varphi(U_s^f)}{s} = \mathcal{E}_{\mathrm{tc}}(\Gamma_f) = G_\alpha^{\mathrm{big}}(f) \qquad (s > 0). \tag{28.11}$$

*For each component $E$ of its representing model $W$ and every sufficiently large shift $c$, a minimal current $T$ in $B = \rho^*\alpha + c_1(D) + c\, c_1(W_0)$ satisfies*

$$\nu_E(T) = b_E\big(f(v_E) - F_{\Gamma_f}(v_E)\big). \tag{28.12}$$

***Proof*** We first construct a compact extension. Fix $L \geq \|f\|_\infty$ as in (28.10), and choose $0 < r_1 < r_2 < 1$ and a radial function $\chi$ on $\mathbb{P}^1 \setminus \{0\}$ which equals $A \log |t|^2 + B_0$ for $|t| \leq r_1$, where $A > L$. Take $B_0$ so that $\chi \leq L \log |t|^2$ near zero, and extend $\chi$ to be positive in a collar of $|t| = r_2$ and smooth at infinity. Choose a nonnegative smooth form $\lambda$ of mass one on the parameter line, vanishing on a disk about zero and strictly positive where the smooth curvature of $\chi$ away from zero can be negative. Then $c\lambda + dd^c\chi \geq 0$ off zero for all sufficiently large $c$. On $|t| < r_2$ take $\widetilde{U} = \max\{U^f, p_X^*\varphi + \chi\}$ and use $p_X^*\varphi + \chi$ outside. The choices of $\chi$ give agreement on the gluing collar and $\widetilde{U} = U^f$ near zero. Thus $\widetilde{U}$ is psh in the background $p_X^*\theta + c\lambda$, where base forms are pulled back. Its difference from $p_X^*\varphi$ is bounded on compact parameter sets away from zero.

Choose $\ell_D$ and $\gamma_D$ with $dd^c\ell_D = [D] - \gamma_D$. On $W$ the current

$$T = q^*(p_X^*\theta + c\lambda + dd^c\widetilde{U}) + [D]$$

is positive by (28.9) and divisorial decomposition. Its potential in $\rho^*\theta + c\pi^*\lambda + \gamma_D$ is the psh representative of $\widetilde{U} \circ q + \ell_D$ off the central fibre. It has minimal singularities. To prove this, let $w \leq 0$ be an extremal potential in that background. Over a small disk where $\lambda = 0$, subtracting $\ell_D$ from $w$ gives a $p_X^*\theta$-psh potential downstairs. Indeed subtracting $[D]$ from the positive current adds an effective divisor. Off zero we have the corresponding potential on the product, bounded above near zero since $\ell_D$ is bounded below when $D \leq 0$. It extends across zero by the bounded-above extension theorem [Xia26a, Theorem 8.4.1]. The pullback and the original potential agree as psh representatives. The extended potential satisfies the Lelong constraints since its pullback current contains $[-D]$.

By fibre restriction and the same uniform upper bound, this descended potential is at most $p_X^*\varphi + C$ on a smaller disk. Subtract a constant large enough that it is below $\varphi + L \log |t|^2$ on the boundary collar of that smaller disk. Its maximum with this barrier, extended by the barrier outside, is an admissible disk potential. The greatest potential $U^f$ bounds it above, so near the central fibre the potential of $T$ bounds $w$ above up to an additive constant. On the compact complement, the same fibre restriction bound applied to $w - \ell_D$ and the bounded difference of $\widetilde{U}$ from $\varphi$ give the comparison. This proves minimality. Along each $E$, divisorial decomposition and the jet value formula give (28.12).

Write $e(t) = E_\theta^\varphi(\widetilde{U}_t)$ on the punctured parameter line. Its curvature as a distribution is

$$dd^c e = \frac{1}{n+1}\pi_*\langle T^{n+1}\rangle - cV\lambda \qquad (t \neq 0). \tag{28.13}$$

To verify (28.13), use on the compact product the background $p_X^*\theta + c\lambda$ and the two potentials $\max\{\widetilde{U}, p_X^*\varphi - R\}$ and $p_X^*\varphi$, for a finite $R > 0$. Their difference is bounded. Push to the parameter line the mixed non-pluripolar sum defining the relative energy with this difference as weight. The fibre formula [Xia26a, Corollary 15.5.1] identifies the result with the fibre energy as a distribution. Apply $dd^c$ to it by testing against a smooth parameter function. The integration by parts formula [Xia26a, Theorem 17.1.3] moves $dd^c$ onto the difference of potentials, and the mixed sum

telescopes to the difference of the two top powers, divided by $n+1$. The reference top power is $(n+1)c\lambda$ times the pulled-back top measure of $\varphi$, so its pushforward is $(n+1)cV\lambda$. For each compact set of parameter tests away from zero, take $R$ so large that the truncation agrees with $\widetilde{U}$ on a neighborhood of that set. Off zero the product model and $W$ are isomorphic. This gives (28.13).

By geodesicity, the relative energy is affine in $s$, and the bound at $s=0$ sets its intercept equal to zero [Xia26a, Theorem 17.7.3]. Write its slope as $g$. Near zero, $e(t) = -g\log|t|^2$, which is locally integrable and has distributional $dd^c$ there equal to $-g\delta_0$. The non-pluripolar top product of $T$ gives no mass to $W_0$, and $\lambda$ vanishes near zero. Integrating the global distributional $dd^c e$ against 1 yields

$$0 = \frac{\mathrm{vol}_W(B)}{n+1} - cV - g.$$

This is the shifted obstacle volume of (28.3). The other equality in (28.11) is the relative ray formula from Theorem 25.8.1 and Corollary 25.9.1. □

**Theorem 28.4.1** *For every continuous real obstacle $f$ on $X^{\square}$ there is a greatest compatible test-curve family $\Gamma_f$ such that $F_{\Gamma_f} \le f$. Write $P_\alpha(f) = F_{\Gamma_f}$ in this big-class construction. The family has finite energy and*

$$\mathcal{E}_{\mathrm{tc}}(\Gamma_f) = G^{\mathrm{big}}_\alpha(f), \qquad \mu^{\mathrm{big}}_\alpha(f)\big(\{P_\alpha(f) < f\}\big) = 0. \tag{28.14}$$

*For real continuous $f, g$ and $\delta = \|f-g\|_\infty$, one has $P_\alpha(f)-\delta \le P_\alpha(g) \le P_\alpha(f)+\delta$. For nef and big $\alpha$, the envelope equals the perturbative nef envelope.*

***Proof*** A common translation permits us to consider $f \le 0$. Indeed translation of the cutoff adds that constant to the jet function, and adds the constant times $V$ to the energy by (25.3). It has the same effect on $G^{\mathrm{big}}_\alpha$ and leaves its derivative measure invariant. Thus the model results apply to all real model obstacles by translation.

For continuous $f \le 0$, take nonpositive model $f_j$ converging uniformly to $f$ and write $\delta_j = \|f_j - f\|_\infty$. Consider the upper regularized supremum $U$ of the nonpositive circle-invariant disk potentials $Q \le p_X^*\varphi$ with $J_Q \le f$. This family contains $U^{f_j} - \delta_j s$. For each $Q$ the shift $Q - \delta_j s$ satisfies the $f_j$ constraints by the jet values on the model, so $Q - \delta_j s \le U^{f_j}$. Taking the supremum gives, on the punctured disk,

$$U^{f_j} - \delta_j s \le U \le U^{f_j} + \delta_j s.$$

The right inequality passes to the upper regularization because the right side is upper semicontinuous there. The upper envelope is psh, and the relative bounds of the first comparison and $U \le \varphi$ give a common linear lower bound. Since $U - \delta_j s \le U^{f_j}$, the same inequality holds for the jet functions by multiplier-ideal monotonicity and translation. Passing to the limit gives $J_U \le f$. Hence $U$ is the greatest admissible potential. By Lemma 28.3.1, $J_U = F_{\Gamma_f}$ for a finite-energy family, since the jet function is computed from the space-time ideals along the central fibre.

For any compatible family $\Gamma$ with $F_\Gamma \le f$, the model maximality for $f_j + \delta_j$ gives $F_\Gamma \le F_{\Gamma_{f_j}} + \delta_j$. The left comparison for $U$ bounds $F_{\Gamma_{f_j}}$ by $F_{\Gamma_f} + \delta_j$. Let

$j \to \infty$ and use the order embedding to get the greatest family. For two obstacles, translating either family down by $\delta$ makes it admissible for the other obstacle, which proves the two inequalities for their envelopes. The comparison of rays above bounds the difference of their relative energies at time $s$ by $sV\delta_j$. Using (28.11) and the continuous-obstacle Lipschitz bound proves the energy identity.

For contact at a model obstacle, use (28.12) on its representing model after normalizing by a constant. If $f(v_E) > F_{\Gamma_f}(v_E)$, the minimal coefficient $\nu_E(T)$ in the shifted class $B$ is positive. For small $r > 0$, $T - r[E]$ is positive in the big class $B - rc_1(E)$ and has the same non-pluripolar top mass, since the products ignore the divisor. Volume monotonicity in the reverse comparison gives $\mathrm{vol}(B - rc_1(E)) = \mathrm{vol}(B)$. The derivative in Proposition 28.2.1 therefore gives $M_B(E) = 0$. The finite support formula (28.8) proves contact for model $f$.

For an arbitrary continuous obstacle the gap $f - P_\alpha(f)$ is nonnegative and lower semicontinuous. On the open set where this gap is $> \eta > 0$, the gap for $f_j$ is positive for large $j$ by the two envelope inequalities and uniform convergence. Hence the measures of $f_j$ give this open set zero mass. Weak convergence (Theorem 28.2.1) and the portmanteau lower bound for open sets give zero limiting mass there. Taking the union over positive rational $\eta$ proves contact. For nef and big backgrounds the images of the compatible families are exactly the perturbative cone by Theorem 26.1.1. The greatest minorant characterizations agree, so they give the same envelope. □

**Corollary 28.4.1** *If two continuous obstacles have the same big-class envelope $u$, their measures agree. Write $\mu_u$ for this measure and write $\mathcal{E}_{\mathrm{tc}}(u)$ for the energy of the family corresponding to $u$. For every finite-energy compatible-family function $w$,*

$$\mathcal{E}_{\mathrm{tc}}(w) \le \mathcal{E}_{\mathrm{tc}}(u) + \int (w - u)\, \mathrm{d}\mu_u. \tag{28.15}$$

*In particular $w$ is integrable against $\mu_u$.*

***Proof*** The minimum $k = \min(f, g)$ of two obstacles with the same envelope $u$ still has envelope $u$. All three $G_\alpha^{\mathrm{big}}$-values equal the same energy. By contact, $k = f$ almost everywhere for $\mu_\alpha^{\mathrm{big}}(f)$, since $u \le k \le f$. Concavity and first variation of $G_\alpha^{\mathrm{big}}$ give its supporting inequality at $f$. Apply the inequality to $k + th$ for both signs of $t$. The constant term is $G_\alpha^{\mathrm{big}}(k)$ by the energy identity and the contact just proved. Differentiating at zero gives $\int h\, \mathrm{d}\mu_\alpha^{\mathrm{big}}(k) = \int h\, \mathrm{d}\mu_\alpha^{\mathrm{big}}(f)$ for every continuous $h$. The same comparison with $g$ proves independence of the obstacle.

Let $u = P_\alpha(f)$. The test-curve energy is increasing for the order on compatible families. Indeed extend their mass functions by zero beyond their respective cutoffs. Order of slices gives order of the masses by non-pluripolar mass monotonicity [Xia26b, Theorem 2.4.3]. Compute both energies with the same upper endpoint in (25.3). The extension by zero compensates the change in cutoff term. Now take continuous majorants $g$ of $w$. Maximality gives $w \le P_\alpha(g)$, so the energy and supporting inequality give

$$\mathcal{E}_{\mathrm{tc}}(w) \le G_\alpha^{\mathrm{big}}(g) \le \mathcal{E}_{\mathrm{tc}}(u) + \int (g - f)\, \mathrm{d}\mu_u.$$

The continuous majorants form a decreasing net with pointwise limit $w$ by Lemma 4.1.1. On a tail they are bounded above by a fixed constant. Use decreasing Radon integration (Theorem 4.2.1) and $u = f$ almost everywhere for $\mu_u$, by contact. The difference $w - u$ is taken on this set where $u$ is finite. This proves (28.15). Its finite left side forces $\int w\, \mathrm{d}\mu_u > -\infty$, while $w$ is bounded above by its cutoff. □

## 28.5 Upper semicontinuous obstacles in big classes

Let $\alpha$ be a big class, with $V = \mathrm{vol}_X(\alpha)$. An intrinsic finite-energy potential in $\alpha$ is a full-space function $F_\Gamma$ of a compatible family $\Gamma$ with $\mathcal{E}_{\mathrm{tc}}(\Gamma) > -\infty$. For an upper semicontinuous $f : X^\beth \to \mathbb{R} \cup \{-\infty\}$ define the relaxed energy

$$\overline{G}_\alpha^{\mathrm{big}}(f) = \inf_{\substack{g \in C(X^\beth, \mathbb{R}) \\ g \ge f}} G_\alpha^{\mathrm{big}}(g). \tag{28.16}$$

The majorants are directed downwards and decrease pointwise to $f$, by Lemma 4.1.1. Compactness gives an upper constant majorant.

**Proposition 28.5.1** *The energy in (28.16) is finite if and only if $f$ has a finite-energy compatible-family minorant. In that case there is a greatest compatible-family minorant $P_\alpha(f)$, of finite energy, and*

$$\mathcal{E}_{\mathrm{tc}}(P_\alpha(f)) = \overline{G}_\alpha^{\mathrm{big}}(f).$$

*The envelopes of any decreasing net of continuous majorants converge at divisor orders and at $v_0$ to $P_\alpha(f)$, provided the majorants decrease to $f$. For a decreasing net of upper semicontinuous obstacles $f_i \downarrow f$, the relaxed energies decrease to $\overline{G}_\alpha^{\mathrm{big}}(f)$, with $-\infty$ allowed.*

***Proof*** If $w \le f$ is a finite-energy compatible-family function, it is below every majorant envelope. Monotonicity of energy and Theorem 28.4.1 give the lower bound $\mathcal{E}_{\mathrm{tc}}(w)$ in (28.16).

Suppose that the infimum is finite. Let $g_i \downarrow f$ be the net of continuous majorants, or any such net with this infimum pointwise. On a tail let $A$ be a common upper bound for their cutoffs $M_i = P_\alpha(g_i)(v_0)$, using one upper bound for $g_i$. Write $C$ for a finite lower bound for their energies. Since $G_\alpha^{\mathrm{big}}(g_i) \le VM_i$, their decreasing cutoffs have a finite infimum $M$. Extend each slice mass $m_i$ by zero at and above $M_i$. The energy formula gives

$$\int_{-\infty}^{A} (V - m_i(\tau))\, \mathrm{d}\tau = VA - G_\alpha^{\mathrm{big}}(g_i) \le VA - C.$$

The deficits are nonnegative and increasing in $\tau$. Fix $\tau_0 < C/V$ so far to the left that $(VA-C)/(A-\tau_0) < V/2$. Then $m_i(\tau_0) \geq V/2$ on the tail. The decreasing test-curve lemma Lemma 25.7.1 gives an ordinary $\mathcal{I}$-model limit curve in $\alpha$, of cutoff $M$, with convergence of the masses at all interior slopes. Its enlargement gives a compatible family.

This limit is below the majorant families by their slice order, hence its full function is at most $f$. If a compatible family is below $f$, it is below each majorant envelope. At divisor orders and $v_0$, its values are therefore at most the decreasing evaluations of the test-curve limit. Order recovery (Proposition 25.4.1 and Theorem 25.11.1) gives the same inequality on the full space. This proves maximality.

The extended masses decrease to the extended mass function of the limit at all slopes except possibly $M$. On a finite slope interval, the integrals converge by upper and lower sums of these bounded monotone functions. A subdivision can avoid the exceptional slope, and the two sums differ by at most $V$ times the mesh. The left tail is then handled by taking the supremum of the integrals of the nonnegative deficits over finite left intervals. Interchange this supremum with the increasing limit over $i$. The energy of the limit is the limit of the majorant energies and is finite.

For the scalar continuity assertion, let $g$ be any continuous majorant of $f$, and $\varepsilon > 0$. The increasing open sets $\{f_i < g+\varepsilon\}$ cover the compact space, so one contains it. Then $\overline{G}^{\mathrm{big}}_{\alpha}(f_i) \leq G^{\mathrm{big}}_{\alpha}(g)+V\varepsilon$. Taking the infimum over $g$ and letting $\varepsilon \downarrow 0$ proves the asserted decrease, by monotonicity. This argument works for arbitrary extended values at the limit. In particular it identifies the energy just computed with (28.16) for any of the indicated majorant nets. □

**Proposition 28.5.2** *For an intrinsic finite-energy potential $w$ one has $\overline{G}^{\mathrm{big}}_{\alpha}(w) = \mathcal{E}_{\mathrm{tc}}(w)$.*

***Proof*** Apply Proposition 28.5.1 with obstacle $w$. Its greatest minorant is $w$. □

**Theorem 28.5.1** *Let $f$ be an upper semicontinuous obstacle of finite relaxed energy. For any decreasing continuous majorants $g_i \downarrow f$, the measures $\mu^{\mathrm{big}}_{\alpha}(g_i)$ converge weakly to the same positive Radon measure $\overline{\mu}^{\mathrm{big}}_{\alpha}(f)$ of mass $V$. For each real continuous $h$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \overline{G}^{\mathrm{big}}_{\alpha}(f+th) = \int h \,\mathrm{d}\overline{\mu}^{\mathrm{big}}_{\alpha}(f).$$

*If $h$ is a real model function, the remainder satisfies*

$$\left|\overline{G}^{\mathrm{big}}_{\alpha}(f+th) - \overline{G}^{\mathrm{big}}_{\alpha}(f) - t\int h \,\mathrm{d}\overline{\mu}^{\mathrm{big}}_{\alpha}(f)\right| \leq C_h t^2 \tag{28.17}$$

*for all sufficiently small $|t|$, uniformly in $f$. The measure depends only on the greatest minorant. It varies weakly continuously along decreasing upper semicontinuous obstacles whose limit has finite relaxed energy. It agrees with the nef obstacle measure for bounded obstacles when $\alpha$ is nef and big.*

***Proof*** The relaxed energy is monotone and has the constant translation rule of the continuous energy, since translation bijects the continuous majorants. Thus $f - \delta \leq g \leq f + \delta$ implies $|\overline{G}_\alpha^{\mathrm{big}}(f) - \overline{G}_\alpha^{\mathrm{big}}(g)| \leq V\delta$ when either energy is finite. In particular, the perturbations $f + th$ have finite energy.

The quadratic remainder for continuous obstacles in Theorem 28.2.1 is independent of the obstacle. For fixed $t$, the continuous majorants $g_i + th$ decrease to $f + th$, so the two terms of their difference quotients converge by the scalar continuity in Proposition 28.5.1. For $t \neq 0$, the difference quotients approximate the integrals against $\mu_\alpha^{\mathrm{big}}(g_i)$ to within $C|t|$ on a tail. First choosing small $|t|$ and then letting the net advance proves that the integrals are Cauchy. Comparison with the same difference quotients shows independence of the net. Model density and the fixed positive mass give weak convergence to a Radon measure of mass $V$. Passing to the limit in the estimate gives (28.17). Approximate a continuous direction uniformly by model directions. The Lipschitz bound controls the difference quotients and the measure mass controls the integrals, proving the derivative formula.

To prove independence from the obstacle, note first that the relaxed energy is concave on upper semicontinuous obstacles of finite relaxed energy. Indeed, take decreasing continuous majorants of two obstacles and use concavity on their convex combinations. Pass to the limit by the scalar continuity of Proposition 28.5.1. If $f$ and $g$ have the same greatest minorant, their minimum $k$ has it as well. Their three energies are therefore equal. For $0 < s < 1$, put $f_s = (1 - s)k + sf$. Its energy is still the same, by order. For a continuous $h$, concavity gives

$$\overline{G}_\alpha^{\mathrm{big}}(f_s + sth) \geq (1 - s)\overline{G}_\alpha^{\mathrm{big}}(k) + s\,\overline{G}_\alpha^{\mathrm{big}}(f + th).$$

Divide the increments by $st$ and let $t$ tend to zero from both sides. The derivatives at $f_s$ and $f$ agree. Applying the same argument to $f_s + (1 - s)th$ gives equality with the derivative at $k$. Repeat with $g$ in place of $f$. The measures agree by testing all continuous $h$.

For decreasing upper semicontinuous $f_i$ with finite-energy limit, each relaxed energy is finite by monotonicity. The same difference quotient argument, now using (28.17), gives weak convergence. In the nef and big case the continuous energies agree by Theorem 28.1.1. Continuous majorants decreasing to them recover the bounded upper semicontinuous energy by Theorem 27.1.1 and Proposition 28.5.1. Hence their directional derivatives agree. □

**Corollary 28.5.1** *For any intrinsic finite-energy potential $u$ define $\mathrm{MA}_\alpha(u) = \overline{\mu}_\alpha^{\mathrm{big}}(u)$. It has mass $V$, agrees with the envelope measure on continuous-obstacle envelopes, and is weakly continuous along decreasing intrinsic finite-energy potentials with finite-energy limit. For nef and big $\alpha$ it is the intrinsic nef measure.*

***Proof*** The obstacle $u$ has relaxed energy $\mathcal{E}_{\mathrm{tc}}(u)$ by Proposition 28.5.2, and is its own greatest minorant. The measure construction, independence of the obstacle, and decreasing continuity follow from Theorem 28.5.1. For the nef comparison, the finite-energy domains agree by Theorem 26.5.1. The truncations $\max(u, -k)$ are bounded

members of the perturbative cone and decrease to $u$. The two measures agree at each truncation by the bounded comparison in Theorem 28.5.1. Their weak limits agree by decreasing continuity here and in Theorem 26.6.1. □

**Proposition 28.5.3** *For a positive Radon measure $\nu$ of mass $V$, put*

$$\mathcal{E}_\alpha^\vee(\nu) = \sup_w \left( \mathcal{E}_{\mathrm{tc}}(w) - \int w \, \mathrm{d}\nu \right),$$

*where $w$ ranges over intrinsic finite-energy potentials. Suppose that this number is finite. For every maximizing net $u_i$, the measures $\mathrm{MA}_\alpha(u_i)$ converge weakly to $\nu$. In particular, an attained maximum gives a solution of the Monge–Ampère equation.*

***Proof*** Finiteness implies that every finite-energy potential is integrable against $\nu$, since such a potential is bounded above. Put $\Delta_i = \mathcal{E}_\alpha^\vee(\nu) - \mathcal{E}_{\mathrm{tc}}(u_i) + \int u_i \, \mathrm{d}\nu$. For any real model $h$ and real $t$, the envelope $w_{i,t} = P_\alpha(u_i + th)$ has finite energy by the translation and monotonicity bounds. By the dual definition and $w_{i,t} \le u_i + th$,

$$\overline{G}_\alpha^{\mathrm{big}}(u_i + th) - \overline{G}_\alpha^{\mathrm{big}}(u_i) \le \Delta_i + t \int h \, \mathrm{d}\nu.$$

Apply (28.17) with both signs of $t$ to obtain, for small $t > 0$,

$$\left| \int h \, \mathrm{d}(\mathrm{MA}_\alpha(u_i) - \nu) \right| \le \frac{\Delta_i}{t} + C_h t.$$

First pass along the maximizing net and then let $t \downarrow 0$. Model density and fixed mass give weak convergence. Apply this result to the constant net at a maximizer to prove the last claim. □

*Example 28.5.1* Suppose $\dim X \ge 1$ and let $f$ be one at the trivial valuation $v_0$ and zero elsewhere. Then $f$ is upper semicontinuous and $\overline{\mu}_\alpha^{\mathrm{big}}(f) = V\delta_{v_0}$, while $P_\alpha(f)(v_0) = 0$.

Take a nontrivial base divisor order $v$, for instance the order at a point on a curve or the exceptional order after blowing up a point in higher dimension. The points $rv$ tend to $v_0$ as $r \downarrow 0$ by the ideal evaluations in the topology of $X^{\beth}$. For any compatible family, $F_\Gamma(rv) \to \Gamma_{\max} = F_\Gamma(v_0)$. The upper bound is the cutoff and the lower bound follows by testing $\tau - rv(\Gamma_\tau^\omega)$ at any fixed interior $\tau$ in (25.1), then letting $\tau$ increase to the cutoff. For a continuous $h$ and real $t$, a family below $f + th$ is below $th$ on that ray. Continuity of $h$ forces its value at $v_0$ to be at most $th(v_0)$. Thus the same families are below $f + th$ and below $th$, so the greatest minorants agree. Their energies are the same by Proposition 28.5.1. Differentiation identifies the measure at $f$ with that of the zero obstacle. On the product model the measure of zero is $V\delta_{v_0}$ by (28.8). The constant-in-time ray $\varphi = V_\theta$ is allowed by the zero obstacle. It satisfies the model constraint on the product and already attains the upper bound $\varphi$. Its cutoff is zero. Hence the measure gives full mass to the point with a gap.

## 28.6 Prescribing masses on an equivariant model

Fix a big class $\alpha$ on $X$, with $V = \mathrm{vol}_X(\alpha)$. Let $q : W \to X \times \mathbb{P}^1$ be a smooth refinement with strict normal crossings central fibre $W_0 = \sum_{j=1}^{r} b_j E_j$. In this section assume that rotation of the parameter lifts to $W$. Denote the normalized base tests of the components by $v_j$.

**Lemma 28.6.1** *Every real divisor supported on $W_0$ represents a real model function on $W$. The tests $v_1, \dots, v_r$ are distinct and are flag-model tests.*

***Proof*** It suffices to treat an integral vertical divisor $D$ and then take real linear combinations. The second assertion of Lemma D.4.1, applied to $q$ and a product Kähler form whose factor on $\mathbb{P}^1$ has mass one, gives an effective rational exceptional divisor $F$ for which

$$\rho^*[\omega] + c_1(W_0) - c_1(F)$$

is Kähler, with $\omega$ the factor on $X$ and $\rho$ the projection to $X$. For large integers $m$, its $m$-fold multiple plus $c_1(D)$ is still Kähler. Subtract a smooth representative of $m\, c_1(W_0)$ pulled back from the parameter line and supported away from zero. This leaves a smooth representative of $\rho^*[m\omega] + c_1(D - mF)$ positive near $W_0$. The argument with $D = 0$ gives the same positivity for the divisor $-mF$.

Apply the relative-ideal construction Lemma D.6.1 to each of these two rational divisors. Its ideals are pushforwards of the invertible sheaves of suitably shifted and scaled divisors. Every central component is invariant under the connected circle, so both pushforwards are invariant. They contain a power of the ideal of zero. In the quotient modulo this power, the Vandermonde argument of Theorem 21.4.1 decomposes them into homogeneous parts. The ideals are therefore flag ideals. The two divisors are represented by their flag model functions after rescaling and adding the respective fibre constants. Subtracting represents $D$. For $D = b_i E_i$, the test values are one at $v_i$ and zero at the other tests by (21.2). Hence the tests are distinct.

The flag ideal obtained for $-mF$ also realizes $W$ as a flag model. Its pullback to $W$ is invertible, so $W$ maps to its normalized blowup $M$ by Lemma D.1.1. The class of the pulled-back ideal is a positive multiple of $-c_1(F)$ plus a multiple of $c_1(W_0)$, by (D.2). Consequently the Kähler class $\rho^*[\omega] + c_1(W_0) - c_1(F)$ restricts to zero on each fibre of $W \to M$. Indeed the ideal class is pulled back from $M$, and the other classes are pulled back from the product. A positive-dimensional compact analytic component of a fibre would have positive volume for this Kähler form, contradicting the vanishing intersection on its integration cycle. Thus $W \to M$ is a proper finite-to-one modification between normal spaces. It is an isomorphism by the finite-map argument in the proof of Lemma D.1.2. □

**Lemma 28.6.2** *Fix a Kähler form $\Omega$ on $W$. Let $B = \rho^*\alpha + c_1(D) + c\,c_1(W_0)$ be as in (28.3). If $\int_{E_j} B[\Omega]^{n-1} < 0$, then $M_B(E_j) = 0$.*

***Proof*** Suppose the generic Lelong number of a minimal current in $B$ on $E_j$ is zero. Regularize its potential from above with gentle analytic singularities [Xia26a, Theorem 13.4.5]. Each approximant has generic Lelong number zero on $E_j$ by monotonicity. Its analytic pole set therefore omits a general point of $E_j$. Adding the vanishing curvature correction $\varepsilon[\Omega]$ gives a positive current whose local potentials restrict to $E_j$, since they are not identically $-\infty$ there. Its restricted class has nonnegative degree against $(\Omega|_{E_j})^{n-1}$. Letting $\varepsilon$ decrease to zero contradicts the degree hypothesis.

Thus the minimal current has positive generic Lelong number. Subtract a sufficiently small positive multiple of $[E_j]$. The result is positive by divisorial decomposition [Xia26a, Theorem 14.3.7] and its top mass is unchanged off $E_j$. Neither top measure charges that divisor. Volume monotonicity and maximality give equality of the two volumes, so the derivative in Proposition 28.2.1 vanishes in the direction $E_j$. □

**Theorem 28.6.1** *For any $m_j \geq 0$ with $\sum_{j=1}^{r} m_j = V$, there exists a real model obstacle $f$ represented on $W$ such that its envelope $u = P_\alpha(f)$ has measure*

$$\mu_u = \sum_{j=1}^{r} m_j\, \delta_{v_j}.$$

*This $u$ maximizes $\mathcal{E}_{\mathrm{tc}}(w) - \int w\, \mathrm{d}\mu_u$ among all finite-energy test-curve potentials.*

***Proof*** In the coordinates $d = (d_1, \dots, d_r) \in \mathbb{R}^r$, let $f_d$ be the model function of $D(d) = \sum_j b_j d_j E_j$, using Lemma 28.6.1. Choose these representatives linearly, by using one for each basis divisor. The values are $f_d(v_j) = d_j$. Define $G(d) = G_\alpha^{\mathrm{big}}(f_d)$. The first variation and atom formula Theorem 28.2.1 give

$$\nabla G(d)_j = \mu_\alpha^{\mathrm{big}}(f_d)\{v_j\}.$$

These coordinates are nonnegative and sum to $V$. Translation by equal constants in all coordinates adds that constant times $V$ to $G$.

Fix a Kähler form $\Omega$ on $W$. Write $a_j = b_j \int_{E_j} \rho^*\alpha[\Omega]^{n-1}$ and for $i \neq j$ put $w_{ij} = b_i b_j \int_{E_i \cap E_j} \Omega^{n-1}$. The numbers $w_{ij}$ are symmetric and are positive precisely when the components meet. Let $L$ be the graph Laplacian $(Ld)_j = \sum_{i \neq j} w_{ij}(d_j - d_i)$. For the class of the shifted divisor define

$$Q_j(d) = b_j \int_{E_j} (\rho^*\alpha + c_1(D(d)) + c\,c_1(W_0))[\Omega]^{n-1}.$$

Since the fibre class restricts to zero on $E_j$, Lemma D.2.2 on $E_j$ gives $Q_j(d) = a_j - (Ld)_j$. Indeed the off-diagonal intersections give $w_{ij}$, and the identity for the full fibre determines the diagonal coefficient as their negative sum.

The component graph is connected, as proved in Lemma 23.5.1. Thus $L$ is positive definite on the zero-sum subspace, since $d \cdot Ld = \sum_{i<j} w_{ij}(d_i - d_j)^2$ and its kernel consists of constants. Choose $A > 0$ with $\sum_j a_j + rA > 0$. On the zero-sum subspace the constraints $Q_j(d) \geq -A$ define a nonempty compact polytope. The sum of the $Q_j$ is fixed, and $d \mapsto Q(d)$ is an affine bijection to that fixed-sum hyperplane. The degree vector with all coordinates equal is strictly feasible, and the lower bounds and fixed sum bound all coordinates above.

Let $d_*$ maximize $G(d) - \sum_j m_j d_j$ on this polytope. Set $y = \nabla G(d_*) - (m_1, \dots, m_r)$, a zero-sum vector. If $Q_j(d_*) = -A$, then Lemma 28.6.2 gives $\nabla G(d_*)_j = 0$, hence $y_j = -m_j \leq 0$. If $y \neq 0$, solve $Lh = y$ in the zero-sum subspace. In each active constraint, changing $d_*$ to $d_* + th$ for $t > 0$ changes $Q_j$ by $-ty_j \geq 0$. All inactive constraints remain satisfied for sufficiently small $t$. The derivative of the objective in this feasible direction is $y \cdot h = h \cdot Lh > 0$, contrary to maximality. Therefore $y = 0$. The first variation and atom formula now give the target measure for $f = f_{d_*}$. The intrinsic supporting inequality (28.15) gives the last assertion, including integrability of every finite-energy competitor. □

**Corollary 28.6.1** *Let $\nu$ be a positive measure of mass $V$ with finite support in the central tests of flag models. Then it is the intrinsic measure of a model-obstacle envelope. Such envelope measures are weakly dense in the positive Radon measures of mass $V$.*

***Proof*** Choose the finitely many flag ideals that give the models of the marked tests. In the common smooth refinement construction of Lemma D.4.2, principalize their product with the ideal of the central fibre. The ideals are rotation invariant, so each centre of this canonical principalization is invariant, by the first assertion of Lemma D.4.1. The rotations lift to each blowup by its universal property. The remaining strictification centres are defined by the number of local branches of the reduced central fibre and of its strict transforms, which are rotation invariant, so these centres are invariant too. This gives a smooth equivariant common refinement. Each marked component has a strict transform with the same test value map, by Lemma D.1.2. Put zero target mass on the extra components and apply Theorem 28.6.1.

The flag-model tests are dense by Proposition 21.6.1. For a finite collection of continuous test functions and a prescribed $\varepsilon > 0$, cover the compact space by finitely many open sets on which every test oscillates by at most $\varepsilon$. Take a disjoint Borel partition subordinate to this cover and put the mass of each part at one flag-model test in its covering open set. The resulting measure has mass $V$ and its integrals against the chosen tests differ by at most $V\varepsilon$ from the original integrals. Each such measure is realized by the first assertion, which proves weak density. □

# Chapter 29
# The variational equation in big classes

In the point blowup of Example 28.2.1, the pullback of the point model function of any depth $c \leq a$ has obstacle energy zero, its value for the zero obstacle. Energies in a big class are normalized at the minimal singularities. The envelope $u_* = P_\alpha(0)$ of the zero obstacle is the jet of the constant ray of the minimal potential $\varphi = V_\theta$. Its value at the test of a vertical divisor $E$ is minus the generic Lelong number of that ray along $E$, divided by the fibre multiplicity of $E$. Its energy is zero, and every normalized finite-energy potential lies below it (Section 29.1). We therefore also measure the dual problem against $u_*$. For a positive Radon measure $\nu$ of mass $V$, the relative dual energy (29.1) integrates $u - u_*$ on $\Xi = \{u_* > -\infty\}$. It differs from the absolute dual energy by $\int (-u_*)\, \mathrm{d}\nu$, by (29.2).

Throughout this chapter $X$ is a connected compact Kähler manifold of dimension $n$, $\alpha$ is a big class with smooth representative $\theta$, $V = \mathrm{vol}_X(\alpha)$, and $\varphi = V_\theta$. We use the intrinsic finite-energy potentials with their energy $\mathcal{E}_{\mathrm{tc}}$, the envelopes $P_\alpha$ and the operator $\mathrm{MA}_\alpha$ of Chapter 28. A potential $u$ is normalized if $u(v_0) = 0$, and weak convergence of potentials means convergence at the tests $\mathcal{T}$ of vertical divisors on smooth local product models.

Normalized energy balls are weakly compact with upper semicontinuous energy (Theorem 29.1.1), and normalized maximizing nets for the absolute or the relative dual objective lie in one ball on a tail whenever the corresponding dual energy is finite (Proposition 29.2.1). This solves the equation for countable sums of Dirac masses at flag-model tests with summable lower bounds (Corollary 29.2.1), and measures with finite support in these tests are strongly dense among those of finite absolute dual energy (Theorem 29.2.1). For general measures we use the rooftop distance $d_{1,\alpha}$. It is the slope at infinity of the analytic distance between the relative rays and is complete (Theorem 29.3.1 and Corollary 29.3.1). Normalized energy balls are uniformly integrable against each measure of finite absolute dual energy (Theorem 29.3.2), which solves the equation for such measures carried by countably many tests (Corollary 29.3.2). On energy balls, ordered extrapolation bounds the integral of $|u - v|$ against a measure of finite relative dual energy by a power of $d_{1,\alpha}(u, v)$ (Proposition 29.4.1). With the midpoint estimate Proposition 29.4.2, every positive Radon measure of mass $V$ and finite relative dual energy is the Monge–Ampère measure

of an intrinsic finite-energy potential, and its relative dual supremum has a unique normalized maximizer (Theorem 29.4.1). Convergence in $d_{1,\alpha}$ is convergence on tests together with convergence of energy (Theorem 29.5.1), and $\mathrm{MA}_\alpha$ is weakly continuous for this topology (Theorem 29.5.2).

Chapter 30 identifies the image of $\mathrm{MA}_\alpha$ with the measures of finite relative dual energy and proves that the normalized operator is a homeomorphism onto them. It uses the attainment theorem, the rooftop topology and the local test bound Lemma 29.5.1.

## 29.1 Weak compactness on an energy ball

Retain the big class $\alpha$, the form $\theta$, and the minimal potential $\varphi$ of Section 28.3. Let $\mathcal{T}$ consist of the normalized restrictions $v_E$ of all vertical divisor orders on smooth local product models, with normalization by $b_E = \mathrm{ord}_E(t)$. These tests include $v_0$, and contain the positive rational multiples of every base divisor order by the crossing construction after Lemma 25.10.1. They determine the order on compatible families by convex continuity along each positive divisor ray and Proposition 25.4.1 and Theorem 25.11.1. In this section weak convergence of potentials means pointwise convergence on $\mathcal{T}$.

Normalize a finite-energy potential $u$ by $u(v_0) = 0$. It then satisfies $u \le 0$, by the jet formula for the ray with cutoff zero. Write $u_* = P_\alpha(0)$. Its disk ray is $p_X^*\varphi$: the constraint for the zero obstacle on the product is automatic, and Lemma 28.3.2 takes the greatest potential below $p_X^*\varphi$. Hence

$$u_*(v_E) = -\frac{v_E(p_X^*\varphi)}{b_E}, \qquad \mathcal{E}_{\mathrm{tc}}(u_*) = 0.$$

A normalized $u$ lies below $u_*$ by the greatest-minorant property. Since $u_*(v_0) = 0$, it is the greatest normalized potential. The relative logarithmic barrier $\varphi - Ls$ has test values $u_*(v_E) - L$. For a finite-energy potential $u$, write $U^u$ for the ordinary relative ray in $\theta$ of its finite-energy family (Theorem 25.8.1 and Corollary 25.9.1).

**Lemma 29.1.1** *Let $f$ be a continuous real obstacle with $u = P_\alpha(f) \le 0$. Then $u = P_\alpha(\min(f, 0))$, and for some finite $L \ge 0$,*

$$\varphi - Ls \le U_s^u \le \varphi \qquad (s > 0).$$

***Proof*** Since $u \le \min(f, 0) \le f$, maximality of both envelopes gives $u \le P_\alpha(\min(f, 0)) \le P_\alpha(f) = u$. For the nonpositive continuous obstacle $\min(f, 0)$, the proof of Theorem 28.4.1 constructs the greatest admissible disk potential $U$. It lies below $p_X^*\varphi$, and in that proof $U \ge U^{f_j} - \delta_j s$ for a model ray $U^{f_j}$. Hence (28.10) gives a lower bound $\varphi - Ls$ with $L$ finite. By Lemma 28.3.1, $U$ is a relative weak geodesic ray based at $\varphi$ with $\mathcal{I}$-model inverse curve, and $J_U = u$. By order recovery this inverse curve is the ordinary curve of $u$, and Legendre inversion gives $U = U^u$.□

**Lemma 29.1.2** *Let $H$ be a circle-invariant $p_X^*\theta$-psh potential on the disk product with $\varphi - Ls \le H_s \le \varphi$ for some finite $L \ge 0$. There is a greatest such disk potential below $p_X^*\varphi$ with vertical generic Lelong numbers at least those of $H$. It dominates $H$ and has its vertical generic Lelong numbers on all local models. Its jet is an intrinsic finite-energy potential.*

***Proof*** Take the upper regularized supremum of the circle-invariant potentials below $p_X^*\varphi$ satisfying the inequalities at each vertical prime in every local smooth model. The family contains $H$. The point Lelong-number calculation in Lemma 22.7.1, with local potentials for $\theta$ added before pullback, gives the formula for the generic numbers on each local model. Indeed the pullback of the supremum agrees with the supremum of the pullbacks on the open set where the model map is biholomorphic. The regularized expressions agree everywhere as psh representatives, since that open set has full measure. Thus the inequalities persist, with equality because $H$ itself is a competitor. The upper regularization is radial and bounded above by $p_X^*\varphi$. The vertical numbers are determined by the germs of the multiplier ideals along the central fibre by the divisorial order estimate [Xia26a, Corollary 12.6.8], so Lemma 28.3.1 applies. □

**Lemma 29.1.3** *Let $u$ be an intrinsic finite-energy potential and let $f$ be an upper semicontinuous obstacle. If $u(v) \le f(v)$ for every $v \in \mathcal{T}$, then $u \le f$.*

***Proof*** First let $f$ be real model and translate both functions by a constant so that $f \le 0$. The cutoff of $u$ is then nonpositive by the inequality at $v_0$. The ordinary relative ray in $\theta$ of its finite-energy family is below $p_X^*\varphi$, extends over the fibre, and has vertical slopes $u(v_E)$ by Lemma 25.10.1 and Theorem 25.11.1. On the representing model of $f$, the test inequalities give the constraints of Lemma 28.3.2. Thus this ray is dominated by the ray of $P_\alpha(f)$. Inequalities of the vertical slopes and order recovery give $u \le P_\alpha(f) \le f$. For an arbitrary upper semicontinuous $f$, take continuous majorants, and approximate each of those uniformly from above by real model functions using Proposition 21.5.1 and an upward constant shift. Their infimum is $f$ by Lemma 4.1.1, so the model case applies. □

**Theorem 29.1.1** *For each $R \ge 0$ the set of normalized intrinsic finite-energy potentials with $\mathcal{E}_{\mathrm{tc}} \ge -R$ is weakly compact. Its energy is weakly upper semicontinuous.*

***Proof*** Take a net $(u_i)$ in this set with a limit $a$ of its test coordinates in the product $[-\infty, 0]^{\mathcal{T}}$. For each tail $j \ge i$, take the regularized supremum on the disk of the ordinary rays of $u_j$ and the barrier $p_X^*\varphi + k \log|t|^2$, where $k$ is a positive integer. The rays are below $p_X^*\varphi$, since their cutoffs are zero. The barrier gives a logarithmic lower bound. By Lemmas 22.7.1 and 29.1.2 the saturated supremum gives a finite-energy potential $w_{i,k}$ with values

$$w_{i,k}(v) = \max\Big\{\sup_{j \ge i} u_j(v),\ u_*(v) - k\Big\} \qquad (v \in \mathcal{T}).$$

In particular it is normalized. By order recovery it majorizes the tail and decreases when $i$ or $k$ increases. Its energy is therefore at least $\sup_{j\ge i}\mathcal{E}_{\mathrm{tc}}(u_j)\ge -R$.

Use the ordinary curves of $w_{i,k}$ in $\theta$, all with cutoff zero and left mass $V$. For each curve the mass deficit is nonnegative and increasing in $\tau<0$, and its integral over $(-\infty,0)$ is at most $R$. Consequently the deficit at any $\tau_0<-2R/V$ is at most $R/(-\tau_0)<V/2$ if $R>0$, and is zero if $R=0$. The decreasing evaluation lemma Lemma 25.7.1 now gives a limit ordinary curve and convergence on divisor rays and at $v_0$. Enlarge this curve to its compatible family and denote its function by $w$. To identify the limit at every test in $\mathcal{T}$, choose a single Kähler perturbation of $\theta$. All the $w_{i,k}$ and $w$ belong to its decreasing cone by Lemma 25.12.1 and independence of the jet function. The nested decreasing closure in Theorem 23.5.1 puts the pointwise infimum of the $w_{i,k}$ in the same cone, since its value at $v_0$ is zero. By Theorem 25.12.1 this infimum and $w$ agree everywhere, as their values agree on all divisor rays and at $v_0$.

The masses at every fixed interior slope decrease to those of the limiting curve, by the analytic decreasing-slice theorem used in Lemma 25.7.1. Since the masses are monotone in the slope and bounded by $V$, their integrals on compact subintervals converge by bounding them with finite upper and lower sums. Then take the supremum over bounded intervals for the nonnegative deficits. This shows that the energy of the limit is the limit of the energies of $w_{i,k}$, and hence is at least $\limsup_i \mathcal{E}_{\mathrm{tc}}(u_i)$. At each test the infimum in $(i,k)$ of the displayed maximum is $a(v)$, since the original coordinates converge. Thus the limit lies in the energy ball and has the asserted energy inequality. This proves that the ball is closed in the compact product, and order recovery identifies its points. □

## 29.2 Dual energy and approximation in big classes

Keep the normalization of Section 29.1 and write $E=\mathcal{E}_{\mathrm{tc}}$. Put $j(u)=-E(u)$ for a normalized finite-energy potential. Thus $j\ge 0$ and $j(u_*)=0$.

Let $\Xi=\{u_*>-\infty\}$. For a positive Radon measure $\nu$ of mass $V$ carried by $\Xi$, the relative dual energy is

$$D_{\mathrm{rel}}(\nu)=\mathcal{E}^{\vee}_{\alpha,\mathrm{rel}}(\nu)=\sup_{u}\left(E(u)-\int_{\Xi}(u-u_*)\,\mathrm{d}\nu\right). \tag{29.1}$$

The supremum runs over the intrinsic finite-energy potentials. Set $D_{\mathrm{rel}}(\nu)=+\infty$ if $\nu(X^{\beth}\setminus\Xi)>0$. Write $D_{\mathrm{abs}}(\nu)=\mathcal{E}^{\vee}_{\alpha}(\nu)$ for the absolute dual energy. Both objectives are unchanged by adding constants to $u$, since $\nu$ has mass $V$. After normalizing $u$, both deficits $u_*-u$ and $-u_*$ are nonnegative on $\Xi$, and $-u$ is their sum there. Taking suprema gives

$$D_{\mathrm{abs}}(\nu)=D_{\mathrm{rel}}(\nu)+\int_{X^{\beth}}(-u_*)\,\mathrm{d}\nu. \tag{29.2}$$

If $\nu$ charges $X^{\beth} \setminus \Xi$, the same identity holds with both sides infinite by testing at $u_*$.

We use monotone integration for relative differences. If nonpositive potentials $u_i$ decrease pointwise to $u$ and $\nu$ is carried by $\Xi$, then the integrals of $u_* - u_i$ increase to that of $u_* - u$ on $\Xi$. Indeed, restrict first to the compact sets $K_m = \{u_* \geq -m\}$. On $K_m$ the increasing net of nonnegative lower semicontinuous functions $-u_i$ satisfies Theorem 4.2.1, and $\int_{K_m} u_* \, \mathrm{d}\nu$ is finite. Take the supremum over $m$ and interchange the two suprema, using $u_i, u \leq u_*$ and $\bigcup_m K_m = \Xi$.

**Lemma 29.2.1** *There is a constant $C_n$ such that for every normalized intrinsic finite-energy potential $u$ and every $0 \leq a \leq 1$, the obstacle $(1-a)u_* + au$ has a normalized finite-energy envelope $q$ with*

$$j(q) \leq C_n a^2 j(u). \tag{29.3}$$

***Proof*** The endpoint cases $a = 0$ and $a = 1$ are immediate. First take $u$ to be a normalized envelope of a continuous obstacle, so its ordinary ray $U = U^u$ has a relative logarithmic lower bound by Lemma 29.1.1. For $0 < a < 1$, apply Lemma 29.1.2 to the potential $H = (1-a)p_X^*\varphi + aU$, and let $W$ denote the greatest ray given there. Linearity of the generic Lelong numbers and equality of the vertical numbers of $H$ and $W$ give $J_W(v) = (1-a)u_*(v) + au(v)$ on $\mathcal{T}$. By Lemma 29.1.3, $J_W \leq (1-a)u_* + au$ everywhere. The greatest minorant $q$ therefore exists by Proposition 28.5.1. It is normalized since $J_W(v_0) = 0$, and its energy is at least that of $J_W$.

For a time $s > 0$ write

$$A_\ell(s) = \int_X (\varphi - U_s)\langle(\theta + dd^c U_s)^\ell (\theta + dd^c \varphi)^{n-\ell}\rangle, \qquad 0 \leq \ell \leq n.$$

These numbers are nonnegative, and their sum is $(n+1)sj(u)$ by the relative energy and ray formulas. By the cutoff formula, $\sup_X U_s = 0$ [Xia26b, Corollary 9.2.2]. The reference measure with $\ell = 0$ has bounded density by [Xia26a, Corollary 16.2.32]. The normalized $L^1$ bound [Xia26a, Proposition 14.2.1] gives $A_0(s) \leq C_0$ for all $s > 0$, since $0 \leq \varphi - U_s \leq -U_s$. Expanding the mixed terms of the relative energy of $H_s = (1-a)\varphi + aU_s$ gives

$$-E_\theta^\varphi(H_s) = \frac{a}{n+1} \sum_{m=0}^{n} \sum_{\ell=0}^{m} \binom{m}{\ell} a^\ell (1-a)^{m-\ell} A_\ell(s) \leq aC_0 + C_n a^2 s j(u).$$

Indeed the coefficients for $\ell = 0$ have sum at most $a$, and for $\ell \geq 1$ each has two factors of $a$. Monotonicity gives the same upper bound for $-E_\theta^\varphi(W_s)$, since $W_s \geq H_s$. Divide by $s$, apply the relative ray formula, and let $s$ tend to infinity. This proves the bound for $J_W$, hence for $q$.

For a general normalized $u$, take decreasing continuous majorants $g_i$ with limit $u$. The envelopes of $\min(g_i, 0)$ are normalized: they dominate $u$ and are at most zero. They decrease pointwise to $u$, being squeezed between $u$ and the obstacles, and their energies converge by Proposition 28.5.1. Apply the bound to their convex combinations with $u_*$. The obstacle combinations decrease to $(1-a)u_* + au$. By

decreasing continuity of relaxed energy the limit has finite energy with the same bound. Its greatest minorant dominates $u$, since $u \le u_*$, and therefore is normalized. This proves (29.3). □

**Proposition 29.2.1** *Let $\nu$ be a positive Radon measure of mass $V$. Take $b = 0$ or $b = u_*$, with $D_b$ denoting the absolute or relative dual energy, respectively. Write $\Xi_b = \{b > -\infty\}$. Then $D_b(\nu) < \infty$ if and only if $\nu$ is carried by $\Xi_b$ and every finite-energy potential has integrable difference $u - b$ there. If $D_b(\nu) < \infty$, then*

$$\int_{\Xi_b} (b-u)\,\mathrm{d}\nu \le \bigl(D_b(\nu) + C_n\bigr)\sqrt{1+j(u)} \tag{29.4}$$

*for normalized $u$. Hence every normalized maximizing net for either objective lies in one energy ball on a tail.*

***Proof*** Test at $u_*$ to obtain $D_b(\nu) \ge 0$. Finiteness implies the asserted concentration and integrability, since each test has finite energy and $u - b$ is bounded above on $\Xi_b$. For $a = (1 + j(u))^{-1/2}$ take the envelope $q$ in Lemma 29.2.1. Since $b \ge u_*$, one has $b - q \ge a(b - u)$ on $\Xi_b$. Testing at $q$ proves (29.4). It bounds the objective above by $-j(u) + (D_b(\nu) + C_n)\sqrt{1 + j(u)}$, which tends to $-\infty$ as $j(u) \to \infty$.

Conversely, assume concentration and integrability as in the statement. The integrals of $b - u$ must be bounded on the normalized ball $j \le 1$. Otherwise choose $u_k$ in it with $\int_{\Xi_b}(b - u_k)\,\mathrm{d}\nu \ge 2^k$, and let

$$f_m = \sum_{k=1}^{m} 2^{-k} u_k + 2^{-m} u_*.$$

They decrease since $u_{m+1} \le u_*$. Concavity of relaxed energy gives $\overline{G}_\alpha^{\mathrm{big}}(f_m) \ge -1$. By Proposition 28.5.1 the limit has a greatest finite-energy minorant $w$. On $\Xi_b$ it satisfies $b - w \ge \sum_{k=1}^{m} 2^{-k}(b - u_k)$ for each $m$, which contradicts integrability. Write $B$ for the bound on the ball and apply Lemma 29.2.1 with $a = (1 + C_n j(u))^{-1/2}$. Its normalized envelope $q$ lies in that ball and $b - q \ge a(b - u)$. Thus $\int_{\Xi_b}(b - u)\,\mathrm{d}\nu \le B\sqrt{1 + C_n j(u)}$. The supremum of the objective is finite. □

**Corollary 29.2.1** *Fix a Kähler form $\eta$ such that $\theta + \eta$ is Kähler. At each flag-model test $\nu$, choose a lower bound $-B_\nu \le 0$ for the normalized potentials of the Kähler cone in $[\theta + \eta]$, as in Lemma 23.5.1. If $\nu = \sum_{r \ge 1} m_r \delta_{\nu_r}$ has mass $V$, $m_r \ge 0$, and $\sum_r m_r B_{\nu_r} < \infty$, the big-class variational supremum is attained and there is an intrinsic finite-energy solution of $\mathrm{MA}_\alpha(u) = \nu$. Any weak cluster point of a normalized maximizing net is a maximizer. Along each weakly convergent maximizing subnet the energies converge to the energy of its limit.*

***Proof*** A normalized compatible family belongs to the Kähler cone in $[\theta + \eta]$ at this coordinate by jet independence and Lemma 25.12.1. Thus the chosen bounds control all the terms of the integral on the entire normalized finite-energy cone in $\alpha$. It follows

that the dual supremum is finite and that the integral is weakly continuous on each energy ball, as a uniform limit of its finite partial sums. A normalized maximizing net stays in one energy ball by Proposition 29.2.1. Take a weak cluster point using Theorem 29.1.1. Upper semicontinuity of the energy and continuity of the integral give attainment. The equation at a maximizer follows from Proposition 28.5.3. For a convergent maximizing subnet, add its convergent integral to the objective. The limit of the energies is then the dual energy plus the integral of the limit, which is the energy of that maximizer. □

### 29.2.1 Approximation for the absolute dual energy

If $f$ is continuous and $\lambda = \mu_\alpha^{\mathrm{big}}(f)$, then Corollary 28.4.1 and contact give

$$\mathcal{E}_\alpha^\vee(\lambda) = G_\alpha^{\mathrm{big}}(f) - \int f \, \mathrm{d}\lambda. \tag{29.5}$$

Indeed, the supporting inequality bounds the dual objective above by its value at $P_\alpha(f)$, and on the contact set that potential equals $f$. Here strong convergence of measures means weak convergence together with convergence of $D_{\mathrm{abs}}$.

**Theorem 29.2.1** *Let $\nu$ have mass $V$ and finite absolute dual energy. There is a sequence of measures of continuous-obstacle envelopes converging to $\nu$ in total variation and in absolute dual energy. Measures with finite support in the flag-model tests are strongly dense among measures of mass $V$ and finite absolute dual energy.*

***Proof*** Write $F(f) = G_\alpha^{\mathrm{big}}(f) - \int f \, \mathrm{d}\nu$ for continuous $f$. The dual energy is the supremum of $F$. To see that it bounds all these values, test the dual at $P_\alpha(f)$, which is below $f$ and has energy $G_\alpha^{\mathrm{big}}(f)$. Conversely, for each intrinsic finite-energy potential use a decreasing net of continuous majorants. Their obstacle energies decrease to its energy by Proposition 28.5.2; their integrals decrease to its integral by Theorem 4.2.1 after subtracting a common upper bound on a tail. This proves the reverse inequality. Uniform model density and continuity of $F$ show that the same supremum is obtained with real model obstacles.

Write $S = \sup F$. Given $\varepsilon > 0$, choose $g$ with $F(g) \geq S - \varepsilon$. Put $R = 1 + \|g\|_\infty$ and $k = \varepsilon / R$. Set $g_0 = g$, and for each $j \geq 0$ use the closed set

$$A_j = \{f : F(f) \geq F(g_j) + k\|f - g_j\|_\infty\}.$$

Choose $g_{j+1} \in A_j$ with value within $2^{-j}\varepsilon$ of the supremum on $A_j$. Each $A_j$ contains $g_j$ and $A_{j+1} \subset A_j$ by the triangle inequality. Summing the distance inequalities gives

$$\sum_{j \geq 0} \|g_{j+1} - g_j\|_\infty \leq (S - F(g))/k \leq R.$$

The sequence converges uniformly to some $h \in \bigcap_j A_j$. If $F(f) \geq F(h)+k\|f-h\|_\infty$, then $f \in A_j$ for each $j$ by the triangle inequality. The choice of $g_{j+1}$ and continuity of $F$ therefore give $F(f) \leq F(h)$, so $f = h$. It follows that for every continuous $b$ and real $t$ one has $F(h+tb) - F(h) \leq k|t|\|b\|_\infty$.

Use the first variation of $G_\alpha^{\mathrm{big}}$ for both signs of $t$, and set $\lambda = \mu_\alpha^{\mathrm{big}}(h)$. The bound on the derivative for every $b$ gives $\|\lambda - \nu\|_{\mathrm{TV}} \leq k$. Since $F(g) \leq F(h) \leq S$ and $\|h\|_\infty \leq 2R$, (29.5) implies

$$|\mathcal{E}_\alpha^\vee(\lambda) - \mathcal{E}_\alpha^\vee(\nu)| \leq |F(h) - S| + \|h\|_\infty \|\lambda - \nu\|_{\mathrm{TV}} \leq 3\varepsilon.$$

Taking successive $\varepsilon$ tending to zero proves the assertion about continuous obstacles.

For fixed $h$, take real model functions uniformly approximating $h$. Their measures converge weakly by Theorem 28.2.1, and their dual energies converge by (29.5), using the common mass and uniform approximation in its integral. Represent each on a smooth equivariant refinement as in the proof of Corollary 28.6.1. Then (28.8) gives finite support in the tests of that flag model by Lemma 28.6.1. Given finitely many continuous tests and a positive tolerance, first approximate $\nu$ by the measure of a continuous obstacle, then approximate that obstacle by a model function to the requested accuracy for the tests and the dual value. This proves strong density, with a net indexed by the finite families of tests and positive tolerances. □

## 29.3 Rooftops and uniformly integrable energy balls

Fix the big class $\alpha$ and retain the reference $\varphi = V_\theta$, its jet $u_* = P_\alpha(0)$, and the tests $\mathcal{T}$ of Section 29.1. Write $E(u) = \mathcal{E}_{\mathrm{tc}}(u)$ in this section, and $U^u$ for the ordinary relative ray in $\theta$ of a finite-energy potential $u$, as in Section 29.1. Set $u \wedge v = P_\alpha(\min(u, v))$ whenever the obstacle has finite relaxed energy; this is the *rooftop* of $u$ and $v$. At an analytic time slice, write $P_\theta(\min(a, b))$ for the complex envelope. We use the analytic $d_1$ without dividing by the mass, as in [Xia26b, Definition 4.3.1].

**Theorem 29.3.1** *For all intrinsic finite-energy potentials $u, v$, their rooftop $u \wedge v$ exists and has finite energy. If $u, v \leq 0$, then*

$$E(u \wedge v) \geq E(u) + E(v). \tag{29.6}$$

*The function*

$$d_{1,\alpha}(u, v) = E(u) + E(v) - 2E(u \wedge v) = \lim_{s\to\infty} s^{-1} d_1(U_s^u, U_s^v) \tag{29.7}$$

*is a metric on the intrinsic finite-energy domain.*

***Proof*** First take $u, v \leq 0$ that are envelopes of continuous obstacles. Write $U = U^u$ and $Q = U^v$. They lie below $\varphi$ and have a common lower barrier $\varphi - Ls$, by Lemma 29.1.1. Each ray is the greatest circle-invariant disk potential below $p_X^*\varphi$

with vertical orders at least its own. To see this, saturate the ray by Lemma 29.1.2. The saturated ray has the same jet on $\mathcal{T}$, hence the same compatible family by order recovery. Legendre duality then identifies the rays.

At time $T > 0$ put $\psi_T = P_\theta(\min(U_T, Q_T))$. It satisfies $\varphi - LT \le \psi_T \le \varphi$. Let $H^T$ be the Perron segment from $\varphi$ to $\psi_T$ on $[0,T]$, as in [Xia26a, Theorem 17.6.8]. Endpoint comparison with $U$ and $Q$ gives $H^T_s \le \min(U_s, Q_s)$ for $s \le T$. Also $H^T_s \ge \varphi - Ls$ by the affine competitor. If $T' > T$, then $H^{T'}_T \le \psi_T$ by the envelope property at that time. The same endpoint comparison gives $H^{T'}_s \le H^T_s$ for $s \le T$. Thus the segments have a decreasing limit $H$ on every bounded time interval. The bounds give a nontrivial circle-invariant psh potential, extending over the central fibre by its common upper bound.

This limit is the greatest circle-invariant disk potential below both rays. Indeed, if $B$ is a competitor, take its maximum with $\varphi - Ls$. It remains below both rays, tends to $\varphi$ in $L^1$ at time zero, and its value at $T$ is at most $\psi_T$. Its interior time slices are $L^1$-continuous: off the pole set of $\varphi$ they are finite convex functions of time and their differences from $\varphi$ are uniformly bounded on bounded intervals. Hence the clipped function is a Perron competitor on $[0,T]$. Letting $T$ increase gives the assertion. Saturate $H$. Its vertical orders remain at least those of $U$ and $Q$, since $H \le U, Q$. The two saturation descriptions imply that the saturation of $H$ still lies below both. By maximality it equals $H$. In particular $J_H$ is a finite-energy compatible potential below $u, v$ by test order. If $r$ is any finite-energy compatible minorant of $u, v$, its ray is below $p_X^*\varphi$ and satisfies both sets of vertical constraints. Thus its ray is below $H$, and testing gives $r \le J_H$. This proves $J_H = u \wedge v$.

At time $T$, the triangle inequality for $d_1$ with middle point $\varphi$ gives

$$E^\varphi_\theta(\psi_T) \ge E^\varphi_\theta(U_T) + E^\varphi_\theta(Q_T).$$

Indeed the distances to $\varphi$ are the negatives of the energies, while the distance between $U_T$ and $Q_T$ is the sum of their energies minus twice the rooftop energy [Xia26b, Definition 4.3.1 and Lemma 4.3.1]. For a fixed $s > 0$, affinity along $H^T$ [Xia26a, Theorem 17.7.3] and decreasing continuity [Xia26b, Proposition 3.1.20] give

$$E^\varphi_\theta(H_s) = s \lim_{T\to\infty} T^{-1} E^\varphi_\theta(\psi_T).$$

The two inputs have time energies $TE(u)$ and $TE(v)$. The ray formula for $H$ yields (29.6). Substituting the same limit into the slice formula for $d_1(U_T, Q_T)$ gives (29.7).

For arbitrary $u, v \le 0$, take decreasing continuous majorants clipped above by zero, and their envelopes $u_i, v_i$, with the two families on one product directed set. These decrease to $u, v$ pointwise, since each lies between its limit and its obstacle, and their energies converge by Proposition 28.5.1. The relaxed energies of $\min(u_i, v_i)$ have a common finite lower bound by the inequality just proved and decrease to the relaxed energy of $\min(u, v)$. This proves existence and (29.6) in general. The ordinary curves are ordered when their potentials are ordered, by the divisor-ray recovery, so their Legendre rays are ordered as well. The radial distances from $U^{u_i}$ to $U^u$, and from $U^{v_i}$ to $U^v$, are therefore the corresponding energy differences [Xia26b,

Proposition 4.3.6]. They tend to zero. The triangle inequality for the radial distance and decreasing continuity for the rooftop energy give (29.7) in the limit.

A common constant translation reduces arbitrary $u, v$ to the nonpositive case. The rooftop translates by the same constant. On the analytic side the rays translate by that constant times the time, and the slice $d_1$ is unchanged by equal translations. Finally the radial distance is a metric by [Xia26b, Theorem 4.3.4], and the rays distinguish the compatible potentials by Legendre inversion. □

**Corollary 29.3.1** *The metric $d_{1,\alpha}$ is complete, and taking the rooftop with a fixed potential is a contraction.*

***Proof*** For any $u, a$ in the finite-energy domain, put $\psi_s = P_\theta(\min(U_s^u, U_s^a))$. The ray of $u \wedge a$ is at most $\psi_s$ at time $s$, by order of the rays. Its $d_1$-distance from $\psi_s$ is therefore the ordered energy difference. By (29.7) and the formula for the slice metric,

$$\lim_{s\to\infty} s^{-1} d_1(U_s^{u\wedge a}, \psi_s) = 0.$$

Apply the analytic rooftop contraction [Xia26b, Proposition 4.3.1] with third entry $U_s^a$. Dividing by $s$ and using the triangle inequality on both sides gives $d_{1,\alpha}(u \wedge a, v \wedge a) \le d_{1,\alpha}(u, v)$.

Take a Cauchy sequence and choose a subsequence $(u_k)$ with $d_{1,\alpha}(u_k, u_{k+1}) \le 2^{-k}$. The cutoffs are bounded above. Indeed the estimate [Xia26b, Lemma 4.3.3 and Corollary 9.2.2] gives $Vsu_k(v_0) \le Ad_1(\varphi, U_s^{u_k}) + B$ for constants $A, B$ independent of $k, s$. Divide by $s$ and use the bounded radial distances to $u_*$.

For $m \ge k$ let $w_{k,m}$ be the rooftop of $u_k, \dots, u_m$. Contraction bounds the distance from $w_{k,m}$ to $w_{k,m+1}$ by that from $u_m$ to $u_{m+1}$. Thus $d_{1,\alpha}(u_k, w_{k,m}) \le 2^{1-k}$. Since $w_{k,m} \le u_k$, this distance is the energy difference. The finite minimum obstacles decrease, and their energies are bounded below. By Proposition 28.5.1, their infimum has an envelope $w_k$, and $d_{1,\alpha}(u_k, w_k) \le 2^{1-k}$ by the ordered energy difference again.

The $w_k$ increase, with cutoffs bounded above by those of $u_k$. Their ordinary curves have a regularized supremum in the fixed class $\theta$, with cutoff the supremum of the cutoffs, by [Xia26b, Proposition 9.4.3]. Its left limit is $\varphi$, since every curve has this left limit. At each slope below the limiting cutoff, the slice masses on a tail increase to the mass of the resulting slice by [Xia26b, Corollary 2.4.1]. Extend the masses by zero at and above each cutoff. Their deficits from $V$ are dominated by the deficit of the first curve, integrated up to one common upper bound for the cutoffs. This bound is integrable by the finite-energy formula. Thus the new curve has finite energy, and the energies of $w_k$ tend to its energy. Write $w$ for its compatible potential. Since $w_k \le w$, the ordered distance $d_{1,\alpha}(w_k, w)$ tends to zero. Therefore $u_k$ tends to $w$, and so does the original Cauchy sequence. □

**Theorem 29.3.2** *For any positive Radon measure $\nu$ of mass $V$ and finite absolute dual energy, each normalized energy ball of intrinsic finite-energy potentials*

$$\mathcal{B}_R = \{u : u(v_0) = 0,\ -E(u) \le R\} \qquad (R \ge 0)$$

*is uniformly integrable against $\nu$. In particular, for Borel sets $A$, the integrals $\int_A (-u)\,\mathrm{d}\nu$ tend uniformly to zero on $\mathcal{B}_R$ as $\nu(A) \to 0$.*

***Proof*** Suppose this last uniform absolute continuity fails. There are $\varepsilon > 0$, Borel sets of arbitrarily small measure, and corresponding $u \in \mathcal{B}_R$ with $\int_A (-u)\,\mathrm{d}\nu \ge \varepsilon$. Choose a sequence of such pairs $(u_k, A_k)$. After choosing each pair, choose $\delta_k > 0$ so that $\int_B (-u_k)\,\mathrm{d}\nu \le \varepsilon/2$ whenever $\nu(B) \le \delta_k$. This is possible by integrability. At stage $j$ impose $\nu(A_j) \le 2^{k-j}\delta_k$ for all $k < j$. These are finitely many conditions. The sets $B_k = A_k \setminus \bigcup_{j>k} A_j$ are disjoint and satisfy $\int_{B_k} (-u_k)\,\mathrm{d}\nu \ge \varepsilon/2$, since the union after $k$ has measure at most $\delta_k$.

Set $a_k = 1/k$ and let $q_k$ be the envelope of $(1 - a_k)u_* + a_k u_k$. By Lemma 29.2.1, $q_k \le a_k u_k \le 0$ and $-E(q_k) \le C_n a_k^2 R$. A potential is below a finite minimum if and only if it is below each entry. Thus iterating (29.6) gives envelopes of $\min(q_1, \dots, q_m)$ with energy at least $-C_n R \sum_k a_k^2$. The minimum obstacles decrease, so Proposition 28.5.1 gives a finite-energy minorant $w$ of their infimum. On the disjoint sets $B_k$ one has $-w \ge a_k(-u_k)$. Countable additivity gives $\int (-w)\,\mathrm{d}\nu \ge (\varepsilon/2)\sum_k a_k = +\infty$. This contradicts finite absolute dual energy.

The coercive estimate (29.4) bounds the integrals of $-u$ on $\mathcal{B}_R$. Thus the sets $\{-u > M\}$ have uniformly small measure as $M \to \infty$. The uniform absolute continuity just proved then bounds the integrals on those sets uniformly by quantities tending to zero. This is uniform integrability. □

**Corollary 29.3.2** *Let $\nu$ be a positive Radon measure of mass $V$ and finite absolute dual energy carried by a countable subset of $\mathcal{T}$. Then its intrinsic variational supremum is attained, and it is the Monge–Ampère measure of an intrinsic finite-energy potential. Every weak cluster point of a normalized maximizing net is a maximizer, and along any weakly convergent maximizing subnet the energies converge to the energy of its limit.*

***Proof*** On a normalized energy ball, integration against each finite subsum of the atomic measure is weakly continuous by definition of the test topology. The tails have mass tending to zero and their integrals tend uniformly to zero on the ball by Theorem 29.3.2. Hence the full integral is weakly continuous there. By Proposition 29.2.1 the normalized maximizing net lies in one energy ball on a tail. Use Theorem 29.1.1 and upper semicontinuity of the energy to get a maximizer at every weak cluster point. The equation follows from Proposition 28.5.3. Along a maximizing subnet converging to $u$, the objective converges to the dual value and the integral converges to $\int u\,\mathrm{d}\nu$, so the energy converges to the energy of $u$. □

This contains Corollary 29.2.1. Flag-model tests lie in $\mathcal{T}$, since they are tests of central components of smooth refinements (see the proof of Corollary 28.6.1), and

the summability hypothesis of Corollary 29.2.1 gives finite absolute dual energy, by its proof. The present corollary also applies to every measure of mass $V$ with finite support in $\mathcal{T}$. Indeed each $V\delta_v$ with $v \in \mathcal{T}$ has finite absolute dual energy by Lemma 29.5.1, and the absolute dual energy is convex in the measure, being a supremum of affine functions.

**Corollary 29.3.3** *Fix a measure $\sigma$ of mass $V$ and finite absolute dual energy. On the positive Radon measures of mass $V$ dominated by a fixed multiple of $\sigma$, the absolute dual energy is continuous for the total variation norm.*

***Proof*** Suppose $\nu \le C\sigma$, where $C$ is fixed. It has finite absolute dual energy by the integrability criterion in Proposition 29.2.1. For normalized $u$, its objective is bounded above by $-j(u) + C\int(-u)\,\mathrm{d}\sigma$. The coercive estimate for $\sigma$ makes this negative for $j(u) > R$ with one common $R$. Testing at $u_*$ gives a nonnegative value. Thus all these suprema may be taken on $\mathcal{B}_R$. For two such measures $\nu, \lambda$,

$$|\mathcal{E}^\vee_\alpha(\nu) - \mathcal{E}^\vee_\alpha(\lambda)| \le \sup_{u\in\mathcal{B}_R} \int (-u)\,\mathrm{d}|\nu - \lambda|.$$

For any $M > 0$ split $-u$ at level $M$. The bounded part contributes at most $M\|\nu-\lambda\|_{\mathrm{TV}}$. Since $|\nu - \lambda| \le 2C\sigma$, the other part is at most $2C\sup_{u\in\mathcal{B}_R}\int_{\{-u>M\}}(-u)\,\mathrm{d}\sigma$. First choose $M$ using uniform integrability, then let the total variation distance go to zero. This proves continuity. □

## 29.4 Variational attainment in a big class

Retain $\alpha, \theta, \varphi, u_*$ and the tests $\mathcal{T}$ from Section 29.1. Write $E(u) = \mathcal{E}_{\mathrm{tc}}(u)$ for intrinsic energy and $\mathscr{E} = E^\varphi_\theta$ for analytic relative energy. All energies and measures retain the mass $V$. As in Section 29.3, $U^u$ denotes the relative ray of a finite-energy potential $u$, and $d_{1,\alpha}$ is the complete rooftop distance.

### 29.4.1 Ordered extrapolation and integration

**Lemma 29.4.1** *Let $u, v$ be nonpositive finite-energy potentials whose rays have lower bounds of the form $\varphi - Ls$. For $0 < a < 1$, the saturation of $(1-a)U^u + aU^v$ has jet equal to $(1-a)u + av$ on $X^{\sqsupset}$.*

***Proof*** The mixed ray has a common relative logarithmic lower bound and lies below $p_X^*\varphi$, so Lemma 29.1.2 applies. Additivity of the vertical generic Lelong numbers makes its saturated jet equal to $(1-a)u + av$ on $\mathcal{T}$. To pass to the full space choose one Kähler perturbation of $\theta$. The jet of every compatible family lies in its decreasing cone by comparison in Lemma 25.12.1. This cone is closed under pointwise convex

combinations: take decreasing continuous semipositive approximants to both inputs on the product directed set and take their convex combinations. The continuous semipositive cone is convex by the common-refinement description of model positivity and uniform approximation. Thus both the saturated jet and the pointwise combination belong to the Kähler decreasing cone. Their equality on divisor rays and at $v_0$ gives equality everywhere by Theorems 25.11.1 and 25.12.1. □

**Lemma 29.4.2** *Let $w \le u \le 0$ be intrinsic finite-energy potentials whose rays have relative logarithmic lower bounds. For $a > 1$ there is a finite-energy potential $r \le w$ with*

$$a^{-1}r + (1-a^{-1})u \le w, \qquad E(u) - E(r) \le (n+1)a^{n+1}\big(E(u) - E(w)\big). \tag{29.8}$$

***Proof*** Write $U = U^u$ and $W = U^w$. By order of their ordinary curves and Legendre transformation, $W_s \le U_s \le \varphi$. Choose $L$ with $W_s \ge \varphi - Ls$. At time $T$, let $\psi_T$ be the largest $\theta$-psh potential with $a^{-1}\psi_T + (1-a^{-1})U_T \le W_T$. The barrier $\varphi - aLT$ is a competitor. The ordered extrapolation of [Xia26a, Lemma 16.4.3] identifies $\psi_T$ with the envelope of $aW_T - (a-1)U_T$, evaluated off the common pole set. Its measure is carried by the contact set and there is bounded by $a^n$ times the measure of $W_T$. On the contact set $U_T - \psi_T = a(U_T - W_T)$. The ordered energy difference formula [Xia26a, Proposition 17.2.3] and the concavity of energy therefore give

$$\begin{aligned}
\mathscr{E}(U_T) - \mathscr{E}(\psi_T) &\le \int_X (U_T - \psi_T)\,\mathrm{d}\mu_{\psi_T} \\
&\le a^{n+1}\int_X (U_T - W_T)\,\mathrm{d}\mu_{W_T} \\
&\le (n+1)a^{n+1}\big(\mathscr{E}(U_T) - \mathscr{E}(W_T)\big),
\end{aligned}$$

where $\mu$ denotes the analytic time-slice measure.

Let $H^T$ be the Perron segment from $\varphi$ to $\psi_T$ on $[0,T]$. The subgeodesic $a^{-1}H^T + (1-a^{-1})U$ has endpoints dominated by those of $W$. It lies below $W$ by the Perron description [Xia26a, Theorem 17.6.8]. In particular $H^T \le W$, and $H^T_s \ge \varphi - aLs$ by the affine competitor. For $T' > T$ the inequality with $W_T$ at time $T$ puts $H^{T'}_T$ below $\psi_T$. Segment comparison gives $H^{T'}_s \le H^T_s$ for $s \le T$. The decreasing limit $H$ is a radial disk potential with the same bounds, using bounded-above extension over the central fibre. It satisfies $a^{-1}H + (1-a^{-1})U \le W$.

Saturate $H$, and let $r$ be the resulting jet. The disk inequality yields $a^{-1}r(v) + (1-a^{-1})u(v) \le w(v)$ on $\mathcal{T}$, since saturation preserves the vertical orders. The combination on the left is itself an intrinsic potential by Lemma 29.4.1. Test order gives the inequality everywhere. Similarly $r \le w$, since $H \le W$. Affinity on the segments and decreasing continuity give $\mathscr{E}(H_s) = s\lim_T T^{-1}\mathscr{E}(\psi_T)$. The saturated ray dominates $H$, so its time energy is at least this value. Divide the slice estimate by $T$ and use $\mathscr{E}(U_T) = TE(u)$ and $\mathscr{E}(W_T) = TE(w)$. The relative ray formula then gives the claimed energy bound for $r$. □

**Proposition 29.4.1** *Let $\nu$ be a positive Radon measure of mass $V$ and finite relative dual energy $D = D_{\mathrm{rel}}(\nu)$. For $u, v \le 0$ of finite energy with $-E(u), -E(v) \le B$, one has*

$$\int_{\Xi} |u - v| \, \mathrm{d}\nu \le C_{n,B,D} \, d_{1,\alpha}(u, v)^{1/(n+1)}. \tag{29.9}$$

*In particular the map $u \mapsto u - u_*$ into $L^1(\Xi, \nu)$ is continuous for the finite-energy metric.*

***Proof*** First take $w \le u \le 0$ with relative logarithmic lower bounds. Write $\delta = E(u) - E(w)$. For $a > 1$, test the dual energy at $r$ from Lemma 29.4.2. Since $u \le u_*$, the two inequalities there give

$$a \int_{\Xi} (u - w) \, \mathrm{d}\nu \le \int_{\Xi} (u - r) \, \mathrm{d}\nu \le D - E(u) + (n + 1) a^{n+1} \delta.$$

All differences are evaluated where the potentials are finite, a set of full $\nu$-measure by the integrability criterion. For general ordered potentials choose decreasing continuous majorants $g_i$ of $u$ and $h_i$ of $w$ on a joint directed set. The envelopes of $\min(g_i, 0)$ and $\min(g_i, h_i, 0)$ form ordered approximations to $u, w$. They have relative logarithmic lower bounds by Lemma 29.1.1 and decrease pointwise to the respective limits. Energies converge by decreasing continuity of relaxed energy. Their differences from $u_*$ converge in integral by the monotone integration rule following (29.2). Thus the displayed bound holds for all ordered pairs and every fixed $a > 1$. If $0 < \delta < 1$, take $a = \delta^{-1/(n+1)}$. For $\delta = 0$ let $a$ increase to infinity. For $\delta \ge 1$ use $\int_{\Xi} (u_* - w) \, \mathrm{d}\nu \le D - E(w)$. On energy-bounded nonpositive data this gives the claimed power estimate in the ordered case.

For two potentials $u, v \le 0$, take $w = u \wedge v$. Then $-E(w) \le 2B$ by (29.6). The ordered estimate for $w \le u$ and $w \le v$ gives (29.9), since both ordered distances are bounded by $d_{1,\alpha}(u, v)$ and their pointwise test differences sum to at least $|u - v|$. For general metric convergence the cutoffs are uniformly bounded above, as in the proof of Corollary 29.3.1, and the energies are bounded by the metric formula. A common downward translation gives a nonpositive energy-bounded family. It preserves the distances and the differences inside the integrals. This proves continuity. □

## 29.4.2 The midpoint estimate

**Lemma 29.4.3** *Let $A, B \le \varphi$ be $\theta$-psh potentials with bounded differences from $\varphi$, and let $P = (A + B)/2$. Write $J = -\mathscr{E}(A) - \mathscr{E}(B)$ and $M = \mathscr{E}(P) - \frac{1}{2}(\mathscr{E}(A) + \mathscr{E}(B))$. With $\gamma = 2^{-n}$ one has*

$$d_1(A, B) \le \int_X |A - B| \, \mathrm{d}\mu_\varphi + C_n J^{1-\gamma} M^{\gamma}. \tag{29.10}$$

***Proof*** For a $\theta$-psh potential $F$ put $T_F = \theta + dd^c F$ and $\mu_F = \langle T_F^n \rangle$. All mixed products below are non-pluripolar. For a product $R$ of $n - 1$ closed positive factors, write

$$Q_R(g,h) = -\int_X g\, dd^c h \wedge R$$

for the pairing of bounded quasi-psh differences. It is symmetric and positive semidefinite by [Xia26a, Theorem 17.1.3 and Lemma 17.1.4], so Cauchy–Schwarz applies. On the common pole set the products give zero mass and weights may be set to zero. For $f = A - B$ put

$$I = \int_X f\, \mathrm{d}(\mu_B - \mu_A) = \sum_{\ell=0}^{n-1} Q_{\langle T_A^{\ell} T_B^{n-1-\ell}\rangle}(f,f) \geq 0.$$

This follows by telescoping the two top powers. Discarding crossed terms and using the energy difference formula from $\varphi$ gives $I \leq (n+1)J$.

Along the segment $A_t = (1-t)A + tB$, write $e(t) = \mathscr{E}(A_t)$. From [Xia26a, Proposition 17.2.3] and multilinearity, $-e''(t) = nQ_{\langle T_{A_t}^{n-1}\rangle}(f,f)$. Thus $M$ is the integral of $-e''$ against the piecewise linear weight $\min(t, 1-t)/2$ on $[0,1]$. On $[1/4, 3/4]$, both coefficients of $T_{A_t}$ are at least $1/4$. Positivity of the mixed squares therefore bounds the integral below by a dimension constant times $I$. Hence $I \leq C_n M$.

To test against the reference, set

$$b_r = Q_{\langle T_\varphi^r T_P^{n-1-r}\rangle}(f,f), \qquad 0 \leq r \leq n-1.$$

Expanding $T_P = (T_A + T_B)/2$ gives $b_0 \leq C_n I$. Define $I(F,G) = \int (F-G)\, \mathrm{d}(\mu_G - \mu_F)$ for two potentials $F, G$ of this type. For $r < n-1$ put $R_0 = \langle T_\varphi^r T_P^{n-2-r}\rangle$ and $h = \varphi - P$. Using $dd^c h = T_\varphi - T_P$ and $dd^c f = T_A - T_B$, one gets

$$b_{r+1} - b_r = Q_{\langle T_A R_0\rangle}(f,h) - Q_{\langle T_B R_0\rangle}(f,h).$$

Here a product with $R_0$ denotes the mixed product with its indicated factors. Since $T_A, T_B \leq 2T_P$, the $f$-squares on the right are bounded by $2b_r$, and the $h$-squares by $2I(\varphi, P)$, by the telescoping expression for $I(\varphi, P)$. As $P \leq \varphi$, the ordered energy formula gives $I(\varphi,P) \leq (n+1)(-\mathscr{E}(P)) \leq (n+1)J/2$. By Cauchy–Schwarz, $b_{r+1} \leq b_r + C_n\sqrt{b_r J}$. Consequently

$$b_r \leq C_n J^{1-2^{-r}} I^{2^{-r}},$$

by induction using $I \leq (n+1)J$. If $J = 0$, all the squares vanish. For $n = 1$ the initial bound is the only one needed.

Let $C = \max(A,B)$ and put $g = C - A$, $k = C - B$. They are bounded quasi-psh differences. Since $gk = 0$, plurifine locality of mixed products [Xia26a, Proposition 15.1.5] gives $Q_R(g,k) = Q_R(k,g) = 0$: on the strict set where one weight is positive, the other has zero Hessian factor by equality of its two defining potentials on that set. Together with $f = k - g$, positivity gives $Q_R(g,g), Q_R(k,k) \leq Q_R(f,f)$. The analytic metric and concavity of energy give

$$d_1(A,B) \le d_1(A,C) + d_1(C,B) \le \int_X g\,\mathrm{d}\mu_A + \int_X k\,\mathrm{d}\mu_B.$$

Telescope $\mu_A - \mu_\varphi$ in the first integral. Each term is a pairing of $g$ with $A - \varphi$ against a product of $T_A, T_\varphi$. The first square is bounded by $C_n b_r$, where $r$ is the number of reference factors, using $T_A \le 2T_P$. The second square is bounded by $I(A,\varphi) \le (n+1)(-\mathscr{E}(A))$. The second integral has the same bounds using $k, B$. Cauchy–Schwarz and the bounds on $b_r$ show that the sum of these errors is at most $C_n J^{1-\gamma} I^\gamma$. The sum of the reference integrals is $\int_X (g+k)\,\mathrm{d}\mu_\varphi = \int_X |A-B|\,\mathrm{d}\mu_\varphi$. Using $I \le C_n M$ proves the estimate. □

**Proposition 29.4.2** *Let $u, v$ be normalized intrinsic finite-energy potentials. For $m = P_\alpha((u+v)/2)$, set*

$$J = -E(u) - E(v), \qquad \mathcal{M} = E(m) - \tfrac{1}{2}(E(u) + E(v)).$$

*Then $m$ has finite energy and $\mathcal{M} \ge 0$. With $\gamma = 2^{-n}$ one has*

$$d_{1,\alpha}(u,v) \le C_n J^{1-\gamma} \mathcal{M}^\gamma. \tag{29.11}$$

***Proof*** First take envelopes of continuous obstacles. Their rays $U, Q$ have relative logarithmic bounds by Lemma 29.1.1, since $u, v \le 0$. Saturate their timewise midpoint $(U+Q)/2$. By Lemma 29.4.1 its jet is the pointwise midpoint, so it gives $m$. The saturated ray dominates the mixed potential. At time $s > 0$, its energy $sE(m)$ thus bounds $\mathscr{E}((U_s + Q_s)/2)$ above. As $\mathscr{E}(U_s) = sE(u)$ and $\mathscr{E}(Q_s) = sE(v)$, the analytic midpoint gap is at most $s\mathcal{M}$. The negative time-slice energies sum to $sJ$. The reference measure has bounded density [Xia26a, Corollary 16.2.32]. The time slices have supremum zero because both cutoffs are zero [Xia26b, Corollary 9.2.2]. Their normalized $L^1$ bound in [Xia26a, Proposition 14.2.1] makes $\int |U_s - Q_s|\,\mathrm{d}\mu_\varphi$ bounded independently of $s$. Apply Lemma 29.4.3, divide by $s$, and pass to the radial distance.

For general $u, v$, use decreasing normalized continuous-obstacle envelopes, on a joint directed set, as in Lemma 29.2.1. They converge in $d_{1,\alpha}$ by the ordered energy formula and decreasing energy continuity. Their midpoints decrease as obstacles to $(u+v)/2$; the relaxed energies have a finite lower bound by concavity. They decrease to $E(m)$ by Proposition 28.5.1. Concavity also gives $\mathcal{M} \ge 0$. Passing to the limit gives the estimate. □

### 29.4.3 Convergence of maximizing nets

**Theorem 29.4.1** *Every positive Radon measure $\nu$ of mass $V$ and finite relative dual energy is the Monge–Ampère measure of an intrinsic finite-energy potential in $\alpha$. The relative dual supremum is attained by a unique normalized potential $u_\nu$. Every normalized maximizing net converges to $u_\nu$ in $d_{1,\alpha}$.*

***Proof*** Write $D = D_{\mathrm{rel}}(\nu)$ and $\Delta_\nu(u) = D - E(u) + \int_\Xi (u - u_*)\,\mathrm{d}\nu$. For two normalized candidates $u, v$ let $m$ be their midpoint envelope. The pointwise inequality $m \le (u+v)/2$ and the definition of $D$ give

$$E(m) - \tfrac{1}{2}(E(u) + E(v)) \le \tfrac{1}{2}(\Delta_\nu(u) + \Delta_\nu(v)).$$

By Proposition 29.2.1, a normalized maximizing net has a uniform energy bound on a tail. Hence (29.11) makes it Cauchy in $d_{1,\alpha}$. Completeness (Corollary 29.3.1) gives a limit $u_\nu$. Completeness for nets follows from sequence completeness by choosing members from tails of successively smaller diameter. Energy is continuous by the metric formula, and integration against $\nu$ is continuous by Proposition 29.4.1. Thus the limit attains $D$. The measure $V\delta_{v_0}$ has absolute dual energy zero: the test-curve formula gives $E(w) \le Vw(v_0)$ for all $w$, with equality at $u_*$. Since $u_*(v_0) = 0$, (29.2) shows that its relative dual energy is zero as well. By Proposition 29.4.1, integration against it is also continuous for the metric, so $u_\nu(v_0) = 0$. If two normalized potentials attain the maximum, the midpoint estimate with zero deficits gives distance zero. This proves uniqueness of the normalized maximizer. For $h \in C^0(X^\beth)$ and $t \in \mathbb{R}$, the obstacle $u_\nu + th$ has a finite-energy minorant $u_\nu - |t|\|h\|_\infty$. Let $q_t$ be its greatest finite-energy minorant. The variational equality at $u_\nu$ and $q_t \le u_\nu + th$ give

$$E(q_t) - E(u_\nu) \le \int_\Xi (q_t - u_\nu)\,\mathrm{d}\nu \le t \int h\,\mathrm{d}\nu.$$

Divide by positive and negative $t$ and use the first variation in Theorem 28.5.1 and Corollary 28.5.1. Then $\mathrm{MA}_\alpha(u_\nu) = \nu$. By (29.2), all maximizing nets and maximizers for a finite absolute dual energy are also those for the relative energy. □

**Corollary 29.4.1** *Let $\nu_i, \nu$ be positive Radon measures of mass $V$ and finite absolute dual energy. If $\nu_i \to \nu$ weakly and $\mathcal{E}^\vee_\alpha(\nu_i) \to \mathcal{E}^\vee_\alpha(\nu)$, then $u_{\nu_i} \to u_\nu$ in $d_{1,\alpha}$. This applies to total variation convergence under fixed domination as in Corollary 29.3.3.*

***Proof*** The dual energies stay bounded on a tail, so Proposition 29.2.1 bounds the energies of $u_{\nu_i}$ there. For the fixed potential $u = u_\nu$ use continuous majorants decreasing to $u$. Weak convergence tested against each majorant, followed by decreasing integration against $\nu$, gives $\limsup_i \int u\,\mathrm{d}\nu_i \le \int u\,\mathrm{d}\nu$. Together with convergence of the dual energies this gives $\limsup_i \Delta_{\nu_i}(u) \le 0$, because by (29.2) each absolute objective has the same deficit as the relative objective. The deficits are nonnegative and $\Delta_{\nu_i}(u_{\nu_i}) = 0$. Apply the midpoint estimate (29.11) to this pair with the deficit bound for $\nu_i$. For dominated total variation convergence, Corollary 29.3.3 supplies the dual-energy convergence, and total variation implies weak convergence. □

## 29.5 The strong topology and continuity of the measure

We retain the notation of Sections 29.1 and 29.4. In particular, $E$ denotes the intrinsic energy in a fixed big class $\alpha$, and the weak topology on potentials uses the local tests $\mathcal{T}$. Strong convergence of potentials means weak convergence together with convergence of their energies.

**Proposition 29.5.1** *The intrinsic finite-energy potentials are closed under pointwise convex combinations and pointwise maxima. Energy is concave on this set.*

***Proof*** First suppose that the rays of both inputs have relative logarithmic lower bounds. Translate both inputs down by a common constant to make their rays lie below the reference. We may add the constant back after forming the combination or maximum. The assertion about a convex combination follows from Lemma 29.4.1. For a maximum, take the maximum of the two relative rays on the disk and saturate it. The vertical orders of the maximum are the smaller of the two vertical orders, by Lemma 22.7.1. Consequently the saturated jet is the pointwise maximum on $\mathcal{T}$. The pointwise maximum belongs to the decreasing cone of one fixed larger Kähler class. Indeed the continuous semipositive cone is stable under finite maxima, by rational model approximation and Lemma 23.1.1; taking decreasing continuous semipositive majorants passes this to the decreasing cone. The saturated jet lies in the same Kähler cone, and equality on the tests gives equality everywhere by Theorem 25.12.1 and order recovery.

For general inputs take two decreasing nets of envelopes of continuous majorants, indexed jointly. After a common downward translation these have relative logarithmic lower bounds by Lemma 29.1.1, and they decrease pointwise to the given potentials. Their energies converge. Their convex combinations and maxima are intrinsic potentials with energies bounded below. For the combinations use concavity of relaxed energy, and for the maxima use monotonicity. Their cutoffs converge to the corresponding combination or maximum of the two original cutoffs, by their values at $v_0$. Subtract these cutoffs to place the nets in one normalized energy ball. At each test their values converge to the normalized pointwise combination or maximum. Weak compactness on that ball gives an intrinsic limit with these values. Add back the limit cutoff. The pointwise combination or maximum belongs to the larger Kähler decreasing cone, since this cone is closed under both operations. Thus test equality identifies it with the intrinsic limit on all of $X^{\beth}$. Energy concavity follows from concavity of relaxed energy on these potential obstacles. □

**Lemma 29.5.1** *For each fixed $v \in \mathcal{T}$, the values $u(v)$ are uniformly bounded below on normalized intrinsic finite-energy potentials. In particular $V\delta_v$ has finite absolute dual energy.*

***Proof*** Fix a smooth local product model $q$ and a vertical prime $F$ representing $v$. Put $b_F = \mathrm{ord}_F(t)$. Let $U = U^u$ for a normalized potential $u$. It has cutoff zero, so its slices have supremum zero [Xia26b, Corollary 9.2.2]. They have a common $L^1(X)$ bound for a smooth fixed volume form by [Xia26a, Proposition 14.2.1]. Fubini therefore bounds the $L^1$ norm of $U$ on each product chart over a fixed smaller parameter disk, uniformly in $u$. The central fibre has product volume zero, so the values there are immaterial to this bound.

Choose a coordinate ball $B$ of radius $r_0$ centered at a point $p$ of $F$, with closure $\overline{B}$ in the chart, and add a smooth local potential for the pulled-back background to $U \circ q$ to obtain a psh function $\Psi$ near $\overline{B}$. It has a fixed upper bound $C$ on $\overline{B}$ since $U \le 0$. The logarithmic growth estimate [Xia26a, Proposition 12.1.1, (12.2)] gives $\Psi(z) \le C + \nu_p(\Psi)\log(|z-p|^2/r_0^2)$ for $0 < |z-p| < r_0$. Hence $\Psi \le C - \nu_p(\Psi)\log 4$ on the punctured concentric ball of radius $r_0/2$. Within the half ball choose one smaller open ball in the biholomorphic locus of $q$ off the central fibre. Its image has fixed positive product volume. After subtracting the fixed smooth local potential, the same estimate, with $C$ replaced by a larger fixed constant, holds for $U$ on that image. If $\nu_p(\Psi)$ were unbounded, this would contradict the uniform product $L^1$ bound. Hence $\nu_p(\Psi)$ is uniformly bounded and so is $\nu_F(U) \le \nu_p(\Psi)$. The test formula Lemma 25.11.1 gives $u(v) = -\nu_F(U)/b_F$, proving the lower bound.

For $V\delta_v$, the dual objective on normalized potentials is $E(u) - Vu(v)$, which is bounded above by the test bound and $E(u) \le 0$. The objective is invariant under constant shifts since the mass is $V$. This proves finite absolute dual energy on the whole domain. □

**Theorem 29.5.1** *For nets of intrinsic finite-energy potentials, convergence in $d_{1,\alpha}$ is equivalent to strong convergence.*

***Proof*** The metric formula (29.7) gives $|E(u) - E(v)| \le d_{1,\alpha}(u,v)$ by monotonicity of energy below both inputs. By Lemma 29.5.1, $D_{\mathrm{abs}}(V\delta_v) < \infty$ for every $v \in \mathcal{T}$. Since $u_*(v) > -\infty$, (29.2) gives $D_{\mathrm{rel}}(V\delta_v) < \infty$, so Proposition 29.4.1 applies to $V\delta_v$. Hence evaluation at $v$ is also continuous in $d_{1,\alpha}$. This proves one direction.

Suppose $u_i$ converges strongly to $u$. First take $u_i, u$ normalized. By Proposition 29.5.1, the midpoints $m_i = (u_i + u)/2$ are intrinsic potentials. Their energies are bounded below by $(E(u_i) + E(u))/2$, and their test values converge to those of $u$. Upper semicontinuity on a normalized energy ball gives $\limsup_i E(m_i) \le E(u)$. Thus their midpoint defects tend to zero and (29.11) gives $d_{1,\alpha}(u_i, u) \to 0$. For unnormalized data, the cutoffs converge since $v_0 \in \mathcal{T}$. Normalize by subtracting them, so energies still converge. Restore them by translation invariance of the metric and the ordered distance $d_{1,\alpha}(w + c, w) = V|c|$. □

**Corollary 29.5.1** *An increasing net of intrinsic finite-energy potentials with cutoffs bounded above has a least intrinsic finite-energy upper bound and converges strongly to it. For a normalized $u$ the relative truncations* $\max(u, u_* - L)$ *decrease strongly to $u$ as $L \to \infty$.*

***Proof*** For an increasing net use its increasing ordinary curves in $\theta$ and take their regularized supremum [Xia26b, Proposition 9.4.3]. Its left limit is still $\varphi$, and its cutoff is the supremum of the cutoffs. At each slope below that cutoff the slice masses increase to the mass of the resulting slice by [Xia26b, Corollary 2.4.1], applied on a tail on which the slope is interior. Extend the mass functions by zero at and above each cutoff. Compute the energies by the deficits from $V$ up to a common upper endpoint. On finite slope intervals the integrals converge by monotonicity in the slope and elementary upper and lower sums. The tails on the left are bounded by the integrable deficit of any fixed member on a tail of the net. This proves that the supremum curve has finite energy and that the energies converge to its energy. Its compatible function dominates the net by curve order. The ordered metric formula thus gives radial convergence, hence convergence at each test. Any intrinsic upper bound dominates this limit on the tests and thus everywhere, giving leastness.

The relative truncations are intrinsic potentials by Proposition 29.5.1, and decrease pointwise to $u$. Their energies decrease to $E(u)$ by Proposition 28.5.1. The ordered metric formula then gives strong convergence. □

**Proposition 29.5.2** *For a continuous real test $h$, finite-energy potentials $u, v$, and $a > 1$, one has*

$$\left|\overline{G}^{\mathrm{big}}_{\alpha}(u+h) - \overline{G}^{\mathrm{big}}_{\alpha}(v+h)\right| \le \frac{2V\|h\|_{\infty}}{a} + (n+1)a^{n} d_{1,\alpha}(u,v). \tag{29.12}$$

***Proof*** First take $u, v$ to be envelopes of continuous obstacles and subtract a common constant so they are nonpositive. Their rays have relative logarithmic lower bounds by Lemma 29.1.1. Write $w = u \wedge v$, $q = P_{\alpha}(u+h)$, and $M = \|h\|_{\infty}$. The bounds $u - M \le q \le u + M$ give $|E(q) - E(u)| \le VM$. The relative ray of $w$ has a logarithmic lower bound, by the common barrier in the rooftop construction. Apply Lemma 29.4.2 to $w \le u$ with parameter $a$, giving $r$. The intrinsic potential

$$(1 - a^{-1})q + a^{-1}r - M/a$$

is below $w + (1 - a^{-1})h - M/a$, hence below $v + h$. By maximality, concavity of energy, and the extrapolation bound,

$$\overline{G}^{\mathrm{big}}_{\alpha}(v+h) \ge (1 - a^{-1})E(q) + a^{-1}E(u) - (n+1)a^{n}(E(u) - E(w)) - VM/a.$$

Subtract $E(q) = \overline{G}^{\mathrm{big}}_{\alpha}(u+h)$ and use $|E(q) - E(u)| \le VM$. Both $E(u) - E(w)$ and $E(v) - E(w)$ are at most the rooftop distance, so reversing $u, v$ gives the estimate. Constant translations of both potentials preserve this estimate.

For general $u, v$, use jointly indexed decreasing envelopes of continuous majorants with their energies converging to $E(u), E(v)$. They converge in the rooftop metric

by the ordered energy differences. Their shifted obstacles decrease to $u + h, v + h$, whose relaxed energies are finite by the constant shift bounds. Passing to the limit by decreasing continuity of relaxed energy proves (29.12). □

**Theorem 29.5.2** *The intrinsic Monge–Ampère operator in a big class is weakly continuous for the strong topology on finite-energy potentials. More precisely, for each fixed real model test $h$ there is a constant $C_h$ such that*

$$\left| \int h \, \mathrm{d}\big(\mathrm{MA}_\alpha(u) - \mathrm{MA}_\alpha(v)\big) \right| \le C_h d_{1,\alpha}(u, v)^{1/(n+2)}. \tag{29.13}$$

***Proof*** Write $d = d_{1,\alpha}(u, v) > 0$. The uniform first-variation remainder (28.17) bounds the left side by

$$2C_h^0 t + \frac{\big|\overline{G}_\alpha^{\mathrm{big}}(u + th) - \overline{G}_\alpha^{\mathrm{big}}(v + th)\big| + |E(u) - E(v)|}{t}$$

for all sufficiently small $t > 0$, where $C_h^0$ is the constant of (28.17) for the direction $h$ and the threshold is independent of the potentials. For sufficiently small $d$, choose $t = d^{1/(n+2)} < 1$ and $a = 1/t$. Apply (29.12) with shift $th$. After division by $t$, its terms are bounded by $2V\|h\|_\infty t + (n + 1)t$, since $a^n d/t = t$. The energy difference divided by $t$ is at most $d/t = t^{n+1} \le t$. This proves (29.13) for small $d$, and for larger $d$ use the common mass bound $2V\|h\|_\infty$. For $d = 0$ the potentials agree. Real model functions are uniformly dense among continuous tests. Their estimate and the fixed mass give weak continuity in $d_{1,\alpha}$, and Theorem 29.5.1 identifies this with strong convergence. □

# Chapter 30
# Modifications and the relative equation

Let $\pi : X \to Y$ be the blowup of a point of a connected compact Kähler manifold of dimension at least two, with exceptional divisor $D$, and let $\alpha = \pi^*\zeta + a\{D\}$ for a Kähler class $\zeta$ on $Y$ and $a > 0$. Every positive current in $\alpha$ has generic Lelong number at least $a$ along $D$, with equality for the pullback of a Kähler form plus $a[D]$ (Example 30.1.1). Removing the divisorial part $N = aD$ leaves the nef class $\pi^*\zeta$. Subtracting the function $e_N(v) = a\,v(\mathcal{O}_X(-D))$ maps the intrinsic finite-energy potentials of $\pi^*\zeta$ onto those of $\alpha$. It preserves energy, rooftop distance and Monge–Ampère measures, and adds $\int e_N \,\mathrm{d}\nu$ to the absolute dual energy of a measure $\nu$ (Theorem 30.1.1). Since the remaining class is nef, the nef theory of Chapter 27 gives the relative equation in $\alpha$ (Corollary 30.1.1). The same holds for every big class on a compact Kähler surface.

Throughout this chapter $X$ is a connected compact Kähler manifold and $\alpha$ is a big class with $V = \mathrm{vol}_X(\alpha)$. We use the intrinsic finite-energy potentials and $\mathrm{MA}_\alpha$ of Chapter 28, and the greatest normalized potential $u_*$, the set $\Xi = \{u_* > -\infty\}$, the relative dual energy (29.1) and the rooftop distance of Chapter 29.

In dimension three the positive part can fail to be nef. On the blowup of $\mathbb{P}^3$ at two points, the class $3H - 2D_0 - 2D_1$ is big, has no divisorial part, and has degree $-1$ on the strict transform $C$ of the line through the two points. After blowing up $C$, the divisorial part of its pullback is the exceptional divisor and the positive part is nef (Example 30.2.1); a product with a nonprojective torus does the same on a nonprojective manifold. Pullback to a modification identifies the intrinsic finite-energy domains, preserving energy and rooftop distance, and the pushforward of the measure of a pulled-back potential is the original measure (Theorem 30.2.1). For a finite sequence of blowups of smooth centres, the induced map is a homeomorphism on the locus of finite order along a suitable exceptional divisor (Proposition 30.2.1). This gives the relative equation whenever some modification has nef divisorial positive part (Theorem 30.2.2 and Corollary 30.2.1).

The last section removes every nefness assumption. Regularized compactified rays make all continuous-obstacle envelopes continuous on a common open locus of finite ideal order (Lemma 30.3.1). Pole tests (Lemma 30.3.2) show that measures of finite relative dual energy give no mass to the infinite-order locus of any nonzero

coherent ideal. The image of $\mathrm{MA}_\alpha$ is the set of measures of mass $V$ and finite relative dual energy (Theorem 30.3.1), and the normalized operator is a homeomorphism onto it in every big class (Theorem 30.3.2). Section 33.5 applies Theorem 30.1.1 and Example 30.1.1 to the blowup of $\mathbb{P}^2$ at a torus-fixed point. This gives an intrinsic Monge–Ampère measure of infinite absolute dual energy (Proposition 33.5.1), whose relative dual energy is finite by Theorem 30.3.1.

## 30.1 The divisorial part of a big class

Let $X$ be a connected compact Kähler manifold and $\alpha$ a big class. For a smooth representative $\theta_\alpha$, let $T_{\min} = \theta_\alpha + dd^c V_{\theta_\alpha}$ be a current with minimal singularities. For each prime divisor $D$ on $X$ let $a_D = \nu_D(T_{\min})$ be its generic Lelong number. Any positive current in $\alpha$ has generic Lelong numbers at least $a_D$, by minimality. In particular these numbers are independent of the choice of minimal current.

**Lemma 30.1.1** *Only finitely many $a_D$ are positive. Put $N = \sum_{a_D>0} a_D D$ and $\beta = \alpha - \{N\}$, where $\{N\}$ denotes the class of the integration current $[N]$. Addition of $[N]$ gives a bijection between positive closed currents in $\beta$ and in $\alpha$. The class $\beta$ is big and has a positive current with zero generic Lelong number along every prime divisor on $X$. If $X$ is a surface, $\beta$ is nef.*

***Proof*** Fix a Kähler form $\omega$. Since $\alpha$ is big there is a current $S \in \alpha$ with $S \geq \delta\omega$ for some $\delta > 0$. Apply the analytic singularity approximation [Xia26a, Theorem 13.4.5] to the potential of $S - \delta\omega$. Choose an approximant with curvature lower bound $-\varepsilon_j\omega$ for $\varepsilon_j < \delta$. Adding back $\delta\omega$ gives a Kähler current in $\alpha$ with analytic singularities. The singular set has finitely many irreducible components by compactness and local finiteness. A divisor with $a_D > 0$ must be one of its codimension-one components, proving finiteness.

By Siu's divisorial decomposition [Xia26a, Theorem 14.3.7], subtracting $[N]$ from any positive current in $\alpha$ gives a positive current. The reverse operation is positive by effectivity. Applying the same decomposition to $S-\delta\omega$, whose generic Lelong numbers equal those of $S$, gives $S - [N] \geq \delta\omega$. Thus $\beta$ is big. The residual current $T_{\min} - [N]$ has zero generic Lelong numbers on $X$.

On a surface take the approximants with analytic singularities of this residual current in [Xia26a, Theorem 13.4.5]. Their curvature lower bounds are $-\varepsilon_j\omega$, with $\varepsilon_j \downarrow 0$. Their Lelong numbers are at most those of the original potential [Xia26a, Lemma 13.4.2], so their singular sets contain no curves. Hence each approximant is smooth off a finite set. Locally at one of these points, add a smooth potential of a representative of $\beta+\varepsilon_j[\omega]$. The resulting function is psh, is smooth on the punctured ball and tends to $-\infty$ at the point. Replace it by a smooth convex regularization of its maximum with a negative constant chosen below its minimum near the boundary of a smaller ball. Use a regularization equal to the maximum away from a small neighborhood of the break. It is constant near the pole and agrees with the original function near the boundary. This follows by convolving $x \mapsto \max(x, 0)$ with a small

smooth symmetric nonnegative kernel, then translating its argument by the constant. Subtract the smooth local potential to obtain a modification in the same class. Do this in disjoint balls about the finitely many poles. The result gives a smooth semipositive representative of $\beta + \varepsilon_j[\omega]$. Thus $\beta + \varepsilon[\omega]$ is Kähler for every $\varepsilon > 0$, proving nefness. □

For each prime $D$ occurring in $N$, use the canonical section of its divisor line bundle and a smooth metric to write its log squared norm $l_D \leq 0$. Set $l = \sum a_D l_D$ and choose smooth representatives such that

$$\theta_\alpha - \theta_\beta + dd^c l = [N].$$

At $v \in X^{\beth}$ set

$$e_N(v) = \sum_{a_D > 0} a_D \, v(\mathcal{O}_X(-D)).$$

This is a nonnegative extended Borel function, since the ideal evaluations are continuous into $[0, +\infty]$, and it vanishes at $v_0$.

**Lemma 30.1.2** *Translation by $l$ induces a bijection between multiplier types of $\theta_\beta$-psh potentials and those of $\theta_\alpha$-psh potentials. If $b$ is the nonpositive $\mathcal{I}$-model representative of a type in $\beta$, its translated model representative $\widehat{b}$ satisfies*

$$b + l \leq \widehat{b} \leq b + l + C \tag{30.1}$$

*for a constant $C$ independent of the type. The translated models have equal top masses, as do the two classes.*

***Proof*** The current bijection yields a potential bijection via addition of $l$. For the inverse subtract $l$ as a distribution and take its $\theta_\beta$-psh representative, since the remaining current is positive. The potential recovered by adding $l$ coincides with the original one as a distribution and hence as a psh representative. At every divisor above $X$ the generic Lelong numbers increase by the pullback order of $N$, by additivity of Lelong numbers and the Poincaré–Lelong formula [Xia26a, Proposition 12.1.4 and Theorem 7.4.5]. Thus equality of types is preserved in both directions by the divisorial criterion [Xia26b, Proposition 3.2.1(5)].

The translated potential $b+l$ is nonpositive, so it is a competitor for the $\mathcal{I}$-envelope $\widehat{b}$ and gives the lower bound. Conversely, if a nonpositive $\theta_\alpha$-psh potential is written $b' + l$, then $b' \leq -l$ off the support of $N$. On one fixed open set $U$ of positive smooth volume with closure disjoint from that support, $-l \leq M_0$ for some constant $M_0$. Normalize $b'$ by its supremum and use the fixed-class $L^1$ bound [Xia26a, Proposition 14.2.1]. If $K$ is this bound, then

$$\sup_X b' \leq M_0 + K/\mathrm{vol}(U).$$

Take $C \geq 0$ to be this uniform bound. In particular the subtraction of $l$ from $\widehat{b}$, followed by subtraction of $C$, gives a nonpositive competitor of the type of $b$. It lies

below $b$, proving (30.1) off the divisor and hence everywhere by uniqueness of psh representatives.

The top mass of $b + l$ in $\alpha$ equals that of $b$ in $\beta$. Their currents agree off $N$ and the non-pluripolar products give no mass to $N$. The bounded correction in (30.1) preserves the top mass by monotonicity in singularity type [Xia26b, Theorem 2.4.3]. Also $V_{\theta_\beta} + l \le V_{\theta_\alpha} \le V_{\theta_\beta} + l + C$ by the same bound on all nonpositive potentials. These two minimal types therefore have the same top mass in the respective classes. This proves $\mathrm{vol}(\alpha) = \mathrm{vol}(\beta)$. □

**Lemma 30.1.3** *For a nonpositive circle-invariant quasi-psh function $\Phi$ on the product disk one has*

$$J_{\Phi + p_X^* l} = J_\Phi - e_N \quad \text{on } X^\sqsupset.$$

***Proof*** Work locally and write $s_D$ for the equation of $D$. For nonnegative integers $k_D$, multiplication by $\prod_D s_D^{k_D}$ takes $\mathcal{I}(m\Phi)$ into the multiplier ideal for the weight $m\Phi + \sum_D k_D \log |s_D|^2$, with the spatial equations pulled back to the product. For the converse inclusion, near a smooth point of any local divisor branch away from the other branches, local integrability forces divisibility to order $k_D$ in the normal coordinate since $\Phi$ is bounded above. A germ in the multiplier ideal is therefore divisible by the entire product. Indeed its quotient is holomorphic off the intersections and singular loci of the reduced divisor, and extends across their codimension-two union by the removable singularity theorem for holomorphic functions. Cancellation then gives the remaining multiplier condition. Hence

$$\mathcal{I}\Big(m\Phi + \sum_D k_D \log |s_D|^2\Big) = \Big(\prod_D s_D^{k_D}\Big)\mathcal{I}(m\Phi).$$

For the real weights $ma_D$ sandwich the ideal between those for $\lfloor ma_D \rfloor$ and $\lceil ma_D \rceil$, using local equations of norm at most one and ignoring bounded smooth terms. Multiplication by the spatial divisor ideals adds their ordinary $v$-orders to $S$ of the product ideal in (25.6), since their pullbacks have the same Taylor coefficient ideal in every degree. Write $S_m = S(\mathcal{I}(m\Phi))(v)$ and $\widetilde{S}_m = S(\mathcal{I}(m(\Phi + p_X^* l)))(v)$. Thus

$$S_m + \sum_D \lfloor ma_D \rfloor\, v(\mathcal{O}_X(-D)) \le \widetilde{S}_m \le S_m + \sum_D \lceil ma_D \rceil\, v(\mathcal{O}_X(-D)).$$

A zero exponent contributes zero even at an infinite spatial order. If $S_m$ is infinite then $\widetilde{S}_m$ is infinite as well. If a spatial order is infinite, the floor coefficient is positive for all sufficiently large $m$ and forces the translated order to be infinite. Otherwise divide the bounds by $m$ and take the dyadic limit in (25.7). This gives the identity with all extended cases included. □

**Theorem 30.1.1** *The map $T_N : u \mapsto u - e_N$ is an order isomorphism from the intrinsic finite-energy domain of $\beta$ onto that of $\alpha$. It preserves energy, rooftop distance and intrinsic Monge–Ampère measures. For a positive Radon measure $\nu$ of mass $V = \mathrm{vol}(\alpha) = \mathrm{vol}(\beta)$,*

$$\mathcal{E}^{\vee}_{\alpha}(\nu) = \mathcal{E}^{\vee}_{\beta}(\nu) + \int e_N \, \mathrm{d}\nu, \tag{30.2}$$

*as a sum of nonnegative extended numbers.*

***Proof*** By Corollary 25.9.1, finite-energy families have ordinary representatives. Translate each of their spatial multiplier types by $l$, then take the nonpositive model representative in the new class. This operation preserves the cutoff. The new slices remain decreasing and concave. To check this, at any divisor above $X$ their generic Lelong numbers are the old numbers plus a constant independent of the slope. The comparisons that encode monotonicity and concavity therefore pass to the new slices. In the concavity comparison the combination of the two endpoint slices is a nonpositive psh function, and its generic orders are at least those of the middle slice. Its maximum with the middle slice has the type of that slice. Maximality of the $\mathcal{I}$-model representative yields the pointwise concavity inequality. The decreasing comparison follows in the same way using the individual slices. This argument applies also to the inverse translation of types.

The slice masses are preserved by Lemma 30.1.2. A translated family has a positive-mass left limit, since its left limit majorizes each interior slice. Its energy is finite and unchanged by the slice formula (25.3), using the common class volume. Thus the new family belongs to the required intrinsic domain. Inverse translation recovers all slices by uniqueness of the model representative of a type.

For cutoff zero, the slice bounds (30.1) give the same bounds for the ordinary rays by taking their Legendre suprema. The $\alpha$-ray differs from the $\beta$-ray plus $p_X^* l$ by a number in $[0, C]$ almost everywhere. The excluded sets are the divisor, the central fibre and poles of the rays. Thus their multiplier ideals on the product agree at every positive integral power, since bounded changes off a null set preserve local integrability. By Theorem 25.11.1 and Lemma 30.1.3 the valuation-space map is $T_N$ everywhere. Both cutoffs translate by constants, giving the same formula in general. The forward map preserves order pointwise. The inverse cancels $e_N$ at the local vertical tests, where the orders of spatial divisors are finite. These tests recover the order by Lemma 29.1.3. The order isomorphism commutes with rooftops and hence is an isometry by (29.7).

For a continuous real $h$, the finite-energy minorants of $u + th$ in $\beta$ correspond to those of $T_N(u) + th$ in $\alpha$. Indeed the forward inequality holds pointwise, and the reverse follows by cancellation on the local tests and Lemma 29.1.3. These obstacles have finite relaxed energy by the constant shift bounds. Thus their largest minorants have equal energy. Differentiating at $t = 0$ by Theorem 28.5.1 gives equality of the two measures. For the dual formula normalize $u$ by its cutoff so that $u \leq 0$ and $E_\beta(u) \leq 0$. Then $-T_N(u) = -u + e_N$ is a sum of nonnegative functions. The dual supremum in $\alpha$ therefore equals the one in $\beta$ plus the integral of $e_N$, which proves (30.2) also when the added term is infinite. □

*Example 30.1.1 (A point blowup)* Let $\pi : X \to Y$ blow up a point of a connected compact Kähler manifold of dimension at least two. The blowup is Kähler by the second assertion of Lemma D.4.1. Let $D$ be its exceptional divisor and let $\zeta$ be a Kähler class on $Y$. For $\alpha = \pi^*\zeta + a\{D\}$ with $a > 0$, the divisor of Lemma 30.1.1 is $N = aD$. Indeed every positive current $S$ in $\alpha$ agrees with $\pi^*\pi_*S$ off $D$. Their difference is a real multiple of $[D]$ by the support theorem [Xia26a, Theorem A.2.17], and pairing with a line of the exceptional projective space shows that this multiple is $a$. The pullback term has zero degree on the line, while $\{D\}$ has degree $-1$ by its tautological normal bundle. Thus $\nu_D(S) \geq a$. The pullback of a smooth Kähler form plus $a[D]$ attains this multiplicity and is smooth off $D$, so minimality gives exactly the asserted generic multiplicities.

**Lemma 30.1.4** *Let $\gamma$ be a big class on a compact Kähler manifold and $G$ any effective real divisor. Define $e_G$ as for $N$. There is an intrinsic finite-energy potential in $\gamma$ that is $-\infty$ on $\{e_G = +\infty\}$.*

***Proof*** Write $l_G \leq 0$ for a sum of log squared norms representing the effective divisor $G$ as above, and fix a smooth representative $\theta_\gamma$. Choose $\varepsilon > 0$ small enough that $\gamma - \varepsilon\{G\}$ is big. Adding $\varepsilon l_G$ to a psh potential in this latter class and subtracting a constant gives a $\theta_\gamma$-psh potential $b \leq \varphi = V_{\theta_\gamma}$ with current at least $\varepsilon[G]$. For $\tau < 0$, let $a(\tau) = (1-\tau)^{-2}$ and take the $\mathcal{I}$-envelope of $(1-a(\tau))\varphi + a(\tau)b$ in $\gamma$. These slices are decreasing and concave because $a$ is increasing and convex, $b \leq \varphi$, and the $\mathcal{I}$-envelope is monotone and concave. The positive mixed expansion for the input current and mass monotonicity under enveloping bound the slice masses below by $(1-a(\tau))^n \operatorname{vol}(\gamma)$. Their mass deficits have finite integral since $1-(1-a)^n \leq na$. This lower bound also implies full mass of the left limit. Thus the ordinary slices give a compatible potential of finite energy and cutoff zero by Corollary 25.9.1.

At each local vertical test $v$, the normalized vertical order of such a slice pulled back to the product is at least $\varepsilon a(\tau) e_G(v)$. In fact the input current contains $\varepsilon a(\tau)[G]$, and the orders of the input and its envelope agree by the product multiplier formula and the divisorial order estimate [Xia26a, Theorem 12.4.1 and Corollary 12.6.8]. The Legendre jet formula bounds the test values of the new potential above by $\Psi(e_G(v))$, where

$$\Psi(r) = \sup_{\tau<0}\{\tau - \varepsilon a(\tau) r\}, \qquad r \geq 0.$$

The function $\Psi$ is continuous on $[0,\infty)$, since it is a supremum of affine functions with a common Lipschitz bound $\varepsilon$. It tends to $-\infty$ as $r \to \infty$. Indeed for $\tau \leq -L$ the supremand is at most $-L$, and for $-L < \tau < 0$ it is at most $-\varepsilon(1+L)^{-2}r$. For each $M > 0$ the continuous function $\Psi(\min(e_G, M))$ bounds the test values above. By Lemma 29.1.3 it bounds the potential above everywhere. Let $M$ increase to obtain the desired poles. □

### 30.1.1 The relative equation when the positive part is nef

Suppose $\beta$ is nef. The zero function lies in the perturbative cone of $\beta$ (Section 26.1), so $P_\beta(0) = 0$ by the nef comparison in Theorem 28.4.1; thus 0 is the greatest normalized potential of $\beta$. Since $T_N$ preserves order and normalization, as $e_N(v_0) = 0$, the greatest normalized potential of $\alpha$ is $u_* = T_N(0) = -e_N$. On $\Xi$ the transported potential $T_N^{-1}(w)$ equals $w - u_*$.

**Corollary 30.1.1** *When $\beta$ is nef, in particular for every big class on a compact Kähler surface, a finite-energy $w$ solves $\mathrm{MA}_\alpha(w) = \nu$ if and only if the relative dual energy is finite and the supremum in (29.1) is attained at $w$. The measure operator is a homeomorphism from the normalized domain with its strong topology onto the positive Radon measures of mass $V$ with finite relative dual energy, equipped with weak convergence and convergence of the relative dual energy. These measures give no mass to $\{u_* = -\infty\}$.*

***Proof*** The pole test Lemma 30.1.4 in $\beta$ shows that a measure of finite absolute dual energy there is carried by $\{e_N < +\infty\} = \Xi$. For a measure carried by $\Xi$, the relative dual objective in $\alpha$ equals the absolute dual objective of the transported potential in $\beta$, by Theorem 30.1.1. If the measure charges the complement, both dual energies are infinite, by definition in $\alpha$ and by the pole test in $\beta$. The nef energy and mass have the same normalization as the big-class ones by Theorem 26.5.1 and Corollary 28.5.1. Dividing them by $V$ gives the variational problem for $\nu/V$ in Proposition 27.2.1 and Theorem 27.4.1. This proves the equivalence, existence and normalized uniqueness.

To compare the strong topologies, note that the local vertical tests include the nef model tests. Conversely, a normalized nef net converging on the model tests and in energy belongs to one energy ball on a tail. For each local test $\nu$, $\delta_\nu$ has finite normalized nef dual energy by Lemma 29.5.1. Integration against it is weakly continuous on the ball by the first step of the proof of Theorem 27.4.1. Hence all local test values converge. Thus the nef strong topology agrees with the rooftop topology by Theorem 29.5.1. The rooftop isometry and the nef strong homeomorphism Corollary 27.4.1 yield the homeomorphism. □

## 30.2 Pullback and the relative equation on a modification

Let $f : Y \to X$ be a proper modification of connected compact Kähler manifolds, and put $p = f^{\sqsupset}$. Fix a smooth representative $\theta$ of a big class $\alpha$ on $X$. Its pullback represents the big class $\widetilde{\alpha} = f^*\alpha$ [Xia26a, Proposition 14.4.7]. We use the scalar energy and the measure of the full intrinsic finite-energy domain on each manifold.

**Lemma 30.2.1** *Spatial pullback identifies the $\mathcal{I}$-model potentials in $\theta$ and $f^*\theta$, preserving their top masses. It identifies the ordinary positive-mass test curves with their slice masses and cutoffs. If $u$ is the jet of such a curve on $X$, the pulled-back curve has jet $u \circ p$.*

***Proof*** Let $A$ be an analytic subset outside which $f$ is an isomorphism. An $f^*\theta$-psh potential descends over $X \setminus A$. It is bounded above because $Y$ is compact. After adding a local potential for $\theta$, the descended function extends psh across $A$ [Xia26a, Theorem 8.4.1]. Its pullback equals the original potential on the complement of $f^{-1}A$, hence everywhere by uniqueness of the psh representative. This is a bijection which preserves order and the upper bound zero, since $f$ is surjective. In particular the minimal potentials correspond.

Equality of multiplier types downstairs implies equality upstairs. Indeed each divisor on a smooth model over $Y$ is also a divisor on a smooth model over $X$ by composition. Apply the divisorial criterion [Xia26b, Proposition 3.2.1(5)]. Conversely, on a small coordinate neighborhood of a point of $X$, test local integrability of $|g|^2 e^{-mb}$ by change of variables off $A$. Upstairs it is equivalent to the local integrability of

$$|g \circ f|^2 \, |\operatorname{Jac} f|^2 \exp(-mb \circ f)$$

along the whole fibre. Indeed properness lets one shrink the coordinate neighborhood so its inverse image lies in finitely many neighborhoods where the integrals are finite. In those charts the Jacobian is holomorphic, so this tests membership of $(g \circ f) \operatorname{Jac} f$ in the local multiplier ideals. Thus equal types upstairs give the same ideals downstairs. The potential bijection and the nonpositive bounds now identify the $\mathcal{I}$-model representatives.

The non-pluripolar top products of corresponding potentials agree over the isomorphism locus and give no mass to its analytic complement on either manifold. This proves equality of the volumes by applying it to the minimal potentials. It also proves equality of all slice masses. Pullback and descent preserve the order and concavity of ordinary model curves, with the same cutoffs. The left limits are nonpositive model potentials and also correspond: any least upper bound in the respective spatial cone is preserved by its order isomorphism. Thus the positive-mass curves correspond.

We check the identification on the entire ideal spectrum. Let $\widetilde{u}$ be the jet upstairs of the pullback curve of $u$. At every divisorial ray upstairs it agrees with $u \circ p$. Indeed the composite model over $X$ computes the same generic Lelong number for the spatial slice, and $p$ evaluates an ideal on $X$ by extending it to $Y$. Thus the slice formula (25.1) gives the stated equality. It holds also at the trivial valuation by the cutoff. To recover the full functions, take a Kähler perturbation on $X$ for which the jet $u$

lies in its decreasing semipositive cone. Pullbacks of its continuous semipositive majorants belong to the continuous cone upstairs for that pulled-back class, by the model pullback in Section 24.2. Add a Kähler form upstairs to this background. The same majorants are semipositive in the larger class and decrease to $u \circ p$. It belongs to that Kähler decreasing cone. The enlarged curve upstairs gives $\widetilde{u}$ in the same cone. Now Theorem 25.12.1 and Proposition 25.4.1 identify these two potentials from their values on divisorial rays and at the trivial valuation. □

**Theorem 30.2.1** *Pullback $u \mapsto u \circ p$ is an order isomorphism of the intrinsic finite-energy domains in $\alpha$ and $\widetilde{\alpha}$. The volumes agree. Pullback preserves energy and rooftop distance, and*

$$p_* \mathrm{MA}_{\widetilde{\alpha}}(u \circ p) = \mathrm{MA}_{\alpha}(u). \tag{30.3}$$

*For a positive Radon measure $\widetilde{\nu}$ of the common mass one has*

$$\mathcal{E}^{\vee}_{\widetilde{\alpha}}(\widetilde{\nu}) = \mathcal{E}^{\vee}_{\alpha}(p_*\widetilde{\nu}). \tag{30.4}$$

***Proof*** By Lemma 30.2.1 the ordinary curves correspond with all their slice masses and their common class volume. The finite-energy criterion Corollary 25.9.1 identifies the finite-energy domains and their energies, and the lemma identifies the full jets.

The order follows from surjectivity of $p$ (Proposition 21.10.1). This gives the rooftop isometry by (29.7). If $h$ is continuous real on $X^{\sqsupset}$, the finite-energy minorants of $u + th$ correspond exactly to those of $u \circ p + th \circ p$. The greatest minorants therefore have equal energy. Apply the first variation for the two obstacles at $t = 0$ by Theorem 28.5.1 to get (30.3). Finally, integrating a pulled-back potential against $\widetilde{\nu}$ is integrating the original against $p_*\widetilde{\nu}$. Taking both dual suprema gives (30.4), with the same convention for an integral equal to $-\infty$. □

### 30.2.1 Finite orders along an exceptional divisor

The restriction of $p$ to a suitable open set has an inverse when the modification has an effective divisor whose negative class is relatively positive. We first show how to read a coherent ideal from an analytic jet.

**Lemma 30.2.2** *For a nonzero coherent ideal $\mathfrak{a}$ on a connected compact Kähler manifold $Y$, there is a global quasi-psh function $\psi_{\mathfrak{a}} \leq 0$ with local expression $\log \sum_i |g_i|^2 + O(1)$ for generators $g_i$ of $\mathfrak{a}$. For its pullback by the spatial projection $p_Y$ of the product disk, the jet is*

$$J_{p_Y^*\psi_{\mathfrak{a}}}(v) = -v(\mathfrak{a}).$$

*The same identity holds for any nonpositive quasi-psh weight of that local type.*

***Proof*** Choose a finite cover by generator neighborhoods and smooth nonnegative functions $\chi_j$ supported in them, with $\sum_j \chi_j^2 > 0$ everywhere. Extend $\chi_j^2 \|g^{(j)}\|^2$ by zero outside its neighborhood and take the logarithm of the sum. Near a point choose an index with $\chi_j > 0$ there. On a smaller neighborhood the other relevant tuples of generators are holomorphic matrix multiples $A_i g^{(j)}$ of this tuple. The sum is $\|g^{(j)}\|_H^2$ for the smooth Hermitian matrix $H = \sum_i \chi_i^2 A_i^* A_i$, with $A_j$ the identity. It is uniformly positive definite on a still smaller neighborhood. This proves the local bounded comparison.

The log of this norm is quasi-psh. In a holomorphic frame normal for $H$ at a point of the neighborhood, the first derivatives of $H$ vanish and the second derivatives give a lower bound $-C\|g^{(j)}\|_H^2$ for their contribution to the complex Hessian of the squared norm in each unit direction. The contributions from differentiating the holomorphic tuple to the Hessian of the logarithm are nonnegative by Cauchy–Schwarz. The same bound holds after adding a positive constant to the squared norm before taking the logarithm. Taking that constant to zero and using compactness gives a global curvature lower bound. Subtract a constant to normalize the weight.

For a weight of this local type, multiplication of generators gives $\mathfrak{a}^k \subseteq \mathcal{I}(k\psi_{\mathfrak{a}})$. The ideal-power inclusion in the proof of [Xia26b, Lemma 1.6.3, proof of (1.23)] gives the reverse inclusion into $\mathfrak{a}^{(k-n+1)_+}$, where $n = \dim Y$ and $x_+ = \max(x, 0)$. Indeed this is the inclusion from division with coefficient one in the local log of the squared generating norm. The bounded remainder changes neither inclusion. The multiplier ideals of the spatial pullback are the extensions of these ideals, by the product formula [Xia26a, Theorem 12.4.1]. Their Gauss orders equal the corresponding ideal evaluations, since a spatial ideal has the same Taylor coefficient ideal in every degree. The two powers therefore bound the quotient of each order by $k$ and give $v(\mathfrak{a})$ in the limit, including $+\infty$. Taking negatives in (25.7) proves the identity. □

**Proposition 30.2.1** *Let $D$ be an effective Cartier divisor on $Y$ such that $Cf^*[\omega] - \{D\}$ is Kähler for a Kähler form $\omega$ on $X$ and $C > 0$. There is a continuous extended nonnegative function $r$ on $X^{\beth}$ with $r \circ p = e_D$, where $e_D(v) = v(\mathcal{O}_Y(-D))$ for $v \in Y^{\beth}$. The map $p$ restricts to a homeomorphism*

$$\Omega_Y = \{e_D < +\infty\} \longrightarrow \Omega_X = \{r < +\infty\}, \qquad \Omega_Y = p^{-1}(\Omega_X).$$

*Such a divisor supported on the exceptional set exists for a finite sequence of blowups of smooth centres of codimension at least two.*

***Proof*** Choose a Kähler form $\kappa$ in $Cf^*[\omega] - \{D\}$. By the $dd^c$-lemma [Xia26a, Theorem 14.1.7], choose a metric on $\mathcal{O}_Y(D)$ with curvature $Cf^*\omega - \kappa$. Let $l_D \leq 0$ be the log squared norm of its canonical section, arranged by scaling the metric. Then

$$dd^c l_D = [D] + \kappa - Cf^*\omega.$$

For $\mathfrak{a}$ as in Lemma 30.2.2, put $\psi = \psi_{\mathfrak{a}} + m l_D$, with $m$ a sufficiently large positive integer. The positive current of $\psi$ in $f^*(mC\omega)$ is

$$m\kappa + dd^c \psi_{\mathfrak{a}} + m[D] \geq \kappa.$$

It has positive top mass by non-pluripolar positivity and expansion with the smooth form $\kappa$. The spatial descent in Lemma 30.2.1 therefore gives a nonpositive potential of positive mass on $X$. Take the $\mathcal{I}$-model representative of its multiplier type and the constant ordinary test curve with that representative at every negative slope and cutoff zero. Its pullback has the multiplier type of $\psi$ at every slice by the same lemma. The Legendre ray is constant, and its jet is the jet of the spatial weight $\psi$, because the multiplier type and hence the product ideals agree. The weight has the local type of the coherent ideal $\mathfrak{a}\mathcal{O}_Y(-mD)$. By Lemma 30.2.2, its jet is

$$-v(\mathfrak{a}) - me_D(v).$$

Thus this function is a pullback from $X^{\sqsupset}$ for each nonzero $\mathfrak{a}$.

Taking $\mathfrak{a} = \mathcal{O}_Y$ shows that $e_D$ descends to $r$. The surjection between compact Hausdorff spaces is a quotient map, so continuity of the extended ideal order $e_D$ gives continuity of $r$. In particular the displayed finite loci are open. If two points of $\Omega_Y$ have the same image, cancelling their common finite $me_D$ in the pulled-back jet shows that their evaluations of $\mathfrak{a}$ agree for every nonzero coherent ideal, also when an order is infinite. The zero ideal has the same value at both, proving injectivity. The restriction is surjective and closed relative to the two open sets, since it is the restriction of a closed map to an open set and its preimage. It is therefore a homeomorphism.

For one blowup $g$, the curvature construction in the proof of Lemma D.4.1 gives a form $\eta$ in the negative exceptional class with $g^*\omega + \varepsilon\eta$ Kähler for small $\varepsilon > 0$. Taking $\varepsilon = 1/k$ and multiplying by $k$ gives the required class for the exceptional divisor. For a tower, suppose $Cf^*[\omega] - \{D\}$ is Kähler at one stage. Blow up at the next stage by $g$, with exceptional divisor $F$. Apply the same construction using a Kähler form of the preceding class. After multiplication by $k$, the resulting class is $kC(fg)^*[\omega] - \{kg^*D + F\}$. The divisor is effective, supported on the exceptional set and contains its divisorial components. Induction proves the assertion, with $D = 0$ for an empty tower. □

For a single nontrivial blowup, the restriction to finite orders is necessary for injectivity. Over any point of its centre, choose two distinct points of the exceptional fibre. A point $y$ defines the ideal valuation with value zero on ideals that are the unit ideal at $y$ and $+\infty$ on all other ideals. These satisfy the sum and product rules because the proper ideals in a local ring lie in its maximal ideal. The two valuations are distinguished by the ideal of one of the two points. Both evaluate the exceptional ideal to $+\infty$, and both push forward to the same point valuation at the image. Indeed the pullback of an ideal is a unit at $y$ exactly when the ideal is a unit at $f(y)$.

### 30.2.2 The relative equation and measure lifting

Suppose $p$ restricts to a homeomorphism from an open set $\Omega_Y = p^{-1}(\Omega_X)$ onto an open set $\Omega_X \subset X^\beth$. Assume $Y^\beth \setminus \Omega_Y \subset \{e_G = +\infty\}$ for some effective divisor $G$ on $Y$. These hypotheses hold in Proposition 30.2.1 with $G = D$.

They also hold for an algebraic modification of smooth projective varieties over $\mathbb{C}$. Indeed choose a proper closed algebraic subset $A \subset X$ over whose complement $f$ is an isomorphism. The projective comparison of Section 21.8 (Lemma D.3.1 and Proposition 21.2.1) identifies $X^\beth$ and $Y^\beth$ with the analytifications over the trivial valuation, compatibly with $f$. Under these identifications, the open analytifications of $X \setminus A$ and $Y \setminus f^{-1}(A)$ are the required sets. The isomorphism of the algebraic opens gives the homeomorphism and the preimage identity. A homogeneous polynomial in a projective embedding of $Y$ which vanishes on the proper closed subset $f^{-1}(A)$ without vanishing on all of $Y$ gives a divisor $G$ containing that subset. Its ideal has infinite value at each point of the analytification supported there.

Assume that the divisorial positive part $\beta$ of $\widetilde{\alpha}$ is nef, and write $N$ for its divisorial negative part. Recall that $u_*$ is the greatest normalized potential in $\alpha$ and $\Xi = \{u_* > -\infty\}$.

**Theorem 30.2.2** *Under these assumptions a finite-energy $u$ solves $\mathrm{MA}_\alpha(u) = \nu$ if and only if the relative dual energy is finite and the supremum in (29.1) is attained at $u$. The normalized operator is a homeomorphism from the normalized intrinsic finite-energy potentials with the strong topology onto the positive Radon measures of mass $V$ and finite relative dual energy, with weak convergence and convergence of this energy. The relative dual energy and these conclusions are independent of the choice of modification satisfying these lifting hypotheses and having nef positive part.*

***Proof*** By Lemma 30.1.4, every measure of finite absolute dual energy in $\beta$ is carried by $\Omega_Y$. Applying that lemma also to $N$ shows that it is carried by $\{e_N < +\infty\}$.

For each $u$ let $z_u = T_N^{-1}(u \circ p)$ be its corresponding finite-energy potential in $\beta$. This is a rooftop isometry by Theorems 30.1.1 and 30.2.1. The normalization and the order are preserved. The greatest normalized potential in the nef class $\beta$ is zero, by Section 30.1.1 applied to $\widetilde{\alpha}$, so $u_* \circ p = -e_N$. On $p^{-1}(\Xi)$,

$$z_u = (u - u_*) \circ p.$$

By (30.3) and divisorial transport, $\mathrm{MA}_\alpha(u) = p_* \mathrm{MA}_\beta(z_u)$.

If the relative dual energy of $\nu$ is finite, the definition gives $\nu(X^\beth \setminus \Xi) = 0$. It also gives $\nu(X^\beth \setminus \Omega_X) = 0$. Indeed take a potential $z$ in $\beta$ with poles on $\{e_G = +\infty\}$ as in Lemma 30.1.4. The corresponding relative difference downstairs is $-\infty$ on $\Xi \setminus \Omega_X$: every lift of such a point has $e_N$ finite and $e_G$ infinite. Positive mass on this set would make the relative objective infinite.

There is a unique Radon lift $\widetilde{\nu}$ carried by $\Omega_Y$, obtained by the inverse homeomorphism on the opens and extended by zero. For any measure $\nu$ carried by $\Xi \cap \Omega_X$ its

lifted absolute dual energy in $\beta$ is $\mathcal{E}^\vee_{\alpha,\mathrm{rel}}(\nu)$, by the equality of corresponding potentials on $p^{-1}(\Xi)$ and of their energies. Conversely a measure in $\beta$ of finite absolute dual energy is carried by $\Omega_Y \cap p^{-1}(\Xi)$, as proved above, and is the unique such lift of its pushforward. Thus the correspondence identifies the two measure domains and all variational objectives. If $\mathrm{MA}_\alpha(u) = \nu$, then $\mathrm{MA}_\beta(z_u)$ has finite absolute dual energy in $\beta$, with the supremum attained at $z_u$, by Proposition 27.2.1. By the first paragraph and $u_* \circ p = -e_N$, it is therefore carried by $\Omega_Y \cap p^{-1}(\Xi)$, so it is the supported lift of $\nu$. Hence, by the identification of the objectives, the relative dual energy of $\nu$ is finite and attained at $u$. The nef variational theorem Proposition 27.2.1 and Theorem 27.4.1 gives existence and uniqueness, and identifies equation solutions with maximizers by the measure pushforward formula and uniqueness of the supported lift.

On these measure domains weak convergence is equivalent to weak convergence of the lifts. One direction is continuity of pushforward. For the other, any subnet of lifts has a weakly convergent subnet by compactness of the positive Radon measures of fixed mass. The cluster measure pushes to the limit $\nu$. It therefore gives no mass to the complement of $\Omega_Y$ and is the unique supported lift of $\nu$. Thus the lifts converge. The equality of dual energies and the rooftop comparisons identify the strong topologies, using the nef strong homeomorphism and its comparison with the rooftop topology in Corollary 30.1.1. Finally the formula (29.1) is intrinsic on $X^\sqsupset$. Its measure domain and the conclusions are independent of $f$. □

**Corollary 30.2.1** *For any proper modification $f : Y \to X$ of connected compact Kähler manifolds, there is a dense open set in $X^\sqsupset$ over which $p = f^\sqsupset$ is a homeomorphism from its preimage. If the divisorial positive part of $f^*\alpha$ is nef, the relative equation in $\alpha$ has all the conclusions of Theorem 30.2.2.*

***Proof*** Hironaka's domination theorem [Xia26b, Theorem B.1.2] gives a tower $h : Z \to X$ of smooth blowups factoring as $f \circ g$ for a proper modification $g : Z \to Y$. Blowups in smooth divisors are isomorphisms and can be omitted. By the second assertion of Lemma D.4.1, $Z$ is compact Kähler. Apply Proposition 30.2.1 to $h$, and use its open set $\Omega_X$. Its preimage in $Z^\sqsupset$ is dense because every valuation from a flag-model component has finite order on $D$ and these valuations are dense by Proposition 21.6.1. Surjectivity of $h^\sqsupset$ gives density of $\Omega_X$. Since $g^\sqsupset$ is surjective, any two points of $p^{-1}(\Omega_X)$ with the same image lift to points of $(h^\sqsupset)^{-1}(\Omega_X)$ with the same image. The lifted points are equal, so the original points are equal. The restriction of the compact surjection $p$ to the open set and its preimage is relatively closed, which gives the claimed homeomorphism.

For the second assertion write $f^*\alpha = \beta + \{N\}$ with $\beta$ nef, and put $\gamma = g^*\beta$. It is big by [Xia26a, Proposition 14.4.7] and nef by pulling back smooth Kähler approximants and adding a Kähler form of arbitrarily small size on $Z$. The spatial descent in Lemma 30.2.1 identifies the positive currents in $h^*\alpha$ with the pullbacks of the currents in $f^*\alpha$. These currents are precisely the sum of $[g^*N]$ and a positive current in $\gamma$, by Lemma 30.1.1 and the same spatial pullback in $\beta$.

The minimal generic divisorial Lelong numbers in a big nef class vanish. In fact, for a Kähler form $\eta$, take a smooth positive representative of $\gamma + \varepsilon[\eta]$, subtract $\varepsilon\eta$,

and mix it with a Kähler current in $\gamma$ dominating $\delta\eta$. The coefficient $\varepsilon/(\varepsilon+\delta)$ of the latter gives a positive current in $\gamma$ with generic Lelong numbers tending to zero along every fixed divisor. Consequently the divisorial negative part of $h^*\alpha$ is $g^*N$ and its positive part is the nef class $\gamma$. Theorem 30.2.2 applies using $h$, whose lifting hypotheses hold by Proposition 30.2.1 and the pole test Lemma 30.1.4. Its conclusion is intrinsic on $X^{\sqsupset}$. □

*Example 30.2.1 (Blowing up a curve to make the positive part nef)* Let $X$ be the blowup of $\mathbb{P}^3_{\mathbb{C}}$ at the coordinate points $p_0, p_1$ with nonzero coordinates $X_0, X_1$, respectively. Write $H$ for the hyperplane pullback and $D_i$ for the exceptional divisors. Let $C$ be the strict transform of the line through the two points. For $\alpha = c_1(3H-2D_0-2D_1)$, the divisorial negative part on $X$ is zero and $\alpha$ is big and non-nef. On the blowup $f : Y \to X$ of $C$, the divisorial negative part of $f^*\alpha$ is its exceptional divisor $G$ with coefficient one, and the positive part is nef.

To compute the base ideal, use the degree-three monomials in $X_0, \dots, X_3$ in which each of $X_0, X_1$ has exponent at most one. Each vanishes to order at least two at the blown-up points and to order at least one on the line, so they give sections on $X$ vanishing on $C$. Off the line and the exceptional divisors, a monomial $X_i^3$ for $i \in \{2,3\}$ is nonzero. At points of the line away from its endpoints, $X_0X_1X_2$ and $X_0X_1X_3$ generate its ideal after trivializing the line bundle. In the chart at $p_0$ with $X_1/X_0 = r$ and $X_i/X_0 = ry_i$ for $i = 2,3$, these same monomials give $y_2, y_3$ after division by $r^2$; these generate the ideal of the strict transform of the line. On the exceptional divisor over $p_0$ at directions with a nonzero transverse coordinate for some $i = 2,3$, the initial quadratic term of $X_0X_i^2$ gives a nonzero value. The argument at $p_1$ uses $X_1X_i^2$ and the chart with line parameter $X_0/X_1$. This proves that the displayed system has base ideal $\mathcal{I}_C$ on $X$.

Hence the transformed system on $Y$ generates the line bundle of class $\beta = f^*\alpha - \{G\}$ by dividing the pullbacks by $G$. Its Fubini–Study form is semipositive and positive on the open chart $X_2 \neq 0$ away from the blowups: the ratios of $X_0X_2^2, X_1X_2^2, X_3X_2^2$ to $X_2^3$ give all the affine coordinates there. Thus $\int_Y \beta^3 > 0$, so $\beta$ is nef and big by [Xia26a, Proposition 15.8.2 and Theorem 17.4.20]. Adding $G$ shows $f^*\alpha$ big and pushing down gives $\alpha$ big by [Xia26a, Proposition 14.4.7]. The original system gives a positive current with singularities along $C$ only, so each minimal generic divisorial number on $X$ is zero. Yet $\alpha \cdot C = 3-2-2 = -1$: the degree of $H$ is one and each exceptional divisor meets $C$ transversely once. Thus $\alpha$ is non-nef.

It remains to identify the coefficient on $Y$. The normal bundle of the starting line is $\mathcal{O}(1)^{\oplus 2}$, since the two transverse coordinates are its two linear equations. Blowing up one point on the line twists the normal bundle of its strict transform by minus that point. In the line-direction chart the new transverse coordinates are the old ones divided by the line parameter, so the conormal frames are twisted by the point and the normal frames by its negative. After two blowups, $N_{C/X} = \mathcal{O}(-1)^{\oplus 2}$. Thus $G = \mathbb{P}(N_{C/X})$ is $C \times \mathbb{P}^1$ and $\mathcal{O}_Y(G)|_G$ is the tautological line subbundle $\mathcal{O}(-1,-1)$. Every curve $\ell$ of the form $C \times \{q\}$ has $G \cdot \ell = -1$.

The positive current consisting of the smooth semipositive representative of $\beta$ plus $[G]$ has generic number one on $G$ and zero on all other divisors. Let $a \leq 1$ be

the minimal generic coefficient on $G$. Subtract $a[G]$ from a minimal current in $f^*\alpha$. The residual current is positive and has generic number zero on $G$. Its approximants with analytic singularities and curvature error $-\varepsilon_j\omega$ from [Xia26a, Theorem 13.4.5] have no pole along the generic point of $G$, by the Lelong upper bound. For each $j$ choose a curve $\ell_j = C \times \{q_j\}$ not contained in the polar set. Such curves cover $G$. After adding $\varepsilon_j\omega$, the approximant restricts to a positive current on $\ell_j$, since its local potentials restrict to psh functions there without being identically $-\infty$. Its integral is the degree of its class. All these curves have the same degree against $\omega$, so passage to the limit gives, for the common curve class $[\ell]$,

$$0 \le (f^*\alpha - a\{G\}) \cdot \ell = -1 + a.$$

Hence $a = 1$, and the nef positive part is $\beta$ as asserted. This is an instance of Theorem 30.2.2 in dimension three where the positive part on the original manifold is non-nef.

The same construction gives nonprojective examples. Let $T$ be the nonprojective Kähler torus from Section 21.9 and $\omega_T$ a Kähler form on it. Use the product modification $f \times \mathrm{id}_T$, which blows up $C \times T$, and the sum of the pullbacks of $\alpha$ and $[\omega_T]$ on $X \times T$. The product is nonprojective since an ample line bundle would restrict to an ample line bundle on a torus fibre. The pullback class has a smooth semipositive part given by the sum of the pullbacks of $\beta$ and $[\omega_T]$. This part is positive on a nonempty open set, so it is nef and big by the same volume argument. The remaining divisor is $G \times T$. The coefficient computation above applies using the curves $C \times \{q\} \times \{t\}$, which cover $G \times T$ and have the same class. The current from the original linear system, plus the torus form, has singularities only along $C \times T$, so there is no fixed divisor on $X \times T$. The degree on $C \times \{t\}$ is still $-1$. Thus the lifting and variational theorems give the relative equation here using a smooth blowup of a nonprojective compact Kähler manifold.

## 30.3 Relative dual energy in an arbitrary big class

We retain the notation of Sections 29.1 and 29.2. For a nonzero coherent ideal $I$ on $X$ put $e_I(v) = v(I)$. This function takes values in $[0, +\infty]$ and is continuous for the extended interval topology.

**Lemma 30.3.1** *There is a nonzero coherent ideal $I_0$ such that all continuous-obstacle envelopes $P_\alpha(f)$ are finite and continuous on $\Omega_0 = \{e_{I_0} < +\infty\}$.*

***Proof*** Fix a Kähler form $\omega$. Regularize the potential of a Kähler current in $\alpha$ in a background reduced by half its strict lower bound, using [Xia26a, Theorem 13.4.5]. An approximant with sufficiently small curvature error gives a potential $b_0 \le 0$ with $\theta + dd^c b_0 \ge \delta\omega$, $\delta > 0$, of gentle analytic type $(c, I_0)$ with $c$ positive rational.

We use the following multiplier-ideal calculation on the product disk. For a weight of analytic type $(a, I)$ with $a = p/q > 0$ rational, $q$ times the weight is of type

$(1, I^p)$. Apply the ideal-power bounds used in Lemma 30.2.2 in dimension $n+1$, with exponents $\lfloor m/q \rfloor$ and $\lceil m/q \rceil$. After applying the order $S$ of (25.6) and dividing by $m$, both bounds converge to $aS(I)$, including at infinite order. The negative of this limit is its jet. Jets preserve inequalities up to bounded constants by inclusion of multiplier ideals. The jet of $p_X^* b_0$ is therefore $-ce_{I_0}$. Since $b_0 \le \varphi$, this yields $-ce_{I_0} \le u_*$, using Section 29.1 and Theorem 25.11.1.

First take a real model obstacle $f \le 0$, represented by $D$ on $W$, and let $U = U^f$ be the ray from Lemma 28.3.2. Use the compactification $\widetilde{U}$ in the proof of Proposition 28.4.1. It agrees with $U$ near zero, and the added parameter form vanishes there. The approximants $U_j \ge \widetilde{U}$ of [Xia26a, Theorem 13.4.5] have curvature error $\varepsilon_j$ with respect to the sum of $\omega$ and a Kähler form on the parameter line. They have analytic type $(m_j^{-1}, I_j)$, where $m_j = 2^j$ and $I_j = \mathcal{I}(m_j \widetilde{U})$. On a disk about zero, choose a smooth potential $\kappa$ of the parameter Kähler form. Then $U_j + \varepsilon_j \kappa$ has curvature error only $\varepsilon_j p_X^* \omega$ in the disk background. On the finitely many components $E$ of $W_0$, [Xia26a, Corollary 12.6.8] gives

$$\nu_E(U_j) \ge \nu_E(U) - K b_E / m_j$$

with one constant $K \ge 0$. Choose an integer $L > \|f\|_\infty + 1$ and rational $\lambda_j \in (0,1)$ tending to zero with $(1-\lambda_j)\varepsilon_j \le \lambda_j \delta$ and $(1-\lambda_j)K/m_j \le \lambda_j$. The function

$$H_j = (1-\lambda_j)(U_j + \varepsilon_j \kappa) + \lambda_j (p_X^* b_0 + L \log |t|^2)$$

is $p_X^* \theta$-psh on the small disk. Its orders satisfy the constraints (28.9), since $L b_E \ge -\operatorname{ord}_E(D) + b_E$ and $b_0$ has nonnegative orders. On each punctured slice, a uniform upper bound for $H_j$ on a smaller disk gives $H_j \le p_X^* \varphi + C_j$ by the definition of $\varphi$. Subtract a constant so it is below $p_X^* \varphi$ throughout the smaller disk and below $p_X^* \varphi + L \log |t|^2$ on a boundary collar. Glue its maximum with that barrier inside the disk to the barrier outside. The maximum retains the Lelong constraints, and the resulting potential is at most $U$ by its defining maximality. Thus $H_j \le U + O(1)$ near zero.

The product rule for $S$ and the rational weights in $H_j$ give

$$(1-\lambda_j) F_j - \lambda_j (c e_{I_0} + L) \le P_\alpha(f) \le F_j \le 0, \qquad F_j = -m_j^{-1} S(I_j).$$

In fact clearing the denominators expresses the mixed jet using the product of powers of $I_j$, the extended ideal $I_0$, and the parameter ideal $(t)$, whose $S$-value is one. This calculation uses only the multiplier order of these weights. For a noninvariant approximant, the same dyadic limit exists by subadditivity and computes its order, without any divisorial identification for a noninvariant ideal. Each $F_j$ is finite on $\{e_{I_0} \le R\}$, because $P_\alpha(f) \ge u_* - \|f\|_\infty$. The continuous truncations in Section 25.11 show that $F_j$ is continuous wherever finite. The last displayed inequality bounds $F_j - P_\alpha(f)$ on $\{e_{I_0} \le R\}$ between zero and $\lambda_j(cR + L)$. Uniform convergence gives continuity on the open set $\{e_{I_0} < R\}$. Let $R$ increase and translate the obstacle to remove its sign restriction. Finally real model functions approximate continuous obstacles uniformly, and their envelopes differ by at most the uniform difference of the obstacles on $\Omega_0$,

by the order and translation properties in Theorem 28.4.1. This proves continuity for all continuous obstacles. □

**Lemma 30.3.2** *For each nonzero coherent ideal* $\mathfrak{a}$ *there are a normalized finite-energy potential* $w$ *and a continuous increasing function* $H : [0,+\infty) \to [0,+\infty)$, *with* $H(r) \to +\infty$, *such that on* $\Xi$

$$u_* - w \;\geq\; H(e_{\mathfrak{a}}).$$

*Here* $H(+\infty) = +\infty$. *Consequently every measure of finite relative dual energy satisfies* $\int H(e_{\mathfrak{a}})\,\mathrm{d}\nu \leq D_{\mathrm{rel}}(\nu) - E(w)$.

***Proof*** Use $b_0$ from the proof of Lemma 30.3.1. Take the weight $\psi_{\mathfrak{a}} \leq 0$ of Lemma 30.2.2 and choose $\varepsilon > 0$ small enough that $b = b_0 + \varepsilon\psi_{\mathfrak{a}}$ is $\theta$-psh. It satisfies $b \leq \varphi$. For $\tau < 0$ let $a(\tau) = (1-\tau)^{-2}$ and take the $\mathcal{I}$-envelope of $(1-a(\tau))\varphi + a(\tau)b$. The monotonicity and concavity argument of Lemma 30.1.4 gives an ordinary model curve. Its slice masses are at least $(1-a(\tau))^n V$ by the positive mixed expansion and mass monotonicity. Thus the integral of its mass deficits is finite and the left limit has mass $V$. The finite-energy criterion Corollary 25.9.1 gives a normalized jet $w$.

At a local vertical test the generic orders of the slice and its input agree by the product multiplier formula and the divisorial order estimate [Xia26a, Theorem 12.4.1 and Corollary 12.6.8]. Since $b_0 \leq \varphi$, additivity of generic Lelong numbers and the ideal weight formula bound the normalized order below by $-u_*(v) + \varepsilon a(\tau) e_{\mathfrak{a}}(v)$. The Legendre jet formula therefore gives $w(v) \leq u_*(v) - H(e_{\mathfrak{a}}(v))$ on the tests, with

$$H(r) = \inf_{\tau<0}\{-\tau + \varepsilon a(\tau) r\}.$$

This infimum is nonnegative, increasing and $\varepsilon$-Lipschitz. For $L > 0$ the expression inside the infimum is at least $L$ if $\tau \leq -L$, and at least $\varepsilon(1+L)^{-2}r$ if $-L < \tau < 0$. This proves the asserted divergence. For $M > 0$, the upper semicontinuous function $u_* - H(\min(e_{\mathfrak{a}}, M))$ bounds $w$ above on the tests. Apply Lemma 29.1.3 and then let $M$ increase. On $\Xi$, $u_*$ is finite so the bound gives the relative difference inequality. Testing the dual objective at $w$ gives the integral estimate. □

**Theorem 30.3.1** *In an arbitrary big class, the image of the intrinsic Monge–Ampère operator is the set of positive Radon measures of mass* $V$ *and finite relative dual energy. Every such measure gives zero mass to* $\{e_{\mathfrak{a}} = +\infty\}$ *for each nonzero coherent ideal* $\mathfrak{a}$. *There is a constant* $C_n$ *such that for every normalized intrinsic finite-energy potential* $u$,

$$D_{\mathrm{rel}}(\mathrm{MA}_\alpha(u)) \leq C_n(-E(u)).$$

***Proof*** Surjectivity onto the finite relative dual domain is Theorem 29.4.1. The pole test Lemma 30.3.2 gives the asserted concentration. To prove the energy bound, first take $z = P_\alpha(f)$ normalized, with $f$ continuous. Write $\mu_z = \mathrm{MA}_\alpha(z)$. The supporting inequality (28.15) shows that $\mu_z$ has finite absolute dual energy and that $z$ attains it. Therefore $z$ also attains the relative dual energy by (29.2). The ordered extrapolation

Lemma 29.4.2 applies to $z \le u_*$, since both rays have relative logarithmic lower bounds: the ray of $z$ by Lemma 29.1.1, and that of $u_*$ is $p_X^*\varphi$. With $a = 2$ it gives $r$ with $r - z \le z - u_*$ on $\Xi$ and $-E(r) \le (n+1)2^{n+1}(-E(z))$. Applying the supporting inequality to $r$ gives

$$D_{\mathrm{rel}}(\mu_z) = E(z) + \int_\Xi (u_* - z)\,\mathrm{d}\mu_z \le 2E(z) - E(r) \le (n+1)2^{n+1}(-E(z)).$$

For general $u$, take normalized continuous-obstacle envelopes $z_i \downarrow u$ using continuous majorants of $u$ cut off above at zero. They satisfy $-E(z_i) \le -E(u)$ by monotonicity. The measures $\mu_{z_i}$ converge weakly to $\mu_u = \mathrm{MA}_\alpha(u)$ by Corollary 28.5.1. Their relative dual energies are bounded by $B = (n+1)2^{n+1}(-E(u))$. For the ideal $\mathfrak{a} = I_0$ in Lemma 30.3.1, use the pole test with its $H, w$. Each continuous truncation $H(\min(e_{I_0}, M))$ has integral at most $B - E(w)$ against $\mu_{z_i}$ and hence against $\mu_u$. Monotone convergence implies $\mu_u$ is carried by $\Omega_0$.

For any normalized finite-energy test $q$, the function equal to $u_* - q$ on $\Omega_0$ and zero outside is nonnegative and lower semicontinuous: on $\Omega_0$ use continuity of $u_*$, and at each point outside use nonnegativity. Its integral against $\mu_u$ is at most the lower limit of its integrals against $\mu_{z_i}$, by taking the supremum of continuous minorants and using Theorem 4.2.1. All these measures are carried by $\Omega_0 \subseteq \Xi$. Adding $E(q)$ and taking the supremum over $q$ gives $D_{\mathrm{rel}}(\mu_u) \le B$. This proves the bound and the characterization of the image. □

**Corollary 30.3.1** *The relative dual energy is weakly lower semicontinuous on the positive Radon measures of mass $V$. If $\nu_i \to \nu$ weakly, all have finite relative dual energy, and $D_{\mathrm{rel}}(\nu_i) \to D_{\mathrm{rel}}(\nu)$, then the normalized variational maximizers satisfy $u_{\nu_i} \to u_\nu$ strongly.*

***Proof*** For a weak limit of measures in a finite relative dual energy sublevel, the pole test passes to the limit by its continuous truncations, so the limit is carried by $\Omega_0$. The lower semicontinuous extensions of $u_* - q$ and the supremum over $q$ then put the limit in the same sublevel. This proves lower semicontinuity. Recall the deficit $\Delta_\nu(u) = D_{\mathrm{rel}}(\nu) - E(u) + \int_\Xi (u - u_*)\,\mathrm{d}\nu$ from the proof of Theorem 29.4.1. For the fixed test $q = u_\nu$, the same extensions give

$$\limsup_i \Delta_{\nu_i}(u_\nu) \le D_{\mathrm{rel}}(\nu) - E(u_\nu) - \int_\Xi (u_* - u_\nu)\,\mathrm{d}\nu = 0.$$

The deficits $\Delta_{\nu_i}(u_\nu)$ are nonnegative, and $\Delta_{\nu_i}(u_{\nu_i}) = 0$. Relative coercivity (Proposition 29.2.1) puts the maximizers in one energy ball on a tail. Apply the midpoint estimate as in Corollary 29.4.1 and identify the metric with the strong topology by Theorem 29.5.1. □

**Theorem 30.3.2** *For any big class $\alpha$, the normalized intrinsic Monge–Ampère operator is a homeomorphism from the strong finite-energy space onto the positive Radon measures of mass $V$ with finite relative dual energy. The measure topology is weak convergence together with convergence of the relative dual energy. A finite-energy potential solves the equation precisely when it maximizes the relative dual objective.*

***Proof*** We first prove a testing property. Suppose $\nu_i \to \nu$ weakly with finite relative dual energies uniformly bounded on a tail. The limit has finite relative dual energy by Corollary 30.3.1, and all the measures are carried by $\Omega_0$. For a normalized continuous-obstacle envelope $z$, the function $z - u_*$ is continuous on $\Omega_0$ by Lemma 30.3.1. It is bounded there by comparison with the envelopes of constants. Extend it to $X^{\sqsupset}$ by a lower constant outside $\Omega_0$, and separately by an upper constant there. The two extensions are lower and upper semicontinuous, respectively, and have equal integrals for every one of the measures. Approximation from below and above by continuous functions proves $\int_\Xi (z - u_*)\,\mathrm{d}\nu_i \to \int_\Xi (z - u_*)\,\mathrm{d}\nu$.

For a general normalized finite-energy $u$ take normalized continuous-obstacle envelopes $z_j \downarrow u$. Their energies converge to $E(u)$, and their rooftop distances to $u$ tend to zero because $z_j \geq u$. The estimate (29.9) bounds the integrals of $|z_j - u|$ uniformly for $\nu_i$ on a tail and for $\nu$. Thus the same testing convergence holds with $u$ in place of $z$. If normalized $u_i \to u$ strongly, their energies converge and their rooftop distances to $u$ tend to zero. Applying the same estimate to the pair $u_i, u$ gives the joint convergence

$$\int_\Xi (u_i - u_*)\,\mathrm{d}\nu_i \longrightarrow \int_\Xi (u - u_*)\,\mathrm{d}\nu. \tag{30.5}$$

For an arbitrary normalized $u$ choose decreasing normalized continuous-obstacle envelopes $z_i$ as above and set $\nu_i = \mathrm{MA}_\alpha(z_i)$, $\nu = \mathrm{MA}_\alpha(u)$. The measures converge weakly and have uniformly bounded relative dual energies by Theorem 30.3.1. At each $z_i$ the supporting equality holds by Corollary 28.4.1 and (29.2). Now (30.5) gives

$$D_{\mathrm{rel}}(\nu_i) = E(z_i) - \int_\Xi (z_i - u_*)\,\mathrm{d}\nu_i \longrightarrow E(u) - \int_\Xi (u - u_*)\,\mathrm{d}\nu.$$

Lower semicontinuity bounds $D_{\mathrm{rel}}(\nu)$ above by the right side, while testing at $u$ bounds it below. This proves the supporting equality at $u$. Together with Theorem 29.4.1 it gives the equation criterion and normalized bijection.

Finally let $u_i \to u$ strongly. The measures converge weakly by Theorem 29.5.2, and their relative dual energies are bounded on a tail by Theorem 30.3.1. Apply (30.5) to their supporting equalities. The relative dual energies converge. The inverse is continuous by Corollary 30.3.1. □

# Part VI
# Applications

On an integral regular projective curve the analytic rays give a direct way to solve the Monge–Ampère equation. The mass beyond each radius prescribes the slope there. Integrating the square of these slopes, weighted by the residue degrees, gives a criterion for the finite energy of the measure. On a split torus the corresponding local calculation uses volumes of polytopes of slopes. The toric models give the same volumes by intersection and make the section weights explicit. The second toric chapter solves the equation for arbitrary measures on the Gauss section, characterizes their finite energy by integrability of the roof, gives moment conditions sufficient for finite energy and for strong convergence, compares mixed measures under finite covers, and on the blowup of the projective plane at a torus-fixed point exhibits a finite-energy potential in a big class whose Monge–Ampère measure has infinite absolute dual energy.

A valuation filters every space of sections by order of vanishing. For a valuation of linear growth on an integral projective variety over an algebraically closed field, or on an integral regular projective curve, the final chapter identifies the limit of the average normalized order of the sections of the powers of an ample line bundle with the energy of the Dirac mass at the valuation, using the section lattices of the global energy theory. The curve and toric chapters supply its two worked cases, in which the invariants become a ray length and the position of a barycentre.

# Chapter 31
# Monge–Ampère on a projective curve

On a regular projective curve every nontrivial valuation lies on a ray indexed by a closed point. The mass beyond a point of a ray determines the slope of the solution there, and squaring these slopes and integrating them with the weight $d_p$ of (20.2) on the ray of $p$ gives exactly the energy condition on the measure. Theorem 31.2.1 proves both formulas, including the case of a measure whose support meets infinitely many rays. Its proof approximates the measure by measures supported at the trivial point and finitely many rational radii, which Lemma 31.1.1 solves by model potentials. The chapter ends with explicit atoms and a decaying tail on one ray.

Fix an integral regular projective curve $C$ over a trivially valued field $k$, and write $d_p = [\kappa(p) : k]$ for the residue degree of a closed point $p$. Let $\omega$ be an ample numerical class and put $V = \deg \omega > 0$. Write $v_0$ for the trivial point and $x_{p,t} = t \operatorname{ord}_p$ for the points of the $p$-ray, with endpoint $x_{p,\infty}$ as in Theorem 1.4.1. By $\mu_u$ we mean the Monge–Ampère measure of $u$ divided by $V$. We use the normalized energy $e$ of Chapter 19 and the dual energy $e^\vee$ of Section 19.1.

Chapter 34 uses Theorem 31.2.1 to compute the expected vanishing orders of valuations on a curve, and Section 33.2 compares it with a toric calculation on the projective line.

## 31.1 Model potentials on the rays

**Lemma 31.1.1** *Suppose $\mu$ is a probability measure supported on $v_0$ and finitely many $x_{p,a/b}$, with $a, b > 0$ integers. There is a semipositive real model potential $u$, normalized by $u(v_0) = 0$, given by*

$$u(x_{p,t}) = -\frac{V}{d_p} \int_{(0,\infty]} \min(t, s)\, \mathrm{d}\mu_p(s).$$

*Here $\mu_p$ is the restriction of $\mu$ to the $p$-ray, expressed in its parameter. Its normalized measure is $\mu_u = \mu$.*

***Proof*** With the notation of Lemma 20.3.1, put

$$u = \sum_{p,s} \frac{V\mu(\{x_{p,s}\})}{d_p} h_{p,s}.$$

This is the stated profile, since the sum is finite. By (20.6), the measure of $(\omega, u)$ has the prescribed mass $V\mu(\{x_{p,s}\})$ at each bend and the remaining mass $V\mu(\{v_0\})$ at $v_0$. It is nonnegative. The semipositivity criterion in Theorem 20.3.1 gives the result; by Theorem 17.4.1, $u$ is $\omega$-psh. □

## 31.2 The solution and its energy

For a Radon probability measure $\mu$ on $C^{\mathrm{an}}$ define its tail on each ray by

$$M_p(r) = \mu_p((r, \infty]) \qquad (0 \le r < \infty).$$

By Theorem 1.4.1 the rays are pairwise disjoint open subsets of $C^{\mathrm{an}}$. Since $\mu$ has mass one, for each integer $N \ge 1$ at most $N$ rays have mass at least $1/N$, so the rays of positive mass form an at most countable set. A compact subset of the open union of the zero-mass rays meets only finitely many rays, again by Theorem 1.4.1, and so has zero mass. By inner regularity this union has zero mass. Sums below are over the positive-mass rays.

**Theorem 31.2.1** *Let $\mu$ be a Radon probability measure on $C^{\mathrm{an}}$, with ray tails $M_p$ as above. Set*

$$Q(\mu) = \sum_p \frac{1}{d_p} \int_0^\infty M_p(r)^2 \, \mathrm{d}r,$$

$$u_\mu(x_{p,t}) = -\frac{V}{d_p} \int_0^t M_p(r) \, \mathrm{d}r, \qquad u_\mu(v_0) = 0.$$

*Then $u_\mu$ is $\omega$-psh. The measure $\mu$ has finite dual energy if and only if $Q(\mu) < \infty$. In this case $u_\mu$ is the unique normalized finite-energy solution $u$ of $\mu_u = \mu$, and*

$$e(u_\mu) = -\frac{V}{2} Q(\mu), \qquad e^\vee(\mu) = \frac{V}{2} Q(\mu).$$

***Proof*** Exhaust the positive-mass rays by increasing finite sets $F_j$. For $p \in F_j$ send $x_{p,s}$ to the point of the same ray with radius

$$s_j = 2^{-j} \left\lfloor 2^j \min(s, j) \right\rfloor.$$

Radius zero means $v_0$. Send all points of the other rays to $v_0$, and fix $v_0$. The push-forward measure $\mu_j$ is supported at finitely many rational breakpoints and at $v_0$. On every positive-mass ray the radii increase to $s$, including when $s = \infty$. The tail functions $M_{j,p}$ for $\mu_j$ increase pointwise to $M_p$: if $s > r$, the approximating radius is

eventually larger than $r$, whereas if $s \le r$ it never is. Pointwise convergence of the images on a set of full $\mu$-mass also implies $\mu_j \to \mu$ weakly.

Let $u_j$ be the model potential of Lemma 31.1.1 for $\mu_j$. Use the identity

$$\min(t, s) = \int_0^t \mathbf{1}_{\{r<s\}} \, \mathrm{d}r$$

and Tonelli's theorem with monotone convergence to see that $u_j$ decreases to $u_\mu$ at each point. The limit is finite at $v_0$, so it is psh by Theorem 6.2.2. The dimension-one case of (17.3), with $\mathrm{MA}(\omega, 0) = V\delta_{v_0}$ by (20.6), averages the integrals against $\delta_{v_0}$ and $\mu_j$. Since $u_j(v_0) = 0$, Tonelli's theorem gives

$$e(u_j) = \frac{1}{2}\int u_j \, \mathrm{d}\mu_j = -\frac{V}{2}\sum_p \frac{1}{d_p}\int_0^\infty M_{j,p}(r)^2 \, \mathrm{d}r.$$

Decreasing continuity of energy (Section 17.7) and monotone convergence give

$$e(u_\mu) = -\frac{V}{2}Q(\mu),$$

with value $-\infty$ allowed.

If $Q(\mu) < \infty$, the solution has finite energy, and decreasing weak continuity of the measures (Theorem 17.8.1) gives $\mu_{u_\mu} = \lim_j \mu_j = \mu$. Again Tonelli's theorem yields $\int u_\mu \, \mathrm{d}\mu = -VQ(\mu)$. Then Proposition 19.2.1 identifies the dual energy. If $v$ is another normalized finite-energy solution, then $\iota(u_\mu, v) = V^{-1}I_\omega(u_\mu, v) = 0$ by the definition of $I_\omega$ in Section 17.11, since the two measures are equal. Uniqueness follows from the strong potential separation in Theorem 19.4.1.

If $Q(\mu) = \infty$, use the bounded potentials $u_j$ as tests in the dual energy. Their integrals involve the product $M_{j,p}M_p$, so $M_{j,p} \le M_p$ gives

$$e(u_j) - \int u_j \, \mathrm{d}\mu \ge \frac{V}{2}\sum_p \frac{1}{d_p}\int_0^\infty M_{j,p}(r)^2 \, \mathrm{d}r.$$

The right-hand side increases to infinity. Thus the dual energy is infinite. □

For example, a unit atom at a finite positive radius $s$ has dual energy $Vs/(2d_p)$. Its solution on the ray is $-(V/d_p)\min(t, s)$ and is zero on the other rays. If $s$ is irrational, the solution has a breakpoint outside the rational divisorial radii of model functions. An atom at an endpoint has constant positive tail on its ray and therefore fails the finite-energy condition. A probability measure with uniform radial density on $[1, 2]$ along one ray has tail $2 - r$ on $[1, 2]$. The solution is quadratic on the interior of that interval.

*Example 31.2.1 (A decaying tail on one ray)* Fix a closed point $p$ and a real number $\alpha > 0$. The density $\alpha(1+r)^{-\alpha-1}\,\mathrm{d}r$ on the finite positive $p$-ray has total mass one and tail $M_p(r) = (1+r)^{-\alpha}$. By Theorem 31.2.1, its dual energy is finite exactly when $\alpha > 1/2$, since

$$\int_0^\infty (1+r)^{-2\alpha}\,\mathrm{d}r = \frac{1}{2\alpha - 1} \qquad (\alpha > 1/2)$$

and the integral diverges for $2\alpha \le 1$. In the finite-energy range the dual energy is $V/(2d_p(2\alpha-1))$. The potential is finite at the endpoint exactly when $\alpha > 1$: its value there is $-V/d_p$ times $\int_0^\infty (1+r)^{-\alpha}\,\mathrm{d}r$. When finite that value is $-V/(d_p(\alpha-1))$.

# Chapter 32
# Toric model measures

Take $P = [0, 2]$ and the affine roof $\gamma(m) = m/2 + 1/4$. The test configuration of Proposition 32.2.1 for this roof has one central component, coming from the cone facet with primitive inward normal $(-2, 4, -1)$. The parameter $t$ vanishes to order 2 along it, and the polarization has degree one on the reduced component, the length of $P$ in the lattice of even integers. The model measure is therefore $2\delta_{\eta_{-1/2}}$. In character coordinates the slopes of the weight fill an interval of length 2 at $w = -1/2$, and the local product gives the same atom.

As on the valuation rays of a curve in Chapter 31, a change of slope contributes mass at the bend. In dimension $n$ the active affine terms at a bend form a polytope of slopes, and $n!$ times its lattice volume is the local mass. A facet of the model gives the same number by intersection, since its multiplicity cancels the index of the lattice in which its degree is computed.

The base field $k$ is trivially valued, and $T = \operatorname{Spec} k[M]$ is a split torus of dimension $n$ with dual lattice $N$ and Gauss section $\eta : N_{\mathbb{R}} \to T^{\mathrm{an}}$. Polytopes are lattice polytopes in $M_{\mathbb{R}}$. They are full-dimensional except for the semiample polarizations of Sections 32.8 and 32.9, and $X_P$ is the polarized toric variety of $P$. Lemma 32.1.1 reduces calibrated integrals on the torus to lattice integrals on $N_{\mathbb{R}}$. Proposition 32.2.1 and Theorem 32.2.1 construct the test configuration of a rational convex roof and identify its model measure with the product (32.2) in character coordinates. The comparison extends to mixed products for polytopes with a common normal fan and to continuous roofs (Theorem 32.3.1 and Corollary 32.3.1). Counting monomial sections gives the energy $-n! \int_P \gamma \, \mathrm{d}m$ and the limiting distribution of the section weights (Theorem 32.4.1).

Sections 32.5 and 32.6 then solve the Monge–Ampère equation on the Gauss section. Theorem 32.5.1 places prescribed masses at finitely many real Gauss points by a maximum of affine roofs, and Theorem 32.6.1 passes to compactly supported measures by uniform approximation. A closed convex roof, finite on the interior of $P$, still defines a psh potential whose energy is given by the same integral, possibly $-\infty$ (Proposition 32.7.1). Its real measure is the image of $n!$ times lattice volume under $-\nabla\gamma$ (Proposition 32.7.2), and the measures of its truncations on the strict sets increase to the Gauss pushforward of this full-mass measure (Proposition 32.7.3).

Semiample polarizations permit different normal fans and polytopes of smaller dimension (Proposition 32.8.1). Mixed truncations retain the full intersection mass, also for roofs of energy $-\infty$ (Theorem 32.9.1 and Example 32.9.1).

Chapter 33 builds on the closed roofs and the gradient formula to solve the equation for every positive Radon measure of mass $n!\,\mathrm{vol}_M(P)$ on $N_{\mathbb{R}}$ and to characterize finite energy. It also pulls back the semiample measures and the mixed truncations to finite covers. Chapter 34 uses the Gauss valuations of Section 32.1, the normality and Veronese arguments in the proof of Proposition 32.2.1, and the vertex charts of Section 32.3 to compute the expected vanishing orders of toric valuations.

## 32.1 Integration in character coordinates

Let $T = \operatorname{Spec} k[M]$ be a split torus of dimension $n$, where $M$ is a lattice and $N$ its dual. Write $w = \operatorname{trop}(x)$ for the restriction of a semivaluation $x$ on $T$ to the characters. The Gauss section $\eta : N_{\mathbb{R}} \to T^{\mathrm{an}}$ sends $w$ to the seminorm $|\sum_m a_m T^m| = \max_{a_m \neq 0} \exp(-\langle m, w\rangle)$. It is continuous by this formula, and $\operatorname{trop} \circ \eta$ is the identity.

**Lemma 32.1.1** *For a smooth $(n,n)$-form $\Omega$ on $N_{\mathbb{R}}$ and a compactly supported ordinary smooth test function $\chi$ on $T^{\mathrm{an}}$, the calibrated integral is*

$$\int_{T^{\mathrm{an}}} \chi \, \operatorname{trop}^* \Omega = \int_{N_{\mathbb{R}}} (\chi \circ \eta)\, \Omega. \tag{32.1}$$

*The integral on the right uses the lattice of $N$.*

***Proof*** Locally a smooth test is a smooth function of the negative log norms of analytic units. We can replace any finite list of these norms near a given point by norms of rational units. To see this, choose a closed evaluation neighborhood inside the analytic neighborhood, using bounds on the Laurent generators and finitely many upper and positive lower bounds on Laurent polynomials. Its algebra is given by the rational affinoid presentations in Chapter 2. The original coordinates and the denominators adjoined in the presentation generate a dense subalgebra of rational functions. Approximate each unit so closely at the chosen point that the error has strictly smaller norm than the unit. This strict inequality holds on a smaller neighborhood by continuity, and the ultrametric inequality gives equality of the two norms there. This is the same argument as in Section 20.1.

Use the joint moment consisting of the original characters and the test coordinates, and project its target back to the original character torus. A cell on which this projection has rank less than $n$ gives zero after pulling back the $(n,n)$-form. On a full-rank cell, compute the calibration using this projection as the coordinate moment. The composite analytic map is the identity on the neighborhood in $T^{\mathrm{an}}$. It is an isomorphism onto that neighborhood, so its local finite-flat degree is one. Thus the integral on this cell is ordinary lattice integration in the projected coordinates. More explicitly, in the rational presentations write each extra coordinate as a ratio of Laurent polynomials. Off the finitely many hyperplanes where two of their monomials

have equal weight, the extra tropical coordinates are integral linear functions of $w$. The lattice of their graph projects isomorphically to $N$. The degree and the lattice both give weight one in the projected coordinates.

Finally, away from the countable union of all proper rational linear hyperplanes, the fibre over $w$ is the single point $\eta_w$. Indeed all distinct Laurent monomials have distinct weights there, so a sum takes the norm of its unique largest term. The ordinary integral of a smooth top form of full projected rank gives zero mass to these hyperplanes. Consequently on the overlap of two test presentations the projected coefficient contributing to the integral is the same, and is read at $\eta_w$. A compact test support admits a finite cover by such presentations. The local formulas integrate the same coefficient on the corresponding subsets of the Gauss section, so they give (32.1). □

For continuous convex functions $H_1, \dots, H_n$ on $N_{\mathbb{R}}$, the product computed by smooth convex approximation in the character coordinates is the measure

$$\nu_T(H_1, \dots, H_n) = \eta_*\big(d'd''H_1 \wedge \cdots \wedge d'd''H_n\big). \tag{32.2}$$

Indeed for smooth inputs Lemma 32.1.1 gives this identity on tests. For arbitrary continuous convex inputs choose smooth convex approximations converging locally uniformly. They exist by convolution in the character coordinates. The polyhedral construction in Theorem 11.2.1 is independent of the choices and is continuous under locally uniform convergence of the inputs. For compactly supported continuous $\chi$ on $T^{\mathrm{an}}$, the pullback $\chi \circ \eta$ is continuous with compact support, since its support is contained in $\mathrm{trop}(\operatorname{supp}\chi)$. The same weak convergence thus holds on the torus. This proves the approximation statement for the product in (32.2). Any ordinary analytic product that agrees with the smooth integrals and is continuous under these approximations gives the measure in (32.2).

## 32.2 Models from rational convex roofs

We compute the intersection measure on a split toric model by counting lattice points in the facets of its defining polyhedron. There are two lattices in the calculation. The section ring is graded by the tensor power, while the divisor order is taken in the field of rational functions, where that degree is zero.

Keep the lattices $M, N$ of Section 11.8. Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope, and let $\gamma : P \to \mathbb{R}$ be a rational convex piecewise affine function. For an integer $d \geq 0$ consider the monomials with triples $(m, \ell, d)$ satisfying

$$m \in dP, \qquad \ell \geq d\gamma(m/d) \quad (d > 0).$$

At $d = 0$ take $m = 0, \ell \geq 0$. In both cases the triples are integral. They form a monoid $S_\gamma$ in $M \oplus \mathbb{Z} \oplus \mathbb{Z}$. In the monoid algebra $R_\gamma = k[S_\gamma]$, the monomial with triple $(0, 1, 0)$ is denoted $t$. The grading is by $d$.

**Proposition 32.2.1** *The projective scheme $\mathcal{X}_\gamma = \operatorname{Proj}_{k[t]} R_\gamma$, with its rational polarization $\mathcal{O}(1)$, is a normal relatively ample test configuration for the polarized toric variety $X_P$ defined by the lattice monoid in the cone over $P$. The components of the special fibre are indexed by the lower facets of the cone over the epigraph of $\gamma$. For a lower facet with primitive inward normal $(q, b, c) \in N \oplus \mathbb{Z} \oplus \mathbb{Z}$, $b > 0$, write $P_F$ for the projection of its $d = 1$ section onto $P$. The component contributes*

$$n!\,\mathrm{vol}_M(P_F)\,\delta_{\eta_{q/b}}$$

*to the model measure, where $\eta_w$ is the torus Gauss valuation of weight $w$.*

***Proof*** The monoid is the intersection of a rational polyhedral cone with the lattice. To see finite generation, take integral generators of the rays of the cone. Write a lattice point of the cone as a nonnegative combination of these, and subtract the integral parts of the coefficients. The remainder belongs to a bounded set of lattice points in the cone. Those remainders together with the ray generators generate the monoid. Its group of differences is the whole lattice, since adding a sufficiently large multiple of an interior lattice point carries any lattice vector into the cone. The ring is normal: in the Laurent monoid algebra of the whole lattice, take the intersection of the nonnegative valuation rings of all the facet orders. A facet order on a Laurent polynomial is the minimum of the pairings of the exponents with the inward normal. Its multiplicativity follows by taking the lowest homogeneous part for this grading. The intersection, taken inside the Laurent algebra, is exactly $R_\gamma$. The Laurent algebra and the valuation rings are integrally closed in the common fraction field.

A sufficiently divisible Veronese is generated in degree one over $k[t]$. For example take a common multiple $h$ of the positive degrees of a finite set of generators. A monomial in them can be written as a product of pure powers of degree $h$ and a remainder of degree less than $sh$, where $s$ is the size of the generating set. In a degree divisible by $sh$, split off blocks of $s$ such pure powers until only a factor of degree $sh$ is left. This proves the assertion. It gives projectivity and defines the ample rational line bundle $\mathcal{O}(1)$ by division of the tautological bundle of that Veronese. The projective charts are normal, as the degree-zero parts of homogeneous localizations of a normal graded ring. They are torsion-free over $k[t]$, hence flat. Inverting $t$ removes the inequality on $\ell$ and gives the marked product for the monoid of the cone over $P$. Its dense torus has coordinate ring $k[M]$, obtained in degree zero after localizing by an interior monomial in that cone. Scaling $t$ gives the action and proves the test-configuration assertion.

The lower-facet condition means exactly that the facet order of $t$ is positive. The radical of $(t)$ is a monomial ideal. Indeed it is homogeneous for the lattice grading, as can be checked by extracting the extremal term of a power with a generic linear ordering of the finitely many exponents. Subtracting that term and repeating gives every component of a polynomial in the radical. A monomial of exponent $u$ lies in the radical if and only if $ju - (0, 1, 0)$ lies in the cone for some positive integer $j$. Equivalently, the order of $u$ is strictly positive at every lower facet. Thus the irreducible components of the reduced central projective fibre have homogeneous coordinate rings $k[S_\gamma \cap F]$ for the lower facets $F$.

Fix a facet $F$ and let $g$ generate $\langle q, M\rangle + b\mathbb{Z}$ in $\mathbb{Z}$, with $g > 0$. Choose a monomial with exponent in the relative interior of $F$, of positive degree. Localize by it and take degree zero. The resulting monoid consists of the pairs $(m, \ell)$ with $\langle q, m\rangle + b\ell \geq 0$. Indeed adding a sufficiently high multiple of the chosen exponent makes every other facet inequality hold, and gives a numerator of the required degree. The homomorphism

$$(m, \ell) \longmapsto (\langle q, m\rangle + b\ell)/g$$

is primitive. Choosing a pair mapping to 1 identifies the chart ring with a polynomial ring in one variable over the Laurent algebra of its kernel. The special component there is the zero divisor of that variable. Consequently $\operatorname{ord}_F(t) = b/g$ on $\mathcal{X}_\gamma$. The normalized order on a character $\chi^m$ is $\langle q/b, m\rangle$. Terms of the same order leave distinct Laurent monomials in the kernel algebra, so the order of a sum of characters is its minimum order. This identifies the attached valuation with $\eta_{q/b}$.

Let $M_F \subset M$ be the sublattice of $m$ with $\langle q, m\rangle \equiv 0 \pmod b$. Its index is $b/g$. For degrees $d$ divisible by $b$, the slice of the facet in degree $d$ projects bijectively on the lattice points of $dP_F$ in $M_F$: the facet equation $\langle q, m\rangle + b\ell + cd = 0$ determines the integral $\ell$. The leading term of this lattice count is $d^n \operatorname{vol}_{M_F}(P_F)$, by Riemann sums. The Hilbert polynomial of the facet ring in divisible degrees therefore gives degree $n! \operatorname{vol}_{M_F}(P_F)$ for $O(1)$ on the reduced component. Since $\operatorname{vol}_{M_F} = \operatorname{vol}_M/(b/g)$, the multiplicity $b/g$ cancels the lattice index in (17.1). The formula may be computed on a model dominating the product and $\mathcal{X}_\gamma$. Push its central fibre cycle to $\mathcal{X}_\gamma$ and use the projection formula. A component contracted in this pushforward contributes zero, while one dominating a lower-facet component has the same valuation after normalizing by the order of $t$. This gives the asserted mass. □

**Theorem 32.2.1** *The top product of the model metric of Proposition 32.2.1, computed in its character coordinates by (32.2), equals the model intersection measure. Both measures are supported on the Gauss section of the dense split torus.*

***Proof*** Choose $m_0 \in P \cap M$ to trivialize the rational polarization on the torus by its monomial section. In a divisible degree the monomials $t^\ell \chi^m$ generate the model bundle, with least $\ell = \lceil d\gamma(m/d)\rceil$ for each $m \in dP \cap M$. The local additive metric weight is therefore the pullback by tropicalization of

$$H(w) = \langle m_0, w\rangle + \max_{m \in P}\{-\langle m, w\rangle - \gamma(m)\}.$$

Indeed the Fubini–Study weights are $-\ell$ by Theorem 5.5.1, and one may take the degree divisible enough to realize all lifted vertices of the linearity cells of $\gamma$. The maximum on each cell is attained at one of those vertices.

The maximizing $m$'s in $H(w)$ span a full-dimensional body precisely when they form the projection of a lower facet of the roof. In that case $w = q/b$, and their gradients in $H$ comprise $m_0 - P_F$, with lattice volume $\operatorname{vol}_M(P_F)$. Use (11.3) and (32.2). The result is the same atomic measure on the Gauss section as in Proposition 32.2.1. □

*Example 32.2.1* For the roof $\gamma(m) = m/2 + 1/4$ on $P = [0, 2]$ from the start of Chapter 32, the lower facet has primitive inward normal $(q, b, c) = (-2, 4, -1)$. Here $b = 4$ and $g = 2$, so $\operatorname{ord}_F(t) = b/g = 2$, and $M_F = 2\mathbb{Z}$ has index 2. The interval $P_F = P$ has length 1 in $M_F$, so $O(1)$ has degree 1 on the reduced component. The model measure is therefore $2\delta_{\eta_{q/b}} = 2\delta_{\eta_{-1/2}}$. The slope of the affine roof also places the entire subgradient interval of length 2 at $w = -1/2$.

## 32.3 Mixed products and continuous roofs

To extend Theorem 32.2.1 to mixed products, let $P_1, \dots, P_r$ be full-dimensional lattice polytopes with the same minimizing normal fan in $N_{\mathbb{R}}$. A cone at a vertex consists of the linear functionals whose minimum is attained at that vertex. We explain the common toric variety and the tensor products before comparing the measures.

On $X_P$, localizing by the monomial of a vertex $m$ and taking degree zero gives the chart with monoid $\mathbb{R}_{\geq 0}(P - m) \cap M$. In fact any lattice vector in this cone belongs to $d(P - m)$ for sufficiently large $d$. The vertex monomials give a projective cover, since a positive multiple of every positive-degree monomial exponent is a sum of the vertex exponents. On the intersection of the charts for $m, m'$, the second vertex monomial becomes the unit $T^{m'-m}$ in the first chart. On monoids this localization leaves exactly the inequalities from the normal cone at $m$ that vanish on $m' - m$. To verify sufficiency, add a large multiple of $m' - m$ to restore the other inequalities. The remaining cone of inequalities is the intersection of the normal cones at $m$ and $m'$. Thus the charts and their intersections glue in the same way for all $P_i$, by the identity on characters. The line bundle of each polytope has the vertex monomial as a frame on the corresponding chart. For the sum of the polytopes the minimizers are sums of the minimizers. The normal fan is therefore the same, and the frame for the sum is the product of the individual frames. This identifies the bundle of the sum with their tensor product on the common variety $X$.

**Theorem 32.3.1** *For $n$ lattice polytopes $P_i$ with this common fan, let $\gamma_i$ be rational convex roofs and let $H_i$ be the associated additive weights in character coordinates. The mixed model measure equals $\nu_T(H_1, \dots, H_n)$. At the Gauss point $\eta_w$ its mass is $n!$ times the mixed lattice volume of the $n$ subgradient polytopes at $w$.*

***Proof*** First identify the sum of any positive number of these model pairs. Write $Q_i$ for the epigraph of $\gamma_i$. The Minkowski sum of the $Q_i$ is the epigraph on $\sum P_i$ of the infimal convolution

$$\gamma(m) = \min_{\sum m_i = m} \sum_i \gamma_i(m_i).$$

The minimum is attained on the compact set of feasible $m_i$. The epigraph is rational polyhedral, since the sum of the convex hulls of finitely many rational vertices plus the upward ray is the corresponding convex hull of the vertex sums plus that ray.

Choose $d$ divisible enough that all the bundles of degree $d$ are generated by graded monomials, and that the vertices of each $Q_i$ become integral in that degree. Choose

the same divisibility for the bundles of all nonempty sums. On a common model the tensor product bundle is generated by products of those monomial section lists. The same products give sections on the epigraph model of $\gamma$. In fact their exponent points in degree $d$, divided by $d$, belong to $\sum Q_i$. They include every vertex of $\sum Q_i$. For a linear functional with a unique minimum at a given vertex of the sum, the minimizing faces of the summands must all be vertices. A direction with this property exists by taking a functional in the interior of the normal cone of the vertex. The functional has positive slope in the vertical direction because its minimum is unique on a polyhedron containing upward rays.

These products generate the bundle of degree $d$ on the model of the sum. Indeed their exponents, together with the exponent of $t$, generate the full cone. For any positive-degree monomial, a power of its exponent is a sum of these exponents with integer nonnegative coefficients. Some term in the sum comes from a product section, since that term has positive degree. Thus a power of the monomial lies in the ideal of the products. The distinguished opens of the product sections cover the projective model. Consequently the two models have the same Fubini–Study weights and section lists on the marked generic variety. By Theorem 5.5.1, the potential of their common class is the sum of the potentials. The weight on the torus is consequently the sum of the $H_i$.

Apply Theorem 32.2.1 to every nonempty subset sum. The wedge products in (32.2) and the intersection measures are symmetric and multilinear. Subtract the subset top powers with signs $(-1)^{n-|I|}$ for subsets $I$ and divide by $n!$. All terms except the mixed term cancel, which proves equality. The mass formula is then Corollary 11.8.1. □

A continuous convex function $\gamma$ on $P$ is a uniform limit of rational convex piecewise affine roofs. Extend it by $+\infty$ off $P$. By Theorem 4.5.1, its affine minorants approximate its value at each point of $P$ from below. Given an error $\varepsilon > 0$, finitely many suffice to approximate throughout $P$, by continuity and compactness. Approximate their coefficients by rationals uniformly on $P$ and subtract a rational constant of size at most $\varepsilon$ to keep the resulting functions below $\gamma$. Their maximum gives the asserted approximation with error tending to zero.

For two rational roofs on the same $P$, the model potentials satisfy

$$\|\varphi_{\gamma_1} - \varphi_{\gamma_2}\|_\infty \le \|\gamma_1 - \gamma_2\|_\infty. \tag{32.3}$$

Indeed in every sufficiently divisible degree $d$ the Fubini–Study section lists use the same monomial sections of $dO(1)$ on $X_P$, with weight $-\lceil d\gamma_i(m/d)\rceil$ at $m$. At any point at least one section is nonzero, and bounds on the difference of weights give the same bounds on the difference of maxima after division by $d$. The error is at most $1/d$. Let $d$ increase through the divisible degrees. Thus $\varphi_\gamma$ for a continuous convex roof has a continuous extension to $X_P^{\mathrm{an}}$ by uniform approximation. It is psh in the class of $O(1)$, by Theorem 6.2.2. More explicitly, choose uniform errors at most $2^{-j-2}$ and add constants $2^{-j}$ to make the approximating sequence decreasing. The constants tend to zero and preserve the class. Passing to the limit in uniform rational approximations, (32.3) holds for all continuous convex roofs on $P$.

**Corollary 32.3.1** *The equality of mixed measures of Theorem 32.3.1 holds for continuous convex roofs on the same polytopes, with the continuous psh extensions defined by (32.3). The measure gives full mass to the Gauss section of the dense torus.*

***Proof*** Take simultaneous uniform approximations by rational roofs. On $X^{\mathrm{an}}$ the model measures converge weakly to the global mixed measure by Theorem 17.5.1. On the dense torus their character weights converge locally uniformly, so their measures converge to (32.2) against compactly supported tests there. It remains to account for the mass near the toric boundary.

For the diagonal case suppose the approximating roofs are bounded in absolute value by $C$, and fix dual Euclidean norms. If $m$ minimizes $\langle m, w\rangle + \gamma_j(m)$ on $P$ and its distance to the boundary is at least $\delta$, take a unit vector $e \in M_{\mathbb{R}}$ with $\langle e, w\rangle = \|w\|$ and compare with $m - \delta e \in P$. The minimality gives $\delta\|w\| \le 2C$. Thus for $\|w\| > R$, the maximizing slope polytope, reflected and translated into $P$, is contained within distance $2C/R$ of its boundary. The full-dimensional cells of $\gamma_j$ in $P$ have disjoint interiors, since each is a region with a different affine formula. By Proposition 32.2.1, the sum of their masses with $\|w\| > R$ is at most $n!$ times the volume of this boundary neighborhood in $P$. This tends to zero uniformly in $j$ as $R$ increases. For the mixed case the positive mixed measure is at most $1/n!$ times the top measure of the sum. The roofs of the sum are bounded by the sum of the individual uniform bounds, by the minimum formula in the proof of Theorem 32.3.1. The same estimate applies on the sum polytope.

The Gauss image of the closed ball of radius $R$ is compact in $X^{\mathrm{an}}$. For weak convergence of Radon measures on this compact Hausdorff space, its mass in the limit is at least the upper limit of the approximating masses there. This follows by testing with continuous majorants of its indicator and using outer regularity. The preceding uniform estimate gives full mass to the increasing union of those compact sets. The equality on the dense torus now proves the asserted equality globally. □

## 32.4 Energy and section weights

Counting the monomial sections of a roof model computes the energy of its potential and the limiting distribution of the section weights.

**Theorem 32.4.1** *Let $\gamma$ be a continuous convex roof on $P$. Put $\theta = [\mathcal{O}_{X_P}(1)]$. For the potential $\varphi_\gamma$ of Section 32.3 the unnormalized energy is*

$$E_\theta(\varphi_\gamma) = -n! \int_P \gamma(m)\, \mathrm{d}m. \tag{32.4}$$

*For a rational roof the measures of scaled section weights, with mass $n!/d^n$ per monomial of degree $d$, converge to $n!(-\gamma)_*\,\mathrm{vol}_M|_P$. This measure extends to continuous convex roofs by uniform approximation, and its first moment is the energy in (32.4).*

***Proof*** Take a rational roof and compactify its model over $\mathbb{P}^1$ by the marking. In sufficiently divisible degree $d$, the sections over the affine parameter line are the graded module generated over $k[t]$ by

$$t^{\lceil d\gamma(m/d)\rceil}\chi^m, \qquad m \in dP \cap M.$$

Here the equality of the section module with the degree-$d$ part of $R_\gamma$ also follows directly from the monomial charts. On the marked dense torus over $t \neq 0$, a monomial frame of the bundle expresses a section as a finite Laurent polynomial in $t$ and the characters. Its summands are monomials of degree $d$ in the fraction ring of $R_\gamma$. To check extension, cover the projective model by distinguished opens of the positive-degree ray monomials, as in the proof of Theorem 32.3.1. A Laurent monomial of degree $d$ extends in the chart of such a monomial if adding a high multiple of that chart exponent carries it into the cone. This requires precisely the inequalities of the facets containing that ray. Every facet contains a positive-degree ray, since its dimension is $n + 1 \geq 2$ and the degree-zero face is the single vertical ray. Thus a monomial extends on the cover exactly when its exponent belongs to the cone. For a Laurent polynomial the condition is termwise, since the chart clearing factors are monomials.

Let $F_\infty$ be the fibre over infinity. Add $cF_\infty$ to the compactified bundle so that it is ample, and take $c > \sup_P |\gamma|$. At infinity the marking imposes the additional bound $\ell \leq cd$ on the powers $t^\ell$. With roof zero the model is the product, and the allowed powers start at zero. Thus the difference of the two section dimensions after adding the same $cF_\infty$ is

$$-\sum_{m\in dP\cap M} \lceil d\gamma(m/d)\rceil.$$

By Riemann sums, division by $d^{n+1}$ tends to $-\int_P \gamma \,\mathrm{d}m$. By the Hilbert polynomial the same limit is $1/(n+1)!$ times the difference of the top intersections of the two compactified bundles after adding $cF_\infty$. The terms from $cF_\infty$ cancel: its square is zero and the two restricted generic bundles have the same top degree on that fibre. The energy is $1/(n+1)$ times the difference of the original top intersections, by the model pairing definition. This proves (32.4) for rational roofs. If $\|u - v\|_\infty \leq \varepsilon$, monotonicity and the translation identity in Section 17.5 bound the difference of energies by $\varepsilon(\theta^n)$. Together with (32.3) this gives the formula for continuous roofs.

The section module modulo $t$ has one basis vector per $m \in dP \cap M$. With the convention of Fubini–Study weights its scaled weight is $-\lceil d\gamma(m/d)\rceil/d$. For a continuous test on the bounded range of weights, the rounding error tends uniformly to zero. Lattice Riemann sums therefore converge to the stated pushforward of volume. The same formula is continuous under uniform approximation of the roof. Integrating the identity function against this pushforward gives (32.4). □

## 32.5 Prescribed point masses on the Gauss section

This section solves the Monge–Ampère equation for finite atomic measures on the Gauss section. Consider the polarized split toric variety $X_P$ of Section 32.2. Write $V = n!\,\mathrm{vol}_M(P)$ for the total mass. The points we prescribe can have arbitrary real weights in $N_{\mathbb{R}}$.

**Theorem 32.5.1** *Take pairwise distinct $w_1, \dots, w_r \in N_{\mathbb{R}}$ and real numbers $a_i > 0$ such that $\sum_i a_i = V$. There exist real constants $c_i$ for which the roof*

$$\gamma(m) = \max_{1 \le i \le r} \{c_i - \langle m, w_i \rangle\} \qquad (m \in P) \tag{32.5}$$

*gives a continuous psh potential with Monge–Ampère measure $\sum_i a_i \delta_{\eta_{w_i}}$. The constants are unique up to adding a common constant. The cell where the $i$-th term attains the maximum has lattice volume $a_i/n!$.*

***Proof*** For $c = (c_1, \dots, c_r)$, denote the maximum in (32.5) by $\gamma_c$. Define

$$G(c) = n! \int_P \gamma_c(m)\,\mathrm{d}m - \sum_{i=1}^{r} a_i c_i.$$

It is convex, since $\gamma_c(m)$ is a maximum of affine functions of $c$, and continuous, since the maximum changes by at most the sup norm of a change in $c$. Adding a constant to all coordinates leaves $G$ unchanged, by the equality of total masses. Normalize $\max_i c_i = 0$ and choose $B$ with $|\langle m, w_i \rangle| \le B$ for all $m \in P$ and all $i$. One of the terms has $c_i = 0$, hence $\gamma_c \ge -B$. With $a_{\min} = \min_i a_i > 0$,

$$G(c) \ge -VB + a_{\min}(-\min_i c_i).$$

On this normalized set the sublevel sets are therefore closed and bounded in a finite-dimensional space. Taking a minimum on the sublevel set containing $c = 0$ gives a minimizer, which is global by translation invariance.

For each $c$, the terms can tie only on finitely many affine hyperplanes, since the $w_i$ are distinct. Off those hyperplanes the partial derivative of $\gamma_c$ with respect to $c_i$ is the indicator of the cell of term $i$. The difference quotients have absolute value at most one. Differentiating under the integral at the minimizer gives

$$0 = \frac{\partial G}{\partial c_i} = n!\,\mathrm{vol}_M(P_i) - a_i,$$

where $P_i$ is the closed cell of term $i$. Every $P_i$ is thus full-dimensional.

On $P$ the minimizers of $m \mapsto \gamma_c(m) + \langle m, w_i \rangle$ are exactly $P_i$. If for some $w$ the analogous minimum set has nonempty interior, then it contains an open subset inside one of the cells. There $\gamma_c$ is affine of slope $-w_i$; constancy of the sum on that subset forces $w = w_i$. On each of the finitely many real polyhedral cells the maximum defining the character weight is attained at a vertex, since the expression in $m$ is affine

there. The character weight is thus a maximum of finitely many affine functions of $w$, with no rationality restriction. At $w_i$, the active gradients have convex hull $m_0 - P_i$, which has the same lattice volume as $P_i$. The character product formula (11.3) and the continuous comparison in Corollary 32.3.1 give precisely the prescribed masses.

The preceding computation applies to every $c$, a cell of empty interior contributing no mass. Writing $P_i(c)$ for the cell of term $i$, the Monge–Ampère measure of the potential of $\gamma_c$ is therefore $\sum_i n!\,\mathrm{vol}_M(P_i(c))\,\delta_{\eta_{w_i}}$. A vector $c$ thus gives the prescribed measure exactly when its cells have the prescribed volumes.

For uniqueness, a vector with the prescribed cell volumes minimizes $G$, since the computed derivative vanishes and $G$ is convex. Let $c, c'$ be two such vectors. Equality in convexity of $G$ at their midpoint gives equality almost everywhere in the pointwise convexity inequality for the two maxima. Equality of these finite maxima at the midpoint requires a common maximizing index at the two endpoints. Off the tie hyperplanes each index is unique. Thus the cells agree up to sets of volume zero. Since every cell is closed, convex and full-dimensional, it is the closure of its interior, and two such cells that agree up to a set of volume zero coincide.

If two distinct cells $P_i, P_j$ meet, take $m$ in their intersection. Both terms maximize there for each vector, which gives

$$c_i - c_j = \langle m, w_i - w_j \rangle = c'_i - c'_j.$$

Their intersection graph is connected. Otherwise two disjoint nonempty finite unions of closed cells would disconnect $P$. Therefore $c_i - c'_i$ is independent of $i$, as asserted.□

## 32.6 Compactly supported measures on the Gauss section

The finite construction of Theorem 32.5.1 also solves the equation for measures supported on any compact part of the Gauss section. Fix the polarized toric variety $X_P$ and a point $m_* \in P$. Give $M_{\mathbb{R}}$ a Euclidean norm and $N_{\mathbb{R}}$ the dual norm.

**Theorem 32.6.1** *Let $\mu$ be a positive Radon measure supported on a compact set $C \subset N_{\mathbb{R}}$, with mass $V = n!\,\mathrm{vol}_M(P)$. There is a unique continuous convex roof $\gamma$ with $\gamma(m_*) = 0$ whose Monge–Ampère measure on $X_P^{\mathrm{an}}$ is $\eta_*\mu$. The roof has Lipschitz constant at most $\max_{w\in C} \|w\|$. For a sequence of such measures supported in the same $C$ and converging weakly to $\mu$, the normalized roofs converge uniformly.*

***Proof*** For each $j \geq 1$, choose a finite Borel partition of $C$ into sets of diameter less than $1/j$. For example, take intersections with a finite cover by sufficiently small balls, and remove the earlier sets from each successive set. Replace the mass of each positive-mass set by a point mass at any point of that set. The resulting measure $\mu_j$ has mass $V$ and converges weakly to $\mu$, by uniform continuity of continuous test functions on $C$.

By Theorem 32.5.1, $\mu_j$ gives a roof $\gamma_j$ that is the maximum of affine functions with slopes in $-C$. Subtract its value at $m_*$. This does not change its measure, since

adding a constant to the roof subtracts the same constant from the character weight. The roof has Lipschitz constant at most $L = \max_{w \in C} \|w\|$, by the bound on the slopes of the affine functions. It is therefore bounded in absolute value by $L \operatorname{diam}(P)$.

Choose a subsequence which converges on a countable dense set in $P$, by successive extraction of convergent subsequences and taking a diagonal. Given a uniform error, use a finite net from the dense set and the common Lipschitz bound to see that the subsequence is uniformly Cauchy on $P$. The limit $\gamma$ is convex, normalized at $m_*$, and has the same Lipschitz bound. The potentials converge uniformly by (32.3), and their global measures converge by Theorem 17.5.1. For a continuous test on $X_P^{\mathrm{an}}$, its pullback by $\eta$ is continuous on $C$. Thus the same measures converge weakly to $\eta_* \mu$. This proves existence.

Two such roofs give continuous psh potentials $u, v$ in the same ample class $\theta = [O_{X_P}(1)]$. Their equal measures force $\iota(u, v) = V^{-1} I_\theta(u, v) = 0$, by the definition of $I_\theta$ in Section 17.11. Theorem 19.4.1 then makes their difference constant. In the monomial frame used in Theorem 32.2.1, the character weights consequently differ by the same constant. After removing the common affine term, they are the convex conjugates evaluated at $-w$ of the two roofs extended by $+\infty$ off $P$. By Theorem 4.5.1 the roofs differ by the opposite constant. Their normalization makes it zero.

For a converging sequence of measures with the stated common support, the proved Lipschitz estimate and normalization hold for every solution. Every subsequence of roofs thus has a further uniformly convergent subsequence. The same weak measure comparison shows that its limit solves the equation for $\mu$. By uniqueness all these limits are $\gamma$. If the full sequence failed to converge uniformly, a subsequence at a fixed positive uniform distance from $\gamma$ would contradict this conclusion. □

## 32.7 Closed convex roofs

Keep the polarization $\theta = [O_{X_P}(1)]$, and assume $n \geq 1$. Write $\lambda = n!\, \mathrm{d}m$ for lattice volume multiplied by $n!$ on $P$, and $V = \lambda(P)$. For a finite-energy potential $u$ write $\mathrm{MA}_\theta(u)$ for $\mathrm{MA}((\theta, u)^n)$.

A *closed convex roof* is a lower semicontinuous convex function $\gamma : P \to \mathbb{R} \cup \{+\infty\}$ that is finite on $P^\circ$. We extend it by $+\infty$ off $P$. In the monomial frame with exponent $m_0$ put

$$H_\gamma(w) = \langle m_0, w \rangle + \sup_{m \in P} \{-\langle m, w \rangle - \gamma(m)\}. \tag{32.6}$$

Every roof is bounded below by compactness and lower semicontinuity. On an interior cube the maximum at its vertices gives an upper bound. Together these bounds give a Lipschitz estimate on any smaller concentric cube, by bounding secant slopes using segments extended to the larger cube. The character weight is finite and convex. It is Lipschitz, with constant at most $\max_{m \in P} \|m_0 - m\|$, by the supremum formula.

**Lemma 32.7.1** *If closed convex roofs satisfying $\gamma_j(a) = \gamma(a) = 0$ at $a \in P^\circ$ converge locally uniformly to $\gamma$ on $P^\circ$, then $H_{\gamma_j} \to H_\gamma$ locally uniformly on $N_{\mathbb{R}}$.*

***Proof*** For a boundary point $m \in P$, convexity and lower semicontinuity give the radial limit $\gamma((1-t)m+ta) \to \gamma(m)$ as $t \downarrow 0$. This also holds for an infinite boundary value by lower semicontinuity. Thus the supremum in (32.6) can be taken over $P^\circ$. Testing each fixed interior $m$ proves the lower bound for the limiting supremum. For the upper bound put $m_t = (1-t)m + ta$, where $0 < t < 1$. Since $\gamma_j(m_t) \le (1-t)\gamma_j(m)$, we have

$$-\langle m, w\rangle - \gamma_j(m) \le \frac{-\langle m_t, w\rangle - \gamma_j(m_t) + t\langle a, w\rangle}{1-t}.$$

The points $m_t$ lie in a compact subset of $P^\circ$. Take the supremum in $m$, use local uniform convergence on that subset, and then let $t \downarrow 0$. This proves pointwise convergence of the weights. The common Lipschitz bound upgrades it to local uniform convergence. □

**Proposition 32.7.1** *Every closed convex roof determines a $\theta$-psh potential $\varphi_\gamma$. On the dense torus its character weight is $H_\gamma$, and its energy is*

$$E_\theta(\varphi_\gamma) = -\int_P \gamma \, \mathrm{d}\lambda. \tag{32.7}$$

*The value $-\infty$ is allowed. The potential is the decreasing limit of the potentials of any increasing sequence of continuous convex roofs converging to $\gamma$.*

***Proof*** For $j \ge 1$ the functions

$$\gamma_j(m) = \min_{q \in P}\{\gamma(q) + j\|m - q\|\}$$

are continuous convex roofs. Compactness gives minimizers, convex combinations of minimizers prove convexity, and the triangle inequality gives a $j$-Lipschitz bound. The sequence increases and is bounded above by $\gamma$. For fixed $m$, if $\gamma_j(m)$ stays bounded then the minimizers tend to $m$ by the fixed lower bound on $\gamma$. Lower semicontinuity gives $\lim_j \gamma_j(m) \ge \gamma(m)$. Thus $\gamma_j$ increases to $\gamma$ everywhere.

The potential reverses the order of continuous roofs. For rational roofs this follows from the common section lists in the proof of (32.3), whose negative rounded weights decrease with the roof. Uniform approximation and translation by constants give the same order for continuous roofs. Thus the $\varphi_{\gamma_j}$ decrease. The roofs converge locally uniformly to $\gamma$ on $P^\circ$, by continuity of finite convex functions there and Proposition 4.1.1. Subtract their values at $a$, which converge to $\gamma(a)$, and apply Lemma 32.7.1. After restoring the constants the weights converge to $H_\gamma$. This argument holds for every such increasing approximation.

The decreasing potential limit is finite on the dense torus, in particular at the trivial valuation, and is psh by Theorem 6.2.2. Its restriction to that torus is $(H_\gamma - H_0)\circ \mathrm{trop}$,

since the zero roof gives the reference metric. Every divisorial point lies over the generic point of $X_P$, hence lies in the dense torus. Divisorial determination (Theorem 6.3.1) gives independence of the approximating sequence on all of $X_P^{\mathrm{an}}$. Apply (32.4) to the continuous roofs, use decreasing continuity of the energy in a fixed ample class (Section 17.7), and use monotone convergence on $P$ after subtracting a common lower bound. This gives (32.7). □

**Proposition 32.7.2** *For every closed convex roof, its real top measure is*

$$(d'd''H_\gamma)^n = (-\nabla\gamma)_*\lambda. \tag{32.8}$$

*The gradient is defined almost everywhere on $P^\circ$. In particular the measure has mass $V$.*

***Proof*** At any interior point $m$, choose affine minorants whose values at $m$ tend to $\gamma(m)$, using Theorem 4.5.1. Evaluating them at $m \pm r e_i$ for a fixed small $r > 0$ bounds their slopes. A limiting minorant gives a subgradient at $m$. The same evaluations centred at nearby points give uniform bounds for all their subgradients.

For each coordinate direction, one-sided derivatives of the restriction to a line disagree at at most countably many interior parameters, by monotonicity. Fubini's theorem gives existence of all partial derivatives on a set of full volume. At each such point every subgradient has the prescribed coordinate derivatives, so the subgradient is a singleton $\{s\}$. Any limit of nearby subgradients is still a subgradient at that point, by continuity and the supporting inequalities. Thus the nearby subgradients tend to $s$. The two supporting inequalities at $m$ and $m+h$ bound the increment of $\gamma$ between $\langle s, h\rangle$ and $\langle s_h, h\rangle$, where $s_h$ is a subgradient at $m + h$. This proves differentiability. The derivatives are measurable by the coordinate difference quotients.

For a rational polyhedral roof, the maximizing points in (32.6) form a polytope. On each cell of the roof the affine objective attains its maximum on the convex hull of its maximizing vertices. Thus the active gradient polytope for $H_\gamma$ consists of $m_0 - m$ for all maximizing $m$. At a differentiability point of the roof, membership in this set is equivalent to $\nabla\gamma = -w$, by the supporting inequality. The real atom formula (11.3) proves (32.8) in this case.

Approximate the closed roof on compact subsets of $P^\circ$ by rational polyhedral roofs, by regularizing it as in Proposition 32.7.1 and then approximating each continuous roof uniformly on $P$. Their weights converge locally uniformly by Lemma 32.7.1, after accounting for the values at $a$. Their gradients converge at every point of differentiability common to the limit and the countably many approximants. In each coordinate, bound the derivative of the approximant by forward and backward secants. Pass to the limit on fixed secants and then let their lengths tend to zero. This gives convergence of each component of the gradient.

The identity on the rational roofs now passes to the limit. The real top products converge on compactly supported continuous tests by Theorem 11.2.1, while the gradient integrals converge by bounded convergence on $P$. The boundary of $P$ has zero volume, so the right-hand side of (32.8) has mass $V$. □

For later use, the gradient convergence argument in the proof of Proposition 32.7.2 applies to any sequence of closed convex roofs converging locally uniformly on $P^\circ$. By (32.8) and bounded convergence, their real measures converge on all bounded continuous tests on $N_{\mathbb{R}}$.

**Corollary 32.7.1** *If $\gamma$ is integrable on $P$, then the unnormalized finite-energy measure is*

$$\mathrm{MA}_\theta(\varphi_\gamma) = \eta_*(d'd''H_\gamma)^n. \tag{32.9}$$

***Proof*** The energy formula gives finite energy. In the decreasing approximation by continuous roof potentials, their global measures converge weakly by Theorem 17.8.1. By Corollary 32.3.1, each of these measures is the pushforward of the real top measure. The extension of the gradient convergence argument recorded after Proposition 32.7.2 gives weak convergence of the real measures against all bounded continuous tests. Every continuous test on $X_P^{\mathrm{an}}$ pulls back by $\eta$ to such a test. This proves (32.9). □

**Proposition 32.7.3** *For any closed convex roof put $u = \varphi_\gamma$ and $u_j = \max(u, -j)$, where $j \geq 1$. Each $u_j$ is the potential of a continuous convex roof. The measures $\mathbf{1}_{\{u>-j\}}\,\mathrm{MA}_\theta(u_j)$ increase to $\eta_*(d'd''H_\gamma)^n$. In particular this truncation measure has mass $V$.*

***Proof*** Write $0_P$ for the zero roof extended by $+\infty$ off $P$. The function $h_j = \max(\gamma^\vee, 0_P^\vee - j)$ is finite convex and decreases to $\gamma^\vee$. If $L = \inf_P \gamma$, then $0_P^\vee - j \leq h_j \leq 0_P^\vee + \max(-L, -j)$. Conjugation shows that $\rho_j = h_j^\vee$ is finite and bounded on $P$ and has value $+\infty$ off $P$. Moreover $\rho_j \uparrow \gamma$, by interchanging the two suprema in the conjugate and using double conjugation.

A bounded lower semicontinuous convex function on a polytope is continuous. At a fixed point $m \in P$, write $P$ by its finite list of facet inequalities. Every unit direction from $m$ to a point of $P$ obeys the active inequalities, and the other inequalities have positive slack at $m$. Therefore a distance $r > 0$ independent of the feasible direction can be taken along it inside $P$. Convexity on this segment and the global upper bound imply upper semicontinuity at $m$. This proves continuity of $\rho_j$.

Double conjugation gives $H_{\rho_j} = \max(H_\gamma, H_0 - j)$. The potential $\varphi_{\rho_j}$ thus agrees with $u_j$ on the dense torus and hence everywhere by divisorial determination. The sequence $u_j$ consists of continuous finite-energy potentials. The strict-contact identity of Theorem 17.10.1, applied to $u_\ell$ and $-j$ for $\ell > j$, makes their measures agree on $\{u > -j\}$. This proves monotonicity of the restricted measures.

All the measures of $u_j$ are supported on the Gauss section by Corollary 32.3.1. On a compact part of that section, $u$ is bounded below by its continuous torus weight, so the restrictions leave continuous tests of compact support in $N_{\mathbb{R}}$ unchanged for large $j$. The roofs $\rho_j$ converge locally uniformly on $P^\circ$ by Proposition 4.1.1. Their real measures thus converge by the gradient formula. Tests of compact support on $N_{\mathbb{R}}$ identify the increasing limit on the Gauss section with $(d'd''H_\gamma)^n$. Its pushforward gives the assertion, since an increasing limit of measures concentrated on the section is still concentrated there. □

## 32.8 Semiample toric polarizations

Let $X = X_Q$ for a full-dimensional lattice polytope $Q \subset M_{\mathbb{R}}$, with $\dim X = n \geq 1$. Write $\Sigma$ for its minimizing normal fan and $\omega = [O_{X_Q}(1)]$. Let $P \subset M_{\mathbb{R}}$ be a nonempty lattice polytope. We allow $P$ to have smaller dimension and assume its minimum support function is linear on every cone of $\Sigma$. The minima of $\langle m, w\rangle$ for $w$ in the interior of a maximal cone $\sigma$ of $\Sigma$ occur at a unique vertex $m_\sigma$ of $P$. On the associated chart $\operatorname{Spec} k[\sigma^\vee \cap M]$ take $T^{m_\sigma}$ as the frame of a line bundle $L_P$. On the intersection of two charts the ratio of their frames is a unit, since $m_\sigma - m_{\sigma'}$ pairs to zero with $\sigma \cap \sigma'$. The weights $m \in P \cap M$ give global monomial sections, because $m - m_\sigma \in \sigma^\vee$ on every chart. The section of weight $m_\sigma$ generates on that chart, so $L_P$ is globally generated.

Conversely, suppose a toric line bundle with monomial frames $T^{m_\sigma}$ has a globally generated positive tensor power $a$. A global section of the power is a Laurent polynomial on the torus. Each of its monomials also extends to every chart, since it lies in the module $T^{am_\sigma} k[\sigma^\vee \cap M]$ in that chart. At the torus fixed point of the chart only the constant monomial of $k[\sigma^\vee \cap M]$ has nonzero value. Generation there thus gives a global section of weight $am_\sigma$. It follows that $m_\tau - m_\sigma \in \sigma^\vee$ for all $\sigma, \tau$. The convex hull of the $m_\sigma$ is a polytope of the stated kind and gives the original frames. This describes the semiample toric line bundles in terms of their polytopes.

Let $\gamma$ be a lower semicontinuous convex roof on $P$ finite on its relative interior. Choose $m_0 \in P \cap M$. Put $M_P = M \cap \operatorname{span}_{\mathbb{R}}(P - P)$ and regard $P - m_0$ as a full-dimensional lattice polytope in $M_{P,\mathbb{R}}$, carrying the translated roof $m \mapsto \gamma(m + m_0)$. The inclusion $M_P \subset M$ gives a map

$$r_P : X \longrightarrow X_{P-m_0}.$$

Indeed the dual map on cocharacter spaces takes each cone of $\Sigma$ into a normal cone of $P - m_0$. The inclusion of the corresponding dual monoids defines the map on the affine charts. The maps agree on intersections since they agree on the dense torus and the target is separated. The frames of the pullback are $T^{m_\sigma - m_0}$. Multiplying all of them by the same rational character identifies the pullback of $O(1)$ with $L_P$.

Set $\theta_P = [L_P]$. Pull back the roof potential from $X_{P-m_0}$ and denote it by $u_\gamma$. If $P$ is a point, the target is a point and the potential is the constant $-\gamma(m_0)$. In positive dimension, Proposition 32.7.1 and the pullback rule of Theorem 6.2.1 give $u_\gamma \in \operatorname{PSH}(X, \theta_P)$. The trivial valuation maps to the trivial valuation, where the roof potential is finite. On the dense torus of $X^{\mathrm{an}}$, $u_\gamma = (H_\gamma - H_0) \circ \operatorname{trop}$, with $H_\gamma$ defined by (32.6) on $P$. Here $H_0$ is its zero-roof weight. Continuous roofs give continuous potentials, and $\max(u_\gamma, -j)$ is the potential of a continuous roof by Proposition 32.7.3, applied on the target of $r_P$.

**Lemma 32.8.1** *For nonempty lattice polytopes $P_i$ and closed roofs $\gamma_i$ finite on* relint $P_i$*, the infimal convolution*

$$\gamma_I(m) = \inf_{\sum_{i\in I} m_i = m} \sum_{i \in I} \gamma_i(m_i) \tag{32.10}$$

*is a closed roof on $P_I = \sum_{i\in I} P_i$, finite on its relative interior. If all the $\gamma_i$ are continuous, so is $\gamma_I$. In additive frames $H_{\gamma_I} = \sum_{i\in I} H_{\gamma_i}$. The real top measure of this weight on $N_{\mathbb{R}}$ has mass $n!\,\mathrm{vol}_M(P_I)$.*

***Proof*** The summands are bounded below. The tuples lie in a compact product, so a finite infimum is attained and limiting tuples prove lower semicontinuity. Combining two minimizing tuples proves convexity. For $m$ in the relative interior of $P_I$, choose $a_i \in \operatorname{relint} P_i$, and put $a = \sum_i a_i$. For small $t > 0$, the point $(m - ta)/(1 - t)$ still belongs to $P_I$. Write it as a sum of $q_i \in P_i$. Then $m = \sum_i((1 - t)q_i + ta_i)$ is a sum of points in the respective relative interiors. This gives finiteness. In the continuous case the summands are bounded above, so the infimal convolution is bounded on $P_I$. The facet argument in the proof of Proposition 32.7.3, applied in the affine hull of $P_I$, gives continuity.

Taking the supremum in (32.6) over the product of the $P_i$ gives the weight identity. If $P_I$ is full-dimensional, its real mass is given by Proposition 32.7.2. Otherwise, in a frame whose exponent lies in $P_I$, the weight factors through the quotient by the annihilator of $P_I - P_I$. Convolution with smooth nonnegative approximate identities preserves this invariance. Each smoothed Hessian has rank less than $n$, so its top product vanishes. Local uniform convergence and Theorem 11.2.1 give the same result for the original weight. The full-dimensional lattice volume of $P_I$ is then zero.□

**Proposition 32.8.1** *Let $P_1, \dots, P_n$ have minimum support functions linear on every cone of $\Sigma$. For continuous convex roofs $\gamma_i$ and the resulting $\theta_{P_i}$-psh potentials $u_i$ on $X$, one has*

$$\mathrm{MA}((\theta_{P_i}, u_i)_{i=1}^n) = \eta_*\big(d'd''H_{\gamma_1} \wedge \cdots \wedge d'd''H_{\gamma_n}\big). \tag{32.11}$$

***Proof*** For an integer $a > 0$, the normal fan of $aP_i + Q$ is $\Sigma$. The minimizing face of a sum in a direction is the sum of the two minimizing faces. For a vertex sum the normal cone is therefore the intersection of the cones of its summands. On each maximal cone of $\Sigma$, both summands have a minimizing vertex and the cone of the vertex of $Q$ is contained in that of the vertex of $P_i$. Hence the maximal cones are exactly those of $\Sigma$. The vertex of the sum gives the product frame, so its polarization has class $a\theta_{P_i} + \omega$.

Take on $aP_i + Q$ the infimal convolution of the roof $a\gamma_i(\cdot/a)$ with the zero roof on $Q$. It is continuous by Lemma 32.8.1. Writing $H_Q$ for the zero-roof weight on $Q$, its weight is $aH_{\gamma_i} + H_Q$. The reference weight is $aH_{0,i} + H_Q$. Its potential on the dense torus thus agrees with $au_i$. Both potentials are $(a\theta_{P_i} + \omega)$-psh, by the nef inclusion in Theorem 6.2.1. The dense torus contains all divisorial points, so Theorem 6.3.1 gives equality everywhere.

Apply Corollary 32.3.1 to the polytopes $aP_i + Q$ with common normal fan $\Sigma$ and divide by $a^n$. This identifies the mixed measure of $(\theta_{P_i} + a^{-1}\omega, u_i)$ with the Gauss pushforward of the real mixed product of the $H_{\gamma_i} + a^{-1}H_Q$. The mixed measure on $X^{\mathrm{an}}$ converges weakly by Theorem 17.5.1. On the real side, expand by multilinearity. Every positive-degree coefficient in $a^{-1}$ is a finite measure, since it is a sum of terms bounded by the top product of $H_Q + \sum_i H_{\gamma_i}$. That top product is finite by

Lemma 32.8.1. The real products thus converge in total variation to the product of the $H_{\gamma_i}$. The pushforwards have the same limit, which proves (32.11). □

For a closed roof $\gamma$ on $P$, let $\Gamma$ be its infimal convolution with the zero roof on $Q$. The same weight identity for $P+Q$ identifies $u_\gamma = \varphi_\Gamma$ in class $\theta_P + \omega$, again by divisorial determination. In particular

$$E_{\theta_P+\omega}(u_\gamma) = -n! \int_{P+Q} \Gamma \, \mathrm{d}m \tag{32.12}$$

by (32.7). Since a potential of finite energy in one ample class also has finite energy in every ample class in which it is psh, this tests membership in $\mathcal{E}^1(X)$ for any of these semiample roof potentials.

## 32.9 Mixed truncations of closed roofs

Keep $X = X_Q$ and take lattice polytopes $P_1, \dots, P_n$ whose minimum support functions are linear on every cone of the fan of $Q$. Write $\theta_i = \theta_{P_i}$ for the associated semiample classes. Fix closed convex roofs $\gamma_i$ finite on the relative interiors, with potentials $u_i = u_{\gamma_i}$. Use compatible monomial frames so that the weights $H_i = H_{\gamma_i}$ and the reference weights $H_{0,i}$ add under tensor products. Put $u_{i,j} = \max(u_i, -j)$ for $j \geq 1$, and use the strict sets

$$G_j = \bigcap_{i=1}^{n} \{u_i > -j\} \subset X^{\mathrm{an}}.$$

**Theorem 32.9.1** *The positive measures*

$$\mathbf{1}_{G_j} \, \mathrm{MA}\big((\theta_1, u_{1,j}), \dots, (\theta_n, u_{n,j})\big)$$

*increase to the mixed measure in character coordinates*

$$\eta_*(d'd''H_1 \wedge \cdots \wedge d'd''H_n).$$

*Its mass is* $(\theta_1 \cdots \theta_n \cdot [X])$. *The unrestricted measures of the continuous truncations also converge in total variation to this measure. If each* $u_i$ *lies in* $\mathcal{E}^1(X)$, *this is the mixed finite-energy measure in the fixed nef classes* $\theta_i$.

***Proof*** All the $u_{i,j}$ are continuous roof potentials by the pullback construction in Section 32.8. In character coordinates the corresponding weights are $H_{i,j} = \max(H_i, H_{0,i} - j)$. By Proposition 32.8.1, the mixed global measure of the truncations is the pushforward of their real mixed product. The strict set pulls back to the open set

$$W_j = \bigcap_{i=1}^{n} \{H_i - H_{0,i} > -j\} \subset N_{\mathbb{R}}.$$

On $W_j$ every $H_{i,j}$ equals $H_i$. Real locality from Theorem 11.2.1 identifies the restricted mixed product with the restriction of the product of the $H_i$ to $W_j$. The weights are finite and continuous on $N_{\mathbb{R}}$, so the $W_j$ increase to that space. This proves the monotone limit on the Gauss section and on $X^{\mathrm{an}}$.

For each nonempty set $I$ of indices, Lemma 32.8.1 gives diagonal mass $n!\,\mathrm{vol}_M(\sum_{i\in I} P_i)$ for the sum of the $H_i$ with $i \in I$. The real mixed product is finite since positivity and multilinearity bound it by $1/n!$ times the top product of the sum of all the $H_i$. Polarize the diagonal masses as in Theorem 32.3.1. The same diagonal masses are obtained for the zero roofs. Their mixed measure has mass $(\theta_1 \cdots \theta_n \cdot [X])$ by Proposition 32.8.1 and Theorem 17.5.1. Polarization gives this mass for the mixed product of the closed roofs.

Write $\nu_j$ and $\nu$ for the real products of the truncated and original weights. They have the same total mass and agree on $W_j$. Thus both give the same mass to the complement, and their difference satisfies

$$\|\nu_j - \nu\|_{\mathrm{TV}} \le 2\nu(N_{\mathbb{R}} \setminus W_j) \longrightarrow 0.$$

Pushforward by $\eta$ does not increase total variation, so the convergence holds on $X^{\mathrm{an}}$. When each $u_i$ belongs to $\mathcal{E}^1(X)$, the continuous truncations decrease in the fixed nef classes to these finite-energy potentials. On each continuous truncation the two constructions agree, since a decreasing rational model approximation with convergent classes is uniform by Proposition 4.1.1. Decreasing continuity in Section 17.9 identifies the same limit. □

*Example 32.9.1 (Two roofs with energy $-\infty$)* Take $P_1 = P_2 = [0,1]^2$, and use the roofs $\gamma_1(m) = 1/m_1$ and $\gamma_2(m) = 1/m_2$, with value $+\infty$ at a zero denominator. Their energies are $-\infty$ by (32.7), since each integral of the reciprocal over the square diverges. The mixed truncation measure has mass two. We compute it in the frame with exponent 0.

Let $A$ be the weight of $1/m$ on $[0,1]$ and let $B(w) = \max(0,-w)$ be the weight of the zero roof. The supremum formula separates the character weights as $A(w_1) + B(w_2)$ and $B(w_1) + A(w_2)$. The derivative of $B$ jumps by one at 0. The slope map of the reciprocal roof is $-\gamma'(m) = m^{-2}$. By (32.8), the curvature of $A$ is the pushforward of length by this map. This gives the probability $\sigma$ with density $\frac{1}{2}w^{-3/2}\mathbf{1}_{\{w>1\}}\,\mathrm{d}w$.

Convolve $A$ and $B$ on the real line with the same nonnegative smooth approximate identity. Their smooth curvature functions are the convolutions of $\sigma$ and $\delta_0$ with the kernel. In the mixed wedge only the two terms using distinct coordinate directions survive. These give the product of the two smoothed copies of $\sigma$ and the product of the two smoothed copies of $\delta_0$, with coefficient one in each case. Taking the limit amounts to convolution by an approximate identity on the plane. Thus the real mixed product is $\sigma \otimes \sigma + \delta_{(0,0)}$. Theorem 32.9.1 pushes this measure to the Gauss section and proves the mass assertion.

*Example 32.9.2 (Two segment polarizations)* On $X = \mathbb{P}^1 \times \mathbb{P}^1$, take $P_1 = [0,1] \times \{0\}$ and $P_2 = \{0\} \times [0,1]$. The corresponding maps $r_{P_i}$ are the two projections, as is seen on the affine charts by the inclusion of the respective coordinate monoids. Use the reciprocal roof on each segment. In the notation of Example 32.9.1, the weights are $A(w_1)$ and $A(w_2)$. The square of either curvature is zero by Lemma 32.8.1. In the mixed product, convolution in the separate variables gives $\sigma \otimes \sigma$, as each factor supplies one coordinate curvature. Its Gauss pushforward has mass one, the intersection of fibres of the two projections. Thus the mixed measure can have positive mass even when both individual top measures vanish.

For $Q = [0,1]^2$, the roof for the first potential in the ample class $\theta_1 + \omega$ is $1/\min(1, m_1)$ on $P_1 + Q = [0,2] \times [0,1]$, with its infinite value at $m_1 = 0$. In (32.10) the minimum uses the largest first coordinate admitting the prescribed sum, namely $\min(1, m_1)$. The integral in (32.12) diverges at zero. The same calculation with the coordinates interchanged applies to the second potential, so both lie outside $\mathcal{E}^1(X)$.

# Chapter 33
# Finite energy, covers and blowups of toric varieties

Except in Section 33.5, the base field $k$ is trivially valued and $T = \operatorname{Spec} k[M]$ is a split torus of dimension $n$ with dual lattice $N$. In Sections 33.1 to 33.3, $X_P$ is the polarized toric variety of a full-dimensional lattice polytope $P \subset M_{\mathbb{R}}$, with $\lambda = n!\,\mathrm{d}m$ and $V = \lambda(P)$, and roofs are the closed convex roofs of Section 32.7, normalized at an interior point.

Theorem 33.1.1 solves the real equation for every positive Radon measure of mass $V$ on $N_{\mathbb{R}}$, and weak convergence of the measures gives locally uniform convergence of the roofs. By Corollary 33.1.1 the normalized Gauss pushforward has finite dual energy exactly when the roof is integrable. The roof potential is then the finite-energy solution, unique up to constants. A finite first moment, or more sharply a finite moment of order $n/(n+1)$, implies integrability. A finite first moment is not necessary: for $P = [0,1]$ the roof $m^{-\alpha}$ with $0 < \alpha < 1$ is integrable, although its measure has infinite first moment (Example 33.2.1). Uniformly bounded first moments, or uniformly small tails of the critical moments, give strong convergence of the potentials (Corollary 33.2.1 and Theorem 33.3.1). Example 33.3.1 shows that bounded critical moments alone do not.

Theorem 33.4.1 compares mixed measures under any finite surjective morphism of normal integral projective varieties. The mixed measure of pulled-back model pairs, or of continuous psh potentials in nef classes, integrates a test through its trace with the degree weights of Proposition 13.12.1. Combined with the local products on finite torus covers in Section 11.9, it identifies the pulled-back roof measures, and the limits of their mixed truncations, with local products in the projected torus coordinates when the cover is flat over the dense torus (Corollaries 33.4.1 and 33.4.2). Section 33.5 works on a complex projective surface with the big classes of Chapters 28 to 30. On the blowup of $\mathbb{P}^2$ at a torus-fixed point, the pulled-back hyperplane class plus the exceptional class carries an intrinsic finite-energy potential whose measure has infinite absolute dual energy. Along a sequence of normalized intrinsic potentials converging strongly to the greatest normalized potential, the absolute dual energies are finite and tend to 1, while the limit has absolute dual energy 0 (Proposition 33.5.1).

The toric calculation in Chapter 34 does not use this chapter. It relies only on the models, charts and Gauss valuations of Chapter 32.

## 33.1 Measures on the Gauss section

Consider the full-dimensional polarized toric variety $X_P$ of Section 32.7, with $\lambda = n!\,\mathrm{d}m$ and $V = \lambda(P)$. Fix $a \in P^\circ$ and normalize a roof by $\gamma(a) = 0$. Fix a Euclidean norm on $M_{\mathbb{R}}$ and the dual norm on $N_{\mathbb{R}}$, as in Section 32.6. Weak convergence on $N_{\mathbb{R}}$ means convergence on all bounded continuous tests. A sequence with this convergence and fixed mass is tight. Indeed, take a compactly supported continuous function between zero and one whose integral is close to the mass of the limit. For large indices its integral gives the same bound up to a small error. Enlarge its support to account for the finitely many earlier terms.

**Theorem 33.1.1** *Every positive Radon measure $\mu$ of mass $V$ on $N_{\mathbb{R}}$ is $(d'd''H_\gamma)^n$ for a unique normalized closed convex roof $\gamma$. If measures of mass $V$ converge weakly, their normalized roofs converge locally uniformly on $P^\circ$.*

***Proof*** For a finite atomic measure of mass $V$ use the continuous roof from Theorem 32.5.1. To approximate $\mu$, send the mass outside the ball of radius $j$ to the origin and partition the ball into Borel sets of diameter at most $1/j$. In each nonempty part of positive mass put one atom of that mass. This sequence converges against bounded continuous tests. For a fixed ball, uniform continuity on a slightly larger ball controls the replacement error there. Off the fixed ball the error is bounded by twice the test norm times the tail mass, which can be made arbitrarily small.

We give a compactness estimate for any sequence of normalized roofs whose real measures are tight. By (32.8) this is a tail bound on the gradients. Fix $K \Subset P^\circ$, and choose $r > 0$ so that the ball of radius $r$ about every $m \in K$ lies in $P^\circ$. If $s$ is a subgradient at $m$, then at almost every other point $q$ in the ball one has

$$\langle q - m, \nabla\gamma_j(q) - s\rangle \geq 0.$$

This follows by adding the two supporting inequalities. If $\|s\| > 2R$, the part of the ball satisfying $\langle q - m, s\rangle \geq \|q - m\|\|s\|/2$ has $\|\nabla\gamma_j(q)\| > R$ almost everywhere. Its lattice volume is a fixed positive number depending only on $n$, $r$ and the norm, by rotation invariance of Euclidean volume. Take $R$ so large that the common tail bound is smaller than this volume multiplied by $n!$. This bounds every subgradient on $K$.

Apply the estimate to compact convex subsets exhausting $P^\circ$ and containing $a$. The supporting inequalities bound the Lipschitz constants there, and normalization bounds the values. Successive extraction on those compact subsets gives a locally uniform subsequential limit. The subgradient bound at $a$ and its supporting inequality give a uniform lower bound on all of $P$. Closing the epigraph of the limit gives a closed roof with the same values on $P^\circ$. The character weights converge locally

uniformly by Lemma 32.7.1. Continuity of the real top product on compactly supported tests (Theorem 11.2.1) identifies the measure of the limiting weight with $\mu$. This proves existence.

To prove uniqueness, use (32.8) for two roofs $\gamma_1, \gamma_2$ with the same measure. There is a probability law of points $x, z \in P^\circ$ such that both marginals are $\lambda/V$ and $\nabla\gamma_1(x) = \nabla\gamma_2(z)$ almost surely. Here is a construction. For each roof consider the graph measure of its gradient with source law $\lambda/V$. The two graph measures have the same gradient marginal on the one-point compactification of $N_{\mathbb{R}}$, a compact metrizable space. Choose a finite Borel partition of this compactification of mesh at most $1/j$. For each cell of marginal mass $b > 0$, take the product of the restrictions of the two graph measures to the cell, divided by $b$. Sum over the cells. The two graph marginals stay fixed. Take a weak cluster point of these probability measures on the product of the compact graph ambient spaces, using compactness from Section 4.3. The two gradient coordinates in the limit agree, since the integral of their distance is at most $1/j$ before taking the limit. The fixed graph marginals exclude infinite gradient values and boundary points of $P$, and identify the finite coordinates as the two gradients. This gives the claimed law.

With $y = \nabla\gamma_1(x) = \nabla\gamma_2(z)$, the supporting inequalities give

$$\gamma_1(z) - \gamma_1(x) \geq \langle z - x, y\rangle \geq \gamma_2(z) - \gamma_2(x).$$

The random variables $(\gamma_1 - \gamma_2)(z)$ and $(\gamma_1 - \gamma_2)(x)$ have the same law. Apply any bounded strictly increasing function to both and integrate. It follows that the two inequalities are equalities almost surely. Equality in the second inequality makes $y$ a subgradient of $\gamma_2$ also at $x$. Differentiability almost everywhere then gives $\nabla\gamma_1 = \nabla\gamma_2$ $\lambda$-almost everywhere on $P^\circ$. Their difference is locally Lipschitz and has zero distributional gradient, by integration along coordinate lines and Fubini's theorem. Convolution on interior balls makes the difference constant there, and overlapping balls give a single constant on $P^\circ$. The radial boundary limit and the normalization give equality on $P$.

Finally, for a weakly convergent sequence of measures the estimate gives the same compactness of the normalized roofs. Every subsequence has a further subsequence whose limit solves the equation for the limit measure. Uniqueness forces local uniform convergence of the whole sequence. □

**Corollary 33.1.1** *Let $\gamma$ solve the real equation of Theorem 33.1.1 for $\mu$, and put $\nu = V^{-1}\eta_*\mu$. Then $\nu$ has finite dual energy if and only if $\gamma$ is integrable on $P$. In that case $\varphi_\gamma$, up to a constant, is the unique finite-energy $\theta$-psh solution of* $\mathrm{MA}_\theta(u) = \eta_*\mu$.

***Proof*** If $\gamma$ is integrable, the energy formula (32.7) and (32.9) give the solution. Its measure has finite dual energy by Proposition 19.2.1. For two solutions the energy difference $\iota$ vanishes by equality of their measures. Uniqueness up to constants follows from Theorem 19.4.1.

Conversely, assume $e^\vee(\nu) < \infty$. For each $j$, restrict $\mu$ to the ball of radius $j$ and send its remaining mass to the origin, obtaining $\mu_j$. Then $\mu_j \to \mu$ weakly, and $\nu_j = V^{-1}\eta_*\mu_j$ satisfies $e^\vee(\nu_j) \leq e^\vee(\nu)$. Indeed in the dual energy tests we may

normalize every potential to have supremum zero at the trivial point. Moving mass to this point only increases its integral.

Let $\gamma_j$ be the normalized roof of $\mu_j$, continuous by Theorem 32.6.1 and uniqueness. Let $u_j = \varphi_{\gamma_j} - \sup \varphi_{\gamma_j}$. This potential maximizes the dual expression for $\nu_j$, so $e^\vee(\nu_j) = V^{-1} J_{\theta,u_j}(0)$. Since $u_j$ vanishes at the trivial point, $-e(u_j) = V^{-1} J_{\theta,0}(u_j)$. The inequalities comparing $J$ to $I$ in Section 17.11, and the symmetry of $I$, give

$$-e(u_j) \le n e^\vee(\nu_j) \le n e^\vee(\nu).$$

At the trivial point the value of $\varphi_{\gamma_j}$ is $-\min_P \gamma_j \ge 0$, by $\gamma_j(a) = 0$. Thus the energies $E_\theta(\varphi_{\gamma_j})$ are uniformly bounded below. By the stability assertion of Theorem 33.1.1, the roofs converge locally uniformly on $P^\circ$, and the subgradient bound at $a$ in its compactness argument gives a common lower bound on $P$. Fatou's lemma applied to the integrals in (32.7) gives $\int_P \gamma \, \mathrm{d}\lambda < \infty$. □

## 33.2 Measures with a finite first moment

The criterion of Corollary 33.1.1 is a condition on the roof of the measure. An integral bound on the slopes gives a sufficient condition stated directly in terms of the measure. It also controls convergence up to the boundary of the polytope in $L^1$.

**Corollary 33.2.1** *Suppose $\mu$ has mass $V$ and $\int \|w\| \, \mathrm{d}\mu(w) < \infty$. Its normalized roof is integrable and gives the finite-energy solution of $\mathrm{MA}_\theta(u) = \eta_* \mu$. If measures $\mu_j$ of mass $V$ converge weakly to $\mu$ with uniformly bounded first moments, then the normalized roofs converge in $L^1(P, \lambda)$ and their potentials converge strongly.*

***Proof*** By (32.8), the first moment equals $\int_P \|\nabla \gamma\| \, \mathrm{d}\lambda$. We estimate the mass of $|\gamma|$ near the boundary. Put $P_\delta = a + (1-\delta)(P-a)$ for $0 < \delta \le 1/2$, and $D = \mathrm{diam}(P)$. On the cone from $a$ to a facet $F$, write $m = a + t(b-a)$ with $b \in F$ and $0 < t < 1$. Lattice volume in these coordinates is $c_F t^{n-1} \, \mathrm{d}t \, \mathrm{d}\sigma_F(b)$ for a constant $c_F > 0$ and Euclidean facet measure $\sigma_F$. Indeed, the differential in $t$ is $b - a$, whose normal component is the height of the facet above $a$. The other $n-1$ differentials are multiplied by $t$. For $n = 1$, facet measure is counting measure. The cones overlap only on sets of zero volume.

For almost every $b$, the roof is locally Lipschitz on the segment up to each $t < 1$ and differentiable as a function on $P^\circ$ at almost every point of the segment, by the volume formula and Fubini's theorem. For $1/2 \le t < 1$, the fundamental theorem of calculus along the segment gives

$$|\gamma(a + t(b-a))| \le \sup_{P_{1/2}} |\gamma| + D \int_{1/2}^{t} \|\nabla \gamma(a + s(b-a))\| \, \mathrm{d}s.$$

Integrate over $t > 1 - \delta$. The first term contributes at most $nV\delta \sup_{P_{1/2}} |\gamma|$. For the second, use $t^{n-1} \le 1$, the interval length $\delta$, and $1 \le 2^{n-1} s^{n-1}$ for $s \ge 1/2$. Sum over

the facets to obtain

$$\int_{P\setminus P_\delta} |\gamma|\,\mathrm{d}\lambda \le \delta\left(nV \sup_{P_{1/2}} |\gamma| + 2^{n-1} D \int_P \|\nabla\gamma\|\,\mathrm{d}\lambda\right). \tag{33.1}$$

Together with local boundedness, this proves integrability. The solution follows from Corollary 33.1.1.

For a weakly convergent sequence with uniformly bounded first moments, the limit has finite first moment by testing with $\min(\|w\|, R)$ and letting $R \to \infty$. The roofs converge locally uniformly on $P^\circ$ by Theorem 33.1.1. Then (33.1) bounds their boundary integrals uniformly by a constant times $\delta$, and gives the same estimate for the limit. Use local uniform convergence on $P_\delta$ and let $\delta \downarrow 0$ to obtain $L^1$ convergence. By (32.7) the energies converge. The character weights converge locally uniformly by Lemma 32.7.1, so the potentials converge at each divisorial point of the torus, which includes all divisorial points of $X_P$. This and energy convergence give strong convergence in the sense of Section 19.4. □

*Example 33.2.1 (An integrable roof with unbounded slopes)* Let $P = [0, 1]$ and take $\gamma(m) = m^{-\alpha}$ for $m > 0$, with $\gamma(0) = +\infty$ and $\alpha > 0$. The measure has infinite first moment for every $\alpha > 0$, whereas the roof is integrable exactly when $0 < \alpha < 1$.

Indeed the map $-\gamma'(m) = \alpha m^{-\alpha-1}$ sends length measure to the real top measure by (32.8). Its tail at radius $r$ is 1 for $0 \le r \le \alpha$ and $(\alpha/r)^{1/(1+\alpha)}$ for $r > \alpha$. This follows by solving $\alpha m^{-\alpha-1} > r$ for $m$ and taking the length of the resulting interval. Integrating the tail gives an infinite first moment. Integrating $m^{-\alpha}$ gives roof energy $-1/(1-\alpha)$ for $0 < \alpha < 1$, and $-\infty$ otherwise, by (32.7).

On the projective line this same measure lies on the ray at the zero of the torus coordinate. Its squared tail integral is finite precisely for $2/(1+\alpha) > 1$. In this case that integral is $\alpha + \alpha(1+\alpha)/(1-\alpha) = 2\alpha/(1-\alpha)$. The dual energy formula of Theorem 31.2.1 therefore gives $\alpha/(1-\alpha)$, in agreement with Corollary 33.1.1. For $\alpha \ge 1$ the truncation measure of the roof still has mass one by Proposition 32.7.3.

## 33.3 A critical moment for strong convergence

The first moment controls the integral of a roof by controlling its slope along segments to the boundary. A large slope at an interior point also forces large slopes on a cone around that point. This gives a sharper condition by weighting each slope by distance to the boundary.

**Theorem 33.3.1** *Let $q = n/(n+1)$. A positive Radon measure $\mu$ on $N_{\mathbb{R}}$ of mass $V$ with finite $q$-moment has an integrable normalized roof. Suppose that measures $\mu_j$ of mass $V$ with finite $q$-moments converge weakly to $\mu$ and*

$$\lim_{R\to\infty} \sup_j \int_{\{\|w\|>R\}} \|w\|^q \, \mathrm{d}\mu_j(w) = 0. \tag{33.2}$$

*Then the normalized roofs converge in $L^1(P,\lambda)$ and the corresponding potentials converge strongly. In particular, a uniform moment bound of any order strictly greater than $q$ is sufficient.*

***Proof*** For a roof $\gamma$, put $g = \|\nabla\gamma\|$ on its set of differentiability and $d(m) = \operatorname{dist}(m, \partial P)$, using a Euclidean norm and its dual. Let $A = \int_P g^q \, \mathrm{d}\lambda$, the $q$-moment by (32.8). At an interior point $m$ with $g(m) > 0$, consider the points $x = m + v$ with $0 < \|v\| < d(m)$ and $\langle \nabla\gamma(m), v\rangle > g(m)\|v\|/2$. These points form a cone sector of volume $c\, d(m)^n$ in $P^\circ$, where $c > 0$ is independent of $m$ and $\gamma$. At each differentiability point in this sector the supporting inequalities at $x$ and $m$ give

$$\langle \nabla\gamma(x), v\rangle \geq \langle \nabla\gamma(m), v\rangle > g(m)\|v\|/2.$$

Thus $A \geq c\, d(m)^n (g(m)/2)^q$. Since $1 - q/n = q$, this gives an almost-everywhere bound

$$d(m)g(m) \leq C A^{1/n} g(m)^q, \tag{33.3}$$

which holds also where $g(m) = 0$.

Use the same radial coordinates $m = a + s(b-a)$ from the proof of Corollary 33.2.1, and let $r = \operatorname{dist}(a, \partial P) > 0$. The ball of radius $(1-s)r$ about $m$ lies in $P$, by taking convex combinations of $b$ and the ball of radius $r$ about $a$. Hence $d(m) \geq (1-s)r$. In the boundary integral in (33.1), a slope at radial parameter $s \geq 1/2$ can contribute only for outer parameters $t \geq \max(s, 1-\delta)$. This interval has length $\min(\delta, 1-s)$. Using the same Jacobian bound and summing over facets gives

$$\int_{P\setminus P_\delta} |\gamma| \, \mathrm{d}\lambda \leq nV\delta \sup_{P_{1/2}} |\gamma| + 2^{n-1} D \int_{P\setminus P_{1/2}} \min(\delta, d/r)\, g \, \mathrm{d}\lambda.$$

In the last integral, the part where $g \leq R$ is at most $VR\delta$. On $g > R$, apply (33.3) to bound the part by

$$(C/r) A^{1/n} \int_{\{g>R\}} g^q \, \mathrm{d}\lambda.$$

For a single roof with $A < \infty$, these bounds give integrability, and its potential has finite energy by (32.7).

For the sequence, (33.2) and the fixed mass bound the $q$-moments uniformly. Testing weak convergence with $\min(\|w\|^q, R)$ and letting $R$ increase proves that $\mu$ has finite $q$-moment. By Theorem 33.1.1, the roofs converge locally uniformly on $P^\circ$, which bounds their suprema in absolute value on $P_{1/2}$. First choose $R$ large in the tail estimate using (33.2), then let $\delta \downarrow 0$. This gives uniformly small integrals on the boundary layers. Together with the interior convergence and the same boundary estimate for the limit, this gives $L^1$ convergence. The character weights converge locally uniformly by Lemma 32.7.1. Their potentials therefore converge at the divisorial points, and their energies converge by (32.7). This is strong convergence as in Section 19.4. For $p > q$, on $\{\|w\| > R\}$ the $q$-moment integrand is bounded by $R^{q-p}\|w\|^p$, proving the final assertion. □

*Example 33.3.1 (Concentration at a vertex)* Take $P = [0,1]^n$ and $S(m) = m_1 + \cdots + m_n$. For $0 < t \le 1$, the region $S < t$ is a simplex of $\lambda$-volume $t^n$. This follows from the determinant of the $n$ coordinate edges of length $t$. Thus the distribution of $S$ has density $nt^{n-1}$ on $(0,1)$.

The roof $S^{-n}$, extended by $+\infty$ at the origin, has slope norm $n\sqrt{n}\,S^{-n-1}$. For any $0 < p < n/(n+1)$ the $p$-moment of its real measure is finite. Indeed, near the vertex its integral is a constant times $\int_0^1 t^{n-1-p(n+1)}\,\mathrm{d}t$, and away from that vertex the slope is bounded. The roof integral contains $\int_0^1 nt^{-1}\,\mathrm{d}t$ and diverges. Subtracting the roof value at $a$ for normalization leaves the slopes and the divergence unchanged.

For $0 < \varepsilon < \min(1, S(a))$ set

$$\gamma_\varepsilon(m) = \varepsilon^{-(n+1)} \max(\varepsilon - S(m), 0).$$

These continuous roofs have value zero at $a$. Their measures are obtained by sending the simplex $\{S < \varepsilon\}$ to $\varepsilon^{-(n+1)}(1, \ldots, 1)$ and its complement to zero, by (32.8). The measures converge weakly to $V\delta_0$, and their $q$-moments equal $\varepsilon^n(\sqrt{n}\,\varepsilon^{-(n+1)})^q = n^{q/2}$. By integrating in the coordinate $S$,

$$\int_P \gamma_\varepsilon \,\mathrm{d}\lambda = \varepsilon^{-(n+1)} \int_0^\varepsilon (\varepsilon - t) nt^{n-1}\,\mathrm{d}t = \frac{1}{n+1}.$$

The limiting measure has normalized roof zero. The fixed energy gap shows why bounded critical moments alone do not give strong convergence.

## 33.4 Measures on finite covers

A finite flat cover of a toric variety can have several analytic points above one Gauss valuation. The local product of pulled-back weights assigns a separate mass to each point. We compare these masses with model intersections on the cover. The intersection argument works on any normal projective base.

**Lemma 33.4.1** *Let $F$ be a complete ultrametric field. Fix a positive real radius for $t$, and complete $F(t)$ for the Gauss absolute value of that radius, obtaining a field $G$. For a finite local $F$-algebra $D$, the algebra $D \otimes_F G$ is local and has the same dimension over $G$ as $D$ over $F$.*

***Proof*** First let $D = L$ be a finite field extension with its extending absolute value. Choose an $F$-basis $b_1, \dots, b_d$ of $L$. Its coefficient maximum norm and the field norm are equivalent, by the norm-equivalence step in the proof of Lemma 2.3.1. Therefore the same two comparison constants compare the Gauss norm of $\sum_i b_i f_i(t)$ with the maximum of the Gauss norms of the $f_i$, first for polynomials over $F$ and then for $f_i \in F(t)$ by a common denominator. The vector space over $F(t)$ spanned by the $b_i$ is all of $L(t)$: the basis remains linearly independent by comparing polynomial coefficients, and its span is a finite-dimensional domain over $F(t)$, hence a field. Completing the vector space for either equivalent norm identifies the completion with $L \otimes_F G$. The Gauss norm on $L(t)$ is multiplicative, and remains so after completion. Thus $L \otimes_F G$ is a field. For general $D$ its maximal ideal is nilpotent, and the quotient is a finite field extension of $F$. After tensoring, that ideal stays nilpotent and the quotient is a field by the first case. This proves locality. Dimension is preserved by scalar extension. □

**Theorem 33.4.1** *Let $f : Y \to X$ be finite and surjective between normal integral projective varieties over $k$. Put $n = \dim X$. For a model pair $\mathcal{P} = (\theta, \varphi)$, put $f^*\mathcal{P} = (f^*\theta, \varphi \circ f^{\mathrm{an}})$. If $\mathcal{P}_1, \dots, \mathcal{P}_n$ are model pairs on $X$, then for every continuous real $\chi$ on $Y^{\mathrm{an}}$,*

$$\int_{Y^{\mathrm{an}}} \chi \, \mathrm{d}\,\mathrm{MA}(f^*\mathcal{P}_1, \dots, f^*\mathcal{P}_n) = \int_{X^{\mathrm{an}}} f_{\#}\chi \, \mathrm{d}\,\mathrm{MA}(\mathcal{P}_1, \dots, \mathcal{P}_n), \tag{33.4}$$

*where $f_{\#}$ uses the degree weights of Proposition 13.12.1. It also holds for pairs of continuous psh potentials in nef classes.*

***Proof*** On an affine open of $X$, apply Proposition 13.12.1 to the finite map and the restrictions of $\chi$ and $-\chi$. The two upper semicontinuous traces are negatives of one another, hence the trace is continuous. The weights agree on overlapping affine opens: the normal extension of the two function fields used to count conjugates is the same, and its normalizations restrict to the normalizations over the smaller opens. This defines $f_{\#}\chi$ on $X^{\mathrm{an}}$.

Choose a common normal model $\mathcal{X}$ of the pairs dominating $X \times \mathbb{A}^1$, with model classes $M_i$. Normalize it in $k(Y)(t)$, obtaining $\mathcal{Y} \to \mathcal{X}$. It is finite. On an affine chart, clear denominators in the monic equations for field generators to make their scaled representatives integral. They generate a finite algebra with fraction field $k(Y)(t)$. Its ordinary normalization is finite [Stacks, Tag 0BXR] and is the integral closure of the original ring by transitivity of integrality. It is marked by $Y \times \mathbb{G}_m$ off the central fibre, since that product is normal and finite over the marked product of $X$. It dominates $Y \times \mathbb{A}^1$: on each affine open of $X$, the functions defining its finite inverse image in $Y$ are integral over the coordinate ring and hence regular on the normalization over

the corresponding open of the model. The action on the punctured marking extends by normalization: the product of $\mathcal{Y}$ with $\mathbb{G}_m$ is normal, so pulling back the integral local equations along the action on the base model gives regular functions on this product. Uniqueness on the punctured marking gives the action identities. Flatness over the parameter line follows from torsion-freeness over $k[t]$. Pulling back the model pairs represents $f^*\mathcal{P}_i$, since their vertical local equations give the pulled-back functions upon evaluation of the Gauss extensions. The pulled-back model classes are relatively nef if the original classes are, by the projection formula on curves.

Let $D$ be a central component on $\mathcal{X}$. It corresponds to $x = v_D \in X^{\mathrm{an}}$, and its divisor order divided by $b_D = \mathrm{ord}_D(t)$ is the Gauss extension of $x$ by Proposition 5.3.1. Write $F = \mathcal{H}(x)$ and $G$ for the completion of $k(X)(t)$ at this extended valuation. Both $x$ and its extensions to $Y$ have generic support, since their valuations restrict to the function field $k(X)$ and $Y \to X$ is finite. The field $G$ is also the completion of $F(t)$ with the Gauss absolute value at radius $e^{-1}$. Indeed coefficients and nonzero denominators in $F(t)$ are approximated by ones from $k(X)(t)$. The fibre algebra $k(Y) \otimes_{k(X)} F$ becomes $k(Y)(t) \otimes_{k(X)(t)} G$ after extending scalars. Both identifications follow from the field basis of $k(Y)$ over $k(X)$, which stays a basis after adjoining a variable. The finite map is flat at the generic point of $X$. There the specialization weights are the local fibre dimensions by Proposition 13.12.1. By Lemma 33.4.1, this extension preserves its local factors and their dimensions. The valuation on $k(Y)(t)$ in the factor over $y$ restricts to $y$ on $k(Y)$, by uniqueness of the extending absolute value on its complete residue field.

To compute the dimension of a factor over $G$, take the finite normalization of the discrete valuation ring at the generic point of $D$ in $k(Y)(t)$. Its maximal ideals are the generic points of $E$ above $D$, and the corresponding local rings are discrete valuation rings. The completion over the base uniformizer splits as the product of the completions of these local rings: for each power of the uniformizer use the Chinese remainder theorem for the primary factors, then take the inverse limit. After inverting the uniformizer, the completed factors are fields whose product is $k(Y)(t) \otimes_{k(X)(t)} G$. This follows because finite modules over the base discrete valuation ring commute with completion and localization. Each completed local ring is a finite torsion-free module over the completed base, hence free. Its rank is the dimension over $\kappa(D)$ of its quotient by the base uniformizer. Filtering by the uniformizer of $E$ makes that dimension $e_{E/D}[\kappa(E) : \kappa(D)]$, where $e_{E/D}$ is the ramification index. In particular there is one $E$ for each local analytic factor above $x$. The valuation of this factor extends $b_D^{-1}\,\mathrm{ord}_D$, so it is $b_E^{-1}\,\mathrm{ord}_E$ with $b_E = e_{E/D} b_D$. By Proposition 5.3.1 its point $y$ is therefore $v_E$, and

$$d_f(y) = e_{E/D}[\kappa(E) : \kappa(D)].$$

On $E$, the multiplicity is $b_E$. Its mixed intersection degree is the one on $D$ multiplied by $[\kappa(E) : \kappa(D)]$, by the projection formula. Since the map of models is finite, every central component of the cover lies over some $D$. Consequently the mass from $D$ in (17.1) is distributed between its lifts with factors $d_f(y)$. Summing gives (33.4).

For continuous psh inputs in nef classes, take decreasing rational Fubini–Study approximations with convergent classes by Proposition 6.2.1. Their potential functions converge uniformly by Proposition 4.1.1. Pullback preserves this convergence, the convergence of the classes, and semipositivity on models as checked above. Thus Theorem 17.5.1 applies on both projective varieties. The continuity of $f_\# \chi$ gives (33.4) in the limit. □

**Corollary 33.4.1** *Let $X = X_Q$ be as in Section 32.8 and $f : Y \to X$ as in Theorem 33.4.1. Assume that $f$ is flat over the dense torus of $X$. For continuous roof potentials in the semiample classes of Proposition 32.8.1, the mixed global measure of the pullback agrees over the inverse image of the torus with the local product in the projected torus coordinates. It gives full mass to the inverse image of the Gauss section in the torus.*

***Proof*** Use Proposition 32.8.1 on $X$, and apply (33.4). On each torus box the finite map is the finite locally free projection used in Section 11.9. In the local frames pulled back from the base, the weights are the corresponding functions of the projected character coordinates. The trace of a continuous test there at the Gauss point is $K_w(\chi)$. Thus the two measures agree over each box interior by (11.4). The Gauss section is the union of its compact parts over a sequence of increasing boxes. Testing (33.4) with a continuous function pulled back from $X^{\mathrm{an}}$ shows that the pushforward of the measure on $Y^{\mathrm{an}}$ is the degree times the measure on $X^{\mathrm{an}}$. Since the latter gives full mass to this countable union, the measure upstairs gives full mass to its inverse image. □

**Corollary 33.4.2** *Under the hypotheses of Corollary 33.4.1, take closed roofs as in Theorem 32.9.1. The strict-set measures of their pulled-back truncations increase to the local product of the original weights in projected torus coordinates, extended by zero. The unrestricted mixed truncations converge to the same measure in total variation. Its mass is $[k(Y) : k(X)](\theta_1 \cdots \theta_n \cdot [X])$.*

***Proof*** For any positive Radon measure $\mu$ on $X^{\mathrm{an}}$ write $f^\# \mu$ for the measure given by the positive functional $\chi \mapsto \int f_\# \chi \, \mathrm{d}\mu$. With $d = [k(Y) : k(X)]$, the trace satisfies $\|f_\# \chi\|_\infty \leq d \|\chi\|_\infty$ and $f_\# 1 = d$. Thus $f^\#$ is positive, multiplies mass by $d$, and has norm at most $d$ on signed measures for total variation. For a Borel set $G \subset X^{\mathrm{an}}$ it satisfies

$$f^\#(\mathbf{1}_G \mu) = \mathbf{1}_{(f^{\mathrm{an}})^{-1}(G)} f^\# \mu.$$

Indeed, for a continuous $\chi$ upstairs and continuous $h$ downstairs, the trace of $(h \circ f^{\mathrm{an}})\chi$ is $h f_\# \chi$. This identifies the pushforward of $\chi f^\# \mu$ with $(f_\# \chi)\mu$ as signed measures, so the identity also holds after testing with $\mathbf{1}_G$.

Pointwise truncation commutes with pullback. The continuous truncation measures upstairs are the pullbacks by $f^\#$ of the truncation measures on $X^{\mathrm{an}}$ by (33.4). Their strict-set restrictions are the corresponding pullbacks by the Borel identity. Positivity and the total variation bound now give both convergence statements by Theorem 32.9.1. The limiting measure is the pullback of the mixed measure on the

Gauss section. On each box its trace is that in (11.4), which applies to the convex weights of the original roofs. The pushforward of the limiting measure by $f^{\mathrm{an}}$ is $d$ times the Gauss measure downstairs. This proves both concentration over the dense torus and the mass formula. □

## 33.5 Absolute dual energy after a toric blowup

On the blowup of the projective plane at a torus-fixed point, the big class $\alpha$ below shows two failures of absolute dual energy. The measure of an intrinsic finite-energy potential in $\alpha$ can have infinite absolute dual energy. Absolute dual energy can also fail to be continuous at the greatest normalized potential, even along solutions whose measures all have finite absolute dual energy (Proposition 33.5.1).

Let $Y = \mathbb{P}^2_{\mathbb{C}}$ and use the triangle

$$P = \{(x, y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\}$$

of the hyperplane bundle. Let $\pi : X \to Y$ blow up the torus fixed point at the vertex 0. The fan of the blowup inserts the ray $(1, 1)$ in the positive normal cone at 0, as seen in the two monomial charts of the blowup with coordinates $(z_1, z_2/z_1)$ and $(z_1/z_2, z_2)$. This is the normal fan of the lattice polytope $Q = \{x, y \geq 0,\ 1 \leq x + y \leq 2\}$. The minimum support function of $P$ is linear on every cone of that fan. Write $D$ for the exceptional curve and set $\beta = \pi^* c_1(\mathcal{O}_Y(1))$ and $\alpha = \beta + \{D\}$, both of volume one. By Example 30.1.1 the fixed part is $D$, so $N = D$ in Theorem 30.1.1. Write $T_D = T_N$ and $e_D = e_N$, so that $e_D(v) = v(\mathcal{O}_X(-D))$.

**Proposition 33.5.1** *In $\alpha$ there is an intrinsic finite-energy potential $u$ such that $\mathcal{E}^\vee_\alpha(\mathrm{MA}_\alpha(u)) = +\infty$. There is also a sequence of normalized intrinsic potentials $u_j$ converging strongly to the greatest normalized potential $u_*$ such that all $\mathcal{E}^\vee_\alpha(\mathrm{MA}_\alpha(u_j))$ are finite and converge to 1, whereas $\mathcal{E}^\vee_\alpha(\mathrm{MA}_\alpha(u_*)) = 0$.*

***Proof*** First record the roof formulas in the nef class $\beta$. If $\gamma \geq 0$ is a continuous convex roof, write $z_\gamma$ for its potential pulled back to $X$. Its intrinsic energy is

$$E_\beta(z_\gamma) = -\int_P \gamma \, \mathrm{d}\lambda, \qquad \lambda = 2\,\mathrm{d}x\,\mathrm{d}y.$$

Indeed the potential is bounded, so the perturbative nef energy agrees with the algebraic energy by Theorem 26.5.1 and Corollary 26.4.1. On rational polyhedral roofs, take the closure of the graph of the map from the punctured product of $X$ to the projective model of the roof. Its compactification is proper and birational over the compactified roof model, and carries the pullbacks of both the model polarization and the reference class from $Y$. A normalization along the puncture gives a test configuration. The projection formula of Section 3.5, as used after Definition 17.1.1, shows that the mixed intersections of these pairs are those on the original $Y$-model. The difference

computing the energy thus gives the roof energy of (32.4). Uniform approximation of a continuous convex roof by rational convex polyhedral roofs passes this to $\gamma$, using the uniform bound on potentials and monotonicity and constant shifts for energy. The toric semiample measure formula and the bounded nef comparison give

$$\mathrm{MA}_\beta(z_\gamma) = \eta_*\big((-\nabla\gamma)_*\lambda\big),$$

where $\eta$ sends a character weight to its Gauss valuation on $X$. Here we used Propositions 32.7.2 and 32.8.1 and Corollary 28.5.1. For a closed nonnegative roof with finite integral, take increasing continuous convex roofs as in Proposition 32.7.1. The bounded potentials decrease under pullback to $z_\gamma$. Their intrinsic energies are bounded below by the stated integral, so $z_\gamma$ is intrinsic of finite energy by (26.5) and Theorem 26.5.1. More explicitly, for any fixed $k$ their truncations at $-k$ are bounded and decrease to $\max(z_\gamma, -k)$, so bounded decreasing continuity gives the lower bound for every truncation energy. The intrinsic decreasing continuity then gives the same energy formula for the limit. The measure formula follows by the gradient convergence in the proof of Proposition 32.7.2 and Corollary 28.5.1.

Put $s = x + y$ and take $\gamma(x, y) = 1/s - 1$, with value $+\infty$ at 0. The pushforward of $\lambda$ by $s$ is $2s\,\mathrm{d}s$ on $(0, 1)$, by integrating over the triangles cut out by $x + y \le s$. Thus $\int_P \gamma\,\mathrm{d}\lambda = 1$. The gradient rule gives the weight $w = (s^{-2}, s^{-2})$. At a Gauss valuation of weight $(r, r)$, the exceptional ideal has order $r$: it is the pullback of the ideal of the point, generated by the two coordinates in the chart at 0, each of value $r$. It follows that $\int e_D\,\mathrm{d}\,\mathrm{MA}_\beta(z_\gamma) = \int_0^1 2s\,s^{-2}\,\mathrm{d}s = +\infty$. Apply Theorem 30.1.1 to $u = T_D(z_\gamma)$ and use (30.2) to prove the first assertion.

For $j \ge 2$, let $\gamma_j$ equal $\gamma$ on $s \ge 1/j$ and its tangent line $2j - j^2 s - 1$ on $s \le 1/j$. These are nonnegative continuous convex roofs with minimum zero and $0 \le \gamma_j \le \gamma$. Put

$$\varepsilon_j = (1 + 2\log j)^{-1}, \qquad z_j = z_{\varepsilon_j\gamma_j}, \qquad u_j = T_D(z_j).$$

Every $z_j$ has cutoff zero and lies below zero. Its ordered rooftop distance to zero is $\varepsilon_j \int_P \gamma_j\,\mathrm{d}\lambda \le \varepsilon_j$. The transport theorem preserves rooftop distance, so $u_j \to T_D(0)$ in rooftop distance, hence strongly by Theorem 29.5.1. Since $e_D(v_0) = 0$, the order isomorphism $T_D$ maps the normalized potentials in $\beta$ onto those in $\alpha$. As $\beta$ is nef, the greatest normalized potential $u_* = P_\alpha(0)$ of $\alpha$ (Section 29.1) is $T_D(0) = -e_D$, by Section 30.1.1. The algebraic and cohomological reference measure of zero in $\beta$ is $\delta_{v_0}$, by the product model formula, and transport gives the same measure for $u_*$. Its absolute dual energy is zero since $e_D(v_0) = 0$ and every normalized potential has value zero at $v_0$ and energy at most zero.

Write $\mu_j = \mathrm{MA}_\alpha(u_j) = \mathrm{MA}_\beta(z_j)$. Its gradient coordinates are $(r, r)$ with $r = \varepsilon_j \min(j^2, s^{-2})$, so

$$\int e_D\,\mathrm{d}\mu_j = 2\varepsilon_j\left(\int_0^{1/j} sj^2\,\mathrm{d}s + \int_{1/j}^1 \frac{\mathrm{d}s}{s}\right) = 1.$$

To compute the nef dual energy, use the supporting identity (27.4) at the bounded solution $z_j$. Since $V = 1$, this nef dual energy is $\mathcal{E}_\beta^\vee$: the finite-energy domains and

the energies agree by Theorem 26.5.1, and $\mu_j$ is the nef measure of $z_j$ by Corollary 28.5.1. For a point in $P^\circ$ with coordinate sum $s$, the corresponding weight $w = (r, r)$ is $-\varepsilon_j \nabla \gamma_j$ there. The affine support inequality for the roof shows that this point attains the supremum in (32.6). The zero-roof weight in the frame at 0 vanishes for $w$ with nonnegative coordinates.

Thus the value of $z_j$ at the Gauss point of $w$ is $-sr - \varepsilon_j \gamma_j(s)$. Substitution gives

$$\mathcal{E}_\beta^\vee(\mu_j) = \int_0^1 2s^2 r \, \mathrm{d}s \le 2\varepsilon_j .$$

Equation (30.2) now gives the second assertion. □

# Chapter 34
# Valuations and expected vanishing orders

On $\mathbb{P}^1_k$ let $p$ be the rational point where the coordinate section $x$ of $\mathcal{O}(1)$ vanishes, and let $y$ be a second coordinate section. The sections $x^i y^{m-i}$, $0 \le i \le m$, form a basis of $H^0(\mathbb{P}^1, \mathcal{O}(m))$, and $\operatorname{ord}_p$ takes the values $0, 1, \dots, m$ on them. Every section vanishing to order at least $\lambda$ at $p$ is a combination of the basis elements with $i \ge \lambda$. The average of these orders divided by $m$ is $1/2$ for every $m$. On the other hand, the potential

$$u = \max\{\log|x| + 1, \log|y|\}$$

equals $\max\{1-t, 0\}$ at the point $x_{p,t} = t \operatorname{ord}_p$ of the $p$-ray (Theorem 1.4.1) and equals 1 on the other rays. It vanishes at $\operatorname{ord}_p = x_{p,1}$, and by Theorem 31.2.1 its measure is the Dirac mass there. Its normalized energy is $1 - \frac{1}{2} = \frac{1}{2}$, the same number as the average order.

This chapter shows that the agreement is general. Two-sided bounds show that the Dirac mass $\delta_v$ has finite energy exactly when $v$ has linear growth. For such a valuation, over an algebraically closed field or on a regular curve, the average order of vanishing of a basis adapted to $v$ converges to the dual energy of $\delta_v$. Under the envelope property the maximizer for $\delta_v$ is the envelope of all potentials that are non-positive at $v$. Its supremum is the supremum of the normalized orders of vanishing of sections, and the energy identities give two-sided bounds between the two invariants. The bounds coincide on curves. On toric varieties they locate the barycentre of a polytope between two hyperplanes.

Throughout the chapter, $X$ is an integral projective scheme of dimension $n \ge 1$ over a trivially valued field $k$, and $L$ is an ample line bundle on $X$. Put $V = (L^n \cdot [X])$ and $N_m = h^0(X, mL)$. We use the normalized energies $e$, $e^\vee$ and measures $\mu_u$ of Chapter 19 in the class $\omega = [L]$, and the section lattices $A_m(f)$ and counts $S_m(f, g)$ of Section 18.2. A *valuation* is a real valuation of $k(X)$, trivial on $k$, with a centre on $X$. It is a point of $X^{\mathrm{an}}$ with generic support. For a section $s$ of $mL$ we write $v(s) = -\log|s|(v)$, computed in a local frame at the centre as in Section 5.5. Thus $v(s) \ge 0$, with equality when $s$ does not vanish at the centre, and $v(s) < \infty$ when $s \ne 0$. The trivial valuation is $v_0$.

Section 34.1 sets up the jumping numbers of a valuation. Proposition 34.2.1 computes the asymptotic section counts of semicontinuous obstacles. Theorems 34.3.1 and 34.3.2 are the main results on Dirac masses, and Theorem 34.4.1 constructs the corresponding solution. The invariants $S$ and $T$, the identity $S(v) = e^{\vee}(\delta_v)$ and the bounds of Theorem 34.4.1 are the inputs needed for valuative thresholds of stability.

## 34.1 The filtration of a valuation

We first record the linear algebra of a single filtered space.

**Lemma 34.1.1** *Let $W$ be a $k$-vector space of finite dimension $N \geq 1$, and let $\chi : W \to [0, +\infty]$ satisfy $\chi(s) = +\infty$ exactly when $s = 0$, $\chi(cs) = \chi(s)$ for $c \in k^\times$, and $\chi(s + s') \geq \min\{\chi(s), \chi(s')\}$. Put $F^\lambda W = \{\chi \geq \lambda\}$ and*

$$\lambda_i = \sup\{\lambda \in \mathbb{R} : \dim F^\lambda W \geq i\} \qquad (1 \leq i \leq N).$$

(1) *There is a basis $(s_i)$ of $W$ with $\chi(s_i) = \lambda_i$ and $\chi(\sum_i c_i s_i) = \min_{c_i \neq 0} \lambda_i$. In particular $\dim F^\lambda W = \#\{i : \lambda_i \geq \lambda\}$.*
(2) *Every basis $(s'_i)$ of $W$ satisfies $\sum_i \chi(s'_i) \leq \sum_i \lambda_i$.*
(3) *One has $\int_0^\infty \dim F^\lambda W \, d\lambda = \sum_i \lambda_i$ and $\sum_{j \geq 1} \dim F^j W = \sum_i \lfloor \lambda_i \rfloor$, the second sum over integers $j$.*
(4) *If $W' \subseteq W$ has codimension $c$ and carries the restriction of $\chi$, its jumping numbers satisfy $\lambda_{i+c} \leq \lambda'_i \leq \lambda_i$ for $1 \leq i \leq N - c$.*

***Proof*** The sets $F^\lambda W$ are subspaces, they decrease with $\lambda$, and $F^\lambda W = W$ for $\lambda \leq 0$. Their intersection over all $\lambda$ consists of vectors of infinite value, so it is zero. A decreasing family of subspaces of a finite-dimensional space stabilizes, so $F^\lambda W = 0$ for large $\lambda$, and all $\lambda_i$ are finite. Moreover $F^\lambda W = \bigcap_{\lambda' < \lambda} F^{\lambda'} W$, and this intersection is attained by some $\lambda' < \lambda$. Hence $\dim F^\lambda W \geq i$ exactly when $\lambda \leq \lambda_i$.

The distinct subspaces $F^\lambda W$ form a finite flag. Choose a basis adapted to it, ordered so that $s_1, \ldots, s_d$ span $F^\lambda W$ whenever $d = \dim F^\lambda W$. Then $s_i \in F^{\lambda_i} W$, so $\chi(s_i) \geq \lambda_i$. If $\chi(s_i) > \lambda_i$, then $s_i$ would lie in $F^{\chi(s_i)} W$, which is spanned by basis vectors of index less than $i$. Thus $\chi(s_i) = \lambda_i$. For a combination with minimum $\mu$ over its nonzero coefficients, the ultrametric inequality gives value at least $\mu$. A larger value would place the combination in a subspace $F^{\mu'} W$ with $\mu' > \mu$. That subspace is spanned by $s_1, \ldots, s_{d'}$ with $d' = \dim F^{\mu'} W$, and these have $\lambda_i \geq \mu'$ by the characterization above. This contradicts the nonzero coefficient at an index with $\lambda_i = \mu$. The formula for $\dim F^\lambda W$ follows, and it implies the identities in the third assertion.

For the second assertion order the basis so that $\chi(s'_1) \geq \cdots \geq \chi(s'_N)$. The vectors $s'_1, \ldots, s'_i$ are independent and lie in $F^{\chi(s'_i)} W$. Therefore $\chi(s'_i) \leq \lambda_i$. For the last assertion, $F^\lambda W' = W' \cap F^\lambda W$ gives $\dim F^\lambda W - c \leq \dim F^\lambda W' \leq \dim F^\lambda W$, and the characterization of the jumping numbers by these dimensions gives the inequalities. □

**Definition 34.1.1** Let $v$ be a valuation. For $m$ with $N_m \geq 1$ let $\lambda_{m,1}(v) \geq \cdots \geq \lambda_{m,N_m}(v)$ be the jumping numbers of $\chi = v$ on $H^0(X, mL)$. Put

$$T_m(v) = \frac{\lambda_{m,1}(v)}{m}, \qquad S_m(v) = \frac{1}{mN_m} \sum_{i=1}^{N_m} \lambda_{m,i}(v),$$

and $T(v) = \sup_m T_m(v)$. If the limit exists, put $S(v) = \lim_{m\to\infty} S_m(v)$. We call $S_m(v)$ and $S(v)$ the *expected vanishing orders* of $v$ with respect to $L$.

The one-argument $S_m(v)$ is not to be confused with the two-argument section count $S_m(f, g)$ of Section 18.2. The number $T(v)$ is the invariant $T_L(v)$ of Section 7.4. By Lemma 34.1.1, $mN_mS_m(v)$ is the largest sum of the values of $v$ on a basis of $H^0(X, mL)$, and $0 \leq S_m(v) \leq T_m(v) \leq T(v)$. All these invariants are positively homogeneous in $v$. Replacing $L$ by $aL$ for a positive integer $a$ multiplies $T$ by $a$, and multiplies $S$ by $a$ whenever the limit exists, since the spaces for $aL$ are those for $L$ in degrees divisible by $a$.

Choose $r_0$ such that $rL$ is generated by global sections for all $r \geq r_0$. For each such $r$ and each valuation $v$, one of the generating sections of $rL$ does not vanish at the centre of $v$. Call it $\sigma_r$. Then $v(\sigma_r) = 0$, and multiplication by $\sigma_r$ is an injection $H^0(X, mL) \to H^0(X, (m + r)L)$ preserving the values of $v$, since $X$ is integral.

**Lemma 34.1.2** *Let $v$ be a valuation.*

(1) *The sequence $T_m(v)$ converges to $T(v)$, possibly infinite.*
(2) *Suppose $T(v) < \infty$ and fix $R \geq r_0$. Then $|S_{m+r}(v) - S_m(v)| \to 0$ as $m \to \infty$, uniformly for $r_0 \leq r \leq R$.*
(3) *Suppose $T(v) < \infty$ and $c \in \mathbb{R}$. If for every $\varepsilon > 0$ there is $m_1$ such that $S_m(v) \geq c - \varepsilon$ for every multiple $m$ of $m_1$, then* $\liminf_m S_m(v) \geq c$. *The same holds with upper bounds and* lim sup.

***Proof*** Fix $m'$ with $N_{m'} \geq 1$ and a section $s$ of $m'L$ with $v(s) = m'T_{m'}(v)$. For $m \geq r_0$ divide $m - r_0$ by $m'$, so that $m = qm' + r$ with $r_0 \leq r < r_0 + m'$. The section $s^q\sigma_r$ of $mL$ has value $qm'T_{m'}(v)$. As $qm'/m \to 1$, we get $\liminf_m T_m(v) \geq T_{m'}(v)$. Taking the supremum over $m'$ and using $T_m(v) \leq T(v)$ proves the first assertion.

For the second, put $\Sigma_m = mN_mS_m(v)$ and apply Lemma 34.1.1 to the image of $\sigma_r H^0(X, mL)$, of codimension $c_{m,r} = N_{m+r} - N_m$ in $H^0(X, (m + r)L)$. The inequality $\lambda'_i \leq \lambda_i$ gives $\Sigma_m \leq \Sigma_{m+r}$, because the jumping numbers are nonnegative. The inequality $\lambda_{i+c} \leq \lambda'_i$ and $\lambda_1 \leq (m + r)T(v)$ give $\Sigma_{m+r} \leq \Sigma_m + c_{m,r}(m + r)T(v)$. By the Hilbert polynomial, $N_m = Vm^n/n! + O(m^{n-1})$, so $c_{m,r} = O(m^{n-1})$ uniformly for $r \leq R$. Write

$$S_{m+r}(v) - S_m(v) = S_m(v)\left(\frac{mN_m}{(m + r)N_{m+r}} - 1\right) + \frac{\Sigma_{m+r} - \Sigma_m}{(m + r)N_{m+r}}.$$

The first term is at most $T(v)$ times a quantity tending to zero, and the second is $O(m^n)/m^{n+1}$. This proves the second assertion.

For the third, let $\varepsilon > 0$ and choose $m_1$. Apply the second assertion with $R = r_0 + m_1$. Every large $m$ can be written as $qm_1 + r$ with $r_0 \le r < r_0 + m_1$, and then $S_m(v) \ge S_{qm_1}(v) - \varepsilon \ge c - 2\varepsilon$ for large $m$. The upper bound is the same argument.□

## 34.2 Section lattices of semicontinuous obstacles

For continuous psh potentials Lemma 18.2.2 identifies the limits of section counts with energies. We now allow a semicontinuous obstacle. As there, a statement holds *in divisibility order* if for every $\varepsilon > 0$ it holds up to $\varepsilon$ for all multiples of some positive integer. Recall that $E_{[L]}(0) = 0$, so the limit of $S_m(u, 0)$ is $E_{[L]}(u)$ for continuous $[L]$-psh $u$.

**Proposition 34.2.1** *Let $f$ be a bounded function on $X^{\mathrm{an}}$.*

(1) *If $f$ is lower semicontinuous, then in divisibility order $S_m(f, 0) \ge E_{[L]}(f) - o(1)$.*
(2) *If $[L]$ has the envelope property, then in divisibility order $S_m(f, 0) \le E_{[L]}(f) + o(1)$.*

*In particular, for a bounded lower semicontinuous obstacle under the envelope property, $S_m(f, 0) \to E_{[L]}(f)$ in divisibility order.*

***Proof*** Lattices are monotone in the obstacle. If $f \le g$ then $A_m(f) \subseteq A_m(g)$, and the additivity of signed lengths gives $S_m(f, 0) \le S_m(g, 0)$.

Assume $f$ lower semicontinuous and let $\varepsilon > 0$. Choose an $[L]$-psh competitor $u \le f$ with $E_{[L]}(u) \ge E_{[L]}(f) - \varepsilon$. Replacing $u$ by its maximum with the constant $\inf f$ preserves both properties. By Proposition 6.4.1 choose continuous $[L]$-psh potentials $u_j$ decreasing to $u$. Fix $\delta > 0$. The sets $\{u_j < f + \delta\}$ are open, since $u_j - f$ is upper semicontinuous. They increase with $j$ and cover the compact space $X^{\mathrm{an}}$. Hence $u_j - \delta \le f$ for some $j$, as in the proof of Proposition 12.1.1. The continuous competitor $u' = u_j - \delta$ satisfies $E_{[L]}(u') \ge E_{[L]}(u) - \delta V$. By Lemma 18.2.2, in divisibility order,

$$S_m(f, 0) \ge S_m(u', 0) \ge E_{[L]}(u') - \varepsilon \ge E_{[L]}(f) - 2\varepsilon - \delta V.$$

This proves the first assertion.

Assume the envelope property and put $p = P_{[L]}(f)^*$. It is $[L]$-psh and bounded between $\inf f$ and $\sup f$. The envelope property in $[L]$ is the weak envelope property (Definition 18.1.1) with the identity morphism, so Theorem 18.1.1 gives $E_{[L]}(f) = E_{[L]}(p)$. Choose continuous $[L]$-psh $\psi_i$ decreasing to $p$. Their energies decrease to $E_{[L]}(p)$ by decreasing continuity of the pairing (Section 17.8). For sufficiently divisible $m$, Lemma 18.2.2 gives $A_m(f) = A_m(P_{[L]}(f)) \subseteq A_m(\psi_i)$. Hence in divisibility order $S_m(f, 0) \le E_{[L]}(\psi_i) + o(1)$ for each $i$. Letting $i$ increase proves the second assertion. □

## 34.3 The energy of a Dirac mass

For a valuation $v$ of linear growth define the *Dirac obstacle*

$$f_v = T(v)\,\mathbf{1}_{X^{\mathrm{an}}\setminus\{v\}},$$

with value 0 at $v$ and $T(v)$ elsewhere. It is bounded and lower semicontinuous, because points of the Hausdorff space $X^{\mathrm{an}}$ are closed. Put

$$\psi_v = \sup\left\{\frac{\log|s| + v(s)}{m} \,:\, m \geq 1,\ 0 \neq s \in H^0(X, mL)\right\}.$$

Each function in the supremum belongs to $\mathcal{H}^{\mathrm{gf}}(L)$, so it is $[L]$-psh, and it vanishes at $v$.

**Lemma 34.3.1** *Let $v$ be a valuation.*

(1) *Every $u \in \mathrm{PSH}(X,[L])$ satisfies $\sup u - T(v) \leq u(v) \leq \sup u$, where $\sup u = u(v_0)$.*
(2) *If $v$ has linear growth, then $P_{[L]}(f_v) = \psi_v = \sup\{u \in \mathrm{PSH}(X,[L]) : u(v) \leq 0\}$. Moreover $0 \leq \psi_v \leq T(v)$, $\psi_v(v) = 0$ and $\psi_v(v_0) = T(v)$.*
(3) *If $v$ has linear growth, then $e^\vee(\delta_v) = e(f_v)$ and*

$$S_m(f_v, 0) = \frac{n!}{m^{n+1}} \sum_{i=1}^{N_m} \lfloor \lambda_{m,i}(v) \rfloor.$$

***Proof*** The upper bound in the first assertion follows from (6.4). For the lower bound, first let $u = m^{-1}\max_j(\log|s_j| + a_j)$ be a rational Fubini–Study potential for $L$. Discard the zero sections. Then $u(v_0) = m^{-1}\max_j a_j$, attained at some $j_0$, and

$$u(v) \geq \frac{a_{j_0} - v(s_{j_0})}{m} \geq u(v_0) - T(v).$$

A general $u$ is a decreasing limit of such potentials by Proposition 6.4.1, and both sides pass to the limit.

For the second assertion, let $u$ be $[L]$-psh with $u(v) \leq 0$. By the first assertion $\sup u \leq T(v)$, so $u \leq f_v$. Conversely every competitor below $f_v$ is nonpositive at $v$. This identifies $P_{[L]}(f_v)$ with the supremum on the right. Now let $u_i$ be rational Fubini–Study potentials decreasing to $u$, written as above with sections $s_j$ of $m_i L$. For each $j$, $a_j - v(s_j) \leq m_i u_i(v)$. Therefore

$$u_i \leq \max_j \frac{\log|s_j| + v(s_j)}{m_i} + u_i(v) \leq \psi_v + u_i(v).$$

Since $u_i(v)$ decreases to $u(v) \leq 0$, we get $u \leq \psi_v$. Each function in the definition of $\psi_v$ is psh and vanishes at $v$, so $\psi_v$ equals the supremum on the right and $\psi_v(v) = 0$.

Each function in the definition of $\psi_v$ is at most $v(s)/m \le T(v)$, since $\log|s| \le 0$, and its value at $v_0$ is $v(s)/m$. This gives $\psi_v \le T(v)$ and $\psi_v(v_0) = T(v)$. At a point $w$, the section $\sigma_{r_0}$ chosen for the centre of $w$ has $\log|\sigma_{r_0}|(w) = 0$, so $\psi_v(w) \ge 0$.

For the third assertion, each finite-energy $u$ may be translated so that $u(v) = 0$. Then $u \le f_v$ by the first assertion, so $e(u) - u(v) = e(u) \le e(f_v)$. Conversely every competitor $u \le f_v$ satisfies $e(u) \le e(u) - u(v) \le e^\vee(\delta_v)$, where we may restrict to competitors of finite energy by taking a maximum with the constant zero. Thus $e^\vee(\delta_v) = e(f_v)$.

For the lattice of $f_v$, let $st^j$ have $s \ne 0$. If $j \ge 0$, the condition $\log|s| \le m f_v + j$ always holds, since $\log|s| \le 0 \le f_v$. If $j = -\lambda < 0$, the condition at $v$ is $v(s) \ge \lambda$, and away from $v$ it is $\log|s| \le mT(v) - \lambda$. The latter holds whenever $v(s) \ge \lambda$, since $\lambda \le v(s) \le mT(v)$. Hence the coefficient space of $A_m(f_v)$ in exponent $-\lambda$ is $F^\lambda H^0(X, mL)$, and $A_m(0)$ is the part in nonnegative exponents. The signed length is $\sum_{\lambda \ge 1} \dim F^\lambda H^0(X, mL)$, and Lemma 34.1.1 gives the formula. □

**Theorem 34.3.1** *For every valuation $v$,*

$$\frac{T(v)}{n+1} \le e^\vee(\delta_v) \le T(v).$$

*In particular $\delta_v$ has finite energy if and only if $v$ has linear growth. If $v$ has linear growth, then $e^\vee(\delta_v) \le \liminf_{m\to\infty} S_m(v)$.*

***Proof*** For the upper bound let $u$ have finite energy. In the difference-of-powers formula of Section 17.11 with the zero potential, $E_{[L]}(0) = 0$, the terms decrease, and the first is $V \sup u$, because the measure of the zero potential is $V\delta_{v_0}$ (Example 19.7.1). Hence $e(u) \le \sup u$. With Lemma 34.3.1,

$$e(u) - u(v) \le \sup u - (\sup u - T(v)) = T(v).$$

For the lower bound fix $m_0 \ge 1$ and a section $s$ of $m_0 L$ with $a = v(s) > 0$, and put $b = a/m_0$. If no such section exists then $T(v) = 0$ and there is nothing to prove. Let $\sigma_1, \dots, \sigma_p$ generate $r_0 L$. The function $r_0^{-1} \max_j \log|\sigma_j|$ is identically zero, since some $\sigma_j$ does not vanish at the centre of any given point. Raising all sections to suitable powers shows that

$$u = \max\left\{\frac{\log|s| + a}{m_0},\, 0\right\}$$

belongs to $\mathcal{H}^{\mathrm{gf}}(L)$ and is finite everywhere. It is therefore continuous and $[L]$-psh, with $u(v) = 0$ and $u \ge 0$.

We estimate $S_m(u, 0)$. Since $u \ge 0$, $A_m(0) \subseteq A_m(u)$, and the coefficient space of the quotient in exponent $-\lambda < 0$ is $G_m^\lambda = \{\sigma \in H^0(X, mL) : \log|\sigma| \le mu - \lambda\}$. For integers $k \ge 0$ with $km_0 \le m$ and a section $\tau$ of $(m - km_0)L$,

$$\log|s^k \tau| \le k \log|s| \le k(m_0 u - a) \le mu - ka.$$

Multiplication by $s^k$ is injective, so $\dim G_m^\lambda \geq N_{m-km_0}$ for $k = \lceil \lambda/a \rceil$ whenever $km_0 \leq m$. Then $m - km_0 \geq m - m_0 - \lambda/b$. The Hilbert polynomial gives a constant $C$ with $N_q \geq Vq^n/n! - C(q^{n-1}+1)$ for all integers $q \geq 0$. Hence $\dim G_m^\lambda \geq V(m - m_0 - \lambda/b)_+^n/n! - C(m^{n-1}+1)$ for every $\lambda \geq 1$. If $km_0 > m$, then $m - m_0 - \lambda/b < 0$ and this bound is trivial. Summing over $1 \leq \lambda \leq mb$, and comparing the decreasing function $x \mapsto (m - m_0 - x/b)_+^n$ with its integral over $[1, \infty)$, we obtain

$$\frac{m^{n+1}}{n!} S_m(u,0) \geq \frac{V}{n!} \frac{b}{n+1} (m - m_0 - 1/b)_+^{n+1} - C' m^n.$$

By Lemma 18.2.2, $S_m(u,0)$ converges to $E_{[L]}(u)$ in divisibility order. Dividing by $m^{n+1}/n!$, letting $m$ tend to infinity along multiples of a suitable integer and dividing by $V$ gives $e(u) \geq b/(n+1)$, and

$$e^\vee(\delta_v) \geq e(u) - u(v) \geq \frac{v(s)}{(n+1)m_0}.$$

The supremum over $m_0$ and $s$ is $T(v)/(n+1)$.

Suppose $v$ has linear growth. By Lemma 34.3.1 and Proposition 34.2.1, in divisibility order

$$V_m S_m(v) \geq S_m(f_v, 0) \geq E_{[L]}(f_v) - o(1) = Ve^\vee(\delta_v) - o(1),$$

where $V_m = n! N_m/m^n \to V$. The first inequality uses $\lfloor \lambda \rfloor \leq \lambda$. The third assertion of Lemma 34.1.2 converts this into the bound on the lim inf. □

The example in Section 7.4 gives a valuation with generic support whose Dirac mass therefore has infinite energy. Every divisorial valuation and the trivial valuation have linear growth, by Lemma 6.3.2. For the trivial valuation $T(v_0) = 0$, in agreement with $e^\vee(\delta_{v_0}) = 0$.

**Theorem 34.3.2** *Suppose there is a finite birational morphism $\nu : X' \to X$ from an integral projective scheme $X'$ such that $[\nu^* L]$ has the envelope property. Then for every valuation $v$ of linear growth the limit $S(v) = \lim_m S_m(v)$ exists and*

$$S(v) = e^\vee(\delta_v).$$

*The hypothesis holds for every integral projective $X$ over an algebraically closed trivially valued field, with $\nu$ the normalization, and for every integral regular projective curve, with $\nu$ the identity.*

***Proof*** First suppose that $[L]$ itself has the envelope property. By Lemma 34.3.1 and Proposition 34.2.1, in divisibility order

$$V_m \left( S_m(v) - \frac{1}{m} \right) \leq S_m(f_v, 0) \leq E_{[L]}(f_v) + o(1) = Ve^\vee(\delta_v) + o(1).$$

Here the first inequality uses $\lfloor \lambda \rfloor \geq \lambda - 1$. Together with Theorem 34.3.1 and Lemma 34.1.2, this proves the convergence.

In general write $L' = \nu^* L$, which is ample since $\nu$ is finite, and use primes for the invariants on $X'$. The valuation $v$ of $k(X) = k(X')$ has a centre on $X'$ because $\nu$ is proper. Pullback embeds $H^0(X, mL)$ in $H^0(X', mL')$ and preserves the values of $v$. The cokernel injects into the sections of $Q \otimes mL$, with $Q = \nu_* O_{X'}/O_X$ supported on a proper closed subset. By its Hilbert polynomial the codimension $c_m$ is $O(m^{n-1})$. The projection formula gives $(L'^n \cdot [X']) = V$.

The conductor $\mathfrak{c}$, the annihilator of $Q$, is a nonzero ideal with $\mathfrak{c}\, \nu_* O_{X'} \subseteq O_X$. Choose $r$ and a nonzero section $\tau$ of $\mathfrak{c} \otimes rL$. For every section $s'$ of $mL'$, $\tau s'$ is a section of $(m+r)L$. This gives $v(s')+v(\tau) \le (m+r)T(v)$, so $T'_m(v) \le (1+r/m)T(v)$. Hence $T'(v) = T(v)$ by Lemma 34.1.2, since pullback of sections gives the reverse inequality. By Lemma 34.1.1, the sums of the jumping numbers in degree $m$ differ by at most $c_m m T(v)$, and the dimensions differ by $c_m$. As in the proof of Lemma 34.1.2, $S'_m(v) - S_m(v) \to 0$. The first case applied on $X'$ shows that $S_m(v)$ converges to $e^\vee_{X'}(\delta_v)$. Also $e^\vee_X(\delta_v) \le e^\vee_{X'}(\delta_v)$ by Theorem 34.3.1.

For the reverse inequality we compare the functions $\psi_v$ and $\psi'_v$. On every point $w$ with generic support, $\nu^* \psi_v(w) = \psi'_v(w)$. Indeed pullback of sections gives one inequality. For the other, take a section $s'$ of $mL'$ and $q \ge 1$. The section $\tau s'^q$ of $(qm + r)L$ gives

$$\psi_v(w) \ge \frac{\log|\tau|(w) + q \log|s'|(w) + v(\tau) + q v(s')}{qm + r},$$

which tends to $(\log|s'|(w) + v(s'))/m$ as $q \to \infty$, because $\log|\tau|(w)$ is finite.

For a finite set $F$ of pairs $(m, s)$ containing the pairs $(r_0, \sigma_j)$ of a generating family, let $G_F$ be the maximum of the functions $(\log|s| + v(s))/m$ over $F$. After raising the sections to a common degree, $G_F$ is a finite generically finite Fubini–Study potential, so it is continuous and $[L]$-psh. It satisfies $G_F(v) = 0$ and $G_F \le T(v)$, so it is a competitor for $f_v$ and $e(G_F) \le e^\vee_X(\delta_v)$ by Lemma 34.3.1. The pullback $\nu^* G_F$ is the potential of the pulled-back sections with the same weights, so it is continuous and $[L']$-psh. The net $\nu^* G_F$ increases to $\nu^* \psi_v$. On $\mathcal{D}(X')$ this equals $\psi'_v = P(f'_v)$, which agrees there with the $[L']$-psh function $P(f'_v)^*$ by Theorem 12.2.1. Pullback by the projective birational morphism $\nu$ preserves energy, as in the proof of Theorem 18.1.1. By Corollary 17.13.1 on $X'$ and Theorem 18.1.1,

$$e_X(G_F) = e_{X'}(\nu^* G_F) \longrightarrow e_{X'}(P(f'_v)^*) = e_{X'}(f'_v) = e^\vee_{X'}(\delta_v).$$

Hence $e^\vee_X(\delta_v) \ge e^\vee_{X'}(\delta_v)$.

Over an algebraically closed field the normalization is finite and birational, as in the proof of Proposition 12.3.1, and it is normal, so Theorem 16.5.1 applies. A regular projective curve has the envelope property by Corollary 20.4.1. □

Over an algebraically closed field, if the normalization of $X$ is not a homeomorphism, then no ample class has the envelope property by Proposition 12.3.1, and the normalized Monge–Ampère operator is not onto $\mathcal{M}^1_\omega$ by Corollary 19.7.1. Theorem 34.3.2 nevertheless identifies $e^\vee(\delta_v)$ as a limit of section data, and the potentials $G_F$ of its proof form a maximizing net for $\delta_v$.

## 34.4 The Green function of a valuation

Under the envelope property the maximizer for a Dirac mass is the function $\psi_v$ itself.

**Theorem 34.4.1** *Suppose $[L]$ has the envelope property, and let $v$ be a valuation of linear growth. The function $u_v = \psi_v$ is $[L]$-psh, bounded, and satisfies*

$$u_v(v) = 0, \qquad \sup u_v = u_v(v_0) = T(v), \qquad \mu_{u_v} = \delta_v, \qquad e(u_v) = S(v).$$

*It is the unique finite-energy solution of $\mu_u = \delta_v$ with $u(v) = 0$. Moreover*

$$\frac{T(v)}{n+1} \le S(v) \le \frac{n}{n+1} T(v).$$

***Proof*** Put $p = \psi_v^*$. By the envelope property and Lemma 34.3.1, $p = P(f_v)^*$ is $[L]$-psh with $0 \le p \le T(v)$. By Theorems 18.1.1 and 34.3.2 and Lemma 34.3.1, $e(p) = e(f_v) = e^\vee(\delta_v) = S(v)$.

By Theorem 19.6.1 there is a finite-energy $w$ with $\mu_w = \delta_v$. The value $w(v)$ is finite because $\delta_v$ integrates finite-energy potentials, so we may translate to $w(v) = 0$. Translates of $w$ maximize $e(u) - u(v)$, so $e(w) = e^\vee(\delta_v) = e(p)$. By Lemma 34.3.1, $w \le \psi_v \le p$. In the difference-of-powers formula for $e(p) - e(w)$ the terms $a_j$ decrease, the last one, $a_n = V \int (p - w)\, \mathrm{d}\mu_p$, is nonnegative since $p \ge w$, and their sum is zero. Hence all of them vanish. The first one is $V(p - w)(v) = Vp(v)$, so $p(v) = 0 = w(v)$. Thus $p \le w$ almost everywhere for $\mu_w$, and the domination principle Corollary 19.4.1 gives $p \le w$. We conclude $w = \psi_v = p$. In particular $\psi_v$ is upper semicontinuous and $u_v$ is psh. The values at $v$ and $v_0$ come from Lemma 34.3.1. Uniqueness is part of Theorem 19.6.1.

For the bounds use the difference-of-powers formula for $e(u_v)$ with the zero potential. Its terms $a_0 \ge a_1 \ge \cdots \ge a_n$ satisfy $a_0 = V \sup u_v = VT(v)$ and $a_n = V \int u_v \, \mathrm{d}\mu_{u_v} = Vu_v(v) = 0$. Therefore

$$\frac{VT(v)}{n+1} = \frac{a_0 + na_n}{n+1} \le E_{[L]}(u_v) \le \frac{na_0 + a_n}{n+1} = \frac{nVT(v)}{n+1}.$$

Divide by $V$. □

The lower bound also follows from Theorem 34.3.1 without the envelope property. This proof of the upper bound uses the solution, but the bound holds under the weaker hypothesis of Theorem 34.3.2. Indeed the proof of that theorem shows that $S(v)$ and $T(v)$ coincide with the invariants of $v$ on $X'$ with respect to $\nu^* L$, where Theorem 34.4.1 applies. In particular $T(v)/(n+1) \le S(v) \le nT(v)/(n+1)$ on every integral projective variety over an algebraically closed field. Both bounds are attained, as the toric examples after Corollary 34.5.1 show. In the terminology of Chapter 12, $u_v$ is the envelope of the Dirac obstacle $f_v$, and Theorem 34.4.1 shows that this envelope needs no regularization.

**Corollary 34.4.1** *Let $C$ be an integral regular projective curve over a trivially valued field, $L$ an ample line bundle of degree $V$, and $p$ a closed point with residue degree $d_p$. Every valuation of $C$ has linear growth, and $S(v) = T(v)/2$. For $v = x_{p,t}$,*

$$T(x_{p,t}) = \frac{tV}{d_p}, \qquad S(x_{p,t}) = \frac{tV}{2d_p},$$

*and $u_v(x_{p,s}) = tV/d_p - V\min\{s,t\}/d_p$ on the $p$-ray, with $u_v = tV/d_p$ on the other rays.*

***Proof*** By Theorem 1.4.1 the valuations are $v_0$ and the points $x_{p,t}$ with $0 < t < \infty$. A nonzero section of $mL$ has an effective divisor of degree $mV$, so $\operatorname{ord}_p(s)\, d_p \le mV$ and $T(x_{p,t}) \le tV/d_p < \infty$. The class has the envelope property by Corollary 20.4.1, and for $n = 1$ the two bounds of Theorem 34.4.1 coincide. The tail function of $\delta_{x_{p,t}}$ is 1 on $[0,t)$ and zero afterwards, so Theorem 31.2.1 gives $e^\vee(\delta_{x_{p,t}}) = tV/(2d_p)$ and the solution $-V\min\{s,t\}/d_p$ on the $p$-ray, zero elsewhere. By Theorem 34.3.2, $S(x_{p,t}) = tV/(2d_p)$, and $T(x_{p,t}) = 2S(x_{p,t})$. Translating the solution to vanish at $x_{p,t}$ gives $u_v$, by the uniqueness in Theorem 34.4.1. □

By Lemma 34.1.2, the equality $T(x_{p,t}) = tV/d_p$ says that the largest order of vanishing at $p$ of a section of $mL$ is $mV/d_p - o(m)$. Here this follows from the energy identity; a direct argument uses Riemann–Roch for the divisors $mL - \lambda p$. For $\mathbb{P}^1$, $O(1)$ and a rational point, it recovers the numbers $T = 1$ and $S = 1/2$ of the opening example.

## 34.5 Toric valuations and barycentres

Keep the lattices $M, N$ and the Gauss points $\eta_w$, $w \in N_{\mathbb{R}}$, of Section 32.1. Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope. Let $R_P$ be the monoid algebra of the lattice points of the cone over $P \times \{1\}$, graded by the last coordinate, and assume that $R_P$ is generated in degree one. Any lattice polytope has a multiple with this property, by the Veronese argument in the proof of Proposition 32.2.1. Put $X_P = \operatorname{Proj} R_P$ and $L = O(1)$. The facet-order argument in the proof of Proposition 32.2.1, applied to the cone over $P \times \{1\}$, shows that $R_P$ is a normal domain, so $X_P$ is a normal integral projective variety of dimension $n$, and $L$ is very ample. Write $\chi^{(u,m)}$ for the monomial of a lattice point $(u,m)$ of the cone.

**Lemma 34.5.1** *For every $m \ge 0$ the monomials $\chi^{(u,m)}$, $u \in mP \cap M$, form a basis of $H^0(X_P, mL)$. For $\xi \in N_{\mathbb{R}}$ let $c_\xi = \min_P\langle\cdot,\xi\rangle$. Then $\eta_\xi$ is a valuation of $k(X_P)$ with a centre on $X_P$, and*

$$\eta_\xi\Big(\sum_u a_u\chi^{(u,m)}\Big) = \min_{a_u\neq 0}\langle u,\xi\rangle - mc_\xi.$$

***Proof*** For a vertex $u_0$ of $P$, the chart $D_+(\chi^{(u_0,1)})$ has ring $k[\mathbb{R}_{\geq 0}(P-u_0)\cap M]$, and these charts cover $X_P$, as explained before Theorem 32.3.1. On it, $\chi^{(u_0,1)}$ is a frame of $L$. Since $X_P$ is integral, a section of $mL$ is determined by its restriction to the dense torus. There it is a Laurent polynomial times the $m$-th power of a vertex frame, that is, a finite sum $\sum_u a_u \chi^{(u,m)}$ with $u \in M$. It is regular on the chart of $u_0$ exactly when $u - mu_0 \in \mathbb{R}_{\geq 0}(P-u_0)$ for every $u$ with $a_u \neq 0$. The intersection of the cones $mu_0 + \mathbb{R}_{\geq 0}(P-u_0)$ over the vertices is $mP$, since a polytope is the intersection of its tangent cones at the vertices. Hence the sections are the stated combinations.

For real $\xi$, the terms of minimal weight in a product are the products of the terms of minimal weight of the factors, and their sum is nonzero since $k[M]$ is a domain. Thus $\eta_\xi$ is a valuation on $k[M]$ and on its fraction field. Let $u_0$ be a vertex minimizing $\langle\cdot,\xi\rangle$ on $P$. Then $\eta_\xi \geq 0$ on the chart ring of $u_0$, which therefore contains the centre. Dividing by the frame $\chi^{(u_0,1)}$ to the power $m$ gives the formula. □

Let $b_P$ be the barycentre of $P$ for Lebesgue measure.

**Theorem 34.5.1** *For $\xi \in N_\mathbb{R}$,*

$$T(\eta_\xi) = \max_P \langle\cdot,\xi\rangle - c_\xi, \qquad S(\eta_\xi) = \langle b_P,\xi\rangle - c_\xi.$$

*If $k$ is algebraically closed, then $e^\vee(\delta_{\eta_\xi}) = \langle b_P,\xi\rangle - c_\xi$.*

***Proof*** By Lemma 34.5.1 the monomial basis is adapted to the filtration, and the jumping numbers of $\eta_\xi$ in degree $m$ are $\langle u,\xi\rangle - mc_\xi$, $u \in mP\cap M$. The largest one is $m(\max_P\langle\cdot,\xi\rangle - c_\xi)$, attained at a vertex, so $T_m$ is constant. For the average,

$$S_m(\eta_\xi) = \frac{\sum_{u\in mP\cap M}\langle u/m,\xi\rangle}{\#(mP\cap M)} - c_\xi.$$

Choose a basis of $M$, use the resulting coordinates and the Euclidean distance on $M_\mathbb{R}$, and for $x \in m^{-1}M$ let $Q_x = x + [0,1/m)^n$. The cubes $Q_x$ with $x \in P$ lie in the $\delta_m$-neighborhood of $P$, where $\delta_m = \sqrt{n}/m$, and they cover the points of $P$ at distance at least $\delta_m$ from the boundary. The volumes of both sets tend to the volume of $P$, whose boundary is a null set. Hence $m^{-n}\#(mP\cap M) \to \mathrm{vol}(P)$, and since $\langle\cdot,\xi\rangle$ is uniformly continuous, the numerator divided by $m^n$ tends to $\int_P \langle u,\xi\rangle\,\mathrm{d}u$. This proves the formula for $S$. The last assertion is Theorem 34.3.2, since $X_P$ is normal. □

**Corollary 34.5.1** *Let $P$ be a full-dimensional rational polytope in $\mathbb{R}^n$ and $\xi$ a linear form. Then*

$$\frac{1}{n+1}\Big(\max_P \xi - \min_P \xi\Big) \leq \xi(b_P) - \min_P \xi \leq \frac{n}{n+1}\Big(\max_P \xi - \min_P \xi\Big).$$

***Proof*** All three quantities are unchanged under translation and scale by the same factor under dilation. Replace $P$ by a lattice polytope with $R_P$ generated in degree one, and take $k$ algebraically closed. Then $X_P$ is normal, so $[L]$ has the envelope property by Theorem 16.5.1. Apply Theorems 34.4.1 and 34.5.1. □

For the simplex $P = \operatorname{conv}(0, e_1, \dots, e_n)$, so that $X_P = \mathbb{P}^n$, the barycentre is $(n+1)^{-1}(1, \dots, 1)$. The form $\xi = e_1^*$ gives $S = 1/(n+1)$ and $T = 1$. On the affine chart of the vertex 0, with coordinates $x_i = \chi^{(e_i,1)}/\chi^{(0,1)}$, the valuation $\eta_\xi$ is the order of vanishing along the hyperplane $x_1 = 0$, and it attains the lower bound. The form $\xi = e_1^* + \cdots + e_n^*$ gives $S = n/(n+1)$ and $T = 1$. It takes the value 1 on every $x_i$ and is the order of vanishing at the origin of the chart. It attains the upper bound.

## 34.6 Uniform approximation

The convergence in Theorem 34.3.2 is proved one valuation at a time. Applications to thresholds defined as infima over all valuations need a uniform form, which we record as an open problem in this book. Appendix G.35 discusses the known case of a big line bundle on a normal complex projective variety.

*Question 34.6.1* Let $X$ be an integral projective scheme of dimension $n \geq 1$ over an algebraically closed trivially valued field, and let $L$ be an ample line bundle. Prove that there are numbers $\varepsilon_m \to 0$ such that $S_m(v) \leq (1+\varepsilon_m)S(v)$ for all sufficiently large $m$ and every valuation $v$ of linear growth. Here $S_m$ and $S$ are the expected vanishing orders of Definition 34.1.1, and the limit $S(v)$ exists by Theorem 34.3.2.

For a fixed valuation the convergence is Theorem 34.3.2. On $X_P$ the uniform estimate holds for the toric valuations $\eta_\xi$. Indeed, by the proof of Theorem 34.5.1, $S_m(\eta_\xi) - S(\eta_\xi) = \langle b_m - b_P, \xi \rangle$, where $b_m$ is the mean of the points $u/m$ with $u \in mP \cap M$, and $b_m \to b_P$. Thus the difference is at most $\|b_m - b_P\|\,|\xi|$ for dual norms. Moreover $S(\eta_\xi) > 0$ for $\xi \neq 0$ because $b_P$ is an interior point of $P$, and $S(\eta_\xi) = \langle b_P, \xi \rangle - c_\xi$ is continuous and positively homogeneous in $\xi$, since $c_\xi$ is the minimum of $\langle \cdot, \xi \rangle$ over the vertices of $P$. Compactness of the unit sphere therefore gives a constant $\kappa > 0$ with $S(\eta_\xi) \geq \kappa|\xi|$ for all $\xi$. Hence $S_m(\eta_\xi) \leq (1+\varepsilon_m)S(\eta_\xi)$ for all $\xi$, where $\varepsilon_m = \|b_m - b_P\|/\kappa \to 0$.

# Appendix A
# Algebraic geometry of projective schemes

This appendix proves the facts about real numerical classes on a projective scheme over an arbitrary field that the text uses from Section 3.5 onward. They are collected in Proposition A.1.1.

## A.1 The ample and nef cones

Let $X$ be projective over a field $k$. As in Section 3.5, $N^1(X)_{\mathbb{R}}$ is the space of real Cartier classes modulo numerical equivalence, that is, modulo equality of degrees on all closed integral curves. The ample cone is the cone of positive real linear combinations of classes of ample invertible sheaves. The nef cone is given by nonnegative degree on every closed integral curve.

We use two standard algebraic results. The theorem of the base implies that the group of numerical classes of invertible sheaves of a proper scheme over an algebraically closed field has finite rank [Kle71, Exposé XIII, Théorème 4.6, pp. 645–646, and Théorème 5.1, p. 650]. The numerical ampleness criterion says that an invertible sheaf $D$ on a projective scheme is ample if and only if $(D^{\dim Z} \cdot [Z]) > 0$ for every closed integral subscheme $Z$ of positive dimension [Deb16, Theorem 4.1]. There the intersection numbers are defined through Euler characteristics, as in [Stacks, Tag 0BEP]. They agree with the Chern class intersection numbers of Section 3.5 by [Stacks, Tag 0BFI]. For a proper curve this is the degree $\chi(D) - \chi(\mathcal{O})$ of [Stacks, Tags 0AYR and 0BEY]. Here is how to apply these results to the real numerical classes used in the text.

**Proposition A.1.1** *Let $X$ be projective over a field $k$.*

(1) *The space $N^1(X)_{\mathbb{R}}$ is finite-dimensional with a rational structure given by rational Cartier classes.*
(2) *If $\bar{k}$ is an algebraic closure of $k$, pullback induces an injective linear map $N^1(X)_{\mathbb{R}} \to N^1(X_{\bar{k}})_{\mathbb{R}}$. Thus numerical equivalence of real classes from $X$ is preserved and detected on $X_{\bar{k}}$.*

(3) *For every closed integral subscheme $Z$ of $X$ of dimension $d$, the mixed degree $(\theta_1 \cdots \theta_d \cdot [Z])$ is a well-defined symmetric multilinear function of $\theta_1, \ldots, \theta_d \in N^1(X)_{\mathbb{R}}$. It is positive when all $\theta_i$ are ample and nonnegative when all $\theta_i$ are nef.*
(4) *The ample cone is open, its closure is the nef cone, and the sum of a nef class and an ample class is ample.*
(5) *A rational line bundle is ample if its numerical class is ample.*

***Proof*** Let $\bar{k}$ be an algebraic closure of $k$. A closed integral curve of $X_{\bar{k}}$ descends to a geometrically integral curve over a finite extension $k'/k$ by finite presentation. Its image in $X$ is an integral curve, since the projection from $X_{k'}$ to $X$ is finite. For such a curve $C'$ over $k'$ and its image $C$, we have

$$[k' : k] \deg_{k'}(D_{k'}|_{C'}) = [k'(C') : k(C)] \deg_k(D|_C)$$

for any invertible sheaf $D$ on $X$. Indeed, every closed point $z$ of $C'$ satisfies $[\kappa(z) : k] = [\kappa(z) : k'][k' : k]$, so the left side is the degree over $k$ of the pullback of $D$ to $C'$. The right side follows from (3.2) for the finite morphism $q : C' \to C$, since $q_*[C'] = [k'(C') : k(C)][C]$. Degree is unchanged by extension from $k'$ to $\bar{k}$, as can be seen by computing it on a proper curve as $\chi(D) - \chi(\mathcal{O})$. In the opposite direction, every integral curve $C$ in $X$ is the image of a geometrically integral curve over some finite $k'/k$, by descending a reduced component of $C_{\bar{k}}$. The same formula applies. Since the formula is linear in $D$, a real class from $X$ is numerically trivial if and only if its pullback to $X_{\bar{k}}$ is. This proves the second assertion. The theorem of the base gives the claimed finite dimension. For a basis chosen among integral classes, there exist finitely many curves on which the corresponding square degree matrix is invertible. Its entries are integers, so every integral class has rational coordinates in this basis.

Choose an ample invertible sheaf $H$ and integral classes $[D_1], \ldots, [D_r]$ forming a real basis. By eventual ampleness of twists by an ample sheaf there is a positive integer $m$ such that $mH + D_j$ and $mH - D_j$ are ample for every $j$. The classes $[H] \pm m^{-1}[D_j]$ are ample and their convex hull contains a neighborhood of $[H]$ when the numerical space has positive dimension. When the space is zero, its only class is ample. This applies to every ample invertible sheaf. If $x = \sum_i a_i[H_i]$ with $a_i > 0$ and ample invertible sheaves $H_i$, and $U$ is such a neighborhood of $[H_1]$ inside the ample cone, then $a_1 U + \sum_{i \geq 2} a_i[H_i]$ is a neighborhood of $x$ inside the cone, since the cone is closed under positive scaling and addition. Hence the ample cone is open.

Let $Z$ be a closed integral subscheme of dimension $d$. The mixed degree $(L_1 \cdots L_d \cdot [Z])$ is multilinear in the Cartier classes $L_i$ by additivity of the first Chern class action [Stacks, Tag 02SP], and symmetric since these actions commute (Section 3.5). It depends only on numerical classes: if one factor is numerically trivial, apply the other factors to $[Z]$ first. The result is a class of one-dimensional cycles, on whose components the remaining factor has degree zero. Extending by linearity, the mixed degree is a well-defined symmetric multilinear function on $N^1(X)_{\mathbb{R}}$.

Mixed products of ample invertible sheaves against an integral subvariety have positive degree in top dimension [Stacks, Tags 0BEV and 0BFI]. We recall the ar-

gument, since it also gives an inequality for nef classes. After taking very ample powers one can take the successive divisors meeting properly. Over a finite field, the sections can be chosen after finite extension, since the conditions for each choice are nonempty open conditions in a space of sections. The pushforward of the base-changed cycle with its multiplicities is the field degree times the original cycle. The projection formula then descends the positivity. The same argument shows that the intersection of $d-1$ ample factors with a subvariety of dimension $d$ pairs nonnegatively with any nef class. Indeed, by the projection formula this intersection is the pushforward of the resulting effective curve cycle over the extension, divided by the field degree. That pushforward is an effective cycle of integral curves of $X$, on which a nef class has nonnegative degree. The statements extend by multilinearity to positive real combinations of ample classes. This proves the positivity in the third assertion. Since top degrees depend only on numerical classes, a rational line bundle whose class is ample has positive top degree on each integral subvariety. It is ample by the numerical ampleness criterion after clearing its denominator.

We next show that every nef class is a limit of ample classes. The converse follows from the inequalities on curves. Set $h = [H]$. Let $\beta$ be nef. Since the ample cone is open and contains $h$, the classes $\beta + th$ are ample for all sufficiently large $t$. Let $\tau \geq 0$ be the infimum of the positive $t$ for which they are ample. All $t > \tau$ work since adding positive multiples of $h$ preserves ampleness. For such a $t$ and an integral $Z$ of positive dimension $d$,

$$((\beta + th)^d - (th)^d) \cdot [Z] = \sum_{i=0}^{d-1} \big(\beta(\beta + th)^i (th)^{d-1-i} \cdot [Z]\big) \geq 0.$$

Every summand is nonnegative, since $\beta$ is nef and the remaining $d-1$ factors are ample, by the nonnegativity proved above. Taking $t$ down to $\tau$ shows that if $\tau > 0$, the class $\gamma = \beta + \tau h$ satisfies

$$(\gamma^d \cdot [Z]) \geq \tau^d (h^d \cdot [Z]).$$

The bound is uniform over $Z$ after division by $(h^d \cdot [Z])$. Use the fixed integral basis above and put $P_j = mH + D_j$. In a mixed intersection of ample classes, replacing one $P_j$ by $2mH$ can only increase the degree, because $2mH - P_j$ is ample. Expanding each $D_j = P_j - mH$, the absolute value of the degree of a product of $d$ basis factors is therefore at most $(3m)^d (h^d \cdot [Z])$. In basis coordinates the normalized top-degree polynomials have uniformly bounded coefficients, with $d \leq \dim X$. Their values are uniformly continuous on a fixed bounded neighborhood of $\gamma$, uniformly in $Z$. Hence all top degrees remain positive in a smaller neighborhood when $\tau > 0$. Every rational class of that neighborhood is ample by the numerical ampleness criterion. Each real point of the neighborhood is in the convex hull of finitely many rational points there, by taking a small surrounding simplex with rational vertices. The neighborhood is therefore ample. It contains $\beta + (\tau - \varepsilon)h$ for small $\varepsilon > 0$, contradicting the definition of $\tau$. Thus $\tau = 0$, and $\beta$ is the limit of the ample classes $\beta + th$ as $t \to 0$. Mixed

degrees are multilinear on the finite-dimensional space $N^1(X)_{\mathbb{R}}$, hence continuous. They are therefore nonnegative on nef classes, which completes the third assertion.

Finally, an open convex cone is stable under addition of its closure. Indeed, if $x$ is in the cone and $y_j$ are in the cone tending to $y$, the point $x + y - y_j$ is in the cone for all sufficiently large $j$. Adding $y_j$ gives $x + y$. □

# Appendix B
# Multiplier ideals and generation on a product

We work in characteristic zero. This appendix supplies the multiplier ideals used for the envelope theorem in Section 12.6. There a large power of the polarization gives an ideal on $X \times \mathbb{A}^1$. A multiplier ideal lowers its weight and retains a uniform bound on divisorial orders (Proposition B.1.1). Global generation is needed after the weight is lowered. We obtain it on a compactification of the product, where a twist in the parameter direction disappears on restriction to the affine line (Proposition B.3.1).

## B.1 Multiplier ideals and divisor orders

Let $S$ be smooth and quasi-projective over a field $k$ of characteristic zero. Let $I \subset \mathcal{O}_S$ be coherent and nonzero on every component. Choose a projective log resolution $\pi : Y \to S$ with $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Here $Y$ is smooth and the union of the support of $F$ and the exceptional divisor is a simple normal crossings divisor. We use the algebraic principalization theorem in characteristic zero in this form [Wło05, Theorem 1.0.1]. For a nonnegative rational number $c$ set

$$\mathcal{M}(I, c) = \pi_* \mathcal{O}_Y(K_{Y/S} - \lfloor cF \rfloor).$$

The floor is taken coefficient by coefficient. The relative canonical divisor is given by pulling back regular forms of top degree. At the generic point of a divisor on $S$, the birational morphism is an isomorphism. Hence the displayed pushforward consists of rational functions with no pole along any divisor on $S$, so it is a coherent ideal on the normal scheme $S$.

For a prime divisor $E$ on a normal birational model $Z$ over $S$, write

$$A_S(\mathrm{ord}_E) = 1 + \mathrm{ord}_E(K_{Z/S}), \qquad A_S(a\,\mathrm{ord}_E) = aA_S(\mathrm{ord}_E).$$

The order of the relative canonical divisor is computed at the generic point of $E$, where $Z$ is smooth. It is the order of the pullback of a local frame of the top forms on $S$, measured in a regular frame upstairs. Thus it is intrinsic to the divisor order, and

$A_S(\mathrm{ord}_E)$ is at least one. For a coherent ideal $J$, $w(J)$ is the minimum of the values of its local generators at the centre.

**Proposition B.1.1** *Locally on $S$, a regular function $g$ belongs to $\mathcal{M}(I,c)$ exactly when*

$$w(g) + A_S(w) > c\, w(I)$$

*for every nontrivial divisorial valuation $w$ centred in the open set under consideration. In particular the ideal is independent of the resolution and satisfies $w(\mathcal{M}(I,c)) \geq c\, w(I) - A_S(w)$.*

***Proof*** The inequalities on the divisors of the resolution are the membership conditions for its defining sheaf, because for integers $a$ one has $a \geq \lfloor x \rfloor$ if and only if $a + 1 > x$.

Suppose these membership conditions hold and take $w = \mathrm{ord}_E$ on any normal model. By properness its valuation ring has a centre on $Y$. Near that centre the components of $F$ are given by $z_1, \ldots, z_r$ among local étale coordinates $z_1, \ldots, z_d$, and $F = \sum_{i=1}^{r} b_i\{z_i = 0\}$. For a local frame $\eta$ of the top forms on $S$, the membership assumption gives

$$g\pi^*\eta = h\Big(\prod_{i=1}^{r} z_i^{\lfloor cb_i \rfloor}\Big)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d$$

with $h$ regular near the centre. Put $a_i = w(z_i)$. Writing $z_i = s^{a_i}u_i$ at the divisor, with $s$ a uniformizer and $u_i$ a unit, shows that $\mathrm{d}z_i/z_i$ is a regular differential plus $a_i\mathrm{d}s/s$. In a wedge there can be at most one factor $\mathrm{d}s/s$. The order of $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d$ is therefore at least $\sum_{i=1}^{r} a_i - 1$, and is nonnegative if all $a_i = 0$. When some $a_i > 0$, this gives

$$w(g) + A_S(w) \;\geq\; \sum_{i=1}^{r}\big(1 + \lfloor cb_i \rfloor\big)a_i \;>\; \sum_{i=1}^{r} cb_i a_i.$$

If all are zero, regularity of the differential gives the same strict inequality with right side zero. Multiplication by a positive scalar handles the remaining divisor orders. Evaluate at local generators to obtain the ideal bound. □

The criterion also gives the following rules. The operation is increasing with respect to $I$, $\mathcal{M}(I,1) \supset I$, and $\mathcal{M}(I^a,c) = \mathcal{M}(I,ac)$ for positive integers $a$. Formation of $\mathcal{M}(I,c)$ commutes with field extension. Indeed a log resolution remains such after base change, the multiplicities and relative top forms pull back, and the flat extension commutes with pushforward. The last fact can be seen by expressing sections on an affine open of $S$ as a kernel over a finite affine cover of its inverse image. In particular an inclusion of multiplier ideals defined over a common subfield can be checked after faithfully flat extension.

We describe the comparison over $\mathbb{C}$ when $S$ is projective. Locally take generators $(f_j)_j$ of $I$. On the complex manifold underlying $S$, the analytification of $\mathcal{M}(I,c)$ is the ideal of holomorphic germs $g$ for which $|g|^2(\sum_j |f_j|^2)^{-c}$ is locally integrable. Indeed on a log resolution the sum pulls back to the squared absolute value of a generator of $\mathcal{O}(-F)$ times a smooth positive factor. The change of variables introduces

the squared absolute value of the relative Jacobian. Integration in each normal crossing coordinate gives the integer floor in the defining pushforward. Properness lets one use finitely many such charts over a small neighborhood of a point. A local section of the analytic pushforward gives a holomorphic function on the isomorphism locus downstairs. Since $S$ is normal, $\pi$ is an isomorphism over the generic point of every divisor, as noted after the definition of $\mathcal{M}(I, c)$. Hence it is an isomorphism over the complement of a closed subset of codimension at least two. Hartogs extension across that subset shows that the pushforward is indeed an ideal of holomorphic functions. For this comparison we use projective algebraic-analytic coherence and cohomology [Ser56, Theorems 1–3]. The analytic integrability ideal is coherent by [Xia26a, Theorem 12.2.7]. The analytic pushforward thus algebraizes. Its identification with the algebraic pushforward follows from adjunction and full faithfulness on both projective schemes.

**Corollary B.1.1** *On a smooth quasi-projective scheme open in a smooth projective scheme in characteristic zero, one has*

$$\mathcal{M}(I, c + c') \subset \mathcal{M}(I, c)\mathcal{M}(I, c') \qquad (c, c' \in \mathbb{Q}_{\geq 0}).$$

***Proof*** Take the scheme-theoretic closure of the zero scheme in the compactification. Its ideal extends the given ideal. Formation of the ideal is local, so it suffices to prove the inclusion on the projective scheme. The scheme and the ideal are defined over a finitely generated subfield of $k$, which may be enlarged to ensure that the descended scheme is smooth and projective. The sheaves and the inclusion can be checked after extending that subfield to $\mathbb{C}$. Over $\mathbb{C}$ the weights are $c \log \sum_j |f_j|^2$ and $c' \log \sum_j |f_j|^2$. Subadditivity for these psh weights is [Xia26a, Theorem 12.4.1]. Projective algebraic-analytic comparison detects the inclusion of the coherent ideals. Faithfully flat base change then gives the assertion over $k$. □

## B.2 A vanishing calculation for section weights

Here is the analytic calculation used for generation. The lower curvature bound must persist as the weights approach the section ideal.

**Lemma B.2.1** *Let $M$ be a compact Kähler manifold and $B$ a holomorphic line bundle. Suppose its local metric weights $\phi$ are the decreasing limits of smooth weights $\phi_j$ on $B$ whose curvatures are bounded below by a fixed positive multiple of a Kähler form. Write $\mathcal{I}(\phi)$ for the ideal of locally square-integrable holomorphic germs with weight $e^{-\phi}$, as in Chapter 25. Then*

$$H^q(M, K_M \otimes B \otimes \mathcal{I}(\phi)) = 0 \qquad (q > 0).$$

***Proof*** Write $d = \dim M$. For a smooth metric of curvature at least $a\omega$, $a > 0$, the Bochner–Kodaira identity bounds the square of the norm of a compactly supported

$(d,q)$-form by $(2\pi a)^{-1}$ times the sum of the squared norms of $\bar\partial$ and its formal adjoint on that form. The Kähler commutator computation of [Xia26a, Lemma 14.6.2] applies in every degree. In a unitary coframe $\theta_1,\dots,\theta_d$ diagonalizing the curvature, with eigenvalues $\lambda_j \ge a$, put $e = \theta_1\wedge\cdots\wedge\theta_d$. For an index set $J$, let $\bar\theta_J$ be the ordered wedge product of the $\bar\theta_j$ with $j\in J$. The curvature commutator multiplies $e\wedge\bar\theta_J$ by $2\pi\sum_{j\in J}\lambda_j$. Indeed, the computation of [Xia26a, Lemma 14.6.1(1)] applies to $e\wedge\bar\theta_J$: its wedge product with the curvature vanishes for degree reasons, the adjoint $\Lambda$ of the Lefschetz operator removes $\theta_j\wedge\bar\theta_j$ for one $j\in J$, and the curvature restores it with the factor $\lambda_j$. Since $\sum_{j\in J}\lambda_j \ge qa \ge a$ for $|J| = q \ge 1$, the same identity gives the stated bound for each $q\ge 1$. The estimate passes to the distributional domains by the graph approximation with smooth forms [Xia26a, Lemma 14.6.3]. The proof works in each degree: the cutoff errors are exterior multiplication by the cutoff derivative and its pointwise adjoint, with bounds independent of degree. On a compact manifold the remaining approximation uses local mollification. On a complete ball with the metric below, the cutoff argument first reduces to compact supports. The Hilbert space $\bar\partial$ argument [Xia26a, Theorem 11.1.4] gives, for each closed $(d,q)$-form $g$ in the smooth weighted space, a solution $\bar\partial u = g$ with

$$\|u\|^2 \le (2\pi a)^{-1}\|g\|^2.$$

For a closed form integrable with weight $e^{-\phi}$, the bound is uniform when solving with $e^{-\phi_j}$, since these densities increase to $e^{-\phi}$. A weak subsequence in the first weighted Hilbert space gives a distributional solution. For each fixed $r$, lower semicontinuity of the norm with weight $e^{-\phi_r}$ gives the same bound, first on relatively compact subsets and then on the whole domain. Monotone convergence gives the estimate in the limiting weight. This proves global exactness in all positive degrees for weight $e^{-\phi}$.

For the sheaf calculation, use $(d,q)$-forms in $B$ that are locally square integrable with this weight and whose distributional $\bar\partial$ is locally square integrable with the same weight. These are fine sheaves, by multiplication by smooth partitions of unity. Their kernel in degree zero is $K_M\otimes B\otimes\mathcal{I}(\phi)$: distributional $\bar\partial$-closed coefficients are holomorphic. To see local exactness, take a smaller coordinate ball of radius $R$ on which the given closed form is square integrable. Use the complete Kähler metric $\sigma = \mathrm{dd}^c\rho$ for $\rho = -\log(1-|z|^2/R^2)$, and add $\rho$ to every weight. The function $\rho$ has bounded gradient in $\sigma$, so cutoffs of $\rho/j$ give the graph approximation just used. The metric $\sigma$ dominates a fixed positive multiple of the Euclidean metric. The top holomorphic degree cancels the change in volume, and the remaining norm on the antiholomorphic factors only decreases up to a fixed constant. Thus the given form is integrable for this complete metric with the added weight, since $e^{-\rho}\le 1$. The smooth weights in the local frame are psh, so after adding $\rho$ the curvature is at least $\sigma$. Apply the smooth estimates and the same weak limit argument. On a smaller ball, $\rho$ is bounded and $\sigma$ is comparable with the Euclidean metric, so these norms are equivalent to the original norms with weight $e^{-\phi}$. This gives the desired local solution. The fine resolution computes the sheaf cohomology, and its global exactness proves the claim. □

## B.3 Generation with a fixed twist

**Proposition B.3.1** *Let $X$ be smooth projective of dimension $n$ over a characteristic zero field, with an ample line bundle $L$ and a very ample line bundle $H$. Put $S = X \times \mathbb{A}^1$, and use additive notation for line bundles pulled back to $S$. Suppose $I(mqL)$ is globally generated on $S$, where $m, q \geq 1$ are integers and $I$ is a coherent ideal nonzero on each component. Then*

$$\mathcal{M}(I, 1/q)(mL + K_X + (n+2)H)$$

*is globally generated on $S$. The twist is independent of $q, m$ and $I$.*

***Proof*** Put $\bar{S} = X \times \mathbb{P}^1$ and let $W$ be the pullback of $O_{\mathbb{P}^1}(1)$. Choose finitely many generators of $I(mqL)$ on $S$. Regarded as sections of $mqL$, they are polynomials in the affine coordinate with coefficients in $H^0(X, mqL)$. Homogenize them to a common sufficiently large degree $b \geq 1$ and let $\bar{I}$ be their ideal of coefficients in local frames of $C = mqL + bW$. It restricts to $I$. Set $D = mL + \lceil b/q \rceil W$ and $G = H + W$, a very ample line bundle on $\bar{S}$.

All these data descend to a finitely generated subfield, and generation can be checked after extending it to $\mathbb{C}$. Indeed extension commutes with the formation of the multiplier ideal and with global sections of the coherent sheaf on the projective scheme. Faithful flatness detects surjectivity of its evaluation. Work over $\mathbb{C}$ from now on. In local frames let $s_j$ be the holomorphic coefficients of the homogenized sections and choose smooth positive weights $\phi_C, \phi_G$ on $C, G$, and a smooth semi-positive weight $\phi_W$ on $W$ pulled back from the line. For an integer $l \geq 1$ the local expressions

$$\frac{1}{q} \log\bigl(\textstyle\sum_j |s_j|^2 + \varepsilon e^{\phi_C}\bigr) + \bigl(\lceil b/q \rceil - b/q\bigr)\phi_W + l\phi_G$$

are smooth weights on $D + lG$. The transition law follows by factoring the same squared frame norm from the two terms in the logarithm, and from $qD = C + (q\lceil b/q \rceil - b)W$. The first summand is psh by convexity of logarithmic sums of exponentials of psh functions. The other terms give a uniform positive curvature bound for $l \geq 1$. Decrease $\varepsilon$ to zero. The multiplier ideal of the limit weight is the analytification of $\mathcal{M}(\bar{I}, 1/q)$ by the change of variables calculation above.

With $d = n + 1$ and $1 \leq i \leq d$, apply Lemma B.2.1 with $l = d + 1 - i$. Using algebraic-analytic comparison this gives

$$H^i\bigl(\bar{S}, \mathcal{M}(\bar{I}, 1/q)(K_{\bar{S}} + D + (d + 1 - i)G)\bigr) = 0.$$

For $i > d$ the same group vanishes by dimension. Let

$$\mathcal{F} = \mathcal{M}(\bar{I}, 1/q)(K_{\bar{S}} + D + (d+1)G).$$

The displayed vanishing says that its pushforward to projective space by the embedding given by $G$ is 0-regular. Hence $\mathcal{F}$ is globally generated [Stacks, Tag 08A8].

Restrict to $S$. There $W$ is trivial and $K_{\bar{S}}|_S = K_X$ via the differential of the affine parameter. This is exactly the asserted twist. □

# Appendix C
# Smooth alterations

To alter a projective variety, we project it to a lower-dimensional base with one-dimensional fibres. Marking each component of each fibre forces the stable pointed model to dominate the original family. After altering the base, the remaining singularities are controlled by equations of the form $uv = \prod_i t_i^{a_i}$.

This appendix gives the projective form of the alteration theorem over an algebraically closed field (Theorem C.7.1), which the envelope theorem uses in Section 16.5. We use the Hilbert scheme and the proper stack of stable pointed curves in the two base-change steps, and prove the remaining steps here. Appendix C.1 projects the variety to a family of curves, and Appendix C.2 flattens the family and provides a stable pointed model. Appendix C.3 constructs the marks, and Appendix C.4 uses them to extend the graph from the stable model to the original family. Appendices C.5 and C.6 resolve the nodal total space over a strict normal crossings divisor, and Appendix C.7 assembles these steps. The organization follows de Jong's construction [Jon96, Sections 2–4].

## C.1 Projecting a variety to a family of curves

Fix an algebraically closed field $k$. A projective alteration of an integral projective $k$-scheme $X$ is a projective dominant generically finite morphism from an integral $k$-scheme. Such a morphism is surjective. The source is projective over $k$, and its dimension is $\dim X$ by the equality of the transcendence degrees of the two function fields. The composition of projective alterations is a projective alteration.

**Proposition C.1.1** *Let $X$ be normal, integral and projective over $k$, with $d = \dim X \geq 2$. Let $Z \subsetneq X$ be either empty or the support of an effective Cartier divisor, endowed with its reduced structure. There is a projective birational morphism $X_1 \to X$ and a projective morphism $f_1 : X_1 \to S$ to a normal integral projective scheme $S$ of dimension $d-1$. The geometric fibres are connected and of pure dimension one. The relative smooth locus is dense on each component of each geometric*

*fibre, and the generic fibre is smooth and geometrically integral. The reduced inverse image of $Z$ is finite and generically étale over $S$.*

***Proof*** Embed $X$ in $\mathbb{P}^N_k$ by a very ample system. Choose a flag of linear spaces $L_0 \subset L_1 \subset H$ of dimensions $N-d-1, N-d, N-d+1$, respectively. The space $L_0$ is empty if $N = d$. We use flags with the following properties. The space $L_0$ is disjoint from $X$, and the projection $\pi : X \to \mathbb{P}^d_k$ from $L_0$ is étale over the point $p$ corresponding to $L_1$. This point is outside $\pi(Z)$. Projection from $L_1$ restricts to a finite generically étale map of $Z$ to $\mathbb{P}^{d-1}_k$. Finally, $X \cap H$ is smooth of dimension one.

Here is why these conditions can be achieved simultaneously. A general $L_0$ is disjoint from $X$. Its projection is finite since the pullback of $O(1)$ is ample and a positive-dimensional projective fibre would contain a curve on which that line bundle is trivial. The map is surjective by dimension. We may require $L_0$ to miss the projective tangent space at one fixed smooth point of $X$. Then the differential of the projection at that point has full rank, so the projection is generically étale. For such $L_0$ the finite map is étale over a nonempty open of $\mathbb{P}^d$, and removing $\pi(Z)$ leaves a nonempty open. Taking $p$ there gives the condition on $L_1$. For each component of $Z$, we can impose that a general $L_1$ miss its tangent space at one smooth point of that component. A general $L_1$ also misses all of $Z$ by dimension. Each component of $Z$ has dimension $d-1$, so the argument with the ample pullback and the differential proves finiteness and generic étaleness of projection on $Z$. Finally a general $H$ of codimension $d-1$ misses the singular locus of the normal $X$, which has dimension at most $d-2$. Successive applications of Bertini smoothness on the smooth locus give a smooth curve section [Stacks, Tag 0FD6]. Once the intersection with the singular locus is excluded, failure of transversality is a closed rank condition on the defining linear forms along the universal intersection. Its image on the flag variety is closed by properness. Thus these are nonempty open conditions on the flag variety. For the condition on $p$, the non-étale locus of the universal finite projection has closed image under that finite map. The flag variety is irreducible, and its projections to the indicated Grassmannians are surjective. The nonempty opens therefore have a common $k$-point.

Let $P_1 \to \mathbb{P}^d_k$ be the blowup at $p$. It is a $\mathbb{P}^1$-bundle over the space $\mathbb{P}^{d-1}_k$ of lines through $p$, with each fibre mapping isomorphically to its line in $\mathbb{P}^d_k$. The fibre product $X_1 = X \times_{\mathbb{P}^d} P_1$ is the blowup of $X$ at the finite smooth points over $p$. Indeed near that finite fibre $\pi$ is étale, so the blowup commutes with the flat pullback. Off the fibre both modifications are the identity. It follows that $X_1$ is integral and normal, and smooth near the exceptional fibre. Write $f : X_1 \to \mathbb{P}^{d-1}$ for the induced map.

For a geometric line $L$ through $p$, the fibre of $f$ is the scheme-theoretic inverse image of $L$ in $X$. Finiteness over $L$ bounds the dimensions of its components by one. The $d-1$ linear equations for $L$ give the opposite bound for each component by the height theorem. Thus every component dominates $L$ and contains a point over $p$. The morphism $f$ is smooth near those points, since $X_1 \to P_1$ is étale there and the bundle projection is smooth. This proves the density of the relative smooth locus in all the fibre components. The line corresponding to $H$ has smooth inverse image contained in the smooth locus of $X$, and its fibre in $X_1$ is contained in the smooth locus of $X_1$. At every point of this fibre, the tangent space of the fibre has dimension

one, so the differential from the smooth $X_1$ to the smooth target has rank $d-1$. Thus $f$ is smooth there by the smoothness criterion for a morphism of smooth varieties. Properness of $f$ then gives a nonempty base open over which it is smooth.

Take the Stein factorization $X_1 \xrightarrow{f_1} S \to \mathbb{P}^{d-1}$ of $f$ [Stacks, Tag 03H0]. The finite scheme $S$ over $\mathbb{P}^{d-1}$ is normal and integral. For an affine open of the base its coordinate ring is the ring of functions on the inverse image in $X_1$, an intersection of integrally closed local rings in $k(X_1)$. We claim that the finite map $S \to \mathbb{P}^{d-1}$ is étale. For any closed point $s$ of $S$, choose a point in the relative smooth locus of $f$ lying above $s$. Such a point exists because the underlying set of the fibre of $f_1$ is a union of components of the fibre of $f$. Indeed over the image of $s$ the finite scheme $S$ has a discrete set of points, so each of its inverse images is open and closed in that fibre. By shrinking the smooth locus about a chosen point, we can exclude the inverse images of the other points of that finite scheme. Use an étale-local section of this smooth open over $\mathbb{P}^{d-1}$, by [Stacks, Tag 055U]. Its composition with $X_1 \to S$ gives a section of the finite map after étale base change, through a lift of $s$. On an integral étale neighborhood of the base point, the finite scheme is still normal since it is étale over $S$. The section is a closed immersion whose image has the full base dimension. It is a whole irreducible component of the normal finite scheme, with the reduced structure. That component is open and closed, so the finite map is étale there. Étale descent shows it is étale at $s$. The étale locus contains all closed points, so it is all of $S$.

In particular $f_1$ has the same relative smooth locus as $f$, as can be seen after étale base change separating the finite étale target. Its geometric fibres are the connected pieces of the fibres of $f$ given by the factorization, hence have pure dimension one and the required density of smooth points. The generic fibre is smooth and geometrically connected, so it is geometrically integral. The inverse image of $Z$ is unchanged by the blowup. Its map to $\mathbb{P}^{d-1}$ is finite and generically étale by the choice of $L_1$. The induced map to $S$ is proper and quasi-finite, hence finite. The function fields of the components of $Z$ are separable over $k(\mathbb{P}^{d-1})$, and thus over the intermediate field $k(S)$. This gives generic étaleness over $S$. □

## C.2 Flat closures and stable pointed models

We keep the algebraically closed field $k$. The two constructions in this section act on the base of a family of curves. First a modification makes the closure of the generic fibre flat. Then a projective alteration supplies a stable pointed model.

### C.2.1 Flat closure after a modification

Let $V \to S$ be projective with $S$ integral and Noetherian, and let $T \to S$ be a dominant morphism with $T$ integral. The schematic closure of the generic fibre in $V \times_S T$ is

obtained by dividing out the torsion over $T$. On an affine chart over a domain $R$ this says that the ideal of the closure is the kernel of localization at $R \setminus \{0\}$. If a flat closed subscheme of the pullback agrees with the generic fibre, it is this closure. Indeed its functions are torsion-free over $R$, so localization is injective on them. Conversely, if the closure is flat with geometrically integral generic fibre, its pullback under any further dominant integral base change is integral and is again the closure of the new generic fibre. For integrality, each nonempty affine chart injects into its generic chart by flatness. The generic fibre is integral and is dense in the total space, which gives the assertion.

**Lemma C.2.1** *Let $V \to S$ be projective with $S$ integral and Noetherian. There is a projective birational morphism $S' \to S$, with $S'$ integral, such that the schematic closure of the generic fibre in $V \times_S S'$ is flat over $S'$. If $S$ is a variety over $k$, $S'$ can be taken normal.*

***Proof*** Choose a relatively very ample line bundle and a nonempty open $U \subseteq S$ where $V$ is flat, by generic flatness. Over this integral $U$ the fibre Hilbert polynomial is fixed. We use representability of the Hilbert scheme of a projective scheme with fixed Hilbert polynomial, in the projective form of [Gro61, Exposé 221, Théorèmes 3.1–3.2]. The family $V_U$ gives a section over $U$ of this projective Hilbert scheme for $V/S$. Take the reduced closure of the section in the Hilbert scheme and call it $S'$. It is integral, is projective over $S$, and agrees with the section over $U$. The universal closed subscheme pulls back to a flat family on $S'$ with the required generic fibre. The torsion argument just given identifies it with the schematic closure. If $S$ is a variety, normalize $S'$, which is finite over $S'$. Flatness persists under pullback and identifies the new flat family with the generic closure by the same argument. □

We record the effect on the fibres in the situation of Proposition C.1.1. For a dominant integral base change, the relative smooth locus of the pullback of the original family lies in the schematic closure. Its open coordinate rings are flat over the new base and thus inject into their generic fibres. On each geometric fibre this smooth open is dense in every component, by the hypothesis and base change. Since the closure is closed in the pullback, it contains the underlying set of every geometric fibre, including all the points in the marked locus. It still has geometrically connected fibres of pure dimension one, with its relative smooth locus dense on each component. For a flat closure with geometrically integral generic fibre every pulled-back effective Cartier divisor that misses the generic point of the total space is still Cartier. Locally its equation is nonzero in the integral total space. A finite marked locus whose generic fibre was étale thus remains the support of a Cartier divisor finite over the new base, with generically étale reduced structure.

### C.2.2 A stable family over a projective alteration

Let $g, n \geq 0$ be integers with $2g-2+n > 0$. A *stable $n$-pointed curve of genus $g$* over a scheme $T$ is a flat proper morphism $C \to T$ with $n$ sections satisfying the following conditions [Knu83, Definition 1.1]. Every geometric fibre is a reduced connected curve of arithmetic genus $g$ with at most ordinary double points. On each geometric fibre the values of the sections are distinct and lie in the smooth locus. For every nonsingular rational component of a geometric fibre, the number of points where it meets the rest of the fibre plus the number of marked points on it is at least three.

**Lemma C.2.2** *Let $f : X \to S$ be projective and flat with $X, S$ integral and projective over $k$, and let its generic fibre be a smooth geometrically integral curve. Suppose there are $n \geq 3$ sections that are distinct on the generic fibre. After a projective alteration $S' \to S$ there is a projective stable $n$-pointed curve $C$ over $S'$ agreeing with $X \times_S S'$ and its sections over a dense open of $S'$.*

***Proof*** Shrink to an open $U \subseteq S$ on which the fibres are smooth and geometrically connected and the sections are disjoint. Such an open exists. By smoothness of the generic fibre and properness, the fibres are smooth over a nonempty open, and the sections are disjoint off their closed equality loci. Over the open of smooth fibres the number of geometric connected components of the fibres is locally constant [Stacks, Tag 0E0N], since the family is proper, flat and of finite presentation with geometrically reduced fibres. This open is irreducible, so the number equals its value one at the generic fibre. Here the smooth $n$-pointed curve of genus $g$ is stable since $2g-2+n > 0$. We use Knudsen's proper algebraic stack of stable pointed curves [Knu83, Theorem 2.7]. Its diagonal is finite. Separatedness gives properness of the diagonal, and a stable pointed curve has unramified automorphisms since there are no infinitesimal automorphisms. To see the latter assertion on a geometric fibre, a vector field on the normalization vanishes at the marked points and both preimages of each node. Every normalized component has negative degree tangent bundle after those zeros are imposed: for nonsingular rational components this is the three-point condition, and for the other components of genus at most one it follows from connectedness and $2g - 2 + n > 0$. Hence all such vector fields vanish. The diagonal is therefore unramified and proper, hence finite.

The stack has a proper surjective cover by a projective scheme $M$ over $k$. Indeed Chow's lemma for algebraic stacks [Stacks, Tag 0CQ8] first gives a proper surjective cover by an algebraic space. That space is proper over $k$ by composition. The algebraic-space version of Chow's lemma [Stacks, Tag 089J] then gives a proper surjective cover by a quasi-projective scheme. Properness makes this scheme projective over $k$.

Use the classifying map from $U$ to the stack. The fibre product $U \times_{\overline{M}_{g,n}} M$ is a scheme, since it is finite over $U \times_k M$ by the finite diagonal. It is surjective over $U$. Take a closed point of its generic fibre over $U$, giving a finite extension $L/k(S)$. Normalize $S$ in $L$, a finite map of projective schemes. The resulting integral scheme has a rational map to $M$ from the chosen generic point of the cover. Take the reduced

closure of its graph in the product with $M$, and call it $S'$. The morphism $S' \to S$ is projective and dominant and has function field $L$, so it is a projective alteration. The map to the moduli stack supplies a pointed stable curve over $S'$. The restriction of its relative log dualizing bundle to each fibre has a very ample third power [Knu83, Corollary 1.9]. By [Stacks, Tag 0D2N] at each point of $S'$ and the locality of relative ampleness on the base [Stacks, Tag 01VJ], this bundle is ample relative to $S'$. Since $S'$ is projective over $k$, the family is therefore projective over $S'$ [Stacks, Tag 0B45]. By the definition of the fibre product the generic families and their sections agree, and their isomorphism spreads out over a dense open of $S'$. □

## C.3 Finite multisections meeting all fibre components

This section constructs marks on a projective family of curves. They supply the three points on each fibre component used by Proposition C.4.1.

**Proposition C.3.1** *Let $k$ be algebraically closed, and let $f : X \to S$ be a projective surjective morphism of integral projective $k$-schemes. Suppose its geometric fibres are of pure dimension one and the relative smooth locus is dense on each fibre component. There is a finite union of effective Cartier divisors on $X$ whose reduced union $Z$ is finite and generically étale over $S$. Every irreducible component of a geometric fibre meets $Z$ in at least three distinct points in the relative smooth locus.*

***Proof*** Fix a very ample line bundle $L$ on $X$. Choose an integer $m > \max(\dim S, 2)$, and use the embedding defined by $L^m$, with dual projective space $P$ parametrizing hyperplanes. Let $N = \dim P$. For a geometric fibre component $B$, take two sections of $L$ whose quotient is nonconstant on $B$. They exist because the complete system of $L$ embeds $X$. The $m + 1$ monomials of degree $m$ in the two sections restrict to linearly independent sections on $B$ over the algebraically closed field of the geometric fibre. After dividing by the $m$-th power of the second section, this follows from the transcendence of a nonconstant rational function over that field. Thus the hyperplanes containing $B$ belong to a linear subspace of $P$ of codimension at least $m + 1$.

The locus in $P \times S$ where a hyperplane meets the corresponding fibre in dimension one is closed, by upper semicontinuity of fibre dimension for the projective incidence morphism. Its geometric fibre over a point of $S$ is the union of the linear subspaces for the finitely many components $B$. It therefore has dimension at most $\dim S + N - m - 1 < N$. Its image in $P$ is closed since $S$ is proper, and is a proper closed subset by the dimension bound. Every hyperplane outside this subset cuts a divisor finite over $S$. The intersection with any component has dimension zero and is nonempty by ampleness.

Now fix a closed point $s \in S$. The complement of the relative smooth locus in $X_s$ has finite support because the smooth locus is dense on every component of this projective curve. Require also that the chosen hyperplane avoid this finite set. At every smooth point of the fibre, hyperplanes that contain its tangent line form a codimension-two linear locus in $P$. Their incidence has dimension at most $N - 1$,

since the smooth points have dimension one, so a general hyperplane has no tangencies there. Choose a hyperplane satisfying all these conditions. The resulting effective Cartier divisor $H_s$ is finite over $S$. At each of its points above $s$ the cutting section has nonzero relative differential, so the relative smooth morphism cut by that section is étale there. By closedness of a finite morphism we may shrink to an open neighborhood $U_s$ of $s$ over which all of $H_s$ is étale and lies in the relative smooth locus of $X$. For a geometric point of $U_s$, all intersections of $H_s$ with a fibre component $B$ are consequently distinct transverse points and lie in that smooth locus. Their number is $\deg_B(L^m) = m \deg_B L$, at least three.

The union of the $U_s$ contains every closed point of $S$ and hence is all of $S$, since any nonempty closed subset has a closed point. Choose a finite subcover and take the reduced union of the associated divisors. It is finite over $S$. Each irreducible component of any chosen divisor has dimension $\dim S$, so it dominates $S$ by finiteness. Each divisor is étale on a nonempty base open. Its generic reduced algebra is therefore a finite product of separable fields. Taking the reduced union does not alter this property of the generic algebra. Thus $Z$ is generically étale. The finite subcover gives the asserted number of smooth marked points on every fibre component. □

**Corollary C.3.1** *Assume the hypotheses of Proposition C.3.1 and in addition that $f$ is flat with smooth geometrically integral generic fibre. Let $Z$ be a reduced union of effective Cartier supports, finite and generically étale over $S$. After a finite surjective base extension $S' \to S$ with $S'$ normal and integral, the reduced inverse image of $Z$ is a union of sections of $X \times_S S' \to S'$.*

***Proof*** Choose a finite normal separable extension of $k(S)$ splitting the generic algebra of $Z$. Take $S'$ to be the normalization of $S$ in this field. It is finite over $S$. The closure of each generic point of the pulled-back marked locus is finite and birational to $S'$, hence isomorphic to $S'$. We check that these closures exhaust the marked locus.

The total space $X \times_S S'$ is integral by flatness over the integral $S'$ and integrality of its generic fibre. For each effective Cartier divisor whose support occurs in $Z$, its local equations remain nonzero after this dominant base change. Its pullback is therefore an effective Cartier divisor on the new total space. Every component has codimension one, and its finite image in $S'$ has dimension $\dim S'$. Thus all components dominate $S'$. The union of the closures of their generic points is the whole underlying marked locus, and reduction gives the claimed union of sections. □

## C.4 Extending the graph of a pointed curve

Let $k$ be algebraically closed and let $S$ be a normal integral $k$-scheme of finite type. A *nodal family of curves* is a flat proper morphism of finite presentation with connected one-dimensional nodal geometric fibres. Such a morphism is at-worst-nodal of relative dimension one [Stacks, Tags 0C59 and 0C5A]. In the construction of an alteration, the stable model over $S$ must map to the original family. Proposition C.4.1

uses three marks on each fibre component to show that no component of a fibre of the graph closure contracts to a point of the stable model, so that the rational map extends.

**Lemma C.4.1** *Let $R$ be a normal domain. If $a \in R$ is nonzero, then the quotient $R[u,v]/(uv-a)$ is normal.*

***Proof*** Embed this ring in $R[u,u^{-1}]$ by $v = a/u$. To see injectivity, use the free $R$-module basis of the node algebra given by $1, u, u^2, \ldots, v, v^2, \ldots$. Localizing at $u$ and at $v$ gives the normal rings $R[u,u^{-1}]$ and $R[v,v^{-1}]$, respectively. Inside $\mathrm{Frac}(R)(u)$ their intersection is precisely the given algebra. Indeed for a Laurent polynomial $\sum b_i u^i$ in $R[u,u^{-1}]$, expression in the $v$ coordinate amounts to $u = a/v$. The new coefficients belong to $R$ exactly when $b_{-j} \in a^j R$ for every $j > 0$, which is the condition for belonging to $R[u,v]/(uv-a)$. An intersection of integrally closed domains in the same fraction field is integrally closed. □

**Proposition C.4.1** *Let $f : X \to S$ and $c : C \to S$ be projective morphisms of integral schemes with geometrically integral smooth generic fibres of dimension one. Suppose all geometric fibres of $X \to S$ are of pure dimension one, and the relative smooth locus is dense on each component of each of them. Let $C \to S$ be a nodal family with disjoint sections $\tau_1, \ldots, \tau_n$ in its relative smooth locus. Let $\sigma_1, \ldots, \sigma_n$ be sections of $X \to S$ such that each component of each geometric fibre contains at least three distinct $\sigma_i$-values in the relative smooth locus.*

*Suppose $C$ and $X$ are isomorphic over a dense open of $S$, respecting all the sections, and let $G$ be the reduced closure of the graph in $C \times_S X$. If $G \to S$ is flat, the graph projection $G \to C$ is an isomorphism. In particular the original rational map extends to a projective birational morphism $C \to X$.*

***Proof*** The scheme $C$ is normal. At smooth points of $c$ this follows from normality of $S$. At the nodes use the étale local form $uv = a$ over a normal étale neighborhood of the base [Stacks, Tag 0CBY]. If $a = 0$ on the local base domain, the node locus in the chart has an open image in the corresponding node locus $u = v = 0$ over that base. This would give a node in the generic fibre. Thus $a \neq 0$. The chart is normal by Lemma C.4.1, and normality is étale local.

Write $q_C : G \to C$ and $q_X : G \to X$ for the projections. Both are projective and birational. Each geometric fibre of $G \to S$ has dimension at most one. Indeed projectivity and flatness give a constant fibre Hilbert polynomial over the integral base. Its degree is one by the generic graph. Fix a geometric point $\bar{s}$ of $S$, working over its algebraically closed field. Each irreducible component of $X_{\bar{s}}$ is dominated by exactly one component of $G_{\bar{s}}$. To see this, note first that the positive-dimensional fibres of $q_X$ on $G_{\bar{s}}$ lie above a finite set. Every such fibre contains a one-dimensional component of $G_{\bar{s}}$ contracted to a point. Choose on a prescribed component of $X_{\bar{s}}$ a point in the relative smooth locus and outside this finite set and the other components. Since the graph map has a finite fibre at that point, projectivity gives an open neighborhood of its image in $X$ over which the map is finite. The target is normal near that point because $X \to S$ is smooth there. A finite birational map to that normal open is an

isomorphism. Its scalar extension at $\bar{s}$ is thus an isomorphism over a nonempty open of the prescribed component. Surjectivity of the proper birational graph map gives a component dominating it, and the isomorphism over the open gives uniqueness. The same argument applies to each component of $C_{\bar{s}}$, using normality of $C$.

Let $B$ be the component of $G_{\bar{s}}$ that dominates a prescribed component $X_0$ of $X_{\bar{s}}$. If its image in $C_{\bar{s}}$ is a point, call it $b$. Choose three indices with distinct values $x_i = \sigma_i(\bar{s})$ in $X_0$ in the relative smooth locus. The corresponding fibres of $q_X$ in $G_{\bar{s}}$ are connected. In fact over the relative smooth locus the finite factor in the Stein factorization of $q_X$ is finite birational to a normal target and hence an isomorphism. The fibres there are geometrically connected by [Stacks, Tag 03H0]. Each of the three fibres meets $B$, by properness and surjectivity of $B \to X_0$. Each contains the point $(\tau_i(\bar{s}), x_i)$, because the section pair lies in the graph on the dense open of $S$ and $G$ is closed.

Consequently their images in $C_{\bar{s}}$ are connected closed subsets each containing $b$ and $\tau_i(\bar{s})$. Whenever these two points differ, an irreducible curve component of that image contains $b$. Indeed an isolated point component in a finite union of curve components and points would be an open and closed subset of the image. The images of distinct fibres cannot share a curve component, since each curve component of $C_{\bar{s}}$ is dominated by only one component of $G_{\bar{s}}$. If $b$ differs from all three marked points, at least three components go through $b$, which is impossible on a nodal curve. If $b$ equals one of the marked points, the other two images give two components through a smooth point, also impossible. Thus $B \to C_{\bar{s}}$ is nonconstant.

If $q_C$ had a positive-dimensional fibre, a component of some $G_{\bar{s}}$ would contract to a point of $C_{\bar{s}}$. That component cannot also contract to a point of $X_{\bar{s}}$, since $G$ is a closed subscheme of the product over $S$. It would therefore dominate a component of $X_{\bar{s}}$, contrary to what was just proved. Thus $q_C$ is quasi-finite and proper, hence finite. Since it is birational and $C$ is normal, it is an isomorphism. □

## C.5 Resolving a split nodal family over a crossing divisor

In this section let $k$ be algebraically closed, let $S$ be a smooth integral $k$-scheme of finite type, and let $D = \sum_i D_i$ be a reduced strict normal crossings divisor on $S$. Thus the components $D_i$ are smooth and their local equations through any point are part of a regular system of parameters. Recall from Appendix C.4 that a nodal family of curves is a flat proper morphism of finite presentation with connected one-dimensional nodal geometric fibres. It is split if the irreducible components of every fibre are smooth and geometrically irreducible over the residue field and the nodes are rational over that field. We only consider families with smooth geometrically integral generic fibre.

**Lemma C.5.1** *Let $f : X \to S$ be a nodal family of this kind, smooth over $S \setminus D$. Étale locally at a node over $s \in S$ its equation has the form*

$$uv = \prod_{i=1}^{r} t_i^{a_i}, \qquad a_i \geq 0,$$

*where $t_1, \ldots, t_r$ are local equations for the components of $D$ through $s$. The total space is regular at this node if and only if $\sum_i a_i = 1$.*

***Proof*** The étale local form of a nodal morphism [Stacks, Tag 0CBY] gives $uv = h$ over an étale neighborhood of $s$. The node scheme in the model is cut out by $u, v$, and its image on the base is $V(h)$. In the chosen neighborhood of the node this image lies in $D$. More precisely, the étale chart restricted to the node scheme is étale over $V(h)$, with image containing the point in question, so we may discard from the base all components of $V(h)$ not meeting its germ. Thus in the regular local ring of the étale base, $V(h)$ is contained in $V(\prod_i t_i)$. Smoothness at the generic fibre gives $h \neq 0$. The local base ring is factorial by [Stacks, Tag 0AG0], so the containment of divisors factors $h$ as a unit times $\prod_i t_i^{a_i}$. Absorb the unit in $v$ and shrink the chart so that this factorization holds throughout it. The cotangent space of the hypersurface at $u = v = 0$ gives the regularity criterion. The equation has nonzero linear part exactly when one $a_i = 1$ and all others are zero. □

**Lemma C.5.2** *Let $E$ be a reduced irreducible component of $X \times_S D_i$ for a split family. The blowup of $X$ along $E$ is again a split nodal family, smooth off $D$, and is an isomorphism at every regular point of $X$. At any singular point $x$ of $X$ it has at most one singular point over $x$. If $E$ contains $x$ and its local ideal is one of the two branch ideals at $x$, the sum of the exponents in the local node equation decreases by one at that successor, when a successor exists. At all other singular points the blowup is an isomorphism.*

***Proof*** The components of $X \times_S D_i$ all dominate $D_i$. Indeed it is flat over the integral scheme $D_i$, so a minimal prime lies over the generic point by going down for flat morphisms. The fibres have pure dimension one, and $E$ has codimension one in $X$. At any regular point the ideal of $E$ is invertible by factoriality of the regular local ring, hence its blowup is an isomorphism.

At a node $x$ as in Lemma C.5.1, localize on an étale chart on which the components of $D$ through the image of $x$ are given by the $t_j$. If the chart has $a_i = 0$, then the equation modulo $t_i$ is $uv = \prod_{j \neq i} t_j^{a_j}$ with nonzero right-hand side. Over the regular local base $t_i = 0$, it is a normal domain by Lemma C.4.1. Normality persists étale locally, so $X \times_S D_i$ has just one local branch, cut out by $t_i$. If $a_i > 0$, the two branches on that divisor have ideals $(u, t_i)$ and $(v, t_i)$. They belong to distinct global components of $X \times_S D_i$. In fact they specialize from the two branches of the node over the generic point of $D_i$, by the étale local model with node scheme $u = v = t_i = 0$ along $D_i$. The étale node scheme maps openly to $D_i$ there, so the generic point is in its image after shrinking the base. The two branches over that generic point lie on different irreducible fibre components, because every such component is smooth and geometrically irreducible by splitness. These components are the generic fibres of the global ones. Thus the ideal of $E$ is locally invertible or one of the two branch ideals.

We compute when the ideal is $(u, t_i)$. Interchanging $u, v$ treats the other case. Put $M = \prod_j t_j^{a_j}$ in the affine model $uv = M$. The blowup has charts

$$\begin{aligned} t_i &= uT, \quad v = T(M/t_i), \text{ on the } u\text{-chart},\\ u &= t_i U, \quad Uv = M/t_i, \quad \text{on the } t_i\text{-chart}. \end{aligned}$$

Each displayed chart ring is a domain. It embeds in the localization of the original ring by its chart generator and is generated by the original ring and the displayed ratio, giving the actual chart ring of the blowup. The calculation commutes with the flat étale base changes. The first chart has regular total space since its only remaining equation is $uT = t_i$. In the second chart the smoothing monomial has one fewer factor. A singular point of the second chart over the old node has $U = v = 0$, so is unique if it exists. For the split node we may take the étale local model with the original residue field. Indeed the node is rational and its two branches lie on distinct smooth fibre components, so it is a split node in the sense of [Stacks, Tag 0CBV]. For such a node the first proof of [Stacks, Tag 0CBY] uses elementary étale neighborhoods [Stacks, Tag 0GDX]. Write $g = M/t_i$. When $g$ is a unit, $t_i = uv/g$ shows that $(u, t_i) = (u)$, so the blowup is an isomorphism. When $g$ vanishes in the residue field, the exceptional fibre is given by the affine coordinates $T$ and $U$, glued by $U = 1/T$. It is a projective line over the field of the fibre, since the old node is rational by splitness. The two axes of the old fibre meet this line at its two rational endpoints. Thus the new fibres still have smooth geometrically irreducible components and rational nodes. To check flatness in the standard node charts, note that for any base coefficient $g$ the algebra $R[x, y]/(xy - g)$ is free as an $R$-module on $1, x, x^2, \ldots, y, y^2, \ldots$ by monomial reduction. The blowup is proper, and its fibres as a map to the old curves remain connected since the modification is an isomorphism or inserts that rational component at each modified node. This proves the lemma. □

**Proposition C.5.1** *Let $f : X \to S$ satisfy the hypotheses of Lemma C.5.1 and be split. There is a projective modification $X' \to X$, an isomorphism over the regular locus of $X$, such that $X'$ is smooth over $k$, $X' \to S$ is again a split nodal family, and the reduced inverse image of $D$ on $X'$ is a strict normal crossings divisor.*

***Proof*** At each node use the sum of the exponents of its smoothing monomial as an integer invariant. It is intrinsic. The relative nonsmooth subscheme in a node chart is the first Fitting subscheme of $\Omega^1_{X/S}$, cut out by $u, v$. On an étale base on which the branch of this subscheme through the node is a closed immersion, its defining ideal $(h)$ is the smoothing ideal. Changing the chart preserves the relative nonsmooth subscheme. On common étale refinements its base ideal therefore remains the smoothing ideal, so the order of that ideal in the regular local base ring is independent of the chart. This order is the sum in question. In any chart where $h$ is a fixed monomial times a unit, this invariant is the sum of the $a_i$'s for the components through the base point. It is therefore upper semicontinuous. The node charts cover the nonsmooth subscheme, and it is quasi-compact. Thus the invariant has a finite upper bound. For each integer $q \geq 2$ the nodes where the invariant is at least $q$ form a closed subset of

$X$ because the relative nonsmooth locus is closed and the inequality defines a closed set in each étale node chart.

Suppose the largest invariant of a singular point is $q \geq 2$. Each blowup in Lemma C.5.2 preserves this bound. Its closed locus with invariant $q$ maps properly to the old one and has at most one point over each old point. If the current locus is nonempty, pick a point of it, choose a factor $t_i$ of its smoothing monomial, and blow up one of the two global components of $X \times_S D_i$ through that point. The lemma removes the chosen point from the image of the next closed locus of invariant $q$. To see that finitely many steps suffice, take the images of all the successive loci in the first locus of invariant $q$. Each is closed by properness. Each map of loci has fibres of cardinality at most one, so removal of the chosen point removes its image in the first locus as well. The images form a strictly descending sequence of closed subsets as long as they are nonempty. Noetherianity ends that sequence. Now reduce $q$ and repeat until no singular nodes remain. Points off the node locus are smooth over the smooth base, and hence are regular from the beginning.

Finally, near a remaining node the smoothing monomial has precisely one factor, $uv = t_i$. The reduced inverse image of $D$ is then locally given by $uv \prod_{j \neq i} t_j = 0$, with the factors part of a regular system of parameters. The two branches $u, v$ are on different global components by splitness as in Lemma C.5.2. At a smooth point of the family, the pullbacks of the boundary parameters themselves are part of a regular system of parameters. In particular all global boundary components and their intersections are smooth of the required codimensions, which can be checked in the étale charts. This is the strict normal crossings condition. □

## C.6 Resolving a nodal family over a crossing divisor

Continue with a smooth integral base $S$ and a strict normal crossings divisor $D$ over an algebraically closed field $k$. Let $f : X \to S$ be a nodal family as in Lemma C.5.1, smooth over $S \setminus D$. We allow the fibre components to have self-intersections and the branches of a node to be interchanged over residue field extensions. We also allow finitely many disjoint sections in the relative smooth locus. Write $B$ for the reduced union of their images and the inverse image of $D$.

### C.6.1 Removing repeated smoothing factors

Write $N$ for the relative nonsmooth subscheme, defined by the first Fitting ideal of $\Omega^1_{X/S}$. In the local node equation $uv = h$, its ideal is $(u, v)$, and over the base of the chart it is étale over $V(h)$. In particular $N$ is finite over $S$, since it is proper and its fibres are finite.

For a component $D_i$, consider a reduced irreducible component $E$ of $N \times_S D_i$ dominating $D_i$. Near a point of $E$, the smoothing parameter is divisible by the corre-

sponding local equation $t_i$. Indeed, if it were not, the node chart over $D_i$ would have image in the proper intersection with the other components of $D$. The étale chart has open image in the relative node locus. On $E$ that image contains its generic point, which would contradict dominance over $D_i$. The étale local model over $D_i$ therefore has a single branch through the point, with ideal $(u, v, t_i)$. Its support is smooth, being étale over the smooth $D_i$. Consequently two components of $N \times_S D_i$ dominating $D_i$ cannot share a point. Each $E$ is smooth and has the displayed reduced ideal in the chart. Conversely, a factor $t_i$ at a node gives one such component $E$ through it. The branch $(u, v, t_i)$ maps étale to $D_i$ on the base chart, so its image contains the generic point of the base divisor. This argument uses the node as a centre without choosing one of the two branches of the curve.

The integer $a_i$ in the smoothing monomial is constant along $E$. It is the multiplicity of $D_i$ in the local equation of the relative nonsmooth subscheme over the étale base at this node. To see the independence of the chart, pass to the strict henselization of the base at the image of the point. The factor of the finite unramified node scheme corresponding to the chosen geometric node is a closed subscheme of this base, defined by the smoothing ideal $(h)$ in the node chart. Thus the orders of $h$ along the pullbacks of the $D_i$'s are intrinsic. In a fixed étale chart the factorization of $h$ by the $t_j$'s holds on a neighborhood and gives the same multiplicity at all points of this branch over $D_i$. This proves local constancy on $E$, hence constancy by irreducibility. The same charts show that

$$q(x) = \sum_{i:\, t_i(f(x))=0} a_i$$

is upper semicontinuous and bounded on the node locus. The locus of nodes with some repeated smoothing factor is closed there. In a chart it is the union of the conditions $t_i = 0$ for the indices with $a_i \geq 2$.

Suppose $a_i \geq 2$ along a component $E$ as above. Blow up its ideal. Étale locally it is the blowup of $(u, v, t_i)$ in the hypersurface

$$uv = t_i^{a_i} P, \qquad P = \prod_{j \neq i} t_j^{a_j}.$$

On the $u$-chart write $v = uV$ and $t_i = uT$. The equations after cancelling $u^2$ are

$$t_i = uT, \qquad V = u^{a_i - 2} T^{a_i} P.$$

The $v$-chart is symmetric. On the $t_i$-chart the equations are

$$u = t_i U, \qquad v = t_i V, \qquad UV = t_i^{a_i - 2} P.$$

These are the chart rings since they embed in the localizations by the chart generators and are generated there by the indicated ratios. For example the first chart, after eliminating $V$, has equation $uT = t_i$, a domain by Lemma C.4.1. On the last chart use

the same argument with its nonzero right-hand side. These computations commute with the étale coordinate changes.

The first two charts have regular total space, because $t_i$ is part of a regular system of parameters of $S$. They are flat with at-worst-nodal fibres over $S$, as is the last chart by its node equation. Indeed a ring of the form $R[x,y]/(xy-g)$ is free over $R$ on 1 and the positive powers of $x$ and $y$. The blowup is an isomorphism off $E$. Over the old node the exceptional fibre on a geometric chart is the plane conic with homogeneous equation $UV = \gamma T^2$. Here $\gamma$ is the image of $P$ if $a_i = 2$ and is zero if $a_i > 2$. This follows by setting the old coordinates and $t_i$ to zero in the chart equations. The conic is connected also when it is a union of two lines. Thus the proper surjective map on geometric fibres has connected point fibres, so the new fibres are connected. The resulting morphism is again a nodal family, smooth off $D$. The sections are unchanged since their images miss the centre.

At most one singular point of the total space can lie over an old point of $E$, and only on the $t_i$-chart. Its $q$ has decreased by two. All other points are unchanged. To terminate, take the largest $q$ among nodes with a repeated factor and blow up a component $E$ meeting that locus. The closed locus of nodes with that value of $q$ and a repeated factor has proper image in the old one, missing the selected point. In later blowups the maps of these maximum loci have at most one point over each old point. Their images in the first maximum locus therefore strictly decrease until empty. This takes finitely many steps by Noetherianity, as in the termination argument of Proposition C.5.1. Repeat with the smaller largest value of $q$. No new repeated factor can be created by the chart formula. At the end all smoothing monomials are squarefree.

### C.6.2 The squarefree case

Every singular point now has, étale locally, a model

$$uv = t_1 \cdots t_r, \qquad r \geq 2,$$

and the reduced boundary has equation $t_1 \cdots t_\ell = 0$ with $\ell \geq r$. The boundary equations $t_1, \ldots, t_\ell$ are part of a regular parameter system on the smooth base. For this calculation one may work at closed points. Over $k$ these parameters can be extended to étale coordinates on the base. At regular points the boundary has normal crossings étale locally. At a remaining relative node of the regular total space one has $uv = t_i$, so the two branches of $t_i = 0$ are coordinate hyperplanes. At a relative smooth point the base boundary coordinates pull back to part of a regular system of parameters, and a marked section contributes at most one more independent parameter.

The reduced singular locus of the total space in a squarefree node chart has branches with ideals $(u, v, t_i, t_j)$ for $1 \leq i < j \leq r$. Each such branch dominates a component of $D_i \cap D_j$. There is only one local branch over a specified pair $(i,j)$, by the étale node chart. Moreover distinct pairs have distinct codimension-two images at their generic points because $D$ has strict normal crossings. It follows that a

global irreducible component of the singular locus has only one branch in the chart at any point on it. These branches are smooth, since they are étale locally cut out by coordinates. Thus every irreducible component of the singular locus is smooth. We will preserve these local properties while improving the total space.

Blow up one reduced irreducible component $E$ of the singular locus. At a point of $E$ relabel the local coordinates so that its ideal is $(u, v, t_1, t_2)$. On the $u$-chart use the coordinates $v = uV$, $t_1 = uA$, $t_2 = uC$. Cancelling $u^2$ gives

$$V = ACt_3 \cdots t_r .$$

This chart is regular. Its reduced boundary is the union of the coordinate hyperplanes of $uACt_3 \cdots t_\ell = 0$. The $v$-chart is symmetric. On the $t_1$-chart write $u = t_1U$, $v = t_1V$, $t_2 = t_1C$. The equation and reduced boundary become

$$UV = Ct_3 \cdots t_r, \qquad t_1Ct_3 \cdots t_\ell = 0.$$

In the latter expression $t_1$ is a boundary coordinate independent of the smoothing monomial. The $t_2$-chart has the same form with the indices reversed. These are the chart rings by the ratio calculation in the localizations, since the transformed presentations are domains. They hold on the coordinate models and after the étale base changes. In particular all singular charts again have a squarefree equation in part of a list of independent boundary coordinates.

The remaining singular branches in the $t_1$-chart are the pairs among $C, t_3, \ldots, t_r$, with $U = V = 0$. Each branch meets the open $t_1 \neq 0$. A pair $C, t_j$ there is the old pair $t_2, t_j$ and a pair $t_j, t_l$ is the old pair with those indices. Thus each new branch is the strict transform of an old one, and distinct new branches come from distinct old local branches. The $t_2$-chart accounts for the pairs with $t_1$ in the same way. There are no singular branches in the other charts. Each old global component apart from $E$ survives by strict transform since its generic point was outside the centre. Its transform has only one branch in each chart, which is smooth by the new coordinate description. Therefore the number of irreducible components of the singular locus drops by one and all the inductive local properties persist. After finitely many blowups the total space is smooth with a reduced boundary that has normal crossings étale locally. The strictification in the next lemma completes the construction.

### C.6.3 Making the crossings strict

**Lemma C.6.1** *Let $W$ be a smooth variety over $k$ and let $B$ be a reduced divisor with normal crossings étale locally. There is a sequence of blowups in smooth centres such that the reduced inverse image of $B$ is a strict normal crossings divisor.*

***Proof*** For $j \geq 2$ let $B_{\geq j}$ be the reduced closed locus where at least $j$ local branches meet. This locus is defined locally in the étale topology as the union of the coordinate intersections of size $j$. Its support is where the order of a local equation of $B$ is at

least $j$. Étale local extension preserves that order, since it extends the maximal ideal and is faithfully flat. It also preserves reducedness, so the reduced strata in the charts give these global centres. Blow up the nonempty strict transforms in decreasing order of $j$.

Check the construction on an étale chart at a point with $r$ local branches $x_1, \dots, x_r$. The operations with $j > r$ have empty centre on this chart after shrinking it. The first centre is given by all these coordinates. In its $x_i$-chart the coordinates are $y_i = x_i$ and $y_j = x_j/x_i$ for $j \neq i$; the reduced inverse image is $\prod_j y_j = 0$. For a proper subset $I$ of $\{1, \dots, r\}$ the strict transform of $x_j = 0$ for $j \in I$ is absent from this chart if $i \in I$, since every point there with $x_i = 0$ maps to the centre. If $i \notin I$, its strict transform is $y_j = 0$ for $j \in I$. Thus the remaining process on this chart is the same procedure on the $r - 1$ coordinates with $j \neq i$, with the free exceptional coordinate $y_i$. Induction proves that at each step the centre is smooth and that the final reduced inverse image has normal crossings.

Give a boundary branch the label $j$ if it was first created by the blowup of the size-$j$ locus. Label the strict transform of the original divisor by 1. This label is constant on each global component. The same chart induction shows that at most one branch of a given label occurs through each point. In fact the first new exceptional variable has label $r$ and all remaining labels occur among the other $r - 1$ variables. Consequently no global component has two branches through a point. Each component and each intersection is smooth of the appropriate codimension, by the final coordinate charts. This is the strict normal crossings condition. □

**Theorem C.6.1** *Under the standing hypotheses of this section, there is a projective birational morphism from a smooth integral $k$-scheme to $X$ such that the reduced inverse image of $B$ is a strict normal crossings divisor.*

***Proof*** The successive blowups above first remove repeated smoothing factors and then resolve the squarefree singularities. The reduced boundary on the resulting smooth scheme has normal crossings in étale coordinates. Apply Lemma C.6.1. All centres are contained in proper closed subsets and all blowups are projective and birational. □

## C.7 The alteration theorem

**Theorem C.7.1** *Let $X$ be an integral projective scheme over an algebraically closed field $k$ of arbitrary characteristic, and let $Z \subsetneq X$ be closed. There is a projective alteration $Y \to X$ with $Y$ smooth and integral over $k$, such that the reduced inverse image of $Z$ is a strict normal crossings divisor on $Y$.*

***Proof*** We argue by induction on $d = \dim X$. In dimension zero the scheme is a point and $Z$ is empty. In dimension one use the normalization, which is finite and smooth over the perfect field $k$. A proper closed subset of a regular curve has a reduced inverse image consisting of smooth points.

Suppose $d \geq 2$. If $Z \neq \emptyset$, blow up its reduced ideal, making its inverse image the support of an effective Cartier divisor on the integral blowup. In either case, normalize. This is a projective birational modification, and the pullback of the divisor remains effective Cartier. We may therefore assume that $X$ is normal and $Z$ is either empty or a Cartier support. Projection by Proposition C.1.1 gives a projective family of curves with geometrically connected fibres over an integral projective base $S$ of dimension $d-1$, after a projective birational modification of $X$. Replace this family by the flat closure of its generic fibre after a projective modification of the base, using Lemma C.2.1. The discussion after that lemma retains pure fibre dimension one, the dense relative smooth loci and the finite generically étale marked support $Z$. The generic fibre is smooth and geometrically integral.

Enlarge the finite marked locus by Proposition C.3.1 to give at least three distinct marks in the relative smooth locus on every component of each geometric fibre. The union is still a Cartier support, finite and generically étale over $S$. By Corollary C.3.1, after a finite surjective base change it is the union of sections that are distinct at the generic point. There are at least three such sections. Apply Lemma C.2.2 to get a stable pointed family $C$ agreeing with the given family on a dense open after a projective alteration of the base. By flatness and geometric integrality the pullbacks of the old family and the stable family are integral. The property that the sections supply three distinct smooth marks on each old fibre component persists under these base changes.

Take the reduced closure of the generic graph from $C$ to the old family. Apply Lemma C.2.1 to this graph to make it flat over a normal base by a projective modification. The old family and $C$ are already flat and remain integral, so their pullbacks are the generic closures. The flat closure of the graph, as a subscheme of their product, is the graph closure for the new base. The sections still give the required smooth marks on every old fibre component. By Proposition C.4.1, the generic map from $C$ extends to a projective birational morphism to the old family. The composite map from $C$ to the original $X$ is therefore a projective alteration.

Choose a dense open $U$ of the present base over which this map of families is an isomorphism, $C$ is smooth over the base, and the inverse image of the original $Z$ is contained in the marked sections. Such an open exists because these conditions hold generically, and the comparison and the sections spread out as above. Put $D = S \setminus U$ on the present base $S$. By the induction hypothesis, there is a projective alteration $S' \to S$ with $S'$ smooth projective and the reduced inverse image $D'$ of $D$ a strict normal crossings divisor. The new curve family $C \times_S S'$ is flat and nodal. Its marked sections are disjoint in the relative smooth locus, and the family is smooth over $S' \setminus D'$. It is integral by flatness and geometric integrality at the generic point. Now Theorem C.6.1 gives a projective birational map from a smooth integral scheme $Y$ with strict normal crossings reduced boundary. The composite $Y \to X$ is a projective alteration.

This boundary contains the inverse image of the original $Z$, since over $U$ that inverse image is in the marks and above $D$ it lies in the vertical boundary. If $Z \neq \emptyset$, its inverse image on $Y$ is a Cartier support by the initial blowup and dominance of all

subsequent maps. A reduced Cartier support contained in a strict normal crossings divisor is a union of some of its components and has strict normal crossings. □

# Appendix D
# Analytic modifications and model intersections

We collect the analytic facts needed to construct flag models on a complex space and compare their intersections with algebraic intersection degrees. They are used in the transcendental theory from Chapter 21 onward. Appendix D.1 constructs normalized blowups of coherent ideals, Appendix D.2 integrates Chern forms over analytic cycles, and Appendix D.3 compares the analytic and algebraic constructions on projective schemes. The last three sections construct smooth Kähler refinements of flag models, positive forms near a union of divisors, and relative ideals of vertical divisors whose classes are positive near the central fibre. We use the finite normalization theorem and the dimension and proper mapping theorems for complex spaces [Dem12, Ch. II, Theorem 7.12 and § 8]. For the integration argument we use the resolution of a compact reduced complex space by finitely many blowups in smooth centres lying over its singular locus, after which the space is smooth [BM97, Theorem 13.2(1) and Theorem 1.6]; the composite is therefore an isomorphism over the regular locus.

## D.1 The normalized blowup

**Lemma D.1.1** *Let $Y$ be a normal irreducible complex space and $I$ a nonzero coherent ideal. There is a proper normal modification $Y_I \to Y$, isomorphic over $Y \setminus V(I)$, on which $I$ becomes invertible. Every normal modification making $I$ invertible factors uniquely through $Y_I$. If a complex Lie group acts holomorphically with the same pullback of $I$ under the action and projection, the action lifts holomorphically to $Y_I$. For two nonzero coherent ideals $I, J$, the model $Y_{IJ}$ dominates both $Y_I$ and $Y_J$.*

***Proof*** On a neighborhood with generators $f_1, \dots, f_r$ of $I$, take the reduced closure of the graph of $[f_1 : \cdots : f_r]$ in the product with $\mathbb{P}^{r-1}$. It is analytic: in the locus defined by $f_i \xi_j = f_j \xi_i$, keep the irreducible components that meet the complement of the zero set of the ideal. There that locus is precisely the graph. The graph is dense in each retained component and is disjoint from each component lying entirely over the zero set. Local finiteness of the irreducible components gives the stated reduced analytic closure. The projection is proper by closedness in the local projective product. On

the chart $\xi_i \neq 0$, the pullback of $I$ is generated by $f_i$, a nonzerodivisor because it is nowhere zero on the graph in that chart.

If a modification $T \to Y$ makes $I$ invertible, divide the $f_j$ by a local generator of the invertible ideal on $T$. The resulting tuple generates the unit ideal. Its projective coordinates extend the graph map and land in the closed graph by density. This gives the unique factorization and proves that the constructions glue. Take the finite normalization of the glued blowup. The map from a normal modification lifts: a meromorphic function integral over the target local ring pulls back to one integral over the source local ring, hence holomorphic there. This gives the same universal property after normalization. To lift the action, work over the product of the group with $Y$. The automorphism $(g, y) \mapsto (g, g \cdot y)$ preserves the extended ideal. The graph construction on this product is the product of the group with the blowup, and its normalization is the product with $Y_I$, since taking a product with a complex manifold preserves normality. Lifting the automorphism gives a holomorphic action map by projection to $Y_I$. The group law holds by uniqueness.

On $Y_{IJ}$ the product of the pulled-back ideals is principal in every local ring, generated by a nonzero element $u$. In the fraction field, the product of the first ideal with $u^{-1}$ times the second is the unit ideal. Some pair of generators has product a unit in the local ring. Its first member generates the first ideal, since the product of each element of that ideal with the chosen second member is in the local ring. The same argument gives a generator for the second ideal. Both ideals are therefore invertible, and the lifts give the two dominations. □

**Lemma D.1.2** *A proper modification of normal irreducible complex spaces is an isomorphism over an open set meeting every irreducible divisor of the target in a dense open.*

***Proof*** The subset of the target over which the fibre has positive dimension is closed by the upper semicontinuity of fibre dimension and properness. Over its complement the map is finite, and a finite modification of a normal space is an isomorphism by integral closure. For any irreducible divisor $E$ of the target, the inverse image is a proper analytic subset of the source, so has dimension at most $\dim E$. In a relatively compact neighborhood of a point of $E$, its irreducible components are locally finite over the target. Those whose images are contained in proper subsets of $E$ can be omitted on a dense open of $E$, by properness and the dimension theorem for analytic images. On each component that dominates $E$, the dimensions of the two spaces agree. On its smooth locus the differential has full rank on a dense open. The remaining set has smaller dimension and its image can be omitted by properness and the rank bound. After shrinking to a dense open of $E$ all fibres are therefore zero-dimensional. The open over which the map is an isomorphism meets $E$ as claimed. □

## D.2 Intersections of analytic cycles

We use smooth forms on complex spaces that are locally given by smooth ambient forms in holomorphic embeddings. Their restrictions are understood by pullback to smooth test manifolds mapping into the space. Two representatives are identified if all these pullbacks agree. This gives pullback, wedge product and exterior derivative, and includes the curvatures of smooth Hermitian metrics. Smooth partitions of unity on local embeddings give smooth Hermitian metrics. Their Chern forms differ by the differential of a smooth form, by the curvature formula in holomorphic frames.

On a reduced compact irreducible analytic space $Z$ of dimension $d$, integrate a smooth form of degree $2d$ on $Z_{\mathrm{reg}}$, using its complex orientation. The integral is finite and equals the integral of its pullback to a smooth compact resolution. Indeed the latter map is an isomorphism on a dense open, and proper analytic subsets of smooth complex manifolds have zero measure for smooth forms of top degree. The pullback is smooth, so it also bounds the integral of the absolute value. By Stokes the integral of a product of closed forms does not change when one form is changed by an exact form. We use the same convention over cycles by linearity.

**Lemma D.2.1** *Let $f : Z' \to Z$ be a proper holomorphic map of reduced compact irreducible complex spaces with $\dim Z' = d$, and let $\eta$ be a smooth form of degree $2d$ on $Z$. If $f$ is bimeromorphic, then $\int_{[Z']} f^*\eta = \int_{[Z]} \eta$. If $\dim f(Z') < d$, then $\int_{[Z']} f^*\eta = 0$.*

***Proof*** If $f$ is bimeromorphic, it restricts to a biholomorphism between dense opens of the regular loci whose complements are proper analytic subsets. These subsets have zero measure, as above, which gives the first identity. If the image has dimension less than $d$, the pullback of a form of degree $2d$ vanishes on a dense open by the rank bound and has zero integral. □

**Lemma D.2.2** *Let $Z$ be a normal compact irreducible complex space of dimension $d$, let $D$ be a real Cartier divisor on $Z$, and let $\eta$ be a product of closed smooth forms of total degree $2d-2$. Represent $c_1(D)$ by the corresponding real combination of Chern forms of smooth Hermitian metrics. Then*

$$\int_{[Z]} c_1(D) \wedge \eta = \sum_F \operatorname{ord}_F(D) \int_{[F]} \eta, \tag{D.1}$$

*where $F$ runs over the irreducible divisors with nonzero coefficient in $D$.*

***Proof*** By linearity it suffices to treat a Cartier divisor $D$. Pull back to a smooth compact resolution and apply the divisor formula for a meromorphic section of a line bundle there. That formula follows by applying Poincaré–Lelong to local equations and integrating the exact term by Stokes [Xia26a, Theorem 7.4.5]. By Lemma D.1.2, the strict transform of each $F$ has the same coefficient and is bimeromorphic to $F$, so it contributes $\operatorname{ord}_F(D) \int_{[F]} \eta$ by Lemma D.2.1. The other divisors of the pullback map to smaller-dimensional sets and contribute zero by the same lemma. □

**Lemma D.2.3** *Let $Z$ be an integral projective scheme over $\mathbb{C}$ of dimension $d$, with associated compact complex space $Z_{\mathbb{C}}$, and let $L_1, \dots, L_d$ be invertible sheaves on $Z$. For the Chern forms of arbitrary smooth Hermitian metrics on the associated holomorphic line bundles,*

$$(L_1 \cdots L_d \cdot [Z]) = \int_{[Z_{\mathbb{C}}]} c_1(L_1) \wedge \cdots \wedge c_1(L_d).$$

*By multilinearity the same holds for real Cartier classes.*

***Proof*** The right side does not depend on the metrics, by Stokes. By multilinearity, twist each line bundle by a high power of a very ample bundle and take a difference. For very ample bundles, choose general sections with a finite transverse intersection in the smooth locus, avoiding the singular locus by dimension. On a smooth resolution of the associated complex space, the pulled-back linear systems are still basepoint-free. At each step the general section cuts the preceding smooth intersection transversely: the universal zero locus is smooth because evaluation in the section variable is surjective, and the regular values of its projection form a full-measure set by Sard's theorem. These choices can be made together with the algebraic proper-intersection and avoidance conditions. Use a resolution isomorphic over the regular locus. The last intersection on the resolution then consists of the same reduced points. Repeated Poincaré–Lelong and Stokes on these smooth intersections gives the number of points as the Chern integral. This is the algebraic intersection degree. The dimension-zero case assigns degree one to a point. □

## D.3 Projective comparison for ideals

**Lemma D.3.1** *For a projective complex scheme, analytification identifies algebraic and analytic coherent ideals, preserving sums, products and extension along morphisms of projective schemes. If the scheme is normal and integral and the ideal nonzero, the normalized algebraic blowup analytifies to the normalized analytic blowup.*

***Proof*** The exact equivalence on coherent sheaves, with the local-ring flatness comparison, is the projective coherence theorem [Ser56, § 2, Proposition 3 and Corollary 1, § 3, Proposition 10 and Theorems 2–3]. For a closed subscheme of projective space, first apply the equivalence to the coherent pushforward on the ambient projective space. Faithful exactness detects that the defining ideal of the subscheme annihilates that module, and the local-ring comparison restricts by taking the quotients by the defining ideals. This gives the same comparison for the closed subscheme. An inclusion of a coherent sheaf into the analytic structure sheaf algebraizes to a map into the algebraic structure sheaf. Its kernel vanishes by the faithful exact comparison, so it is an inclusion. Sums and products are images of coherent maps and commute with this comparison. Extension of an ideal is the image of its pullback in

the structure sheaf, and pullback commutes with analytification by the local tensor product description.

For the blowup, the algebraic chart with homogeneous coordinate $\xi_i \neq 0$ is the graph closure in that chart. If $f_i$ is the corresponding generator, its relations are the kernel of the localization map after imposing the equations $f_j = f_i\xi_j/\xi_i$. Flatness of analytic local rings over the corresponding algebraic local rings preserves that kernel, so $f_i$ is a nonzerodivisor on the analytified chart. The algebraic chart is integral, so its analytification is reduced [Ser56, § 2, Proposition 4]. The analytic chart is thus the reduced graph closure on the same open chart, giving the analytic blowup.

For the normalization comparison, first let $Z$ be normal and projective. The finite analytic normalization of its analytic space is defined by its coherent algebra over $\mathcal{O}_{Z_{\mathbb{C}}}$, as can be checked locally using generators integral over the base. The coherent equivalence algebraizes that algebra and its multiplication. It therefore algebraizes the normalization map to a finite map over $Z$. The map is an isomorphism over a dense open of $Z$: it is so analytically along the smooth locus, and exactness of the coherent comparison detects this for the finite algebra. The algebraic map is birational with reduced source, with no components over the complement since the smooth inverse image is analytically dense. It is an isomorphism by normality. Thus the associated complex space of a normal projective scheme is normal. The algebraic normalization of the algebraic blowup now analytifies to a finite normal modification of the analytic blowup. Uniqueness of normalization proves the comparison. □

## D.4 Smooth Kähler refinements of flag models

We first record two properties of blowups of compact complex manifolds in smooth centres.

**Lemma D.4.1** *Let $V$ be a compact complex manifold.*

(1) *Let $I$ be a coherent ideal on $V$ that is nonzero on every connected component. There is a finite sequence of blowups in smooth centres, each lying over the zero set of $I$, after which the pullback of $I$ is the ideal of an effective divisor with normal crossings support. The centres are determined canonically by $I$: every biholomorphism of $V$ preserving $I$ preserves each centre and lifts to a biholomorphism of each stage.*

(2) *Suppose that $V$ is Kähler with Kähler form $\Omega$, and let $p : V' \to V$ be the composite of finitely many blowups in smooth centres. Let $\widetilde{E}_j$ be the pullback to $V'$ of the exceptional divisor of the $j$-th blowup. If positive rational numbers $\varepsilon_j$ are chosen successively, each sufficiently small in terms of the preceding ones, then $p^*[\Omega] - c_1(F)$ is a Kähler class for $F = \sum_j \varepsilon_j \widetilde{E}_j$. The divisor $F$ is effective, with positive coefficient along the strict transform of every exceptional divisor. In particular $V'$ is Kähler.*

***Proof*** For the first assertion use analytic principalization of a coherent ideal on a smooth complex manifold by blowups in smooth centres [BM97, Theorem 1.10 and

Remarks 1.8, 1.18]. The leading invariant of the centres is the order of the weak transform. It is positive at every nontrivial stage, so the centres avoid the locus where the weak transform is the unit ideal. Off the inverse image of the zero set of $I$ the blowups are isomorphisms and the weak transform is the unit ideal. Hence every centre lies over that zero set. Each centre is the maximum locus of the invariant, which depends only on the local isomorphism class of the transformed ideal together with the exceptional hypersurfaces [BM97, § 1]. A biholomorphism preserving $I$ therefore preserves the first centre and lifts to its blowup by the universal property. The lift preserves the weak transform and the exceptional hypersurfaces, and induction treats the later stages.

For the second assertion consider first the blowup $b$ of one smooth centre in a compact Kähler manifold with Kähler form $\omega$. On the blowup choose a smooth Hermitian metric on $O(-E)$, where $E$ is the exceptional divisor, whose curvature $\gamma$ is positive on the fibres of $E$ over the centre. Locally the blowup embeds in the product with a projective space of normal directions, and on that product the Fubini–Study metric gives this curvature. Patch the logarithmic weights of these metrics near $E$ with a partition of unity from the base, so the curvature on each fibre is a convex combination of its local positive curvatures. Extend the metric off a neighborhood of $E$ using a cutoff. Then $b^*\omega + \varepsilon\gamma$ is Kähler for sufficiently small $\varepsilon > 0$. Otherwise, by compactness, unit tangent vectors violating positivity for a sequence $\varepsilon \to 0$ would have a limit in the kernel of $b^*\omega$ at which $\gamma$ is nonpositive. This kernel consists of the tangent directions to the exceptional fibres. The form $b^*\omega+\varepsilon\gamma$ represents $b^*[\omega] - \varepsilon c_1(E)$. Apply this to the successive blowups with rational $\varepsilon = \varepsilon_j$, starting from $\Omega$. Pulling back the classes from the earlier stages gives the Kähler class $p^*[\Omega] - c_1(F)$. Each $\widetilde{E}_j$ is effective and contains the strict transform of the $j$-th exceptional divisor with coefficient one, which gives the assertion about the coefficients of $F$. □

Let $X$ be a compact Kähler manifold. Flag ideals, flag models and the ideal $\mathfrak{t}$ of the fibre over zero of $X \times \mathbb{P}^1$ are as in Section 21.5. A smooth refinement is obtained from $X \times \mathbb{P}^1$ by a finite sequence of blowups in smooth centres over the fibre at zero. We require the reduced central fibre of the result to have strict normal crossings.

**Lemma D.4.2** *Every finite collection of flag models of $X$ admits a common smooth refinement. Every smooth refinement is a compact Kähler manifold. Two smooth refinements admit a common smooth refinement dominating both.*

***Proof*** On the compact product $X\times\mathbb{P}^1$, apply Lemma D.4.1 to the product of the given flag ideals and $\mathfrak{t}$. The zero set of this product lies in the fibre at zero, and so do all the centres. On the resulting manifold every factor of the product becomes invertible, by the local ideal argument in Lemma D.1.1. Hence the manifold dominates all the given normalized blowups. The fibre at zero is supported on the normal crossings zero divisor of the product ideal. Its support $B$ has normal crossings in holomorphic coordinates. If an irreducible component of $B$ has more than one local branch through some point, apply the procedure in the proof of Lemma C.6.1, blowing up the strict transforms of the reduced order loci $B_{\geq j}$ of $B$, in decreasing order of $j$. That lemma

is stated for varieties, but its chart computation applies verbatim in holomorphic coordinates, where these loci are closed analytic subsets of the central fibre. It gives smooth centres and strict normal crossings.

Every smooth refinement is Kähler by Lemma D.4.1, since $X \times \mathbb{P}^1$ is Kähler.

For the last assertion start with one smooth refinement and the sequence of smooth blowup centres producing the other. Lift the map through that sequence one blowup at a time. At each step principalize the pullback of the next centre's ideal by Lemma D.4.1. This ideal is the unit ideal off the central fibre, so the new centres lie over zero. Then use the universal property of the blowup. At the end principalize the ideal of the central fibre to ensure normal crossings, then strictify as above. This sequence still consists of smooth blowups supported over zero. □

## D.5 Positive forms near a union of divisors

For a smooth real closed $(1,1)$-form $\beta$ on a complex manifold, we use its real Bott–Chern class: two representatives of the same class differ by $dd^c$ of a smooth real function.

**Lemma D.5.1** *Let $W$ be a compact Kähler manifold, and let $S$ be a union of finitely many smooth compact divisors. If the restriction of $[\beta]$ to every component of $S$ is Kähler, there is a smooth real function $\psi$ on $W$ such that $\beta + dd^c\psi$ is Kähler on a neighborhood of $S$.*

***Proof*** For one component $B$, take a potential $\psi_B$ with $\beta|_B + dd^c\psi_B > 0$. Extend it smoothly and add $C\,\mathrm{dist}(\,\cdot\,, B)^2$ for a smooth Hermitian metric. Along $B$ the Hessian of the squared distance is positive on the normal block and zero in the tangent and mixed blocks. The tangent block is already positive. The Schur complement and compactness give one $C > 0$ yielding positivity near $B$. Extend by a cutoff.

Suppose we have treated a union $A$ and want to adjoin another component $B$. Let $\psi_A, \psi_B$ give positive forms near $A, B$, respectively. If $A \cap B$ is empty, the potentials glue on disjoint neighborhoods. Otherwise, using defining sections of the two effective divisors with smooth Hermitian metrics, set

$$\ell = \log\bigl(\|s_A\|^2 + \|s_B\|^2\bigr).$$

This function is smooth off $A \cap B$ and tends uniformly to $-\infty$ near that intersection. Its curvature is bounded below by $-C\Omega$, where $\Omega$ is a fixed Kähler form. To see this, add a common local potential of $C\Omega$ to the two logarithmic norms and apply the convex increasing function $(x, y) \mapsto \log(e^x + e^y)$. Off their zero sets the Hessian rule gives the bound, and the resulting local plurisubharmonic functions extend across the zero sets by upper regularization.

Take $g = \psi_A + \delta\ell$, with $\delta > 0$ small enough for positivity near $A$, and $h = \psi_B - M$. Choose a neighborhood $V_B$ of $B$ with compact closure inside the positivity neighborhood of $\psi_B$. For $M$ large, $g > h + 2$ near $A \cap \partial V_B$, which is compact and

disjoint from $B$. For this $M$ we have $h > g + 2$ on a neighborhood of $A \cap B$. Choose neighborhoods $U_A, U_B$ of $A, B$ small enough that $U_B \subset V_B$, the first inequality holds near $U_A \cap \partial V_B$, and the second holds on $U_A \cap U_B$. Such shrinking is possible by compactness. Use $g$ in $U_A \setminus V_B$, a smooth regularized maximum of $g, h$ in $U_A \cap V_B$, and $h$ in $U_B$.

Convolve the real maximum in two variables with a smooth nonnegative kernel of integral one, barycentre zero and support of radius less than $1/2$. The result equals the larger entry for gaps greater than one. It is convex and its nonnegative first derivatives sum to one by translation equivariance. The chain rule gives a positive weighted sum of the two forms plus a semipositive Hessian term. The separation inequalities glue the formulas smoothly, including near the poles of $g$. Extend by a cutoff and iterate.□

## D.6 Relative ideals of vertical divisors

Let $X$ be a compact Kähler manifold, let $Y = X \times \mathbb{P}^1$, and let $q : W \to Y$ be a smooth refinement as in Appendix D.4. We use the projections $\rho : W \to X$ and $\pi : W \to \mathbb{P}^1$ and write $W_0 = \pi^*[0]$. A divisor on $W$ is vertical if it is supported on $W_0$.

**Lemma D.6.1** *Let $B$ be a rational vertical divisor and $\alpha$ a Kähler class on $X$. Suppose $\rho^*\alpha + c_1(B)$ has a smooth representative on $W$ that is Kähler near $W_0$. There are a rational number $c$, a positive integer $m$ and a coherent ideal $I$ on $Y$ containing a power of $\mathfrak{t}$ such that*

$$I\mathcal{O}_W = \mathcal{O}_W\big(m(B - cW_0)\big). \tag{D.2}$$

***Proof*** Since $B$ is supported on $W_0$ and every component of $W_0$ has positive coefficient, we may take $c$ rational with $F = cW_0 - B \geq 0$. Let $L_m = \mathcal{O}_W(-mF)$ for $m$ clearing denominators. Denote the divisor of the holomorphic Jacobian of $q$ by $K_{W/Y}$. It is effective and supported on $W_0$ since $q$ is a biholomorphism off $W_0$.

We first show that the line bundles $L_m$ and $L_m \otimes \mathcal{O}_W(-K_{W/Y})$ are both generated over neighborhoods of the fibres by local sections over the base, for sufficiently large divisible $m$. Write $\omega_B$ for the given representative on $W$, and let $V$ be a neighborhood of $W_0$ on which $\omega_B$ is Kähler. Let $U$ be a coordinate ball on $Y$ centred at a point of $X \times \{0\}$, with the inverse image of its closure in $V$. Such balls exist by properness. Choose smooth representatives $\omega \in \alpha$ and $\omega_{\mathrm{FS}}$ of the point class on $\mathbb{P}^1$. By the $dd^c$-lemma [Xia26a, Theorem 14.1.7], a metric on $L_m$ can be chosen with curvature

$$m\big(\omega_B - \rho^*\omega - c\pi^*\omega_{\mathrm{FS}}\big).$$

Over $q^{-1}(U)$, add the pullback of a smooth weight on $U$ to cancel the two base terms. The curvature becomes $m\omega_B$. For the second bundle tensor with a fixed smooth metric for the twist. On the compact inverse image of the closure of $U$, $\omega_B$ dominates a positive multiple of a fixed Kähler form $\Omega$ on $W$; the twist has bounded curvature.

For a ball of radius $R$, use the exhaustion $\psi = -\log(1 - |z|^2/R^2)$ on $U$. Direct differentiation gives $dd^c\psi \geq c_0 i\partial\psi \wedge \bar{\partial}\psi$ for some $c_0 > 0$. The form $\sigma = \Omega +$

$dd^c(\psi \circ q)$ on $q^{-1}(U)$ is Kähler. The inverse image of a sublevel of $\psi$ is compact by properness. Cutoffs of $(\psi \circ q)/j$, equal to one on sublevels up to $j$ and zero beyond $2j$, have derivative bounded by $C/j$ in the $\sigma$-norm. The derivative bound follows from $\sigma \geq c_0 i\partial(\psi \circ q) \wedge \bar\partial(\psi \circ q)$.

Fix a smaller ball $U' \Subset U$ and an interpolation point $p \in q^{-1}(\overline{U'})$. Let $G$ be either of the two bundles and let $N = \dim W$. Multiply a holomorphic frame of $G$ near $p$ by a smooth cutoff equal to one near $p$. This gives a smooth section $s$ of $G$ whose $\bar\partial s$ has compact support away from $p$. Regard $\bar\partial s$ as an $(N,1)$-form with values in $G \otimes K_W^{-1}$. Use a fixed smooth metric for $K_W^{-1}$. Add weights $\psi \circ q$ and $N\chi \log(|z - z(p)|^2 + \delta)$ to the metric of the latter bundle, where the second weight is in coordinates near $p$, smoothly cut off by $0 \leq \chi \leq 1$ equal to one around $p$. Off the cutoff transition the second weight has semipositive Hessian or is zero. On the transition its derivatives are bounded uniformly for $0 < \delta < 1$. For large $m$ the total curvature thus dominates a positive multiple of $\sigma$. The compactly supported data have uniformly bounded norms for these weights. Using $\sigma$ as Hermitian background, the cutoffs just constructed allow the application of [Xia26a, Lemma 14.6.5]. It gives a solution $s'_\delta$ of $\bar\partial s'_\delta = \bar\partial s$ with uniformly bounded weighted square integrals.

Take a sequence of $\delta$'s decreasing to zero. Extract one weak subsequence in the first, least singular, weighted space. Each fixed later weight is equivalent to the first because the radius regularization differs only by a bounded function on the compact coordinate support. The bounds in each fixed weight pass to the same limit $s'$ by weak lower semicontinuity, and then to the pole weight by monotone convergence. The equation passes to the distributional limit. Near $p$, $s'$ is holomorphic, and square integrability against $|z-z(p)|^{-2N}$ forces $s'(p) = 0$. The section $s - s'$ is holomorphic and nonzero at $p$.

The size of $m$ can be fixed uniformly for $p \in q^{-1}(\overline{U'})$. Finitely many relatively compact coordinate charts in $q^{-1}(U)$ suffice there. Place the moving pole in a smaller ball in each chart and use a cutoff of fixed radius, with a smaller inner radius and a larger outer radius. The curvature loss on its transition is uniformly bounded by a multiple of $\Omega$. The same choice works for both twists after increasing $m$. For each such $m$ finitely many sections suffice over $q^{-1}(\overline{U'})$, since their nonvanishing opens cover that compact set. Taking finitely many pairs $U' \Subset U$ around the central product gives a common $m$.

The inclusion $L_m \subset \mathcal{O}_W$ gives $I = q_* L_m \subset \mathcal{O}_Y$. Indeed a holomorphic function upstairs descends off the fibre, and is locally bounded near the fibre by properness. Its negative Laurent coefficients in the parameter vanish, so it extends holomorphically. To check coherence, let $g_1, \ldots, g_r$ be the functions on $U'$ descended in the same way from the chosen generators of $\mathcal{O}_W(-mF - K_{W/Y})$. Upstairs $\sum_i |g_i \circ q|^2$ is comparable locally to the squared modulus of one equation for $mF + K_{W/Y}$, since the remaining factors generate the unit ideal. A germ $h$ downstairs belongs to the multiplier ideal of $\log \sum_i |g_i|^2$ precisely when the pulled-back square-integrability condition holds. In change of variables off the fibre the squared Jacobian cancels the factor for $K_{W/Y}$. The remaining condition in strict normal crossings coordinates is integrability of $|h \circ q|^2/|s_{mF}|^2$. Polar integration at a general point of each component requires the vanishing orders to be at least the corresponding integer coefficients of $mF$. These

inequalities also suffice, by division by the local monomial for $mF$. The germ condition may be checked on a neighborhood of the full fibre, and properness permits shrinking $U'$ for this check. Thus the restriction of $I$ to $U'$ is the multiplier ideal of $\log \sum_i |g_i|^2$, which is coherent by [Xia26a, Theorem 12.2.7]. Off zero $I = \mathcal{O}_Y$. The generation of $L_m$ proves (D.2). Finally, $\mathfrak{t}^a \subset I$ if $aW_0 \geq mF$. □

# Appendix E
# Gluing affinoid domains

The Laurent pair gives the basic exact sequence for gluing. Iterating such pairs treats finite rational covers. To pass to arbitrary affinoid domains, we approximate their coordinates in completed residue fields and use a strict spectral bound to obtain local finiteness. These arguments give the structure sheaf on the G-topology of an affinoid space, for the affinoid and analytic domains of Chapter 2.

## E.1 Rational covers and residue fields

Throughout this appendix, $K$ is a complete ultrametric field. For a $K$-affinoid algebra $A$, an affinoid domain $V$ of $\mathcal{M}(A)$ and a finite $A$-module $M$, we write $M|_V = M \otimes_A A_V$. By Proposition 2.2.2, Proposition 2.5.1 and the factorization discussion in Section 2.5, the algebra (2.3) of a rational presentation represents its rational domain. Hence different presentations of the same rational domain have canonically isomorphic algebras.

**Proposition E.1.1** *Let $A$ be $K$-affinoid, let $M$ be a finite $A$-module, and let $f \in A$ and $r > 0$. For the two Laurent domains $V_+ = \{|f| \leq r\}$ and $V_- = \{|f| \geq r\}$, the augmented two-set Čech sequence*

$$0 \longrightarrow M \longrightarrow M|_{V_+} \oplus M|_{V_-} \longrightarrow M|_{V_+ \cap V_-} \longrightarrow 0$$

*is exact.*

***Proof*** Give $M$ its finite module quotient norm. We use weighted series with coefficients in $M$, with the same convergence condition as for algebra coefficients. Such series are exact in finite modules for a short exact sequence. Indeed the submodule norm is equivalent to the induced norm, so the kernel is computed coefficientwise. For surjectivity lift each coefficient with a fixed factor of norm increase in the quotient norm. Lemma 2.3.2 gives these norm equivalences. In particular a module of weighted series is the tensor product of the scalar series algebra with $M$. To see this,

use a finite presentation of $M$ and lift the corresponding series coefficientwise. The kernel consists of the series in the relation submodule, which again have coefficientwise lifts to a finite free module by the same norm comparison.

Put $P = M\{r^{-1}T\}$, $Q = M\{rT^{-1}\}$, and let $W$ be the module of Laurent series $\sum_{n\in\mathbb{Z}} m_n T^n$ with $\|m_n\|r^n$ tending to zero away from finite subsets.

The three localizations are, respectively,

$$P/(T-f)P, \qquad Q/(1-fT^{-1})Q, \qquad W/(T-f)W,$$

where for the first two we use the presentations $|f| \le r$ and $|1| \le r^{-1}|f|$. For the overlap, introduce $T$ and $U$ with radii $r$ and $r^{-1}$ and the relations $TU = 1$, $T = f$. In $A\{r^{-1}T, rU\}$ the ideals $(TU-1, T-f)$ and $(fT-f^2, fU-1)$ coincide: modulo the second, $f$ is a unit with inverse $U$. The latter is the rational presentation of $V_+ \cap V_-$ with denominator $f$ and numerators $f^2$ and $1$ of radii $r$ and $r^{-1}$. By Proposition 2.2.2, Proposition 2.5.1 and the factorization discussion in Section 2.5, its algebra is canonically isomorphic to that of any rational presentation of this domain. The Laurent expansion of Section 2.6 at equal radii identifies the quotient by $TU-1$ with the series defining $W$. The same monomial reduction works for module coefficients.

Multiplication by $T-f$ on $W$ is injective. First suppose $M = A/\mathfrak{p}$ with $\mathfrak{p}$ prime. The spectral seminorm of any nonzero element of the integral affinoid algebra $A/\mathfrak{p}$ is positive, by Corollary 2.5.1, since a nonzero element of a domain is not nilpotent. Thus a nonzero coefficient of a nonzero Laurent series is detected at a spectral point. Evaluating the coefficients there gives a nonzero series over a complete valued field. The Gauss Laurent algebra over that field is an integral domain, since its norm is multiplicative (Section 2.4.1 and Proposition 2.1.1). Since $T - f(x)$ is nonzero, their product is nonzero. This proves the assertion when $M = A/\mathfrak{p}$.

For general finite $M$ use a finite filtration by submodules with quotients $A/\mathfrak{p}$. Such a filtration can be constructed by choosing in each nonzero quotient a nonzero element with maximal annihilator, which is prime. The process terminates by the ascending chain condition. Exactness of Laurent series in finite modules now proves injectivity for all $M$, by induction on the filtration.

The difference map $P \oplus Q \to W$ is surjective by splitting the Laurent series into positive and nonpositive powers. Its kernel is the common constants. Suppose $p \in P$ and $q \in Q$ have the same class in $W/(T-f)W$, so $p - q = (T-f)h$ for some $h \in W$. Write $h = u - T^{-1}v$ with $u \in P$ and $v \in Q$, by grouping nonnegative and strictly negative exponents. Then $p - (T-f)u = q - (1-fT^{-1})v$ is a common constant. This proves exactness in the middle after the quotients. If a common constant has zero image on both charts, write it as $(T-f)u = (1-fT^{-1})v$. Injectivity on $W$ gives $u = T^{-1}v$, forcing both sides to be zero by their disjoint exponent supports. This proves injectivity at the start. Surjectivity at the end follows from the surjective difference map before quotients. □

**Proposition E.1.2** *Let $A$ be $K$-affinoid and let $(U_i)_{i=1}^q$ be a finite rational cover of $\mathcal{M}(A)$. For every finite $A$-module $M$, the augmented Čech complex of the rational localizations of $M$ for this cover is exact.*

***Proof*** Write $X = \mathcal{M}(A)$. A rational domain given by $d, h_1, \ldots, h_p$ and radii $r_j > 0$ consists of the points where $|h_j| \leq r_j|d|$ for all $j$, with $d, h_1, \ldots, h_p$ generating the unit ideal. Pulling back by a bounded map of affinoid algebras gives the rational domain defined by the images of the same elements and the same radii. Finite intersections of rational domains in $X$ are rational by the intersection formula in Section 2.2. The universal property of rational localization holds at the prescribed radii, by Proposition 2.2.2, Proposition 2.5.1 and the factorization discussion in Section 2.5. For rational domains $U, U'$ of $X$, it identifies the restriction to $U$ followed by the restriction to the pullback of $U'$ with the single restriction to $U \cap U'$. Consequently the same identities hold for the tensor products defining the restrictions of $M$.

We first reduce the topology of the cover to comparisons within a single family. Write

$$U_i = \{x : w_{ij}(x) \leq w_{i0}(x) \text{ for every } j\}, \qquad w_{ij}(x) = c_{ij}|f_{ij}(x)|,$$

where the $c_{ij}$ are positive real numbers and $c_{i0} = 1$. For a tuple $J = (j_1, \ldots, j_q)$ having at least one $j_i = 0$ use the function $h_J = \prod_i f_{ij_i}$ with weight $c_J = \prod_i c_{ij_i}$. At each point of $X$ there is a row whose maximum is attained at index zero. Each row has positive maximum, since the elements of a row generate the unit ideal and so do not all vanish at a point. Choosing maximum entries, with index zero in a row whose maximum is attained there, shows

$$\max_J c_J|h_J(x)| = \prod_i \max_j w_{ij}(x) > 0.$$

If $J$ attains the maximum on the left, each of its entries attains the maximum in its row, since all the factors on the right are positive. Hence the set on which that $J$ attains the maximum lies in $U_i$ for any $i$ with $j_i = 0$. These maximum sets are rational. The $h_J$ generate the unit ideal, since a proper ideal they generate would lie in a maximal ideal and its quotient affinoid algebra would have a spectral point with every $h_J$ zero. This uses the existence of points and closedness of ideals from Theorems 2.1.1 and 2.2.1. Thus the sets just described give a finite rational refinement.

Consider any such family of weights $c_J$ and functions $h_J$ with positive maximum everywhere. By compactness choose $\varepsilon > 0$ strictly less than $\min_{x \in X} \max_J c_J|h_J(x)|$ if $X$ is nonempty. Successively split $X$ by the two Laurent inequalities

$$|h_J| \leq \varepsilon/c_J, \qquad |h_J| \geq \varepsilon/c_J$$

for every $J$. On each resulting nonempty chart, at least one of the upper-value choices $|h_J| \geq \varepsilon/c_J$ occurs. The other entries have weighted values at most $\varepsilon$ and can never attain the maximum there. The functions for the upper-value choices are units in the chart algebra, since the Laurent localization explicitly adjoins their inverses. For ev-

ery pair $J, J'$ of such entries, further split by the two Laurent inequalities comparing $|h_J/h_{J'}|$ with $c_{J'}/c_J$. At a nonempty leaf the chosen comparisons orient every pair, an arrow indicating a weak inequality from larger to smaller. A finite tournament has a vertex from which every other vertex can be reached. Indeed a vertex with the largest reachable set has this property, since an outside vertex would have arrows toward the entire set and could reach a strictly larger set. Along every oriented path the weak inequalities compose. Thus one entry gives the maximum throughout that leaf, even if some of the equalities hold everywhere on it. The leaf therefore lies in a single $U_i$. This argument uses arbitrary real radii throughout, regardless of the value group of the ground field.

It remains to descend exactness through the finite succession of Laurent splits. At a chart $Y$ occurring in the succession use the augmented complex of the fixed cover restricted to $Y$, with $M|_Y$ in degree $-1$. Its intersections are rational in $Y$, by pullback and finite intersection as above. A split into $Y_+, Y_-$ gives a termwise short exact sequence of augmented complexes

$$0 \to C^\bullet(Y) \to C^\bullet(Y_+) \oplus C^\bullet(Y_-) \to C^\bullet(Y_+ \cap Y_-) \to 0.$$

For degree $-1$ use Proposition E.1.1 on $Y$. In every other degree use it on each intersection of the restricted cover, with the restricted finite module and the image of the splitting function. The maps commute with the Čech differentials. At a leaf lying in one $U_i$, the augmented complex is contractible. In alternating cochains a contracting homotopy inserts the index $i$ first, using that intersection with $U_i$ on this chart changes no domain. If a chart has empty spectrum its algebra is zero and the complex is exact there also.

Now induct on the depth of the succession of splits. The children have strictly smaller depth, and the overlap has a refinement by a tree of smaller depth as well, by pulling back the succession for the plus child to the overlap. Pullback retains both Laurent inequalities at each split and the leaf-containment condition. The long exact cohomology sequence of the displayed short exact sequence proves exactness at the parent. At the root this is the assertion. □

**Proposition E.1.3** *Let $A$ be $K$-affinoid, let $V$ be an affinoid domain in $\mathcal{M}(A)$, and let $y \in \mathcal{M}(A_V)$ map to $x \in V$. Then the canonical isometric embedding $\mathcal{H}(x) \to \mathcal{H}(y)$ is an isomorphism. If $V' \subset V$ is a further affinoid domain in $\mathcal{M}(A)$ and $y' \in \mathcal{M}(A_{V'})$ maps to $y$ under the map induced by the restriction $A_V \to A_{V'}$ given by the domain property of $V$, then $\mathcal{H}(y) \to \mathcal{H}(y')$ is an isometric isomorphism.*

***Proof*** Write $B = A_V$. We first recall the affinoid algebra describing a fibre product. For two bounded maps out of $A$, take Banach algebra presentations of their targets and use the algebra on the combined variables with the target relations imposed and the two images of $A$ identified. Its quotient norm has the universal property for jointly bounded maps. This is the completed tensor product over $A$. Equivalently, for a presentation $B = A\{r^{-1}T\}/I$ with the quotient topology and an affinoid algebra $C$ over $A$, use $C\{r^{-1}T\}/IC\{r^{-1}T\}$. Such a presentation over $A$ is obtained by adjoining a finite presentation over the ground field and imposing its compatibility relations with

$A$. The same construction defines a base change to an arbitrary complete valued field $F$ with a bounded map $A \to F$, by taking $F$ in place of $C$. Using the combined presentations shows that imposing the relations twice or in reverse order gives the same quotient. The ideals are closed by Theorem 2.1.1.

In the present situation, multiplication gives an isomorphism $B\widehat{\otimes}_A B \simeq B$. Indeed the two maps from $B$ to this affinoid algebra extend the same map from $A$, whose spectral image is in $V$ since it already factors through $B$. Uniqueness in the domain property makes the maps equal. Multiplication is then an inverse of either map by the universal property of the completed tensor product.

For $x$ and $y$ as in the statement, take $F = \mathcal{H}(x)$. The fibre algebra $D = B\widehat{\otimes}_A F$ is $F$-affinoid and nonzero, since it maps to $\mathcal{H}(y)$. Base changing the multiplication isomorphism shows $D\widehat{\otimes}_F D \simeq D$ by multiplication. The map $D \otimes_F D \to D\widehat{\otimes}_F D$ is injective. To see this in the form needed here, the bounded remainder section for a Tate presentation of $D$ in Theorem 2.1.1 realizes its Banach space as a subspace of a space of weighted series. On any finite-dimensional subspace of $D$, its coefficient functionals separate points and thus span the dual of that subspace. Hence a nonzero algebraic tensor is detected by a product of continuous linear functionals, each a finite linear combination of coefficient functionals. Such a product is a bounded bilinear form and acts on the completed Banach tensor product by continuity. Over a complete field the affinoid construction of the tensor product agrees with this Banach completion. Indeed the two Tate series spaces have completed tensor product the series space in the combined variables by finite truncation and the Gauss norm. Passing to quotients by the two ideals gives the Banach quotient by the closure of their algebraic images, which is precisely the closed ideal generated by the two sets of relations.

Since $d \otimes 1 - 1 \otimes d$ goes to zero under multiplication for every $d \in D$, injectivity forces $d \in F \cdot 1$. Thus $D = F$. The map $B \to \mathcal{H}(y)$ factors through this fibre, so the dense fraction field whose completion is $\mathcal{H}(y)$ lies in the image of $F$. The image of $F$ is complete and isometric. It is therefore all of $\mathcal{H}(y)$, as claimed.

For $V' \subset V$, the domain property of $V$ gives the restriction map $A_V \to A_{V'}$ over $A$. Hence the embeddings $\mathcal{H}(x) \to \mathcal{H}(y) \to \mathcal{H}(y')$ compose to the embedding $\mathcal{H}(x) \to \mathcal{H}(y')$. The first and the composite are isomorphisms by the case just proved, and so is $\mathcal{H}(y) \to \mathcal{H}(y')$. □

**Proposition E.1.4** *Let $A$ be $K$-affinoid and $X = \mathcal{M}(A)$.*

(1) *If $V$ and $W$ are affinoid domains in $X$, then $V \cap W$ is an affinoid domain in $X$ with algebra $A_V \widehat{\otimes}_A A_W$.*
(2) *Let $V$ be an affinoid domain in $X$, identified with $\mathcal{M}(A_V)$. A subset of $V$ is an affinoid domain in $X$ if and only if it is an affinoid domain in $\mathcal{M}(A_V)$, with the same algebra. For affinoid domains $V'' \subset V' \subset V$ in $X$, the restriction map $A_V \to A_{V''}$ given by the domain property of $V$ is the composite of the restriction maps $A_V \to A_{V'} \to A_{V''}$.*
(3) *If $V$ is an affinoid domain in $X$ and $U$ is the rational domain of $X$ defined by $g, f_i, s_i$ as in (2.2), then $V \cap U$ is the rational domain of $\mathcal{M}(A_V)$ defined by the images of $g, f_i$ and the same radii.*
(4) *Finite intersections of analytic domains in $X$ are analytic domains in $X$.*

***Proof*** For (1) write $B = A_V$ and $C = A_W$, and form $B\widehat{\otimes}_A C$ by the combined presentation in the proof of Proposition E.1.3. Each point $x$ of $V \cap W$ defines maps from $B$ and $C$ to the same $\mathcal{H}(x)$, by Proposition E.1.3. The maps agree on $A$, and substitution in the combined presentation gives a spectral point of $B\widehat{\otimes}_A C$ over $x$. Conversely the restrictions of a spectral point of $B\widehat{\otimes}_A C$ give points of $\mathcal{M}(B)$ and $\mathcal{M}(C)$ over the same point $x$ of $X$, which therefore lies in $V \cap W$. Their evaluations on $B$ and $C$ take values in the canonical copy of $\mathcal{H}(x)$, again by Proposition E.1.3. They determine the seminorm on the dense algebra of finite sums of their products, hence on the completed tensor product. Thus its spectrum maps to $V \cap W$ by a continuous bijection, and compactness makes this a homeomorphism. A bounded map from $A$ to an affinoid algebra with spectral image in $V \cap W$ factors uniquely through $B$ and through $C$. The universal property of the combined presentation for jointly bounded maps gives the unique factorization through $B\widehat{\otimes}_A C$.

For (2), subsets of $V$ and of $\mathcal{M}(A_V)$ correspond under the homeomorphism, and a bounded map $A_V \to E$ to an affinoid algebra has spectral image in a subset $V'$ if and only if the composite $A \to E$ does. Suppose first that $V'$ is an affinoid domain in $\mathcal{M}(A_V)$ with algebra $B'$. Then $\mathcal{M}(B') \to V'$ is a homeomorphism. A bounded map $A \to E$ with spectral image in $V'$ factors uniquely through $A_V$, and the resulting map $A_V \to E$ factors uniquely through $B'$. Two factorizations $B' \to E$ of $A \to E$ agree after composition with $A_V \to B'$, by uniqueness for $V$, and hence agree by uniqueness for $V'$. Thus $A \to A_V \to B'$ represents $V'$ in $X$. Conversely suppose that $V'$ is an affinoid domain in $X$. The domain property of $V$ gives a map $A_V \to A_{V'}$ over $A$. A bounded map $A_V \to E$ with spectral image in $V'$ restricts to a map $A \to E$, which factors uniquely through $A_{V'}$. The given map and the composite through $A_{V'}$ are factorizations of $A \to E$ through $A_V$, hence equal. Uniqueness over $A_V$ follows from uniqueness over $A$. The composite $\mathcal{M}(A_{V'}) \to \mathcal{M}(A_V) \to V$ is the homeomorphism onto $V'$, so the first map is a homeomorphism onto the subset of $\mathcal{M}(A_V)$ corresponding to $V'$. Thus $A_V \to A_{V'}$ represents $V'$ in $\mathcal{M}(A_V)$. For $V'' \subset V' \subset V$, both maps $A_V \to A_{V''}$ in the last assertion are factorizations of $A \to A_{V''}$ through $A_V$, hence equal.

For (3), the relation expressing 1 in the ideal generated by $g, f_1, \ldots, f_m$ maps to $A_V$, so the images generate the unit ideal. A point of $\mathcal{M}(A_V)$ satisfies the image inequalities exactly when its image in $X$ lies in $U$.

For (4), let $D, D'$ be analytic domains in $X$ and $x \in D \cap D'$. Choose finitely many affinoid domains in $D$ containing $x$ whose union is a neighborhood of $x$ in $D$, and similarly for $D'$. Their pairwise intersections are affinoid domains by (1). They contain $x$ and lie in $D \cap D'$, and their union is the intersection of the two unions, a neighborhood of $x$ in $D \cap D'$. □

## E.2 Finite rational refinement

**Lemma E.2.1** *Let $A\{r^{-1}T\} \twoheadrightarrow B$ be an admissible presentation over an affinoid algebra $A$, with $r_i > 0$. Let $t_i$ be the image of $T_i$. If $\sup_{\mathcal{M}(B)} |t_i| < r_i$ for every $i$, then the associated morphism of affinoid spaces is finite, with the quotient module topology on $B$.*

***Proof*** Write $P = A\{r^{-1}T\}$ and let $\pi : P \to B$ be the quotient map. By the spectral radius formula Theorem 2.2.1 and admissibility, for each $i$ there are $m_i \geq 1$ and $G_i \in P$ such that $\pi(G_i) = t_i^{m_i}$ and $\|G_i\| < r_i^{m_i}$. Indeed the norms of $t_i^m$ divided by $r_i^m$ tend to zero, and an admissible quotient gives lifts bounded by a fixed multiple of the norm.

Let $P_0$ be the finite free $A$-module of polynomials supported on $0 \leq \alpha_i < m_i$ for all $i$. For $F \in P$, put the terms in this box into $R(F)$. Assign every other term to the first index $i$ with $\alpha_i \geq m_i$ and divide it by $T_i^{m_i}$. The resulting series $Q_i(F)$ satisfy

$$F = R(F) + \sum_i T_i^{m_i} Q_i(F), \qquad \|R(F)\| \leq \|F\|, \qquad r_i^{m_i} \|Q_i(F)\| \leq \|F\|.$$

Define $S(F) = \sum_i G_i Q_i(F)$. Then $\|S(F)\| \leq c\|F\|$ for some $c < 1$ independent of $F$, and $\pi(F) = \pi(R(F) + S(F))$. Iterating gives

$$\pi(F) = \pi\left(\sum_{j \geq 0} R(S^j F)\right).$$

The series converges in $P_0$ and has norm at most $\|F\|$. Thus every element of $B$ has a lift to this finite module, with norm bounded by a fixed multiple of its norm in $B$, again by admissibility. The map $P_0 \to B$ is bounded. These two bounds give the quotient module topology, and the indicated monomials generate $B$ as an $A$-module. □

**Lemma E.2.2** *If an affinoid domain in $\mathcal{M}(A)$ is finite over $\mathcal{M}(A)$, its algebra is $A/(e)$ for some idempotent $e \in A$. The domain is $\{|e| = 0\}$.*

***Proof*** Let $B$ be its algebra. The ordinary tensor product $B \otimes_A B$ is finite over $A$, so it is affinoid with its quotient module topology by Proposition 2.3.1. Every $A$-linear map of finite modules with these topologies is bounded: express it on a finite set of generators and then use the infimum over lifts. In particular the two maps $B \to B \otimes_A B$ are bounded. They agree on $A$, and this common map has spectral image in the domain. Uniqueness in the domain property gives equality of the two maps. It follows that multiplication $B \otimes_A B \to B$ is an isomorphism.

At every maximal ideal $\mathfrak{m}$ of $A$ the multiplication on $B/\mathfrak{m}B$ is then an isomorphism from its tensor square over $A/\mathfrak{m}$. Its dimension $d$ satisfies $d^2 = d$, so the unit generates this fibre if it is nonzero. The cokernel of $A \to B$ is finite and its fibres at maximal ideals vanish. Nakayama's lemma implies that the cokernel vanishes. Write $B = A/I$, with its quotient topology since 1 now generates it as an $A$-module.

The map $A \to A/I^2$ has the same spectral image as $A \to A/I$, since a multiplicative seminorm kills the square-zero ideal $I/I^2$. It therefore factors through $B$, proving $I = I^2$. Write generators of the ideal $I$ as linear combinations of themselves with coefficients in $I$. The determinant of the identity minus that coefficient matrix annihilates $I$ and has the form $1 - e$ with $e \in I$, by the adjugate identity. Consequently $e^2 = e$ and $I = (e)$. Points of $\mathcal{M}(A)$ factor through $A/(e)$ exactly when $|e| = 0$, which proves the assertion. □

**Theorem E.2.1** *Let $A$ be $K$-affinoid and $V$ an affinoid domain of $X = \mathcal{M}(A)$. Every point of $V$ has a relatively open neighborhood in $V$ contained in a rational domain of $X$ lying in $V$. In particular $V$ is a finite union of rational domains of $X$, with arbitrary positive real radii permitted.*

***Proof*** Write $B = A_V$. Choose a presentation $K\{r^{-1}T\} \twoheadrightarrow B$ with its quotient topology and let $b_i$ be the images of its variables. The map $A \to B$ is bounded, and the monomials in $b_i$ have norms bounded by a fixed multiple of their Gauss weights. Thus substitution also gives a bounded map $A\{r^{-1}T\} \to B$. The lifts from $K\{r^{-1}T\}$ already have norms bounded by a fixed multiple of the corresponding norms in $B$. Hence the relative presentation is surjective and admissible. Fix a point $x \in V$ and let $y \in \mathcal{M}(B)$ correspond to it. By Proposition E.1.3 both completed residue fields identify with $\mathcal{H}(x)$. The fractions from $A/\ker(x)$ are dense in that field. Choose $a_i, g \in A$, with the same denominator $g$ nonzero at $x$, such that

$$\left|b_i(y) - a_i(x)/g(x)\right| < r_i \qquad \text{for all } i.$$

The common denominator can be taken to be the product of the individual denominators.

Take $0 < c < |g(x)|$. Because $\mathcal{M}(B) \to V$ is a homeomorphism, there is an open neighborhood of $x$ in $X$ on which $|g| > c$ and the strict error inequalities hold on its intersection with $V$. Choose a rational neighborhood $W$ of $x$ contained in this neighborhood, by the closed neighborhood basis of Section 2.2.

Define the set

$$U = W \cap \{\, |a_i| \le r_i |g| \text{ for every } i, \;\; |1| \le c^{-1} |g| \,\}.$$

This is rational in $X$: the second system includes the numerator 1, and rational domains are closed under finite intersections. On $V \cap W$ the strict error inequalities and $|b_i| \le r_i$ imply $|a_i/g| \le r_i$. Hence $V \cap U = V \cap W$, which contains a relative neighborhood of $x$.

In the algebra $A_W$ the element $g$ is a unit because it vanishes at no spectral point. We may use for $A_U$ the rational presentation over $A_W$ with denominator 1 and numerators $a_i/g$ at radii $r_i$. Indeed this presentation and the ambient rational one represent the same domain by Proposition 2.2.2, Proposition 2.5.1 and the composition of domain inclusions in Proposition E.1.4. The quotient norms in this presentation satisfy $\|a_i/g\|_{A_U} \le r_i$. The inverse image of $U$ in $\mathcal{M}(B)$ is rational, by substituting the images of the defining elements into the inequalities. Its algebra $B_U$ has the affinoid base change presentation from the proof of Proposition E.1.3. It represents $V \cap U$ as a domain in $U$: a map from $A_U$ with spectral image in that intersection factors through $B$ over $A$, and the two factorizations give the required unique map from $B_U$. Its relative presentation over $A_U$ is again admissible.

Translate each relative variable $T_i$ by $a_i/g$. This is a bounded automorphism of the Tate algebra over $A_U$ with bounded inverse, since the substituted elements have norm at most $r_i$. The images of the new generators are $b_i - a_i/g$. They have spectral suprema strictly below $r_i$ on $\mathcal{M}(B_U)$, by the strict error inequalities and compactness of $V \cap W$. Now Lemmas E.2.1 and E.2.2 give an idempotent $e \in A_U$ such that $V \cap U$ is the zero locus of $e$ in $U$.

If $d \in A$ is the denominator of an ambient rational presentation for $U$, then $A[d^{-1}]$ has dense image in $A_U$ by polynomial truncation in the rational-localization variables. Choose $h/d^m$ with $h \in A$, $m \ge 0$, within norm $< 1$ of $1 - e$. On $U$ its absolute value equals 1 on $V \cap U$ and is strictly below 1 on the complement. Put $\delta = \min_{z \in U} |d(z)| > 0$. It follows that

$$V \cap U = U \cap \{\, |d^m| \le |h|, \quad |1| \le \delta^{-m} |h| \,\}.$$

The right-hand side is rational in $X$, since the last system uses denominator $h$ and includes 1 among its numerators. Thus $V \cap W$ is rational in $X$. The relative neighborhoods for all $x \in V$ cover the compact domain $V$, so finitely many of the resulting rational domains suffice. □

## E.3 The structure sheaf

**Theorem E.3.1** *Let $X = \mathcal{M}(A)$ for a $K$-affinoid algebra $A$. The algebras of the affinoid domains of $X$ and their restriction maps determine a unique sheaf of $K$-algebras on the G-topology of analytic domains of $X$. For every finite cover of $X$ by affinoid domains, the augmented Čech complex of the restrictions of each finite $A$-module is exact.*

***Proof*** By Proposition E.1.4, finite intersections of affinoid domains are affinoid domains, the restriction maps for inclusions of affinoid domains compose, and finite intersections of analytic domains are analytic domains.

Consider now a finite affinoid cover $(V_i)_i$ of $X$. Choose a finite ambient rational refinement $(U_j)_j$ by Theorem E.2.1. Let $M$ be a finite $A$-module. Use the double Čech complex of the two finite covers, with terms $M|_{V_{i_0}\cap\cdots\cap V_{i_p}\cap U_{j_0}\cap\cdots\cap U_{j_q}}$. Over each intersection $V'$ of the $V_i$, the $U_j$ restrict to a rational cover of $\mathcal{M}(A_{V'})$, by the third assertion of Proposition E.1.4. By its second assertion, restricting $M|_{V'}$ to the members of this cover and their intersections gives the corresponding restrictions of $M$. The augmented complex of this cover is exact by Proposition E.1.2. Over an intersection of the $U_j$, one of the $V_i$ contains the entire intersection, since the cover by $U_j$ refines the one by $V_i$. The augmented $V_i$-complex there is contractible by inserting that $i$ as in the rational proof. Taking cohomology in the two directions of this finite double complex identifies the cohomology of the two unaugmented Čech complexes, compatibly with the maps from $M$. Since the augmented rational complex is exact, this proves exactness for $(V_i)_i$.

For a G-cover of an affinoid space by affinoid domains, the finitely many domains in each local covering family cover a neighborhood of their point. Compactness gives a finite subcover by such domains. The finite-cover assertion for $M = A$ gives a unique glued section for that subcover. To recover the prescribed section on any other member, apply separation to its finite cover by intersections with the selected subcover. Thus the same equalizer property holds for an arbitrary G-cover by affinoid domains.

On an analytic domain $D$ in $X$, define a section to be a family of elements of $A_W$ for all affinoid domains $W \subset D$, compatible under restriction. This gives $A_D$ on an affinoid $D$, by reading the entry at $D$. Suppose $(D_i)_i$ G-covers $D$ and sections on the $D_i$ agree on their overlaps. For any affinoid $W \subset D$, use the cover of $W$ by all $W \cap B$ with $B$ affinoid in some $D_i$. This is a G-cover: at $x \in W$, take a finite local family of $D_i$ containing $x$, and in each take a finite affinoid family containing $x$ and covering a relative neighborhood there. Intersect the corresponding ambient neighborhoods and the neighborhood of the first family to obtain a neighborhood of $x$ in $W$ covered by these intersections. The matching sections glue uniquely on $W$ by the affinoid equalizer just proved. For $W' \subset W$, restriction to intersections with the same cover gives compatibility by uniqueness on $W'$. The resulting family on $D$ has the given restrictions and is unique. Conversely, any sheaf on the G-topology with these values and restriction maps on affinoid domains consists of such compatible families: an

analytic domain is G-covered by the affinoid domains it contains, and their pairwise intersections are affinoid by Proposition E.1.4. This proves uniqueness. □

The theorem concerns a single affinoid space $X = \mathcal{M}(A)$, which is the setting of this appendix. Gluing these sheaves along an affinoid atlas requires compatibility arguments for the gluing data that this appendix does not carry out, and the book does not use them.

# Appendix F
# Separation on finite Gauss charts

Over the torus with coordinates $t$, $s$, in characteristic different from two, the equations $z_1^2 = t$ and $z_2^2 = t(1+s)$ define a finite cover. Over a Gauss point with $s$-weight $v_s > 0$ it has two points of degree two, at which $z_1$ and $z_2$ have the same norms. The unit $z_1 z_2/t - 1$ distinguishes them (Example F.3.2). This appendix constructs such units on the finite free charts of Theorem 9.4.1 in the cases listed below. Together with the moment coordinates, their log norms embed the Gauss preimage of a chart near its central point, with the weighted polyhedral description of Lemma 9.1.2.

Throughout, $k$ is trivially valued and $f : Y \to T_Q$ is a moment of an affinoid space of dimension at most $n$ into an $n$-dimensional annular box, as in Chapter 9. The analytic statements concern a one-point finite free chart at a relative interior point over a Gauss point. The algebraic statements concern finite extensions of the rational torus field $F = \mathrm{Frac}(k[M])$.

For a moment of an affinoid space of dimension at most one, the Gauss preimage is locally an interval or a finite star, embedded by one further unit (Theorem F.1.1). In higher dimension, powering seminorms makes the Gauss set a local product away from weight zero (Proposition F.2.1). Near a point over weight zero, the Gauss preimage of finitely many rays through zero is again a star (Theorem F.2.1). On Kummer and Laurent Artin–Schreier charts, tests on rational approximations of the coefficients count the points and give separating units (Theorem F.3.1 and Proposition F.3.1).

For a general chart, the separable factor of Proposition F.4.1 reduces separation to the prolongations of the Gauss values to a finite separable extension of $F$ (Corollary F.4.2). These are separated when the Galois closure has degree invertible in $k$ (Theorem F.5.1), for Artin–Schreier extensions (Theorem F.5.2), and for abelian Galois groups and direct products of an abelian $p$-group with a group of order prime to $p$ (Theorem F.6.1 and Corollary F.6.1). In characteristic zero every chart therefore has a finite separating family of units. In positive characteristic so does a chart at a point of degree at most two (Corollary F.5.1). Other Galois groups of order divisible by the characteristic are not treated. The main text uses only the power test of Lemma F.3.1, in an example of Section 11.11.

## F.1 Local skeleta of one-dimensional moments

Let $k$ be trivially valued, let $t$ be a split torus coordinate and let $Q \subset \mathbb{R}$ be a closed interval with nonempty interior. A moment $f : Y \to T_Q$ from an affinoid space of dimension at most one has a finite fibre over each Gauss point. We describe the local topology of the union of these fibres at relative-interior points and find analytic coordinates that distinguish its branches. The local degrees are those of (9.4), which are invariant under chart restriction by Proposition 9.4.1.

**Theorem F.1.1** *Let $y$ be a relative-interior point of $f$ above $\eta_w$, with $w \in Q^\circ$. There is a finite free chart $V \to U$ at $y$ with a single point over $\eta_w$. On a sufficiently small closed interval $I \subset Q^\circ$ about $w$, the Gauss section lies in $U$, and its preimage in $V$ has the following description.*

(1) *If $w \neq 0$, projection to $I$ is a homeomorphism and the local degree is constant.*
(2) *If $w = 0$, the preimage is a finite union of closed intervals with a single common endpoint $y$ over zero. Each interval projects homeomorphically to one of the two halves of $I$. The local degree is constant on each open branch, and the sum of the degrees on either side equals $d_f(y)$.*

*After shrinking the chart if needed, two analytic units, the pullback of $t$ and one further unit, give a joint log norm map that embeds this preimage. The log norm of any analytic unit of the chart is affine with rational slope on each branch in its projected weight coordinate.*

***Proof*** If $w \neq 0$, then $w$, as a single weight, is linearly independent over $\mathbb{Q}$. By Corollary 9.2.1 there is a finite free chart over a small annular interval of the same sign. Write $t_a = t$ on the positive side and $t_a = t^{-1}$ on the negative side. The annular algebra is $k((t_a))$ as a ring. In fact all norms of $t_a$ on this interval are bounded above by a number strictly less than one and below by a positive number. The Laurent series with coefficients in $k$ that converge uniformly are exactly the series with finitely many negative powers. At any Gauss point of the interval, the completed residue field is this same field with $|t_a| = \exp(-|v|)$, where $v \in I$.

The finite source algebra over $k((t_a))$ is local Artinian, since the fibre at $w$ has just one point. The fibre algebra at any $v \in I$ is this same algebra over the field with the rescaled norm. On its residue field the norm at $v$ is the $|v|/|w|$-th power of the norm at $w$, by uniqueness of finite extension of the absolute value. There is one point over each such $v$ with the same degree. Evaluation of each element is continuous in $v$, so the section is continuous. For a nonzero residue value, the normalized log norm for $|t_a| = e^{-1}$ is rational. In a nonzero polynomial relation over $k((t_a))$, at least two nonzero terms attain the same largest norm, and the coefficient norms have integer logarithms. Equating these two terms gives the rationality. The norm of a unit therefore has affine logarithm in $v$, and $t$ alone distinguishes the points. We can take the other unit to be 1.

Suppose $w = 0$. Take a finite free chart at $y$ by Theorem 9.4.1. Restrict its base to a Laurent neighborhood $U$ in the original annulus with all bounds strict at $\eta_0$. The

source over $U$ remains finite free. By Lemma 9.4.3, the base algebra $A = O(U)$ is a localization of $k[t, t^{-1}]$.

Put $B = O(V)$ and $F = k(t)$. The completed residue field at $\eta_0$ is $F$ with the trivial absolute value. Thus $B_F = B \otimes_A F$ is local Artinian, by the one-point construction. Write $\mathfrak{m}$ for its maximal ideal, $E$ for its residue field, and $\ell$ for its length as a module over itself. In particular

$$d_f(y) = \dim_F B_F = \ell[E : F]. \tag{F.1}$$

Indeed the successive quotients by powers of $\mathfrak{m}$ are $E$-vector spaces, and $\ell$ is the sum of their dimensions over $E$.

Let $C$ be the integral closure of $\mathbb{P}^1_k$ in $E$. The map $C \to \mathbb{P}^1_k$ is finite. To see this on either affine chart with coordinate $t_a$, clear denominators in monic equations of field generators of $E/F$. After multiplication by nonzero polynomials they are integral over $k[t_a]$ and generate a finite domain with fraction field $E$. Its normalization is finite by [Stacks, Tag 0BXR]. By transitivity of integrality this normalization is also the integral closure of $k[t_a]$ in $E$. The two algebras glue because integral closure commutes with localization, by the same denominator argument in monic equations. The resulting $C$ is a normal integral curve, since it is finite dominant over $\mathbb{P}^1_k$.

Continuity of the Gauss section gives an interval $I$ about zero with image in $U$. For $v > 0$ in $I$ put $a = 0$, and for $v < 0$ put $a = \infty$. The completed residue field at $\eta_v$ is $F_a = k((t_a))$ with the absolute value $|t_a| = \exp(-|v|)$. The fibre algebra is the algebraic tensor product $B_F \otimes_F F_a$ for the field embedding $F \to F_a$. Modulo the extended nilpotent ideal $\mathfrak{m}(B_F \otimes_F F_a)$, it is the product of the fields $E_P$ from (8.2). Thus its points $y_{P,v}$ are indexed by $P \mapsto a$. The dimensions of the local factors are

$$d_f(y_{P,v}) = \ell\, e_P [\kappa(P) : k]. \tag{F.2}$$

Lemma 9.1.1 gives the degree as $\ell[E_P : F_a]$. By (8.2), the sum of the field degrees over either $a$ is $[E : F]$. The corresponding sums of local degrees are both (F.1).

For each $b \in B$, let $\bar{b}$ be its image in $E$. If $\bar{b} = 0$, the seminorm values are zero on every indicated fibre. Otherwise

$$|b(y_{P,v})| = \exp\bigl(-|v| \operatorname{ord}_P(\bar{b})/e_P\bigr). \tag{F.3}$$

This follows from the normalized field norm in Lemma 8.7.1. It is continuous for $v$ of the indicated sign and tends to $1 = |b(y)|$ as $v \to 0$. This proves continuity of every branch with its endpoint $y$ in the spectrum topology. Different branches are disjoint off zero by the fibre decomposition. The space obtained by gluing their intervals at the common endpoint is compact. Its continuous bijection to the Gauss preimage in the Hausdorff space $V$ is a homeomorphism. For a unit, (F.3) gives the asserted affine log norm.

It remains to choose a unit separating the branches. In the affine preimage of $\mathbb{P}^1_k \setminus \{1\}$ in $C$, all the points above 0 and $\infty$ are represented by distinct maximal ideals. For every such $P$, choose a function $c_P$ of order one at $P$ and order zero at all the other points of this finite set. The Chinese remainder theorem gives one by

prescribing a nonzero class in $\mathfrak{m}_P/\mathfrak{m}_P^2$ and the residue 1 at the other points. The prescribed class exists because the localization at $P$ is a discrete valuation ring. Take distinct positive integers $N_P$ and put $c = \prod_P c_P^{e_P N_P} \in E^\times$. Then $\mathrm{ord}_P(c)/e_P = N_P$. Lift $c$ to $B_F$ and clear its denominators by an element $h \in A \setminus \{0\}$ to obtain $b \in B$ with $\bar{b} = hc$. The normalized orders of $h$ are $\mathrm{ord}_a(h)$, common to the points above each $a$. The resulting slopes for $b$ remain distinct on each side of zero.

Since $|b(y)| = 1$ and $y$ is the sole lift over zero in $V$, the compact zero locus of $b$ has image avoiding $\eta_0$. Restrict to a Laurent base neighborhood avoiding that image and shrink $I$ within it. Then $b$ is a unit by Theorem 2.2.1. The smaller Laurent base is again algebraic as above and has the same fraction field, so the formulas also apply to every unit after restriction. The first coordinate $t$ distinguishes different weights. By (F.3) the second coordinate $b$ distinguishes points over each nonzero weight. The endpoint over zero is unique. Compactness of the interval union gives a topological embedding by the two log norm coordinates. □

For the quadratic cover in Example 9.1.1, the two branches on the positive side have degree one and the one on the negative side has degree two. They form a three-edge star at the point over zero, and the log norms of $t, z$ distinguish its points by the calculation in that example. Adjoining a variable $\varepsilon$ with $\varepsilon^r = 0$ for an integer $r \geq 1$ leaves this star unchanged and multiplies its degrees by $r$. Each fibre factor is tensored with the rank-$r$ algebra with basis $1, \varepsilon, \ldots, \varepsilon^{r-1}$, and its residue field is unchanged.

### F.1.1 Slices over rational lines

**Corollary F.1.1** *Let $f : Y \to T_Q$ be a moment of an affinoid space of dimension at most $n$ into an $n$-dimensional annular box over $k$. Let $u$ be a primitive vector of the cocharacter lattice. At a point $y$ in the relative interior of $f$ above a weight $w = \lambda_0 u \in Q^\circ$, restrict the Gauss preimage to the line $\mathbb{R}u$. Locally at $y$ this restricted preimage has the interval or star description of Theorem F.1.1, in the coordinate $\lambda$ on the line. Its degrees are those of $f$. The separating units are analytic on the corresponding one-dimensional affinoid slice over a trivially valued extension of $k$.*

***Proof*** Choose a basis $m_1, \ldots, m_n$ of the character lattice with $\langle m_1, u\rangle = 1$ and $\langle m_i, u\rangle = 0$ for $i > 1$. This is possible since $u$ is primitive. Put $t_i = T^{m_i}$. Project to the last $n-1$ coordinates, with target a box containing their norm image and zero. Over its Gauss point at weight zero the completed residue field is $K' = k(t_2, \ldots, t_n)$ with its trivial norm. Indeed its annular algebra is the Laurent polynomial ring, since all presentation radii are at least one. For $n = 1$ this projection has target $\mathcal{M}(k)$.

The fibre of the projection on $T_Q$ is the $K'$-annulus with interval $J = \{\lambda : \lambda u \in Q\}$. In the relative presentation of this fibre, the $t_i$ for $i > 1$ are now constants of norm one. Each original character coordinate becomes $\gamma_j t_1^{u_j}$ with $\gamma_j \in K'^\times$ and $|\gamma_j| = 1$. The original bounds on the character coordinates and their inverses consequently impose exactly the bounds described by $J$ on $t_1$. These are the monomial inequalities

of Section 8.4. To check the identification on algebras, on a $K'$-affinoid test algebra either set of bounds implies the other on its spectrum, after the monomial substitution. Proposition 2.5.1 makes those bounds sufficient for bounded substitution. The inequality-domain property for the presentations then gives inverse maps, since they are inverse on the monomial generators.

The fibre $Y'$ of the composite projection on $Y$ is $K'$-affinoid. By Proposition 9.3.1 it has dimension at most one, since the projection base point has invariant $d_k = n-1$. An admissible relative presentation for $Y$ over $T_Q$ with strict bounds at $y$ gives, by base change, one for $Y'$ over $T_J$ with the same bounds at its lifted point. This point is in the relative interior of the slice map. The $\lambda$-Gauss norm over $K'$ restricts to the $\lambda u$-Gauss norm over $k$, by collecting the Laurent terms with a fixed $t_1$-power. The two completed residue fields are the same, because the common dense field is $k(t_1, \dots, t_n)$ with this norm. The relative fibre presentations of $f$ and of the slice map at these points agree by imposing the same scalar relations and the same relative equations. Their full local factors, and hence their degrees, are therefore equal.

The homeomorphism with the topological fibre in Proposition 9.3.1 identifies the Gauss preimage of the slice with the restricted subset of $Y$. Apply Theorem F.1.1 over $K'$, with the one-dimensional weight coordinate $\lambda$. This gives the indicated description and the analytic units on the slice. □

## F.2 Radial directions of a Gauss set

Let $f : Y \to T_Q$ be a moment from an affinoid space of dimension at most $n$ into a full-dimensional box of an $n$-dimensional split torus over a trivially valued field $k$. Write $S$ for its Gauss set, and $w(z) = f_{\mathrm{trop}}(z)$. We work at points $y \in S$ in the relative interior of $f$, with $w(y) \in Q^\circ$.

### F.2.1 Local products away from zero

**Proposition F.2.1** *Suppose $w(y) \neq 0$. Let $\gamma$ be a real linear functional with $\gamma(w(y)) = 1$. There is a relatively open neighborhood $D$ of $y$ in $S \cap \{\gamma(w) = 1\}$ and an open interval $J \subset \mathbb{R}_{>0}$ about $1$ such that positive powering of seminorms gives a homeomorphism*

$$D \times J \longrightarrow S, \qquad (z, s) \longmapsto z^s$$

*onto a relatively open neighborhood of $y$. It satisfies $w(z^s) = sw(z)$, and $d_f(z^s)$ is constant in $s$ for each $z \in D$.*

***Proof*** Use an admissible presentation over the annular algebra with strict generator bounds at $y$. Combining it with the annular presentation gives a Tate presentation of $O(Y)$ with all bounds strict there. Indeed the annular coordinate and inverse-coordinate bounds are strict since $w(y) \in Q^\circ$. The coefficient lifts in Theorem 2.1.1

give the quotient topology in the combined presentation. Let $W \subset Y$ be the open set where all these bounds are strict. Its points are in the relative interior of $f$.

Write $b_i$ for the combined generators and $r_i$ for their radii. On the open set of pairs

$$(z, s) \in W \times \mathbb{R}_{>0}, \qquad |b_i(z)|^s < r_i \quad \text{for all } i,$$

the rule $|b(z^s)| = |b(z)|^s$ defines a point of $W$. On polynomials the inequalities give the Gauss norm bound, because the norm of each ground-field coefficient is unchanged by the positive power. For a series use its polynomial truncations. Their evaluations at $z$ converge in $\mathcal{H}(z)$, so the norms raised to $s$ satisfy the same bound in the limit. Taking the infimum over lifts gives a bounded seminorm on the quotient algebra. The rule is jointly continuous, by evaluation on each $b \in \mathcal{O}(Y)$. Where successive powers are defined, they multiply, by the same evaluation formula. On the Gauss set the target seminorm is the Gauss norm at $sw(z)$, first by the Laurent formula on polynomials and then by density.

Put $H(z) = \gamma(w(z))$. Choose an ordinary open neighborhood $W_0 \subset W$ of $y$ and an interval $J$ about 1 so that the powers are defined on $W_0 \times J$. Put $D = S \cap W_0 \cap \{H = 1\}$. The image of $D \times J$ consists of the points $x \in S \cap W$ with $H(x) \in J$ for which $x^{1/H(x)}$ is defined in $W$ and belongs to $W_0$. These conditions are open in $S \cap W$, by continuity and the strict bounds. The inverse has coordinates $x^{1/H(x)}$ and $H(x)$, hence is continuous.

Fix $z \in D$ and $s_0 \in J$. Choose a finite free chart $V \to U$ with a single point $z^{s_0}$ over $\eta_{s_0 w(z)}$, by Theorem 9.4.1. For $s$ near $s_0$ the point $z^s$ lies in $V$, and $\eta_{sw(z)}$ lies in $U$. All the lifts in $V$ of that Gauss point lie in any prescribed ordinary neighborhood of $z^{s_0}$ when $s$ is sufficiently close: the complement in the compact space $V$ has compact image avoiding the starting point. Choose this neighborhood so that each lift can be powered by $s_0/s$ and stays in $V$ after powering. This is possible by joint continuity and the ordinary neighborhood property of $V$ at $z^{s_0}$. The powered point lies over $\eta_{s_0 w(z)}$, so it is $z^{s_0}$; powering back shows that the original lift is $z^s$. Its degree is therefore the free rank of the chart, by Proposition 9.4.1. The degree is locally constant in $s$, hence constant on $J$. □

**Corollary F.2.1** *Choose a primitive character $m$ with $a = \langle m, w(y)\rangle \neq 0$. For $\gamma = \langle m, \cdot\rangle/a$ in Proposition F.2.1, the transversal is the Gauss set, locally at $y$, of a moment on an affinoid fibre of dimension at most $n-1$ over $\mathcal{H}(\eta_a)$. The fibre degrees on the transversal agree with those of $f$.*

***Proof*** Project the box by the character $t = T^m$ to a one-dimensional annular box containing its image. The completed residue field $K' = \mathcal{H}(\eta_a)$ has nontrivial absolute value with $|t| = \exp(-a)$. The fibre $Y'$ of the composite projection at $\eta_a$ is $K'$-affinoid of dimension at most $n-1$, by Proposition 9.3.1 and $d_k(\eta_a) = 1$.

Complete $m$ to a basis of the character lattice. In the corresponding target fibre, $t$ is now a coefficient in $K'$. The original annular bounds on the characters and their inverses become monomial inequalities on the remaining $n-1$ coordinates, with constants determined by $|t|$. Their weight polytope is the affine slice $\langle m, w\rangle = a$ of $Q$, in these coordinates. The identification on algebras is by bounded substitution

in the relative presentation, as in the proof of Corollary F.1.1. The strict relative presentation at $y$ also base changes.

The Gauss points of this monomial domain over $K'$ restrict to precisely the original Gauss points on the affine slice. On Laurent polynomials this follows by grouping the powers of the other characters and taking the $t$-Gauss norms of the resulting coefficients. The completed residue fields are the same because the rational functions over $k$ are dense for the resulting Gauss norm, by approximation of the coefficients in $K'$. Thus both relative fibre constructions impose the same equations with the same completed residue coefficients, giving the same local factors and degrees. The homeomorphism with the topological fibre in Proposition 9.3.1 identifies the Gauss set of $Y'$ with the transversal subset of $S$. □

### F.2.2 Finitely many rays through zero

**Theorem F.2.1** *Suppose $w(y) = 0$, and fix a nonempty finite list of nonzero directions $u_1, \dots, u_q$ in the real cocharacter space, on distinct half-lines through zero. For a one-point finite free chart $V \to U$ at $y$, as in Theorem 9.4.1, and a sufficiently small $\varepsilon > 0$, the preimage in $V$ of the Gauss section over $\bigcup_i [0, \varepsilon] u_i$ is a finite union of intervals with the single common endpoint $y$. Each interval projects homeomorphically to one of the prescribed half-segments. Its degree is constant off $y$, and the sum of the degrees over each open half-segment is $d_f(y)$. Every unit on the chart has linear log norm on each interval in the parameter $s$ of $su_i$. After shrinking the chart and $\varepsilon$, one additional analytic unit, together with the original moment coordinates, embeds the whole finite union.*

***Proof*** Restrict the target $U$ of the given chart to a Laurent neighborhood with strict bounds at $\eta_0$, and $V$ to its preimage. For small $\varepsilon$ this does not change the preimage of the Gauss section. By Lemma 9.4.3 the algebra $A$ of the new target is a localization of $k[M]$. Put $B = \mathcal{O}(V)$, $F = \mathrm{Frac}(k[M])$, and $D_0 = B \otimes_A F$. The field $\mathcal{H}(\eta_0)$ is $F$ with its trivial norm. Thus $D_0$ is local Artinian. Write $E$ for its residue field and $\ell$ for its length.

For each $i$, let $L_i$ be the completion of $F$ for the Gauss norm of weight $u_i$. For $s > 0$ the Gauss norm of weight $su_i$ is the $s$-th power of that norm. Their Cauchy sequences are the same. Thus for all $s > 0$ on the segment the fibre algebra is the fixed algebraic tensor $D_0 \otimes_F L_i$, with the corresponding powered absolute value on its base field. The interval can be taken short enough that all its target Gauss points belong to $U$, since their section is continuous. The points of a fibre are the maximal ideals of its Artinian algebra. They are therefore indexed by the local factors $E_{i,j}$ of $E \otimes_F L_i$. By Lemma 9.1.1, their degrees are

$$\ell \dim_{L_i} E_{i,j}. \tag{F.4}$$

Their sum over $j$ is $\ell[E : F] = d_f(y)$, using Proposition 9.4.1. This argument also covers a nonreduced $E \otimes_F L_i$.

Let $|\cdot|_{i,j}$ be the extending absolute value on the residue field of $E_{i,j}$ for $s = 1$, and write $\bar{b}$ for the image of $b \in B$ in $E$. At the point $y_{i,j}(s)$ of the indicated branch,

$$|b(y_{i,j}(s))| = |\bar{b}|_{i,j}^{s}. \tag{F.5}$$

This follows from uniqueness of extension of the complete absolute value to a finite field extension. If $\bar{b} \neq 0$, its image in every residue field is nonzero, so the expression tends to $1 = |b(y)|$ as $s \downarrow 0$. If $\bar{b} = 0$ it is zero throughout, including at $y$. Thus each closed interval maps continuously to $V$, and the intervals meet only at $y$ by the fibre descriptions and the distinct half-lines. The induced map from their compact union with common endpoint to the Gauss preimage is a continuous bijection into a Hausdorff space, hence a homeomorphism. Formula (F.5) gives the log norms on each interval.

For each $i$, the diagonal image of $E$ is dense in $E \otimes_F L_i$ in the topology of the finite-dimensional $L_i$-algebra: take an $F$-basis of $E$ and use density of $F$ for each coefficient. Distinct spectral points are continuous seminorms and so differ on some element of $E^\times$. Choose a list $c_1, \dots, c_g \in E^\times$ that detects every pair of distinct local factors over each of the finitely many $i$. For each such pair, equality of the log norms of $\prod_l c_l^{e_l}$ imposes a proper linear hyperplane on $(e_l) \in \mathbb{R}^g$. There is an integer vector avoiding all these hyperplanes: $(1, N, \dots, N^{g-1})$ lies in each only for finitely many $N$. With these exponents the product $c$ separates all the specified pairs. If there are no pairs take $c = 1$.

Lift $c$ to $D_0$ and clear denominators with $h \in A \setminus \{0\}$ to obtain $b \in B$ with $\bar{b} = hc$. Multiplication by $h$ changes all log norms over the same Gauss point by the same amount. Since $b(y)$ has norm one and $y$ is the only point over zero, the compact zero locus of $b$ has image avoiding $\eta_0$. Restrict the target to a smaller Laurent neighborhood disjoint from that image and shorten the half-segments. The element $b$ is a unit there by Theorem 2.2.1. The new target is again a localization of $k[M]$ and the generic algebra is unchanged. Thus the norm formulas apply on this smaller chart. The original moment coordinates distinguish different weights. At the same nonzero weight the unit $b$ distinguishes different lifts, by (F.5). This proves the embedding by compactness. □

## F.3 Separating lifts of finite Gauss fibres

A finite free chart can have several Gauss lifts of the same weight, and their number can change as the weight varies. For a Kummer chart, initial Laurent fractions determine the number of lifts. When the chart has several equations, products of powers of its coordinates distinguish lifts that the individual equations cannot detect (Theorem F.3.1). There is a corresponding calculation using sums of coordinates on Laurent Artin–Schreier charts (Proposition F.3.1).

### F.3.1 Powers in Gauss fields

Write $F = \mathrm{Frac}(k[M])$, where $M$ is the character lattice of the torus. For a real weight $v$, extend the initial part from Laurent polynomials to $F^\times$ by

$$\mathrm{in}_v(P_1/P_2) = \mathrm{in}_v(P_1)/\mathrm{in}_v(P_2) \ \in F^\times.$$

Multiplication of initial parts in Section 8.2 shows that this expression is independent of the fraction presentation and multiplicative. The numerator of $R - \mathrm{in}_v(R)$ in a common denominator has its lowest terms canceled.

Consequently, in the Gauss norm,

$$|R - \mathrm{in}_v(R)|_v < |R|_v, \qquad |\mathrm{in}_v(R)|_v = |R|_v. \tag{F.6}$$

For the same reason, if $|R - S|_v < |R|_v$, then $\mathrm{in}_v(R) = \mathrm{in}_v(S)$: after multiplying by a common denominator the two numerators have the same lowest part.

**Lemma F.3.1** *Let $q \geq 2$ be invertible in $k$, let $H = \mathcal{H}(\eta_v)$, and let $a \in H^\times$. Suppose $R \in F^\times$ satisfies $|a - R| < |R|$. Then $a$ is a $q$-th power in $H$ if and only if $\mathrm{in}_v(R)$ is a $q$-th power in $F$. For each root $b \in H$ of $b^q = a$, there is a root $c \in F$ of $c^q = \mathrm{in}_v(R)$ with $|b - c| < |b|$.*

***Proof*** If $b^q = a$, choose $c' \in F$ with $|b - c'| < |b|$, using density in $H$. The difference of the $q$-th powers is bounded by $|b - c'||b|^{q-1}$, so $|(c')^q - R| < |R|$. Multiplicativity of initial parts gives $c^q = \mathrm{in}_v(R)$ for $c = \mathrm{in}_v(c')$. By (F.6), $|c' - c| < |c'| = |b|$, which gives the assertion about $|b - c|$.

Conversely let $c^q = \mathrm{in}_v(R)$. By (F.6), $h = a/c^q - 1$ has norm less than one. To solve $(1 + x)^q = 1 + h$, iterate the map

$$x \longmapsto \frac{1}{q}\left(h - \sum_{j=2}^{q} \binom{q}{j} x^j\right)$$

on the closed ball $|x| \leq |h|$. The coefficient $q$ is a unit of absolute value one, and the binomial coefficients have norm at most one. The map preserves the ball. The difference of its values at $x, x'$ has norm at most $|h||x - x'|$, by factoring the differences of powers. Thus iteration converges to a fixed point by the completeness of $H$. Then $c(1 + x)$ is a $q$-th root of $a$. □

### F.3.2 Kummer charts

**Theorem F.3.1** *Let $V \to U$ be a finite free chart at $y$ with one point over $\eta_w$, as in Theorem 9.4.1. Suppose its algebra over $A = \mathcal{O}(U)$ is*

$$A[z_1, \dots, z_s]/(z_i^{m_i} - a_i : 1 \le i \le s),$$

*where each $m_i \ge 2$ is invertible in $k$ and $\mu_{m_i}(k)$ has order $m_i$. After shrinking, a fixed finite family of analytic units, together with the original moment coordinates, embeds the Gauss preimage over a closed weight box about $w$ by joint log norms. Its image has the weighted polyhedral description of Lemma 9.1.2.*

*Let $I$ consist of the indices with $a_i(\eta_w) \neq 0$. Choose strict rational approximations $R_i$ to $a_i$, $i \in I$, as in Lemma 9.4.5. For each tuple $e = (e_i)_{i \in I}$ with $0 \le e_i < m_i$, let $q_e$ be the order of the character $(\zeta_i) \mapsto \prod_i \zeta_i^{e_i}$ on $\prod_{i \in I} \mu_{m_i}(k)$. Define*

$$R_e = \prod_{i \in I} R_i^{q_e e_i / m_i}, \qquad E_v = \{e : \mathrm{in}_v(R_e) \in F^{q_e}\}. \tag{F.7}$$

*At each weight $v$ of the box there are $|E_v|$ lifts, each of degree $\left(\prod_{i=1}^{s} m_i\right)/|E_v|$. These numbers are constant on the relative interiors of a finite polyhedral subdivision.*

***Proof*** After shrinking, all $a_i$ with $i \notin I$ are zero, and $|a_i - R_i| < |R_i|$ for each $i \in I$, by Lemma 9.4.5. Over any completed Gauss field $H = \mathcal{H}(\eta_v)$, the variables outside $I$ are nilpotent. Adjoining them leaves the points unchanged and multiplies all local algebra dimensions by $\prod_{i \notin I} m_i$. We therefore first consider the equations indexed by $I$.

Write $\Gamma = \prod_{i \in I} \mu_{m_i}(k)$. Choose a tuple $r = (r_i)_{i \in I}$ of roots in an algebraic closure of $H$, and put $L = H(r_i : i \in I)$. Each polynomial is separable and splits in $L$, so $L/H$ is finite Galois. Its group embeds in $\Gamma$ by $\sigma \mapsto (\sigma(r_i)/r_i)_i$. Denote the image by $D$. The geometric points are the tuples $\gamma r$ for $\gamma \in \Gamma$; their Galois orbits are the cosets of $D$. Indeed evaluation at any such tuple has image $L$, and two evaluations have the same kernel precisely when related by an $H$-automorphism of $L$. The algebra is reduced since it becomes a product of fields over $L$ by the Chinese remainder theorem. Its local factors are thus copies of $L$, each of degree $|D|$.

Write $z^e = \prod_{i \in I} z_i^{e_i}$. The value of $z^e$ at $r$ belongs to $H$ exactly when the character indexed by $e$ is trivial on $D$. Its $q_e$-th power is $a_e = \prod_{i \in I} a_i^{q_e e_i / m_i}$. All $q_e$-th roots of unity belong to $k$: the character takes values in the roots of unity and its image has order $q_e$. Hence $z^e(r) \in H$ if and only if $a_e \in H^{q_e}$. In fact, if $a_e$ has a root in $H$, its ratio with $z^e(r)$ is a root of unity in $H$. The strict inequalities for the $R_i$ give $|a_e - R_e| < |R_e|$, by multiplying the factors $a_i/R_i$, each strictly within one of 1. For $q_e > 1$ we can apply Lemma F.3.1; the index $e = 0$ satisfies the power condition with $q_e = 1$. Thus $E_v$ is exactly the set of characters trivial on $D$.

Over $L$, the coordinate algebra of the root tuples has the basis $z^e$ for all permitted $e$. The action of $D$ by multiplying the coordinates, fixing $L$, scales each basis element by its character. Its invariant subspace has the basis indexed by $E_v$. Under the identification with functions on $\Gamma$, these are the functions constant on each $D$-

coset. They separate distinct cosets, and their dimension is $[\Gamma : D]$. Consequently $|E_v| = [\Gamma : D]$, and some $z^e$ with $e \in E_v$ has different values on any two given cosets. This proves the point count, and adjoining the omitted nilpotents gives degree $\left(\prod_{i=1}^{s} m_i\right)/|E_v|$. By multiplicativity of initial parts the test (F.7) uses only the initial numerators and denominators of the $R_i$. Their monomial comparison hyperplanes give the asserted finite subdivision, including its lower-dimensional cells.

For each $e \neq 0$ and each possible $\mathrm{in}_v(R_e)$, take every $q_e$-th root $c$ of that fraction in $F$. There are finitely many. On a common target restriction their denominators are invertible, since every nonzero rational function has nonzero finite Gauss norm at $\eta_w$. Consider the functions $z^e - c$ on the source. At $w$ the one-point hypothesis gives $D = \Gamma$, so no nontrivial character has its value in $H$. Thus each candidate is nonzero at the central point. On an affinoid restriction its zero locus is compact. The images of all these zero loci avoid $\eta_w$. Shrinking the target to an affinoid neighborhood avoiding their union makes every candidate an analytic unit.

At any weight in that neighborhood, choose $e \in E_v$ distinguishing two cosets, and let $\beta, \beta' \in H$ be the corresponding distinct values of $z^e$. They differ by a root-of-unity factor, so $|\beta' - \beta| = |\beta|$. By Lemma F.3.1 some candidate $c$ has $|\beta - c| < |\beta|$. Then $|\beta' - c| = |\beta|$, which proves that the norm of $z^e - c$ separates the two lifts. The original coordinates separate different weights. A closed box in this weight neighborhood has compact Gauss preimage, so the continuous injective log norm map is an embedding there. The graph and multiplicity assertion now follows from Corollary 9.4.1, after a further restriction of the box if needed. □

**Corollary F.3.1** *For a single equation $z^m = a$ as in Theorem F.3.1, suppose $a$ is nonzero at the central Gauss point and let $R$ be its strict rational approximation. The number of lifts at $v$ is*

$$b(v) = \max\{q : q \mid m,\ \mathrm{in}_v(R) \in F^q\}, \tag{F.8}$$

*and their degree is $m/b(v)$.*

***Proof*** A subgroup of $\mu_m(k)$ of order $d$ equals $\mu_d(k)$, since all its elements solve $X^d = 1$. For a divisor $q \mid m$ the character $\zeta \mapsto \zeta^{m/q}$ is trivial on that subgroup exactly when $d \mid m/q$. Its power test in (F.7) is $\mathrm{in}_v(R) \in F^q$. The largest such $q$ is $m/d$, the number of cosets. The degree then follows from the theorem. □

**Corollary F.3.2** *Let $y$ be a relative-interior Gauss fibre point of a moment as in Theorem 9.4.1. Assume $d_f(y) \le 2$. If $d_f(y) = 2$ and $\operatorname{char} k = 2$, suppose that the fibre algebra at $y$ has just one point after algebraic closure of its scalar field. Then there is a finite family of analytic units on a chart at $y$ whose joint log norms with the moment coordinates embed its Gauss preimage over a neighborhood of weights. The image has the weighted polyhedral description of Theorem F.3.1.*

***Proof*** A one-point finite free chart has rank $d_f(y)$. In rank one every fibre has a single point, and the moment already gives the required embedding. In rank two, choose an element $b$ of the source algebra so that $(1, b)$ is a basis of the central fibre over $\mathcal{H}(\eta_w)$. Such an element can be taken from a basis of the free module, since 1 is nonzero in the fibre. The change-of-basis determinant is nonzero there. Inverting it on an affinoid neighborhood makes $(1, b)$ a basis throughout, with relation $b^2 - cb + d = 0$ over $A$. The map from $A[X]/(X^2 - cX + d)$ to the given algebra sending $X$ to $b$ is an isomorphism, since it maps a basis to a basis.

If two is invertible, take $z = b - c/2$. The resulting equation is $z^2 = c^2/4 - d$, to which Theorem F.3.1 applies. If the characteristic is two and the central fibre has one geometric point, its quadratic polynomial has a double root over the algebraic closure. This forces $c(\eta_w) = 0$. By Lemma 9.4.5, $c$ is identically zero on a smaller chart. Each Gauss fibre then has the equation $b^2 = -d$ and a single geometric point, hence just one spectral point of degree two. The original coordinates embed the Gauss preimage as in rank one, with constant degree two. □

### F.3.3 Laurent Artin–Schreier charts

Suppose $k$ has characteristic $p > 0$. Write $\wp(x) = x^p - x$.

**Lemma F.3.2** *Every $P \in k[M]$ has a decomposition*

$$P = \wp(C) + P_0, \qquad P_0 = c_0 + \sum_{u \neq 0} c_u T^u, \tag{F.9}$$

*where $C \in k[M]$, and no nonzero term of $P_0$ has both nonzero exponent in $pM$ and coefficient in $k^p$. In the completed Gauss field at $v$, $P$ is in the image of $\wp$ exactly when $c_0 \in \wp(k)$ and every nonconstant term of $P_0$ has strictly positive $v$-weight.*

***Proof*** Subtract $\wp(bT^u)$ to remove a term $b^p T^{pu}$ with $u \neq 0$. The only exponent introduced is $u$, with smaller divisibility by $p$. All exponents encountered are among the original exponents and their successive divisions by $p$ in $M$. Processing this finite set in decreasing order of divisibility gives (F.9).

Put $H = \mathcal{H}(\eta_v)$. If $P_0$ has norm greater than one, an element $b \in H$ with $\wp(b) = P_0$ would have norm greater than one. Choose $r \in F$ with $|r - b| < |b|$. In characteristic $p$ this gives

$$|\wp(r) - P_0| \le \max\{|r - b|^p, |r - b|\} < |b|^p = |P_0|.$$

The initial part of $\wp(r)$ is $\mathrm{in}_v(r)^p$, because $|r|^p > |r|$. Hence $\mathrm{in}_v(P_0)$ would be a $p$-th power in $F$. A Laurent polynomial with a $p$-th root in $F$ has a Laurent polynomial root by integral closedness of $k[M]$. Its $p$-th power has all exponents in $pM$ and coefficients in $k^p$. The lowest terms of $P_0$ here have negative weight and therefore nonzero exponent, contradicting (F.9).

If $P_0$ has norm at most one, any root of $\wp(b) = P_0$ has norm at most one. Let $L_v \subset M$ be the kernel of the weight. The residue field of $H$ is $\mathrm{Frac}(k[L_v])$. Indeed each element of norm one has a strict rational approximation within one. The initial fraction of that approximation has weight zero, and dividing its numerator and denominator by a common monomial of their weight expresses it as a fraction in $k[L_v]$. It has the same residue by (F.6). Conversely the norm on $\mathrm{Frac}(k[L_v])$ is trivial, so this subfield injects into the residue field.

The residue of $P_0$ consists of its weight-zero terms. A root of the residue equation is integral over $k[L_v]$ and therefore lies in $k[L_v]$. If the root is nonconstant, take a real linear functional on $L_v$ that is positive on a nonzero exponent of its support. Perturb it to attain a unique maximum on the support. The maximum is positive, so the largest term of its Artin–Schreier image has nonzero exponent in $pL_v$ and coefficient in $k^p$. This is again excluded by (F.9). Thus the residue equation forces the nonconstant terms to have strictly positive $v$-weight and $c_0 = \wp(d)$ for some $d \in k$.

Conversely under these conditions $P_0 - c_0$ has norm less than one. For any $h \in H$ with $|h| < 1$, the series

$$-\sum_{j\geq 0} h^{p^j}$$

converges to an element of norm less than one with $\wp$-image $h$, by telescoping. Apply this to $h = P_0 - c_0$ and add $d + C$ to obtain a root of $\wp(x) = P$. □

**Proposition F.3.1** *Let $V \to U$ be a one-point finite free Gauss chart at $y$ over $\eta_w$, with algebra*

$$A[z_1, \dots, z_s]/(\wp(z_i) - a_i : 1 \leq i \leq s).$$

*Suppose for every $i$ a Laurent polynomial $P_i \in k[M]$ satisfies $|a_i - P_i|(\eta_w) < 1$. For $\lambda \in \mathbb{F}_p^s$ choose a decomposition (F.9) for $P_\lambda = \sum_i \lambda_i P_i$. Let $E_v$ consist of the $\lambda$ for which the resulting constant is in $\wp(k)$ and each resulting nonconstant term has strictly positive $v$-weight. After shrinking the chart, $E_v$ is a linear subspace of $\mathbb{F}_p^s$ at each weight $v$ of a small closed box about $w$. There are $p^{\dim E_v}$ Gauss lifts, all of degree $p^{s-\dim E_v}$.*

*A fixed finite family of analytic units, together with the original moment coordinates, embeds the Gauss preimage of this box by joint log norms, with the weighted polyhedral description of Lemma 9.1.2. For $s = 1$, one additional unit suffices.*

***Proof*** Shrink so that all $|a_i - P_i| < 1$. Put $a_\lambda = \sum_i \lambda_i a_i$. In any completed Gauss field of this neighborhood the difference $a_\lambda - P_\lambda$ has norm less than one. The series in the proof of Lemma F.3.2 shows that $a_\lambda \in \wp(H)$ if and only if $P_\lambda \in \wp(H)$.

At a weight $v$, choose a root tuple $r$ over $H = \mathcal{H}(\eta_v)$. The extension $L/H$ generated by its coordinates is Galois since the other roots of each equation are translates

by $\mathbb{F}_p$ and the equations are separable. Its group embeds in $\mathbb{F}_p^s$ by $\sigma \mapsto (\sigma(r_i) - r_i)_i$; write $D$ for the image. The geometric tuples are $r + d$ for $d \in \mathbb{F}_p^s$, and the evaluation at each tuple has image $L$. Evaluations have the same kernel if and only if the tuples differ by $D$, as in Theorem F.3.1. The algebra is reduced by factorization over $L$, so it has $p^{s-\dim D}$ factors of degree $p^{\dim D}$.

The coordinate $z_\lambda = \sum_i \lambda_i z_i$ has value in $H$ at a tuple exactly when the linear form $\lambda$ annihilates $D$. It satisfies $\wp(z_\lambda) = a_\lambda$. All roots differ by constants in $\mathbb{F}_p$, so this is equivalent to $a_\lambda \in \wp(H)$. By Lemma F.3.2 the annihilator is $E_v$. It has dimension $s - \dim D$, and its linear forms distinguish different $D$-cosets. This proves the point and degree counts. The tests are constant on the relative interiors of the cells cut out by the weight-zero hyperplanes of the finitely many exponents in the normal forms.

Choose one representative $\lambda$ of each one-dimensional subspace of $\mathbb{F}_p^s$. When its normal form has constant $c_\lambda = \wp(d_\lambda)$ with $d_\lambda \in k$, fix such a root and put $b_\lambda = z_\lambda - C_\lambda - d_\lambda$. Each $b_\lambda - j$, $j \in \mathbb{F}_p$, is nonzero at the central point, since $D = \mathbb{F}_p^s$ there, and a zero would put $z_\lambda$ in $H$. Shrink away from the compact images of their zero loci as in Theorem F.3.1, so all these functions are units.

For a chosen $\lambda \in E_v$, $b_\lambda$ satisfies an Artin–Schreier equation with right-hand side

$$(a_\lambda - P_\lambda) + (P_{\lambda,0} - c_\lambda)$$

of norm less than one. Its roots have the form $h+j$ for $j \in \mathbb{F}_p$, with $h$ the root of norm strictly below one. The unit $b_\lambda$ ensures $|h| > 0$. Define one unit for this direction by

$$u_\lambda = \prod_{j=1}^{p-1} (b_\lambda - j)^j.$$

At the value $h + j$ of $b_\lambda$ its norm is $|h|^j$, with $j = 0, \dots, p-1$ viewed as integer exponents. The norms are distinct. For two distinct cosets, some linear form in $E_v$ takes different values. Its chosen representative still does, so the associated $u_\lambda$ separates the lifts. For $s = 1$ there is only one direction. The compactness argument and graph criterion of Theorem F.3.1 finish the proof. □

In characteristic two, this includes rank-two charts at independent weights. In the notation of the proof of Corollary F.3.2, if $c$ vanishes at the Gauss point, the geometric fibre has one point. Otherwise shrink to invert $c$ and use $z = b/c$ to obtain an Artin–Schreier equation. At an independent weight the Gauss field is the Laurent completion by Section 9.2.1, so the coefficient has a strict Laurent approximation of absolute error less than one.

### F.3.4 Examples

*Example F.3.1 (A change of degree across a hyperplane)* Let $t$ be one torus coordinate, let $k$ contain the $m$-th roots of unity with $m \geq 2$ invertible in $k$, and consider the finite cover $z^m = 1 + t$. On each Gauss fibre with $t$-weight $v_t > 0$ there are $m$ points of degree one. At $v_t \leq 0$ there is one point of degree $m$. This description is independent of the other weights.

The Gauss norm of $1 + t$ is always at least one, so a Laurent restriction $|1 + t| \geq c$ with $0 < c < 1$ contains the whole Gauss section of the box. The cover is finite free there. The orbit calculation giving Corollary F.3.1 applies at every weight with $R = 1 + t$. For $v_t > 0$, the initial coefficient is 1, and every divisor of $m$ is permitted in (F.8). For $v_t < 0$ the initial coefficient is $t$, and for $v_t = 0$ it is $1 + t$. The orders at the respective prime divisors $t = 0$ and $t = -1$ are one, so neither is a nontrivial power in the rational torus field. This proves the count and the degrees.

Each $z - \zeta$, $\zeta \in \mu_m(k)$, is a unit, since their product is $t$. On the positive side, the lift reducing to $\zeta$ has $t$-weight $v_t$ as the log norm of $z - \zeta$; the other $z - \xi$ have norm one. Indeed the root near $\zeta$ comes from Lemma F.3.1, and distinct roots of unity differ by units of norm one. The product formula then gives the claimed log norm. On the negative side, $|z| > 1$ and each $z - \zeta$ has log norm $v_t/m$. At $v_t = 0$, $|z| = 1$ and all factors in the product $t$ have norm at most one, so all those log norms are zero. The joint log norm graphs glue along the hyperplane $v_t = 0$ and separate the lifts.

*Example F.3.2 (Two equations with a common square class)* Let $t, s$ be two torus coordinates and suppose $k$ has characteristic different from two. Consider the finite cover

$$z_1^2 = t, \qquad z_2^2 = t(1 + s).$$

When $v_s \leq 0$, the Gauss fibre has one lift of degree four. When $v_s > 0$, it has two lifts of degree two, and the two coordinate norms $|z_1|, |z_2|$ are the same at both lifts. The unit $z_1 z_2/t - 1$ separates them.

To compute the degrees, write $S = \mathrm{in}_v(1 + s)$. The three nontrivial character tests in (F.7) ask if $t$, $tS$ and $t^2 S$ are squares in $F$. The first two are never squares, since $S$ depends only on $s$ and the order at $t = 0$ is one. The last is a square exactly when $S$ is one: for $v_s < 0$, $S = s$ has order one at $s = 0$, and for $v_s = 0$, $S = 1 + s$ has order one at $s = -1$. Thus the fibre at $v = 0$ is a field of degree four by the orbit calculation of Theorem F.3.1. The theorem applies on a neighborhood of that weight, and the same field calculation gives the stated counts at all weights. The repeated coordinate norms follow from $|z_1|^2 = |t|$ and $|z_2|^2 = |t(1 + s)|$.

Put $\beta = z_1 z_2/t$. Its square is $1 + s$, so $(\beta - 1)(\beta + 1) = s$, and both factors are units. At $v_s > 0$ the two lifts give opposite square roots in the Gauss field, by the nontrivial character test. One of them satisfies $|\beta - 1| < 1$ by Lemma F.3.1; there $|\beta + 1| = |2| = 1$, so $|\beta - 1| = |s|$. At the other lift $|\beta - 1| = 1$. On $v_s < 0$, $|\beta| > 1$ and $-\log|\beta - 1| = v_s/2$. At $v_s = 0$ both factors have norm at most one and their product

has norm one, so their log norms are zero. The two graphs on the positive side meet the degree-four graph along $v_s = 0$.

## F.4 Separable factors of finite Gauss charts

A point of a Gauss fibre can carry a nonreduced local algebra or an inseparable residue field extension. A separable factor of the finite free chart accounts for the points of every nearby fibre (Proposition F.4.1). The rank over that factor accounts for the remaining multiplicity (Corollary F.4.1).

### F.4.1 Lifting a simple root

Let $V \to U$ be a one-point finite free chart over $\eta_w$, with central point $y$, as in Theorem 9.4.1. Write $A = \mathcal{O}(U)$, $B = \mathcal{O}(V)$, $H = \mathcal{H}(\eta_w)$, and $F = \mathrm{Frac}(k[M])$.

**Lemma F.4.1** *Let $G \in A[Z]$, $z \in B$, and $r > 0$. Write $G(z + X) = \sum_{j=0}^{d} c_j X^j$. Suppose at $y$ that $c_1(y) \neq 0$ and*

$$|c_0(y)| < r|c_1(y)|, \qquad |c_j(y)|r^{j-1} < |c_1(y)| \quad (2 \le j \le d).$$

*After shrinking the target, there is $\tau \in B$ with $G(\tau) = 0$ and $|\tau(y) - z(y)| < r$.*

***Proof*** Choose positive numbers $s_j > |c_j(y)/c_1(y)|$ for $j = 0, 2, \dots, d$ so that $s_0 < r$ and $s_j r^{j-1} < 1$ for $j \ge 2$. The nonvanishing of $c_1$ and the strict bounds with $s_j$ hold on a source neighborhood of $y$. The image of its closed complement is compact and misses $\eta_w$, so the preimage of a smaller affinoid target neighborhood satisfies all the bounds. There $c_1$ is a unit by Theorem 2.2.1. Put $b_j = -c_j/c_1$ for these indices.

For an affinoid Banach norm $\|\cdot\|$ on $B$, define the vector norm

$$\|u\|_* = \sup_{\alpha} \frac{\|(\prod_j b_j^{\alpha_j})u\|}{\prod_j s_j^{\alpha_j}},$$

where $\alpha$ ranges over the nonnegative multiindices for $j = 0, 2, \dots, d$. The strict spectral bounds make every sequence $\|b_j^n\|/s_j^n$ bounded by Theorem 2.2.1. The displayed norm is therefore equivalent to $\|\cdot\|$, with the lower bound furnished by $\alpha = 0$. Multiplication by $b_j$ has operator norm at most $s_j$, by shifting the $j$-th index. The operator norm of multiplication defines an equivalent complete algebra norm on $B$. Commutativity bounds it above by $\|u\|$. Evaluation at 1 bounds it below by $\|u\|_*/\|1\|_*$.

In that algebra norm the map $x \mapsto b_0 + \sum_{j=2}^{d} b_j x^j$ sends the closed ball of radius $r$ into the ball of radius $\max(s_0, \max_{j\ge2} s_j r^j) < r$. Factoring differences of powers shows that its Lipschitz constant is at most $\max_{j\ge2} s_j r^{j-1} < 1$, with an empty maximum taken to be zero. Iterating the map gives a fixed point $x$ of norm strictly less

than $r$. Since absolute values at points are bounded by the equivalent algebra norm, by taking powers, $\tau = z + x$ has the required value at $y$. The fixed point equation gives $G(\tau) = 0$. □

### F.4.2 Extracting the separable factor

**Proposition F.4.1** *Let $L = \mathcal{H}(y)$, let $L_s$ be its maximal separable subextension over $H$, and put $d_s = [L_s : H]$. After shrinking $U$, there is a factorization of finite affinoid maps*

$$V \longrightarrow W = \mathcal{M}(C) \longrightarrow U, \qquad C = A[Z]/(G),$$

*where $G \in F[Z]$ is monic separable irreducible of degree $d_s$, with coefficients analytic on $U$ and discriminant invertible there. The map $C \to B$ is injective and $B$ is free over $C$ of rank $\ell = \operatorname{rank}_A(B)/d_s$. Every field fibre of $C \to B$ has exactly one geometric point.*

*The image of $C$ in $B$ is unique as a germ among such factorizations inducing $L_s$ at the central point. In characteristic zero, every $b \in B$ differs from some $c \in C$ by a nilpotent. In characteristic $p > 0$, there is one $p$-power $q$ with $b^q \in C$ for every $b \in B$. When $B \otimes_A H$ is a separable field, $C = B$.*

***Proof*** The central algebra $B_H = B \otimes_A H$ is local with residue field $L$ by Theorem 2.3.1. Choose a primitive element $\theta$ of $L_s/H$, with monic separable polynomial $P \in H[Z]$. In a fixed $H$-basis of $L$, a minor of the first $d_s$ powers of $\theta$ is nonzero. By finite-dimensional norm equivalence in Lemma 2.3.1, these powers stay independent for all elements in a small ball about $\theta$. Choose its radius $r > 0$ small enough also to satisfy the strict Taylor bounds of Lemma F.4.1 at $(P, \theta)$.

Approximate the coefficients of $P$ by elements of $F$ to get a monic polynomial $G \in F[Z]$ of degree $d_s$. Choose $z \in B$ with value approximating $\theta$ by taking $F$-linear combinations of images of a free $A$-basis in $L$. Those images span $L$ over $H$, and $F$ is dense in $H$. The finitely many denominators in the two approximations are invertible on a common target neighborhood. The Taylor bounds are open in the coefficients and the value of $z$, so both approximations can be chosen to satisfy them and $|z(y) - \theta| < r$. We can also keep the discriminant nonzero at $\eta_w$, since it is a polynomial in the coefficients and the discriminant of $P$ is nonzero. Invert it on a smaller target.

By Lemma F.4.1, there is a root $\tau \in B$ of $G$ with $|\tau(y) - \theta| < r$. The powers of $\tau(y)$ up to exponent $d_s - 1$ are independent over $H$, so $G$ is the minimal polynomial of $\tau(y)$ over $H$. It is separable, and $\tau(y)$ generates $L_s$. The map $H[Z]/(G) \to B_H$ evaluates at $\tau$; it embeds a field that maps isomorphically onto $L_s$ in the residue field. In particular $G$ is irreducible over $F$.

Put $e_1 = 1$ and choose $e_2, \ldots, e_\ell$ among the elements of an $A$-basis of $B$ so that the images of $e_1, \ldots, e_\ell$ form a basis of $B_H$ over this field. This is possible because the images of an $A$-basis span $B_H$. The determinant of the products $\tau^j e_i$ for $0 \le j < d_s$ is nonzero at $\eta_w$. Inverting it on the target makes these products an $A$-basis of $B$.

This proves that evaluation at $\tau$ identifies $A[Z]/(G)$ with the subalgebra $C$ and that $B$ has $C$-basis $(e_i)$. The finite module norm from this $C$-basis is equivalent to the finite $A$-module norm using the products $\tau^j e_i$ and Proposition 2.3.1.

In characteristic zero, $C \otimes_A H$ maps onto $L$. For each $i$, take $a_i \in C$ to be the trace of multiplication by $e_i$ on the free $C$-module $B$, divided by $\ell$. In the central algebra the operator is the residue scalar plus a nilpotent operator. Thus $e_i - a_i$ is nilpotent in $B_H$, and its $\ell$-th power is zero there, since the dimension over the central field is $\ell$. The coefficients of $(e_i - a_i)^\ell$ in the $A$-basis vanish at $\eta_w$. They vanish on a smaller Laurent target by Lemma 9.4.5. For any $b = \sum_i u_i e_i$, $u_i \in C$, the difference $b - \sum_i u_i a_i$ lies in the nilpotent ideal generated by the finitely many $e_i - a_i$.

In characteristic $p > 0$, a sufficiently large $p$-power sends each residue of $e_i$ into $L_s$. Indeed the minimal polynomial of any element of $L$ is a separable polynomial in a power of its variable. Lifting the resulting scalars in $L_s$ to $C \otimes_A H$, a further $p$-power kills the nilpotent differences in $B_H$. This gives one $q$ such that the central images of all $e_i^q$ belong to $C \otimes_A H$. Their coefficients along the complementary basis groups, those indexed by $i > 1$, vanish at $\eta_w$. Shrinking by Lemma 9.4.5 makes them vanish on the target. Thus every $e_i^q$ belongs to $C$, and so does $b^q$ for any $b \in B$, by the Frobenius identity.

After extension to an algebraically closed field over $C$, the images of all $e_i$ at a point are determined by the preceding relations. In characteristic zero they equal the images of the $a_i$. In characteristic $p$ each has its prescribed $q$-th power. The images of the $e_i$ generate the algebra. The fibre has at least one point because it is free of positive rank over the field, so it has exactly one geometric point.

For uniqueness, the two central subfields of $B_H$ are identical. Indeed two lifts of an element of $L_s$ that solve its separable minimal polynomial in $B_H$ have difference zero. Factoring the difference of polynomial values yields their difference times an element with nonzero residue, hence a unit. For each of the two subalgebras, its generator therefore has zero complementary coefficients in a basis adapted to the other. Once more, vanishing at $\eta_w$ spreads to a Laurent neighborhood. This gives both inclusions there. If the central algebra is a separable field, $\ell = 1$, so the relative basis consists of 1 and $C = B$. □

**Corollary F.4.1** *The map $V \to W$ of Proposition F.4.1 is a homeomorphism that takes rational subsets to rational subsets in both directions. Write $N_{B/C}(b)$ for the determinant of multiplication by $b$. If $x \in V$ corresponds to $x_s \in W$, then*

$$|N_{B/C}(b)(x_s)| = |b(x)|^{\ell} \qquad (b \in B). \tag{F.10}$$

*Each local degree over $U$ on $V$ is $\ell$ times the corresponding degree on $W$. After dividing the fibre measures in (9.2) by their respective total masses, they correspond under the homeomorphism.*

***Proof*** The finite fibre theorem and the single geometric point condition give a continuous bijection between compact Hausdorff spectra, hence a homeomorphism. In a fibre over $x_s$, after extension to an algebraic closure of its completed field, multiplication by $b$ is the value of $b$ at the unique geometric point plus a nilpotent operator. Its determinant is the $\ell$-th power of that value. Its absolute value gives (F.10). A rational domain on $V$ given by $|f_i| \le r_i|g|$ becomes the rational domain on $W$ defined by

$$|N_{B/C}(f_i)| \le r_i^{\ell}|N_{B/C}(g)|.$$

The norms generate the unit ideal. Otherwise their quotient algebra would have nonempty spectrum by Theorems 2.1.1 and 2.2.1, giving a common zero by (F.10). Conversely pullback takes rational domains to rational domains by their defining inequalities.

For a point $z \in U$, the algebra $C \otimes_A \mathcal{H}(z)$ is a product of fields since the discriminant is invertible. The field factors give idempotents in $B \otimes_A \mathcal{H}(z)$. Each corresponding $B$-factor is local because its fibre has just one point. It is free of rank $\ell$ over the field factor, so its dimension over $\mathcal{H}(z)$ is the field degree times $\ell$. Both the weights and their total mass are multiplied by $\ell$, proving the assertion about normalized measures. □

### F.4.3 Reduction to rational torus fields

**Corollary F.4.2** *Put $E = F[Z]/(G)$ in Proposition F.4.1. Over a weight $v$ in the chart, the lifts in $V$ are indexed by the real absolute values of $E$ extending the Gauss value of $F$. The degree of a lift is $\ell$ times the degree of the corresponding completed field over $\mathcal{H}(\eta_v)$.*

*Finite separation by analytic units on a weight neighborhood of $w$ is equivalent, after shrinking the neighborhood, to finite separation by a subset of $E^\times$. If these conditions hold, the joint log norms with the moment coordinates embed a compact Gauss preimage with the weighted polyhedral description of Lemma 9.1.2.*

***Proof*** For $H_v = \mathcal{H}(\eta_v)$, the algebra $H_v \otimes_F E$ is a product of fields. Each field is complete by Lemma 2.3.1 and has a dense copy of $E$. Indeed the powers of $Z$ through $d_s - 1$ span the field factor over $H_v$, and coefficients in $H_v$ are approximated by $F$. Conversely a prolongation gives a map from $H_v \otimes_F E$ to the completion of $E$ for that value. Its image is a field finite over $H_v$, hence complete. It contains the dense subfield $E$ and therefore is the completion, identifying one field factor. These constructions are inverse. Now Corollary F.4.1 gives the degree statement.

Represent a finite family in $E^\times$ by polynomials in $Z$ with coefficients in $F$, and invert their denominators on a common target neighborhood. Their values at the sole central point of $W$ are nonzero, since $H \otimes_F E$ is a field. Shrink away from the images of their compact zero loci to obtain analytic units. A separating family on the field extends to one on the chart by pullback.

Conversely suppose analytic units in $B$ separate the lifts. Their norms in $C$ are units, and by (F.10) separate the corresponding points of $W$. For each such unit $a \in C$, write it in the $A$-basis of powers of $Z$ and approximate its coefficients by elements of $F$. The approximating element $a_0 \in E$ can be chosen to satisfy $|a - a_0| < |a|$ at the central point of $W$, by density. Its denominators become analytic units on a target neighborhood. The strict inequality continues to hold on the preimage of a smaller target neighborhood by compactness and the one-point condition. It implies $|a_0| = |a|$ there. Doing this for the finite list gives the required family in $E^\times$.

Over a small closed weight box the joint log norm map is a continuous injection from a compact set into Euclidean space. Corollary 9.4.1 gives its weighted polyhedral image. □

*Example F.4.1 (Inseparable and nilpotent multiplicities)* Suppose $\operatorname{char} k = p > 0$. Over an annular neighborhood of the weight-zero Gauss point of the $t$-torus, take the finite free chart with equations

$$z^p - z = t, \qquad u^p = z, \qquad \varepsilon^2 = 0.$$

At $t$-weight $v_t \le 0$ the Gauss fibre has one lift of degree $2p^2$. At $v_t > 0$ there are $p$ lifts, each of degree $2p$. The points and separating coordinates come from the first equation, and the other two equations multiply the local degrees by $2p$.

To verify the central algebra, first apply Lemma F.3.2 to $t$. At $v_t = 0$ it has no Artin–Schreier root in $k(t)$. As the Galois group of a root embeds in the translations by $\mathbb{F}_p$, the polynomial is irreducible of degree $p$. The resulting field is $k(z)$, with $t = z^p - z$. In $k(z)$, $z$ has order one at $z = 0$. In particular $z$ is not a $p$-th power. A $p$-th root of $z$ thus has purely inseparable degree $p$, and adjoining $\varepsilon$ gives a local algebra with nilpotent ideal $(\varepsilon)$ of square zero. Its separable residue subfield over $k(t)$ is $k(z)$. This verifies the one-point hypothesis and identifies $z^p - z = t$ as the separable factor.

At $v_t > 0$ the first equation has $p$ roots in the completed Gauss field, namely $h + j$ for $j \in \mathbb{F}_p$, where the small root $h = -\sum_{a \ge 0} t^{p^a}$ has $|h| = |t|$. At $v_t \le 0$ it has no root by the same Laurent criterion and is irreducible of degree $p$. The remaining two equations always give a free algebra of rank $2p$ with one geometric point over each field factor. This gives the stated degrees by Corollary F.4.1.

Each $z - j$ is a unit since $\prod_{j \in \mathbb{F}_p} (z - j) = t$. Thus the unit $g = \prod_{j=1}^{p-1} (z - j)^j$ separates the $p$ lifts at $v_t > 0$, taking log norm $j v_t$ at the lift with $z = h + j$, where $j$ is represented by $0, \dots, p - 1$. At $v_t < 0$, the relation forces $|z| = |t|^{1/p} > 1$ and each $|z - j|$ equals $|z|$, so the log norm of the unit is $(p - 1)v_t/2$. At $v_t = 0$ all those norms are one because they are at most one and their product is one. The $p$ graphs and the single negative graph thus meet at the weight-zero lift, as in Figure F.1.

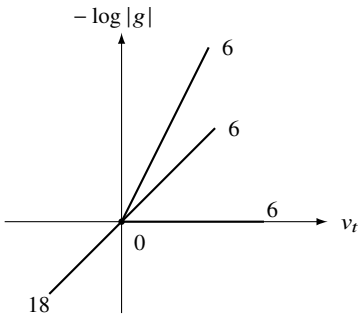



**Fig. F.1** Separating log norms for $p = 3$ in Example F.4.1. The labels give the local fibre degrees on the open branches.

### F.4.4 Descent of separating families

**Lemma F.4.2** *Let $E'/E/F$ be a tower of finite separable extensions, and let $S$ be any set of weights. If a finite subset of $(E')^\times$ separates the prolongations to $E'$ of the Gauss value at each weight in $S$, then a finite subset of $E^\times$ separates the prolongations to $E$ at every weight in $S$.*

***Proof*** Choose a separating list $b_1, \dots, b_m$ in $(E')^\times$, with $m \geq 1$ by adjoining 1 if needed. Put $\delta = [E' : E]$. Take $1 + (m-1)\binom{2\delta}{2}$ distinct positive integers $j$. For each such $j$ let $P_j \in E[X]$ be the characteristic polynomial of multiplication on $E'$ by

$$B_j = \prod_{i=1}^{m} b_i^{j^{i-1}}.$$

We show that the finite list of nonzero coefficients of the $P_j$ separates.

Fix a Gauss weight in $S$ and two distinct prolongations $x, x'$ to $E$. Each has at least one and at most $\delta$ prolongations to $E'$, indexed by the field factors of $\mathcal{H}(x) \otimes_E E'$ and $\mathcal{H}(x') \otimes_E E'$, where these completed fields are the completions of $E$ at $x, x'$. This is the same completion argument as in Corollary F.4.2. The sets of vectors $(-\log|b_i|)_{i=1}^m$ on the two sets of prolongations are disjoint by separation on $E'$. Their union has at most $2\delta$ points. For two distinct vectors $u, u'$, the real polynomial $\sum_{i=1}^m (u_i - u_i')X^{i-1}$ has at most $m-1$ roots. Hence one of the chosen $j$ makes the projection $u \mapsto \sum_i u_i j^{i-1}$ injective on that union. The projected values over $x$ and $x'$ are disjoint.

The absolute values of the roots of $P_j$ at $x$ are exactly the values of $|B_j|$ on the prolongations to $E'$, counted with their local factor dimensions, by the field factor calculation of Proposition 9.1.1. The same holds at $x'$. The projected log norms are the negatives of their logarithms. Since the two lists are different, (9.1) shows that some coefficient of $P_j$ has different absolute values at $x, x'$. This proves separation by the claimed list. □

In particular, separation for all finite Galois extensions of the rational torus field implies separation for all finite separable extensions, by taking normal closures and applying Lemma F.4.2.

## F.5 Separation on toric cones

A finite set of Laurent polynomials determines cones on which their leading terms have the same monomial factors. Such cones organize the tame ramification calculation (Theorem F.5.1) and the reduction of rational Artin–Schreier coefficients (Theorem F.5.2).

### F.5.1 A common choice of coordinates

Write $F = \mathrm{Frac}(k[M])$ for the rational torus field and $N = \mathrm{Hom}(M, \mathbb{Z})$.

**Lemma F.5.1** *Given finitely many nonzero elements of $F$, there is a finite fan covering $N_{\mathbb{R}}$ with the following property. For each cone $\sigma$ of dimension $r$, there is a character basis $x_1, \dots, x_n$ with $x_i$ of positive weight for $1 \le i \le r$ on* relint $\sigma$ *and of weight zero for $i > r$. Put $K_0 = k(x_{r+1}, \dots, x_n)$. Each given element is a monomial in $x_1, \dots, x_r$ times a unit of*

$$R_\sigma = K_0[x_1, \dots, x_r]_{(x_1, \dots, x_r)}.$$

***Proof*** Write the elements as Laurent fractions. Take the fan of all monomial comparison hyperplanes in these numerators and denominators, together with the coordinate hyperplanes from a lattice basis. The latter make all cones pointed.

We refine to a fan whose primitive ray generators in each cone extend to a lattice basis. First choose a rational interior ray for each nonzero cone. Subdivide in order of dimension by coning the subdivided boundary to that ray, using a transverse polytope section of each pointed cone. The shared choices on faces give a simplicial fan. For a maximal cone its index is the absolute determinant of the primitive generators. If an index is greater than one, take a cone of largest index and a nonzero lattice point in the half-open parallelepiped of its generators. The primitive vector of its ray has coefficients in $[0, 1)$ in those generators. Subdivide the fan by inserting that ray in the smallest face containing it. In each maximal cone containing that face, the new cones replace one generator of the face by the inserted ray. The determinant is multiplied by the coefficient of the replaced generator, which is strictly between zero and one. The indices decrease in the cones affected by the subdivision. Thus the pair consisting of the largest index and the number of maximal cones having that index decreases lexicographically. This process stops with index one everywhere.

For each cone take characters dual to an extended basis of its primitive generators. Each Laurent polynomial used in the comparisons has a monomial minimizing the weight simultaneously on all rays of the cone. Indeed a monomial minimizing at a relative interior weight has nonpositive weight difference with every other monomial on the whole cone, by the choice of the original fan. Divide by its monomial in $x_1, \dots, x_r$. The quotient is a polynomial over $K_0$ with nonzero constant coefficient. Terms of zero exponent in these first $r$ coordinates have distinct remaining exponents,

so that coefficient cannot cancel. Taking the quotient of the two expressions gives the claim. □

### F.5.2 Extensions of degree invertible in the ground field

**Theorem F.5.1** *Let $E/F$ be a finite separable extension whose Galois closure has degree invertible in $k$. There is a finite subset of $E^\times$ that separates the prolongations to $E$ of every Gauss value of $F$.*

***Proof*** By Lemma F.4.2, it suffices to prove this when $E/F$ is Galois of degree $m$ invertible in $k$. Choose a monic primitive polynomial $P \in F[Z]$, and apply Lemma F.5.1 to its nonzero coefficients and discriminant. The weight zero has just one prolongation, since an absolute value extending the trivial one to an algebraic extension is trivial. Fix a nonzero cone and write $R = R_\sigma$. After inverting $x_1 \cdots x_r$, the polynomial has coefficients in the resulting ring and its discriminant is a unit there. Its finite étale algebra is normal, since the base is regular. It is the normalization in $E$ on that open set. Indeed it embeds in $E$ by monic division and has fraction field $E$, and integral closedness identifies the integral closure.

Adjoin variables $s_i$ with $s_i^m = x_i$ for $1 \le i \le r$. Set $F' = K_0(s_1, \ldots, s_r)$ and $R' = K_0[s_1, \ldots, s_r]_{(s_1,\ldots,s_r)}$. The polynomial algebra in the $s_i$ is finite free over the polynomial algebra in the $x_i$. The fibre over $(x_1, \ldots, x_r)$ has just one point, so after localization over $R$ the algebra is $R'$. Since $R'$ is integrally closed, it is the normalization of $R$ in $F'$. The ramification index over the divisor $x_i = 0$ is $m$ at $s_i = 0$, by the relation $x_i = s_i^m$.

Fix a compositum $E' = EF'$ in an algebraic closure. Let $B'$ be the normalization of $R'$ in $E'$. The required normalizations are finite. For a normal Noetherian domain and a finite separable field extension, choose a field basis of integral elements by clearing denominators. An integral element pairs with each basis element under the field trace to give an element of the base ring, by normality. The integral closure is therefore contained in the finite module spanned by the trace-dual basis, and Noetherianity gives finiteness.

We claim that $B'$ is finite étale over $R'$. It is so away from the coordinate divisors by base change from the finite étale algebra already constructed. In fact $E'$ is one factor of $E \otimes_F F'$, and each component of the étale algebra over this open set is normal and is the normalization in its fraction field. At a coordinate divisor of $R$, the normalization of the discrete valuation ring is finite free of rank $m$. The Galois action is transitive on its primes above the closed point. To see this, if two primes are in distinct orbits, prime avoidance and incomparability give an element of the first prime outside the orbit of the second. Its Galois norm belongs to the first prime's contraction and avoids the second prime, a contradiction. Thus all the ramification indices and residue degrees are equal to $e$ and $f$, respectively. Reducing the free algebra modulo a uniformizer and filtering each local factor by its maximal ideal gives $m = gef$, where $g$ is the number of factors. Hence $e$ and $f$ divide $m$. Since $m$

is invertible in $k$, $e$ is prime to the characteristic exponent of $k$, and the inseparable degree of each residue extension, a power of that exponent dividing $f$, equals one. Thus the residue extensions are separable.

At $s_i = 0$ the base change has ramification index divisible by $e$. Abhyankar's lemma and the discrete valuation criterion for unramifiedness [Stacks, Tags 0BRM and 09E7] give ramification index one and separable residue extensions above this prime. The extended maximal ideal then equals the maximal ideal upstairs. Together with separability and finite type this gives unramifiedness in codimension one. Heights of the primes and their contractions agree by going down over the normal domain $R'$ and incomparability. Purity of the branch locus [Stacks, Tag 0BMB] applies to the finite map with normal source and regular target, and yields the claim.

We now relate this normalization to the Gauss values in the relative interior of the cone. There is a unique prolongation to $F'$ of such a value on $F$. The Gauss value in the new variables with weights $v(x_i)/m$ is one prolongation. For it the monomials $s^\alpha$, $0 \le \alpha_i < m$, give an orthogonal $F$-basis. After clearing a common denominator in $F$, the Laurent supports of the summands for distinct $\alpha$ are disjoint modulo the original lattice, so their norms combine by the maximum. This basis therefore gives the full degree $m^r$ in the completion. Since the completed scalar extension has total dimension $m^r$, the factor corresponding to this completion accounts for the entire fibre. The correspondence with prolongations is the one in the proof of Corollary F.4.2.

Let $H'$ be that completion of $F'$. The positive weights of the $s_i$ and the trivial norm on $K_0$ give a local map $R' \to O_{H'}$ to the valuation ring. Its residue field is the fraction field of the Laurent ring over $K_0$ with exponent lattice $\ker(v|_{\mathbb{Z}^r})$. Indeed a norm-one element is approximated strictly by a rational function of the same norm. The leading terms of its numerator and denominator have the same weight, so division by a common monomial gives a quotient supported in the kernel lattice, with the same residue. Conversely these fractions have trivial norm. Thus the Gauss residue field is purely transcendental over $K_0$, even at dependent weights.

The field factors of $B' \otimes_{R'} K_0$ stay fields over this purely transcendental extension. The ring $O_{H'}$ is henselian. For a polynomial with a simple residue root, start from a lift $b$ of that root. The derivative at $b$ has norm one and the other Taylor coefficients have norm at most one. If the polynomial vanishes at $b$, the root is already lifted. Otherwise its norm there is $\varepsilon < 1$, and we solve for the increment by the contraction in the proof of Lemma F.4.1 on the closed ball of radius $\varepsilon$ in $H'$. This verifies the simple-root criterion. The equivalence for finite étale algebras over a henselian local ring then applies [Stacks, Tags 04GG and 04GK]. In particular the residual field factors of $B' \otimes_{R'} O_{H'}$ lift to distinct finite étale factors. Each lifted factor has field generic fibre. To verify this, lift a primitive element of its separable residue field. Its powers form a basis by Nakayama's lemma and freeness over the local valuation ring. The monic relation is irreducible over $H'$ because its reduction is irreducible of the same degree and any monic factors would have integral coefficients. This identifies the prolongations to $E'$ with the original residue field factors of $B' \otimes_{R'} K_0$.

The Chinese remainder theorem gives elements of $B'$ equal to one modulo one maximal ideal and zero modulo the others. They have norm one at the corresponding prolongation and strictly smaller norm at every other prolongation, for all weights

in the relative interior of the cone. Here their values are integral over the Gauss valuation ring, hence of norm at most one, and their residues are read in the lifted factors. This is a finite family of nonzero elements of $E'$. Descend it to $E$ on this set of weights by Lemma F.4.2. The union over the finitely many cones finishes the proof. □

In characteristic zero every one-point finite free chart of Proposition F.4.1 has a finite family of analytic units separating its Gauss lifts on a weight neighborhood. Indeed its separable rational factor has finite Galois closure, so Theorem F.5.1 and Corollary F.4.2 apply.

### F.5.3 Rational Artin–Schreier coefficients

Assume that $k$ has characteristic $p > 0$, and write $\wp(b) = b^p - b$. Fix a nonzero cone of Lemma F.5.1 and use its coordinates and field $K_0$. Put $R^\circ_\sigma = R_\sigma[1/(x_1 \cdots x_r)]$. Elements of this ring have expansions in $K_0[[x_1, \dots, x_r]][1/(x_1 \cdots x_r)]$. In this subsection a term is called polar if at least one exponent coordinate is negative.

**Lemma F.5.2** *For $a \in R^\circ_\sigma$ there are $C, a_0 \in R^\circ_\sigma$ with $a = \wp(C) + a_0$, such that every nonzero polar term of $a_0$ with exponent in $p\mathbb{Z}^r$ has coefficient outside $K_0^p$. At any relative interior weight $v$, $a$ is in $\wp(\mathcal{H}(\eta_v))$ if and only if the constant coefficient $c_0$ of $a_0$ lies in $\wp(K_0)$ and every nonconstant term has strictly positive weight.*

***Proof*** Choose a $K_0^p$-linear projection of $K_0$ onto $K_0^p$, for instance using a vector basis containing 1. On Laurent series with lower exponent bounds in each coordinate define $\Pi$ by retaining exponents in $p\mathbb{Z}^r$ and applying that projection to their coefficients. For $a = P/D$, with $P$ Laurent and $D$ polynomial with $D(0) \neq 0$, multiplication by $D^p$ commutes with $\Pi$. Every term of $D^p$ has exponent divisible by $p$ and coefficient in $K_0^p$. Consequently

$$\Pi(a) = \frac{\Pi(PD^{p-1})}{D^p}$$

is a $p$-th power in $R^\circ_\sigma$, by taking the Laurent polynomial root of the projected numerator.

Taking the polar part of an expansion preserves $R^\circ_\sigma$. For one variable $x_i$, first multiply by a power $x_i^n$ to clear negative $x_i$-exponents of $P$. The negative terms of $P/D$ in this variable are determined by the first $n$ Taylor coefficients of $x_i^n P/D$. They belong to the analogous ring in the other variables. Indeed coefficient division in $x_i$ uses only powers of $D|_{x_i=0}$, which has nonzero constant coefficient in the other variables. The negative projection in each variable thus preserves rationality and the form of the denominator. The projection onto terms negative in at least one variable is obtained by subtracting the product of the complementary projections from the identity. It also preserves $R^\circ_\sigma$.

The polar part of $\Pi(a)$ is therefore $C_1^p$, where $C_1$ is the polar part of its rational $p$-th root. Subtract $\wp(C_1)$ from $a$. By additivity, the polar part of the new projection is the polar part of $\Pi(C_1)$. If the projected polar part had lower exponent bounds $-b_i$ with $b_i \geq 0$, its new bounds are $-\lfloor b_i/p \rfloor$, since the exponents of the root were divided by $p$. After finitely many repetitions the polar projection vanishes. Summing the corrections gives $C$ and $a_0$ as asserted, since the coefficient projection fixes $K_0^p$.

Now fix $v$ in the relative interior. The expansions converge in $\mathcal{H}(\eta_v)$ because all $x_i$ have positive weight and there are only finitely many terms below any fixed weight. They agree with the rational functions by expanding the inverse of $D$ as a geometric series about $D(0)$. The least terms of a nonzero series form its initial Laurent polynomial. In particular an initial Laurent polynomial of $a_0$ of negative weight consists of polar terms. It cannot be a $p$-th power in $F$. A root would be integral over the Laurent polynomial ring over $K_0$, and hence would lie in that ring. Its $p$-th power would have all exponents divisible by $p$ and all coefficients in $K_0^p$.

If $a_0 = \wp(b)$ has norm greater than one, approximate $b$ by $d \in F$ with $|d-b| < |b|$. The initial fraction of $\wp(d)$ is $\mathrm{in}_v(d)^p$, since $|d|^p > |d|$. The strict approximation gives $|\wp(d) - a_0| < |a_0|$, so this is also the initial Laurent polynomial of $a_0$. This contradicts the preceding observation.

If $|a_0| \leq 1$, let $a_{0,v}$ be its finite sum of terms of weight zero. The difference $a_0 - a_{0,v}$ has norm strictly less than one and belongs to $\wp(\mathcal{H}(\eta_v))$, by the convergent small-root series in Lemma F.3.2. Every nonconstant term of $a_{0,v}$ is polar, since nonnegative exponents other than zero have positive weight. Thus $a_{0,v}$ is already in the Laurent normal form of that lemma over the trivially valued field $K_0$. It belongs to the image of $\wp$ exactly when it is the constant $c_0 \in \wp(K_0)$. Together these arguments prove the criterion. □

**Theorem F.5.2** *Let $E = F(z_1, \ldots, z_s)$, where $\wp(z_i) = a_i \in F$. There is a finite subset of $E^\times$ separating the prolongations to $E$ of every Gauss value on $F$.*

***Proof*** The extension is Galois with group $\Gamma \subseteq \mathbb{F}_p^s$ acting by translations on the tuple $(z_i)$. For $\lambda = (\lambda_i) \in \mathbb{F}_p^s$ put $z_\lambda = \sum_i \lambda_i z_i$ and $a_\lambda = \sum_i \lambda_i a_i$. Choose the fan of Lemma F.5.1 for the nonzero $a_\lambda$. The weight zero has a single prolongation. On any other relative interior, apply Lemma F.5.2 to each $a_\lambda$ with $z_\lambda \notin F$. Write $a_\lambda = \wp(C_\lambda) + a_{\lambda,0}$. If the constant term of $a_{\lambda,0}$ is $\wp(d_\lambda)$ with $d_\lambda \in K_0$, include all the elements

$$z_\lambda - C_\lambda - d_\lambda - j, \qquad j \in \mathbb{F}_p,$$

in the separating family for this cone. They are nonzero because $z_\lambda \notin F$.

For a Gauss field $H = \mathcal{H}(\eta_v)$, choose an embedding of $E$ into an algebraic closure of $H$ over $F$. The image of the Galois group of the compositum over $H$ is a subgroup $D \subseteq \Gamma$. Fibre points are the $D$-orbits on the embeddings of $E$ over $F$, as in the Galois calculation of Proposition F.3.1. The embeddings differ by $\Gamma$. Given distinct orbits, there is a linear form $\lambda$ killing $D$ and taking different values on those two cosets. It is nonzero on $\Gamma$, so $z_\lambda \notin F$. The values of $z_\lambda$ at these embeddings belong to $H$ and are distinct. Thus $a_\lambda \in \wp(H)$, and by the normal form on the cone the corrected values $z_\lambda - C_\lambda - d_\lambda$ satisfy an Artin–Schreier equation with right side of norm strictly less

than one. They have the form $h + j$ for the root $h$ of norm less than one and distinct $j \in \mathbb{F}_p$. Subtract the $j$ for one of the two values. The resulting element of $E^\times$ has norm less than one there and norm one at the other. This proves separation by the finite union over the cones. □

**Corollary F.5.1** *A one-point finite free Gauss chart with equations $\wp(z_i) = a_i$ in analytic coefficients admits a finite separating family of units near its central weight. In particular all relative-interior Gauss fibre points of degree at most two admit such a family.*

***Proof*** Choose rational coefficients $a_i' \in F$ with $|a_i - a_i'| < 1$ at the central Gauss point. Shrink the target so the $a_i'$ are analytic and the strict inequalities hold throughout. The spectral radius of each $h_i = a_i - a_i'$ is then less than one. Thus $-\sum_{j\geq 0} h_i^{p^j}$ converges in the affinoid algebra, by the spectral radius formula. If $b_i$ is this sum, then $\wp(b_i) = h_i$. Put $u_i = z_i - b_i$, so $\wp(u_i) = a_i'$. The corresponding finite separable algebra over $F$ is a field, since its extension to the central Gauss field is a field by the one-point hypothesis and separability. The separating elements of Theorem F.5.2 are polynomials in the $u_i$ with rational coefficients. Invert their denominators on a smaller target. At the central point their values are nonzero because the central algebra is a field. Shrink away from the images of their compact zero loci to make all of them units. At each nearby Gauss weight the fibre algebra is the corresponding completed scalar extension of the rational field, so the units separate.

In degrees one and two use Corollary F.3.2 for the cases treated there. In the remaining characteristic-two case, the quadratic presentation $b^2 - cb + d = 0$ in its proof has $c \neq 0$ at the central Gauss point. After inverting $c$, scaling by $c$ gives an Artin–Schreier equation, so the first assertion applies. □

*Example F.5.1 (An Artin–Schreier denominator)* On the two-dimensional torus with coordinates $s, t$, consider the equation

$$\wp(z) = \left(\frac{s}{1+t}\right)^p .$$

Its Gauss fibre has $p$ points of degree one when $v_s > \min(0, v_t)$. At every other weight it has one point of degree $p$.

Put $a = s/(1+t)$ and $u = z - a$, so $\wp(u) = a$. The Gauss norm of $a$ is less than one precisely in the displayed region, where the small-root series solves the equation. If $|a| > 1$, its least part is $s/\mathrm{in}_v(1+t)$. Its order at $s = 0$ is one, so it is not a $p$-th power in $k(s,t)$. The negative-weight argument of Lemma F.5.2 therefore precludes a root. At $|a| = 1$, test the residue. If $v_t > 0$ the residue field is $k(s)$ and the residue is $s$. If $v_t < 0$ that field is $k(s/t)$ and the residue is $s/t$. In each case the residue has a pole of order one at infinity, so it cannot be an Artin–Schreier image. Indeed a pole of an Artin–Schreier image has order divisible by $p$, by dominance of the $p$-th power at a pole. If $v_t = 0$, then $v_s = 0$ and the residue is $s/(1+t)$ in $k(s,t)$. Its order-one pole at $t = -1$ gives the same obstruction. When there is no root, the translation Galois group has order $p$, so the fibre is irreducible of degree $p$.

On the split region the small root satisfies $|h| = |a|$. Each $u - j$, $j \in \mathbb{F}_p$, is a unit near the Gauss preimage since their product is $a$. At the point $h + j$ it has norm $|a|$, and at the other points it has norm one. These functions give a fixed family separating the fibres across both boundary rays of the region.

## F.6 Witt equations and abelian Gauss fibres

In a cyclic extension of degree $p^d$, the number of Gauss lifts is a power of $p$ bounded by $p^d$. The successive Witt coordinates detect the cyclic quotients and distinguish their lifts (Theorem F.6.1). We first construct the additive operations used in this argument.

### F.6.1 Additive Witt coordinates

Fix a prime $p$. For a ring $A$ and an integer $d \geq 1$, write $W_d(A) = A^d$ in Witt coordinates, with $W_0(A) = 0$. For $0 \leq j < d$ the ghost polynomials are

$$g_j(X) = \sum_{i=0}^{j} p^i X_i^{p^{j-i}}. \tag{F.11}$$

Over a ring without $p$-torsion they determine the coordinates inductively.

**Lemma F.6.1** *There are addition and inverse polynomials over $\mathbb{Z}$ making the ghost maps additive. They give $W_d(A)$ a functorial abelian group structure. The $j$-th addition coordinate is $X_j + Y_j$ plus a polynomial in the lower coordinates. It is just $X_j + Y_j$ if the lower coordinates of $Y$ are zero. Truncation and the shift $V$ that inserts a zero first coordinate and increases the length are additive.*

*When $A$ has characteristic $p$, the map $\mathsf{F}(X) = (X_i^p)_i$ is additive, and multiplication by $p$ equals $V\mathsf{F}$, using truncation on the right. For any field $A$ of characteristic $p$, the kernel of $\wp_W = \mathsf{F} - 1$ is $W_d(\mathbb{F}_p)$, cyclic of order $p^d$. Truncation to length $j$ on this kernel is the quotient of order $p^j$.*

***Proof*** We use the following divisibility observation. If $a - b$ is divisible by $p^s$ for $s \geq 1$, then $a^p - b^p$ is divisible by $p^{s+1}$, by expanding powers. Work in a polynomial ring over $\mathbb{Z}$ with an endomorphism $\phi$ that fixes integers and sends each variable to its $p$-th power. A list $h_j$ in this ring is the ghost list of a vector if and only if

$$h_j \equiv \phi(h_{j-1}) \pmod{p^j} \quad (j \geq 1). \tag{F.12}$$

For a ghost list, compare the term of index $i < j$ on both sides. The difference of $x_i^p$ and $\phi(x_i)$ is divisible by $p$. Raising to the power $p^{j-1-i}$ and multiplying by $p^i$ makes

the difference divisible by $p^j$. Conversely solve for coordinate $j$ after coordinates $0, \dots, j-1$ are found. The ghost sum of those earlier coordinates is congruent to $\phi(h_{j-1})$ modulo $p^j$ by the same calculation. The numerator in the recursive division by $p^j$ is therefore divisible by $p^j$, as required.

The sum and the negative of ghost lists satisfy (F.12), giving integral polynomials for the group operations. The group identities follow over universal polynomial rings from injectivity of the ghost map there, and then specialize to all rings. The recursion for the sum gives its triangular form. If the lower coordinates of $Y$ are zero, the sum has lower coordinates equal to those of $X$, and comparison of the $j$-th ghost coordinates gives $X_j + Y_j$. Truncation is additive by the same formulas. The shift is additive since its ghost list has first entry zero and the subsequent entries are $pg_{j-1}(X)$.

Raising coordinates to $p$-th powers commutes with addition in characteristic $p$, since the addition polynomials have coefficients in the prime field. To compute the sum of $p$ copies of $X$ modulo $p$, compare its ghost list with that of the integral polynomial vector $V((X_i^p)_i)$. By (F.11), the difference at index $j$ is $p^{j+1}X_j$. If two such ghost lists agree modulo $p^{j+1}$ at each $j$, the vectors agree coordinatewise modulo $p$. Indeed, by induction the contributions of coordinates $i < j$ to the difference at $j$ are divisible by $p^{j+1}$, by the power divisibility observation. Dividing the remaining term by $p^j$ gives equality modulo $p$ at coordinate $j$. This proves the formula for multiplication by $p$.

The kernel condition is coordinatewise $X_i^p = X_i$, so over a field its elements are exactly $W_d(\mathbb{F}_p)$. For $[1] = (1, 0, \dots, 0)$, the formula for multiplication by $p$ gives $p^{d-1}[1] = V^{d-1}[1] \neq 0$ and $p^d[1] = 0$. This element generates the entire group of $p^d$ elements. Truncation sends it to the analogous generator of the shorter group. □

Write $[c] = (c, 0, \dots, 0)$ also for other scalars $c$. From the triangular form in Lemma F.6.1, coordinate $j$ of $\wp_W(X)$ is

$$X_j^p - X_j + P_j(X_0, \dots, X_{j-1}), \tag{F.13}$$

where $P_j$ has zero constant coefficient. Indeed the inverse has coordinate $-X_j$ plus a polynomial in the lower coordinates, by solving the triangular addition equations. Applying addition to $\mathsf{F}(X)$ and the inverse of $X$ gives (F.13). In particular, over an algebraically closed field any Witt equation $\wp_W(X) = a$ can be solved successively by scalar equations.

### F.6.2 Cyclic extensions as Witt equations

**Lemma F.6.2** *Let $E/F$ be cyclic of degree $p^d$ in characteristic $p$. There is $z \in W_d(E)$ whose coordinates generate $E$ over $F$, with $\wp_W(z) \in W_d(F)$. The Galois action on $z$ is by the constant translations $W_d(\mathbb{F}_p)$, through an isomorphism with the Galois group. The first $j$ coordinates generate the cyclic subextension of degree $p^j$.*

***Proof*** We first check additive descent. For a finite Galois extension $E/F$ with group $G$, choose $b \in E$ with $\mathrm{Tr}_{E/F}(b) = 1$. For example, if $\theta$ has monic separable minimal polynomial $P$ of degree $m$ and generates $E$, take $b = \theta^{m-1}/P'(\theta)$. The leading coefficients in the interpolation formula for $T^{m-1}$ at the roots of $P$ give trace one. For a family $c_\sigma \in E$ satisfying $c_{\tau\sigma} = c_\tau + \tau(c_\sigma)$, the element $y = -\sum_\sigma c_\sigma \sigma(b)$ satisfies $\tau(y) - y = c_\tau$. Indeed applying $\tau$ replaces the summand coefficient by $c_{\tau\sigma} - c_\tau$, and the trace of $b$ is one.

The same assertion holds for such families in $W_d(E)$, with its additive group law. Project to length $d-1$, use induction to solve there, and lift the resulting vector to length $d$. Subtract its differences from the family. The remaining differences lie in the kernel of truncation, identified with the additive group of $E$ in the last coordinate by Lemma F.6.1. Apply the trace argument there and add the resulting correction to the lift.

Now choose an isomorphism from $G$ to $W_d(\mathbb{F}_p)$ and use its values as the family $c_\sigma$, since the constants are fixed by $G$. The preceding argument gives $z$ with $\sigma(z) - z = c_\sigma$. Its image under $\wp_W$ is fixed by the group. No nonidentity element fixes all coordinates of $z$, so they generate $E$. The kernel on a truncation is the kernel of the constant quotient map in Lemma F.6.1. Galois correspondence gives the final assertion. □

### F.6.3 Solvability of the truncations

Suppose $k$ has characteristic $p > 0$, and now let $F = \mathrm{Frac}(k[M])$. Take a fan as in Lemma F.5.1 for the nonzero coordinates of $a \in W_d(F)$. Fix a nonzero cone $\sigma$, with ring $R^\circ_\sigma$ and coefficient field $K_0$ as in Lemma F.5.2. For a relative interior weight $v$, let $\mathfrak{m}_v$ be the maximal ideal of the valuation ring of $H_v = \mathcal{H}(\eta_v)$.

**Lemma F.6.3** *For each $j$ for which the equation $\wp_W(z_{<j}) = a_{<j}$ is solvable in $W_j(H_v)$ at some relative interior weight, there is $c^{(j)} \in W_j(R^\circ_\sigma)$ such that*

$$a_{<j} - \wp_W(c^{(j)}) \in W_j(\mathfrak{m}_v)$$

*at every such weight. Here a subscript $< j$ denotes the first $j$ coordinates, and $W_j(\mathfrak{m}_v)$ denotes the vectors all of whose coordinates are in $\mathfrak{m}_v$. The set of solvable weights is obtained by successive applications of the scalar normal form. It is cut out in* relint $\sigma$ *by finitely many strict rational linear inequalities if nonempty.*

***Proof*** On $W_j(\mathfrak{m}_v)$, the equation $\wp_W(x) = b$ has a unique solution in $W_j(\mathfrak{m}_v)$ for each $b \in W_j(\mathfrak{m}_v)$. In fact (F.13) reduces it successively to scalar equations with right sides in $\mathfrak{m}_v$. Each scalar equation has a small root by the convergent series in Lemma F.3.2, and it is unique because two scalar roots differ by a constant in $\mathbb{F}_p$. All Witt solutions in $H_v$ are its translates by $W_j(\mathbb{F}_p)$, by Lemma F.6.1.

Induct on $j$, starting with length zero, when every weight is allowed. Given a correction for length $j$, extend it by zero to length $j+1$ and subtract its Witt image under $\wp_W$ from $a_{<j+1}$. The lower right sides are now small on the set of weights already admitted. On this set the new equation of length $j+1$ is solvable if and only if its last right-side coordinate $A_j$ belongs to $\wp(H_v)$, where $\wp(x) = x^p - x$. For necessity, subtract a constant vector from any solution so its first $j$ coordinates are the small root of the truncated equation. The shorter constant vector that makes this adjustment lifts to length $j+1$ by surjectivity of truncation on constants. The final coordinate equation differs from $\wp(x_j) = A_j$ by a small term, by (F.13). Small terms belong to $\wp(H_v)$. For sufficiency start from the small solution of the truncated equation and solve the scalar last equation by the same observation.

All the right-side coordinates lie in $R^\circ_\sigma$, since the Witt operations are polynomial. Apply Lemma F.5.2 to $A_j$. If its normalized constant is not in $\wp(K_0)$, no weights remain at length $j+1$ or higher. Otherwise denote a constant root by $d_j$ and the rational correction by $C_j$. The condition on the newly admitted weights is that the normalized coefficient minus its constant has norm less than one. Write that difference as a Laurent numerator divided by a polynomial with nonzero constant coefficient. On the relative interior the denominator has norm one. Thus the condition is positivity of the weights of the finitely many numerator monomials, with no condition for a zero numerator. Add $V^j[C_j + d_j]$ to the previous correction at length $j+1$. Its Witt coboundary has first $j$ coordinates zero and last coordinate $\wp(C_j + d_j)$. By the triangular addition formula the subtraction changes only that last right-side coordinate. The scalar test makes it small at exactly the newly admitted weights. This completes the induction. □

### F.6.4 Separating the cyclic and abelian fibres

**Theorem F.6.1** *Suppose $k$ has characteristic $p > 0$. For every cyclic extension $E/F$ of degree $p^d$, there is a finite subset of $E^\times$ separating the prolongations of all Gauss values of $F$. With the equation of Lemma F.6.2, let $j(v)$ be the largest length whose truncation is solvable in $H_v$, with length zero included. There are $p^{j(v)}$ lifts and their completed degrees over $H_v$ are $p^{d-j(v)}$.*

***Proof*** Write $G = \mathrm{Gal}(E/F)$. For each $v$, choose an embedding of $E$ in an algebraic closure of $H_v$ and let $D \subset G$ be the image of the Galois group of its compositum with $H_v$. The completed field factors are the orbits on the $F$-embeddings of $E$ under $D$. Their number is $[G : D]$ and their degrees are $|D|$, since the embeddings are parametrized by $G$ and the action is free. This is the completion correspondence of Corollary F.4.2.

Under the constant translation action, the length-$j$ truncation belongs to $H_v$ exactly when $D$ maps to zero in $W_j(\mathbb{F}_p)$. It belongs to $H_v$ if the truncated equation has any solution there, since the two solutions differ by a constant vector. Thus the largest such $j$ is the integer with $D = p^j G$, which proves the count and degrees.

The weight zero has one lift. For the remaining weights, use the finitely many cones and corrections of Lemma F.6.3. For each available correction $c^{(j)}$ on a cone include the nonzero rational functions in $E$ of the form

$$(z_{<j} - c^{(j)})_i - b, \qquad 0 \le i < j, \quad b \in \mathbb{F}_p.$$

For a weight with largest split length $j > 0$, the corrected truncated solutions are all integral and reduce to the distinct constants of $W_j(\mathbb{F}_p)$. Indeed they are the constant translations of the unique small solution, and Witt addition commutes with residue. Two different lifts correspond to different constants because the translation kernel at this length is $D$. At some coordinate the residues differ. Subtracting the residue of that coordinate at the first lift gives a function of norm less than one there and norm one at the second lift. It is nonzero in $E$ because of its value at the second lift. The finite union of the lists separates all the fibres. □

**Corollary F.6.1** *For every finite abelian Galois extension $E/F$ there is a finite subset of $E^\times$ separating the prolongations to $E$ of every Gauss value of $F$. In characteristic $p > 0$ the same holds when the Galois group is a direct product of an abelian $p$-group and a group of order prime to $p$.*

***Proof*** Characteristic zero follows from Theorem F.5.1. In positive characteristic first suppose that the group $G$ is an abelian $p$-group. For two distinct cosets of a decomposition subgroup $D$, some cyclic quotient of $G/D$ separates them, by decomposition of a finite abelian group into cyclic groups. Its corresponding subextension is cyclic of $p$-power degree, and the two lifts restrict to distinct prolongations there. Indeed the decomposition subgroup acts trivially on that quotient, while the two cosets have different images. Take the union of the finite separating families in Theorem F.6.1 for all cyclic quotients of $G$.

For a direct product of groups of coprime orders, every subgroup is the product of its intersections with the two factors. For an element of the subgroup, powers with exponents congruent to one modulo one order and zero modulo the other give the two components separately. Thus two distinct decomposition cosets differ after projection to one of the two factors. Apply the preceding $p$-group result and Theorem F.5.1 to the two corresponding Galois subextensions. This also covers every finite abelian group by its primary decomposition. □

In these cases the analytic chart of Proposition F.4.1 has a finite separating family as soon as its rational separable field is Galois with one of these groups. This follows by transferring the family through Corollary F.4.2.

*Example F.6.1 (Two successive splits)* Over the two-dimensional torus with coordinates $s, t$ in characteristic $p > 0$, take a Witt equation of length two with right side

$$(t, s/(1+t)).$$

Its Gauss fibre has one point of degree $p^2$ for $v_t \le 0$, $p$ points of degree $p$ for $v_t > 0$ and $v_s \le 0$, and $p^2$ points of degree one for $v_t > 0$ and $v_s > 0$.

Over the rational function field, there is no root of the first equation. The pole of $t$ at infinity has order one and a pole of an Artin–Schreier image has order divisible by $p$. Thus the first equation is irreducible by the translation group calculation for a scalar equation. The successive scalar equations are separable, and the field generated by one solution contains all its constant translates. It is therefore Galois with group acting by translations in $W_2(\mathbb{F}_p)$. Its image on length one is surjective by the irreducibility of the first equation and Galois correspondence. A subgroup of the cyclic group of order $p^2$ surjecting onto length one is the full group. The field generated by one solution therefore has degree $p^2$.

At a Gauss weight, the Laurent test in Lemma F.3.2 shows that the first component splits exactly when $v_t > 0$. There it admits a small root, so the criterion for extending to length two in Lemma F.6.3 is the solvability of the scalar equation $\wp(u) = s/(1+t)$. Its calculation in Example F.5.1 gives exactly $v_s > 0$ on that region. The three possible values of the maximal split length in Theorem F.6.1 give the stated degrees, as shown in Figure F.2.

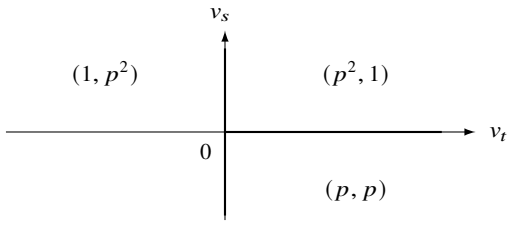



**Fig. F.2** The number of points and the degree of each point in Example F.6.1. On $v_t = 0$ the left-hand pair applies. On $v_s = 0$ with $v_t > 0$ the lower pair applies.

# Appendix G
# Source notes

## G.1 Gradient transport and toric measures

The convex transport statement underlying Theorem 33.1.1 is a special case of McCann's main theorem [McC95, p. 310]. The source probability is $\lambda/V$ on $P$, and the target probability is $(-\mathrm{id})_*(\mu/V)$. Convex transport in that theorem allows targets without moment restrictions.

Finite atomic measures give polyhedral partitions. Tightness bounds the slopes on compact interior sets, using a uniform positive volume bound for cones there. Coupling the two gradients at a common value gives uniqueness without integrating an unbounded cost.

The Gauss pushforward of the real top product has mass $V$. By Proposition 32.7.3 it computes the truncation measure also for roofs of energy $-\infty$.

## G.2 Decreasing limits on polyhedral charts

The sheaf used in Section 11.4 uses the sequence closure of Chambert-Loir and Ducros [CLD25, Definition 8.1.1] in the induced smooth structure of a real chart. The successor step in their proof [CLD25, Proposition 8.1.11, pp. 197–198] identifies local decreasing and uniform approximation by continuous functions of the preceding stage. Membership in the psh sheaf at the successor stage initially supplies a decreasing sequence of psh functions of the preceding stage. The missing implication is the replacement of that sequence by continuous predecessors when its limit is continuous.

Proposition 11.5.1 and Theorem 11.6.1 give approximation on fixed neighborhoods using a common smooth trace or a common affine support. Corollary 11.6.1 identifies the mixed products on those charts.

## G.3 Extended energy and orthogonality

Boucksom and Jonsson define the weak envelope property and prove that a bounded obstacle and the upper regularization of its envelope have the same energy in an ample class, on an irreducible projective variety over an algebraically closed field [BJ23b, Definition 3.2 and Theorem B]. Theorem 18.1.1 is stated over every trivially valued field. As in their proof, an increasing family of competitors is pulled back to the birational model, and the limit of the energies is controlled by increasing convergence, here Corollary 17.13.1. Their energy is normalized by the volume, so it is the energy of this book divided by $(\omega^n \cdot [X])$.

The restriction estimate of Proposition 18.2.1 follows the proof of [BGM22, Lemma 3.2], which compares the relative volume $\mathrm{vol}(L, \varphi + f, \varphi)$ with $\int f\,(dd^c\varphi + dd^c\psi_1)^n$ for a continuous psh metric $\varphi$ on an ample line bundle $L$ and a difference $f = \psi_1 - \psi_2$ of continuous psh metrics on an auxiliary ample line bundle, over an arbitrary non-Archimedean field, by restricting sections to a vertical divisor. Over the affine line the torsion quotients are finite-dimensional vector spaces, and the Hilbert polynomial on the divisor replaces their asymptotic Riemann–Roch theorem [BGM22, Theorem 2.4]. Lemma 18.2.2 shows that a section lattice is unchanged when the obstacle is replaced by its envelope. The corresponding identity between the volume of the sup-norms of a continuous function and its extended energy appears in the proof of [BJ22, Lemma 8.7].

Boucksom, Gubler and Martin prove differentiability of relative volumes [BGM22, Theorem 3.1]. Under continuity of envelopes for an ample line bundle this gives differentiability of the energy of the envelope and orthogonality, which they show to be equivalent [BGM22, Theorem 3.4 and Lemma 3.5]. Boucksom and Jonsson prove uniform differentiability of the extension $E^{\downarrow}$ of the energy by continuous majorants along real PL directions at every potential of finite energy in an ample class [BJ22, Theorem 8.5], using [BJ22, Lemmas 8.7 and 8.8]. At continuous psh potentials, [BJ22, Corollary 8.6] gives differentiability of the extended energy along continuous directions without the envelope property. Proposition 18.3.1, and with it Theorem 18.3.1, assumes the envelope property only in the real target class. Its proof transfers the section-count estimate from rational ample classes converging to that class, in which the envelopes may be discontinuous. Proposition 18.3.2 corresponds to [BGM22, Lemma 3.5], and Corollary 18.3.1 to [BJ22, Theorem 13.1], which assumes an irreducible variety. Lemma 18.1.1 quantifies, for bounded obstacles, the continuity in the class of [BJ22, Proposition 8.1(iii)].

## G.4 Toric mixed products and moment bounds

Burgos Gil, Philippon and Sombra express the measures of semipositive toric metrics by real Monge–Ampère measures of concave weights [BGPS14, Theorem 4.8.11 and Corollary 4.8.12]. Their discretely valued formula uses a section of tropicalization to recover the analytic measure. In Corollary 32.3.1 the weight convention is convex and

the section is the trivial-valuation Gauss section. Proposition 32.8.1 uses an auxiliary ample polytope to compare several semiample polarizations. The diagonal product uses $n!$ times lattice volume, which vanishes for a lower-dimensional polytope.

Burgos Gil, Gubler, Jell and Künnemann construct the non-pluripolar truncation operator on the tropical compactification and solve the full-mass equation for measures carried by the dense part of the skeleton [BG+21, § 6.3.5 and Theorems 6.3.10 and 6.3.13]. In Theorem 32.9.1, polarization of the masses on Minkowski sums gives the mixed total mass. Locality on the increasing strict sets then gives the total variation estimate.

The dimension-one moment estimate in [Com+18, Proposition 3.11] uses monotonicity of the derivative of the dual convex function. The cone estimate in Theorem 33.3.1 gives integrable roofs at the exponent $n/(n+1)$ in dimension $n$ and strong convergence under uniform control of its tails. Example 33.3.1 computes the effect of concentration at a single vertex.

## G.5 Envelopes of cohomological potentials

Mesquita-Piccione proves the divisorial maximum principle and regularizes the ray transform [MP25, Proposition 4.3.6 and Theorems 4.3.8 and 5.1.4]. In Theorem 22.6.1, homogeneous flag ideals compute the normalized orders on smooth refinements by the minimum rule. Exceptional degrees give the obstacle comparison.

The ray proof of envelope continuity for continuous obstacles in a Kähler class is in [MPWN25, Theorem 1.4.1 and § 4.1]. In Proposition 22.7.1, a constant model candidate supplies the lower bound for compactifying the ray. The nonpositive models decrease after rescaling because their factors increase.

Lemma D.5.1 uses Demailly–Păun's gluing argument [DP04, Proposition 3.3(i)–(ii)]. In Proposition 22.7.2 the positive representative and exceptional perturbation give approximate ray realizations with uniform errors tending to zero.

The proofs of the obstacle-gap estimates in [MPWN25, Proposition 4.2.3] and [WN25, Proposition 7.4] each assert that a representative has a Lelong number at most the infimum minus a positive error. Theorem 23.2.1 obtains equality after a large fibre shift using rays and positive currents. Homogeneous generators control rotations even where the action does not lift to the smooth refinement.

## G.6 Volume loss along a divisor

Vu bounds volume loss by applying the complex envelope contact formula to a rounded logarithmic obstacle [Vu23, Lemma 3.1]. Lemma 23.4.4 keeps the curvature bound as a form so that its mixed integrals can be estimated on all the smooth refinements. In Theorem 23.4.1, the effective divisor used for subtraction is the complement of the approximation error in a fixed multiple of the central fibre. The square

of the fibre class is zero. The resulting first-order term is the intersection of the approximation error with the positive class.

Mesquita-Piccione and Witt Nyström prove orthogonality via numerical restricted volumes [MPWN25, Theorem 1.5.1 and § 4.2]. Vu's divisorial derivative formula [Vu23, Theorem 1.1] and the general Morse inequality of Tosatti [Tos26, Theorem 1.1 and Corollaries 1.2–1.3] also relate these restricted volumes to the positive intersection product in big classes.

## G.7 The Kähler variational equation

Mesquita-Piccione uses a fixed component intersection graph for divisorial bounds [MP25, Theorem 4.3.3]. Lemma 23.5.1 uses the positive real intersection weight of each edge, with a Kähler form on the fixed smooth refinement. The product component starts the path estimate at the trivial valuation.

The bounded ray supremum and compactness arguments in Theorem 23.5.1 are related to [MPWN25, Theorem 3.1.1 and Proposition 3.1.2]. The compact product argument uses convergent subnets. The variational theorem and the strong homeomorphism correspond to [MPWN25, Theorem B and § 6]. Here (23.10) follows from the continuous obstacle derivative and an energy estimate on ordered potentials. For the uniform testing bound (23.11), the midpoint and Cauchy–Schwarz estimates are computed on models. The uniqueness argument tests first against model measures and then against every fixed divisorial Dirac mass.

## G.8 Change of class and relative positivity

The local dual-energy bound of Proposition 23.7.2 is the Kähler counterpart of the global ample-class estimate in [BJ22, Theorem 9.24]. Class independence of the strong measure topology corresponds to [BJ22, Proposition 10.16]. The equation with varying classes uses a vanishing variational deficit and uniform testing in the limiting class.

Mesquita-Piccione imposes nefness relative to the parameter line [MP25, Definition 4.1.2 and Theorem 4.1.5]. The lift in Proposition 24.1.1 uses a base cutoff on a fixed smooth refinement. Example 24.1.1 distinguishes the full divisorial orders used in its coefficient bound.

## G.9 Maximal rays and test curves

Mesquita-Piccione constructs maximal geodesic rays and states the finite-energy test-curve comparison [MP25, Theorem 5.1.7 and § 5.2]. We use $s = -\log|t|^2$, twice the

parameter in [MP25, Equation (5.1.1)]. In Lemma 24.3.1 the smooth energy polynomial and Stokes give the cohomological intersection as the circular mean slope. Dyadic multiplier approximations and a bound on the full coefficients of smooth model corrections give the upper and lower energy inequalities. The decreasing limit in Theorem 24.4.1 uses strict energy monotonicity for ordered potentials.

The unnormalized test-curve energy is the radial energy of [DXZ24, Theorem 3.17] multiplied by the class volume. Theorem 25.10.1 matches it with the cohomological energy in a Kähler class. Product multiplier ideals identify the full divisorial ray data, and the two-coordinate blowups recover every rational scaling of a base divisor order.

The maximal ray with arbitrary finite-energy initial value in Theorem 24.4.2 recovers its endpoint by the ordered complex energy difference and domination. The mixed slope comparison [MP25, Theorem 5.3.4] uses smooth compatible and maximal rays. Here Theorem 24.5.1 applies to maximal rays in Kähler classes with finite-energy initial values. The estimate (24.7) controls the passage from continuous cohomological data to finite energy uniformly for $s \geq 1$ after division by $s$.

## G.10 Full evaluation and nef perturbations

The divisorial evaluation of test curves and its projective comparison appear in [DXZ24, Theorems 3.12 and 3.14]. Theorem 25.11.1 uses the finite Taylor coefficient ideals to obtain an upper semicontinuous function on the whole complex valuation space. The local multiplier criterion compares both the vertical and horizontal orders of the space-time weights. For the Kähler comparison, Theorem 25.12.1 uses decreasing bounded truncations and tests the slice orders after pullback to the compact product.

For an arbitrary numerical class $\theta$, [BJ22, Definition 4.1] defines $\theta$-psh functions as decreasing limits of generically finite Fubini–Study functions in rational classes converging to $\theta$, as in Definition 6.2.1. Theorem 26.1.1 instead intersects the decreasing model cones of the Kähler perturbations of a nef class. In Theorem 26.2.1 the Kähler obstacle-volume identity determines the limiting scalar energy, and the ordered energy test makes its first variation uniform in the class. The perturbative operator on bounded potentials recovers this measure at the envelope. Comparison with fixed-nef models uses the membership hypothesis of Corollary 26.3.2.

The expansion in Theorem 26.4.1 parallels the algebraic pairing and measure construction in [BJ22, §§ 7.1–7.3]. Its polynomial dependence on the backgrounds gives total variation convergence for fixed potentials. Theorem 26.5.1 identifies the truncation energy with the extended test-curve energy. The ordered estimate gives the intrinsic measures in Theorem 26.6.1.

## G.11 Locality and removal of poles

The strict comparison in Theorem 26.8.1 has an algebraic counterpart in [BJ22, Theorem 7.40]. On a model, the first slot agrees in a neighborhood of every component with strict inequality. For general bounded potentials the integration formula and decreasing continuity of the scalar pairings pass the weighted identity to the limit. Total variation for fixed bounded data then permits the change to nef backgrounds.

The restriction in Theorem 26.9.1 is analogous to the use of strict superlevels in [Bou+10, §§ 1.1–1.2]. Here the products being restricted are the bounded cohomological mixed measures on the valuation space. Finite energy in the sum class bounds the discarded mass, and Example 26.9.1 shows that separate intrinsic energies in classes of zero volume do not control that mass.

## G.12 The intrinsic equation in a nef class

In the ample theory, uniform differentiation and weak convergence along maximizing nets are proved in [BJ22, Theorems 8.5 and 9.11]. The integrability criterion for irreducible varieties is [BJ22, Theorem 9.20]. For Kähler backgrounds see [MPWN25, Theorem B and § 6]. The big test configurations in that reference still have a Kähler class on the general fibre. In Theorem 27.4.1 the background on $X$ is itself nef and big.

The uniform remainder passes to bounded upper semicontinuous obstacles by decreasing approximation in Theorem 27.1.1. Model obstacle envelopes supply the finite test support used in the existence and uniqueness proof. Continuous majorants in a fixed larger Kähler class give the uniform testing bound (27.11). For strong convergence of the inverse solution map, upper semicontinuity of each fixed bounded envelope gives the required upper bound on the variational deficit.

## G.13 Moving nef and big backgrounds

For an irreducible projective variety, [BJ22, Theorem 9.24] compares dual energies under two-sided multiplicative bounds on ample classes. The Kähler comparison Theorem 23.7.1 also has a fixed-reference criterion. In Theorem 27.6.1 the dual energies are evaluated in the moving classes.

On normal projective varieties over an algebraically closed trivially valued field of characteristic zero, [RW26, Theorem C] proves continuity over the big cone for probabilities with the same finite quasimonomial support. The quotient pseudometric uses the limiting class. Corollary 27.6.1 uses model tests and moving nef energies and allows vanishing weights.

## G.14 First variation over big backgrounds

Lemma 28.1.1 proves affinity in the full fibre direction by gluing minimal potentials to a product potential on the complement of a tube. In Lemma 28.2.2, the positive approximations can have singularities outside the central fibre. Their divisorial parts are bounded by those of a reference current whose smooth remainder absorbs the fixed test curvature. This gives the uniform quadratic estimate on refinements used for the extension to continuous obstacles.

Proposition 28.2.1 gives the positive-product pairing for vertical directions. It agrees with the numerical restricted-volume formula of [Vu23, Theorem 1.1] and with the positive-product expression in [Tos26, Corollaries 1.2–1.3]. On a smooth equivariant model with simple normal crossings central fibre and a Kähler background, take the big test configuration of class $B = \rho^*\alpha + c_1(D) + c\,c_1(W_0)$ as in that proposition, with an admissible fibre shift $c$. Its Monge–Ampère measure [WN25, Definition 3.3], a probability measure by [WN25, Theorem 3.4], gives each central component $E$ the mass $V^{-1}b_E$ times the restricted volume of $B$ along $E$. Placing these masses at the normalized tests $v_E$ gives $V^{-1}\mu_\alpha^{\mathrm{big}}(f)$ by (28.8), once the restricted volume is identified with $M_B(E)$. Apply the two derivative formulas in each component's direction to see this identification. Theorem 28.2.1 uses $V = \mathrm{vol}_X(\alpha)$ for an arbitrary big background and also allows continuous obstacles.

For rational line-bundle data on a smooth projective model, the volume in (28.3) also equals the asymptotic section volume. Fujita's approximate Zariski decompositions, as stated in [Bou02, Theorem 1.1], approximate the section volume by masses of positive currents with analytic singularities. Conversely, the approximating top masses in (28.6) are masses of such currents, hence integrals of the top powers of their absolutely continuous parts. They are bounded by the section volume by [Bou02, Theorem 1.2].

## G.15 Relative rays of big-class envelopes

The compatible test-curve families and their divisorial evaluation appear in [DXZ24, Definition 3.3 and equation (24)]. [DXZ24, Theorems 3.12 and 3.14] compare with algebraic potentials for big line bundles and pseudoeffective rational line bundles, respectively. Lemma 28.3.2 realizes the greatest minorant of a real model obstacle by finite constraints on the disk potential. The comparison with perturbed coordinates in Proposition 28.3.1 works even when a competing family has zero limiting left mass in the original class.

For Kähler backgrounds, [WN25, Theorems 7.2–7.3 and Proposition 7.4] relate orthogonality and the obstacle gap to restricted volumes and generic Lelong numbers on the model. Proposition 28.4.1 computes the gap on an arbitrary big background from a compact extension of the relative ray. Its non-pluripolar mass supplies the ray's energy slope. In Theorem 28.4.1 the divisorial volume derivative gives zero

mass on components with a positive gap. The full-space contact formula then follows by approximation of the obstacle.

## G.16 The discrete variational equation on a model

The estimate in Lemma 28.2.1 uses the rank-one form from the rounded logarithmic norm. In Lemma 28.2.2 the remaining error is computed against the fixed test divisor and bounded on the fibre. This removes dependence on the lower bound of the obstacle. Theorem 28.5.1 can therefore differentiate at every obstacle of finite relaxed energy, even when its continuous majorants have no common norm bound.

For Kähler backgrounds, [WN25, Lemma 6.2 and the proof of Theorem A] prescribes the component masses using divisors constructed from graph distances and a map of simplices. In Theorem 28.6.1 the weighted fibre graph supplies the restriction-degree coordinates of a compact polytope. At an active inequality the atom vanishes by Lemma 28.6.2. Inverting the graph Laplacian supplies a feasible direction of strict increase unless the derivative already gives the prescribed masses. The argument uses the big-background first variation and allows zero prescribed masses.

## G.17 Weak compactness and dual approximation in big classes

In Theorem 29.1.1, the logarithmic lower bound for the supremum of a tail is taken relative to the minimal current on the product, whose horizontal singularities can persist in a big class. The limiting curve is obtained using a common positive lower mass at one left slope, as in the decreasing-limit construction of [Xia26b, Propositions 9.4.9 and 13.4.5]. The quadratic scaling estimate in Lemma 29.2.1 comes from a convex combination of the analytic rays. The term with the reference measure is bounded independently of time by the contact formula [Xia26a, Corollary 16.2.32].

Under the orthogonality property, [BJ23c, Proposition 2.19 and Theorem 2.23] use uniform differentiability and mixed energy estimates to prove convergence in dual energy along maximizing sequences. Theorem 29.2.1 chooses continuous obstacles by a norm penalty. The first variation then controls total variation, and contact identifies the dual values. Uniform approximation of each chosen obstacle gives finite model support with weak convergence and convergence in dual energy.

## G.18 Relative rooftops and integrability on energy balls

For the complex background the $d_1$ metric of relative potentials and the radial metric of rays are given in [Xia26b, Definitions 4.3.1 and 4.3.5 and Theorem 4.3.4]. [Gup24, Theorem 5.7] proves a complete geodesic metric theorem for analytic rays based at

the minimal potential in a big class, for all finite $p \geq 1$. In Theorem 29.3.1 the Perron segments ending at analytic rooftops decrease to a disk potential. Saturation preserves the vertical orders and identifies that potential with the intrinsic rooftop. The distance identity transfers contraction from the analytic slice metric. The completeness proof takes successive finite rooftops and the increasing limit of their tail envelopes, using the ordinary test-curve supremum.

Theorem 29.3.2 combines the rooftop inequality with the quadratic scaling bound. If negative integrals concentrate on sets of small measure, the proof retains their integrals on disjoint subsets and combines scaled tests of summable energy cost. The resulting finite-energy minorant would have infinite negative integral. This gives the uniform tail estimate used for countably supported measures in Corollary 29.3.2.

## G.19 Variational attainment and the radial midpoint estimate

In the analytic big-class theory, [Ber+09, Theorems 4.1 and 4.7] characterize solutions as variational maximizers. The existence proof first treats measures dominated by capacity, then passes to general finite-energy measures. In Theorem 29.4.1 the coercive bound and midpoint estimate give Cauchy convergence directly in the complete radial metric. The unique normalized maximizer has the prescribed measure.

The analytic extrapolation obstacle satisfies the contact bound of [Xia26a, Lemma 16.4.3]. In Lemma 29.4.2 Perron segments transfer it to the radial energy. The convex combination of jets is identified by testing in one larger Kähler class. The estimate for integration follows by testing against the extrapolated potential. The midpoint estimate uses symmetric integration by parts for bounded differences [Xia26a, Theorem 17.1.3]. By locality, the estimate for each positive part follows from the same squared Dirichlet pairing.

## G.20 The strong topology in a big class

[BJ22, Definition 12.1 and Theorem 12.4] define strong convergence by combining convergence on divisorial valuations with convergence of energy. For an ample class on an irreducible projective variety they metrize it by an energy quasi-metric. Theorem 29.5.1 identifies the same convergence criterion on the local tests of the complex valuation space using the rooftop distance. The uniform bound for each test comes from the normalized slice $L^1$ bound and the local Lelong estimate.

The ample-class mixed-energy comparison in [BJ22, Proposition 12.7] gives a bi-Lipschitz embedding modulo constants for the potential and measure energy quasi-metrics. Theorem 29.5.2 uses radial continuity of the scalar energy of a shifted obstacle and the uniform first-variation remainder to obtain weak continuity of the measures in arbitrary big classes.

## G.21 The divisorial reduction of the relative equation

[Bou04, Proposition 3.6, Definition 3.7, Proposition 3.8, and Remark 2 in § 2.7] identifies the divisorial multiplicities with those of a minimal current in a big class, constructs the modified nef positive part, and compares nef and modified nef classes on surfaces. Lemma 30.1.1 uses Siu decomposition and regularization to extract the fixed part. On a surface it smooths the remaining isolated poles locally with a regularized maximum.

Theorem 30.1.1 concerns the finite-energy domains on the valuation space. It transfers multiplier types between the two classes and compares the nonpositive model representatives within a bounded correction. The equality of jets also holds at infinite divisor orders, by sandwiching the spatial multiplier ideals between integral divisor powers. The comparison of obstacle derivatives gives the same intrinsic measure. In Corollary 30.1.1 the resulting relative duality has the full nef variational theory whenever the positive part is nef.

## G.22 The intrinsic equation after modification

[DN13, Theorem 2.1 and Proposition 2.3] compares analytic non-pluripolar products under bimeromorphic maps and transfers analytic energy classes when the pushforward maps the positive currents in the given class onto all positive currents in its pushforward class. In Theorem 30.2.1 the analytic change of variables gives the equality of test-curve slice masses in a pulled-back class. The order test in a common Kähler enlargement identifies the pulled-back valuation potential, and the intrinsic obstacle derivatives give the valuation-space measure identity.

[MP25, Proposition 2.1.6, formulas (2.1.3)–(2.1.4)] uses relative generation of twisted ideals and cancellation of the exceptional order on valuations finite on every nonzero ideal. Proposition 30.2.1 works on the open locus of finite exceptional order, allowing other ideal orders to be infinite. The analytic weights identify the negative of the twisted ideal order with a pulled-back jet. The multiplier-ideal inclusion in the proof of [Xia26b, Lemma 1.6.3, (1.23)] and the passage to Gauss orders give that jet identity.

The measure correspondence in Theorem 30.2.2 uses the open set on which the valuation map is a homeomorphism. On projective manifolds it can be obtained from the algebraic isomorphism locus. The weight argument supplies an open set for towers of smooth blowups, including nonprojective manifolds. The finite-energy pole test rules out mass on the complement for measures of finite dual energy in the nef positive part. The relative dual objective downstairs then agrees with the nef one under the unique supported lift.

The domination argument in Corollary 30.2.1 uses the smooth blowup tower of [Xia26b, Theorem B.1.2]. The nef positive part pulls back to the tower. Thus the relative equation applies whenever a proper modification has nef positive part.

## G.23 Relative duality without a nef positive part

The analytic variational functional in [Ber+09, Definition 4.3 and Theorem 4.7] tests finite-energy potentials relative to the extremal potential. On the valuation space, subtracting the minimal jet removes the extra ideal-order moment in (29.2). For a model obstacle, Lemma 30.3.1 regularizes the compactified ray on the product. The fixed-form curvature error is absorbed by a Kähler current on the base and the losses at the finitely many vertical constraints by a logarithm of the parameter. This gives local uniform approximation on the same finite ideal-order locus for all continuous obstacles. The pole test shows that each measure of finite relative dual energy is carried by that locus.

The strong homeomorphism in Theorem 30.3.2 uses continuity of testing on relative dual sublevels. The absolute-dual counterexamples in Section 33.5 concern the extra moment of the greatest normalized potential $u_{*}$. On the locus of finite exceptional order, the transported equation from Theorem 30.2.2 agrees with the intrinsic one.

## G.24 Rational refinement and affinoid gluing

[Tem05, Proposition 2.1 and § 3, Theorem 3.1] uses multiplication on the fibre algebra to identify completed residue fields for affinoid monomorphisms, and proves the Gerritzen–Grauert theorem at arbitrary real radii using graded reduction. The residue-field argument in Appendix E uses that multiplication map and the bounded coefficient functionals of a Tate presentation. For finite rational refinement it approximates all the chosen relative coordinates in the common completed field. Strict errors persist on a neighborhood in the domain. Translating the coordinates then gives a contracting division with a finite module of remainders. The finite domain inclusion is cut out by an idempotent, whose rational approximation gives a rational domain of the original space near the point.

The rational covers and the G-cover convention use [CLD25, §§ 11.1.5–11.1.8]. The proof of exactness starts with a split at one real radius. A finite sequence of such splits places every final chart in a member of the prescribed rational cover. Refinement and the double Čech complex then give exactness for finite affinoid covers. The structure sheaf follows by gluing on affinoid subdomains of each analytic domain.

## G.25 Degrees under two projections of an analytic moment

[CLD25, Theorem 14.2.7 and Proposition 14.3.1] compares degrees on the common interior skeleton when the log norm maps agree up to translation, and identifies the calibrations of two full-rank projections almost everywhere on a top-dimensional source cell. The proof of Theorem 14.2.7 interpolates between nearby tuples of units

over a parameter disc. In Lemma 10.6.1 we carry out the interpolation in the local algebra of an exact Gauss fibre with rationally independent coordinates. Contracting division proves finiteness of the interpolated map, and the leading coefficient at the initial parameter proves torsion-freeness over the one-variable base. The dimensions at the two endpoints agree.

The common exact-norm domain in Theorem 10.6.1 retains the local factors of both finite fibres, including their nilpotents. Isogenies with integral matrices reduce the coordinate change to equal weights. The two lattice indices give the normalized weight equality. Corollaries 10.6.1 and 11.10.2 compare the resulting integrals on a common cell and the mixed products on open subsets contained above that cell. Constructing the finite projection neighborhoods for a general moment is a separate analytic step.

[CLD25, Proposition 14.2.2 and § 14.3.2] uses projection degrees on top-dimensional cells of a paracompact skeletal subset in a moment's characteristic polyhedron. A combined moment gives independence of choices. Theorem 10.8.1 compares smooth top forms on a common subdivision of two finite charts. It treats a projection of deficient rank by the vanishing of the pulled-back top form. For continuous convex weights, Theorem 11.11.1 uses the continuous Gauss fibre traces to extend the degree comparison to dependent weights when one rational linear transition holds on the Gauss preimages. This gives equality of the mixed measures on faces even when the real mixed measure charges those faces.

## G.26 Finite neighborhoods and algebraic tropical weights

[Ber93, Proposition 3.1.4] characterizes the local finiteness of a morphism between good analytic spaces by fibre isolation and relative interior. Proposition 9.2.1 gives the affinoid argument in terms of presentations. A rational source neighborhood isolates the point, and shrinking the target removes the compact image of the loci of equality in the generator bounds. Strict spectral division then gives finiteness.

[CLD25, Lemma 11.6.5 and Proposition 14.2.2] obtains finite flat neighborhoods using the dimension of the fibre at an interior skeletal point with reduced target. In the annular setting of Proposition 9.2.2, once finiteness is available, the Gauss kernels are zero. Generic freeness therefore gives a rational domain of freeness around the whole Gauss section. At an independent weight it contains a smaller annular box.

The top-cell degree description in [CLD25, § 14.5.2 and Remark 14.5.4] uses the finite flat degrees of projections and their lattice indices. In Theorem 10.7.1 a closed box adjoining the top cell gives a finite chart: the transverse characters have norm one throughout that domain, strictly inside the chosen bounds. Division in the equal-radius fibre counts the remainder monomials by the transverse initial algebra. This matches the algebraic multiplicity. Continuous fibre traces then compare the two degree sums also at dependent weights in the relative interior of that cell.

## G.27 Affinoid dimension and maximal Gauss fibres

The dimension formula for analytic points is [Ber93, Lemma 2.5.2]. The affinoid argument of Theorem 9.3.1 uses contraction to obtain finite normalization in the strict case, and the Gauss lift to recover the maximum after strictification. The lift preserves the value-group and residue-field contribution together, including when the auxiliary radii become dependent over a completed residue field.

The dimension and fibre statements used in the skeletal construction occur in [CLD25, Equation (11.2.3.1) and Lemma 11.5.1]. Proposition 9.3.1 derives the affinoid inequality directly by applying (9.3) over the ground field and over the completed residue field of the base point. Every Gauss norm on a torus contributes the full dimension of that torus to the invariant. At dependent weights a lower value-group rank is accompanied by a larger residue transcendence degree. This gives isolated fibres for Corollary 9.3.1.

## G.28 Free charts and intrinsic Gauss degrees

[CLD25, Lemmas 11.6.4–11.6.6 and Proposition 14.2.2] obtains finite flat neighborhoods at relative-interior points over maximal-dimensional Gauss points of reduced targets. It proves degree constancy on the interiors of top-dimensional cells satisfying its convenience condition, when the tropical projection is injective on the cell image. For an annular target, Lemma 9.4.2 establishes reducedness of the required neighborhoods by formal elimination after complete scalar extension. Lemma 9.4.4 uses exactness on finite modules and the Gauss lift to identify the localized target ring as a field. This gives the finite free chart of Theorem 9.4.1.

The fibre-algebra degree of (9.4) is the local degree of [CLD25, § 13.3.1] for zero-dimensional fibres. The argument of Proposition 9.4.1 keeps the whole local Artinian factor under domain restriction. Corollary 9.4.1 uses rational approximations in the completed Gauss field to obtain piecewise affine coefficient norms on these charts, then specializes characteristic polynomials. It gives the joint norm graphs for each prescribed finite list of units.

## G.29 Curve skeleta and separating functions

[Duc12, Theorems 2.8 and 5.1] establishes finite separation of extensions of Gauss valuations and the piecewise linear structure of skeleta of moments. For one-dimensional moments over a trivially valued field, Theorem F.1.1 constructs the pieces explicitly. At weight zero the strict Laurent inequalities give an algebraic localization. Completing the normalized curve at its points over zero and infinity gives the branches and their degrees. The Chinese remainder theorem supplies one additional analytic unit separating the branches on each side.

The degree in (F.2) includes the length of the generic local algebra, so the topology is unchanged under nilpotent thickening while its weights record the length. Corollary F.1.1 applies the same construction over the trivial norm on a rational function field, by forming an analytic fibre of the other character coordinates.

## G.30 Radial products and finite sets of directions

The scaling action on algebraic analytifications is described in [BJ22, § 1.1.4]. For a general affinoid source, Proposition F.2.1 uses strict presentation bounds to raise seminorms to nearby positive powers. Isolating one fibre point in a finite free chart makes its local degree constant along the resulting interval. Corollary F.2.1 realizes the angular slice as an affinoid fibre over the completed Gauss field of a character.

For the finitely many directions of Theorem F.2.1, the algebraic Laurent chart at weight zero has one generic local factor. Along each open ray its fibre algebra stays fixed as a ring and the absolute value on the base field is raised to a positive power. The proof uses density in these finitely many algebras and a single product of separating elements. [Duc12, Theorem 2.8] gives the broader finite separation theorem for the prolongations over all Gauss weights of a rational function field.

## G.31 Finite separation by initial Laurent fractions

[Duc12, Theorem 2.8] gives a uniform finite separating family over Gauss valuations for arbitrary finite algebras over a rational function field. The replacement argument on pp. 47–48 of [Duc13] supplies the finite diagonalizing bases needed in the universal separation step on p. 36.

In Theorem F.3.1, initial fractions detect the characters trivial on the fibre Galois subgroup. Subtracting roots of those fractions from coordinate products gives separating units. The degree formula includes nilpotent lengths and all subdivision faces. In Example F.3.2, the separate coordinate norms agree at both lifts. The normalized product of the coordinates minus one separates them.

For Proposition F.3.1 the approximations are Laurent polynomials with absolute error strictly less than one. After removing Artin–Schreier images, the term weights and the constant coefficient test which linear combinations split. A product of translates gives one separating norm per tested direction in the space of linear forms.

## G.32 Separable charts and the algebraic skeleton problem

[Duc03, Lemma 1.9(iii) and §§ 0.13–0.14] factors the map at an interior point on strictly affinoid neighborhoods. The target is reduced, the image point has total rank

equal to the target dimension, and the source dimension is at most that dimension. The two maps are finite flat radicial and finite étale. In Proposition F.4.1 the separable polynomial is constructed on the original torus target by lifting a simple root. The vanishing of the complementary coefficients on a Laurent neighborhood extends the nilpotence or power relations of the central algebra. Corollary F.4.1 tracks the relative rank in the norm and the local degree, including the contribution of nilpotents.

[DT26, §§ 5.2.3–5.2.4] algebraizes a moment after reduction to a smooth algebraic chart over a perfect field. Approximating its coordinates preserves the Gauss locus by preserving their graded residues. Finite separation then uses [Duc12, Theorem 2.8(1)]. The local statement in Corollary F.4.2 reduces finite separation to the prolongations of the Gauss values on one finite separable extension of the rational torus field. The relative rank multiplies the completed-field degrees.

## G.33 Toric cones and the ramification of Gauss values

The general uniform separation theorem for a finite rational function field extension is [Duc12, Theorem 2.8], with the corrected proof of the universal step in [Duc13]. Theorem F.5.1 uses a different argument in its range. The ramification of the normalization in codimension one is eliminated on a monomial root cover, and purity produces a finite étale algebra near each generic toric orbit. The Chinese remainder idempotent residues there distinguish the prolongations, since the extra Gauss residue variables are purely transcendental.

In Theorem F.5.2 the same subdivision organizes the Laurent expansions of the rational coefficients. Selecting $p$-divisible exponents and taking the principal parts of a rational root keeps the correction rational. Iteration reduces the pole bounds. The resulting constants and least terms detect splitting at every weight in a cone's relative interior, including dependent weights.

## G.34 Witt coordinates and successive splitting

The additive ghost calculation in Lemma F.6.1 uses the prime-power case of the ghost congruence criterion, as in [Haz09, Lemma 9.93]. The presentation in Lemma F.6.2 is the cyclic part of Artin–Schreier–Witt theory. Compare the presentation theorem and the discussion of additive Hilbert 90 in [Haz09, Theorem 7.11 and § 7.12].

The separation proof in Theorem F.6.1 applies the rational scalar reduction of Lemma F.5.2 successively on a toric cone. Where a truncation splits, its corrected small root has zero residue in every coordinate. Constant translations then give separate residues for the cyclic quotient. The longest split truncation reads the decomposition subgroup and hence the local degrees.

## G.35 Expected vanishing orders and Dirac masses

The invariants $T_m$, $S_m$, $T$ and $S$ of Definition 34.1.1 follow Blum and Jonsson [BJ20, §§ 2.4–2.6 and 3.2]. Their characterization of $S_m(v)$ as a maximum over bases is [BJ20, Lemma 3.5], which is the content of the first two assertions of Lemma 34.1.1. For a big line bundle on a normal complex projective variety they prove the existence of $S(v)$ for valuations of linear growth, deducing the limit from the theorem of Boucksom and Chen [BJ20, Theorem 2.8 and Lemma 2.9]. In the same setting they prove, with Okounkov bodies, a uniform estimate $S_m \leq (1+\varepsilon)S$ of the kind asked in Section 34.6 [BJ20, Corollaries 2.10 and 3.6]. The open problem there asks for the estimate in the generality of the chapter, for possibly non-normal varieties over any algebraically closed trivially valued field, which that citation does not cover verbatim. The bounds $T/(n+1) \leq S \leq T$ are [BJ20, (3.1)]. For ample $L$ and divisorial $v$ they record $S \leq nT/(n+1)$ in [BJ20, Proposition 3.11], attributing it to Fujita. Fujita proves the upper bound for a prime divisor over a log Fano pair, with $L = -(K_X + \Delta)$ [Fuj19, Proposition 2.1], and reproves there, for log Fano pairs, the lower bound $T/(n+1) \leq S$ of Fujita and Odaka [Fuj19, Lemma 2.2]; Blum and Jonsson note that the proof of the upper bound does not use $L = -K_X$. By [BJ20, Remarks 2.7 and 3.12] the upper bound holds for every valuation of linear growth, by the energy functionals of the first version of [BJ22].

Boucksom and Jonsson characterize the valuations of linear growth as the points whose Dirac masses have finite energy [BJ22, Theorem B]. On an irreducible projective variety over an algebraically closed field they prove $T(v)/(n+1) \leq e^\vee(\delta_v) \leq nT(v)/(n+1)$ [BJ22, Proposition 11.1], without the envelope property, with the bound $e^\vee \leq T$ obtained as in Theorem 34.3.1. The potentials $m^{-1}\max_s(\log|s| + v(s))$ form a maximizing sequence for $\delta_v$ [BJ22, Lemma 11.17], and the potentials $G_F$ in the proof of Theorem 34.3.2 form a maximizing net. Their companion paper proves $S(v) = e^\vee(\delta_v)$ for all valuations of linear growth on reduced irreducible projective varieties with an ample $\mathbb{Q}$-line bundle over algebraically closed fields of every characteristic, without an envelope hypothesis [BJ23a, Theorem 7.22], and identifies the Fubini–Study potential of the filtration with the solution for $\delta_v$ under the envelope property [BJ23a, Example 7.14]. These are the results of Theorem 34.3.2 and Theorem 34.4.1.

The book's proofs differ in the following respects. Theorem 34.3.1 holds over every trivially valued field, and the identity with $S$ is also proved for regular curves over arbitrary fields. The arguments use the section lattices of Section 18.2 and the monotone terms of the difference-of-powers formula, and no Okounkov body. The upper bound $S \leq nT/(n+1)$ is first proved under the envelope property, then transferred to every integral projective variety over an algebraically closed field through the normalization, as explained after Theorem 34.4.1. Under the hypothesis of Theorem 34.3.2, the existence of the limit $S(v)$ follows from the identity with $e^\vee(\delta_v)$, which bounds the lower limit from below and the upper limit from above.

Corollary 34.5.1 is the case of rational polytopes of the inequality $w(K,u)/(n+1) \leq h(K,u) \leq n\,w(K,u)/(n+1)$, where $h(K,\cdot)$ is the support function and $w(K,u) = h(K,u)+h(K,-u)$ is the width of an $n$-dimensional convex body $K \subset \mathbb{R}^n$

with centroid at the origin; take $K = P - b_P$ and $u = -\xi$, identified with a vector by the standard inner product. The inequality goes back to Minkowski [Sch14, § 5.4, Note 6, p. 320]. An equivalent form is $-K \subset nK$ [Sch14, Lemma 2.3.3, p. 91]. The corollary is included as a check on the constants, and its proof here is not the elementary one.

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