Nonuniqueness for degenerate Hermitian Monge–Ampère equations

Abstract

We give an explicit counterexample to uniqueness for a degenerate Hermitian Monge–Ampère equation. More precisely, on $X = \mathbf{P}^3$ we construct a positive nonclosed Hermitian form ω, a Hermitian reference form ω_X, a smooth function $f \geq 0$, $f \not\equiv 0$, and two distinct smooth normalized ω-plurisubharmonic functions φ and ψ with minimal singularities such that

$$(\omega + \mathrm{dd^c}\varphi)^3 = f\omega_X^3 = (\omega + \mathrm{dd^c}\psi)^3.$$

The density vanishes on a nonempty open set. This disproves uniqueness of the normalized potential in Theorem D of Boucksom–Guedj–Lu, even for a Hermitian background on a connected projective manifold.

Keywords. Complex Monge–Ampère equation, Hermitian manifold, degenerate density, uniqueness.

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1 Introduction

Let X be a compact complex manifold of dimension n, let ω_X be a reference Hermitian form, and let θ be a smooth real $(1,1)$-form, not necessarily closed. Boucksom, Guedj and Lu proved that if θ is big and $0 \leq f \in L^p(X)$ for some $p > 1$, with $f \not\equiv 0$, then there exist a normalized θ-psh function u with minimal singularities and a constant $c > 0$ satisfying

$$(\theta + \mathrm{dd^c}u)^n = cf\omega_X^n. \tag{1.1}$$

They also proved that c is uniquely determined, and left uniqueness of u open even when θ is a Hermitian form [BGL25, Theorem D and the discussion following it]. Kołodziej and Nguyen established uniqueness under the additional assumption that the density is uniformly bounded away from zero [KN19].

Our main result shows that this positivity assumption on the density cannot simply be omitted. The counterexample occurs on projective three-space, so neither disconnectedness nor failure of the bounded mass property is involved.

Theorem A. *Let $X = \mathbf{P}^3$. There exist Hermitian forms $\omega > 0$ and $\omega_X > 0$, with $d\omega \neq 0$, a function $f \in C^\infty(X)$ satisfying $f \geq 0$ and $f \not\equiv 0$, and two distinct functions*

$$\varphi, \psi \in C^\infty(X) \cap \mathrm{PSH}(X, \omega)$$

such that

$$\sup_X \varphi = \sup_X \psi = 0, \qquad (\omega + \mathrm{dd^c}\varphi)^3 = f\omega_X^3 = (\omega + \mathrm{dd^c}\psi)^3.$$

Both potentials have minimal singularities. Moreover, f vanishes on a nonempty open set.

The mechanism is elementary but genuinely three-dimensional. On a dense open subset of $\mathbf{P}^3$ we exhibit three independent rank-one semipositive forms P, Q, R. Cubes of their linear combinations are therefore coefficient products. Suitable one-variable coefficients produce a smooth semipositive form ω_0 and a nonconstant smooth function w such that

$$\left(\omega_0 - \frac{1}{2}\mathrm{dd^c}w\right)^3 = \left(\omega_0 + \frac{1}{2}\mathrm{dd^c}w\right)^3.$$

The form ω_0 degenerates along two real hypersurfaces. We then construct a smooth function ρ for which

$$\omega := \omega_0 + \mathrm{dd^c}\rho > 0.$$

Replacing the two potentials by $-w/2 - \rho$ and $w/2 - \rho$ leaves the two Monge–Ampère forms unchanged. This exact change of representative is what turns the semipositive construction into a counterexample with a Hermitian background.

We use throughout the convention

$$d^c = i(\bar\partial - \partial), \qquad \mathrm{dd^c} = 2i\partial\bar\partial.$$

2 A rank-one frame on projective three-space

Write

$$X = \mathbf{P}(\mathbf{C}^2_A \oplus \mathbf{C}^2_B) = \mathbf{P}^3$$

and use homogeneous coordinates $[A : B]$, with $A, B \in \mathbf{C}^2$. On the open set

$$U := \{A \neq 0,\ B \neq 0\}$$

consider the smooth function

$$r := \log\frac{|A|^2}{|B|^2}.$$

The maps $[A : B] \mapsto [A]$ and $[A : B] \mapsto [B]$ from U to $\mathbf{P}^1$ pull the Fubini–Study form back to smooth semipositive forms P and Q. Equivalently,

$$P = \mathrm{dd^c}\log|A|^2, \qquad Q = \mathrm{dd^c}\log|B|^2.$$

Set

$$R := 2i\partial r \wedge \bar\partial r = dr \wedge d^c r.$$

Lemma 2.1. *The forms P, Q, R are semipositive and have rank one. On U one has*

$$\mathrm{dd^c}r = P - Q, \qquad \mathrm{dd^c}G(r) = G'(r)(P - Q) + G''(r)R \tag{2.1}$$

for every smooth real function G. Moreover,

$$P^2 = Q^2 = R^2 = 0, \qquad P \wedge Q \wedge R > 0. \tag{2.2}$$

Proof. The chain rule and the identities $P^2 = Q^2 = R^2 = 0$ follow directly from the definitions. To check strict positivity of the mixed product, use local coordinates

$$A = (1, a), \qquad B = \tau(1, b).$$

The forms P and Q are positive in the a- and b-directions, respectively, while

$$\partial r = \frac{\bar a\, da}{1 + |a|^2} - \frac{d\tau}{\tau} - \frac{\bar b\, db}{1 + |b|^2}$$

has a nonzero component in the remaining vertical direction. Thus the three rank-one directions are independent. □

It follows that for all $x, y, z \geq 0$,

$$(xP + yQ + zR)^3 = 6xyz\, P \wedge Q \wedge R. \tag{2.3}$$

3 The one-variable transition

Choose an even function $\beta \in C_c^\infty((-1,1))$ which is positive on $(-1,1)$, strictly increasing on $(-1,0)$, and normalized by $\int_{-1}^1 \beta = 2$. For instance, one may normalize

$$\beta(r) = \exp\left(-\frac{1}{1-r^2}\right), \qquad |r| < 1.$$

Define

$$F(r) = -1 + \int_{-1}^r \beta(t)\,dt \quad (-1 < r < 1)$$

and extend F by -1 on $r \le -1$ and by 1 on $r \ge 1$. The function F is smooth and odd. Put

$$p := \frac{F'}{2}, \qquad q := \frac{F''}{2}. \tag{3.1}$$

Then $p > 0$ on $(-1,1)$, $q > 0$ on $(-1,0)$, and $q < 0$ on $(0,1)$.

Fix $L > 0$. Choose nonnegative smooth functions a, b on $[-1,1]$ such that

$$a = L \text{ near } -1, \quad \operatorname{supp} a \subset [-1,0), \quad b(r) = a(-r), \quad \operatorname{supp} a \cap \operatorname{supp} b = \varnothing. \tag{3.2}$$

Choose $\chi \ge 0$, supported away from $\operatorname{supp} a \cup \operatorname{supp} b$ and positive near 0, and set

$$z := (q^2 + \chi^2)^{1/2}. \tag{3.3}$$

The function z is smooth. Indeed, near the endpoints $\chi = 0$ and q has a fixed sign, while $\chi > 0$ near the only interior zero of q. Consequently,

$$z = q \text{ on } \operatorname{supp} a, \qquad z = -q \text{ on } \operatorname{supp} b, \qquad z > 0 \text{ on } (-1,1). \tag{3.4}$$

On the closed band $-1 \le r \le 1$, define

$$\omega_0 := (p+a)P + (p+b)Q + zR. \tag{3.5}$$

This form is positive on the open band and has endpoint values

$$\omega_0|_{r=-1} = LP, \qquad \omega_0|_{r=1} = LQ. \tag{3.6}$$

All coefficients other than the displayed rank-one term are flat at the corresponding endpoint.

The function $w = F(r)$ on U extends smoothly to all of X, since it is constant near the loci $A = 0$ and $B = 0$. By Lemma 2.1,

$$H := \frac{1}{2}\mathrm{dd^c} w = p(P - Q) + qR. \tag{3.7}$$

Proposition 3.1. *On the open band $-1 < r < 1$, the forms*

$$A_0 := \omega_0 - H, \qquad B_0 := \omega_0 + H$$

are semipositive and satisfy

$$A_0^3 = B_0^3 = 0.$$

Proof. Using (3.5) and (3.7), we obtain

$$\begin{aligned} A_0 &= aP + (2p+b)Q + (z-q)R, \\ B_0 &= (2p+a)P + bQ + (z+q)R. \end{aligned} \tag{3.8}$$

All coefficients are nonnegative. Formula (2.3) gives

$$\begin{aligned} A_0^3 &= 6a(2p+b)(z-q)P \wedge Q \wedge R, \\ B_0^3 &= 6(2p+a)b(z+q)P \wedge Q \wedge R. \end{aligned}$$

On $\operatorname{supp} a$ one has $b = 0$ and $z = q$; on $\operatorname{supp} b$ one has $a = 0$ and $z = -q$; away from these supports one has $a = b = 0$. Hence both products vanish identically. $\square$

4 Globalization

We next extend ω_0 across the two ends of the transition band. Fix a Fubini–Study form ω_X on X.

Proposition 4.1. *The form ω_0 defined by (3.5) extends to a smooth semipositive real $(1,1)$-form on X which is positive outside the two real hypersurfaces*

$$M_- := \{r = -1\}, \qquad M_+ := \{r = 1\}. \tag{4.1}$$

The globally defined forms

$$A_0 := \omega_0 - H, \qquad B_0 := \omega_0 + H$$

are semipositive and satisfy

$$A_0^3 = B_0^3 \not\equiv 0. \tag{4.2}$$

Proof. Choose a smooth function s_- on $[-3/2, -1]$ which is flat and zero at -1, positive on $[-3/2, -1)$, and identically one near $-3/2$. On this collar set

$$\omega_0 := LP + s_-(r)(Q + R). \tag{4.3}$$

It is positive for $r < -1$ and matches (3.5) to infinite order at $r = -1$.

On the compact band $-2 \le r \le -3/2$, the form $LP + Q + R$ is positive. Let λ_- be a smooth function of r which is zero near -2 and one near $-3/2$, and define

$$\omega_0 := (1 - \lambda_-)\omega_X + \lambda_-(LP + Q + R).$$

This is positive and matches both adjacent definitions. Set $\omega_0 = \omega_X$ on $r \le -2$. The same construction on the right, starting from $LQ + s_+(r)(P + R)$, defines ω_0 for $r \ge 1$ and makes it equal to ω_X on $r \ge 2$. Since all interpolations take place in compact subbands of U, the resulting form is globally smooth. Its only degeneracies are the two endpoint values in (3.6).

The function w is constant outside $-1 < r < 1$, so $H = 0$ there and $A_0 = B_0 = \omega_0$. Proposition 3.1 proves equality of the cubes on the transition band, while outside it both cubes equal ω_0^3. The common measure is nonzero because ω_0 is positive on the outer region. □

5 Passage to a Hermitian background

The following step is the essential change of representative.

Proposition 5.1. *There exists $\rho \in C^\infty(X)$ such that*

$$\omega := \omega_0 + \mathrm{dd^c}\rho > 0. \tag{5.1}$$

Moreover, $d\omega \neq 0$.

Proof. Near M_- and M_+, respectively, consider

$$g_- := -r + C(r+1)^2, \qquad g_+ := r + C(r-1)^2,$$

where $C > 0$. Formula (2.1) gives

$$\mathrm{dd^c} g_-|_{M_-} = -P + Q + 2CR, \qquad \mathrm{dd^c} g_+|_{M_+} = P - Q + 2CR. \tag{5.2}$$

Thus, for $0 < \varepsilon < L$,

$$\omega_0 + \varepsilon \mathrm{dd^c} g_-|_{M_-} = (L - \varepsilon)P + \varepsilon Q + 2C\varepsilon R > 0, \tag{5.3}$$

$$\omega_0 + \varepsilon \mathrm{dd^c} g_+|_{M_+} = \varepsilon P + (L-\varepsilon)Q + 2C\varepsilon R > 0. \tag{5.4}$$

We record a uniform version of this observation. Write $t = r+1$ near M_-. On the transition side, after shrinking the neighborhood so that $a = L$ and $b = \chi = 0$, the coefficients of $\omega_0 + \varepsilon \mathrm{dd^c} g_-$ in the P, Q, R frame are

$$p + L + \varepsilon(-1+2Ct), \qquad p + \varepsilon(1-2Ct), \qquad q + 2C\varepsilon.$$

On the collar side they are

$$L + \varepsilon(-1+2Ct), \qquad s_- + \varepsilon(1-2Ct), \qquad s_- + 2C\varepsilon.$$

Choose $\delta > 0$ with $2C\delta < 1/2$ and then choose $0 < \varepsilon < L/2$. All these coefficients are strictly positive for $|t| < \delta$. The same argument applies near M_+.

Choose disjoint cutoffs $\kappa_\pm$ which equal one on the δ-neighborhoods of $M_\pm$ and are supported in slightly larger neighborhoods. Their derivative supports are compact and disjoint from $M_- \cup M_+$, so there exists $m > 0$ such that $\omega_0 \geq m\omega_X$ there. The forms $\mathrm{dd^c}(\kappa_- g_-)$ and $\mathrm{dd^c}(\kappa_+ g_+)$ are uniformly bounded with respect to ω_X. After decreasing ε once more, the function

$$\rho := \varepsilon(\kappa_- g_- + \kappa_+ g_+) \tag{5.5}$$

therefore satisfies (5.1) globally.

It remains to check nonclosedness. Since $R = dr \wedge d^c r$, we have

$$dR = -dr \wedge \mathrm{dd^c} r = -dr \wedge (P - Q).$$

On a central subband where $a = b = 0$, differentiating (3.5) yields

$$d\omega_0 = dr \wedge \big((q-z)P + (q+z)Q\big). \tag{5.6}$$

At $r = 0$ one has $q = 0 < z$, and the right-hand side is nonzero by the independence in Lemma 2.1. Finally, $d\mathrm{dd^c}\rho = 0$, so $d\omega = d\omega_0 \neq 0$. □

6 Proof of the main theorem

Proof of Theorem A. Let ρ be given by Proposition 5.1, and define

$$\widetilde{\varphi} := -\frac{1}{2}w - \rho, \qquad \widetilde{\psi} := \frac{1}{2}w - \rho. \tag{6.1}$$

Equations (3.7) and (5.1) give

$$\omega + \mathrm{dd^c}\widetilde{\varphi} = A_0, \qquad \omega + \mathrm{dd^c}\widetilde{\psi} = B_0. \tag{6.2}$$

Set

$$\varphi := \widetilde{\varphi} - \sup_X \widetilde{\varphi}, \qquad \psi := \widetilde{\psi} - \sup_X \widetilde{\psi}.$$

Subtracting constants does not change (6.2); hence φ and ψ are normalized smooth ω-psh functions. They are distinct even modulo constants, because $\widetilde{\psi} - \widetilde{\varphi} = w$ is nonconstant.

Define

$$f := \frac{A_0^3}{\omega_X^3} = \frac{B_0^3}{\omega_X^3}. \tag{6.3}$$

By Proposition 4.1, this is a smooth nonnegative function which is not identically zero. Proposition 3.1 shows that f vanishes on the open band $-1 < r < 1$. Equations (6.2) and (6.3) yield

$$(\omega + \mathrm{dd^c}\varphi)^3 = f\omega_X^3 = (\omega + \mathrm{dd^c}\psi)^3.$$

Finally, the extremal potential V_ω is identically zero. Indeed, $0 \in \mathrm{PSH}(X, \omega)$ and all competitors in the definition of V_ω are at most zero. Since φ and ψ are smooth and bounded, they differ from V_ω by bounded functions and therefore have minimal singularities. Proposition 5.1 gives $d\omega \neq 0$, completing the proof. $\square$

Remark 6.1. Both solutions correspond to the same normalizing constant $c = 1$, so the example does not affect uniqueness of the constant in (1.1). It also shows that the bounded mass property does not restore uniqueness: the underlying manifold $\mathbf{P}^3$ is Kähler, hence belongs to the Fujiki class and has the bounded mass property. The essential feature is instead the vanishing of the density on an open set.

References

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