Generic Nonuniqueness for Subcritical Positive Kähler–Einstein Dirichlet Problems

Abstract

Let $X = \mathbb{C}^3/\{\pm 1\}$ with its isolated klt point p, and let m be the natural adapted measure on X. We exhibit a nonempty open set of smoothly bounded Stein neighborhoods Ω of p on which the normalized positive Kähler–Einstein Dirichlet problem $(\mathrm{dd}^c u)^3 = e^{-\gamma u}\mu_\Omega / \int_\Omega e^{-\gamma u}\, d\mu_\Omega$, $u|_{\partial\Omega} = 0$, admits two distinct bounded solutions for one and the same exponent γ, which is strictly smaller than the Moser–Trudinger critical exponent of (Ω, p). The domains are obtained by joining three balls by thin necks, the two solutions are produced by a Leray–Schauder degree argument, and a uniform Moser–Trudinger inequality keeps the exponent subcritical on the whole family. Consequently uniqueness cannot hold for generic Stein neighborhoods of p in the C^∞ topology on domains, even after imposing countably many prescribed transversality conditions on the boundary.

1 Introduction

Let X be a complex space with an isolated log terminal singularity p, let $\Omega \Subset X$ be a strongly pseudoconvex neighborhood of p, and let μ_p be an adapted probability measure on Ω in the sense of [GT25, Definitions 2.9–2.10]. Guedj and Trusiani [GT25] study the positive Kähler–Einstein Dirichlet problem

$$(\mathrm{dd}^c u)^n = \frac{e^{-\gamma u}\,\mu_p}{\int_\Omega e^{-\gamma u}\, d\mu_p}, \qquad u|_{\partial\Omega} = \phi, \tag{1.1}$$

and prove that it admits a solution, continuous on $\overline{\Omega}$ and smooth on $\Omega \setminus \{p\}$, whenever γ is smaller than a critical exponent $\gamma_{\mathrm{crit}}(X, p)$ defined through a Moser–Trudinger inequality [GT25, Theorem C and Definition 3.12]. They write: "We expect the solution to be unique, at least when Ω is a generic Stein neighborhood of p" [GT25, discussion following Theorem C].

The purpose of this note is to show that, for the simplest isolated quotient singularity, uniqueness fails on an open set of Stein neighborhoods of p in the C^∞ topology on domains, and therefore cannot hold generically in any reasonable sense. Throughout we use the conventions

$$d^c = \frac{i}{2}(\bar\partial - \partial), \qquad \mathrm{dd}^c = i\partial\bar\partial;$$

the translation to the convention $dd^c_{\mathrm{GT}} = \frac{i}{2\pi}\partial\bar\partial$ of [GT25] is explained in Remark 1.3.

The setting

Let $q\colon \mathbb{C}^3 \to \mathbb{C}^6$, $q(z) = (z_1^2, z_2^2, z_3^2, z_1z_2, z_1z_3, z_2z_3)$, and let $X = q(\mathbb{C}^3)$, so that $X \simeq \mathbb{C}^3/\{z \sim -z\}$ is a normal affine threefold whose only singular point $p = q(0)$ is an isolated klt point (Proposition 2.1). Set

$$m = \tfrac{1}{2}\, q_*(dV),$$

where dV is Lebesgue measure on $\mathbb{C}^3$. This is an adapted measure on X in the sense of [GT25] (Proposition 2.1); for a relatively compact neighborhood Ω of p we write

$$\mu_\Omega = \frac{m|_\Omega}{m(\Omega)}$$

for its probability normalization on Ω. Admissible domains, the test class $\mathcal{T}_0(\Omega)$, the energy E_Ω and the critical exponent $\gamma_{\mathrm{crit}}(\Omega, p)$ are defined in Section 2; they are the objects of [GT25, Definitions 2.4, 3.1 and 3.12] for zero boundary data, written in our normalization of dd^{c}. The space of admissible domains carries the usual C^∞ topology, in which a neighborhood of Ω consists of the domains bounded by small C^∞ graphs over $\partial\Omega$ in a fixed collar (Definition 2.2).

Fix once and for all the two numbers

$$\gamma_* = 8\pi\Big(\frac{999}{2000}\Big)^{1/3}, \qquad G = \frac{8\pi}{2^{1/3}}, \qquad \text{so that} \qquad \Big(\frac{\gamma_*}{G}\Big)^3 = \frac{999}{1000} < 1. \tag{1.2}$$

The main result

Theorem 1.1. *There is a nonempty open set $\mathcal{U}$ of admissible domains, all contained in a prescribed neighborhood of p, with the following properties.*

1. *For every $\Omega \in \mathcal{U}$ and every $0 < a < G$ there is a constant $C_{a,\Omega}$ such that*

$$\Big(\int_\Omega e^{-af}\, d\mu_\Omega\Big)^{1/a} \le C_{a,\Omega}\, e^{-E_\Omega(f)} \qquad \text{for all } f \in \mathcal{T}_0(\Omega). \tag{1.3}$$

   In particular $\gamma_* < G \le \gamma_{\mathrm{crit}}(\Omega, p)$.

2. *For every $\Omega \in \mathcal{U}$ there are two distinct functions $u_1, u_2 \in \mathcal{T}_0(\Omega)$, continuous on $\overline{\Omega}$, smooth and strictly plurisubharmonic on $\overline{\Omega} \setminus \{p\}$, such that*

$$(\mathrm{dd}^{\mathrm{c}} u_i)^3 = \frac{e^{-\gamma_* u_i}\, \mu_\Omega}{\int_\Omega e^{-\gamma_* u_i}\, d\mu_\Omega}, \qquad u_i|_{\partial\Omega} = 0, \qquad i = 1, 2. \tag{1.4}$$

   Their Monge–Ampère measures have total mass one and no atom at p, and $\int_\Omega |u_i| (\mathrm{dd}^{\mathrm{c}} u_i)^3 < \infty$.

3. *Let $(Y_\ell)_{\ell \in \mathbb{N}}$ be a countable family of embedded smooth real submanifolds of $X \setminus \{p\}$. The set of $\Omega \in \mathcal{U}$ such that $\partial\Omega$ is transverse to every Y_ℓ is a dense residual subset of $\mathcal{U}$.*

Since $\mathcal{U}$ is open and nonempty, every dense subset of the space of admissible domains meets $\mathcal{U}$; since $\mathcal{U}$ is moreover a Baire space, every residual subset of the space of admissible domains meets $\mathcal{U}$, even after intersecting it with the residual set in (3). Thus:

Corollary 1.2. *The statement "for every $0 < \gamma < \gamma_{\mathrm{crit}}(\Omega, p)$, the normalized problem* (1.1) *with $\phi = 0$ has at most one solution" fails for every Ω in a nonempty C^∞-open set of Stein neighborhoods of p. Consequently it fails for some member of every dense and of every residual family of such neighborhoods, and imposing any finite or countable list of transversality conditions on $\partial\Omega$ does not change this.*

Remark 1.3. With the convention $dd^c_{\mathrm{GT}} = \frac{i}{2\pi}\partial\bar\partial$ of [GT25, Section 2.1.1], the functions $v_i = 2\pi u_i$ solve

$$(dd^c_{\mathrm{GT}} v_i)^3 = \frac{e^{-\gamma_{\mathrm{GT}} v_i}\, \mu_\Omega}{\int_\Omega e^{-\gamma_{\mathrm{GT}} v_i}\, d\mu_\Omega}, \qquad \gamma_{\mathrm{GT}} = \frac{\gamma_*}{2\pi} = 4\Big(\frac{999}{2000}\Big)^{1/3} \approx 3.17,$$

and the critical exponents in the two conventions are related by $\gamma_{\mathrm{crit,GT}}(\Omega, p) = \gamma_{\mathrm{crit}}(\Omega, p)/2\pi$; see Section 8.5. In that convention $G/2\pi = 4 \cdot 2^{-1/3}$; the exponent G is obtained in Lemma 3.1 from the bound $\gamma_{\mathrm{crit}} \ge n + 1 = 4$ on Euclidean balls [GT25, Theorem 5.1] by passing to the double cover. We do not claim that G equals $\gamma_{\mathrm{crit}}(\Omega, p)$.

Strategy

On a disconnected domain the normalized equation (1.4) has many solutions, because the components are coupled only through the single normalizing constant. We start from the disjoint union $D_0 = B_{-1} \cup B_0 \cup B_1$ of the three unit balls centered at $-3e_1, 0, 3e_1$ in $\mathbb{C}^3$, which is invariant under $z \mapsto -z$. On each ball the Fubini–Study type potentials $A_* \log(1 - t + t|z|^2)$ solve $(dd^c u)^3 = c\, e^{-\gamma_* u}\, dV$ explicitly, and a one-parameter family of even solutions of the normalized equation on D_0 is obtained by matching the constant c on the three balls. The choice of γ_* slightly below G ensures that exactly two members of this family have total Monge–Ampère mass two upstairs, that is, mass one on the quotient (Section 4). We then compute the full linearization of the normalized equation at these two solutions, including the rank one term coming from the normalizing constant, and show that both are nondegenerate with Morse indices one and zero (Lemma 4.3).

The connected domains are obtained by thickening the compact set $K = \overline{D_0} \cup [-2e_1, -e_1] \cup [e_1, 2e_1]$, which is polynomially convex, into sign-invariant smooth strictly pseudoconvex domains D_j shrinking to K (Lemma 5.1). The key analytic fact is that the Dirichlet solution operators of the D_j converge uniformly, on bounded sets of continuous densities, to the componentwise operator of D_0 (Lemma 5.2): the solutions are uniformly small on the necks. The normalized equation on D_j is a fixed point problem for a compact map of the space of even continuous functions, and a Leray–Schauder degree argument transfers the two nondegenerate fixed points from D_0 to every D_j close to K (Lemma 6.1); the index of each fixed point is computed in a Hölder space, where the map is differentiable by the implicit function theorem, and carried over to the space of continuous functions by the commutativity property of the fixed point index. A parabolic comparison argument shows that the continuous solutions obtained in this way are smooth up to the boundary (Lemma 7.1). Finally, the Moser–Trudinger inequality on the quotient balls $q(B_R)$, which follows from the inequality on Euclidean balls by pulling back and rescaling, is transported to all admissible subdomains of $q(B_R)$ by maximal subextension, with a constant that does not depend on the subdomain (Proposition 3.2). This uniformity is what makes the whole family $\mathcal{U}$ subcritical for the same exponent.

Organization

Section 2 collects the definitions and the facts about the quotient. Section 3 proves the Moser–Trudinger estimate. Section 4 treats the model problem on three balls. Section 5 constructs the neck domains and proves the convergence of the Dirichlet operators. Section 6 contains the degree argument, and Section 7 the regularity of continuous solutions. Theorem 1.1 is proved in Section 8.

2 Preliminaries

2.1 The quotient and its adapted measure

Let $q\colon \mathbb{C}^3 \to \mathbb{C}^6$ be the map $q(z) = (z_1^2, z_2^2, z_3^2, z_1 z_2, z_1 z_3, z_2 z_3)$ and $X = q(\mathbb{C}^3)$, $p = q(0)$. We write B_R for the Euclidean ball of radius R centered at $0 \in \mathbb{C}^3$, $B(a, r)$ for the ball of radius r centered at a, dV for Lebesgue measure on $\mathbb{C}^3$, and

$$Q_R = q(B_R), \qquad m = \tfrac{1}{2}\, q_*(dV).$$

Identifying $q(z)$ with the symmetric matrix zz^t, X is the cone of complex symmetric 3×3 matrices of rank at most one.

Proposition 2.1. *The map q realizes X as the normal quotient $\mathbb{C}^3/\{\pm 1\}$, and it is a local biholomorphism $\mathbb{C}^3 \setminus \{0\} \to X \setminus \{p\}$ of degree two. The point p is an isolated klt singularity, $2K_X$ is Cartier and trivial, and m is an adapted measure on X in the sense of [GT25, Definition 2.9]. The measure m has no atom at p, and $0 < m(\Omega) < \infty$ for every relatively compact neighborhood Ω of p.*

Proof. The invariant polynomials of the sign action are generated by the quadratic monomials, so X is the categorical quotient $\mathbb{C}^3/\{\pm 1\}$; it is normal, being the quotient of a normal variety by a finite group. The action is free off the origin, so q is a two-sheeted local biholomorphism there. At p the six quadratic generators are independent modulo the square of the maximal ideal, so the embedding dimension is six and p is singular.

The incidence variety $\widetilde{X} = \{(A, [\ell]) \in X \times \mathbb{P}^2 : \operatorname{im} A \subset \ell\}$ is the total space of $\mathcal{O}_{\mathbb{P}^2}(-2)$, and the projection $\pi \colon \widetilde{X} \to X$ is a resolution which is an isomorphism off p, with exceptional divisor the zero section $E \simeq \mathbb{P}^2$, $E|_E = \mathcal{O}_{\mathbb{P}^2}(-2)$. (Equivalently, $\widetilde{X}$ is the quotient of the blow-up of the origin in $\mathbb{C}^3$, which is the total space of $\mathcal{O}_{\mathbb{P}^2}(-1)$, by the involution acting by -1 on the fibers.) The form $\sigma = (dz_1 \wedge dz_2 \wedge dz_3)^{\otimes 2}$ is invariant, descends to a nowhere vanishing section of the reflexive sheaf $\mathcal{O}(2K_X)$, and generates it, since invariant multiples of σ have invariant coefficients. Hence $2K_X$ is Cartier and trivial. Writing $K_{\widetilde{X}} = \pi^* K_X + aE$, adjunction on E gives $K_E = (a+1)E|_E$, that is, $-3 = -2(a+1)$, so $a = \frac{1}{2} > -1$. Since π is a log resolution, p is klt.

Off p, q is a two-sheeted covering, and the smooth positive measure $(c_3 \sigma \wedge \bar{\sigma})^{1/2}$ on $X \setminus \{p\}$ pulls back to a constant multiple of dV; normalize σ so that this multiple is 1. Taking the hermitian metric h on K_X with $|\sigma|_{h^2} \equiv 1$ in [GT25, Definition 2.9], the adapted measure there is a constant multiple of $\frac{1}{2} q_*(dV) = m$, because q has degree two. The constant is fixed by the normalization [GT25, Definition 2.10], which is our μ_Ω. Finally $q^{-1}(p) = \{0\}$ has Lebesgue measure zero, so $m(\{p\}) = 0$, and $q^{-1}(\Omega)$ is bounded with nonempty interior, so $0 < m(\Omega) < \infty$. □

2.2 Admissible domains, test functions and the critical exponent

Plurisubharmonic functions on X are understood in the sense of [GT25, Definition 2.1]: locally they are restrictions of plurisubharmonic functions of the ambient $\mathbb{C}^6$. Smooth functions on X are locally restrictions of smooth ambient functions.

Definition 2.2. An *admissible domain* is a connected relatively compact open neighborhood Ω of p in X for which there is a smooth strictly plurisubharmonic function ρ on an open neighborhood N of $\overline{\Omega}$ in X with

$$\Omega = \{x \in N : \rho(x) < 0\}, \qquad d\rho \neq 0 \text{ on } \partial\Omega.$$

We call ρ a defining function for Ω. Then $\partial\Omega$ is a smooth strongly pseudoconvex hypersurface in $X \setminus \{p\}$, and $-\log(-\rho)$ is a strictly plurisubharmonic exhaustion of Ω; in particular Ω is a strongly pseudoconvex Stein neighborhood of p in the sense of [GT25, Definition 2.2].

Given an admissible domain Ω with boundary S, fix a collar $S \times (-b, b) \to X \setminus \{p\}$, $(y,t) \mapsto c(y,t)$, with $c(y,0) = y$ and $\rho(c(y,t)) = t$. For $h \in C^\infty(S, \mathbb{R})$ with $\|h\|_{C^0} < b$, let Ω_h be the domain bounded by the graph $\{c(y, h(y))\}$ which coincides with Ω outside the collar. The C^∞ *topology* on the set of admissible domains is the one in which the sets $\{\Omega_h : h \in N\}$, N ranging over the C^∞-neighborhoods of 0 in $C^\infty(S, \mathbb{R})$, form a neighborhood basis of Ω. (Lemma 8.2 shows that Ω_h is admissible when h is small in C^2, so the admissible domains form an open subset of the space of all smooth domains, and the topology does not depend on the choices.)

Definition 2.3. For an admissible domain Ω let

$$\mathcal{T}_0(\Omega) = \left\{ f \in \operatorname{PSH}(\Omega) \cap C(\overline{\Omega}) : f|_{\partial\Omega} = 0,\ \int_\Omega (\mathrm{dd^c} f)^3 < \infty \right\}$$

and

$$E_\Omega(f) = \frac{1}{4}\int_\Omega f\,(dd^c f)^3 \qquad (f \in \mathcal{T}_0(\Omega)). \tag{2.1}$$

Here the Monge–Ampère measure of a continuous plurisubharmonic function on Ω is defined on the regular locus $\Omega \setminus \{p\}$ in the usual sense of Bedford–Taylor and is extended by zero across p; by Lemma 2.6 below this is the same as

$$(dd^c f)^3 = \tfrac{1}{2}\, q_*((dd^c \tilde{f})^3), \qquad \tilde{f} = f \circ q. \tag{2.2}$$

The class $\mathcal{T}_0(\Omega)$ and the functional E_Ω are the test class $\mathcal{T}_\phi(\Omega)$ and the energy E_ϕ of [GT25, Definitions 2.4 and 3.1] for $\phi = \phi_0 = 0$.

Definition 2.4. For an admissible domain Ω the *critical exponent* is

$$\gamma_{\text{crit}}(\Omega, p) = \sup\Big\{a > 0 :\ \exists\, C_a < \infty \text{ with } \Big(\int_\Omega e^{-af}\, d\mu_\Omega\Big)^{1/a} \le C_a\, e^{-E_\Omega(f)} \text{ for all } f \in \mathcal{T}_0(\Omega)\Big\}. \tag{2.3}$$

This is [GT25, Definition 3.12] for $\phi = 0$, written in our normalization of dd^c.

Remark 2.5. Since $f \le 0$ for $f \in \mathcal{T}_0(\Omega)$ by the maximum principle, one has $E_\Omega(f) \le 0$. If $f_j \in \mathcal{T}_0(\Omega)$ decrease to f with $E_\Omega(f_j) \to e > -\infty$, then the Moser–Trudinger inequality for the f_j passes to f with the energy e by monotone convergence. Thus every inequality (2.3) extends automatically to decreasing limits of finite energy; we shall not need this.

2.3 Passing to the double cover

Lemma 2.6. *Let Ω be an admissible domain and $D = q^{-1}(\Omega) \subset \mathbb{C}^3$.*

1. *D is a bounded connected domain, invariant under $z \mapsto -z$, with smooth boundary, and it admits a smooth strictly plurisubharmonic defining function on a neighborhood of $\overline{D}$ which is invariant under $z \mapsto -z$.*
2. *For $f \in \mathcal{T}_0(\Omega)$, the function $\tilde{f} = f \circ q$ is continuous, plurisubharmonic and even on $\overline{D}$, vanishes on ∂D, and $(dd^c \tilde{f})^3$ has finite mass and no atom at the origin. Formula (2.2) holds, $(dd^c f)^3$ has no atom at p, $\int_D (dd^c \tilde{f})^3 = 2\int_\Omega (dd^c f)^3$, and*

$$E_\Omega(f) = \frac{1}{8}\int_D \tilde{f}\,(dd^c \tilde{f})^3. \tag{2.4}$$

3. *Conversely, if $h \in \mathrm{PSH}(D) \cap C(\overline{D})$ is even with $h|_{\partial D} = 0$ and $\int_D (dd^c h)^3 < \infty$, then $h = u \circ q$ for a unique $u \in C(\overline{\Omega})$, and $u \in \mathcal{T}_0(\Omega)$. If h is smooth and strictly plurisubharmonic on $\overline{D}$, then u is smooth and strictly plurisubharmonic on $\overline{\Omega} \setminus \{p\}$.*

Proof. (1) Let ρ be as in Definition 2.2. Then $r = \rho \circ q$ is smooth, even, plurisubharmonic near $\overline{D}$, strictly plurisubharmonic and with $dr \neq 0$ on ∂D (as q is a local biholomorphism there), and $D = \{r < 0\}$ near $\overline{D}$. The function r need not be strictly plurisubharmonic at the origin, where $dq = 0$. Since $p \in \Omega$, $r < 0$ on some ball $B(0, 2\epsilon) \subset D$. Let χ be a radial cutoff equal to 1 on $B(0, \epsilon)$ and supported in $B(0, 2\epsilon)$, and put $r_\eta = r + \eta\chi|z|^2$. For $\eta > 0$ small, $r_\eta < 0$ on $B(0, 2\epsilon)$, r_η is strictly plurisubharmonic on $B(0, \epsilon)$ because r is plurisubharmonic, and on the annulus $\epsilon \le |z| \le 2\epsilon$ the Hessian of $\eta\chi|z|^2$ is dominated by the positive lower bound of the Hessian of r. Thus r_η is a smooth even strictly plurisubharmonic defining function for D. Boundedness and sign invariance are clear. Every connected component of D is open and closed in D, and its image under the proper open map $q|_D \colon D \to \Omega$ is open and closed in the connected set Ω, hence equal to Ω; so every component contains the unique preimage 0 of p, and D is connected.

(2) Since q is a local biholomorphism off the origin, $\tilde f$ is continuous and plurisubharmonic on $D \setminus \{0\}$, and it is plurisubharmonic across the origin because it is locally bounded there [GR56]; alternatively, $\tilde f$ is the pullback of a plurisubharmonic function of $\mathbb{C}^6$ by the holomorphic map q. The Monge–Ampère measure of a bounded plurisubharmonic function puts no mass on pluripolar sets [BT82], so $(\mathrm{dd^c}\tilde f)^3$ has no atom at 0. Off the origin the Monge–Ampère measures correspond under the two-sheeted local biholomorphism q, which gives (2.2) on $D \setminus \{0\}$ and $\Omega \setminus \{p\}$; both sides have no mass at the exceptional point, so (2.2) holds on Ω. Integrating (2.2) against 1 and against f gives the mass identity and (2.4).

(3) The function u is well defined and continuous on $\overline{\Omega}$ because q is a proper quotient map for the sign action, and $\overline{\Omega} = q(\overline{D})$. It is plurisubharmonic on $\Omega \setminus \{p\}$, where q is a local biholomorphism, and it is continuous, hence locally bounded, at p. On a normal complex space a plurisubharmonic function on the complement of a nowhere dense analytic subset which is locally bounded near that subset extends to a plurisubharmonic function [GR56]; so u is plurisubharmonic on Ω. (One can also verify directly the disc criterion of [FN80, Theorem 5.3.1]: a holomorphic disc through p meets p in a discrete set unless it is constant, and continuity removes discrete sets for subharmonic functions.) Formula (2.2), proved as in (2), gives $\int_\Omega (\mathrm{dd^c}u)^3 = \frac{1}{2}\int_D (\mathrm{dd^c}h)^3 < \infty$, so $u \in \mathcal{T}_0(\Omega)$. The last assertion holds because q is a local biholomorphism near every point of $\overline{D} \setminus \{0\}$. □

3 The Moser–Trudinger estimate

For a continuous plurisubharmonic function g on a bounded domain $W \subset \mathbb{C}^3$ with $g|_{\partial W} = 0$ and finite Monge–Ampère mass we write $E_W(g) = \frac{1}{4}\int_W g\,(\mathrm{dd^c}g)^3$, as in (2.1). Note that $E_W(\lambda g) = \lambda^4 E_W(g)$ for $\lambda > 0$.

Lemma 3.1. *Every quotient ball $Q_R = q(B_R)$ is admissible. For every $R > 0$ and every $0 < a < G = 8\pi \cdot 2^{-1/3}$ there is a constant $C_{a,R} \geq 0$ such that*

$$E_{Q_R}(h) + \frac{1}{a}\log \int_{Q_R} e^{-ah}\,dm \leq C_{a,R} \qquad \text{for all } h \in \mathcal{T}_0(Q_R). \tag{3.1}$$

Proof. In the weighted coordinates $Z(q(z)) = (z_1^2, z_2^2, z_3^2, \sqrt{2}z_1z_2, \sqrt{2}z_1z_3, \sqrt{2}z_2z_3)$ one has $|Z(q(z))|^2 = |z|^4$, so $|Z|^2 - R^4$ is a smooth strictly plurisubharmonic function on $\mathbb{C}^6$ whose restriction to X defines Q_R, with nonvanishing differential on ∂Q_R (its pullback $|z|^4 - R^4$ has nonzero differential on $|z| = R$, and q is a local biholomorphism there). Hence Q_R is admissible.

The Moser–Trudinger inequality on the unit ball $B_1 \subset \mathbb{C}^3$ with Lebesgue measure, [GT25, Theorem 5.1] together with the remark following it that $\alpha(\mathbb{C}^n, dV) = n$, states that for every $0 < b_0 < 4$ there is C'_{b_0} with

$$\frac{1}{4}\int_{B_1} \phi\,(dd^c_{\mathrm{GT}}\phi)^3 + \frac{1}{b_0}\log\int_{B_1} e^{-b_0\phi}\,dV \leq C'_{b_0}$$

for all continuous plurisubharmonic ϕ on $\overline{B_1}$ with $\phi|_{\partial B_1} = 0$ and finite Monge–Ampère mass. (Replacing the probability normalization of dV used in [GT25] by dV changes only the constant.) Putting $\phi = 2\pi g$ and $b_0 = b/2\pi$, and using $dd^c_{\mathrm{GT}}(2\pi g) = \mathrm{dd^c}g$, this becomes

$$\log\int_{B_1} e^{-bg}\,dV \leq C_b - b\,E_{B_1}(g) \qquad (0 < b < 8\pi). \tag{3.2}$$

Let $h \in \mathcal{T}_0(Q_1)$ and $\tilde h = h \circ q$. By Lemma 2.6, $\tilde h$ is an admissible test function on B_1 with $E_{B_1}(\tilde h) = 2E_{Q_1}(h)$, and $\int_{Q_1} e^{-ah}\,dm = \frac{1}{2}\int_{B_1} e^{-a\tilde h}\,dV$. Apply (3.2) to $g = 2^{-1/3}\tilde h$ and $b = 2^{1/3}a$;

the condition $a < G$ is exactly $b < 8\pi$, and $bg = a\tilde{h}$, $E_{B_1}(g) = 2^{-4/3}E_{B_1}(\tilde{h}) = 2^{-1/3}E_{Q_1}(h)$. Therefore

$$\log \int_{Q_1} e^{-ah}\, dm = -\log 2 + \log \int_{B_1} e^{-a\tilde{h}}\, dV \le C_b - \log 2 - a\, E_{Q_1}(h),$$

which is (3.1) for $R = 1$.

For general R, the dilation $S_R(q(z)) = q(Rz)$ is a biholomorphism $Q_1 \to Q_R$ with $m(S_R(A)) = R^6 m(A)$. For $h \in \mathcal{T}_0(Q_R)$ one has $E_{Q_1}(h\circ S_R) = E_{Q_R}(h)$ and $\int_{Q_R} e^{-ah}\, dm = R^6 \int_{Q_1} e^{-a(h\circ S_R)}\, dm$, so (3.1) holds with $C_{a,R} = \max\{0, C_{a,1} + \frac{6}{a}\log R\}$. $\square$

Proposition 3.2. *Fix $R > 0$ and $0 < a < G$. For every admissible domain Ω with $\overline{\Omega} \subset Q_R$ and every $f \in \mathcal{T}_0(\Omega)$,*

$$E_\Omega(f) + \frac{1}{a}\log \int_\Omega e^{-af}\, dm \le C_{a,R}, \tag{3.3}$$

with the constant of Lemma 3.1. Consequently, for every $0 < a < G$ there is $C_{a,\Omega}$ such that (1.3) holds, and $\gamma_{\mathrm{crit}}(\Omega, p) \ge G > \gamma_*$.

Proof. Both Ω and Q_R are strongly pseudoconvex neighborhoods of p in the sense of [GT25, Definition 2.2], and m is a fixed adapted measure. By [GT25, Proposition 3.14 and its proof], applied with zero boundary data on both domains, the maximal plurisubharmonic subextension h of f to Q_R with zero boundary values belongs to $\mathcal{T}_0(Q_R)$ and satisfies

$$h \le f \text{ on } \Omega, \qquad E_{Q_R}(h) \ge E_\Omega(f);$$

the energy comparison comes from the fact that $(\mathrm{dd^c}h)^3$ is carried by the contact set $\{h = f\}$, and multiplication of all potentials by 2π converts the convention of [GT25] into ours without affecting either inequality. Hence

$$E_\Omega(f) + \frac{1}{a}\log \int_\Omega e^{-af}\, dm \le E_{Q_R}(h) + \frac{1}{a}\log \int_{Q_R} e^{-ah}\, dm \le C_{a,R}$$

by Lemma 3.1. Since $\int_\Omega e^{-af}\, d\mu_\Omega = m(\Omega)^{-1}\int_\Omega e^{-af}\, dm$, exponentiating (3.3) gives (1.3) with $C_{a,\Omega} = e^{C_{a,R}} m(\Omega)^{-1/a}$. Every $a \in (0, G)$ therefore belongs to the set in (2.3), so $\gamma_{\mathrm{crit}}(\Omega, p) \ge G$, and $G > \gamma_*$ by (1.2). $\square$

The point of Proposition 3.2 is that the constant in (3.3) depends only on the outer ball: no lower bound on the width of the necks of the domains constructed below enters.

4 The model problem on three balls

4.1 Notation

Let $e_1 = (1, 0, 0) \in \mathbb{C}^3$ and, for $i \in \{-1, 0, 1\}$,

$$a_i = 3ie_1, \qquad B_i = B(a_i, 1), \qquad D_0 = B_{-1} \cup B_0 \cup B_1.$$

Let

$$K = \overline{B_{-1}} \cup \overline{B_0} \cup \overline{B_1} \cup \{te_1 : -2 \le t \le -1\} \cup \{te_1 : 1 \le t \le 2\}, \tag{4.1}$$

a connected compact set, invariant under $z \mapsto -z$, contained in $\overline{B_4}$; see Figure 1. For $R > 5$ let

$$\mathcal{E}_R = \{h \in C(\overline{B_R}; \mathbb{R}) : h(-z) = h(z)\}, \qquad \|h\|_\infty = \sup_{\overline{B_R}} |h|,$$

a real Banach space. A continuous function on the closure of a subdomain of B_R with zero boundary values is regarded as an element of $C(\overline{B_R})$ by extending it by zero.

Set

$$h_* = \frac{999}{1000}, \qquad A_* = \frac{4}{\gamma_*}, \qquad \text{so that} \qquad (2\pi A_*)^3 = \frac{2}{h_*}. \tag{4.2}$$

For $r > 1$ put

$$\begin{gathered} N(r) = r^2 + r + 1, \qquad D(r) = r^3 + r^2 + r + 1, \\ x(r) = \frac{N(r)}{D(r)}, \qquad y(r) = \frac{rN(r)}{D(r)}, \qquad H(r) = y(r)^3 + 2x(r)^3. \end{gathered} \tag{4.3}$$

For $0 < t < 1$ let

$$F_t(z) = A_* \log\Big(1 - t + t|z|^2\Big), \qquad |z| \le 1, \tag{4.4}$$

and define $U_r \in \mathcal{E}_R$ by

$$U_r(z) = \begin{cases} F_{y(r)}(z), & z \in \overline{B_0}, \\ F_{x(r)}(z - a_i), & z \in \overline{B_i},\ i = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

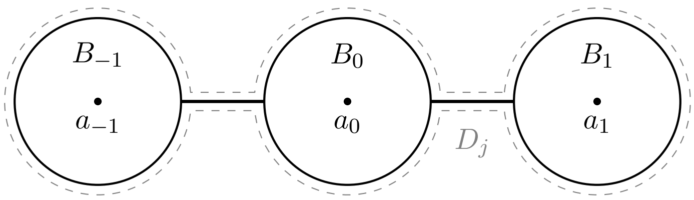

Figure 1: The real slice of the compact set K (three closed unit balls joined by two segments of the real z_1-axis) and of a sign-invariant neck domain D_j (dashed) shrinking to K. The quotient map q identifies B_{-1} with B_1.

4.2 Two explicit solutions on the disconnected domain

Lemma 4.1. *Let h_*, A_*, x, y, H, F_t and U_r be as above.*

1. *The equation $H(r) = h_*$ has a root $r_1 \in (2, 3)$ and a root $r_2 > 3$.*

2. *For $0 < t < 1$, the function F_t is smooth and strictly plurisubharmonic on $\overline{B_0}$, vanishes on ∂B_0, and*

$$(\mathrm{dd^c} F_t)^3 = 48 A_*^3 t^3 (1 - t)\, e^{-\gamma_* F_t}\, dV, \qquad \int_{B_0} (\mathrm{dd^c} F_t)^3 = (2\pi A_*)^3 t^3. \tag{4.5}$$

3. *For $r \in \{r_1, r_2\}$, the function U_r is even, continuous, vanishes on ∂D_0, is smooth and strictly plurisubharmonic on each closed ball, and satisfies*

$$(\mathrm{dd^c} U_r)^3 = \frac{2\, e^{-\gamma_* U_r}}{\int_{D_0} e^{-\gamma_* U_r}\, dV}\, dV \quad \text{on } D_0, \qquad \int_{D_0} (\mathrm{dd^c} U_r)^3 = 2. \tag{4.6}$$

The functions U_{r_1} and U_{r_2} are distinct.

Proof. (1) One computes $H(2) = 686/675 > h_*$, $H(3) = 63713/64000 < h_*$ and $H(r) \to 1 > h_*$ as $r \to \infty$; the intermediate value theorem gives the two roots.

(2) Write $s = |z|^2$ and $d = 1 - t + ts$. The complex Hessian of F_t has the two tangential eigenvalues $A_* t/d$ and the radial eigenvalue $A_* t(1-t)/d^2$, all positive on $\overline{B_0}$. Since $(i\partial\bar\partial F)^3 = 48 \det\left(F_{j\bar k}\right) dV$ and $\gamma_* A_* = 4$, we get $(\mathrm{dd^c} F_t)^3 = 48 A_*^3 t^3 (1-t) d^{-4} dV = 48 A_*^3 t^3 (1-t) e^{-\gamma_* F_t} dV$. Using $dV = \frac{\pi^3}{2} s^2\, ds\, d\sigma$ in polar coordinates ($d\sigma$ the normalized measure on the unit sphere) and $\frac{d}{ds}(s^3 d^{-3}) = 3(1-t)s^2 d^{-4}$,

$$\int_{B_0} (\mathrm{dd^c} F_t)^3 = 24\pi^3 A_*^3 t^3 (1-t) \int_0^1 \frac{s^2\, ds}{(1-t+ts)^4} = (2\pi A_*)^3 t^3.$$

(3) From (4.3), $1 - x = r^3/D$ and $1 - y = 1/D$, hence $x^3(1-x) = y^3(1-y) = r^3 N^3/D^4$. By (4.5) the density coefficient $\theta_r := 48 A_*^3 t^3 (1-t)$ is therefore the same for $t = x(r)$ and $t = y(r)$, so $(\mathrm{dd^c} U_r)^3 = \theta_r\, e^{-\gamma_* U_r}\, dV$ on all of D_0. Its total mass is $(2\pi A_*)^3 (y^3 + 2x^3) = 2H(r)/h_*$, which equals 2 exactly when $H(r) = h_*$. Integrating the equation then identifies $\theta_r = 2/\int_{D_0} e^{-\gamma_* U_r}\, dV$, which is (4.6). Evenness, continuity and the boundary values are clear. Finally $x'(r) = -r^2(r^2 + 2r + 3)/D(r)^2 < 0$, so $x(r_1) \neq x(r_2)$, and the values $U_r(a_1) = A_* \log(1 - x(r))$ distinguish the two functions. ☐

By Lemma 2.6, the descents of U_{r_1} and U_{r_2} to $q(D_0)$ are two distinct solutions of the normalized equation (1.4) on the disconnected open set $q(D_0)$, with Monge–Ampère mass one; here the identification of the two outer balls under q is accounted for by the factor 2 in (4.6).

4.3 The linearized operators on one ball

For $0 < t < 1$ let

$$\omega_t = \mathrm{dd^c} F_t, \qquad \nu_t = \omega_t^3, \qquad \Delta_t = -\operatorname{tr}_{\omega_t} \mathrm{dd^c}, \qquad L_t = \Delta_t - \gamma_*,$$

where $\operatorname{tr}_\omega \mathrm{dd^c} f = \omega^{j\bar k} f_{j\bar k}$, so that $\operatorname{tr}_{\omega_t} \omega_t = 3$. On the closed ball the coefficients of Δ_t are smooth and uniformly elliptic, since the eigenvalues of ω_t are bounded above and below by positive constants there; moreover Δ_t is formally self-adjoint with respect to ν_t, and integration by parts gives, for $f, g \in H_0^1(B_0; \mathbb{R})$,

$$\int_{B_0} g\, \Delta_t f\, d\nu_t = \int_{B_0} \operatorname{Re}\langle \bar\partial f, \bar\partial g\rangle_{\omega_t}\, d\nu_t, \tag{4.7}$$

where the left side is understood weakly. Consider the quadratic form

$$q_t(h) = \int_{B_0} |\bar\partial h|^2_{\omega_t}\, d\nu_t - \gamma_* \int_{B_0} h^2\, d\nu_t, \qquad h \in H_0^1(B_0; \mathbb{R}),$$

and the radial functions (with $s = |z|^2$, $d = 1 - t + ts$)

$$\psi_t = \partial_t F_t = A_* \frac{s-1}{d}, \qquad \kappa_t = \frac{3 - 4t}{t(1-t)}, \qquad R_t = \frac{3(1-t) - ts}{1 - t + ts}. \tag{4.8}$$

Lemma 4.2. *Let $0 < t < 1$.*

1. *$L_t \psi_t = -\kappa_t$, $L_t R_t = 0$, $R_t(1) = 3 - 4t$ (on ∂B_0), and $\psi_t - \kappa_t/\gamma_* = -A_* R_t/(4t(1-t))$.*
2. *If $h \in H_0^1(B_0; \mathbb{R})$ and $\int_{B_0} h\, d\nu_t = 0$, then $q_t(h) \geq 0$, with equality only for $h = 0$.*
3. *If $t \neq 3/4$, the weak equation $L_t h = 0$, $h \in H_0^1(B_0; \mathbb{R})$, has only the zero solution. The negative index of q_t on $H_0^1(B_0; \mathbb{R})$ is 0 if $t < 3/4$ and 1 if $t > 3/4$; in the latter case $q_t(\psi_t) < 0$, and ψ_t is radial. The same statements hold on the closed subspace of even functions.*

Proof. (1) Differentiating the identity $\log\det\Big(F_{t,j\bar k}\Big) = \log(A_*^3t^3(1-t)) - \gamma_*F_t$, which is (4.5), with respect to t gives $\operatorname{tr}_{\omega_t}\mathrm{dd^c}\psi_t = \kappa_t - \gamma_*\psi_t$, that is, $L_t\psi_t = -\kappa_t$. The last identity is a direct computation, and it implies $L_tR_t = 0$ because L_t kills $\psi_t - \kappa_t/\gamma_*$ (constants c satisfy $L_tc = -\gamma_*c$). For later use we record that for a radial function $f(s)$,

$$\Delta_t f = -\frac{d}{A_*t(1-t)}\{sd\,f'' + [3(1-t)+ts]f'\}, \tag{4.9}$$

which follows from the formula $\omega_t^{-1} = \frac{d}{A_*t}(I + \frac{ts}{1-t}P)$, P the orthogonal projection onto $\bar z$.

(2) The substitution $w = \sqrt{t/(1-t)}\,z$ identifies (B_0,ω_t) with a ball in the affine chart of $\mathbb{P}^3$ equipped with the Kähler–Einstein metric $\omega = A_*\,\mathrm{dd^c}\log(1+|w|^2)$, for which $\mathrm{Ric}(\omega) = \frac{4}{A_*}\omega = \gamma_*\omega$. Let λ_1 be the first nonzero eigenvalue of the $\bar\partial$-Laplacian $\Delta = -\operatorname{tr}_\omega\mathrm{dd^c}$ on $(\mathbb{P}^3,\omega)$. The Bochner identity $(\lambda-\gamma_*)\int|\bar\partial G|^2_\omega\omega^3 = \int|D''\bar\partial G|^2_\omega\omega^3$ for an eigenfunction $\Delta G = \lambda G$ [GKY11, Section 2] (Lichnerowicz's estimate) gives $\lambda_1 \ge \gamma_*$, with $D''\bar\partial G = 0$, i.e. $\nabla^{1,0}G$ holomorphic, for eigenfunctions with $\lambda = \gamma_*$. By the variational characterization of λ_1,

$$\int_{\mathbb{P}^3}|\bar\partial G|^2_\omega\,\omega^3 \ge \gamma_*\int_{\mathbb{P}^3}\Big(G - \frac{\int G\,\omega^3}{\int\omega^3}\Big)^2\omega^3 \qquad (G\in H^1(\mathbb{P}^3)), \tag{4.10}$$

with equality only if G is the sum of a constant and a γ_*-eigenfunction. Now extend h by zero to $\mathbb{P}^3$; the extension G lies in H^1, has zero mean, and (4.10) gives $q_t(h)\ge 0$. If $q_t(h) = 0$, then G is a constant plus a function with holomorphic $(1,0)$-gradient; the gradient vanishes on the nonempty open complement of the ball, hence everywhere, so G is constant and therefore zero.

(3) Let $h\in H^1_0(B_0;\mathbb{R})$ with $L_th = 0$ weakly. By interior elliptic regularity h is smooth in B_0. Its average $\bar h$ over $U(3)$ is a radial weak solution of the same equation with zero boundary trace; L_t, ν_t and B_0 are $U(3)$-invariant. By (4.9), a radial solution $h(s)$ satisfies the ordinary differential equation

$$\Big(\frac{s^3}{d^2}h'\Big)' + 4t(1-t)\frac{s^2}{d^4}h = 0,$$

which has a regular singular point at $s=0$. A solution that is smooth at the origin is determined by $h(0)$: if $h(0)=0$, integrating gives $h'(s) = -4t(1-t)d(s)^2s^{-3}\int_0^s\xi^2d(\xi)^{-4}h(\xi)\,d\xi$, hence $\sup_{[0,a]}|h| \le Ca\sup_{[0,a]}|h|$ for small a, so $h\equiv 0$ near 0 and then on $[0,1]$ by uniqueness for the nonsingular equation on $(0,1]$. Since $R_t(0) = 3\neq 0$, every smooth radial solution is a multiple of R_t, whose boundary value $3-4t$ is nonzero for $t\neq 3/4$. Hence $\bar h = 0$, so h has zero ν_t-mean, and testing the equation with h gives $q_t(h) = 0$; by (2), $h = 0$.

The negative index of q_t is at most one, because a two-dimensional negative subspace would contain a nonzero function of zero mean, contradicting (2). If $t < 3/4$, then $R_t > 0$ on $\overline{B_0}$ (it is decreasing in s with $R_t(1) = 3-4t > 0$), and the ground state identity

$$q_t(h) = \int_{B_0}R_t^2\Big|\bar\partial\Big(\frac{h}{R_t}\Big)\Big|^2_{\omega_t}d\nu_t \qquad (h\in H^1_0(B_0;\mathbb{R})),$$

obtained by integrating by parts against $L_tR_t = 0$ (first for $h\in C_c^\infty$, then by density), shows $q_t\ge 0$, with equality only if h/R_t is constant, hence zero; so the index is zero. If $t > 3/4$, then $\kappa_t < 0$, $\psi_t < 0$ in B_0 and $\psi_t|_{\partial B_0} = 0$, so by (1)

$$q_t(\psi_t) = \int_{B_0}\psi_t\,L_t\psi_t\,d\nu_t = -\kappa_t\int_{B_0}\psi_t\,d\nu_t < 0,$$

and the index is one. Since ψ_t and the kernel argument are radial, the same conclusions hold on the even subspace. ☐

4.4 Nondegeneracy of the normalized linearization

Fix a root $r \in \{r_1, r_2\}$ of Lemma 4.1 and write $t_0 = y(r)$ (central ball) and $t_1 = x(r)$ (outer balls). An even function on D_0 is determined by its restrictions to B_0 and B_1; accordingly we put

$$V = H_0^1(B_0; \mathbb{R})_{\text{even}} \oplus H_0^1(B_0; \mathbb{R}),$$

where the second summand represents the outer ball B_1 after translation by a_1. On V define the weighted forms (the weights $\frac{1}{2}$ and 1 account for the central ball and the pair of outer balls identified by q; the total weight $\frac{1}{2}\nu_{t_0}(B_0) + \nu_{t_1}(B_0)$ equals 1 by (4.6))

$$\langle f, g\rangle_r = \frac{1}{2}\int_{B_0} f_0 g_0 \, d\nu_{t_0} + \int_{B_0} f_1 g_1 \, d\nu_{t_1}, \qquad c_r(f) = \langle f, 1\rangle_r,$$

$$\mathcal{E}_r(f, g) = \frac{1}{2}\int_{B_0} \operatorname{Re}\langle \bar\partial f_0, \bar\partial g_0\rangle_{\omega_{t_0}} d\nu_{t_0} + \int_{B_0} \operatorname{Re}\langle \bar\partial f_1, \bar\partial g_1\rangle_{\omega_{t_1}} d\nu_{t_1},$$

and the bounded operator $J_r \colon V \to V^*$,

$$(J_r f)(g) = \mathcal{E}_r(f, g) - \gamma_* \langle f, g\rangle_r + \gamma_* c_r(f) c_r(g). \tag{4.11}$$

By (4.7), $J_r f = 0$ is the weak form of the system

$$\Delta_{t_i} f_i = \gamma_*[f_i - c_r(f)] \text{ in } B_0, \qquad f_i|_{\partial B_0} = 0, \qquad i = 0, 1, \tag{4.12}$$

which is the linearization of the normalized equation (4.6) at U_r (Lemma 6.1, Step 2); the term $c_r(f)$ is the variation of the normalizing integral.

Lemma 4.3. *For $r \in \{r_1, r_2\}$, the operator J_r is an isomorphism $V \to V^*$. The quadratic form $f \mapsto (J_r f)(f)$ has negative index 1 for $r = r_1$ and 0 for $r = r_2$. In particular the system (4.12) has only the zero solution in V.*

Proof. By the Poincaré inequality [GT01, (7.44)] and the bounds on the metrics, $\mathcal{E}_r$ is an inner product on V equivalent to the H^1 inner product. Let $L \colon V \to V^*$, $(Lf)(g) = \mathcal{E}_r(f, g) - \gamma_*\langle f, g\rangle_r$, the weighted direct sum of L_{t_0} on the even factor and L_{t_1} on the other. Since $x < \frac{3}{4} < y$ for $r > 1$ (both equal $\frac{3}{4}$ at $r = 1$, $x' < 0$ and $y' = D'/D^2 > 0$), Lemma 4.2 shows that L has zero kernel and negative index one. Under the Riesz identification $V^* \simeq V$ given by $\mathcal{E}_r$, L is the identity plus a compact self-adjoint operator (the inclusion $V \hookrightarrow L^2$ is compact by Rellich's theorem [GT01, Theorem 7.22]); a kernel-free operator of this form is an isomorphism. The operator $J_r = L + \gamma_* \,\ell \otimes \ell$, $\ell = c_r$, is a rank one perturbation of L, hence again the identity plus a compact self-adjoint operator, and it is an isomorphism as soon as its kernel is zero.

Let $v = L^{-1}\ell \in V$ and $a = \ell(v)$. By Lemma 4.2(1), the components of v are $v_i = -\psi_{t_i}/\kappa_{t_i}$, the unique solutions of $L_{t_i} v_i = 1$ with zero boundary values. If $J_r f = 0$, then $Lf = -\gamma_* \ell(f)\ell$, so $f = -\gamma_* \ell(f) v$ and, applying ℓ, $S\,\ell(f) = 0$ with

$$S = 1 + \gamma_* a = 1 - \gamma_* \Big[\frac{1}{2}\int_{B_0} \frac{\psi_{t_0}}{\kappa_{t_0}} d\nu_{t_0} + \int_{B_0} \frac{\psi_{t_1}}{\kappa_{t_1}} d\nu_{t_1}\Big]. \tag{4.13}$$

Thus J_r is injective if $S \neq 0$.

To determine the sign of S, consider for all $r > 1$ the even function u_r defined like U_r with the parameters $x(r), y(r)$, and the total mass $M(r) = \frac{1}{2}\int_{D_0} (\mathrm{dd^c} u_r)^3 = \frac{(2\pi A_*)^3}{2} H(r)$ of its quotient measure. Its density is $e^{k(r)} e^{-\gamma_* u_r}$ with $k(r) = \log(48 A_*^3 x^3 (1 - x))$, and $k' = \kappa_x x' = \kappa_y y'$. Differentiating the density gives $M' = k'M - \gamma_*\big[\frac{1}{2}\int_{B_0} \partial_r u_r \, d\nu_{t_0} + \int_{B_0} \partial_r u_r \, d\nu_{t_1}\big]$, and on the two types of balls $\partial_r u_r = y'\psi_y = k'\psi_{t_0}/\kappa_{t_0}$, respectively $x'\psi_x = k'\psi_{t_1}/\kappa_{t_1}$. At a root, $M = 1$, so (4.13) reads $S = M'/k'$. Now $k' = \kappa_x x' < 0$ because $x < \frac{3}{4}$ and $x' < 0$, while M' has the sign of

$$H'(r) = \frac{3r^2 N(r)^2\,(r^2 - 2r - 5)}{D(r)^4}$$

(a direct computation using $H = (r^3+2)N^3/D^3$). Thus H decreases on $(1, 1+\sqrt{6})$ and increases on $(1+\sqrt{6}, \infty)$. Since $r_1 \in (2,3)$, $H'(r_1) < 0$; since $H(3) < h_*$ and H decreases on $(3, 1+\sqrt{6})$, the root $r_2 > 3$ lies beyond $1+\sqrt{6}$, so $H'(r_2) > 0$. Consequently $S > 0$ at r_1 and $S < 0$ at r_2; in both cases J_r is an isomorphism.

For the index, consider on $V \oplus \mathbb{R}$ the quadratic form $B(f,s) = (Lf)(f) + 2s\ell(f) - s^2/\gamma_*$. Completing the square in the two possible ways,

$$B(f,s) = (J_r f)(f) - \frac{(s - \gamma_*\ell(f))^2}{\gamma_*} = (L(f+sv))(f+sv) - \frac{S}{\gamma_*}\,s^2,$$

where we used $(Lf)(v) = \ell(f)$ and $(Lv)(v) = a$. Both changes of variables are bounded invertible linear maps, so the first expression shows that B has index $\operatorname{ind}(J_r) + 1$, and the second that B has index $1+1$ if $S > 0$ and $1+0$ if $S < 0$. Hence $\operatorname{ind}(J_r) = 1$ at r_1 and $\operatorname{ind}(J_r) = 0$ at r_2. $\square$

5 Neck domains and convergence of the Dirichlet operators

5.1 Admissible domains shrinking to K

Lemma 5.1. *For every $\epsilon > 0$ there is an admissible domain Ω whose cover $D = q^{-1}(\Omega)$ satisfies*

$$K \Subset D \Subset B_5, \qquad \sup_{z\in\overline{D}} \operatorname{dist}(z, K) < \epsilon.$$

Proof. Step 1: K is polynomially convex. The projection of K to the z_1-line is $L_0 = \bigcup_{i=-1}^{1}\{|\zeta - 3i| \le 1\} \cup [-2,-1] \cup [1,2]$, a compact set with connected complement, so by Mergelyan's theorem [Rud87, Theorem 20.5] every continuous function on L_0 which is constant on each disc is a uniform limit of polynomials on L_0. Let $z \notin K$; we produce a polynomial P with $|P(z)| > \sup_K |P|$, distinguishing three cases according to the position of z_1. If $z_1 \notin L_0$, approximate on $L_0 \cup \{z_1\}$ (still with connected complement) the function equal to 0 on L_0 and 1 at z_1, and take $P(w) = p(w_1)$. If z_1 lies in the closed disc centered at $3i$, then $z \notin \overline{B_i}$, so $d := |z - a_i| > 1$; let $Q(w) = \sum_\ell (w_\ell - (a_i)_\ell)\,\overline{(z_\ell - (a_i)_\ell)}/d$, a linear function with $|Q| \le 1$ on $\overline{B_i}$ and $Q(z) = d$. For $0 < \tau < (d-1)/4$ let χ be continuous on L_0, equal to 1 on this disc, to 0 on the other two, and supported within distance τ of this disc, with values in $[0,1]$. Then $|\chi(w_1)Q(w)| \le 1+\tau$ on K, because $|w - a_i| \le 1+\tau$ on the support of $\chi(w_1)$ in K, while $\chi(z_1)Q(z) = d > 1+\tau$; a polynomial approximation p of χ gives the separator $P(w) = p(w_1)Q(w)$. If z_1 lies in one of the open intervals, then $z_\ell \neq 0$ for some $\ell \in \{2,3\}$; take χ continuous on L_0, supported in a compact subinterval of that interval, with $\chi(z_1) = 1$, so that $\chi(w_1)w_\ell$ vanishes on K and equals $z_\ell \neq 0$ at z, and approximate χ again.

Step 2: even separators, and polynomial convexity of $\mathcal{K} = q(K)$ in $\mathbb{C}^6$. Let P separate z from K and put $A = \max\{|P(z)|, |P(-z)|\} > \varrho := \sup_K |P|$. There are arbitrarily large k with $|P(z)^k + P(-z)^k| \ge A^k/2$: this is clear if $|P(z)| \neq |P(-z)|$, and if $|P(z)| = |P(-z)| \neq 0$ with ratio $e^{i\theta}$, then for every m either $|1 + e^{im\theta}| \ge 1$ or $|1 + e^{2im\theta}| > 1$. Choosing k with $A^k/2 > 2\varrho^k$, the even polynomial $P(w)^k + P(-w)^k$ separates $\{z, -z\}$ from K. An even polynomial is a polynomial in the quadratic monomials, i.e. in the coordinates of $\mathbb{C}^6$ restricted to X, so it separates $q(z)$ from $\mathcal{K}$; a point of $\mathbb{C}^6 \setminus X$ is separated from $\mathcal{K}$ by a 2×2 minor of the symmetric matrix, which vanishes on X. Hence $\mathcal{K}$ is polynomially convex.

Step 3: the domain. Let $V_\epsilon = \{z : \operatorname{dist}(z, K) < \epsilon\}$ with $\epsilon < \frac{1}{2}$, so that $V_\epsilon \subset B_5$, and let $W = \mathbb{C}^6 \setminus (X \setminus q(V_\epsilon))$, an open set with $W \cap X = q(V_\epsilon)$ and $q^{-1}(W) = V_\epsilon$. Fix $T_0 > 1 + \max_{\mathcal{K},\ell} |Z_\ell|$. For each point of the compact set $\{\max_\ell |Z_\ell| \le T_0\} \setminus W$, polynomial convexity gives a polynomial with modulus < 1 on $\mathcal{K}$ and > 1 at the point, hence on a neighborhood of it; by compactness finitely many polynomials $P_1, \dots, P_N$ suffice, and we add the six polynomials Z_ℓ / T_0. Let $\alpha < 1$ bound all $|P_s|$ on $\mathcal{K}$, and choose an integer k and $\delta > 0$ with $(N+6)\alpha^{2k} < \frac{1}{8}$ and $\delta \sup_{\mathcal{K}} |Z|^2 < \frac{1}{8}$. Then

$$F = \sum_s |P_s|^{2k} + \delta |Z|^2$$

is smooth, proper and strictly plurisubharmonic on $\mathbb{C}^6$, $F < \frac{1}{4}$ on $\mathcal{K}$, and $F \ge 1$ outside W; hence $\{F \le \frac{2}{3}\} \Subset W$. By Sard's theorem [Hir76, Theorem 3.1.3] applied to F on the manifold $X \setminus \{p\}$, we may pick a regular value $c \in (\frac{1}{3}, \frac{2}{3})$. Let Ω be the connected component of $\{F < c\} \cap X$ containing the connected set $\mathcal{K} \ni p$. Near each point of the level set $\{F = c\} \cap X$ the sublevel set is a half ball, so $\partial\Omega$ is a union of components of the level set, a smooth hypersurface, and $\rho = F - c$ is a strictly plurisubharmonic defining function for Ω near $\overline{\Omega}$, in the sense of Definition 2.2. Thus Ω is admissible, $\overline{\Omega} \subset W$, and $D = q^{-1}(\Omega)$ satisfies $K \subset D$ and $\overline{D} \subset q^{-1}(W) = V_\epsilon$, which gives the assertion. □

5.2 Convergence of the Dirichlet operators

Let $Q \subset \mathbb{C}^n$ be a bounded domain with a smooth strictly plurisubharmonic defining function near $\overline{Q}$. For $f \in C(\overline{Q})$, $f \ge 0$, let $\mathcal{S}_Q(f)$ denote the unique $u \in \mathrm{PSH}(Q) \cap C(\overline{Q})$ with

$$(\mathrm{dd^c} u)^n = f\, dV \text{ in } Q, \qquad u|_{\partial Q} = 0,$$

which exists by [BT76]; for smooth $f > 0$ it is smooth on $\overline{Q}$ by [CKNS85]. We use repeatedly the comparison principle

$$a, b \in \mathrm{PSH}(Q) \cap C(\overline{Q}), \quad a \le b \text{ on } \partial Q, \quad (\mathrm{dd^c} a)^n \ge (\mathrm{dd^c} b)^n \implies a \le b$$

[BT82], Kołodziej's estimate

$$\|\mathcal{S}_Q(f)\|_\infty \le c(Q, p, n)\, \|f\|_{L^p(Q)}^{1/n} \qquad (p > 1) \tag{5.1}$$

[Koł96] (the power $1/n$ follows from the case $\|f\|_p = 1$ by homogeneity), and the elementary stability estimate

$$\|\mathcal{S}_Q(f) - \mathcal{S}_Q(g)\|_\infty \le \|f - g\|_\infty^{1/n}\, \|\mathcal{S}_Q(1)\|_\infty, \tag{5.2}$$

which follows from the comparison principle: if $\eta = \|f - g\|_\infty$, then $\mathcal{S}_Q(f) + \eta^{1/n} \mathcal{S}_Q(1)$ has Monge–Ampère measure at least $(f + \eta)\, dV \ge g\, dV$ (mixed terms are nonnegative) and zero boundary values, hence lies below $\mathcal{S}_Q(g)$, and symmetrically. We also use the monotonicity $f \le g \Rightarrow \mathcal{S}_Q(f) \ge \mathcal{S}_Q(g)$.

Lemma 5.2. *Let $n \ge 2$, $R > 5$, and let a_i, B_i, D_0, K be as in Section 4.1, with e_1 the first coordinate vector of $\mathbb{C}^n$. Let D_j be bounded domains with smooth strictly plurisubharmonic defining functions such that*

$$K \subset D_j \Subset B_R, \qquad d_j := \sup_{z \in \overline{D_j}} \operatorname{dist}(z, K) \to 0.$$

For $f \in C(\overline{D_j})$, $f \ge 0$, let $E_j \mathcal{S}_{D_j}(f)$ be the zero extension of $\mathcal{S}_{D_j}(f)$ to $\overline{B_R}$, and let $\mathcal{S}_0[f]$ be the function equal to $\mathcal{S}_{B_i}(f|_{\overline{B_i}})$ on each $\overline{B_i}$ and to zero elsewhere. Then for every $C > 0$,

$$\epsilon_j(C) := \sup\{\|E_j \mathcal{S}_{D_j}(f) - \mathcal{S}_0[f]\|_\infty : f \in C(\overline{D_j}),\ 0 \le f \le C\} \longrightarrow 0. \tag{5.3}$$

Consequently, for $\gamma > 0$ and $M \geq 0$, the maps

$$\mathcal{N}_j(h) = E_j \mathcal{S}_{D_j}\Big(\frac{2e^{-\gamma h}}{\int_{D_j} e^{-\gamma h}\, dV}\Big), \qquad \mathcal{N}_0(h) = \mathcal{S}_0\Big[\frac{2e^{-\gamma h}}{\int_{D_0} e^{-\gamma h}\, dV}\Big], \qquad h \in C(\overline{B_R}; \mathbb{R}),$$

satisfy

$$\sup_{\|h\|_\infty \leq M} \|\mathcal{N}_j(h) - \mathcal{N}_0(h)\|_\infty \longrightarrow 0. \tag{5.4}$$

Proof. Step 1: a barrier from Mergelyan's theorem. Let $L = \bigcup_{i=-1}^{1}\{|\zeta - 3i| \leq 1\} \cup [-2,-1] \cup [1,2] \subset \mathbb{C}$, a compact set with connected complement. For $i \in \{-1,0,1\}$ define a continuous function χ_i on L, equal to 1 on the disc centered at $3i$ and to 0 on the other two discs, as follows: on the interval $[a+1, a+2]$ joining the centers a and $a+3$ ($a \in \{-3, 0\}$), with $\theta(x) = \frac{\pi}{2}(x-a-1)$, put

$$\chi_{a/3}(x) = \frac{\cos\theta(x)}{\sqrt{(1+(x-a)^2)/2}}, \qquad \chi_{(a+3)/3}(x) = \frac{\sin\theta(x)}{\sqrt{(1+(x-a-3)^2)/2}},$$

and let the third function vanish there. These functions are continuous on L and holomorphic (constant) in its interior, so by Mergelyan's theorem [Rud87, Theorem 20.5] there are polynomials $P_{i,k} \to \chi_i$ uniformly on L. For $A > 0$ put

$$\Phi_k(z) = A \log\Big[\sum_{i=-1}^{1} |P_{i,k}(z_1)|^2\, \frac{1+|z-a_i|^2}{2}\Big].$$

The bracket is the squared norm of a holomorphic vector-valued function, so Φ_k is plurisubharmonic wherever the bracket is positive. On K the bracket converges uniformly to $(1+|z-a_i|^2)/2 \geq \frac{1}{2}$ on $\overline{B_i}$ and to $\cos^2\theta + \sin^2\theta = 1$ on the intervals. Hence for large k, Φ_k is smooth and plurisubharmonic on a neighborhood of K, and $\Phi_k \to F_K$ uniformly on K, where $F_K = A\log\frac{1+|z-a_i|^2}{2} \leq 0$ on $\overline{B_i}$ and $F_K = 0$ on $K \setminus D_0$. Moreover, for $0 < \tau < 1$, the convergence is in C^2 on the smaller balls $\overline{B(a_i, 1-\tau)}$, by Cauchy's estimates for the $P_{i,k}$ on the smaller discs. The Hessian of $F_i = A\log\frac{1+|z-a_i|^2}{2}$ has eigenvalues $A/(1+\varrho^2)$ ($n-1$ times) and $A/(1+\varrho^2)^2$, $\varrho = |z-a_i|$, so $(\mathrm{dd^c} F_i)^n = 2^n n! A^n (1+\varrho^2)^{-n-1} dV \geq \frac{n!A^n}{2}\, dV$ on $\overline{B_i}$.

Given $C > 0$, fix A with $n!A^n/2 > 4C$. For fixed $0 < \tau < \frac{1}{4}$ choose k so large that, with $\Phi = \Phi_k$,

$$|\Phi - F_K| \leq \tau \text{ on } K, \qquad (\mathrm{dd^c}\Phi)^n \geq 2C\, dV \text{ on } \bigcup_i \overline{B(a_i, 1-\tau)}.$$

Step 2: a correction on the necks. Write $K_s = \{z : \mathrm{dist}(z, K) < s\}$. Choose $s_\tau > 0$ so small that $\overline{K_{2s_\tau}} \subset B_R$ and lies in the region where Φ is smooth and plurisubharmonic, and a smooth cutoff $0 \leq \chi_\tau \leq 1$ equal to 1 on $\overline{K_{s_\tau}} \setminus \bigcup_i B(a_i, 1-\tau)$ and supported in $K_{2s_\tau} \setminus \bigcup_i \overline{B(a_i, 1-2\tau)}$. Put $g_\tau = C\chi_\tau + \tau$ and $\psi_\tau = \mathcal{S}_{B_R}(g_\tau)$. Since the intervals and the spheres have zero volume, the volume of $\operatorname{supp}\chi_\tau$ tends to zero as $\tau \to 0$ (with $s_\tau \to 0$), so by (5.1) with $p = 2$,

$$b_\tau := \|\psi_\tau\|_\infty \leq c(B_R, 2, n)\|g_\tau\|_{L^2}^{1/n} \longrightarrow 0 \qquad (\tau \to 0).$$

For j large, $\overline{D_j} \subset K_{s_\tau}$. Set $c_j = \max\{0, \sup_{\overline{D_j}}\Phi\}$ and $a_j = \Phi + \psi_\tau - c_j$. Then $a_j \leq 0$ on $\overline{D_j}$, and since all mixed terms in the expansion of $(\mathrm{dd^c}\Phi + \mathrm{dd^c}\psi_\tau)^n$ are nonnegative, $(\mathrm{dd^c} a_j)^n \geq (\mathrm{dd^c}\Phi)^n \geq 2C\, dV$ on the smaller balls and $(\mathrm{dd^c} a_j)^n \geq (\mathrm{dd^c}\psi_\tau)^n = g_\tau\, dV \geq C\, dV$ on the rest of D_j, where $\chi_\tau = 1$. The comparison principle gives

$$a_j \leq \mathcal{S}_{D_j}(f) \leq 0 \qquad \text{for } 0 \leq f \leq C.$$

Every limit point of a sequence $z_j \in \overline{D_j} \setminus D_0$ lies in $K \setminus D_0$, where $|\Phi| \leq \tau$; by continuity of Φ and compactness, $\liminf_j \inf_{\overline{D_j}\setminus D_0} \Phi \geq -\tau$, and $\limsup_j c_j \leq \tau$. Therefore

$$\limsup_{j\to\infty} \sup_{0 \leq f \leq C} \sup_{\overline{D_j}\setminus D_0} |\mathcal{S}_{D_j}(f)| \leq 2\tau + b_\tau,$$

and letting $\tau \to 0$ shows that $b_j(C) := \sup_{0\le f\le C} \sup_{\overline{D_j}\setminus D_0} |\mathcal{S}_{D_j}(f)| \to 0$.

Step 3: comparison on the balls. For $0 \le f \le C$ and each i, the functions $u = \mathcal{S}_{D_j}(f)$ and $v = \mathcal{S}_{B_i}(f|_{\overline{B_i}})$ have the same Monge–Ampère measure on B_i, and $-b_j(C) \le u \le 0 = v$ on ∂B_i; the comparison principle gives $v - b_j(C) \le u \le v$ on $\overline{B_i}$. Off D_0 the function $\mathcal{S}_0[f]$ vanishes and $|E_j\mathcal{S}_{D_j}(f)| \le b_j(C)$. Hence $\epsilon_j(C) \le b_j(C) \to 0$, which is (5.3).

Step 4: the normalizing integrals. Let $\|h\|_\infty \le M$, $V_0 = \mathrm{vol}(D_0)$, $v_j = \mathrm{vol}(D_j \setminus D_0) \le \mathrm{vol}(K_{d_j} \setminus D_0) \to 0$, and $Z_j = \int_{D_j} e^{-\gamma h} dV$, $Z_0 = \int_{D_0} e^{-\gamma h} dV$. Then $Z_j \ge Z_0 \ge e^{-\gamma M} V_0$ and $0 \le Z_j - Z_0 \le e^{\gamma M} v_j$, so the densities $f_j = 2e^{-\gamma h}/Z_j$ and $f_0 = 2e^{-\gamma h}/Z_0$ satisfy $0 \le f_j \le C_M := 2e^{2\gamma M}/V_0$ and $\|f_j - f_0\|_{L^\infty(D_0)} \le \eta_j := 2e^{4\gamma M} v_j/V_0^2 \to 0$. By (5.3) and (5.2) on the balls B_i, where $\mathcal{S}_{B_i}(1) = 2^{-1}(n!)^{-1/n}(|z-a_i|^2 - 1)$,

$$\|\mathcal{N}_j(h) - \mathcal{N}_0(h)\|_\infty \le \epsilon_j(C_M) + \Big(\frac{\eta_j}{2^n n!}\Big)^{1/n} \longrightarrow 0$$

uniformly for $\|h\|_\infty \le M$. ☐

Remark 5.3. If D_j is invariant under $z \mapsto -z$ and $n = 3$, then $\mathcal{N}_j$ maps $\mathcal{E}_R$ into itself, by uniqueness of the Dirichlet problem; the same holds for $\mathcal{N}_0$. For even h, the identity $\int_{q(D_j)} e^{-\gamma \bar h}\, dm = \frac{1}{2}\int_{D_j} e^{-\gamma h}\, dV$, $\bar h \circ q = h$, shows that the factor 2 in the definition of $\mathcal{N}_j$ is exactly what makes a fixed point of $\mathcal{N}_j$ the lift of a solution of the normalized equation (1.4) on $q(D_j)$ with γ in place of γ_* (Lemma 2.6).

6 The degree argument

In this section $n = 3$, $\gamma = \gamma_*$, $R > 5$, and $\mathcal{N}_0, \mathcal{N}_j$ are the maps of Lemma 5.2, restricted to the space $\mathcal{E}_R$ of even continuous functions (Remark 5.3). The fixed points of $\mathcal{N}_j$ are the lifts of the solutions we are looking for. We use the fixed point index of compact maps on open subsets of Banach spaces, equivalently the Leray–Schauder degree $\deg(I - F, O, 0)$ for a compact map $F \colon \overline{O} \to E$ without fixed points on ∂O [Gra72], [Dei85, Chapter 8]: its homotopy invariance under compact homotopies without fixed points on ∂O, the existence of a fixed point when the index is nonzero, and the following two facts.

- *The index of a nondegenerate fixed point* [KZ84, §21], [Dei85, Chapter 8]: if F is compact on a neighborhood of a fixed point x_0 in a Banach space E, Fréchet differentiable at x_0, and 1 is not an eigenvalue of $F'(x_0)$, then x_0 is an isolated fixed point and its index is $(-1)^m$, where m is the sum of the algebraic multiplicities of the eigenvalues of the compact operator $F'(x_0)$ which are larger than 1.
- *Commutativity* [Gra72, Section 5, (VI)]: if X, Y are Banach spaces, $U \subset X$, $U' \subset Y$ are open, $f \colon U \to Y$ and $g \colon U' \to X$ are continuous with relatively compact images, and the fixed point sets of $g \circ f$ on $f^{-1}(U')$ and of $f \circ g$ on $g^{-1}(U)$ (which correspond to each other under f) are compact, then the two indices agree.

The second fact allows us to compute the index of $\mathcal{N}_0$ at U_r in a Hölder space, where $\mathcal{N}_0$ is differentiable by the implicit function theorem, while the comparison with $\mathcal{N}_j$ takes place in the uniform norm, where Lemma 5.2 applies.

Throughout, $\beta \in (0,1)$ denotes a Hölder exponent for which the following holds: on each of the fixed domains $Q = B_i$ or $Q = D_j$, and for every $C > 0$, the solutions $\mathcal{S}_Q(f)$ with $0 \le f \le C$ are uniformly bounded in $C^{0,\beta}(\overline{Q})$ [GKZ08], [BKPZ16, Theorem 1.2]. We fix $0 < \alpha < \beta$ and put

$$\mathcal{E}_R^\alpha = \{h \in C^{0,\alpha}(\overline{B_R}; \mathbb{R}) : h(-z) = h(z)\},$$

a Banach space continuously embedded in $\mathcal{E}_R$. Note that the zero extension of a function $v \in C^{0,\beta}(\overline{Q})$ vanishing on ∂Q belongs to $C^{0,\beta}(\overline{B_R})$ with the same seminorm, and that bounded subsets of $\mathcal{E}_R^\beta$ are relatively compact in $\mathcal{E}_R^\alpha$ and in $\mathcal{E}_R$ (Arzelà–Ascoli).

For $r \in \{r_1, r_2\}$ put

$$f_r = \frac{2e^{-\gamma U_r}}{\int_{D_0} e^{-\gamma U_r}\, dV} = \frac{(\mathrm{dd^c} U_r)^3}{dV} \text{ on } D_0, \qquad c_r(h) = \frac{1}{2}\int_{D_0} h\, f_r\, dV,$$

and $\Delta_r = -\operatorname{tr}_{\mathrm{dd^c}U_r} \mathrm{dd^c}$ on each B_i, so that for even h the functional $c_r(h)$ agrees with the one of Section 4. Let T_r be the linear map which sends $h \in \mathcal{E}_R$ to the zero extension of the solution of

$$\Delta_r(T_r h) = \gamma\big[h - c_r(h)\big] \text{ in } B_i, \qquad T_r h|_{\partial B_i} = 0, \qquad i = -1, 0, 1. \tag{6.1}$$

Lemma 6.1. *Let $r \in \{r_1, r_2\}$.*

1. *$\mathcal{N}_0$ is a continuous compact map $\mathcal{E}_R \to \mathcal{E}_R$, and so is $\mathcal{N}_j$ for every j, when D_j is invariant under $z \mapsto -z$. Moreover $\mathcal{N}_0$ maps bounded subsets of $\mathcal{E}_R$ into bounded subsets of $\mathcal{E}_R^\beta$, and it is continuous and compact as a map $\mathcal{E}_R \to \mathcal{E}_R^\alpha$.*

2. *T_r is a compact linear operator on $\mathcal{E}_R$ and on $\mathcal{E}_R^\alpha$, with $\|T_r h\|_{C^{1,1/2}(\overline{B_i})} \le C\|h\|_{L^{12}(D_0)}$, and $\mathcal{N}_0\colon \mathcal{E}_R^\alpha \to \mathcal{E}_R^\alpha$ is continuously Fréchet differentiable on a neighborhood of U_r, with $D\mathcal{N}_0(U_r) = T_r$.*

3. *U_{r_1} and U_{r_2} are isolated fixed points of $\mathcal{N}_0$ in $\mathcal{E}_R$, and for all sufficiently small $a > 0$,*
$$\deg(I - \mathcal{N}_0, B_{\mathcal{E}_R}(U_{r_1}, a), 0) = -1, \qquad \deg(I - \mathcal{N}_0, B_{\mathcal{E}_R}(U_{r_2}, a), 0) = +1.$$

4. *Let D_j be sign-invariant domains as in Lemma 5.2. For all sufficiently small radii $a_1, a_2 > 0$ (depending only on $\mathcal{N}_0$), the disjoint balls $O_i = B_{\mathcal{E}_R}(U_{r_i}, a_i)$, $i = 1, 2$, satisfy, for all sufficiently large j,*
$$\deg(I - \mathcal{N}_j, O_1, 0) = -1, \qquad \deg(I - \mathcal{N}_j, O_2, 0) = +1.$$
In particular $\mathcal{N}_j$ has a fixed point in each O_i; the restriction to D_j of such a fixed point is an even function $h \in \mathrm{PSH}(D_j) \cap C(\overline{D_j})$, $h|_{\partial D_j} = 0$, with
$$(\mathrm{dd^c} h)^3 = \frac{2e^{-\gamma h}}{\int_{D_j} e^{-\gamma h}\, dV}\, dV. \tag{6.2}$$

Proof. Step 1: compactness and continuity. On a bounded subset of $\mathcal{E}_R$ the normalized densities $2e^{-\gamma h}/\int e^{-\gamma h} dV$ are bounded above and below by positive constants, and they depend continuously (in the uniform norm) on h. By the choice of β, the corresponding solutions on the fixed domains are bounded in $C^{0,\beta}$, hence so are their zero extensions; this gives the boundedness of $\mathcal{N}_0$ and $\mathcal{N}_j$ into $\mathcal{E}_R^\beta$, hence their compactness into $\mathcal{E}_R$ and (for $\mathcal{N}_0$) into $\mathcal{E}_R^\alpha$. Continuity in the uniform norm follows from (5.2), and continuity of $\mathcal{N}_0\colon \mathcal{E}_R \to \mathcal{E}_R^\alpha$ then follows from the interpolation inequality $[v]_\alpha \le (2\|v\|_\infty)^{1-\alpha/\beta}[v]_\beta^{\alpha/\beta}$ between Hölder seminorms. This proves (1).

Step 2: the tangent map. On each closed ball, Δ_r is a uniformly elliptic operator with smooth coefficients and no zeroth order term, because the eigenvalues of $\mathrm{dd^c} U_r = \omega_{t_i}$ have positive lower and upper bounds there (Lemma 4.1). By the L^p theory [GT01, Theorem 9.15 and Lemma 9.17], for $g \in L^{12}(B_i)$ the Dirichlet problem $\Delta_r w = g$, $w = 0$ on ∂B_i, has a unique solution $w \in W^{2,12} \cap W_0^{1,12}$, with $\|w\|_{W^{2,12}} \le C\|g\|_{L^{12}}$, and $W^{2,12}(B_i) \hookrightarrow C^{1,1/2}(\overline{B_i})$ since the real dimension is six [GT01, Theorem 7.26]. Taking $g = \gamma[h - c_r(h)]$, where $|c_r(h)| \le C\|h\|_{L^{12}(D_0)}$ as f_r is bounded, gives the estimate in (2), and compactness of T_r on $\mathcal{E}_R$ and on $\mathcal{E}_R^\alpha$ follows since $C^{1,1/2}$ is compactly embedded in $C^{0,\alpha}$; T_r preserves evenness by uniqueness.

For the differentiability, fix a ball $B = B_i$, write U for $U_r|_{\overline{B}}$, and consider

$$\Phi\colon C^{2,\alpha}_0(\overline{B}) \longrightarrow C^{0,\alpha}(\overline{B}), \qquad \Phi(u) = \det\Big(u_{j\bar{k}}\Big),$$

where $C^{2,\alpha}_0$ denotes the functions vanishing on ∂B. Since $C^{0,\alpha}$ is a Banach algebra, Φ is smooth, with $D\Phi(U)v = \det\Big(U_{j\bar{k}}\Big)\, U^{j\bar{k}} v_{j\bar{k}} = \frac{1}{48} f_r\, \Lambda v$, where $\Lambda = \mathrm{tr}_{\mathrm{dd^c}U}\, \mathrm{dd^c} = -\Delta_r$ and we used $(\mathrm{dd^c}U)^3 = 48 \det\Big(U_{j\bar{k}}\Big)\, dV = f_r\, dV$. By Schauder theory [GT01, Theorem 6.14], Λ is an isomorphism $C^{2,\alpha}_0(\overline{B}) \to C^{0,\alpha}(\overline{B})$, so $D\Phi(U)$ is an isomorphism and the implicit function theorem gives neighborhoods $\mathcal{W}$ of U in $C^{2,\alpha}_0$ and $\mathcal{V}$ of $f_r/48$ in $C^{0,\alpha}$ such that $\Phi\colon \mathcal{W} \to \mathcal{V}$ is a diffeomorphism. For $g \in \mathcal{V}$, the function $\Phi^{-1}(g)$ is C^2-close to U, hence strictly plurisubharmonic, so by uniqueness of the Dirichlet problem $\Phi^{-1}(g) = \mathcal{S}_B(48g)$. Consequently $f \mapsto \mathcal{S}_B(f)$ is smooth from a $C^{0,\alpha}$-neighborhood of f_r into $C^{2,\alpha}_0(\overline{B})$, with $\mathcal{S}'_B(f_r)g = \Lambda^{-1}(g/f_r)$. The map $h \mapsto 2e^{-\gamma h}/\int_{D_0} e^{-\gamma h} dV$ is smooth from $\mathcal{E}^\alpha_R$ to $C^{0,\alpha}(\overline{D_0})$ (again because $C^{0,\alpha}$ is a Banach algebra), with derivative $a \mapsto -\gamma f_r[a - c_r(a)]$ at U_r, by the computation

$$\frac{2e^{-\gamma(U_r+\delta a)}}{\int_{D_0} e^{-\gamma(U_r+\delta a)}\, dV} = f_r\{1 - \gamma\delta[a - c_r(a)] + O(\delta^2)\},$$

which uses $\int_{D_0} f_r\, dV = 2$. Finally, zero extension is a bounded linear map from $C^{2,\alpha}_0(\overline{B_i})$ to $C^{0,\alpha}(\overline{B_R})$. Composing, $\mathcal{N}_0$ is smooth on a neighborhood of U_r in $\mathcal{E}^\alpha_R$, and by the chain rule $D\mathcal{N}_0(U_r)a$ is the zero extension of $\Lambda^{-1}(-\gamma[a - c_r(a)])$, that is, $T_r a$. This proves (2).

Step 3: the eigenvalues of T_r and the index in the Hölder space. Let T also denote the restriction of T_r to V. By (4.7) it is characterized by $\mathcal{E}_r(Th, g) = \gamma[\langle h, g\rangle_r - c_r(h)c_r(g)]$ for $g \in V$, so it is self-adjoint for the inner product $\mathcal{E}_r$, compact (the inclusion $V \hookrightarrow L^2$ is compact), and nonnegative (the total weight is one, so $\langle h, h\rangle_r \geq c_r(h)^2$ by Cauchy–Schwarz). Moreover $\mathcal{E}_r((I-T)h, g) = (J_r h)(g)$, so in an orthonormal eigenbasis of T the form $(J_r h)(h)$ is diagonal with entries $1 - \nu$, ν ranging over the eigenvalues of T with multiplicity. Hence the number m of eigenvalues of T greater than 1, counted with multiplicity, equals the negative index of J_r, which is 1 at r_1 and 0 at r_2 by Lemma 4.3; and 1 is not an eigenvalue. The nonzero eigenvalues of T_r on $\mathcal{E}^\alpha_R$ (or on $\mathcal{E}_R$) and on V coincide, with the same algebraic multiplicities: a generalized eigenvector in $\mathcal{E}_R$ for an eigenvalue $\lambda \neq 0$ lies in the range of T_r, hence in $W^{2,12}$ with zero trace on each ball, hence in V; conversely a generalized eigenvector chain $(T - \lambda)h_j = h_{j-1}$, $h_{-1} = 0$, in V consists of continuous functions, because $h_j = \lambda^{-1}(Th_j - h_{j-1})$ and T maps L^2 into $W^{2,2} \subset L^6$, L^6 into $W^{2,6} \subset L^{12}$, and L^{12} into $W^{2,12} \subset C^{1,1/2}$ [GT01, Theorems 9.15 and 7.26]; and self-adjointness on V excludes nontrivial Jordan blocks. In particular 1 is not an eigenvalue of T_r on $\mathcal{E}^\alpha_R$. By (1), (2) and the index formula for nondegenerate fixed points, U_r is an isolated fixed point of $\mathcal{N}_0$ in $\mathcal{E}^\alpha_R$, of index $(-1)^m$: -1 for r_1 and $+1$ for r_2.

Step 4: transfer of the index to $\mathcal{E}_R$. Every fixed point of $\mathcal{N}_0$ in $\mathcal{E}_R$ lies in $\mathcal{E}^\alpha_R$, and if fixed points h_k converged to U_r in $\mathcal{E}_R$, then $h_k = \mathcal{N}_0(h_k) \to \mathcal{N}_0(U_r) = U_r$ in $\mathcal{E}^\alpha_R$ by (1), contradicting isolation in $\mathcal{E}^\alpha_R$. So U_r is isolated in $\mathcal{E}_R$; let $O = B_{\mathcal{E}_R}(U_r, a)$ contain no other fixed point. Apply the commutativity property with $X = \mathcal{E}_R$, $Y = \mathcal{E}^\alpha_R$, $f = \mathcal{N}_0\colon O \to Y$, which has relatively compact image by (1), and g the inclusion of a large ball $U' \subset Y$ containing $\mathcal{N}_0(O)$ into X, which has relatively compact image by Arzelà–Ascoli. Then $g \circ f = \mathcal{N}_0$ on O, and $f \circ g = \mathcal{N}_0$ on $O \cap U' \subset Y$; both have $\{U_r\}$ as fixed point set. Hence the index of $\mathcal{N}_0$ at U_r in $\mathcal{E}_R$ equals its index in $\mathcal{E}^\alpha_R$, which is $(-1)^m$ by Step 3. This proves (3).

Step 5: transfer to the connected domains. Choose disjoint balls $O_i = B_{\mathcal{E}_R}(U_{r_i}, a_i)$ as in (3). By compactness of $\mathcal{N}_0$ and isolation, $d_i := \inf_{\partial O_i} \|h - \mathcal{N}_0(h)\|_\infty > 0$: otherwise a sequence $h_k \in \partial O_i$ with $h_k - \mathcal{N}_0(h_k) \to 0$ would have a subsequence with $\mathcal{N}_0(h_k)$ convergent, hence h_k convergent to a fixed point on ∂O_i. By (5.4), $\sup_{\overline{O_i}} \|\mathcal{N}_j - \mathcal{N}_0\|_\infty < d_i$ for large j, so the compact straight-line homotopy $(1-s)\mathcal{N}_0 + s\mathcal{N}_j$ has no fixed point on ∂O_i: a fixed point would satisfy

$\|h - \mathcal{N}_0(h)\|_\infty = s\|\mathcal{N}_j(h) - \mathcal{N}_0(h)\|_\infty < d_i$. Homotopy invariance gives the two degrees, and a nonzero degree gives a fixed point $h \in O_i$ of $\mathcal{N}_j$. By definition of $\mathcal{N}_j$, h vanishes outside D_j and its restriction to D_j is the continuous plurisubharmonic solution of the Dirichlet problem with density $2e^{-\gamma h}/\int_{D_j} e^{-\gamma h}$, which is (6.2); it is even because $h \in \mathcal{E}_R$. $\square$

7 Regularity of continuous solutions

Because the right-hand side $Ce^{-\gamma h}$ is decreasing in h, the classical theory of [CKNS85] does not directly give the smoothness of a given continuous solution of $(\mathrm{dd^c}h)^n = Ce^{-\gamma h}dV$, and uniqueness fails in general, as this paper shows. We instead use the parabolic Monge–Ampère equation, for which uniqueness and smoothing hold.

Lemma 7.1. *Let $n \geq 2$, and let $D \subset \mathbb{C}^n$ be a bounded domain with a smooth strictly plurisubharmonic defining function ρ near $\overline{D}$. Let $\gamma, C > 0$ and let $h \in \mathrm{PSH}(D) \cap C(\overline{D})$ satisfy*

$$h|_{\partial D} = 0, \qquad (\mathrm{dd^c}h)^n = Ce^{-\gamma h}\, dV \text{ in } D.$$

Then $h \in C^\infty(\overline{D})$ and its complex Hessian is positive definite on $\overline{D}$.

Proof. Choose $\Lambda > 0$ with $(\mathrm{dd^c}(\Lambda\rho))^n \geq C\, dV \geq Ce^{-\gamma h}dV$ on D (recall $h \leq 0$). The comparison principle gives $\Lambda\rho \leq h \leq 0$ on $\overline{D}$, so the function h_{ext} equal to $h = \max\{h, \Lambda\rho\}$ on $\overline{D}$ and to $\Lambda\rho$ on a neighborhood of $\overline{D}$ minus D is continuous, and it is plurisubharmonic by the gluing lemma for plurisubharmonic functions (see e.g. [GZ17, Chapter 1]), since $h = 0 = \Lambda\rho$ on ∂D.

We apply [Do15, Theorem 0.1] to the Cauchy–Dirichlet problem

$$v_t = \log\det(v_{a\bar{b}}) + b(t) \text{ on } D \times (0,T), \qquad v = 0 \text{ on } \partial D \times (0,T), \qquad v(0,\cdot) = h_{\mathrm{ext}},$$

with

$$c = \frac{C}{2^n n!}, \qquad T = \frac{1}{2\gamma}, \qquad a(t) = 1 - \gamma t, \qquad b(t) = -n\log a(t) - \log c.$$

Its hypotheses hold: the initial datum is a bounded plurisubharmonic function on a neighborhood of $\overline{D}$, continuous, and compatible with the lateral datum 0 at $t = 0$; the source $b(t)$ is smooth and does not depend on v. The theorem provides a solution $v \in C^\infty(\overline{D} \times (0,T)) \cap C(\overline{D} \times [0,T))$ whose slices $v(t,\cdot)$ are strictly plurisubharmonic.

On the other hand, $w(t,z) = a(t)h(z)$ solves the same problem in a weak sense: $w(t,\cdot)$ is continuous and plurisubharmonic,

$$(\mathrm{dd^c}w(t,\cdot))^n = 2^n n!\, g_t\, dV, \qquad g_t = c\, a(t)^n e^{-\gamma h}, \qquad w_t = -\gamma h = \log g_t + b(t),$$

and $w = 0$ on $\partial D \times [0,T)$, $w(0,\cdot) = h$. We show $v = w$ on $\overline{D} \times [0,T)$ by a comparison argument which does not differentiate h.

First, two contact inequalities. Let H be continuous and plurisubharmonic near z_0, with $(\mathrm{dd^c}H)^n = 2^n n!\, g\, dV$ for a positive continuous g. If a smooth strictly plurisubharmonic function Q satisfies $Q \geq H$ near z_0 and $Q(z_0) = H(z_0)$, then $\det(Q_{a\bar{b}})(z_0) \geq g(z_0)$: otherwise, on a small ball $B(z_0, \varrho)$ the strict reverse inequality holds, and for small $\eta > 0$ the function $Q_\eta = Q + \eta(|z - z_0|^2 - \varrho^2/2)$ is still strictly plurisubharmonic with $\det\left(Q_{\eta,a\bar{b}}\right) < g$, exceeds H on $\partial B(z_0, \varrho)$ and is smaller than H at z_0; the comparison principle on the nonempty open set $\{Q_\eta < H\}$ then gives $\int_{\{Q_\eta < H\}} g\, dV \leq \int_{\{Q_\eta < H\}} \det\left(Q_{\eta,a\bar{b}}\right) dV$, a contradiction. Symmetrically, if $Q \leq H$ near z_0 with equality at z_0, then $\det(Q_{a\bar{b}})(z_0) \leq g(z_0)$, using $Q_\eta = Q - \eta(|z - z_0|^2 - \varrho^2/2)$ and the set $\{H < Q_\eta\}$.

Now fix $0 < T' < T$ and $\eta > 0$, and suppose $w - v - \eta t$ has a positive maximum on $\overline{D} \times [0, T']$. It is attained at some (t_0, z_0) with $t_0 > 0$ and $z_0 \in D$, because $w - v = 0$ on $\{t = 0\}$ and on $\partial D \times [0,T']$. Then $v(t_0,\cdot) + (w-v)(t_0,z_0)$ is a smooth strictly plurisubharmonic function touching $w(t_0,\cdot)$ from above at z_0, so the first contact inequality gives $\det(v_{a\bar{b}})(t_0,z_0) \geq g_{t_0}(z_0)$, hence $v_t(t_0,z_0) \geq \log g_{t_0}(z_0) + b(t_0) = w_t(t_0,z_0)$. But maximality in t (from the left, also when $t_0 = T'$) gives $w_t - v_t - \eta \geq 0$ at (t_0,z_0), a contradiction. Hence $w - v \leq \eta t$; the symmetric argument with the second contact inequality gives $v - w \leq \eta t$. Letting $\eta \to 0$ and $T' \to T$ yields $v = w$.

Consequently $h = v(t_0,\cdot)/a(t_0)$ for any $t_0 \in (0,T)$, so $h \in C^\infty(\overline{D})$ with $\mathrm{dd^c} h > 0$ in D. On ∂D the Hessian is semipositive by continuity and has determinant $c\, e^{-\gamma h} > 0$, so it is positive definite there as well. $\square$

8 Proof of the main theorem

8.1 Two solutions on all admissible domains close to K

Proposition 8.1. *There are $\epsilon_* > 0$ and disjoint balls $O_1, O_2 \subset \mathcal{E}_6$ around U_{r_1}, U_{r_2} with the following property. Let Ω be an admissible domain whose cover $D = q^{-1}(\Omega)$ satisfies*

$$K \Subset D \Subset B_6, \qquad \sup_{z \in \overline{D}} \operatorname{dist}(z,K) < \epsilon_*. \tag{8.1}$$

Then Ω satisfies conclusions (1) and (2) of Theorem 1.1, and the zero extensions of the lifts $u_i \circ q$ of the two solutions lie in O_1 and O_2 respectively.

Proof. Conclusion (1) is Proposition 3.2 with $R = 6$, since $\overline{\Omega} = q(\overline{D}) \subset Q_6$.

Let O_1, O_2 be balls as in Lemma 6.1(4) for $R = 6$, with radii small enough that $a_1 + a_2 < \|U_{r_1} - U_{r_2}\|_\infty$; they depend only on $\mathcal{N}_0$. We claim that there is $\epsilon_* > 0$ such that for every admissible Ω satisfying (8.1), the map $\mathcal{N}_D$ (the map $\mathcal{N}_j$ of Lemma 5.2 with D in place of D_j, acting on $\mathcal{E}_6$) has degrees -1 and $+1$ on O_1 and O_2. If not, there would be admissible domains $\Omega^{(k)}$ with covers $D^{(k)}$ satisfying $K \Subset D^{(k)} \Subset B_6$ and $\sup_{\overline{D^{(k)}}} \operatorname{dist}(\cdot,K) < 1/k$, for which at least one of the two degrees is undefined or has the wrong value. By Lemma 2.6(1), the sequence $D^{(k)}$ satisfies all hypotheses of Lemma 5.2 and is sign-invariant, so Lemma 6.1(4) gives both degrees for large k, a contradiction.

Now let Ω satisfy (8.1), with cover D. By the claim, $\mathcal{N}_D$ has fixed points $h_1 \in O_1$, $h_2 \in O_2$, which by Lemma 6.1(4) are even continuous plurisubharmonic functions on $\overline{D}$, vanishing on ∂D, with $(\mathrm{dd^c} h_i)^3 = C_i e^{-\gamma_* h_i} dV$, $C_i = 2/\int_D e^{-\gamma_* h_i} dV \in (0,\infty)$. By Lemma 2.6(1) and Lemma 7.1, each h_i is smooth and strictly plurisubharmonic on $\overline{D}$, and $\int_D (\mathrm{dd^c} h_i)^3 = 2$. By Lemma 2.6(3), $h_i = u_i \circ q$ with $u_i \in \mathcal{T}_0(\Omega)$ smooth and strictly plurisubharmonic on $\overline{\Omega} \setminus \{p\}$, and by (2.2)

$$(\mathrm{dd^c} u_i)^3 = \tfrac{1}{2} q_*(C_i e^{-\gamma_* h_i} dV) = \frac{e^{-\gamma_* u_i}\, m|_\Omega}{\int_\Omega e^{-\gamma_* u_i}\, dm} = \frac{e^{-\gamma_* u_i}\, \mu_\Omega}{\int_\Omega e^{-\gamma_* u_i}\, d\mu_\Omega},$$

where we used $\int_\Omega e^{-\gamma_* u_i} dm = \frac{1}{2}\int_D e^{-\gamma_* h_i} dV$ and the fact that the normalization of the measure cancels in the quotient. This is (1.4). The total mass is one, there is no atom at p since $m(\{p\}) = 0$, and $\int_\Omega |u_i| (\mathrm{dd^c} u_i)^3 \leq \|u_i\|_\infty < \infty$. Finally $u_1 \neq u_2$, since $\|h_1 - h_2\|_\infty \geq \|U_{r_1} - U_{r_2}\|_\infty - a_1 - a_2 > 0$. $\square$

8.2 Small perturbations of admissible domains

Lemma 8.2. *Let Ω be an admissible domain with defining function ρ and boundary S, and let $c\colon S \times (-b,b) \to X \setminus \{p\}$ be a collar with $\rho(c(y,t)) = t$. There is $\delta > 0$ such that for every $h \in C^\infty(S,\mathbb{R})$ with $\|h\|_{C^2(S)} < \delta$ the domain Ω_h bounded by the graph $\{c(y,h(y))\}$ is admissible. Moreover, for every compact $K' \subset \Omega$ and every open $W \supset \overline{\Omega}$, one has $K' \subset \Omega_h \subset \overline{\Omega_h} \subset W$ when δ is small enough.*

Proof. Such a collar exists: for a Riemannian metric on $X \setminus \{p\}$, the flow of $\nabla\rho/|\nabla\rho|^2$ starting on the compact hypersurface S has $\frac{d}{dt}\rho = 1$. Fix a smooth cutoff χ on $(-b, b)$, equal to 1 for $|t| < b/2$ and supported in $|t| < 3b/4$, and set

$$\rho_h(c(y,t)) = t - \chi(t)h(y) \text{ in the collar,} \qquad \rho_h = \rho \text{ outside the collar.}$$

Then ρ_h is smooth on a neighborhood of $\overline{\Omega}$, equals ρ near p and outside a compact subset of the collar, and $\rho_h - \rho$ is bounded in C^2 by $C\|h\|_{C^2(S)}$ (in local coordinates of the manifold $X \setminus \{p\}$ on the compact collar). Since the Levi form of ρ has a positive lower bound on the compact collar, ρ_h is strictly plurisubharmonic for small δ. Also $\partial_t\rho_h = 1 - \chi'(t)h(y) > \frac{1}{2}$ for small δ, so each line $t \mapsto c(y,t)$ meets $\{\rho_h = 0\}$ exactly once, namely at $t = h(y)$ (where $\chi = 1$, as $|h| < b/2$); thus $\{\rho_h < 0\}$ coincides near the collar with the domain bounded by the graph of h, and $d\rho_h \neq 0$ there. The map $\Psi_h(y,t) = (y, t + \chi(t)h(y))$ is, for small δ, a diffeomorphism of the collar which is the identity near its ends and carries $\{t < 0\}$ onto $\{\rho_h < 0\}$; extended by the identity, it is a homeomorphism of X taking Ω onto Ω_h, which is therefore connected and relatively compact. Hence $\Omega_h = \{\rho_h < 0\}$ is admissible. The last assertion holds because Ψ_h is C^0-close to the identity. $\square$

In particular the set of admissible domains is open in the set of all smoothly bounded domains, and the C^∞ topology of Definition 2.2 is well defined: near each Ω it is the topology transported from the Fréchet space $C^\infty(S, \mathbb{R})$ by $h \mapsto \Omega_h$.

8.3 Proof of Theorem 1.1

Let ϵ_* be as in Proposition 8.1 and define

$$\mathcal{U}_0 = \Big\{\Omega \text{ admissible}: \ K \Subset q^{-1}(\Omega) \Subset B_6, \quad \sup_{z \in \overline{q^{-1}(\Omega)}} \operatorname{dist}(z, K) < \epsilon_*\Big\}.$$

$\mathcal{U}_0$ is nonempty and open, and it is a Baire space. Lemma 5.1 with $\epsilon < \epsilon_*$ gives an element of $\mathcal{U}_0$. The three conditions defining $\mathcal{U}_0$ are open in the C^0 topology on domains (they say that $\overline{q^{-1}(\Omega)}$ lies in the open set $B_6 \cap \{\operatorname{dist}(\cdot, K) < \epsilon_*\}$ and contains the compact set K; note that $q^{-1}(\overline{\Omega}) = \overline{q^{-1}(\Omega)}$ and that q is proper), hence open in the C^∞ topology, so $\mathcal{U}_0$ is open. By Lemma 8.2, every $\Omega \in \mathcal{U}_0$ has a neighborhood in $\mathcal{U}_0$ of the form $\{\Omega_h : h \in C^\infty(S, \mathbb{R}), \ \|h\|_{C^2(S)} < \delta\}$, which is homeomorphic to an open subset of the Fréchet space $C^\infty(S, \mathbb{R})$; this space is completely metrizable, so every open subset of it is a Baire space by the Baire category theorem, and therefore so is $\mathcal{U}_0$.

Conclusions (1) *and* (2) for $\mathcal{U} = \mathcal{U}_0$ are the content of Proposition 8.1.

Conclusion (3). Fix a countable family (Y_ℓ) of embedded smooth submanifolds of $X \setminus \{p\}$. Cover each Y_ℓ by countably many compact subsets $L_{\ell,s}$ contained in coordinate charts of Y_ℓ, and let $V_{\ell,s}$ be the set of $\Omega \in \mathcal{U}_0$ such that $\partial\Omega$ is transverse to Y_ℓ at every point of $\partial\Omega \cap L_{\ell,s}$. Transversality at x means $T_x\partial\Omega + T_xY_\ell = T_x(X \setminus \{p\})$, i.e. $d(\rho|_{Y_\ell})(x) \neq 0$ for a defining function ρ of Ω.

$V_{\ell,s}$ is open: if $\Omega_{h_k} \to \Omega_h$ in C^1 and $x_k \in \partial\Omega_{h_k} \cap L_{\ell,s}$ are points where transversality fails, then a subsequence $x_k \to x \in \partial\Omega_h \cap L_{\ell,s}$, the defining differentials converge, and transversality fails at x.

$V_{\ell,s}$ is dense, and in fact the set of domains transverse to all Y_ℓ simultaneously is dense: let $\Omega_h \in \mathcal{U}_0$ with defining function ρ_h as in Lemma 8.2, and let N be a neighborhood of Ω_h. For a real number τ with $|\tau|$ small, $\Omega_{h+\tau}$ lies in N, and its boundary is the level set $\{\rho_h = \tau\}$ in the part of the collar where $\chi = 1$. By Sard's theorem [Hir76, Theorem 3.1.3] applied to the restrictions $\rho_h|_{Y_\ell}$ (on the manifolds $Y_\ell \cap \{\chi = 1\}$), the set of critical values of all these restrictions is a countable union of null sets, so we can choose τ arbitrarily small outside it. For such τ,

$d(\rho_h|_{Y_\ell}) \neq 0$ at every point of $\partial\Omega_{h+\tau} \cap Y_\ell$, for every ℓ. Hence the set of transverse domains is dense in $\mathcal{U}_0$; being the countable intersection $\bigcap_{\ell,s} V_{\ell,s}$ of open dense sets, it is a dense residual subset of $\mathcal{U}_0$.

Small neighborhoods of p. For $\sigma > 0$ let $\delta_\sigma(q(z)) = q(\sigma z)$, a biholomorphism of X fixing p. It maps admissible domains to admissible domains, induces a homeomorphism of the space of admissible domains with its C^∞ topology, and preserves transversality to a family (Y_ℓ) if the family is replaced by $(\delta_\sigma^{-1}(Y_\ell))$. Since $m|_{\delta_\sigma(\Omega)} = \sigma^6(\delta_\sigma)_*(m|_\Omega)$, one has $\mu_{\delta_\sigma(\Omega)} = (\delta_\sigma)_*\mu_\Omega$, and for $f^\sigma = f \circ \delta_\sigma^{-1}$

$$E_{\delta_\sigma(\Omega)}(f^\sigma) = E_\Omega(f), \qquad \int_{\delta_\sigma(\Omega)} e^{-af^\sigma}\, d\mu_{\delta_\sigma(\Omega)} = \int_\Omega e^{-af}\, d\mu_\Omega.$$

Therefore $u \mapsto u \circ \delta_\sigma^{-1}$ maps $\mathcal{T}_0(\Omega)$ onto $\mathcal{T}_0(\delta_\sigma(\Omega))$, preserves the inequality (1.3) with the same constants, and carries solutions of (1.4) on Ω to solutions on $\delta_\sigma(\Omega)$, with the same exponent. Thus $\mathcal{U} = \delta_\sigma(\mathcal{U}_0)$ has all the properties of the theorem, and since every $\Omega \in \mathcal{U}_0$ lies in Q_6, the family $\mathcal{U}$ lies in any prescribed neighborhood of p once σ is small. This completes the proof of Theorem 1.1. □

8.4 Proof of Corollary 1.2

If uniqueness held for all subcritical exponents on a family $\mathcal{F}$ of admissible domains, then $\mathcal{F} \cap \mathcal{U} = \emptyset$, because uniqueness fails on every $\Omega \in \mathcal{U}$ for the exponent $\gamma_* < \gamma_{\mathrm{crit}}(\Omega, p)$. A dense family meets the nonempty open set $\mathcal{U}$. If $\mathcal{F}$ is residual, it contains $\bigcap_k G_k$ with G_k open and dense in the space of admissible domains; then $G_k \cap \mathcal{U}$ are open and dense in the Baire space $\mathcal{U}$, and $\bigcap_k (G_k \cap \mathcal{U}) \cap \bigcap_{\ell,s} V_{\ell,s} \neq \emptyset$, where the $V_{\ell,s}$ are the open dense sets of the previous proof for any prescribed countable family of transversality conditions. □

8.5 The convention of [GT25]

With $dd^c_{\mathrm{GT}} = \frac{i}{2\pi}\partial\bar\partial$ one has $dd^c_{\mathrm{GT}}(2\pi u) = \mathrm{dd^c} u$, so $v = 2\pi u$ solves the equation of Remark 1.3 with $\gamma_{\mathrm{GT}} = \gamma_*/2\pi$ whenever u solves (1.4). The energy of [GT25, Definition 3.1] with $\phi = 0$ is $E_{\mathrm{GT}}(v) = \frac{1}{4}\int_\Omega v(dd^c_{\mathrm{GT}} v)^3$, so $E_{\mathrm{GT}}(2\pi f) = 2\pi E_\Omega(f)$ and

$$E_{\mathrm{GT}}(2\pi f) + \frac{1}{a_{\mathrm{GT}}}\log\int_\Omega e^{-a_{\mathrm{GT}}\,2\pi f}\, d\mu_\Omega = 2\pi\Big[E_\Omega(f) + \frac{1}{a}\log\int_\Omega e^{-af}\, d\mu_\Omega\Big], \qquad a_{\mathrm{GT}} = \frac{a}{2\pi}.$$

Since $f \mapsto 2\pi f$ is a bijection of $\mathcal{T}_0(\Omega)$ onto itself, the admissible exponents in the two conventions differ exactly by the factor 2π, and $\gamma_{\mathrm{crit,GT}}(\Omega, p) = \gamma_{\mathrm{crit}}(\Omega, p)/2\pi$. In particular $\gamma_{\mathrm{GT}} < G/2\pi \leq \gamma_{\mathrm{crit,GT}}(\Omega, p)$ for every $\Omega \in \mathcal{U}$.

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