

Subharmonicity on All Affine Complex Lines Forces Upper Semicontinuity

Abstract

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\phi: \Omega \rightarrow [-\infty, \infty)$ be nontrivial. We prove that if ϕ is subharmonic or identically $-\infty$ on every connected component of every affine complex-line section, then ϕ is automatically upper semicontinuous. Thus condition (2) in Definition 1.1.2 of [Xia] is redundant.

1 Statement and two auxiliary lemmas

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\phi: \Omega \rightarrow [-\infty, \infty)$ be not identically $-\infty$. Suppose that for every affine complex line $L \subset \mathbb{C}^n$ and every connected component U of $L \cap \Omega$, the restriction $\phi|_U$ is subharmonic or identically $-\infty$. Then ϕ is upper semicontinuous, hence plurisubharmonic.*

We shall use Arsove's measurability lemma [Ars66, Lemma 1] in the following exact two-block form. If $p, q \geq 2$, $G \subset \mathbb{R}^p \times \mathbb{R}^q$ is open, and $v: G \rightarrow [-\infty, \infty)$ is subharmonic or identically $-\infty$ on every connected component of every section in either block, then v is Borel measurable. No joint local upper bound is required.

For a finite Borel measure μ on $\mathbb{C}^n$ and $0 < s < 2n$, write

$$I_s(\mu) = \iint |x - y|^{-s} d\mu(x) d\mu(y).$$

Lemma 1.2 (High-value star). *Let $n \geq 2$, let $\bar{B}(x, R) \Subset D \subset \mathbb{C}^n$, and let $u: D \rightarrow \mathbb{R}$ be Borel and subharmonic on every affine complex-line section. If $A = u(x)$, there is a Borel probability measure μ carried by*

$$E(x, A, R) = \{y : R/2 \leq |y - x| \leq R, u(y) \geq A - 1\}$$

such that

$$I_{2n-2}(\mu) \leq C_{n,R}. \tag{1.1}$$

The constant is independent of u, x , and A .

Proof. Put $P = \mathbb{CP}^{n-1}$, with normalized invariant measure σ and Fubini–Study distance d_{FS} . For every $r \in [R/2, R]$ and $\xi \in P$, the submean inequality on the line $x + \xi$ gives a unit vector e spanning ξ for which $u(x + re) \geq A$. The set

$$\Gamma = \{(r, \xi, e) : r \in [R/2, R], q(e) = \xi, u(x + re) \geq A - 1\},$$

where $q: S^{2n-1} \rightarrow P$ is the Hopf map, is Borel. Its sections in e are nonempty and compact: compactness follows from upper semicontinuity on the line. The Arsenin–Kunugui theorem [Kec95, Theorem 18.18] therefore gives a Borel selector $e = e(r, \xi)$.

Let m be normalized product measure on $[R/2, R] \times P$ and set $\mu = (x + re(r, \xi))_{\#} m$. This is a Borel probability carried by $E(x, A, R)$. If e, e' span ξ, ξ' , respectively, then

$$|re - r'e'|^2 = (r - r')^2 + rr'|e - e'|^2 \geq (r - r')^2 + c_n R^2 d_{\text{FS}}(\xi, \xi')^2.$$

Indeed, the actual relative phase can only increase the minimum $\inf_{|\alpha|=1} |e - \alpha e'|$, which is uniformly comparable with $d_{\text{FS}}(\xi, \xi')$. The parameter space has real dimension $2n - 1$; hence, in finitely many coordinate charts,

$$I_{2n-2}(\mu) \leq C_{n,R} \iint |s - t|^{-(2n-2)} dm(s) dm(t) < \infty,$$

because the radial singularity is $\rho^{2n-2} \rho^{-(2n-2)} d\rho$. This proves (1.1). $\square$

Lemma 1.3 (Incidence). *Let $K \subset \mathbb{C}^n$ be compact, $n \geq 2$, and let $E_j \subset K$ be Borel. If there are Borel probability measures μ_j with*

$$\mu_j(E_j) = 1, \quad I_{2n-2}(\mu_j) \leq C,$$

then one affine complex line meets infinitely many E_j .

Proof. Let $\mathcal{H} = \text{Gr}_{\mathbb{C}}(n-1, n)$ be the Grassmannian of complex hyperplanes, with normalized invariant measure ν , and let P_H denote orthogonal projection onto H . For a finite measure μ , put $\mu_H = (P_H)_\# \mu$. Fourier disintegration over $\mathcal{H}$ gives

$$\int_{\mathcal{H}} \left\| \frac{d\mu_H}{dm_H} \right\|_{L^2(H)}^2 d\nu(H) = c_n I_{2n-2}(\mu), \quad (1.2)$$

whenever the right-hand side is finite. Here m_H is Lebesgue measure on H . For completeness, this follows from

$$\int_{\mathcal{H}} \int_H g(\zeta) dm_H(\zeta) d\nu(H) = a_n \int_{\mathbb{R}^{2n}} g(\zeta) |\zeta|^{-2} d\zeta$$

and Plancherel, since $\widehat{\mu_H} = \widehat{\mu}|_H$ and the Fourier transform of $|x|^{-(2n-2)}$ is a positive multiple of $|\zeta|^{-2}$.

Choose M with $K \subset \overline{B}(0, M)$ and put

$$\mathcal{L} = \{(H, c) : H \in \mathcal{H}, c \in H, |c| \leq M + 1\}, \quad d\lambda(H, c) = d\nu(H) dm_H(c).$$

Thus (H, c) parametrizes the affine line $c + H^\perp$. Define a probability measure η_j on $\mathcal{L}$ by

$$\int_{\mathcal{L}} G d\eta_j = \int_{\mathcal{H}} \int_H G(H, c) d(\mu_j)_H(c) d\nu(H).$$

Equation (1.2) implies

$$d\eta_j = f_j d\lambda, \quad \|f_j\|_{L^2(\lambda)}^2 \leq c_n C, \quad (1.3)$$

with f_j jointly measurable. One quick justification of joint measurability is to convolve each $(\mu_j)_H$ with the same radial approximate identity on H : the resulting densities are jointly Borel, bounded in $L^2(\lambda)$ by (1.2), and a weakly convergent subsequence yields (1.3).

Let

$$\mathcal{I}_j = \{(H, c) \in \mathcal{L} : \text{some } y \in E_j \text{ satisfies } P_H y = c\}.$$

This is the continuous image of the Borel set $\mathcal{H} \times E_j$, hence is analytic and therefore measurable for the completion of λ . Moreover $\eta_j(\mathcal{I}_j) = 1$. By Cauchy–Schwarz and (1.3),

$$1 \leq \|f_j\|_2 \lambda(\mathcal{I}_j)^{1/2}, \quad \lambda(\mathcal{I}_j) \geq (c_n C)^{-1} =: \delta > 0.$$

Since $\lambda(\mathcal{L}) < \infty$, the decreasing tail unions satisfy

$$\lambda\left(\limsup_{j \rightarrow \infty} \mathcal{I}_j\right) = \lim_{m \rightarrow \infty} \lambda\left(\bigcup_{j \geq m} \mathcal{I}_j\right) \geq \delta.$$

Any (H, c) in this limsup gives one line $c + H^\perp$ meeting infinitely many E_j . $\square$

2 Proof of the theorem

We first treat finite nonnegative functions.

Proposition 2.1. *Let $D \subset \mathbb{C}^n$ be a domain and let $u: D \rightarrow [0, \infty)$ be finite. If u is subharmonic on every connected component of every affine complex-line section, then the given pointwise representative u is plurisubharmonic.*

Proof. We argue by induction on n . The case $n = 1$ is classical. For $n \geq 2$ write $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$. With the last coordinate fixed, the inductive hypothesis makes u plurisubharmonic, hence real-subharmonic, in the first block; with the first block fixed, it is subharmonic in the last complex variable. Arsove's lemma, applied with block dimensions $(2n-2, 2)$, shows that u is Borel.

We claim that u is locally bounded above. Otherwise there are $x_j \rightarrow p \in D$ with $A_j := u(x_j) \rightarrow \infty$. Choose $R > 0$ with $\overline{B}(p, 3R) \Subset D$ and, after discarding finitely many terms, $|x_j - p| < R/2$. Lemma 1.2 supplies probability measures of uniformly bounded I_{2n-2} -energy carried by

$$E_j = \{y : R/2 \leq |y - x_j| \leq R, u(y) \geq A_j - 1\} \subset K := \overline{B}(p, 2R).$$

By Lemma 1.3, one affine line L meets infinitely many E_j . Choose $y_j \in L \cap E_j$ along that subsequence. The compact set $L \cap K$ lies in one component of $L \cap D$ because it is contained in the disc $L \cap B(p, 3R)$. But $u(y_j) \geq A_j - 1 \rightarrow \infty$, contradicting local boundedness above of the subharmonic function $u|_{L \cap D}$ on that compact set.

Now $u \in L^1_{\text{loc}}(D)$. Fix $a \in \mathbb{C}^n$. Fubini's theorem, applied to the parallel lines $w + \mathbb{C}a$, and positivity of their one-variable distributional Laplacians give

$$\sum_{k,\ell=1}^n a_k \overline{a_\ell} \frac{\partial^2 u}{\partial z_k \partial \overline{z_\ell}} \geq 0 \quad (2.1)$$

as a distribution. Polarization (first for the vectors $e_k + i^r e_\ell$, $r = 0, 1, 2, 3$) shows that the coefficient distributions are measures and form a positive semidefinite matrix measure. Hence $\text{dd}^c u \geq 0$. The distributional characterization of plurisubharmonic functions [Xia, Theorem 1.1.2] gives a unique $\tilde{u} \in \text{PSH}(D)$ with $u = \tilde{u}$ almost everywhere.

It remains to recover the original representative. Fix $z \in D$ and let $N = \{u \neq \tilde{u}\}$. Polar disintegration about z ,

$$dV = C_n |t|^{2n-2} dA(t) d\sigma([v]), \quad w = z + tv,$$

shows that for almost every direction $[v] \in \mathbb{CP}^{n-1}$ the set $N \cap (z + \mathbb{C}v)$ has planar measure zero near z . On such a line, u and $\tilde{u}$ are subharmonic and agree almost everywhere; uniqueness of the subharmonic representative implies equality everywhere, in particular at z . Thus $u = \tilde{u}$ pointwise, proving the proposition. $\square$

Proof of Theorem 1.1. For $m \geq 1$, set

$$\phi_m = \max\{\phi, -m\}, \quad u_m = \phi_m + m.$$

The function u_m is finite and nonnegative. On each line component, ϕ_m is either the maximum of a subharmonic function and a constant, or the constant $-m$. Proposition 2.1 therefore shows that every ϕ_m is plurisubharmonic in its given pointwise representative. Since $\phi_m \downarrow \phi$, for every $a \in \mathbb{R}$ we have

$$\{\phi < a\} = \bigcup_{m \geq 1} \{\phi_m < a\}.$$

The right-hand side is open, so ϕ is upper semicontinuous. Together with the assumed line restrictions, this says exactly that ϕ is plurisubharmonic. $\square$

Remark 2.2. Wiegerinck’s example [Wie88] only has separate subharmonicity for a fixed product splitting. The all-directions hypothesis above rules out its spike mechanism: Lemmas 1.2 and 1.3 force infinitely many spikes to recur on one fixed affine complex line.

References

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