Distinct Polar and Non-Bounded Loci on a Projective Fourfold

Abstract

We prove that a big real class of type $(1,1)$ on a smooth projective fourfold can have a non-bounded locus strictly larger than its polar locus, giving a negative answer to Question 42 in the survey of Dinew–Guedj–Zeriahi.

1 Introduction

Background

The polar locus of a potential records its values equal to minus infinity, whereas its non-bounded locus also records failure of boundedness near points where the potential itself is finite. For a normalized envelope in a pseudoeffective class, the distinction is a property of the least singular potentials allowed by the class. Eyssidieux–Guedj–Zeriahi ask whether the two loci always coincide for a big class [DGZ16, Question 42]: with the notation of Definition 2.2, *does* P_θ *always coincide with* NB_θ*?* The authors of [DGZ16] note that examples of individual quasi-plurisubharmonic functions with distinct loci are easy to produce, but that no example of a big class was known. We give a negative answer.

Marked cuspidal curves give pseudoautomorphisms of projective blowups with expanding divisor eigenclasses [PZ14]. We use an explicit member of this construction as a pseudoeffective base example. Its negative curves approach a point where an exact positive eigencurrent remains finite.

1.1 A big class with distinct singular loci

The example is determined by the nine points and the class in Setup 3.3. The normalized envelopes V_b, their polar loci P_b, and their non-bounded loci NB_b are defined in Definition 2.2. All these definitions use canonical plurisubharmonic representatives.

Theorem A. *Let X, β, and v_0 be the smooth projective threefold, real $(1,1)$-class, and point of Setup 3.3. Let θ be any smooth closed real $(1,1)$-form representing β. Then*

$$v_0 \in \mathrm{NB}_\theta \setminus P_\theta.$$

Choose an ample line bundle A with a smooth positively curved metric of curvature a such that $\theta + \frac{1}{2}a$ is Kähler. Let

$$\pi\colon Y = \mathbf{P}_X(\mathcal{O}_X \oplus A) \longrightarrow X$$

parametrize one-dimensional quotients, and let $s\colon X \to Y$ be the section induced by the quotient onto $\mathcal{O}_X$. Let η be the curvature of the induced quotient metric on $\mathcal{O}_Y(1)$, and put

$$\Theta = \eta + \pi^*\theta.$$

Then Y is a smooth projective fourfold, $[\Theta]$ is big, and

$$s(v_0) \in \mathrm{NB}_\Theta \setminus P_\Theta.$$

The construction identifies a specific point of the discrepancy. The base class is pseudoeffective. The passage to Y preserves both singular loci exactly along the distinguished section and makes the envelope locally bounded away from that section.

Sketch of the proof

The negative curves C_n lie on quadric surfaces whose parameters tend to zero. Negative degree forces every admissible potential to be minus infinity on every C_n, by Proposition 3.1. The non-bounded locus therefore contains their ordinary closure.

There are two further steps. Theorem 4.1 constructs a positive current in the expanding eigenclass whose pullback satisfies the eigenvalue equation exactly. Proposition 5.1 supplies ordinary accumulation at the vertex of the limiting cone: on each nearby smooth quadric, a vanishing sphere has intersection number of absolute value one with the corresponding negative curve, and the spheres shrink to the vertex.

A regular fixed point makes the restriction of the eigencurrent to the resolved cone nontrivial, and the possible error in the surface eigenvalue equation, supported on two curves, is killed by their intersection matrix. Theorem 6.1 then makes the restricted potential continuous. Both this theorem and the finiteness at the fixed point rest on one elementary fact, Lemma 2.4: an expanding functional equation for a potential is incompatible with a pole, because poles of plurisubharmonic functions decay at most logarithmically on balls. This proves finiteness at the vertex, and Theorem 7.1 gives the fourfold.

Structure of the paper

Section 2 fixes the conventions for potentials and proves Lemma 2.4. Section 3 develops the marked blowup, its negative curves, and the resolved limiting cone. Section 4 proves the exact eigencurrent theorem and the finiteness at a fixed point. Section 5 proves ordinary vertex accumulation. Section 6 establishes continuity of the surface trace and the base discrepancy. Section 7 proves the projective-bundle identities and Theorem A.

Remark 1.1. The main result of this paper was obtained with the assistance of the Danus system, an automated proof-search and verification system. Because automated systems have limitations, it is possible that we have missed related references in the literature, and we welcome comments from experts.

2 Preliminaries

Notation 2.1. We use

$$\mathrm{dd^c} = \frac{i}{2\pi}\partial\bar{\partial}.$$

All manifolds are connected unless otherwise stated. Closures without an algebraic qualifier are taken in the ordinary complex-manifold topology. On compact Kähler manifolds, real $(1,1)$-classes are identified with their Bott–Chern representatives by the $\partial\bar{\partial}$-lemma [Dem12, Chapter VI, Section 8, Lemma 8.6].

Definition 2.2. Let M be a complex manifold and b a smooth closed real $(1,1)$-form. The space $\mathrm{PSH}(M,b)$ consists of functions

$$u\colon M \longrightarrow [-\infty,\infty)$$

that are not identically minus infinity and such that $u+\psi$ is plurisubharmonic whenever ψ is a smooth real local potential with $\mathrm{dd^c}\psi = b$. The plurisubharmonic functions here are their canonical representatives, rather than arbitrary upper-semicontinuous representatives of distributions.

On a compact M, when this space is nonempty, set

$$V_b = \sup\{u \in \mathrm{PSH}(M,b) \mid u \le 0\}, \qquad P_b = \{V_b = -\infty\},$$

and

$$\mathrm{NB}_b = \{x \in M \mid V_b \text{ is not bounded on any neighborhood of } x\}.$$

The class $[b]$ is pseudoeffective if it contains a closed positive $(1,1)$-current. On a compact Kähler manifold this is equivalent to $\mathrm{PSH}(M,b) \neq \varnothing$. A real $(1,1)$-class is big if it contains a closed positive current dominating a Kähler form.

Lemma 2.3. *Let M be compact and let b be a smooth closed real $(1,1)$-form with $\mathrm{PSH}(M,b) \neq \varnothing$. The pointwise supremum V_b in Definition 2.2 belongs to $\mathrm{PSH}(M,b)$, is nonpositive, and is not identically minus infinity. For every $u \in \mathrm{PSH}(M,b)$,*

$$V_b \ge u - \sup_M u.$$

Moreover, $P_b \subseteq \mathrm{NB}_b$, the set NB_b is closed, and P_b is locally pluripolar.

Proof. Every $u \in \mathrm{PSH}(M,b)$ is bounded above by compactness, so $u - \sup_M u$ is a competitor; hence the defining family is nonempty and $V_b \ge u - \sup_M u$. After addition of a smooth local potential of b, the family is locally uniformly bounded above, so the upper-semicontinuous regularization V_b^* is b-plurisubharmonic [Dem12, Chapter I, Section 5, Theorem 5.7]. It is nonpositive and dominates a nontrivial competitor, hence is itself a competitor, and $V_b^* \le V_b \le V_b^*$. The remaining assertions are immediate: a point where $V_b = -\infty$ has no neighborhood on which V_b is bounded; the complement of NB_b is open; and P_b is locally the minus-infinity set of a nontrivial plurisubharmonic function. □

The following elementary estimate is the engine of Lemma 4.2 and Theorem 6.1: on a ball of radius s, a plurisubharmonic function which is not identically minus infinity has supremum at least $-A - B\log(1/s)$, with constants uniform in the center.

Lemma 2.4. *Let $\Omega \subset \mathbf{C}^n$ be open, let $K \subset \Omega$ be compact, and let $R_0 > 0$ be such that the closed $2R_0$-neighborhood K' of K is contained in Ω. Let F be plurisubharmonic on Ω with $F \in L^1(K')$, and put*

$$A_0 = \int_{K'} |F|\, dV, \qquad B_0 = \max\Big\{0, \sup_{K'} F\Big\}, \qquad A = \frac{A_0}{\operatorname{Vol} B(0,R_0)}, \qquad B = \frac{A + B_0}{\log 2}.$$

Then for every $z \in K$ and every $0 < s \le R_0$,

$$\sup_{\overline{B}(z,s)} F \ge -A - B \log \frac{R_0}{s}.$$

Proof. Fix $z \in K$ and put $M(s) = \sup_{\overline{B}(z,s)} F$ for $0 < s \le 2R_0$. For a unit vector e, the function $s \mapsto \sup_{|t|=s} F(z+te)$ is convex in $\log s$ by Hadamard's three-circles theorem applied to the subharmonic function $t \mapsto F(z+te)$ (or identically $-\infty$); by the maximum principle on discs, $M(s)$ is the supremum of these functions over e, hence convex in $\log s$. Moreover $M(2R_0) \le B_0$, and $M(R_0) \ge -A$, since otherwise $F < -A$ on $B(z,R_0)$ and $\int_{B(z,R_0)} |F| > A_0$. For $s < R_0$, convexity between $\log s < \log R_0 < \log 2R_0$ gives

$$M(s) \ge M(R_0) - \frac{M(2R_0) - M(R_0)}{\log 2} \log \frac{R_0}{s} \ge -A - \frac{B_0 + A}{\log 2} \log \frac{R_0}{s}. \qquad \square$$

3 The cuspidal quartic and its negative curves

In this section, we construct the threefold used in Theorem A and identify the geometry needed for both the eigencurrent restriction and the ordinary accumulation argument.

3.1 Negative degree and canonical potentials

Proposition 3.1. *Let M be a connected compact Kähler manifold, let θ be a smooth closed real $(1,1)$-form with $\mathrm{PSH}(M,\theta) \neq \varnothing$, and put $\beta = [\theta]$. Let $C \subset M$ be an irreducible reduced compact complex curve. If $\nu \colon \widetilde{C} \to C$ is its normalization and $F \colon \widetilde{C} \to M$ is the induced map, define*

$$\beta \cdot C = \int_{\widetilde{C}} F^*\theta.$$

If $\beta \cdot C < 0$, every $u \in \mathrm{PSH}(M,\theta)$ is identically minus infinity on C.

Consequently, if $(C_j)_{j\geq 1}$ is a sequence of such curves with negative degree and

$$K = \overline{\bigcup_{j\geq 1} C_j},$$

then

$$\bigcup_{j\geq 1} C_j \subseteq P_\theta, \qquad K \subseteq \mathrm{NB}_\theta,$$

and every $x \in K$ at which some $u \in \mathrm{PSH}(M,\theta)$ is finite satisfies $x \in \mathrm{NB}_\theta \setminus P_\theta$.

Proof. Fix $u \in \mathrm{PSH}(M,\theta)$ and suppose that $u(x_*) > -\infty$ for some $x_* \in C$; choose $y_* \in \widetilde{C}$ over x_*. The normalization $\widetilde{C}$ is a connected compact Riemann surface.

If $B \subset \widetilde{C}$ is a coordinate disk whose image lies in a coordinate neighborhood $\Omega \subset M$ with $\theta = \mathrm{dd^c}\psi$, then $(u+\psi)\circ F$ is subharmonic or identically minus infinity on B, being the composition of a plurisubharmonic function with a holomorphic map. It is nontrivial on a disk containing y_*, and nontriviality propagates across nonempty overlaps of such disks, because a nontrivial subharmonic function is finite almost everywhere. Since $\widetilde{C}$ is connected, $v = u \circ F$ is a locally integrable function on $\widetilde{C}$ which is locally the sum of a subharmonic function and a smooth function. Hence

$$R = F^*\theta + \mathrm{dd^c} v$$

is a positive measure on $\widetilde{C}$: on each disk as above it equals $\mathrm{dd^c}((u+\psi)\circ F)$. Stokes' theorem for the distribution v gives

$$0 \leq \int_{\widetilde{C}} R = \int_{\widetilde{C}} F^*\theta + \langle \mathrm{dd^c} v, 1\rangle = \beta \cdot C < 0,$$

a contradiction. Thus $u \equiv -\infty$ on C.

Applied to every competitor in the definition of V_θ, this gives $V_\theta \equiv -\infty$ on each C_j, so $\bigcup_j C_j \subseteq P_\theta \subseteq \mathrm{NB}_\theta$, and $K \subseteq \mathrm{NB}_\theta$ because NB_θ is closed (Lemma 2.3). Finally, if $u \in \mathrm{PSH}(M,\theta)$ is finite at x, then $V_\theta(x) \geq u(x) - \sup_M u > -\infty$ by Lemma 2.3, so $x \notin P_\theta$. □

Remark 3.2. The last assertion of Proposition 3.1 is the only route to a discrepancy that we use; note that K is contained in the countable union of the curves C_j and of their limit set, which in our example lies on a surface (Proposition 3.4), so K is pluripolar and no soft argument can replace the finite value. In Lesieutre's example [Les14, Theorem 1.1] the negative curves are Zariski dense, and we make no assertion about finite admissible values on their ordinary closure; the special position of the nine points below is what places the limit set on a surface where such a value can be produced.

3.2 The marking and the explicit map

Our example is a member of the family of pseudoautomorphisms constructed by Perroni and Zhang from marked cuspidal curves [PZ14, Theorem 3.1 and Section 4]; the eigenvalue below appears in [PZ14, Corollary 1.2(2)]. We recall the lattice-theoretic framework only to explain where the numbers in Setup 3.3 come from. Every property of the example that is used later is verified directly in Proposition 3.4, so the reader may take Setup 3.3 as a definition.

Consider the lattice with basis $h, e_1, \dots, e_9$ and pairing

$$h^2 = 2, \qquad e_i^2 = -1, \qquad h \cdot e_i = e_i \cdot e_j = 0 \quad (i \neq j),$$

which is the Picard lattice of the blowup of $\mathbf{P}^3$ at nine points with the Mukai pairing, and let $\kappa = 4h - 2\sum_i e_i$ be the anticanonical class. Let R be the reflection in $h - e_1 - e_2 - e_3 - e_4$, the class action of the standard cubic Cremona involution centered at the first four points; let P fix h and send e_i to e_{i+1} cyclically; and put $W = PR$, a Coxeter element of the Weyl group of type $T_{2,4,5}$. A pseudoautomorphism means a birational self-map that is an isomorphism between the complements of analytic subsets of complex codimension at least two; its action on divisor classes is functorial.

The marking of [PZ14, Section 2] realizes a leading eigenvector $v = \xi h + \sum_i \eta_i e_i \in \kappa^\perp$ of W by nine points with parameters $-\eta_i - \xi/2$ on the cuspidal quartic

$$p(t) = [1 : t : t^2 : t^4],$$

the image of the complete linear system $|4p(0)|$ on the cuspidal cubic; the pseudoautomorphism is a linear map composed with the cubic Cremona involution, and its pushforward on divisor classes is W.

Setup 3.3. Let λ be the unique real root greater than 1 of

$$t^8 - t^5 - t^4 - t^3 + 1,$$

and put $d = 1 + \lambda^2$. Define

$$\begin{aligned} m_1 &= 1 - \frac{\lambda^3}{d}, & m_2 &= \frac{1}{d}, & m_3 &= 1 - \frac{\lambda}{d}, \\ m_4 &= \frac{\lambda^2}{d}, & m_5 &= 1 - \frac{1}{\lambda d}, & m_i &= \lambda^{5-i} m_5 \quad (6 \leq i \leq 9). \end{aligned}$$

Set $t_i = m_i - \frac{1}{2}$ and $P_i = p(t_i)$. Let X be the blowup of $\mathbf{P}^3$ at these nine points. Write H for the pullback hyperplane class and E_i for the exceptional divisor over P_i, and put

$$\beta = H - \sum_{i=1}^{9} m_i E_i.$$

Let M be the matrix whose i-th column is $(1, t_i, t_i^2, t_i^4)^T$, for $1 \leq i \leq 4$. Set

$$D_i = \prod_{\substack{1 \leq j \leq 4 \\ j \neq i}} (t_i - t_j),$$

and let the i-th column of B be

$$\frac{(1, t_{i+1}, t_{i+1}^2, t_{i+1}^4)^T}{D_i^2}, \qquad 1 \leq i \leq 4.$$

Define

$$J([z_1 : z_2 : z_3 : z_4]) = [z_2 z_3 z_4 : z_1 z_3 z_4 : z_1 z_2 z_4 : z_1 z_2 z_3].$$

The lift of BJM^{-1} will be denoted by f, and its inverse by g.

For $i < j$, let L_{ij} be the strict transform of the secant P_iP_j. Put

$$C_0 = L_{35}, \qquad C_n = f^n(C_0), \qquad r_n = (\lambda-1)^2\lambda^{-2n-2}.$$

All images of curves are strict transforms. With homogeneous coordinates $[X_1 : X_2 : X_3 : X_4]$, put

$$Q_1 = X_2^2 - X_1X_3, \qquad Q_2 = X_3^2 - X_1X_4.$$

For finite r, let S_r be the strict transform of $\{Q_2 = rQ_1\}$, and let S_∞ be the strict transform of $\{Q_1 = 0\}$.

Finally, let $v_0 \in X$ be the unique lift of $[0 : 1 : 0 : 0]$, and let $q \in X$ be the unique lift of $p(-\frac{1}{2})$.

Proposition 3.4. *The points in Setup 3.3 are distinct, and X is smooth projective. The map f is a pseudoautomorphism. Its inverse satisfies*

$$g^*\beta = \lambda\beta.$$

The class β is pseudoeffective.

Every C_n is an irreducible compact curve. Every iterate used to define these curves is a local biholomorphism along the curve on which it is applied. Moreover,

$$\beta \cdot C_n = -(\lambda-1)\lambda^{-n-1} < 0, \qquad C_n \subset S_{r_n}, \qquad f(S_r) = S_{r/\lambda^2} \tag{3.1}$$

as strict transforms.

The set

$$K_0 = \bigcap_{N\geq 0} \overline{\bigcup_{n\geq N} C_n}$$

is nonempty and compact, and $K_0 \subset S_0$. The point q belongs to S_0, is fixed by f and g, and both maps are biholomorphic on neighborhoods of q.

Proof. We verify the marking, identify the displayed map, and then follow the secants through the pencil. The lattice pairing used below is the Mukai pairing recalled above, rather than the intersection product of two divisors on a threefold. Finitely many nonvanishing conditions on the parameters t_i are needed; each of them is a rational function of λ with rational coefficients, and it is nonzero exactly when the numerator is not divisible by the irreducible polynomial $t^8 - t^5 - t^4 - t^3 + 1$. All such conditions asserted below were checked in this exact form (they also hold numerically by a wide margin).

Step 1. The coefficients in Setup 3.3 form the required leading eigenvector.

Reflection in $\alpha = h - e_1 - e_2 - e_3 - e_4$ is given by

$$R(x) = x + (x\cdot\alpha)\alpha.$$

For $v = h - \sum_i m_ie_i$, substitution gives

$$\sum_{i=1}^{4} m_i = 3 - \lambda,$$

together with

$$\lambda m_1 = m_9, \qquad \lambda m_{i+1} = m_i + \lambda - 1 \quad (1 \leq i \leq 4), \qquad \lambda m_{i+1} = m_i \quad (5 \leq i \leq 8).$$

The closing recurrence is equivalent to

$$(\lambda-1)(\lambda^8-\lambda^5-\lambda^4-\lambda^3+1)=\lambda^9-\lambda^8-\lambda^6+\lambda^3+\lambda-1=0.$$

These identities give $Wv=\lambda v$. Since W fixes κ and preserves the pairing, $\lambda\neq 1$ and $\lambda^2\neq 1$ imply

$$v\cdot\kappa=0,\qquad v^2=0.$$

Thus

$$\sum_{i=1}^{9} m_i=4,\qquad \sum_{i=1}^{9} m_i^2=2. \tag{3.2}$$

Eliminating the coefficients in the same recurrences, with an indeterminate t, gives the characteristic polynomial

$$(t-1)(t+1)(t^8-t^5-t^4-t^3+1).$$

Its leading root is the specified λ, also identified in [PZ14, Corollary 1.2(2)]. For uniqueness among real roots greater than one, divide the degree-eight factor by t^4 and set $s=t+t^{-1}$. The result is

$$s^4-4s^2-s+1.$$

Its value at $s=2$ is -1, and its derivative is positive for $s\geq 2$. Evaluation at rational endpoints gives

$$\frac{32}{25}<\lambda<\frac{129}{100}.$$

In particular,

$$\lambda^3<d<\lambda^4. \tag{3.3}$$

The parameters $t_i=m_i-\frac{1}{2}$ are pairwise distinct, so the nine points are distinct and X is smooth projective. Moreover, no four of the nine points are coplanar: for four parameters a,b,c,e,

$$\det\begin{pmatrix}1 & a & a^2 & a^4\\ 1 & b & b^2 & b^4\\ 1 & c & c^2 & c^4\\ 1 & e & e^2 & e^4\end{pmatrix}=(a+b+c+e)\prod(\text{differences}),$$

so $p(a),p(b),p(c),p(e)$ are coplanar exactly when $a+b+c+e=0$, and no four of the t_i sum to zero (the smallest absolute value of such a sum is about 0.016). For the same reason no plane through three of the points contains the cusp $p(\infty)=[0:0:0:1]$, and no three of the points are collinear, since a secant of the quartic meets it in only two points (see Step 3).

Step 2. The matrices in Setup 3.3 give the fixed linear identification after Cremona.

Put

$$\sigma=t_1+t_2+t_3+t_4=1-\lambda,\qquad A_0=M^{-1}.$$

The i-th coordinate of $A_0p(t)$ is

$$\frac{\prod_{j\neq i}(t-t_j)(t+\sigma-t_i)}{\sigma D_i}.$$

The expression has the required values at $t_1,\ldots,t_4$, and its cubic coefficient vanishes, so it belongs to the span of $1,t,t^2,t^4$. These four interpolation conditions determine it.

After reciprocation and removal of a common rational factor, the coordinates become

$$q_i(t)=\sigma D_i(t-t_i)\prod_{j\neq i}(t+\sigma-t_j).$$

These quartics form a basis of the space in which the cubic coefficient is 2σ times the quartic coefficient. Define

$$\alpha(t) = \frac{t + \sigma/2}{\lambda}.$$

The same space is spanned by $1, \alpha(t), \alpha(t)^2, \alpha(t)^4$. At $t = t_i - \sigma$, all q_j with $j \neq i$ vanish and $q_i = -\sigma^2 D_i^2$. Polynomial interpolation therefore yields

$$BJA_0 p(t) = p(\alpha(t)) \tag{3.4}$$

as rational maps.

The coefficient recurrences give

$$\alpha(t_i) = t_{i+1} \quad (5 \leq i \leq 9)$$

with cyclic indexing, and

$$\alpha(t_i - \sigma) = t_{i+1} \quad (1 \leq i \leq 4).$$

Thus the rational map BJM^{-1} sends P_i to P_{i+1} for $5 \leq i \leq 8$, sends P_9 to P_1, and contracts the plane through $\{P_1, \dots, P_4\} \setminus \{P_i\}$, which meets the quartic at the additional point $p(t_i - \sigma)$, to P_{i+1} for $1 \leq i \leq 4$.

We now lift this map to X. The cubic Cremona involution J lifts to a pseudoautomorphism of the blowup of $\mathbf{P}^3$ at the four coordinate vertices, which is an isomorphism off the strict transforms of the six edges of the coordinate tetrahedron and exchanges the exceptional divisor over a vertex with the strict transform of the opposite coordinate plane. Conjugating by the linear maps M^{-1} and B, we obtain an isomorphism from the blowup of $\mathbf{P}^3$ at $P_1, \dots, P_4$ minus the six secants L_{ij}, $i < j \leq 4$, onto the blowup at $P_2, \dots, P_5$ minus the six secants L_{ij}, $2 \leq i < j \leq 5$. This isomorphism sends $P_5, \dots, P_9$ to $P_6, \dots, P_9, P_1$, and these points lie off the six secants on either side because no three marked points are collinear; J is a local biholomorphism at $M^{-1}P_i$ for $i \geq 5$ because P_i lies on none of the planes through three of $P_1, \dots, P_4$. Blowing up these five points on both sides, we obtain f with $f_* = W$ and $g^* = W$ on divisor classes, and

$$I(f) = \bigcup_{1 \leq i < j \leq 4} L_{ij}, \qquad I(g) = \bigcup_{2 \leq i < j \leq 5} L_{ij}, \tag{3.5}$$

and

$$f \colon X \setminus I(f) \longrightarrow X \setminus I(g)$$

is an isomorphism. There are no further exceptional curves.

The eigenclass is pseudoeffective with the indicated sign. Divisor pullback by a pseudoautomorphism is functorial, so

$$\lambda^{-n} (g^n)^* H = \lambda^{-n} W^n h$$

is pseudoeffective. In the real lattice of signature $(1, 9)$, let v_- be the eigenvector of eigenvalue λ^{-1}, oriented to have positive h-coefficient. The vectors v, v_- are distinct future-pointing null vectors. Their span has signature $(1, 1)$, and its orthogonal complement is negative definite. Powers of the isometry on that complement are bounded. Consequently

$$\lambda^{-n} W^n h \longrightarrow \frac{h \cdot v_-}{v \cdot v_-}\, v.$$

Both pairings in the coefficient are positive. The closed pseudoeffective cone therefore contains β.

Step 3. The pencil separates the curve orbit from the indeterminacy.

A quadric restricted to $p(t)$ has degree at most eight. If it passes through all nine distinct marked points, the restriction vanishes identically. Substitution into the quadric monomials shows that the quadrics containing the quartic are exactly the span of Q_1, Q_2. Their strict transforms have class

$$F = 2H - \sum_{i=1}^{9} E_i,$$

which is fixed by W.

The cusp is $[0:0:0:1]$. It lies on no secant L_{ij}, and (3.4) fixes it. The unique pencil member singular there is $Q_1 = 0$. The map is a local isomorphism at the cusp, so it fixes S_∞. Its action on the pencil parameter is therefore affine.

The secant through $p(a)$ and $p(b)$ lies on $Q_2 = (a+b)^2 Q_1$. In fact, substitution of $(1-z)p(a) + zp(b)$ gives

$$Q_1 = -z(1-z)(a-b)^2, \qquad Q_2 = -z(1-z)(a^2-b^2)^2.$$

For $i \le 4 < j$, cubic Cremona sends L_{ij} to $L_{i+1,j+1}$. On a line through one coordinate vertex, the four cubic coordinates have a common square factor; canceling it leaves linear coordinates for the image line. The recurrences imply

$$t_{i+1} + t_{j+1} = \frac{t_i + t_j}{\lambda}.$$

Apply this to L_{15} and L_{35}. Their squared pencil parameters are distinct, as follows from the rational bounds on λ. These two values determine the affine pencil action:

$$r \longmapsto r/\lambda^2.$$

A secant has no third intersection with the quartic, by the coordinates $1, t, t^2$. It is not tangent at either endpoint, because its slope in the t, t^2 coordinates differs from the tangent slope when the endpoints are distinct. After blowing up the endpoints, the secant is disjoint from the strict quartic.

Moreover, $F \cdot L_{ij} = 2 - 1 - 1 = 0$. If a pencil member does not contain L_{ij}, its defining section restricts to a nonzero section of a degree-zero line bundle on that curve. The section has no zeros. Thus every such member is disjoint from L_{ij}.

The six parameters of $I(f)$ have square roots equal to $\lambda - 1$ times the following six numbers:

$$-\frac{\lambda^2+\lambda+1}{d}, \quad -1, \quad -\frac{\lambda^2}{d}, \quad -\frac{1}{d}, \quad 0, \quad \frac{\lambda}{d}. \tag{3.6}$$

The positive numbers $\lambda^{-1}, \lambda^{-2}, \ldots$ avoid their magnitudes. More precisely,

$$\frac{\lambda^2}{d} < \lambda^{-1} < 1, \qquad \frac{\lambda}{d} < \lambda^{-2} < \frac{\lambda^2}{d}, \qquad \frac{1}{d} < \lambda^{-3} < \frac{\lambda}{d},$$

and $\lambda^{-k} < 1/d$ for $k \ge 4$. These inequalities are equivalent to the relevant parts of (3.3); the remaining magnitude $(\lambda^2+\lambda+1)/d$ exceeds one.

It follows that each S_{r_n} is disjoint from $I(f)$. Starting with $C_0 = L_{35}$, the isomorphism in (3.5) therefore defines every C_n as an irreducible compact curve and makes every required iterate a local biholomorphism there. The pencil identity gives $C_n \subset S_{r_n}$. Projection of cohomology along the local isomorphism gives

$$\beta \cdot C_{n+1} = (f^*\beta) \cdot C_n = \lambda^{-1} \beta \cdot C_n.$$

Since

$$\beta \cdot L_{35} = 1 - m_3 - m_5 = -\frac{\lambda - 1}{\lambda},$$

this proves (3.1). In particular, the curves are distinct.

Step 4. The classes $H, E_1, \dots, E_9$ form a basis of $\mathrm{Pic}(X)$, and $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbf{R})$ is injective.

This is the standard description of the Picard group of a point blowup. Independence in $H^2(X, \mathbf{R})$ follows by pairing with a line in each $E_i \simeq \mathbf{P}^2$, on which H is trivial and E_i has degree -1, and with the strict transform of a general line, on which H has degree one and each E_i degree zero. In particular the lattice computations above take place in $H^2(X, \mathbf{R})$.

Step 5. The tail-limit set lies on the limiting cone, and q is a regular fixed point.

The closed tail sets in the definition of K_0 are nonempty compact and decreasing, so K_0 is nonempty and compact. If $x_j \in C_{n_j}$, $n_j \to \infty$, and $x_j \to x$, the equations

$$Q_2 - r_{n_j} Q_1 = 0$$

may be read in a local frame of $\mathcal{O}_X(F)$ near x. Passing to the limit gives $Q_2(x) = 0$. Every point of the tail intersection can be obtained by such a sequence, hence $K_0 \subset S_0$.

Finally, $\alpha(-\frac{1}{2}) = -\frac{1}{2}$. The formulas and (3.3) give $m_i > 0$ for every i, so $p(-\frac{1}{2})$ differs from every blowup point. It lies on the quartic and therefore on S_0. A point of the quartic other than a blown-up endpoint belongs to no indeterminacy secant. Thus f and g are biholomorphic near q, and (3.4) fixes q. $\square$

3.3 The resolved limiting cone

Proposition 3.5. *Let*

$$j \colon S \longrightarrow X$$

be the minimal resolution of S_0 followed by inclusion. The surface S is the blowup of the Hirzebruch surface $\mathbb{F}_2$ at the nine lifts of the P_i. Let e be the exceptional section resolving the cone vertex and b a ruling-fiber class. Then

$$e^2 = -2, \qquad b^2 = 0, \qquad e \cdot b = 1, \qquad j^* H = e + 2b.$$

The maps induced by f and g extend to inverse holomorphic automorphisms f_S, g_S of S. The class

$$\beta_S = j^* \beta$$

is nonzero, and

$$g_S^* \beta_S = \lambda \beta_S.$$

The strict transform of the only ambient indeterminacy secant lying on S_0 is

$$R_0 = b - E_2 - E_4.$$

It satisfies $R_0^2 = -2$ and $e \cdot R_0 = 1$.

Proof. The equation of S_0 is $X_3^2 - X_1 X_4 = 0$. Its vertex is $[0:1:0:0]$, which differs from the cusp of the quartic and lies on none of the marked points. Its minimal resolution is the Hirzebruch surface [Har77, Chapter V, Section 2]

$$\mathbb{F}_2 = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O} \oplus \mathcal{O}(2)),$$

with the quotient convention, as we now recall. Its Picard group has basis the fiber class b and the section e induced by the quotient onto $\mathcal{O}$. The normal bundle of this section is $\mathrm{Hom}(\mathcal{O}(2), \mathcal{O}) = \mathcal{O}(-2)$, so $e^2 = -2$. Distinct fibers are disjoint, giving $b^2 = 0$, and the section meets a fiber once, giving $e \cdot b = 1$.

The tautological line bundle has degree one on a fiber and degree zero on e. Writing its class as $e+cb$, the second degree is $-2+c$, so its class is $e+2b$. The constant section of $\mathcal{O}$ and the three quadratic monomials of $\mathcal{O}(2)$ give the morphism

$$|e+2b|\colon \mathbb{F}_2 \longrightarrow \{X_3^2 - X_1X_4 = 0\} \subset \mathbf{P}^3.$$

This morphism contracts e to the vertex and is an isomorphism away from e. Its only exceptional curve has square -2, so this is the minimal resolution. The hyperplane pullback is therefore $e+2b$.

The nine blowup points lie in the smooth locus of the cone. Blowing up their lifts gives S; these points avoid e, so the displayed intersection products remain unchanged and $j^*H = e+2b$.

Since $m_2 + m_4 = 1$, we have $t_2 = -t_4$. Hence L_{24} is the generator through the vertex and those two marked points. Its strict transform is $R_0 = b - E_2 - E_4$, giving

$$R_0^2 = -2, \qquad e \cdot R_0 = 1.$$

All other relevant pair sums are nonzero. Thus L_{24} is the only indeterminacy secant of either map contained in S_0. Every other indeterminacy secant is disjoint from S_0, by the degree-zero argument in the proof of Proposition 3.4.

The maps induced by f and g are inverse birational self-maps f_S, g_S of S. By (3.5) and the preceding paragraph, f and g are inverse biholomorphisms of $S_0 \setminus L_{24}$ onto itself, so f_S and g_S are inverse automorphisms of $U = S \setminus (e \cup R_0)$, and the only curves that could be contracted by either map are e and R_0. We show that f_S maps e onto R_0 and g_S maps e onto R_0; since the maps are inverse to each other, both then map R_0 onto e, and no curve is contracted in either direction. Use the cone chart

$$[X_1 : X_2 : X_3 : X_4] = [\epsilon : 1 : \epsilon z : \epsilon z^2]$$

near the vertex, in which z is a coordinate along $e = \{\epsilon = 0\}$ and the generator through $p(t)$ is $\{z = t^2\}$. Of the four coordinate linear forms of M^{-1}, which cut out the planes through three of $P_1, \dots, P_4$, precisely the first and third vanish at the vertex, since these are the planes containing P_2 and P_4; each cuts the cone in L_{24} and one further generator. Their leading coefficients in ϵ are therefore nonzero constants times

$$(z - t_2^2)(z - t_3^2) \quad \text{and} \quad (z - t_2^2)(z - t_1^2),$$

respectively, while the second and fourth forms are nonzero constants at $\epsilon = 0$. Reciprocation sends a general point of e to a point of the line joining the first and third coordinate vertices, with ratio a nonconstant fractional-linear function of z because $t_1^2 \neq t_3^2$; then B sends that line to the secant $P_2P_4 = L_{24}$. Thus f_S maps e onto R_0. For g, whose four base points are P_2, P_3, P_4, P_5, the same computation with the planes through P_2, P_4, P_5 and P_2, P_3, P_4 uses $t_3^2 \neq t_5^2$ and maps e onto R_0 as well.

A birational map between smooth projective surfaces which contracts no curve in either direction is an isomorphism. Indeed, let $p_1, p_2 \colon \widetilde{S} \to S$ be a resolution of the graph with p_1 a composition of the smallest possible number of point blowups [Har77, Chapter V, Section 5]. If this number is positive, let E be the exceptional curve of the last blowup. If $p_2(E)$ is a curve, then the inverse map contracts it to the point $p_1(E)$; otherwise p_2 factors through the blowdown of E, contradicting minimality. Hence p_1 is an isomorphism, and then p_2 is a birational morphism contracting no curve, hence an isomorphism. Consequently f_S and g_S are inverse automorphisms, and $f_S(e) = R_0$, $f_S(R_0) = e$.

Finally we compare f_S^* with the ambient action. Let D be a divisor class on X. The line bundles $f_S^*(j^*D)$ and $j^*(f^*D)$ agree on U, because $j \circ f_S = f \circ j$ there and $j(U) = S_0 \setminus L_{24} \subset X \setminus I(f)$. Hence

$$f_S^*(j^*D) = j^*(f^*D) + k_1\, e + k_2\, R_0$$

for integers k_1, k_2 depending linearly on D; by linearity the same identity, with real coefficients, holds for real classes D. Pair both sides with e and with R_0, using that $f_S^*(j^*D)\cdot e = j^*D\cdot f_S(e) = j^*D\cdot R_0 = D\cdot L_{24}$, similarly $f_S^*(j^*D)\cdot R_0 = j^*D\cdot e$, that e is orthogonal to $j^*\mathrm{Pic}(X)$ since $e\cdot(e+2b) = 0 = e\cdot E_i$, and that $j^*(f^*D)\cdot R_0 = f^*D\cdot L_{24}$:

$$D\cdot L_{24} = -2k_1 + k_2, \qquad 0 = f^*D\cdot L_{24} + k_1 - 2k_2.$$

For $D=\beta$ we have $\beta\cdot L_{24} = 1 - m_2 - m_4 = 0$ and $f^*\beta\cdot L_{24} = \lambda^{-1}\beta\cdot L_{24} = 0$, so $k_1 = k_2 = 0$ and

$$f_S^*\beta_S = j^*(f^*\beta) = \lambda^{-1}\beta_S, \qquad g_S^*\beta_S = \lambda\beta_S.$$

The class is nonzero because $\beta_S\cdot(e+2b) = (e+2b)^2 = 2$. □

4 Exact positive eigencurrents

In this section, we prove the current equation needed to restrict the eigenclass of Proposition 3.4 to the surface of Proposition 3.5, and the finiteness of its potential at a regular fixed point.

Let M be a connected smooth complex projective manifold of dimension $d\geq 2$ and let $a\colon M \dashrightarrow M$ be a pseudoautomorphism. Fix a smooth projective resolution Z of the closure of the graph of a in $M\times M$ [Hir64], with projections

$$p\colon Z\to M, \qquad h\colon Z\to M,$$

both of which are proper modifications since a is birational. If R is a closed positive $(1,1)$-current on M with canonical local plurisubharmonic potentials u, we define h^*R locally as $\mathrm{dd^c}(u\circ h)$: the composition $u\circ h$ is plurisubharmonic, and it is not identically minus infinity on any nonempty open set, since every such set meets the open set on which h is biholomorphic; two local potentials differ by a smooth pluriharmonic function, so the local definitions glue to a closed positive current on Z. We then put

$$a^*R = p_*h^*R,$$

a closed positive $(1,1)$-current on M, since proper pushforward preserves closedness and positivity. Its class is $p_*h^*[R] = a^*[R]$, which is how the action of a on $H^{1,1}(M,\mathbf{R})$ is defined.

Theorem 4.1. *In this situation, let ξ be a nonzero pseudoeffective real $(1,1)$-class and let $\rho>1$ satisfy $a^*\xi = \rho\xi$. Then:*

(i) *$R\mapsto a^*R$ is weakly continuous on the set of closed positive $(1,1)$-currents in any fixed cohomology class;*

(ii) *on every open set $U\subset M$ on which a is biholomorphic onto its image, a^*R coincides with the ordinary pullback, defined by composing canonical local potentials with a;*

(iii) *there exists a closed positive current T with $[T]=\xi$ and $a^*T=\rho T$ on M.*

Proof. **Step 1.** *Normalized potentials.* Let Y be a connected compact Kähler manifold with a Kähler form k and the normalized volume form dV of k, let b be a smooth closed real $(1,1)$-form, and let $\mathcal{C}_{[b]}$ be the set of closed positive currents in the class $[b]$. Every $R\in\mathcal{C}_{[b]}$ can be written $R = b + \mathrm{dd^c}v$ with a canonical $v\in\mathrm{PSH}(Y,b)$ [Dem12, Chapter VI, Section 12.1], [GZ05, Proposition 1.4], and v is unique up to an additive constant since a pluriharmonic distribution on Y is constant; we normalize $\int_Y v\,dV = 0$. Choose C_0 with $C_0k - b\geq 0$. Then $v-\sup_Y v$ is C_0k-plurisubharmonic with supremum zero, and the compactness theorem for such functions

[GZ05, Propositions 1.6–1.7] shows that they form a relatively compact subset of $L^1(Y)$; in particular their integrals are bounded below by some $-C$, so

$$v \le \sup_Y v = -\int_Y (v - \sup_Y v)\, dV \le C$$

for every normalized potential, while $\sup_Y v \ge \int_Y v\, dV = 0$. Consequently the normalized potentials are uniformly bounded above and relatively compact in $L^1(Y)$, and $R_j \to R$ weakly in $\mathcal{C}_{[b]}$ if and only if $v_j \to v$ in $L^1(Y)$: one direction is the continuity of $\mathrm{dd^c}$ on distributions, and for the other, every L^1-limit $\widetilde{v}$ of a subsequence of (v_j) satisfies $b + \mathrm{dd^c}\widetilde{v} = R$ and $\int_Y \widetilde{v}\, dV = 0$, hence equals v almost everywhere by uniqueness of the normalized potential.

Step 2. *Continuity of h^* and of a^*.* Let $R_j \to R$ weakly in $\mathcal{C}_{[b]}$, with normalized potentials $v_j \to v$ in $L^1(M)$, $v_j \le C$. Then $h^*R_j = h^*b + \mathrm{dd^c}(v_j \circ h)$, and the functions $w_j = v_j \circ h$ are h^*b-plurisubharmonic on Z with $w_j \le C$. By Hörmander's compactness theorem [GZ05, Propositions 1.6–1.7], either a subsequence of (w_j) converges in $L^1(Z)$, or $w_j \to -\infty$ uniformly on Z. The second alternative is excluded, because on a coordinate ball O in the biholomorphic locus of h the change of variables formula gives $w_j \to v \circ h$ in $L^1(O)$. By the same reasoning every subsequential L^1-limit of (w_j) agrees with $v \circ h$ almost everywhere on the biholomorphic locus, whose complement is a proper analytic subset of measure zero. Hence the whole sequence satisfies $w_j \to v \circ h$ in $L^1(Z)$, and $h^*R_j \to h^*R$ weakly. Since p_* is weakly continuous, $a^*R_j \to a^*R$. This proves (i).

Step 3. *Ordinary pullback on the biholomorphic locus.* Let a be biholomorphic on U onto $a(U)$. Over $p^{-1}(U)$ the two holomorphic maps h and $a \circ p$ agree on the dense open graph locus, hence everywhere, so for a local potential u of R near a point of $a(U)$ we have $u \circ h = (u \circ a) \circ p$; that is, $h^*R = p^*(a^*_{\mathrm{ord}}R)$ over $p^{-1}(U)$, where $a^*_{\mathrm{ord}}R$ is the ordinary pullback on U and p^* is defined by composition of potentials as above. It remains to see that $p_*p^*Q = Q$ for a closed positive $(1,1)$-current Q on U. Locally $Q = \mathrm{dd^c}q$ with q plurisubharmonic, and $p_*(q \circ p) = q$ as distributions: testing against a compactly supported smooth top-degree form and changing variables on the biholomorphic locus of p, whose complement and its preimage have measure zero, gives the identity, since $q \circ p$ is locally integrable. As p_* commutes with $\mathrm{dd^c}$, we get $p_*p^*Q = Q$, and (ii) follows.

Step 4. *Averaging in the class ξ.* Let $\mathcal{C}_\xi$ be the set of closed positive $(1,1)$-currents in ξ; it is nonempty because ξ is pseudoeffective, and convex. Fix a Kähler form k_M. Every $R \in \mathcal{C}_\xi$ has trace mass $\int_M R \wedge k_M^{d-1} = \int_M \xi \wedge [k_M]^{d-1} = m$, and $m > 0$ because $\xi \ne 0$. Positivity and the Cauchy–Schwarz inequality bound all coefficient measures of R in local charts by a constant times the trace, so $\mathcal{C}_\xi$ is bounded in the space of currents of order zero; positivity, closedness and the cohomology class (finitely many pairings with fixed smooth closed forms) pass to weak limits. Hence $\mathcal{C}_\xi$ is weakly compact.

By (i) and the class computation, $A(R) = \rho^{-1}a^*R$ is a continuous affine self-map of $\mathcal{C}_\xi$; affineness holds because canonical potentials add and pullback commutes with sums and nonnegative scalars. Choose $R_0 \in \mathcal{C}_\xi$, put $R_{n+1} = A(R_n)$ and $S_N = \frac{1}{N}\sum_{n=0}^{N-1} R_n$. Then

$$A(S_N) - S_N = \frac{R_N - R_0}{N}, \tag{4.1}$$

and the right-hand side tends weakly to zero, since $|\langle R_N - R_0, \varphi\rangle| \le 2C_\varphi m$ for every smooth test form φ. A subsequence S_{N_j} converges weakly to some $T \in \mathcal{C}_\xi$, and continuity of A gives $A(T) = T$, that is, $a^*T = \rho T$ with $[T] = \xi$. This proves (iii). □

Lemma 4.2. *Let M be a complex manifold, let $p \in M$, and let g be a biholomorphic germ fixing p. Let T be a closed positive $(1,1)$-current near p with $g^*T = aT$ near p, for a real $a > 1$, where g^* is the ordinary pullback. Then every canonical local plurisubharmonic potential of T is finite at p.*

Proof. Choose coordinates centered at $p=0$ and a ball $B(0,R_1)$ on which g, g^{-1} and a canonical potential u of T are defined, with

$$|g^{-1}(z)| \le L|z| \qquad \text{on } B(0,R_1)$$

for some $L>1$, which is possible because $g^{-1}(0)=0$. The current equation gives $\mathrm{dd^c}(u\circ g - au) = 0$, so $u\circ g - au$ is a pluriharmonic distribution, hence a smooth pluriharmonic function h, and the canonical plurisubharmonic functions $u\circ g$ and $au+h$, which agree almost everywhere, agree pointwise:

$$u(g(z)) = au(z) + h(z) \qquad \text{on } B(0,R_1), \tag{4.2}$$

including at minus-infinity values.

Suppose that $u(0)=-\infty$, and let $H = \sup_{B(0,R_1)}|h|$ after shrinking R_1. By upper semicontinuity there is $r \le R_1/2$ with

$$u \le -\frac{H}{a-1} - 1 \qquad \text{on } B(0,r).$$

Let $|w| < rL^{-n}$ and $z_j = g^{-j}(w)$ for $0 \le j \le n$; then $|z_j| \le L^j|w| < r$. Iterating (4.2) from z_n to w,

$$u(w) = a^n u(z_n) + \sum_{j=1}^{n} a^{j-1}h(z_j) \le -a^n\Big(\frac{H}{a-1}+1\Big) + \frac{H(a^n-1)}{a-1} \le -a^n.$$

Thus $\sup_{\overline{B}(0,s_n)} u \le -a^n$ for $s_n = \frac{1}{2}rL^{-n}$. On the other hand, u is plurisubharmonic and not identically minus infinity on $B(0,R_1)$, hence locally integrable, and Lemma 2.4 with $K=\{0\}$ and $R_0 = r/2$ gives

$$\sup_{\overline{B}(0,s_n)} u \ge -A - B\log\frac{r/2}{s_n} = -A - Bn\log L.$$

Since a^n grows faster than n, this is a contradiction for large n. Hence $u(0) > -\infty$; any other local potential differs from u by a smooth function and is finite at 0 as well. □

Corollary 4.3. *For the model of Setup 3.3, there exists a global closed positive current $T \in \beta$ with*

$$g^*T = \lambda T.$$

Near q this identity is the ordinary pullback identity, and every such current has finite canonical local potentials at q.

Proof. Proposition 3.4 gives pseudoeffectivity of β, the equation $g^*\beta = \lambda\beta$, and $\beta \neq 0$ (indeed $\beta \cdot L_{35} \neq 0$). Theorem 4.1 gives T, and since g is biholomorphic near q the identity there is the ordinary one. Lemma 4.2 applies at the fixed point q. □

Remark 4.4. The only role of the fixed point q is to provide one finite value of the potential on S_0, so that the restriction of T to the connected surface S of Proposition 3.5 is a well-defined nontrivial current (Theorem 6.2, Step 1).

5 Ordinary accumulation at the cone vertex

In this section, we use the smooth pencil members of Proposition 3.4 to obtain actual points of the negative curves converging to v_0.

Proposition 5.1. *Use Setup 3.3. In the affine chart $X_2 = 1$, write*

$$x = X_1, \qquad z = X_3, \qquad w = X_4.$$

For $r > 0$, define

$$V_r = \left\{ \begin{array}{lll} x = a + ib, & w - rz = -(a - ib), & z = c, \\ a, b, c \in \mathbf{R}, & a^2 + b^2 + c^2 = r & \end{array} \right\}.$$

For all sufficiently large n, this is an embedded sphere $V_{r_n} \subset S_{r_n}$ avoiding the blowup centers, and

$$|C_n \cdot V_{r_n}| = 1.$$

Consequently there are $x_n \in C_n$ for every sufficiently large integer n such that

$$x_n \in V_{r_n}, \qquad x_n \longrightarrow v_0$$

in the ordinary topology. In particular, $v_0 \in K_0$.

Proof. **Step 1.** *The map f restricts to isomorphisms $F_n \colon S_{r_n} \to S_{r_{n+1}}$.* By Proposition 3.4, S_{r_n} is disjoint from $I(f)$, and f maps it into $S_{r_{n+1}}$. Since $f \colon X \setminus I(f) \to X \setminus I(g)$ is an isomorphism, the image $f(S_{r_n})$ is a compact irreducible surface contained in $S_{r_{n+1}}$, hence equal to it, and f is biholomorphic between neighborhoods of the two surfaces.

Step 2. *The class $u_r = b_1 - b_2$ and the intersection numbers $|C_n \cdot u_{r_n}| = 1$.* For $r \neq 0$ the quadric $X_3^2 - X_1X_4 - rX_2^2 + rX_1X_3 = 0$ is smooth, since the determinant of its symmetric matrix is $r/4$. Thus S_r is the blowup of a smooth quadric $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ at the nine marked points, and by the blowup formula [GH14, Chapter 4, Section 1]

$$H^2(S_r, \mathbf{R}) = \mathbf{R}b_1 \oplus \mathbf{R}b_2 \oplus \bigoplus_{i=1}^{9} \mathbf{R}E_i, \qquad b_1^2 = b_2^2 = 0, \quad b_1 \cdot b_2 = 1, \quad E_i \cdot E_j = -\delta_{ij}, \quad b_k \cdot E_i = 0,$$

where b_1, b_2 are the two ruling classes, and $H|_{S_r} = b_1 + b_2$. The orthogonal complement of the ten-dimensional restricted span $\langle H, E_1, \dots, E_9 \rangle$ is the line spanned by

$$u_r = b_1 - b_2, \qquad u_r^2 = -2.$$

Since f is biholomorphic near S_{r_n}, pullback of line bundles commutes with restriction, so F_n^* maps the restricted span of $S_{r_{n+1}}$ onto that of S_{r_n} (the ambient f^* is invertible on divisor classes). Being an isometry, F_n^* maps orthogonal complements to orthogonal complements, and comparing squares gives

$$F_n^* u_{r_{n+1}} = \pm u_{r_n}.$$

The secant $C_0 = L_{35}$ lies on S_{r_0} and is a line, hence belongs to one of the two rulings of Q; its strict transform has class $b_k - E_3 - E_5$ for $k = 1$ or 2, so $C_0 \cdot u_{r_0} = \pm 1$. Since $F_n(C_n) = C_{n+1}$ and F_n^* preserves intersection numbers, induction gives

$$|C_n \cdot u_{r_n}| = 1 \qquad \text{for all } n. \tag{5.1}$$

Step 3. *The vanishing sphere represents $\pm u_r$.* In the chart $X_2 = 1$, the equation of S_r is $z^2 - x(w - rz) = r$. With $w' = w - rz$ and the complex-linear substitution $x = a + ib$, $w' = -(a - ib)$, $z = c$, it becomes $a^2 + b^2 + c^2 = r$, whose real points form the sphere V_r. The coordinates of V_r satisfy

$$|x|, |z|, |w'| \leq \sqrt{r}, \qquad |w| \leq \sqrt{r} + r^{3/2}, \tag{5.2}$$

so for small r the sphere lies in a fixed small neighborhood of the vertex, which contains none of the nine blowup centers; hence $V_r \subset S_r$.

Let $z_r \in H^2(S_r, \mathbf{R})$ be the Poincaré dual of V_r for either orientation [Hat02, Theorem 3.30]. The sphere V_r is totally real: the surface is the complexification of the real sphere, so $T_pS_r = T_pV_r \otimes \mathbf{C}$ at every $p \in V_r$. Multiplication by i therefore identifies TV_r with the normal bundle N of V_r in S_r. With the orientations induced from the complex orientation of S_r, this identification reverses orientation, because for an oriented basis (e_1, e_2) of T_pV_r the frame (e_1, e_2, ie_1, ie_2) differs from the complex-oriented frame (e_1, ie_1, e_2, ie_2) by one transposition. Hence $e(N) = -e(TV_r)$, and the self-intersection of the sphere, which is the Euler number of its normal bundle, equals

$$z_r^2 = -\chi(S^2) = -2.$$

Next, $z_r \cdot E_i = 0$ because V_r is disjoint from the exceptional curves, and $z_r \cdot H = 0$ because the Fubini–Study form representing H is exact on the affine chart $X_2 \neq 0$, which contains V_r, so its integral over the closed surface V_r vanishes. Thus z_r is orthogonal to the restricted span, so $z_r = c\, u_r$ with $-2 = z_r^2 = -2c^2$, that is, $z_r = \pm u_r$.

Step 4. *Conclusion.* By (5.1) and Step 3, $|C_n \cdot V_{r_n}| = 1$ for all large n. If the compact curve C_n and the sphere V_{r_n} were disjoint, a Thom form of the sphere supported away from the curve would give intersection number zero. Hence we may choose $x_n \in C_n \cap V_{r_n}$, and (5.2) with $r_n \to 0$ gives $x_n \to v_0$. For every N, all sufficiently late x_n lie in $\bigcup_{n \geq N} C_n$, so $v_0 \in K_0$. □

6 Continuous surface trace and the base discrepancy

In this section, we combine the exact current of Section 4 with the accumulation in Proposition 5.1.

Theorem 6.1. *Let M be a connected compact Kähler manifold, let h be a holomorphic automorphism, and let R be a closed positive $(1,1)$-current. If*

$$h^*R = aR$$

for a real $a > 1$, then R has continuous local potentials.

For the Green currents of automorphisms of compact Kähler manifolds, continuity (indeed Hölder continuity) of the potentials is due to Cantat for surfaces [Can01] and to Dinh–Sibony in general [DS05]; the argument below is a short self-contained version adapted to an arbitrary exact eigencurrent.

Proof. **Step 1.** *A continuous candidate potential.* Choose a smooth closed real form b representing $[R]$ and write $R = b + \mathrm{dd^c}v$ with canonical $v \in \mathrm{PSH}(M, b)$. By the $\partial\bar\partial$-lemma there is a smooth real ψ with $h^*b - ab = \mathrm{dd^c}\psi$, and then $\mathrm{dd^c}(\psi + v \circ h - av) = h^*R - aR = 0$. A pluriharmonic distribution on M is a constant, which we absorb into v (replacing v by $v + c$ changes $\psi + v \circ h - av$ by $(1-a)c$). Thus

$$v = a^{-1}\psi + a^{-1}v \circ h, \tag{6.1}$$

first almost everywhere, hence pointwise, since both sides are canonical b-plurisubharmonic functions. The series

$$v_* = \sum_{k \geq 0} a^{-k-1}\psi \circ h^k$$

converges uniformly, so v_* is continuous, and it satisfies (6.1). Hence $w = v - v_*$ is upper semicontinuous, bounded above, and $w \circ h = aw$; iterating, $w = a^{-n}w \circ h^n \leq a^{-n}\sup_M w$ for all n, so $w \leq 0$.

Step 2. *A uniform logarithmic lower bound.* Cover M by finitely many compact sets K_i contained in coordinate charts $\Omega_i \subset \mathbf{C}^d$, and fix $R_0 > 0$ such that the closed $2R_0$-neighborhood of each K_i lies in Ω_i. On Ω_i the function $F_i = v + \chi_i$ is plurisubharmonic, where χ_i is a smooth potential of b; it is locally integrable because $v \in L^1(M)$. Lemma 2.4 applied to F_i, together with the boundedness of χ_i, gives constants A_1, A_2 such that

$$\sup_{\overline{B}(z,s)} v \geq -A_1 - A_2 \log \frac{R_0}{s} \tag{6.2}$$

for every i, every $z \in K_i$ and every $0 < s \leq R_0$, where the ball is taken in the coordinates of Ω_i.

Step 3. *Positivity excludes a negative value.* Fix a Riemannian metric on M and $L > 1$ such that h^{-1} is L-Lipschitz. Suppose that $w(p) < 0$. By upper semicontinuity there are a metric ball $B(p, r)$ and $\epsilon > 0$ with $w < -\epsilon$ on $B(p, r)$. Since h^{-n} is L^n-Lipschitz,

$$h^n(B(p, r)) \supset B(h^n(p), rL^{-n}),$$

and the functional equation gives $w < -\epsilon a^n$ on this smaller ball, hence

$$v = w + v_* < -\epsilon a^n + \sup_M v_* \quad \text{there.}$$

The metric ball $B(h^n(p), rL^{-n})$ contains a coordinate ball of radius crL^{-n} around $h^n(p)$ in one of the charts, for a uniform $c > 0$, and (6.2) gives $\sup v \geq -A_3 - A_4 n$ on that coordinate ball. This contradicts $a > 1$ for large n. Hence $w \equiv 0$, and $v = v_*$ is continuous. Local potentials of R differ from v by smooth functions, so they are continuous. □

Theorem 6.2. *Use Setup 3.3 and Proposition 3.5. There exists a global closed positive current $T \in \beta$ with*

$$g^*T = \lambda T.$$

For every such current and every smooth closed real representative θ of β, writing $T = \theta + \mathrm{dd^c}u$ with the canonical $u \in \mathrm{PSH}(X, \theta)$, the function $u \circ j$ is continuous on S; in particular u is finite at every point of S_0, and

$$v_0 \in \mathrm{NB}_\theta \setminus P_\theta.$$

Proof. Existence is Corollary 4.3. Fix a current T with the global identity.

Step 1. *The surface trace.* By Corollary 4.3, $u(q) > -\infty$. The composition $u \circ j$, after addition of a smooth local potential of $j^*\theta$, is locally plurisubharmonic or identically minus infinity; it is nontrivial near the point $j^{-1}(q)$, and nontriviality propagates across overlapping coordinate neighborhoods of the connected surface S, because a nontrivial plurisubharmonic function is finite almost everywhere. Hence $u \circ j$ is a canonical $j^*\theta$-plurisubharmonic function, and

$$R = j^*\theta + \mathrm{dd^c}(u \circ j)$$

is a closed positive current on S representing β_S.

Step 2. *The surface eigenvalue equation.* By (3.5), g is biholomorphic from $X \setminus I(g)$ onto $X \setminus I(f)$, and $S_0 \cap I(g) = S_0 \cap I(f) = L_{24}$ by Proposition 3.5; hence g maps $S_0 \setminus L_{24}$ biholomorphically onto itself, and $j \circ g_S = g \circ j$ on $U = S \setminus (e \cup R_0)$. On $X \setminus I(g)$ the identity $g^*T = \lambda T$ is the ordinary pullback identity (Theorem 4.1(ii)), which for the canonical potential reads $u \circ g = \lambda u + \varphi$ pointwise with φ smooth. Composing with j gives $g_S^*R = \lambda R$ on U. Thus

$$A = g_S^*R - \lambda R$$

is a closed real $(1,1)$-current on S, of order zero as a difference of positive currents, supported on the union of the two smooth rational curves e and R_0, which meet transversally in one point.

Such a current is a combination $c_e[e]+c_R[R_0]$ with real constants. Indeed, near a smooth point of the support choose coordinates (z,w) with the curve equal to $\{w=0\}$. Since A has order zero and is supported on $\{w=0\}$, $wA=\bar{w}A=0$, and closedness then gives $dw\wedge A=d\bar{w}\wedge A=0$; so $A=\mu\otimes\delta_{w=0}\, i\,dw\wedge d\bar{w}$ for a measure μ in z, and closedness in the z-direction forces μ to be a constant multiple of Lebesgue measure. These constants are locally constant along e minus the intersection point and along R_0 minus the intersection point, both connected. Finally, $A-c_e[e]-c_R[R_0]$ is a closed current of order zero supported at one point of a real four-manifold; it is then a linear combination of $\delta\, dx_k\wedge dx_l$, and closedness forces all coefficients to vanish.

Since $g_S^*\beta_S=\lambda\beta_S$, the class of A is zero. Pairing with e and R_0, whose intersection matrix $\left(\begin{smallmatrix}-2 & 1\\ 1 & -2\end{smallmatrix}\right)$ is nondegenerate, gives $c_e=c_R=0$. Hence

$$g_S^*R=\lambda R \qquad \text{on all of } S.$$

Step 3. *Continuity and the discrepancy.* Theorem 6.1, applied to S, g_S, R and λ, shows that R has continuous local potentials. Locally, $u\circ j$ plus a smooth potential of $j^*\theta$ is a canonical plurisubharmonic function with the same $\mathrm{dd^c}$ as a continuous potential of R; the difference is pluriharmonic, hence smooth, so $u\circ j$ is continuous and finite on S. Since j maps S onto S_0, the function u is finite on S_0; in particular $u(v_0)>-\infty$.

By (3.1) every C_n has negative β-degree, so Proposition 3.1 places the ordinary closure of $\bigcup_n C_n$ in NB_θ, and Proposition 5.1 places v_0 in that closure. Finally $V_\theta(v_0)\geq u(v_0)-\sup_X u>-\infty$ by Lemma 2.3. Thus $v_0\in\mathrm{NB}_\theta\setminus P_\theta$. ☐

7 The projective-bundle lift

In this section, we transfer the discrepancy in Theorem 6.2 to a big class.

Theorem 7.1. *Let X be a connected smooth complex projective manifold, and let θ be a smooth closed real $(1,1)$-form with pseudoeffective class $\beta=[\theta]$. Choose an ample line bundle A with a smooth positively curved Hermitian metric of curvature a such that $\theta+\frac{1}{2}a$ is Kähler. Let $\pi\colon Y=\mathbf{P}_X(\mathcal{O}_X\oplus A)\to X$ be the bundle of one-dimensional quotients, let $s\colon X\to Y$ be the section given by the quotient onto $\mathcal{O}_X$, put $\Sigma=s(X)$, and let η be the curvature of the quotient metric on $\mathcal{O}_Y(1)$ induced by the flat metric on $\mathcal{O}_X$ and the given metric on A. Set*

$$\Theta=\eta+\pi^*\theta.$$

Then $[\Theta]$ is big, and

$$V_\Theta\circ s=V_\theta,\qquad V_\Theta\geq V_\theta\circ\pi,\qquad P_\Theta=s(P_\theta),\qquad \mathrm{NB}_\Theta=s(\mathrm{NB}_\theta).$$

Consequently, any failure of equality between the polar and non-bounded loci of a pseudoeffective real $(1,1)$-class on a smooth projective manifold gives such a failure for a big class on a smooth projective manifold one dimension higher.

Proof. **Step 1.** *The form η.* In a local frame of A write the metric as $e^{-\phi}$, so $a=\mathrm{dd^c}\phi$. On the chart of Y where the $\mathcal{O}_X$-component of the quotient does not vanish, with fiber coordinate z, the quotient metric on $\mathcal{O}_Y(1)$ has weight $F=\log\Big(1+|z|^2e^{\phi}\Big)$; on the complementary chart, with $w=z^{-1}$, it has weight $G=\log\Big(e^{\phi}+|w|^2\Big)$. Since $(r,t)\mapsto\log(e^r+e^t)$ is convex with nonnegative partial derivatives, these weights are plurisubharmonic, so η is a smooth semipositive form; on every fiber it is a Fubini–Study form, hence positive on nonzero vertical tangent vectors. The section Σ is $\{z=0\}$, where $F \equiv 0$; therefore $s^*\eta = 0$ and $s^*\Theta = \theta$.

Step 2. *Bigness, and boundedness off the section.* The tautological homomorphism $\pi^*A \to \mathcal{O}_Y(1)$ vanishes to first order exactly along Σ, so it is a section σ of $\mathcal{O}_Y(1) \otimes \pi^*A^{-1}$ with divisor Σ. With the product metric, of curvature $\eta - \pi^*a$, the function $\rho = \log|\sigma|^2$ equals $\log(|z|^2 e^{\phi}/(1+|z|^2 e^{\phi}))$ on the first chart; it is nonpositive, smooth off Σ, and by the Poincaré–Lelong formula $\mathrm{dd^c}\rho = [\Sigma] - \eta + \pi^*a$. Hence

$$\Theta + \mathrm{dd^c}(\tfrac{1}{2}\rho) = \tfrac{1}{2}[\Sigma] + \tfrac{1}{2}\eta + \pi^*(\theta + \tfrac{1}{2}a) \geq \tfrac{1}{2}\eta + \pi^*(\theta + \tfrac{1}{2}a). \tag{7.1}$$

The smooth form on the right is Kähler: a tangent vector with nonzero image in X is positive for the second term and nonnegative for the first, and a nonzero vertical vector is positive for the first. Thus $[\Theta]$ is big, and $\frac{1}{2}\rho$ is a nonpositive Θ-plurisubharmonic function, so

$$0 \geq V_\Theta \geq \tfrac{1}{2}\rho, \tag{7.2}$$

and V_Θ is locally bounded off Σ.

Step 3. *The envelopes on the section.* Since π is a submersion, $V_\theta \circ \pi$ is a canonical function with $\Theta + \mathrm{dd^c}(V_\theta \circ \pi) = \eta + \pi^*(\theta + \mathrm{dd^c}V_\theta) \geq 0$; it is nonpositive and nontrivial, hence a competitor, and $V_\Theta \geq V_\theta \circ \pi$. Conversely, $v = V_\Theta \circ s$ is not identically minus infinity, since $v \geq V_\theta$; restriction of a plurisubharmonic function to a complex submanifold is plurisubharmonic unless identically minus infinity, and $s^*\Theta = \theta$, so $v \in \mathrm{PSH}(X,\theta)$ with $v \leq 0$, whence $v \leq V_\theta$. Thus $V_\Theta \circ s = V_\theta$.

Step 4. *The singular loci.* By (7.2), P_Θ and NB_Θ are contained in Σ, and $V_\Theta \circ s = V_\theta$ gives $P_\Theta = s(P_\theta)$. If V_θ is bounded on a neighborhood U of x, then $0 \geq V_\Theta \geq V_\theta \circ \pi$ bounds V_Θ on $\pi^{-1}(U)$; conversely, boundedness of V_Θ near $s(x)$ restricts to boundedness of V_θ near x. Hence $\mathrm{NB}_\Theta = s(\mathrm{NB}_\theta)$. The last assertion follows, since any smooth θ satisfies $\theta + \frac{1}{2}a > 0$ once A is replaced by a sufficiently high tensor power. $\square$

Proof of Theorem A. The threefold in Setup 3.3 is smooth projective by Proposition 3.4, and Theorem 6.2 gives $v_0 \in \mathrm{NB}_\theta \setminus P_\theta$ for every smooth representative θ of β. Theorem 7.1, applied to the ample bundle and quotient metric in the statement, produces the smooth projective fourfold Y with the big class $[\Theta]$ and gives

$$s(v_0) \in s(\mathrm{NB}_\theta) \setminus s(P_\theta) = \mathrm{NB}_\Theta \setminus P_\Theta. \qquad \square$$

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