A GRAM-CONE OBSTRUCTION TO POSITIVE MIRROR COORDINATES

ABSTRACT. We examine the positive mirror map associated in [Pop20] with the four-dimensional essential deformation space of the Iwasawa manifold. After arranging the deformation parameters into a matrix $T \in M_2(\mathbb{C})$, the map factors through the Gram matrix T^*T. It is therefore constant on every left $U(2)$-orbit, has zero differential at the central fibre, and is not locally injective. This gives an explicit counterexample to the positive-image coordinate correspondence asserted in [Pop20, Theorem 7.3(v)]. We identify the exact local replacement: the Gram map identifies the quotient by left $U(2)$ with the cone of positive-semidefinite Hermitian 2×2 matrices, on which the induced map is locally the restriction of a real-analytic diffeomorphism. We also record the independent dimension and parity obstruction to the metric-side coordinate construction used in the same assertion.

1. Introduction

Let X be the Iwasawa manifold. The essential Kuranishi slice used in [Pop20] is a neighbourhood of the origin in a four-dimensional complex vector space. Write its parameters as

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in M_2(\mathbb{C}).$$

The family of canonical Gauduchon metrics constructed there determines the positive mirror map

$$M_+(T) = [(\omega_T^{1,1})^2]_A \in H_A^{2,2}(X, \mathbb{R}),$$

which is denoted by M in [Pop20, Definition 7.1]. The paper subsequently introduces a different, holomorphic map $\widetilde{M}$ with values in the complexified Aeppli space. These two maps must be kept distinct.

The positive-map coefficients displayed in [Pop20, equation (85)] contain only squared norms, Hermitian products, and $|\det T|^2$. Their matrix form yields the following result.

Theorem A. *There is a real-linear isomorphism $\Phi : \mathrm{Herm}_2 \to H_A^{2,2}(X, \mathbb{R})$ such that the positive mirror map factors as*

$$M_+(T) = \Phi\big(P(T^*T)\big), \qquad P(H) = 2(1 - \det H)(I - H). \tag{1.1}$$

The isomorphism Φ is given explicitly by the real Aeppli basis occurring in [Pop20, equation (85)]. Consequently:

(i) *$M_+(UT) = M_+(T)$ for every $U \in U(2)$;*
(ii) *$d(M_+)_0 = 0$, and M_+ is not injective on any neighbourhood of the origin;*
(iii) *after shrinking to a $U(2)$-invariant parameter neighbourhood, the fibres of M_+ are precisely the left $U(2)$-orbits; under the Gram identification with Herm_2^+, the induced map is the restriction of a real-analytic local diffeomorphism at the vertex.*

In particular, the coordinate correspondence on the positive image asserted in [Pop20, Theorem 7.3(v)] is false.

2. The Gram factorisation

Fix the invariant coframe α, β, γ of the central Iwasawa manifold and set

$$A_1 = [i\alpha \wedge \bar{\alpha} \wedge i\gamma \wedge \bar{\gamma}]_A, \qquad A_2 = [i\beta \wedge \bar{\beta} \wedge i\gamma \wedge \bar{\gamma}]_A,$$
$$C = [i\alpha \wedge \bar{\beta} \wedge i\gamma \wedge \bar{\gamma}]_A, \qquad \bar{C} = [i\beta \wedge \bar{\alpha} \wedge i\gamma \wedge \bar{\gamma}]_A.$$

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The classes $A_1, A_2, C+\bar{C}, (C-\bar{C})/i$ form a real basis of the four-dimensional Aeppli space occurring in the positive image. Hence

$$\Phi : \mathrm{Herm}_2 \longrightarrow H_A^{2,2}(X,\mathbb{R}), \qquad \begin{pmatrix} a & q \\ \bar{q} & b \end{pmatrix} \longmapsto aA_1 + bA_2 + qC + \bar{q}\bar{C}$$

is a real-linear isomorphism onto that space.

For $T \in M_2(\mathbb{C})$, write

$$G(T) = T^*T = \begin{pmatrix} a & q \\ \bar{q} & b \end{pmatrix}, \tag{2.1}$$

where

$$a = |t_{11}|^2 + |t_{21}|^2, \qquad b = |t_{12}|^2 + |t_{22}|^2, \qquad q = \bar{t}_{11}t_{12} + \bar{t}_{21}t_{22}.$$

Its determinant is

$$\det G(T) = ab - |q|^2 = |t_{11}t_{22} - t_{12}t_{21}|^2. \tag{2.2}$$

Substituting (2.1) and (2.2) into [Pop20, equation (85)] gives

$$M_+(T) = 2(1-\det G(T))\big((1-a)A_1 + (1-b)A_2 - qC - \bar{q}\bar{C}\big). \tag{2.3}$$

Equivalently,

$$M_+(T) = \Phi(P(G(T))), \qquad P(H) = 2(1-\det H)(I-H). \tag{2.4}$$

This proves the factorisation in Theorem A.

Since $G(UT) = G(T)$ for every $U \in U(2)$, (2.4) gives

$$M_+(UT) = M_+(T). \tag{2.5}$$

Moreover, the entries of $G(T)$ vanish to second order at $T = 0$. Therefore

$$M_+(T) = M_+(0) + O(\|T\|^2), \qquad d(M_+)_0 = 0. \tag{2.6}$$

3. The coordinate counterexample

For $r > 0$, define

$$T_r = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_r' = iT_r. \tag{3.1}$$

For every sufficiently small r, these are distinct points of the essential Kuranishi slice. Since $T_r' = (iI)T_r$, equation (2.5) yields

$$M_+(T_r) = M_+(T_r'). \tag{3.2}$$

As r can be arbitrarily small, M_+ is not locally injective at the origin.

Corollary 3.1. *The bijection in [Pop20, Theorem 7.3(v)] between the four holomorphic deformation coordinates and four holomorphic coordinates on the positive image is not well-defined as stated.*

Proof. The coordinates $z_1, \dots, z_4$ on the Kuranishi slice distinguish the distinct points in (3.1). Any functions $w_1, \dots, w_4$ defined on the positive image must take equal values at points with the same image, so (3.2) prevents the claimed correspondence from being injective. Conversely, if the w_j distinguish T_r from T_r', they depend on a choice of preimage and do not descend to functions on the positive image. ∎

The common-phase pair in (3.1) is only a one-dimensional part of the degeneracy. Equation (2.5) collapses the full left $U(2)$-orbit of every parameter matrix.

4. The exact local quotient

The unitary invariance accounts for every sufficiently small fibre.

Lemma 4.1. *For $S, T \in M_2(\mathbb{C})$, one has $S^*S = T^*T$ if and only if $S = UT$ for some $U \in U(2)$.*

Proof. The reverse implication is immediate. Suppose $S^*S = T^*T$. Then $\ker S = \ker T$, and the map

$$U_0 : \operatorname{im} T \longrightarrow \operatorname{im} S, \qquad U_0(Tx) = Sx,$$

is well-defined. For $x, y \in \mathbb{C}^2$,

$$\langle Sx, Sy\rangle = \langle S^*Sx, y\rangle = \langle T^*Tx, y\rangle = \langle Tx, Ty\rangle,$$

so U_0 is an isometry. Extending it across the orthogonal complements gives a unitary map $U \in U(2)$ satisfying $S = UT$. □

Proposition 4.2. *After replacing the parameter neighbourhood by a sufficiently small $U(2)$-invariant ball about the origin, the positive mirror map induces a homeomorphism from its orbit space onto its image. The Gram map identifies this orbit space with a neighbourhood of the vertex in Herm_2^+, and under this identification the induced map is the restriction of the real-analytic local diffeomorphism $\Phi \circ P$.*

Proof. The determinant on Herm_2 is quadratic, hence

$$dP_0(K) = -2K.$$

Thus $d(\Phi \circ P)_0 = -2\Phi$ is a real-linear isomorphism. The real-analytic inverse-function theorem shows that $\Phi \circ P$ is injective on a neighbourhood of $0 \in \mathrm{Herm}_2$.

The Gram map has image in Herm_2^+. Conversely, every sufficiently small $H \in \mathrm{Herm}_2^+$ is T^*T for $T = H^{1/2}$. The factorisation (2.4) and Lemma 4.1 identify the local fibres with the left $U(2)$-orbits. The induced Gram map on the orbit space is a homeomorphism onto a neighbourhood of the vertex in Herm_2^+: its inverse is $H \mapsto [H^{1/2}]$, which is continuous. Composing with the restriction of $\Phi \circ P$ proves the assertion. □

The marked point corresponds to the vertex of Herm_2^+, not to an interior point of a complex four-dimensional manifold. The positive map therefore parametrises Gram data, or equivalently left-unitary orbits.

5. The metric-side obstruction and the holomorphic map

Let B be the two-dimensional Albanese torus used in the setup preceding [Pop20, Proposition 6.16]. Its underlying real manifold is a four-torus, so $\dim_{\mathbb{R}} H^2(B, \mathbb{R}) = \binom{4}{2} = 6$. The ten displayed classes $\eta_{0,B}, \ldots, \eta_{4,B}, \nu_{0,B}, \ldots, \nu_{4,B}$ therefore cannot be a basis. Moreover, the bilinear form used there satisfies

$$Q_B(u, v) = -\int_B u \wedge v = Q_B(v, u) \qquad (u, v \in H^2(B, \mathbb{R}))$$

by graded commutativity. Thus Q_B is symmetric, not symplectic, and the odd-degree symplectic argument cannot yield the printed metric-side coordinates.

The counterexample leaves the separately defined map $\widetilde{M}$ of [Pop20, Definition 7.2(iii) and Theorem 7.3(i)] untouched. That map is affine-linear in T and is a biholomorphism onto its image. It does not identify the essential deformation space with the positive image: by (2.6), M_+ has zero differential at the origin and loses each left-unitary orbit, whereas $d\widetilde{M}_0$ is an isomorphism onto its image. The holomorphic affine map and the positive Gram-cone quotient must therefore be stated as distinct constructions.

References

[Pop20] D. Popovici. Non-Kähler Mirror Symmetry of the Iwasawa Manifold. *International Mathematics Research Notices* 23 (2020), pp. 9471–9538.