A DETERMINANT-ZERO OBSTRUCTION TO POINTWISE STABILITY

Abstract. We show that the strict pointwise stability condition of [Pop26, Definition 3.4] is empty on compact Riemann surfaces for holomorphic vector bundles of rank at least two. Indeed, every such bundle contains a line subbundle L, while the coherent subsheaf $L(-p) \subset E$ has a determinant section with a simple zero at p. Its stability function therefore vanishes at p for every smooth Hermitian metric on E. An explicit irreducible $SU(2)$ representation of a genus-two surface group yields a slope-stable flat rank-two bundle, giving counterexamples to [Pop26, Theorem 1.4(2)] and to the decomposition assertion of [Pop26, Theorem 6.1]. The example remains uniformly semi-stable. On curves the obstruction disappears after restricting the test class to saturated subsheaves, whereas in higher dimension even a saturated inclusion may have a vanishing determinant.

1. Introduction

Let $E \to X$ be a holomorphic vector bundle of rank r on a compact complex manifold. In the formalism of [Pop26], a coherent subsheaf $\mathcal{F} \subset \mathcal{O}(E)$ of rank $s \in \{1, \dots, r-1\}$ determines a holomorphic morphism

$$\det \mathcal{F} = (\Lambda^s \mathcal{F})^{\vee\vee} \longrightarrow \Lambda^s E$$

and hence a section

$$\sigma_{\mathcal{F}} \in H^0\big(X, \Lambda^s E \otimes (\det \mathcal{F})^*\big).$$

The pointwise stability function of [Pop26, Definition 3.2] is obtained by evaluating a smooth curvature endomorphism on $\sigma_{\mathcal{F}}$. In particular, for every smooth Hermitian metric h on E,

$$\sigma_{\mathcal{F}}(x) = 0 \quad \Longrightarrow \quad Z^{(m)}_{\omega,\Omega,h}(\mathcal{F})(x) = 0. \tag{1.1}$$

The strict conditions in [Pop26, Definition 3.4] require this function to be positive at every point for every proper coherent subsheaf.

The obstruction is already visible in complex dimension one. A sheaf injection may lose rank on a fibre even when its source is locally free. If $L \subset E$ is a line subbundle and $p \in X$, then the inclusion $L(-p) \subset E$ has generic image L but vanishes on the fibre over p. Thus the determinant section can be nonzero as a global section without being nowhere zero. In particular, the inference from [Pop26, Fact 3.1] to $|\sigma_{\mathcal{F}}|^2 > 0$ everywhere, made immediately after formula (51) in the proof of [Pop26, Theorem 6.1], is invalid.

The following is the principal result of this note.

Theorem A. *Let X be a compact connected Riemann surface and let $E \to X$ be a holomorphic vector bundle of rank at least two. Fix a Kähler form ω, put $m = 1$, and take $\Omega = 1$. There exist a proper locally free coherent subsheaf $\mathcal{F} \subset \mathcal{O}(E)$ of rank one and a point $p \in X$ such that*

$$Z^{(1)}_{\omega,1,h}(\mathcal{F})(p) = 0$$

for every smooth Hermitian metric h on E. Consequently, E is neither 1-positively stable nor uniformly 1-positively stable in the sense of [Pop26, Definition 3.4].

The proof uses only a one-point modification of a line subbundle. To turn Theorem A into counterexamples to the two cited main statements, we construct an irreducible unitary representation of the fundamental group of a genus-two curve. Its associated bundle is stable of degree zero and carries a flat Hermite–Einstein metric. It cannot satisfy the strict pointwise condition by Theorem A, and it is indecomposable by slope stability. The same bundle therefore contradicts both the implication from slope stability and the asserted Hermite–Einstein decomposition.

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The argument does not contradict the semistability assertion in [Pop26, Theorem 6.1]. It detects a defect specific to strict pointwise positivity: zeros of $\sigma_{\mathcal{F}}$ force zeros of the stability function, independently of its curvature coefficient.

2. Determinant zeros on curves

The obstruction begins with the following local calculation.

Lemma 2.1. *Let X be a Riemann surface, let $E \to X$ be a holomorphic vector bundle, and let $L \subset E$ be a holomorphic line subbundle. For a point $p \in X$, set*

$$\mathcal{F} = L(-p) \subset L \subset E.$$

Then $\mathcal{F}$ is a locally free coherent subsheaf of rank one, while its determinant section $\sigma_{\mathcal{F}} \in H^0(X, E \otimes \mathcal{F}^*)$ vanishes to order one at p.

Proof. Choose a coordinate z centred at p and a holomorphic frame $e_1, e_2, \ldots, e_r$ of E such that e_1 spans L. If f is a local frame of the abstract line bundle $\mathcal{F} = L(-p)$, its inclusion into E is

$$f \longmapsto z e_1.$$

The map is injective on stalks because z is a non-zero-divisor in $\mathcal{O}_{X,p}$, but its fibre map at p is zero. Since $\det \mathcal{F} = \mathcal{F}$, the induced determinant section is

$$\sigma_{\mathcal{F}} = z e_1 \otimes f^*.$$

It has a simple zero at p. ☐

Corollary 2.2. *In the setting of Lemma 2.1,*

$$Z^{(1)}_{\omega,1,h}(\mathcal{F})(p) = 0$$

for every smooth Hermitian metric h on E and every Kähler form ω on X.

Proof. The stability function is the value on $\sigma_{\mathcal{F}}$ of the Hermitian quadratic form displayed in [Pop26, Definition 3.2]. The conclusion follows from (1.1) and Lemma 2.1. ☐

The same calculation also shows that the literal normalized expression $Z^{(1)}_{\omega,1,h}(\mathcal{F})/|\sigma_{\mathcal{F}}|^2$ is not defined at p. Thus normalization does not remove the obstruction on the full class of coherent subsheaves.

To complete the proof of Theorem A, it remains to show that the construction of Lemma 2.1 is available for every vector bundle of rank at least two.

Lemma 2.3. *Every holomorphic vector bundle E on a compact Riemann surface contains a holomorphic line subbundle.*

Proof. Choose an effective divisor D of sufficiently large degree. Riemann–Roch gives

$$\chi(E(D)) = \deg E + \mathrm{rk}(E) \deg D + \mathrm{rk}(E)(1-g) > 0,$$

so $H^0(X, E(D)) \neq 0$. A nonzero section yields an injective sheaf morphism $\mathcal{O}_X(-D) \to \mathcal{O}(E)$. Saturate its image in $\mathcal{O}(E)$. On a smooth curve a torsion-free coherent sheaf is locally free, so the saturated image is the sheaf of sections of a holomorphic line subbundle $L \subset E$. ☐

Proof of Theorem A. Choose a line subbundle $L \subset E$ by Lemma 2.3, fix any point $p \in X$, and put $\mathcal{F} = L(-p)$. Since $\mathrm{rk}\, E \geq 2$, this is a proper coherent subsheaf. By Corollary 2.2, its stability function vanishes at p for every smooth metric h on E.

Part (a) of [Pop26, Definition 3.4] allows the metric to depend on $\mathcal{F}$, but still requires strict positivity at every point. No choice of metric meets that requirement for the present $\mathcal{F}$. Part (b), which asks for one metric working for all $\mathcal{F}$, fails a fortiori. ☐

The obstruction disappears if one replaces this particular subsheaf by its saturation, but the stated definition tests the unsaturated subsheaf as well. Indeed, the saturation of $L(-p)$ in E is L, and

$$L/L(-p) \simeq \mathbb{C}_p.$$

The strict pointwise condition sees this length-one quotient through the zero of $\sigma_{L(-p)}$, although $L(-p)$ and L determine the same line in the generic fibre.

3. A stable flat counterexample

To test the implication in [Pop26, Theorem 1.4(2)], let X be a compact Riemann surface of genus two, with presentation

$$\pi_1(X) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle,$$

where $[S, T] = STS^{-1}T^{-1}$. Set

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{in } SU(2),$$

and define

$$\rho(a_1) = A, \qquad \rho(b_1) = B, \qquad \rho(a_2) = B, \qquad \rho(b_2) = A.$$

Lemma 3.1. *The map ρ defines an irreducible unitary representation $\pi_1(X) \to SU(2)$.*

Proof. The identities $A^2 = B^2 = -\operatorname{Id}$ and $AB = -BA$ give

$$[A, B] = [B, A] = -\operatorname{Id}.$$

Thus $[A, B][B, A] = \operatorname{Id}$, so the surface relation is satisfied. The only complex lines invariant under A are $\mathbb{C}e_1$ and $\mathbb{C}e_2$, while B exchanges these two lines. Hence no complex line is invariant under the image of ρ. □

Let

$$E_\rho = (\widetilde{X} \times \mathbb{C}^2)/\pi_1(X)$$

be the associated holomorphic bundle. Its flat unitary connection is the Chern connection of the induced Hermitian metric h_ρ.

Proposition 3.2. *The bundle E_ρ has degree zero, is slope stable, and h_ρ is a Hermite–Einstein metric with Einstein factor zero.*

Proof. Flatness gives $\deg E_\rho = 0$ and zero Chern curvature, hence the Hermite–Einstein assertion. Let $M \subset E_\rho$ be a holomorphic line subbundle and let β_M be its second fundamental form with respect to h_ρ. The curvature formula for a holomorphic subbundle of a flat unitary bundle gives

$$\deg M = -c_\omega \|\beta_M\|^2_{L^2} \leq 0 \tag{3.1}$$

for a positive normalization constant c_ω. Equality in (3.1) forces $\beta_M = 0$, so M is parallel for the flat connection. Its fibre would then be a line invariant under ρ, contradicting Lemma 3.1. Therefore $\deg M < 0 = \mu(E_\rho)$ for every line subbundle M. Every rank-one coherent subsheaf is contained in its saturation, which is a line subbundle of degree at least as large. The same strict inequality therefore holds for every rank-one coherent subsheaf, proving slope stability in rank two. □

Corollary 3.3. *The implication in [Pop26, Theorem 1.4(2)] is false.*

Proof. The bundle E_ρ is slope stable by Proposition 3.2. By Theorem A, no smooth Hermitian metric on E_ρ makes it uniformly 1-positively stable. □

The obstruction is not tied to the particular metric or to the chosen flat bundle. The role of E_ρ is to provide an explicit object satisfying the slope-stability hypothesis. The failure of the strict pointwise conclusion holds for every rank-at-least-two bundle on every compact Riemann surface.

The same bundle also obstructs the decomposition assertion.

Theorem 3.4. *The flat Hermite–Einstein bundle (E_ρ, h_ρ) does not admit the decomposition asserted in [Pop26, Theorem 6.1]. Consequently, the decomposition assertion of that theorem is false.*

Proof. By Theorem A, (E_ρ, h_ρ) is not uniformly stable. Suppose first that $E_\rho = E_1 \oplus E_2$ is a nontrivial holomorphic direct sum. Slope stability gives

$$\mu(E_1) < \mu(E_\rho) \quad \text{and} \quad \mu(E_2) < \mu(E_\rho),$$

whereas the weighted-average identity for the slope of a direct sum gives

$$\mu(E_\rho) = \frac{\operatorname{rk}(E_1)\mu(E_1) + \operatorname{rk}(E_2)\mu(E_2)}{\operatorname{rk}(E_1) + \operatorname{rk}(E_2)}.$$

This is impossible. Thus E_ρ is indecomposable. A decomposition with only one summand would require E_ρ itself to be uniformly stable, which has already been excluded. □

By contrast, the uniform semistability assertion in [Pop26, Theorem 6.1] remains valid for this example. Indeed, for every rank-one coherent subsheaf $\mathcal{F} \subset \mathcal{O}(E_\rho)$, slope stability gives $\deg \mathcal{F} < 0$, while formula (51) of that paper has the form

$$Z^{(m)}_{\omega,\Omega,h}(\mathcal{F}) = c_{E,\mathcal{F}} |\sigma_{\mathcal{F}}|^2.$$

For the flat metric on E_ρ, the coefficient $c_{E_\rho,\mathcal{F}}$ is a positive multiple of $-\deg \mathcal{F}$. Hence the right-hand side is nonnegative everywhere, even when $\sigma_{\mathcal{F}}$ vanishes. It is the passage from a pointwise zero of the product to $c_{E,\mathcal{F}} = 0$ which fails when determinant zeros are allowed.

4. Repair boundary

On a curve, the smallest repair is to restrict the test class to saturated subsheaves.

Proposition 4.1. *Let E be a slope-stable holomorphic vector bundle on a compact Riemann surface, and let h be a Hermite–Einstein metric on E. If the strict pointwise condition of [Pop26, Definition 3.4] is tested only on proper saturated coherent subsheaves, then h satisfies that condition.*

Proof. On a smooth curve, a saturated coherent subsheaf $\mathcal{F} \subset \mathcal{O}(E)$ and its quotient are locally free. Thus $\mathcal{F}$ is the sheaf of sections of a holomorphic subbundle, and $\sigma_{\mathcal{F}}$ is nowhere zero. For $s = \operatorname{rk} \mathcal{F}$, formula (51) of [Pop26] becomes

$$Z^{(1)}_{\omega,1,h}(\mathcal{F}) = c\, s(\mu(E) - \mu(\mathcal{F}))|\sigma_{\mathcal{F}}|^2$$

with $c > 0$ depending only on the curvature normalization and the volume of X. Slope stability makes the coefficient positive, so the right-hand side is positive everywhere. □

Saturation is not sufficient in higher dimension. On $\mathbb{P}^2$, with homogeneous coordinates $[x : y : z]$ and $p = [0 : 0 : 1]$, the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(x,y)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{(-y,x)} \mathcal{I}_p(1) \longrightarrow 0 \tag{4.1}$$

is exact; it is the Koszul sequence of the regular sequence (x, y), twisted by $\mathcal{O}_{\mathbb{P}^2}(1)$. The quotient $\mathcal{I}_p(1)$ is torsion-free, so the inclusion is saturated, but its determinant section (x, y) vanishes at p. A higher-dimensional repair must therefore either restrict to genuine holomorphic subbundles with locally free quotient or replace the smooth pointwise formalism by one adapted to reflexive determinants and singular subsheaves. The present argument establishes neither of those extended theories; it identifies the boundary which any such extension must respect.

References

[Pop26] D. Popovici. *m-Positive Stability of Holomorphic Vector Bundles and Moduli Spaces*. July 19, 2026. arXiv: 2607.17203v1 [math.DG]. URL: https://arxiv.org/abs/2607.17203v1 (visited on 09/19/2026).