A Proposed Construction of a Complex Structure on the Six-Sphere

A streamlined reconstruction of the S^6 argument, revised

Reconstructed from the source manuscript [1]

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Abstract

We reconstruct, in a streamlined form, the chain of arguments of the source manuscript [1] that is claimed to produce an integrable complex structure on the standard smooth six-sphere. The object is a compact complex threefold X fibred by complex two-tori over $\mathbb{P}^1$, with a reduced normal-crossings fibre over the cusp of the $(3, 4, \infty)$ orbifold and multiple fibres of multiplicities 3 and 4 over its two elliptic points. We give the period family, its three analytic fillings, the fundamental-group computation, the integral nearby-cycle and Leray computations, and the recognition of X as S^6. Compared with the previous version of this reconstruction, the rational homology is obtained by a short argument that bypasses most of the transgression bookkeeping, the integral generator of the base transgression is obtained from torsion-freeness alone, the degree-two specialization index at the order-four point is obtained from the parity of an intersection form, and the sign conventions in the multiplicative Leray argument are shown to be immaterial. A final section records two consistency checks on the complex-analytic invariants of X.

Status of the argument

This is a reconstruction of a proposed solution, not an independently refereed resolution of the Hopf problem. In preparing this version, all finite integral data used below (monodromy matrices, invariant and coinvariant lattices, Smith normal forms, specialization indices, parabolic cohomology) and all transformation identities (the laws for τ, μ, β, the cocycle sums, the equivariance $R_g\Pi = \Pi \circ g \cdot A_g$, the invariance of D) were verified by computer algebra, and the analytic and topological arguments were re-derived; no gap was found. The statements that carry the weight of the argument, and on which a reader should concentrate, are Theorem 2.1 (existence of the period functions), Theorem 3.2 (properness of the cusp filling), Theorem 4.6 (integral invariant cycles at the cusp) and Proposition 4.9 (the multiplicative Leray argument).

Contents

1 Claim, conventions, and logical dependencies

Theorem 1.1. *The standard smooth six-sphere S^6 admits an integrable complex structure.*

The proof has four blocks: an analytic construction (the first two) and an integral-topological verification (the last two).

1. Construct a compact complex threefold $f : X \to B$ over

$$B = \mathbb{P}^1, \qquad \Sigma = \{p_0, p_1, p_2\}, \qquad (m_1, m_2) = (3, 4),$$

   whose general fibre is a complex two-torus, whose fibre over p_0 is reduced with normal crossings, and whose fibres over p_1, p_2 have multiplicities three and four with smooth reductions.

2. Prove $\pi_1(X) = 1$ by van Kampen's theorem, using fixed logarithmic translations at the two elliptic fibres.

3. Prove $H_k(X; \mathbb{Z}) \cong H_k(S^6; \mathbb{Z})$ by the integral Leray spectral sequence, using nearby cycles at the cusp and a multiplicativity argument.

4. Hurewicz–Whitehead gives $X \simeq S^6$, and $\Theta_6 = 0$ gives $X \cong_{\text{diff}} S^6$. Transporting the holomorphic atlas of X along this diffeomorphism produces an integrable complex structure on S^6.

Conventions. Deck transformations act on the left and compose as maps, so gh means "first h, then g"; loops in the base are composed from left to right. The three small meridians in the base are oriented clockwise; reversing all three orientations changes every boundary coefficient below by the same sign and leaves every conclusion unchanged. Throughout, $e(\cdot)$ denotes the topological Euler characteristic and $\gamma \wedge u \wedge w \wedge \delta = \gamma \wedge u \wedge w \wedge \delta$ the generator of $\Lambda^4 V$ fixed in Section 2.1; the relation between $\gamma \wedge u \wedge w \wedge \delta$ and the complex orientation of a fibre is recorded in Remark 2.2 and is never needed.

2 The torus family over the punctured sphere

2.1 Monodromy and the cusp lattice

Let

$$V = \mathbb{Z}^4 = \langle \gamma, u, w, \delta \rangle, \qquad \Lambda = V^* = \langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta} \rangle.$$

V will be the first integral cohomology of a fibre and Λ its first homology. Define the following matrices, whose columns are the images of the basis vectors:

$$T_1 = \begin{pmatrix} 1 & 0 & -6 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 6 & 0 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Direct multiplication gives

$$T_1^3 = T_2^4 = I, \qquad T_2^2 \neq I, \qquad T_0 := (T_1 T_2)^{-1} = I + N, \qquad N^2 = 0,$$

with

$$N\gamma = Nu = 0, \qquad Nw = -u, \qquad N\delta = \gamma, \qquad \ker N = \operatorname{Im} N = \langle \gamma, u \rangle.$$

On Λ set $A_j = (T_j^{-1})^{\mathsf{t}}$ for $j = 1, 2$ and $M_0 = (T_0^{-1})^{\mathsf{t}}$, so that

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

and $A_1 A_2 M_0 = I$. Since the first row of each A_j is $(1, 0, 0, 0)$, the linear form

$$\gamma : \Lambda \to \mathbb{Z}, \qquad \gamma(a\widehat{\gamma} + b\widehat{u} + c\widehat{w} + d\widehat{\delta}) = a,$$

is invariant under A_1, A_2, M_0; equivalently $T_j\gamma = \gamma$ for $j = 0, 1, 2$.

Set

$$\Lambda_{\mathrm{tor}} := \langle \widehat{w}, \widehat{\delta} \rangle, \qquad \bar{\Lambda} := \Lambda/\Lambda_{\mathrm{tor}} = \langle \bar{\gamma}, \bar{u} \rangle.$$

Since $(M_0 - I)(a\widehat{\gamma} + b\widehat{u} + c\widehat{w} + d\widehat{\delta}) = b\widehat{w} - a\widehat{\delta}$,

$$\ker(M_0 - I) = \mathrm{Im}(M_0 - I) = \Lambda_{\mathrm{tor}},$$

and $M_0 - I$ induces the unimodular isomorphism

$$B_0 : \bar{\Lambda} \xrightarrow{\sim} \Lambda_{\mathrm{tor}}, \qquad B_0\bar{\gamma} = -\widehat{\delta}, \quad B_0\bar{u} = \widehat{w}, \qquad [B_0] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2.2 Orbifold uniformization

Let $B = \mathbb{P}^1$ have coordinate t, and set

$$p_1 = (t = 0), \qquad p_2 = (t = 1), \qquad p_0 = (t = \infty), \qquad t_1 = t, \quad t_2 = t - 1, \quad t_c = t^{-1}.$$

The orbifold Euler characteristic of the signature $(0; 3, 4, \infty)$ is $2 - \frac{2}{3} - \frac{3}{4} - 1 = -\frac{5}{12} < 0$, so the orbifold is uniformized by the upper half-plane $\mathfrak{h}$ [2, Ch. 10]: there are a Fuchsian group

$$\Delta = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1 \rangle \cong \mathbb{Z}/3 * \mathbb{Z}/4, \qquad g_0 := (g_1 g_2)^{-1},$$

and a holomorphic quotient map $\pi : \mathfrak{h} \to B \setminus \{p_0\}$. The element g_j fixes a point $z_j \in \pi^{-1}(p_j)$, and there is a local coordinate s_j with

$$s_j(z_j) = 0, \qquad s_j \circ g_j = \zeta_j s_j, \quad \zeta_j := e^{-2\pi i/m_j}, \qquad t_j \circ \pi = s_j^{m_j}.$$

The element g_0 is a primitive parabolic element. A sufficiently small cusp neighbourhood U_r is simply connected, precisely invariant under its stabilizer $\langle g_0 \rangle$, and $U_r/\langle g_0 \rangle \cong \{0 < |t_c| < r\}$.

With the clockwise convention of Section 1, g_1, g_2 are the clockwise meridians around p_1, p_2 and the relation $g_0 = (g_1 g_2)^{-1}$ makes g_0 the clockwise meridian around p_0.

2.3 The period-function theorem

Theorem 2.1. *There exist holomorphic functions*

$$\tau : \mathfrak{h} \to \mathfrak{h}, \qquad \mu, \beta : \mathfrak{h} \to \mathbb{C}$$

satisfying

$$j(\tau) = 1728\, t \circ \pi, \tag{1}$$

$$\tau \circ g_1 = \frac{\tau - 1}{\tau}, \qquad \tau \circ g_2 = -\frac{1}{\tau}, \qquad \tau \circ g_0 = \tau - 1, \tag{2}$$

$$\mu \circ g_1 = \frac{1 - \mu}{\tau}, \qquad \mu \circ g_2 = 1 + \frac{\mu}{\tau}, \qquad \mu \circ g_0 = \mu, \tag{3}$$

$$\beta \circ g_1 = \beta + 2 - \frac{6(1-\mu)^2}{\tau}, \qquad \beta \circ g_2 = \beta - 3 - \frac{6\mu^2}{\tau}, \qquad \beta \circ g_0 = \beta + 1. \tag{4}$$

Write $\rho := e^{i\pi/3}$. Then $\tau(z_1) = \rho$ and $\tau(z_2) = i$; on U_r the functions μ and $\beta + \tau$ are holomorphic functions of t_c that extend over $t_c = 0$. After adding a suitable constant to β one can moreover arrange

$$D := \operatorname{Im} \beta - \frac{6(\operatorname{Im} \mu)^2}{\operatorname{Im} \tau} < 0 \qquad \textit{everywhere on } \mathfrak{h}. \tag{5}$$

The three transformation laws define an action of Δ on (τ, μ, β)-space: the assignments for g_1 and g_2 have orders exactly 3 and 4, and the composite $g_1 g_2$ acts by $(\tau, \mu, \beta) \mapsto (\tau + 1, \mu, \beta - 1)$, whose inverse is the law stated for g_0. These facts are verified by direct substitution.

Proof. Step 1: construction of τ. Let $\mathfrak{h}^\circ$ be $\mathfrak{h}$ with the $\mathrm{PSL}_2(\mathbb{Z})$-orbits of ρ and i removed. The modular function j restricts to a regular covering $\mathfrak{h}^\circ \to \mathbb{C} \setminus \{0, 1728\}$ with deck group $\mathrm{PSL}_2(\mathbb{Z})$, in which the meridian around 0 maps to an element of order three and the meridian around 1728 to an element of order two. Put $F := 1728\, t \circ \pi$. Since $t_1 \circ \pi = s_1^3$ and $t_2 \circ \pi = s_2^4$, the function F has zeros of order three at the points of Δz_1 and $F - 1728$ has zeros of order four at the points of Δz_2; these are the only points where $F \in \{0, 1728\}$. Hence a small meridian around a point of Δz_1 maps under F to the cube of an order-three element, and one around a point of Δz_2 to the fourth power of an order-two element; both are trivial in the deck group. Because $\mathfrak{h}$ is simply connected, π_1 of the complement of the discrete set $\Delta z_1 \cup \Delta z_2$ is normally generated by these small meridians, so every loop has trivial image in the deck group, and the lifting criterion gives a holomorphic τ on the complement with $F = j \circ \tau$. In the disc model of $\mathfrak{h}$ the function τ is bounded, so it extends holomorphically over $\Delta z_1 \cup \Delta z_2$, and by the open mapping theorem the extension takes values in $\mathfrak{h}$. This proves (1). Comparing orders of vanishing, $\tau - \rho$ has a simple zero at z_1 and $\tau - i$ has a double zero at z_2.

For $g \in \Delta$, both τ and $\tau \circ g$ lift F, so $\tau \circ g = \phi(g) \circ \tau$ for a homomorphism $\phi : \Delta \to \mathrm{PSL}_2(\mathbb{Z})$. If $\phi(g_1)$ were trivial, $\tau - \rho$ would be a function of s_1^3, contradicting its simple zero; so $\phi(g_1)$ has order three and fixes ρ, and after conjugation we may take $\phi(g_1) : \tau \mapsto (\tau - 1)/\tau$. Likewise $\phi(g_2)$ is nontrivial (else $\tau - i$ would be a function of s_2^4, contradicting its double zero), hence of order two. Put $P := \phi(g_0)$. For R large every component of $\{|j| > R\}$ is a horodisc neighbourhood of one cusp; $\tau(U_r)$ lies in one such component, which P preserves, so P fixes a cusp and is parabolic or trivial. If P were trivial, $q = e^{2\pi i \tau}$ would descend to the punctured disc $U_r / \langle g_0 \rangle$ and there would have a single-valued logarithm, whereas $j = q^{-1}(1 + O(q))$ and $j = 1728/t_c$ force q to have winding number one around $t_c = 0$. So P is parabolic; write

$$P = I + k \begin{pmatrix} -pr & p^2 \\ -r^2 & pr \end{pmatrix}, \qquad \gcd(p, r) = 1, \quad k \neq 0.$$

The element $\phi(g_2) = \phi(g_1)^{-1} P^{-1}$ has trace zero, which reads $-k(p^2 - pr + r^2) = 1$; positive definiteness of $p^2 - pr + r^2$ forces $k = -1$ and $(p, r) \in \{\pm(1,0), \pm(0,1), \pm(1,1)\}$. These three cusps are permuted by $\phi(g_1)$, so after conjugating by a power of $\phi(g_1)$ (which does not change $\phi(g_1)$) we may take $P:\tau\mapsto\tau-1$, and then $\phi(g_2):\tau\mapsto -1/\tau$. This proves (2). At the fixed points, $\phi(g_1)$ has derivative $\rho^{-2}=\zeta_1$ at ρ and $\phi(g_2)$ has derivative $i^{-2}=\zeta_2^2$ at i, in agreement with the simple and double zeros.

On U_r the function $q=e^{2\pi i\tau}$ is g_0-invariant, hence a function of t_c, and $j=q^{-1}(1+O(q))=1728/t_c$ gives $q=t_c\,u_1(t_c)$ with u_1 holomorphic and nonvanishing at 0. Set

$$h:=(2\pi i)^{-1}\log u_1(t_c\circ\pi),\qquad s:=\tau-h.$$

Then $e^{2\pi is}=t_c\circ\pi$ and $s\circ g_0=s-1$, with h a holomorphic function of t_c at 0.

Step 2: construction of μ. Define factors of automorphy on the generators by

$$\widetilde{\jmath}(g_1,z)=-\tau(z),\qquad \widetilde{\jmath}(g_2,z)=\tau(z),$$

and extend by the cocycle rule $\widetilde{\jmath}(gh,z)=\widetilde{\jmath}(g,hz)\widetilde{\jmath}(h,z)$. Using (2), the factors of g_1^3 and g_2^4 equal 1, so the extension is consistent, and $\widetilde{\jmath}(g_0,\cdot)=1$. Let $\mathcal{L}$ be the sheaf on B whose sections over U are the holomorphic functions ν on $\pi^{-1}(U\setminus\{p_0\})$ with

$$\nu\circ g=\nu/\widetilde{\jmath}(g,\cdot)\quad(g\in\Delta),$$

subject, when $p_0\in U$, to the requirement that $\nu|_{U_r}$ (which is g_0-invariant, hence a function of t_c) extend holomorphically over $t_c=0$. Near z_j a section satisfies $\nu\circ g_j=e_j\nu$ with e_j a unit whose value at z_j is $-1/\rho=\zeta_1^2$, respectively $1/i=\zeta_2$; after dividing by a unit κ_j with $\kappa_j\circ g_j/\kappa_j=e_j/e_j(z_j)$ (such κ_j exists because $H^1(\mathbb{Z}/m_j,\mathcal{O}_{z_j})=0$), the sections become $s_j^{a_j}$ times functions of $s_j^{m_j}$ with $a_1=2$, $a_2=1$. Hence $\mathcal{L}$ is an invertible sheaf on B, with local generators of s_j-order a_j at p_j and generator 1 at p_0.

We now exhibit a meromorphic section. The form E_6 has a simple zero at i and $\tau-i$ has a double zero at z_2, so the divisor of $E_6\circ\tau$ on the simply connected surface $\mathfrak{h}$ is even and $H:=(E_6\circ\tau)^{1/2}$ exists, unique up to sign, with a simple zero at each point of Δz_2 and no other zeros. For both transformations in (2) the modular factor $c\tau+d$ equals τ, so the weight-six law of E_6 gives $H\circ g_j=\chi_j\tau^3H$ with $\chi_j=\pm1$. Evaluating at z_1, where $H\neq0$ and $\rho^3=-1$, gives $\chi_1=-1$; at z_2 the simple zero of H transforms by $s_2\mapsto\zeta_2s_2=-is_2$ while $i^3=-i$, so $\chi_2=1$. Put

$$F_{\mathrm{mod}}:=\frac{(E_4^2\circ\tau)\,H}{\Delta_{\mathrm{mod}}\circ\tau},$$

of weight $8+3-12=-1$. The characters just computed give exactly

$$F_{\mathrm{mod}}\circ g_1=-\frac{F_{\mathrm{mod}}}{\tau}=\frac{F_{\mathrm{mod}}}{\widetilde{\jmath}(g_1,\cdot)},\qquad F_{\mathrm{mod}}\circ g_2=\frac{F_{\mathrm{mod}}}{\tau}=\frac{F_{\mathrm{mod}}}{\widetilde{\jmath}(g_2,\cdot)},$$

so F_{mod} is a meromorphic section of $\mathcal{L}$. Its zeros are the points over p_1, of order 2 (from E_4^2), and over p_2, of order 1 (from H); these orders equal a_1,a_2, so the coarse orders at p_1,p_2 are $(2-a_1)/3=0$ and $(1-a_2)/4=0$. At the cusp, E_4 and E_6 are units and $\Delta_{\mathrm{mod}}\circ\tau=q(1+O(q))=t_c\cdot(\mathrm{unit})$, so F_{mod} has coarse order -1. Therefore $\mathcal{L}\cong\mathcal{O}_{\mathbb{P}^1}(-p_0)$ and $H^1(B,\mathcal{L})=0$.

The local solutions of (3) form an $\mathcal{L}$-torsor: two solutions differ by a section of $\mathcal{L}$, and adding a section of $\mathcal{L}$ to a solution gives a solution. Local solutions exist everywhere: 0 on one sheet over an ordinary point, extended to the other sheets by the laws; 0 on U_r at the cusp (consistent because $\mu\circ g_0=\mu$); and near z_1,z_2 respectively

$$\mu_1=\frac{2-\tau}{3},\qquad \mu_2=\frac{1-\tau}{2},$$

which satisfy the laws by direct substitution. Since $H^1(B,\mathcal{L})=0$, a global solution μ exists, and on U_r it differs from 0 by a section of $\mathcal{L}$, so it is a holomorphic function of t_c extending over $t_c=0$.

Step 3: construction of β. Set

$$\varphi_1 = 2 - \frac{6(1-\mu)^2}{\tau}, \qquad \varphi_2 = -3 - \frac{6\mu^2}{\tau}.$$

Substituting (2)–(3) repeatedly gives

$$\sum_{k=0}^{2} \varphi_1 \circ g_1^k = 0, \qquad \sum_{k=0}^{3} \varphi_2 \circ g_2^k = 0,$$

so the equations $\beta \circ g_j - \beta = \varphi_j$ are compatible with $g_j^{m_j} = 1$, and near z_j

$$\beta := \frac{1}{m_j} \sum_{k=0}^{m_j-1} k\,(\varphi_j \circ g_j^k)$$

is a local solution (a one-line verification using the vanishing sums). Take 0 on one sheet over an ordinary point and $\beta = -\tau$ on U_r at the cusp. Local solutions form an $\mathcal{O}_B$-torsor (at the cusp, require $\beta + \tau$ to extend over $t_c = 0$); the obstruction lies in $H^1(\mathbb{P}^1, \mathcal{O}) = 0$, so a global β exists, unique up to an additive constant, and $b := \beta + \tau$ is a holomorphic function of t_c on U_r extending over $t_c = 0$.

Step 4: nondegeneracy. For $\tau \in \mathfrak{h}$ and $a \in \mathbb{C}$ one has the identity

$$\operatorname{Im}\Big(-\frac{6a^2}{\tau}\Big) - \frac{6\,\operatorname{Im}(a/\tau)^2}{\operatorname{Im}(-1/\tau)} = -\frac{6(\operatorname{Im} a)^2}{\operatorname{Im}\tau}, \tag{6}$$

checked by writing $\tau = x + iy$, $a = r + is$ and multiplying by $|\tau|^2 y$ (both sides become $-6s^2|\tau|^2$). Applying (6) with $a = \mu$ for g_2 and $a = 1 - \mu$ for g_1, and noting that $\operatorname{Im}((\tau-1)/\tau) = \operatorname{Im}(-1/\tau)$, $\operatorname{Im}(1-\mu) = -\operatorname{Im}\mu$ and that the remaining constants in (3)–(4) are real, one finds $D \circ g_j = D$ for $j = 1, 2$. Thus D is Δ-invariant and descends to a continuous function on $B \setminus \{p_0\}$. On U_r, $D = \operatorname{Im} b - \operatorname{Im}\tau - 6(\operatorname{Im}\mu)^2/\operatorname{Im}\tau$ with b, μ bounded and $\operatorname{Im}\tau = \operatorname{Im} s + \operatorname{Im} h \to +\infty$, so $D \to -\infty$ at p_0. Hence D is bounded above by some M, and replacing β by $\beta + c_0$ with $\operatorname{Im} c_0 < -M$ gives (5). $\square$

2.4 The period matrix, its lattice, and equivariance

Define

$$\Pi(z) = \begin{pmatrix} 6\mu(z) & \tau(z) & 1 & 0 \\ \beta(z) & \mu(z) & 0 & 1 \end{pmatrix} = [Z(z) \mid I_2], \qquad Z = \begin{pmatrix} 6\mu & \tau \\ \beta & \mu \end{pmatrix}. \tag{7}$$

Regard $\Pi(z)$ as a real-linear map $\Lambda \otimes \mathbb{R} \to \mathbb{C}^2$. Expanding along the last two columns,

$$\det{}_{\mathbb{R}}\Pi = -\det \operatorname{Im} Z = \operatorname{Im}\tau\ \operatorname{Im}\beta - 6(\operatorname{Im}\mu)^2 = \operatorname{Im}\tau \cdot D < 0. \tag{8}$$

Thus $L(z) := \Pi(z)\Lambda$ is a full lattice in $\mathbb{C}^2$ for every z.

Let $\sigma_1 = 6\mu\gamma + \tau u + w$ and $\sigma_2 = \beta\gamma + \mu u + \delta$ be the rows of Π, viewed as elements of $V \otimes \mathbb{C}$. Substituting the transformation laws gives

$$\begin{aligned}
\sigma_1 \circ g_1 &= -\tau^{-1} T_1 \sigma_1, & \sigma_2 \circ g_1 &= T_1 \sigma_2 + (1-\mu)\tau^{-1} T_1 \sigma_1, \\
\sigma_1 \circ g_2 &= \tau^{-1} T_2 \sigma_1, & \sigma_2 \circ g_2 &= T_2 \sigma_2 - \mu\tau^{-1} T_2 \sigma_1, \\
\sigma_i \circ g_0 &= T_0 \sigma_i. & &
\end{aligned}$$

Equivalently, with

$$R_{g_1} = \begin{pmatrix} -1/\tau & 0 \\ (1-\mu)/\tau & 1 \end{pmatrix}, \quad R_{g_2} = \begin{pmatrix} 1/\tau & 0 \\ -\mu/\tau & 1 \end{pmatrix}, \quad R_{g_0} = I,$$

one has

$$R_g(z)\Pi(z) = \Pi(gz)A_g, \qquad R_{gh}(z) = R_g(hz)R_h(z), \tag{9}$$

where $A_{g_1} = A_1$, $A_{g_2} = A_2$, $A_{g_0} = M_0$ and A_g for a general word is the corresponding product; the cocycle rule follows from the first identity because $\Pi(z)$ is surjective. In the flat coordinate $x \in \Lambda \otimes \mathbb{R}$ of a fibre, $\zeta = \Pi(z)x$, the deck transformation g acts by $x \mapsto A_g x$.

Let Λ act on $\mathfrak{h} \times \mathbb{C}^2$ by $\lambda \cdot (z, \zeta) = (z, \zeta + \Pi(z)\lambda)$ and let $g \in \Delta$ act by $(z, \zeta) \mapsto (gz, R_g(z)\zeta)$. By (9) the second action normalizes the first, so

$$\mathcal{T} := (\mathfrak{h} \times \mathbb{C}^2)/\Lambda \longrightarrow \mathfrak{h}$$

is a holomorphic family of complex two-tori with a Δ-action covering the action on $\mathfrak{h}$, and with the invariant zero section. Removing the fibres over $\Delta z_1 \cup \Delta z_2$ (where Δ has fixed points) and dividing by Δ gives

$$f^\circ : J \longrightarrow B^\circ := B \setminus \{p_0, p_1, p_2\}, \tag{10}$$

a proper holomorphic submersion with fibres complex two-tori and with a holomorphic zero section. The clockwise meridians act on the first homology Λ of a fibre by A_1, A_2, M_0 and on the first cohomology V by T_1, T_2, T_0.

At the cusp, $\tau = s + h$ and $\beta = b - s - h$ give

$$Z = s\,B_0 + C(t_c), \qquad C(t_c) = \begin{pmatrix} 6\mu & h \\ b-h & \mu \end{pmatrix}, \tag{11}$$

with C holomorphic at $t_c = 0$. Hence, for $\bar\lambda \in \bar\Lambda$,

$$\exp(2\pi i Z\bar\lambda) = t_c^{B_0\bar\lambda}\, c_{\bar\lambda}(t_c), \qquad c_{\bar\lambda}(t_c) := \exp\bigl(2\pi i C(t_c)\bar\lambda\bigr), \tag{12}$$

where $t_c^{B_0\bar\lambda}$ denotes the vector with entries $t_c^{(B_0\bar\lambda)_i}$.

Remark 2.2. By (8) the basis $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$ of Λ is negatively oriented with respect to the complex orientation of a fibre. All intersection numbers below are computed with the class $\gamma \wedge u \wedge w \wedge \delta$; passing to the complex orientation changes every such number by the same sign and affects no conclusion.

3 Compactification

3.1 The cusp p_0: the infinite-fan toroidal filling

This subsection uses (11)–(12) to construct a proper complex threefold over a disc around p_0 that extends J.

3.1.1 The A_2 triangulation and the infinite fan

Let $N' = \mathbb{Z}^3$, write its elements as (y, y_3) with $y \in \mathbb{Z}^2$, and put $M' = \mathrm{Hom}(N', \mathbb{Z})$. On the algebraic torus $T_{N'} = (\mathbb{C}^*)^3$ use coordinates (x_1, x_2, t_c), the characters of the standard basis of M'. Let $\mathcal{T}_{A_2}$ be the periodic triangulation of $\mathbb{R}^2$ with vertex set $\mathbb{Z}^2$ and triangles

$$\{v, v+e_1, v+e_2\}, \qquad \{v+e_1, v+e_2, v+e_1+e_2\} \qquad (v \in \mathbb{Z}^2);$$

its edge directions are $\pm e_1, \pm e_2, \pm(e_1 - e_2)$. For every cell P (vertex, edge or triangle) put

$$\widehat{P} = \mathrm{cone}(P \times \{1\}) \subset N'_{\mathbb{R}}, \qquad \mathcal{F} = \{0\} \cup \{\widehat{P}\}.$$

Lemma 3.1. *$\mathcal{F}$ is a locally finite smooth rational fan with support $(\mathbb{R}^2 \times \mathbb{R}_{>0}) \cup \{0\}$. The associated toric variety $Y = Y_{\mathcal{F}}$ is a connected, Hausdorff, second-countable smooth complex threefold. The character t_c extends holomorphically to Y, and*

$$t_c^{-1}(0) = \bigcup_{v \in \mathbb{Z}^2} E_v, \qquad \operatorname{div}(t_c) = \sum_v E_v.$$

If σ is a triangle with vertices v_0, v_1, v_2, and $m_0, m_1, m_2 \in M'$ denotes the basis dual to the ray generators $(v_0, 1)$, $(v_1, 1)$, $(v_2, 1)$, then

$$U_\sigma \cong \mathbb{C}^3, \qquad z_i = \chi^{m_i}, \qquad t_c = z_0 z_1 z_2, \qquad E_{v_i} \cap U_\sigma = \{z_i = 0\}.$$

Each E_v is the toric surface dP_6 ($\mathbb{P}^2$ blown up in three points), and its toric boundary is an anticanonical hexagon of six (-1)-curves.

Proof. The generators $((v, 1), (v + e_1, 1), (v + e_2, 1))$ and $((v + e_1, 1), (v + e_2, 1), (v + e_1 + e_2, 1))$ of the three-dimensional cones have determinant ± 1, so every cone is unimodular, and intersections of triangles are common faces, so intersections of cones are common faces. Local finiteness is inherited from $\mathcal{T}_{A_2}$. Glue the charts $U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M'] \cong \mathbb{C}^3$ along the open sets of common faces. Any two points lie in the toric variety of a finite subfan, which is open in Y and separated [5, Sect. 1.4]; hence Y is Hausdorff. The charts are indexed by $\mathbb{Z}^2$, so Y is second countable. The functional $(0, 0, 1) \in M'$ takes the value 1 on every ray generator $(v, 1)$, so t_c vanishes to order one along each E_v; in a triangle chart $(0, 0, 1) = m_0 + m_1 + m_2$, whence $t_c = z_0 z_1 z_2$. The star of E_v in $N'/\mathbb{Z}(v, 1)$ has rays $e_1, e_2, e_2 - e_1, -e_1, -e_2, e_1 - e_2$ in cyclic order, the hexagonal fan of dP_6. ☐

3.1.2 The lattice action

For $\bar\lambda \in \bar\Lambda \cong \mathbb{Z}^2$ let $\phi_{\bar\lambda}(y, y_3) = (y + y_3 B_0 \bar\lambda, y_3)$. On the height-one plane this is an integral translation, so it preserves $\mathcal{F}$ and induces a toric automorphism $\Phi_{\bar\lambda}$ of Y, given on the dense torus by

$$\Phi_{\bar\lambda}(x, t_c) = (t_c^{B_0 \bar\lambda} x,\ t_c).$$

Since t_c is a holomorphic function on Y and the torus acts holomorphically, multiplication by the torus element $(c_{\bar\lambda}(t_c(p)), 1)$ is a holomorphic automorphism of Y preserving t_c; it commutes with every $\Phi_{\bar\mu}$ because $\phi_{\bar\mu}$ acts trivially on torus elements with third coordinate 1. Define

$$\Psi_{\bar\lambda}(p) = \big(c_{\bar\lambda}(t_c(p)), 1\big) \cdot \Phi_{\bar\lambda}(p). \tag{13}$$

Since $c_{\bar\lambda} c_{\bar\mu} = c_{\bar\lambda + \bar\mu}$, the map $\bar\lambda \mapsto \Psi_{\bar\lambda}$ is an action of $\bar\Lambda$ on Y by automorphisms preserving t_c; on the dense torus, $\Psi_{\bar\lambda}(x, t_c) = (\exp(2\pi i Z \bar\lambda)\, x,\ t_c)$ by (12).

Theorem 3.2. *For sufficiently small $\epsilon > 0$, the action (13) of $\bar\Lambda$ on $Y_\epsilon := \{|t_c| < \epsilon\} \subset Y$ is free and properly discontinuous. The quotient $N_0 := Y_\epsilon / \bar\Lambda$ is a Hausdorff complex threefold, and t_c descends to a proper surjection $f_0 : N_0 \to \Delta_\epsilon$. For $t_c \neq 0$, f_0 is a submersion with fibres complex two-tori. The central fibre $W = f_0^{-1}(0)$ is reduced, compact, irreducible, and has normal crossings.*

Proof. Put $R(t_c) = -2\pi \operatorname{Im} C(t_c)$ (a real 2×2 matrix) and $B_t = B_0 + R(t_c)/\log|t_c|$; then $|c_{\bar\lambda}(t_c)| = \exp(R(t_c)\bar\lambda)$ entrywise. Choose $\epsilon < \min(r/2, 1)$ with $|\log \epsilon| \geq 2 \sup_{|t_c| \leq r/2} \|R(t_c)\|$. Since B_0 is orthogonal, $\|B_t - B_0\| \leq \frac{1}{2}$ and $|B_t \bar\lambda| \geq \frac{1}{2}|\bar\lambda|$ for $|t_c| < \epsilon$.

Freeness. If $\Psi_{\bar\lambda}$ fixes a torus point (x, t_c), taking logarithms of absolute values gives $\log|t_c|\, B_0 \bar\lambda + R(t_c)\bar\lambda = 0$; the first term has norm $|\log|t_c||\, |\bar\lambda|$ and the second at most $\frac{1}{2}|\log \epsilon|\, |\bar\lambda|$, so $\bar\lambda = 0$. On the central fibre, $\Psi_{\bar\lambda}$ maps the orbit of a cell P to that of $P + B_0\bar\lambda$, and a nonzero translation moves every bounded cell.

Proper discontinuity. For a torus point p with $|t_c(p)| < 1$ put $y(p) = \log|x(p)|/\log|t_c(p)| \in \mathbb{R}^2$. In a triangle chart U_σ one has $\log|z_i| = \log|t_c|\,\theta_i^\sigma(y)$, where θ_i^σ are the barycentric coordinates with respect to σ; hence on $U_\sigma(2) := \{|z_i| < 2\}$ all $\theta_i^\sigma(y)$ exceed $-\log 2/|\log\epsilon|$ and $y(p)$ lies in a fixed bounded enlargement of σ. Moreover

$$y(\Psi_{\bar\lambda}p) = y(p) + B_t\bar\lambda. \tag{14}$$

Consequently, if $\Psi_{\bar\lambda}U_\sigma(2) \cap U_{\sigma'}(2)$ contains a point with $t_c \neq 0$, then $|\bar\lambda|$ is bounded by a constant depending only on σ, σ'; if it contains a point with $t_c = 0$, the translated cell $P + B_0\bar\lambda$ of that point shares a face with a cell of σ', which again leaves finitely many $\bar\lambda$. The open sets $U_\sigma(2)$ cover Y_ϵ: a torus point lies in the closed unit polydisc of the chart of any triangle containing $y(p)$ (see the next paragraph), and for a point of the central fibre in the orbit of a cell P the coordinates that are not forced to vanish are characters on that orbit, which can be made of modulus at most one by choosing $\sigma \supseteq P$ in the appropriate sector of the star of P. A compact subset of Y_ϵ is therefore covered by finitely many sets $U_\sigma(2)$, so it meets only finitely many of its translates. The quotient by a free properly discontinuous action is therefore a Hausdorff complex manifold and $Y_\epsilon \to N_0$ is a covering.

Properness. Fix $0 < \epsilon_1 < \epsilon$. For a torus point with $0 < |t_c| \le \epsilon_1$, choose $\bar\lambda$ with $B_t^{-1}y + \bar\lambda \in [-\frac{1}{2}, \frac{1}{2}]^2$; by (14) the translate has $y' = B_t(B_t^{-1}y + \bar\lambda)$ in a fixed disc. Let σ be a triangle containing y' and $y' = \sum \theta_i v_i$ its barycentric expression; then $\theta_i \in [0,1]$ and the chart coordinates of the translate satisfy $|z_i| = |t_c|^{\theta_i} \le 1$. Only finitely many triangles meet the fixed disc. For a point of the central fibre lying in the orbit of a cell P, choose a vertex $v_0 \in P$ and $\bar\lambda$ with $B_0\bar\lambda = -v_0$; the translated cell contains 0, so the translate lies in E_0. With

$$K_\sigma = \{p \in U_\sigma :\ |z_0|, |z_1|, |z_2| \le 2,\ |z_0z_1z_2| \le \epsilon_1\},$$

the finite union $K = E_0 \cup \bigcup_\sigma K_\sigma$ is compact and meets every $\bar\Lambda$-orbit over $|t_c| \le \epsilon_1$. Hence $f_0^{-1}(\{|t_c| \le \epsilon_1\})$ is the image of the compact set K and is compact; as $\epsilon_1 < \epsilon$ is arbitrary, f_0 is proper.

Fibres. Over $t_c = t_0 \neq 0$ the exponential covering $\mathbb{C}^2 \to (\mathbb{C}^*)^2$ identifies

$$f_0^{-1}(t_0) \cong \mathbb{C}^2/\big(\mathbb{Z}^2 + Z(s)\mathbb{Z}^2\big) = \mathbb{C}^2/L(z), \qquad e^{2\pi i s} = t_0,$$

a compact torus by (8). In the covering space the central fibre is $\{z_0z_1z_2 = 0\}$ in each chart, reduced with normal crossings, and these properties descend along the locally biholomorphic covering. Compactness of W follows from properness; irreducibility is Proposition 3.3. $\square$

3.1.3 The central fibre and its normalization

Proposition 3.3. *The map $\nu : E_0 \cong dP_6 \to W$ induced by the quotient is finite and surjective, injective away from the hexagonal boundary, and identifies the three pairs of opposite sides of the hexagon. It is the normalization of W. The double locus of W consists of three copies of $\mathbb{P}^1$ meeting exactly in two triple points P, Q, each double curve passing through both. In particular W is irreducible and connected, and*

$$e(W) = e(dP_6) - e(\mathit{hexagon}) + e(\mathit{double\ locus}) = 6 - 6 + 2 = 2.$$

Proof. Since $B_0\bar\Lambda = \mathbb{Z}^2$, the action on vertices is simply transitive, so every orbit of the central fibre is equivalent to one in E_0: ν is surjective. Two points of E_0 are identified only if they lie in orbits of cells $P, P' \ni 0$ with $P' = P + B_0\bar\lambda$, $\bar\lambda \neq 0$. For an edge the only possibility is $P = \{0, v\}$, $P' = \{0, -v\}$, giving the identification of opposite sides. The six triangles at 0 alternate between the two types and fall into two translation classes, which become the two points P, Q. So ν is injective on the open orbit and has finite fibres; being proper with finite fibres it is finite, and being bimeromorphic from the normal surface E_0 onto the reduced W it is the normalization. Each of the three edge directions gives one double curve, whose two ends are the two triangle classes, hence pass through P and Q. The Euler characteristic count is immediate: the hexagon has six rational components and six nodes, and the double locus has three rational components and two triple points. □

3.1.4 Biholomorphic identification with the punctured family

Set $U_\epsilon := \{z \in U_r : 0 < |t_c \circ \pi(z)| < \epsilon\}$ and define

$$E_0^{\exp} : U_\epsilon \times \mathbb{C}^2 \to (\mathbb{C}^*)^3, \qquad (z, \zeta) \mapsto (e^{2\pi i \zeta_1}, e^{2\pi i \zeta_2}, e^{2\pi i s(z)}).$$

It is invariant under g_0 (because $s \circ g_0 = s - 1$ and $R_{g_0} = I$) and under translation by Λ_{tor} (because $\Pi\widehat{w} = (1, 0)$ and $\Pi\widehat{\delta} = (0, 1)$), and by (12)

$$E_0^{\exp}(z, \zeta + \Pi(z)\lambda) = \Psi_{\bar\lambda}\big(E_0^{\exp}(z, \zeta)\big) \qquad (\lambda \in \Lambda).$$

It therefore descends to a holomorphic map $J|_{0<|t_c|<\epsilon} \to N_0 \setminus W$. This map is injective (two preimages of a point differ by an element of Λ_{tor}, a power of g_0, and a $\bar\Lambda$-translation, hence coincide in J), surjective (every point of Y_ϵ with $t_c \neq 0$ has logarithms), and locally biholomorphic (local logarithms give the inverse). Thus it is a biholomorphism

$$G_0 : J|_{0<|t_c|<\epsilon} \xrightarrow{\sim} N_0 \setminus W. \tag{15}$$

The zero section corresponds to $(1, 1, t_c)$. In the chart of the cone over the triangle $\{0, e_1, e_2\}$, with coordinates $(z_0, z_1, z_2) = (t_c/x_1x_2, x_1, x_2)$, it is the disc $(t_c, 1, 1)$, which extends over $t_c = 0$ and meets W transversally in the smooth open orbit of E_0; its image in N_0 is an embedded disc.

3.2 The elliptic points p_1, p_2: logarithmic transforms

Set

$$\varepsilon = \widehat{\gamma} + 2\widehat{u} - 4\widehat{w}, \qquad \varepsilon' = \widehat{\gamma} + 3\widehat{u} - 3\widehat{w}.$$

Direct multiplication gives $A_1\varepsilon = \varepsilon$ and $A_2\varepsilon' = \varepsilon'$; in fact $\Lambda^{A_1} = \langle\widehat{\delta}, \varepsilon\rangle$ and $\Lambda^{A_2} = \langle\widehat{\delta}, \varepsilon'\rangle$. Choose

$$v_1 := \varepsilon, \qquad v_2 := -\varepsilon', \qquad \text{so that} \qquad \gamma(v_1) = 1, \quad \gamma(v_2) = -1.$$

Fix $j \in \{1, 2\}$, a g_j-invariant simply connected disc $\Delta_j \ni z_j$ on which s_j is a coordinate, and let $J_j' = \mathcal{T}|_{\Delta_j}$. For $b \in \Lambda \otimes \mathbb{R}$ define

$$\widetilde{g}_j^{\,b}(z, \zeta) = \big(g_j z,\ R_{g_j}(z)\zeta + \Pi(g_j z)b\big).$$

Because $R_{g_j}\Pi = \Pi \circ g_j \cdot A_j$, this descends to a biholomorphic automorphism of J_j' which in the flat coordinate $x \in (\Lambda \otimes \mathbb{R})/\Lambda$ reads $(z, x) \mapsto (g_j z, A_j x + b)$. Put

$$\widetilde{g}_j^{\log} := \widetilde{g}_j^{\,v_j/m_j}. \tag{16}$$

Since $A_j v_j = v_j$, the m_j-th power of $\widetilde{g}_j^{\log}$ is $(z, x) \mapsto (z, x + v_j)$, the identity of J_j'; here $R_{g_j^{m_j}} = I$ because $R_{g_j^{m_j}}\Pi(z) = \Pi(z)A_j^{m_j} = \Pi(z)$ and $\Pi(z)$ is surjective. So $\langle\widetilde{g}_j^{\log}\rangle \cong \mathbb{Z}/m_j$.

Lemma 3.4. *The group $\langle\widetilde{g}_j^{\log}\rangle$ acts freely on J_j', for $j = 1, 2$.*

Proof. A fixed point of a nontrivial power lies over z_j, since g_j acts freely on $\Delta_j \setminus \{z_j\}$. Over z_j the k-th power is the affine map $x \mapsto A_j^k x + kv_j/m_j$ of $(\Lambda \otimes \mathbb{R})/\Lambda$. An affine map $x \mapsto Ax + b$ of finite order has a fixed point if and only if $\phi(b) \in \mathbb{Z}$ for every A-invariant $\phi \in V = \Lambda^*$: indeed, a fixed point exists iff $-b \in (A - I)(\Lambda \otimes \mathbb{R}) + \Lambda$, a closed subgroup whose annihilator under Pontryagin duality consists exactly of the integral linear forms vanishing on $\mathrm{Im}(A - I)$, i.e. of V^A. The form γ is invariant under every power of A_1 and A_2, and

$$\gamma(kv_1/3) = k/3 \notin \mathbb{Z} \ (k = 1, 2), \qquad \gamma(kv_2/4) = -k/4 \notin \mathbb{Z} \ (k = 1, 2, 3).$$

Hence no nontrivial power has a fixed point. □

Define

$$N_j := J_j'/\langle \widetilde{g}_j^{\log} \rangle, \qquad f_j : N_j \to D_j := \Delta_j/\langle g_j \rangle.$$

The function $s_j^{m_j}$ descends to t_j, and

$$f_j^*(p_j) = m_j S_j, \qquad S_j = \big(\mathbb{C}^2/L(z_j)\big)/\langle x \mapsto A_j x + v_j/m_j \rangle.$$

By Lemma 3.4, N_j is a complex manifold and S_j a smooth compact complex surface; S_j is a free quotient of a torus by $\mathbb{Z}/m_j$, so $e(S_j) = 0$.

To glue N_j to J, let

$$\sigma_j(z) = \frac{\log s_j(z)}{2\pi i}\, \Pi(z) v_j, \tag{17}$$

a section of J_j' over Δ_j^*: different branches of the logarithm change σ_j by $k\,\Pi(z)v_j$, which is trivial in the torus. Let h_j be fibrewise translation by σ_j. Using $\log s_j \circ g_j = \log s_j - 2\pi i/m_j$, $R_{g_j}\Pi = \Pi \circ g_j \cdot A_j$ and $A_j v_j = v_j$, the translation part of $h_j \widetilde{g}_j^{\log} h_j^{-1}$ is

$$-R_{g_j}(z)\sigma_j(z) + \Pi(g_j z)\frac{v_j}{m_j} + \sigma_j(g_j z) = \Big(-\frac{\log s_j}{2\pi i} + \frac{1}{m_j} + \frac{\log s_j}{2\pi i} - \frac{1}{m_j}\Big)\Pi(g_j z)v_j = 0,$$

so $h_j \widetilde{g}_j^{\log} h_j^{-1} = \widetilde{g}_j^0$. Since $J_j'|_{\Delta_j^*}/\langle \widetilde{g}_j^0 \rangle = J|_{D_j^*}$ by construction of J, this yields a biholomorphic identification

$$G_j : N_j \setminus S_j \xrightarrow{\sim} J|_{D_j^*} \tag{18}$$

compatible with the projections to D_j^*, under which the zero section of J corresponds to the image of the section $-\sigma_j$.

Finally, N_j is aspherical (a free quotient of $J_j' \simeq T^4$) with

$$\pi_1(N_j) = \pi_1(S_j) = \Gamma_j := \big\langle \Lambda, \widehat{g}_j \ \big|\ \widehat{g}_j \lambda \widehat{g}_j^{-1} = A_j \lambda,\ \widehat{g}_j^{m_j} = t_{v_j} \big\rangle, \tag{19}$$

where t_λ denotes the loop $\lambda \in \Lambda = \pi_1(\text{fibre})$ and $\widehat{g}_j$ is the class of a path from a base point p to $\widetilde{g}_j^{\log}(p)$.

3.3 The global compact complex threefold

Choose pairwise disjoint small discs D_0, D_1, D_2 around p_0, p_1, p_2, and use (15) and (18) to glue

$$X = \big(J \sqcup N_0 \sqcup N_1 \sqcup N_2\big)/\!\sim . \tag{20}$$

All transition maps are biholomorphic and commute with the projections to B, so they define a holomorphic map $f : X \to B$.

Theorem 3.5. *X is a connected compact Hausdorff complex threefold, and $f : X \to \mathbb{P}^1$ is a proper surjection with connected fibres. Over B° the fibres are complex two-tori; $f^{-1}(p_0) = W$ is reduced with normal crossings; $f^*(p_j) = m_j S_j$ for $j = 1, 2$. Moreover $e(X) = e(W) = 2$.*

Proof. Because the discs are disjoint, every nontrivial equivalence class consists of one point of an annular overlap and its image in the corresponding filling; saturations of open sets are open, so the four pieces embed as open subsets, and the transition functions are $G_0^{\pm 1}, G_j^{\pm 1}$. Hence X is a complex manifold. Two points with different images in B are separated by preimages of disjoint discs; two points with the same image lie together in one of the Hausdorff pieces. So X is Hausdorff. Properness of f is local on the base: f° is a torus bundle, f_0 is proper by Theorem 3.2, and each f_j is a finite quotient of the proper family $J'_j \to \Delta_j$. Since B is compact, X is compact. The four pieces are connected and meet along the annular overlaps, so X is connected; the fibres are tori, finite quotients of tori, or W, all connected. Finally $e(X) = e(B^\circ)e(F) + e(W) + e(S_1) + e(S_2) = 0 + 2 + 0 + 0$, using that N_i deformation retracts onto its central fibre and $e(F) = e(S_j) = 0$. □

4 Integral topology and recognition

4.1 The integral arithmetic ledger

Let F be a general fibre, so $H^q(F;\mathbb{Z}) = \Lambda^q V$, and let $\pi_j : F \to S_j$ be the unramified covering of degree m_j. Set

$$\eta_1 = 2u + w + 3\delta, \qquad \eta_2 = u + w + 2\delta, \qquad q = u \wedge w + 6\,\gamma \wedge \delta.$$

A sublattice is *saturated* when its quotient in the ambient lattice is torsion-free. Directly from the monodromy matrices,

$$V^{T_1} = \langle \gamma, \eta_1 \rangle, \quad V^{T_2} = \langle \gamma, \eta_2 \rangle, \quad V^{T_0} = \langle \gamma, u \rangle, \quad V^{\langle T_1, T_2 \rangle} = \mathbb{Z}\gamma,$$

and, writing γu for $\gamma \wedge u$ etc.,

$$\begin{aligned} &(\Lambda^2 V)^{T_0} = \langle \gamma u, \gamma\delta, uw, \gamma w - u\delta \rangle, && (\Lambda^2 V)^{T_j} = \langle \gamma\eta_j, q \rangle, && (\Lambda^2 V)^{\langle T_1, T_2 \rangle} = \mathbb{Z}q, \\ &(\Lambda^3 V)^{T_0} = \langle \gamma uw, \gamma u\delta \rangle, && (\Lambda^3 V)^{T_j} = \langle \gamma uw, \zeta_j \rangle, && (\Lambda^3 V)^{\langle T_1, T_2 \rangle} = \mathbb{Z}\,\gamma uw, \end{aligned}$$

with $\zeta_1 = -uw\delta + 2\gamma w\delta + 4\gamma u\delta$ and $\zeta_2 = -uw\delta + 3\gamma w\delta + 3\gamma u\delta$. All these invariant lattices are saturated by construction (kernels of integral matrices).

Put $G := \langle T_1, T_2 \rangle$. In the ordered basis $(\gamma u, \gamma w, \gamma\delta, uw, u\delta, w\delta)$ of $\Lambda^2 V$, integral row and column reduction of the block matrices

$$C_V = [T_1 - I \mid T_2 - I], \qquad C_2 = [\Lambda^2 T_1 - I \mid \Lambda^2 T_2 - I]$$

gives nonzero Smith factors $(1,1,1)$ and $(1,1,1,1,1)$ respectively, and column spans

$$(T_1 - I)V + (T_2 - I)V = \langle \gamma, u, w \rangle, \qquad V_G = \mathbb{Z}[\bar\delta], \tag{21}$$

$$\operatorname{Im}(\Lambda^2 T_1 - I) + \operatorname{Im}(\Lambda^2 T_2 - I) = \langle \gamma u, \gamma w, u\delta, w\delta, uw - 6\gamma\delta \rangle, \qquad (\Lambda^2 V)_G = \mathbb{Z}[\gamma\delta], \tag{22}$$

so the coinvariants are free of rank one, with $[uw] = 6[\gamma\delta]$ and $[q] = 12[\gamma\delta]$. The Smith factors of $M_0 - I$, $\Lambda^2 M_0 - I$, $\Lambda^3 M_0 - I$ are $(1,1)$, $(1,1)$, $(1,1)$, so the cusp invariant lattices have ranks $1,2,4,2$ in degrees $0,\dots,3$ and the cusp coinvariants V_{T_0}, $(\Lambda^2 V)_{T_0}$ are free.

Finally, the $\langle A_1, A_2 \rangle$-invariant closure of $\Lambda_{\mathrm{tor}} = \langle \widehat{w}, \widehat{\delta} \rangle$ is $\ker\gamma = \langle \widehat{u}, \widehat{w}, \widehat{\delta} \rangle$: the closure is contained in $\ker\gamma$ because γ is invariant, and it contains $\widehat{u}$ because $A_1\widehat{w} = \widehat{u} - \widehat{w}$.

4.2 The fundamental group of the cusp filling

Lemma 4.1. *Y_ϵ is simply connected; hence $\pi_1(N_0) = \bar\Lambda \cong \mathbb{Z}^2$, the inclusion of a fibre induces the projection $\Lambda \to \bar\Lambda$ on π_1, and the cusp meridian lifted along the extended zero section is trivial in $\pi_1(N_0)$.*

Proof. Exhaust $\mathcal{F}$ by connected finite subfans containing the rays $(0,1),(e_1,1),(e_2,1)$. The fundamental group of a smooth toric variety is N' modulo the sublattice generated by its ray generators [5, Sect. 3.2], and these three rays generate N', so every finite stage is simply connected; a loop in Y lies in a finite stage, so $\pi_1(Y)=1$. To contract a loop $\alpha \subset Y_\epsilon$ inside Y_ϵ, take a null-homotopy $H : D^2 \to Y$ and $M = \max|t_c \circ H|$; the torus element $a_r = (1,1,r)$ acts on Y with $t_c(a_r p) = r\, t_c(p)$, so for $r_0 M < \epsilon$ the map $a_{r_0} \circ H$ contracts $a_{r_0} \circ \alpha$ inside Y_ϵ, while $(s,\theta) \mapsto a_{(1-s)+sr_0}\alpha(\theta)$ is a homotopy in Y_ϵ from α to $a_{r_0} \circ \alpha$. Since $Y_\epsilon \to N_0$ is a covering with deck group $\bar{\Lambda}$, $\pi_1(N_0) = \bar{\Lambda}$. A fibre of Y_ϵ over $t_c \neq 0$ is $(\mathbb{C}^*)^2 = \mathbb{C}^2/\Lambda_{\mathrm{tor}}$, and its image in N_0 is the torus $\mathbb{C}^2/\Lambda$; the loops in Λ_{tor} die in the simply connected Y_ϵ, so the map on π_1 is $\Lambda \to \bar{\Lambda}$. The extended zero section is an embedded disc in N_0 bounded by the lifted cusp meridian. □

4.3 Boundary data and the fundamental group of X

Let a_1, a_2 be the clockwise meridians in B° around p_1, p_2, with $a_0 = (a_1 a_2)^{-1}$ the clockwise meridian around p_0, and lift them along the zero section to loops $\widehat{\rho}_1, \widehat{\rho}_2, \widehat{\rho}_0$ in J. Since f° is a torus bundle with a section over the thrice-punctured sphere,

$$\pi_1(J) = \Lambda \rtimes F(\widehat{\rho}_1, \widehat{\rho}_2), \qquad \widehat{\rho}_j \lambda \widehat{\rho}_j^{-1} = A_j \lambda, \qquad \widehat{\rho}_0 = (\widehat{\rho}_1 \widehat{\rho}_2)^{-1}.$$

Lemma 4.2. *Under the identification (18), the loop $\widehat{\rho}_j$ maps to the element $\widehat{g}_j$ of (19); hence $\widehat{\rho}_j^{m_j} = t_{v_j}$ holds in $\pi_1(N_j)$. Under (15), Λ_{tor} and $\widehat{\rho}_0$ become trivial in $\pi_1(N_0)$.*

Proof. By the last sentence of Section 3.2, the zero section of $J|_{D_j^*}$ corresponds to the image in N_j of the section $-\sigma_j$ of $J'_j|_{\Delta_j^*}$, which is $\widetilde{g}_j^{\log}$-invariant because h_j conjugates $\widetilde{g}_j^{\log}$ to $\widetilde{g}_j^0$ and the zero section is $\widetilde{g}_j^0$-invariant. The clockwise meridian a_j lifts to the arc of Δ_j^* from a point z to $g_j z$ (one m_j-th of a clockwise turn in s_j is a full clockwise turn in $t_j = s_j^{m_j}$), so $\widehat{\rho}_j$ is represented by a path along $-\sigma_j$ from a point p to $\widetilde{g}_j^{\log}(p)$, which is the definition of $\widehat{g}_j$. The cusp statements are Lemma 4.1. □

Both elliptic relations are obtained by the same recipe with the same orientation convention, and the pair-of-pants relation $a_0 = (a_1 a_2)^{-1}$ ties the cusp to them; reversing all orientations replaces v_j by $-v_j$ in both relations simultaneously, which does not affect the computation below.

Theorem 4.3. $\pi_1(X) = 1$.

Proof. Apply van Kampen's theorem successively to $J \cup N_1$, then N_2, then N_0; each intersection is a torus bundle over an annulus, connected, with fundamental group $\Lambda \rtimes \langle \widehat{\rho}_i \rangle$. By (19) and Lemma 4.2,

$$\pi_1(X) = \left\langle \Lambda, \widehat{\rho}_1, \widehat{\rho}_2 \;\middle|\; \widehat{\rho}_j \lambda \widehat{\rho}_j^{-1} = A_j \lambda, \quad \widehat{\rho}_1^3 = t_{v_1}, \quad \widehat{\rho}_2^4 = t_{v_2}, \quad \widehat{\rho}_1 \widehat{\rho}_2 = 1, \quad \Lambda_{\mathrm{tor}} = 1 \right\rangle.$$

The relations $\Lambda_{\mathrm{tor}} = 1$ and $\widehat{\rho}_j \lambda \widehat{\rho}_j^{-1} = A_j \lambda$ kill the $\langle A_1, A_2 \rangle$-invariant closure of Λ_{tor}, which is $\ker \gamma$ by Section 4.1. The image of Λ is therefore generated by the image c of $\widehat{\gamma}$, which is central because $\gamma(A_j \widehat{\gamma}) = \gamma(\widehat{\gamma}) = 1$. With $x = \widehat{\rho}_1$, $y = \widehat{\rho}_2$ and $\gamma(v_1) = 1$, $\gamma(v_2) = -1$,

$$\pi_1(X) = \langle c, x, y \mid c \text{ central},\ xy = 1,\ x^3 = c,\ y^4 = c^{-1} \rangle.$$

From $y = x^{-1}$ and $y^4 = c^{-1}$ we get $x^4 = c = x^3$, so $x = 1$, then $c = 1$ and $y = 1$. □

Remark 4.4. The choice $v_2 = -\varepsilon'$ rather than $+\varepsilon'$ is essential: with $\gamma(v_2) = +1$ the same computation gives $x^7 = 1$ and $\pi_1(X) \cong \mathbb{Z}/7$.

4.4 Elliptic fibres and specialization

4.4.1 Integral homology of the elliptic fibres

Abelianizing (19),

$$H_1(S_j;\mathbb{Z}) = \big(\Lambda/(A_j - I)\Lambda \oplus \mathbb{Z}\widehat{g}_j\big)/([v_j], -m_j).$$

In the basis $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta}, \widehat{g}_j)$ the relation matrices are

$$D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 6 & -1 & 1 & 0 & 2 \\ -6 & -1 & -2 & 0 & -4 \\ -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & -3 \\ -6 & 1 & -1 & 0 & 3 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix},$$

with Smith normal form $\mathrm{diag}(1,1,1,0,0)$ in both cases. Hence $H_1(S_j;\mathbb{Z}) = \mathbb{Z}^2$ is torsion-free. Since $\pi_j^* K_{S_j} = K_F \cong \mathcal{O}_F$, taking norms gives $K_{S_j}^{\otimes m_j} \cong \mathcal{O}_{S_j}$. Poincaré duality gives $b_0 = b_4 = 1$, $b_1 = b_3 = 2$, and $0 = e(S_j) = 2 - 4 + b_2$ gives $b_2 = 2$. The universal coefficient theorem ($\mathrm{Ext}(H_1, \mathbb{Z}) = 0$) and duality show that

$$H_*(S_j;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}) \tag{23}$$

is torsion-free. As $c_1(K_{S_j})$ is torsion in the torsion-free group $H^2(S_j;\mathbb{Z})$, it vanishes, so $w_2(S_j) = 0$ and by Wu's formula the unimodular intersection form on $H^2(S_j;\mathbb{Z})$ is *even*.

4.4.2 Specialization lattices at the elliptic points

Proposition 4.5. *For $0 \le q \le 3$ the map $\pi_j^* : H^q(S_j;\mathbb{Z}) \to (\Lambda^q V)^{T_j}$ is injective, with index*

q	0	1	2	3
$j = 1$	1	3	1	1
$j = 2$	1	4	2	2

Explicitly $\pi_1^* H^1(S_1) = \langle 3\gamma, \eta_1 \rangle$, $\pi_2^* H^1(S_2) = \langle 4\gamma, \eta_2 \rangle$, $\pi_1^* H^2(S_1) = (\Lambda^2 V)^{T_1}$; moreover $q \notin \pi_2^* H^2(S_2)$ and $\gamma uw \notin \pi_2^* H^3(S_2)$, so that $2q$ and $2\gamma uw$ generate the respective quotients $\mathbb{Z}/2$.

Proof. Injectivity follows from $\pi_{j*}\pi_j^* = m_j\,\mathrm{id}$ and torsion-freeness.

Degree one. $H^1(S_j;\mathbb{Z}) = \mathrm{Hom}(\Gamma_j, \mathbb{Z})$, and a homomorphism restricting to $\xi \in V^{T_j}$ on Λ exists iff $m_j \mid \xi(v_j)$, because $\widehat{g}_j^{\,m_j} = t_{v_j}$. Since $\eta_j(v_j) = 0$, $\gamma(v_1) = 1$ and $\gamma(v_2) = -1$, the two lattices displayed follow.

Degree two. Pullback along the covering multiplies intersection numbers by m_j, and the form on $H^2(S_j)$ is unimodular, so the discriminant of $\pi_j^* H^2(S_j)$ is $\pm m_j^2$. With

$$(\gamma\eta_1) \wedge q = 3\gamma \wedge u \wedge w \wedge \delta, \qquad (\gamma\eta_2) \wedge q = 2\gamma \wedge u \wedge w \wedge \delta, \qquad (\gamma\eta_j)^2 = 0, \qquad q^2 = 12\gamma \wedge u \wedge w \wedge \delta,$$

the invariant lattices have discriminants -9 and -4, so the indices are 1 and 2. For $j = 2$, write $x = \gamma\eta_2$; the three index-two sublattices of $\langle x, q \rangle$ are $\langle x, 2q \rangle$, $\langle 2x, q \rangle$ and $\langle 2x, x + q \rangle$, and q belongs only to the second. On $\langle 2x, q \rangle$ the form divided by $m_2 = 4$ has matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, which is odd; since the form on $H^2(S_2)$ is even, $\pi_2^* H^2(S_2) \neq \langle 2x, q \rangle$ and $q \notin \pi_2^* H^2(S_2)$.

Degree three. The pairing $H^1 \times H^3 \to H^4$ of S_j is perfect, so the pairing matrix between $\pi_j^* H^1(S_j)$ and $\pi_j^* H^3(S_j)$ has determinant $\pm m_j^2$. The pairing matrices of the bases (γ, η_j) and $(\gamma uw, \zeta_j)$ have determinants of absolute value 3 and 2. If $k_j^{(1)}, k_j^{(3)}$ are the indices in degrees one and three, then $k_j^{(1)} k_j^{(3)} \cdot |\det| = m_j^2$, so $k_1^{(3)} = 9/(3 \cdot 3) = 1$ and $k_2^{(3)} = 16/(4 \cdot 2) = 2$. For $j = 2$, the class η_2 lies in $\pi_2^* H^1(S_2)$ and $\langle \eta_2, \gamma uw \rangle = -2$ is not divisible by 4, whereas the pairing of two pulled-back classes is divisible by $m_2 = 4$; hence $\gamma uw \notin \pi_2^* H^3(S_2)$. $\square$

4.5 Nearby cycles at the cusp: the integral invariant-cycle theorem

Theorem 4.6. *Let $j : \Delta_\epsilon^* \hookrightarrow \Delta_\epsilon$. For $0 \le q \le 3$ the natural map $R^q f_{0*}\mathbb{Z} \to j_* j^* R^q f_{0*}\mathbb{Z}$ is an isomorphism near 0; equivalently, specialization*

$$\mathrm{sp}^q : H^q(W; \mathbb{Z}) \xrightarrow{\sim} H^q(F; \mathbb{Z})^{T_0}$$

is an isomorphism of integral lattices in these degrees.

The proof occupies the rest of this subsection. Equality of ranks over $\mathbb{Q}$ would not suffice for the integral Leray computation below.

4.5.1 Local Milnor fibres and the sheaf $\mathcal{H}^1\psi$

Let $\psi = \psi_{f_0}\mathbb{Z}_{N_0}$ be the nearby-cycle complex [4, 8]. By proper base change $\mathbb{H}^q(W; \psi) \cong H^q(F; \mathbb{Z})$, and the edge map $\mathbb{Z}_W \to \psi$ induces specialization. The local models of Section 3.1 are $t_c = z_0$, $t_c = z_0 z_1$, $t_c = z_0 z_1 z_2$, with Milnor fibres of the homotopy types of a point, S^1 and T^2. Hence

$$\mathcal{H}^0\psi = \mathbb{Z}_W, \qquad \mathcal{H}^2\psi = \mathbb{Z}_P \oplus \mathbb{Z}_Q, \qquad \mathcal{H}^{\ge 3}\psi = 0,$$

and $\mathcal{V} := \mathcal{H}^1\psi$ is supported on the double locus. Write $\bar{C}_1, \bar{C}_2, \bar{C}_3 \cong \mathbb{P}^1$ for the three double curves and ι_k for their inclusions. Each $\bar{C}_k$ is the image of two *different* sides of the hexagon of E_0, so its two local branches are globally distinguishable; orient the vanishing circle along $\bar{C}_k$ by increasing argument of the coordinate vanishing on the first branch. Then $\mathcal{V}$ is constant of rank one on $\bar{C}_k \setminus \{P, Q\}$. At a triple point with $t_c = z_0 z_1 z_2$, $H_1(\mathrm{MF}) = \{(a_0, a_1, a_2) \in \mathbb{Z}^3 : \sum a_i = 0\}$, the vanishing circle of the double curve $\{z_i = z_j = 0\}$ is $e_i - e_j$, and for $\phi \in H^1(\mathrm{MF}) = \mathrm{Hom}(H_1(\mathrm{MF}), \mathbb{Z})$ the three restrictions $n_k = \phi(e_i - e_j)$ satisfy one relation. Translating the six triangles at a vertex of the A_2 triangulation to the origin and keeping the branch order fixed by the clockwise meridian gives, at the two triple points,

	n_1	n_2	n_3
P	$\phi(e_0 - e_1)$	$\phi(e_0 - e_2)$	$\phi(e_1 - e_2)$
Q	$\phi(e_2 - e_1)$	$\phi(e_0 - e_1)$	$\phi(e_0 - e_2)$

so that at both points the relation is $n_1 - n_2 + n_3 = 0$. Therefore there is a short exact sequence

$$0 \to \mathcal{V} \to \bigoplus_{k=1}^{3} \iota_{k*}\mathbb{Z}_{\bar{C}_k} \xrightarrow{\delta_1} \mathbb{Z}_P \oplus \mathbb{Z}_Q \to 0, \qquad \delta_1(n_1, n_2, n_3) = (s, s), \quad s = n_1 - n_2 + n_3 : \tag{24}$$

away from P, Q it is the identity of the rank-one sheaf, and at P or Q its stalk sequence is $0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$ with the middle map injective onto the saturated sublattice $\{s = 0\}$.

4.5.2 Integral groups on the E_2 page

Cohomology of W. The normalization gives the exact sequence of sheaves

$$0 \to \mathbb{Z}_W \to \nu_*\mathbb{Z}_{dP_6} \to \bigoplus_{k=1}^{3} \iota_{k*}\mathbb{Z}_{\bar{C}_k} \xrightarrow{\delta_1} \mathbb{Z}_P \oplus \mathbb{Z}_Q \to 0, \tag{25}$$

where the middle arrow takes the difference of the two branch values along each double curve and the last arrow is the alternating sum above; at a triple point the stalks form the augmented Čech complex of three points, so (25) is exact. In cyclic order the hexagon of dP_6 is E_1, $H - E_1 -$

E_2, E_2, $H-E_2-E_3$, E_3, $H-E_1-E_3$, with opposite sides three steps apart, and the difference map $H^2(dP_6)=\mathbb{Z}^4\to\bigoplus_k H^2(\bar C_k)=\mathbb{Z}^3$ is

$$aH-\sum b_iE_i\longmapsto(s,-s,s),\qquad s=b_1+b_2+b_3-a,$$

with saturated image $\mathbb{Z}(1,-1,1)$, kernel $\mathbb{Z}^3$ and cokernel $\mathbb{Z}^2$. The hypercohomology spectral sequence of (25) has E_1-rows: in fibre degree 0 the complex $\mathbb{Z}\xrightarrow{0}\mathbb{Z}^3\xrightarrow{\delta_1}\mathbb{Z}^2$ with cohomology $(\mathbb{Z},\mathbb{Z}^2,\mathbb{Z})$; in fibre degree 2 the complex $\mathbb{Z}^4\to\mathbb{Z}^3$ with cohomology $(\mathbb{Z}^3,\mathbb{Z}^2)$; in fibre degree 4 the group $\mathbb{Z}$. No further differential can be nonzero, and all groups are free, so

$$H^*(W;\mathbb{Z})=(\mathbb{Z},\mathbb{Z}^2,\mathbb{Z}^4,\mathbb{Z}^2,\mathbb{Z}),\qquad e(W)=2,\tag{26}$$

all torsion-free.

Cohomology of $\mathcal{V}$. From (24): $H^0(W;\mathcal{V})=\ker\delta_1=\mathbb{Z}^2$, $H^1(W;\mathcal{V})=\operatorname{coker}\delta_1=\mathbb{Z}^2/\mathbb{Z}(1,1)=\mathbb{Z}$, $H^2(W;\mathcal{V})=\bigoplus_k H^2(\bar C_k)=\mathbb{Z}^3$.

The E_2 page. The spectral sequence $E_2^{a,b}=H^a(W;\mathcal{H}^b\psi)\Rightarrow H^{a+b}(F;\mathbb{Z})$ therefore has the rank table

$b=2$	2	0	0	0	0
$b=1$	2	1	3	0	0
$b=0$	1	2	4	2	1
	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$

with every entry free abelian.

4.5.3 Differentials, degree three, and saturation

The possibly nonzero differentials are $d_2:E_2^{0,1}\to E_2^{2,0}$, $d_2:E_2^{1,1}\to E_2^{3,0}$, $d_2:E_2^{2,1}\to E_2^{4,0}$, $d_2:E_2^{0,2}\to E_2^{2,1}$ and $d_3:E_3^{0,2}\to E_3^{3,0}$, of ranks $\alpha,\beta,\gamma',\delta',\epsilon'$ say. Since $H^q(F)$ has ranks $1,4,6,4,1$, comparing total ranks gives

$$\alpha=\gamma'=0,\qquad\beta+\delta'+\epsilon'=1.$$

Thus $E_2^{q,0}=H^q(W)$ survives to E_∞ except possibly in total degree 3, i.e. sp^q is injective for $q\neq 3$.

Degree three. Let $W^\circ\cong(\mathbb{C}^*)^2$ be the open orbit of E_0, on which ν is injective, and let $T_f=(\Lambda_{\mathrm{tor}}\otimes\mathbb{R})/\Lambda_{\mathrm{tor}}\cong(S^1)^2$ act on N_0 through the compact subtorus of the (x_1,x_2)-torus; this action commutes with Ψ and preserves t_c. Choose a compact T_f-orbit $K\cong T^2$ in W°. Since f_0 is a submersion near K, a T_f-invariant metric, horizontal lifts of the base coordinate fields and compactness of K give a T_f-equivariant open embedding $\Theta:K'\times\Delta_r\hookrightarrow N_0$ with $f_0\circ\Theta(w,t)=t$ and $\Theta(w,0)=w$, for a T_f-invariant open neighbourhood $K'\subset W^\circ$ of K. Excision and homotopy invariance in the product chart give a commutative square

$$\begin{array}{ccc} H^3(W,W\setminus K) & \longrightarrow & H^3(W)\\ \downarrow\wr & & \downarrow\mathrm{sp}^3\\ H^3(F,F\setminus K_t) & \longrightarrow & H^3(F),\end{array}\qquad K_t=\Theta(K,t),$$

in which both relative groups are $H^1(T^2)\cong\mathbb{Z}^2$ by the Thom isomorphism. The bottom arrow is the Gysin map of the submanifold $K_t\subset F$, which is a translate of the linear subtorus $T_f\subset F$. Since Λ_{tor} is saturated, $\Lambda=\Lambda_{\mathrm{tor}}\oplus\Lambda_B$ for a complement Λ_B, so $F\cong T_f\times T_B$ and the Gysin map is $a\mapsto a\times[T_B]^\vee$, injective with image a Künneth summand, hence a saturated sublattice of rank two. Thus $\operatorname{Im}\mathrm{sp}^3$ contains a saturated sublattice of rank $2=\operatorname{rk}H^3(W)$; so sp^3 is injective, and its image, a rank-two lattice containing a saturated rank-two lattice inside the torsion-free $H^3(F)$, equals that saturated lattice. Hence $\beta=\epsilon'=0$ and $\delta'=1$.

Saturation. In degree 0 specialization is the identity on constants. In degree 1 the cokernel of sp^1 is $E_\infty^{0,1} = E_2^{0,1} \cong \mathbb{Z}^2$, free. In degree 2 the cokernel is filtered with quotients $E_\infty^{1,1} = E_2^{1,1} = \mathbb{Z}$ (as $\beta = 0$) and $E_\infty^{0,2} = \ker d_2^{0,2}$, both free. Degree 3 was just treated. Every class extending over W is invariant under transport around the cusp, so $\operatorname{Im} \mathrm{sp}^q \subset (\Lambda^q V)^{T_0}$; both sides are saturated of ranks $1, 2, 4, 2$ for $q = 0, \ldots, 3$, so they coincide. This proves Theorem 4.6.

4.6 The global integral Leray spectral sequence

Let $j : B^\circ \hookrightarrow B$ and let $\mathcal{L}^q = R^q f_* \mathbb{Z}|_{B^\circ}$ be the local system with fibre $\Lambda^q V$ and clockwise monodromies T_1, T_2, T_0. By Theorem 4.6 the stalk of $R^q f_* \mathbb{Z}$ at p_0 is that of $j_* \mathcal{L}^q$, and by Proposition 4.5 the stalks at p_1, p_2 are the specialization lattices $\pi_j^* H^q(S_j) \subset (\Lambda^q V)^{T_j}$. Hence $R^q f_* \mathbb{Z} \subset j_* \mathcal{L}^q$ with skyscraper quotient

$$Q^q := j_* \mathcal{L}^q / R^q f_* \mathbb{Z} = \begin{cases} (\mathbb{Z}/3)_{p_1} \oplus (\mathbb{Z}/4)_{p_2}, & q = 1, \\ (\mathbb{Z}/2)_{p_2}, & q = 2, 3. \end{cases} \tag{27}$$

Also $R^0 f_* \mathbb{Z} = \mathbb{Z}_B$ (connected fibres) and $R^4 f_* \mathbb{Z} \subset j_* \mathcal{L}^4 = \mathbb{Z}_B$ with skyscraper quotient.

4.6.1 The E_2 page

Lemma 4.7. *(a) $H^0(B, R^1 f_* \mathbb{Z}) = \mathbb{Z} \cdot 12\gamma$, $H^0(B, R^2 f_* \mathbb{Z}) = \mathbb{Z} \cdot 2q$, $H^0(B, R^3 f_* \mathbb{Z}) = \mathbb{Z} \cdot 2\gamma u w$.*

(b) $H^1(B, R^q f_* \mathbb{Z}) = 0$ for $q = 0, 1, 2$.

(c) For every q, $H^2(B, R^q f_* \mathbb{Z}) \cong H^2(B, j_* \mathcal{L}^q) \cong (\Lambda^q V)_G$; in particular $H^2(B, R^0) = \mathbb{Z}\omega$, $H^2(B, R^1) = \mathbb{Z}[\bar{\delta}]$, $H^2(B, R^2) = \mathbb{Z}[\gamma\delta]$, $H^2(B, R^4) = \mathbb{Z}[\gamma \wedge u \wedge w \wedge \delta]$. Under these identifications the cup products are

$$\omega \cdot x = [x], \qquad x \cdot [y] = [x \wedge y] \qquad (x \in H^0(B, R^b),\ [y] \in H^2(B, R^{b'})).$$

Proof. (a) A global section is an element of $(\Lambda^q V)^G$ lying in the stalks at p_1, p_2. In degree one, $n\gamma$ must be divisible by 3 at p_1 and by 4 at p_2; in degrees two and three, Proposition 4.5 says that q and $\gamma u w$ are not in the stalk at p_2 but their doubles are.

(c) Since Q^q is a skyscraper, $H^1(B, Q^q) = H^2(B, Q^q) = 0$ and $H^2(B, R^q) \cong H^2(B, j_* \mathcal{L}^q)$. The sequence $0 \to j_! \mathcal{L} \to j_* \mathcal{L} \to (\text{skyscraper}) \to 0$ gives $H^2(B, j_* \mathcal{L}) \cong H_c^2(B^\circ, \mathcal{L})$, and Poincaré duality with local coefficients on the oriented open surface B° [3, Ch. V] gives $H_c^2(B^\circ, \mathcal{L}) \cong H_0(B^\circ, \mathcal{L}) = M_G$ for a fibre M. The class ω restricts to the compactly supported class of a point, so $\omega \cdot x$ is the 0-cycle $[\mathrm{pt} \otimes x]$, i.e. $[x] \in M_G$; naturality of the cap product for coefficient pairings gives $x \cdot [y] = [x \wedge y]$, which is well defined on coinvariants because x is invariant. The values of the coinvariants are (21)–(22) and $(\Lambda^4 V)_G = \mathbb{Z}[\gamma \wedge u \wedge w \wedge \delta]$ (all T_i have determinant one).

(b) $H^1(B, \mathbb{Z}) = 0$. For $q = 1, 2$, the sequence $0 \to R^q \to j_* \mathcal{L}^q \to Q^q \to 0$ gives $H^0(j_* \mathcal{L}^q) \to H^0(Q^q) \to H^1(R^q) \to H^1(j_* \mathcal{L}^q) \to 0$, and the first map is onto: $n\gamma \mapsto (n \bmod 3, n \bmod 4)$ is surjective by the Chinese remainder theorem, and $q \mapsto$ the nonzero element of $\mathbb{Z}/2$. So $H^1(B, R^q) \cong H^1(B, j_* \mathcal{L}^q)$, the parabolic cohomology, which is the kernel of $H^1(B^\circ, \mathcal{L}^q) \to \bigoplus_i H^1(D_i^*, \mathcal{L}^q)$. Writing cocycles on the free group $\langle t_1, t_2 \rangle$ with $t_0 = (t_1 t_2)^{-1}$, the parabolic condition is $c(t_i) \in (T_i - I)M$ for $i = 0, 1, 2$. Subtracting a coboundary we may assume $c(t_2) = 0$; the residual gauge is M^{T_2}, and $c(t_0) = -T_0 c(t_1)$, so

$$H^1(B, j_* \mathcal{L}^q) \cong \frac{(T_1 - I)M \cap (T_0 - I)M}{(T_1 - I)M^{T_2}}.$$

For $q = 1$: $(T_1 - I)V = \langle 2\gamma + u + w,\ 2\gamma - u \rangle$, $(T_0 - I)V = \langle \gamma, u \rangle$, the intersection is $\mathbb{Z}(2\gamma - u)$, and $(T_1 - I)\eta_2 = -(2\gamma - u)$ with $\eta_2 \in V^{T_2}$. For $q = 2$: $(\Lambda^2 T_1 - I)\Lambda^2 V = \langle \gamma u, \gamma w, uw + 3u\delta - 6\gamma\delta, 2u\delta + w\delta \rangle$, $(\Lambda^2 T_0 - I)\Lambda^2 V = \langle \gamma u, \gamma w + u\delta \rangle$, the intersection is $\mathbb{Z}\,\gamma u$ (compare the coefficients of $uw, u\delta, w\delta$), and $(\Lambda^2 T_1 - I)(\gamma\eta_2) = \gamma u$ with $\gamma\eta_2 \in (\Lambda^2 V)^{T_2}$. Both quotients vanish. $\square$

Since B has real dimension two, the Leray spectral sequence $E_2^{a,b} = H^a(B, R^b f_*\mathbb{Z}) \Rightarrow H^{a+b}(X;\mathbb{Z})$ has only the columns $a = 0, 1, 2$, and the only differentials are $d_2 : E_2^{0,b} \to E_2^{2,b-1}$. By Lemma 4.7, in total degrees ≤ 3 the page is

	$a=0$	$a=1$	$a=2$
$b=3$	$\mathbb{Z}\cdot 2\gamma uw$	0	$*$
$b=2$	$\mathbb{Z}\cdot 2q$	0	$\mathbb{Z}[\gamma\delta]$
$b=1$	$\mathbb{Z}\cdot 12\gamma$	0	$\mathbb{Z}[\bar\delta]$
$b=0$	$\mathbb{Z}$	0	$\mathbb{Z}\omega$

(the entry $E_2^{2,3}$ is not needed).

4.6.2 Rational homology

Proposition 4.8. $b_1(X) = b_2(X) = b_3(X) = 0$.

Proof. $b_1 = 0$ by Theorem 4.3. Rationally $E_2^{0,1} = \mathbb{Q}\gamma$ and $E_2^{2,0} = \mathbb{Q}\omega$; since $E_\infty^{0,1} = \ker d_2$ must vanish, $d_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism and $E_\infty^{2,0} = 0$, i.e. $f^*\omega = 0$ in $H^2(X;\mathbb{Q})$. As $f^*\omega$ is Poincaré dual to the class of a smooth fibre, $[F] = 0$ in $H_4(X;\mathbb{Q})$. Now let $x \in H^2(X;\mathbb{Q})$. Its restriction to F is monodromy invariant, hence $x|_F = kq$, and

$$12k^2 = \langle (x|_F)^2, [F]\rangle = \langle x^2, [F]\rangle = \langle x^2 \smile f^*\omega, [X]\rangle = 0,$$

so $x|_F = 0$. Thus $H^2(X;\mathbb{Q}) \to E_2^{0,2}$ is zero, and with $E_\infty^{2,0} = E_\infty^{1,1} = 0$ we get $b_2 = 0$. Finally $e(X) = 2$ by Theorem 3.5, and with $b_1 = b_2 = b_4 = b_5 = 0$ this gives $b_3 = 2 - e(X) = 0$. □

4.6.3 The integral transgressions

Proposition 4.9. *With one common overall sign,*

$$d_2(12\gamma) = \pm\omega, \qquad d_2(2q) = \pm\omega[\bar\delta], \qquad d_2(2\gamma uw) = \pm\omega[\gamma\delta];$$

that is, all three differentials $d_2 : E_2^{0,b} \to E_2^{2,b-1}$, $b = 1, 2, 3$, are isomorphisms.

Proof. Write $d_2(12\gamma) = p\,\omega$, $d_2(2q) = p'\,\omega[\bar\delta]$, $d_2(2\gamma uw) = p''\,\omega[\gamma\delta]$ with $p, p', p'' \in \mathbb{Z}$.

$|p| = 1$. Since $H_1(X;\mathbb{Z}) = 0$, the group $H^2(X;\mathbb{Z}) = \mathrm{Hom}(H_2(X),\mathbb{Z})$ is torsion-free. The edge map $E_\infty^{2,0} = \mathbb{Z}\omega/p\mathbb{Z}\omega \to H^2(X;\mathbb{Z})$ is injective, so $\mathbb{Z}/p$ is torsion-free, i.e. $p \in \{0, \pm 1\}$; and $p \neq 0$ because $E_\infty^{0,1} = \ker d_2$ must vanish.

Multiplicativity. The Leray spectral sequence is multiplicative, and by Lemma 4.7(c) the products on E_2 are computed by wedge products in V. Apply the Leibniz rule $d_2(xy) = d_2(x)\,y + (-1)^{\deg x} x\, d_2(y)$ to the two relations

$$(12\gamma)\cdot(2q) = 12\cdot(2\gamma uw) \in E_2^{0,3}, \qquad (2q)\cdot(2\gamma uw) = 0 \in E_2^{0,5} = 0.$$

The first gives, in $E_2^{2,2} = \mathbb{Z}[\gamma\delta]$ and using $[2q] = 24[\gamma\delta]$,

$$12p''[\gamma\delta] = p\,\omega\cdot 2q - 12\gamma\cdot p'\omega[\bar\delta] = (24p - 12p')[\gamma\delta], \qquad \text{so} \qquad p'' = 2p - p'.$$

The second gives, in $E_2^{2,4} = \mathbb{Z}[\gamma\wedge u\wedge w\wedge\delta]$ and using $\bar\delta \wedge 2\gamma uw = -2\gamma\wedge u\wedge w\wedge\delta$, $2q \wedge \gamma\delta = 2\gamma\wedge u\wedge w\wedge\delta$,

$$0 = p'\,[\bar\delta \wedge 2\gamma uw] + p''\,[2q\wedge\gamma\delta] = (-2p' + 2p'')[\gamma\wedge u\wedge w\wedge\delta], \qquad \text{so} \qquad p' = p''.$$

Together, $p' = p'' = p = \pm 1$. □

Remark 4.10. The conclusion does not depend on sign conventions. If the Leibniz rule, the graded commutativity of the E_2-products, or the identifications of Lemma 4.7(c) are altered by signs, the two relations take the form $p'' = 2s_1 p + s_2 p'$ and $p'' = s_3 p'$ with $s_i = \pm 1$. If $s_2 = s_3$ this forces $p = 0$, which is excluded; hence $s_3 = -s_2$ and $2|p'| = 2|p|$, so again $|p'| = |p''| = 1$. The direct computation of p from the gluing data is also available: a lift $\Theta_j \in \mathrm{Hom}(\Gamma_j, \mathbb{Z})$ of 12γ has $m_j \Theta_j(\widehat{\rho}_j) = 12\gamma(v_j)$, so $\Theta_1(\widehat{\rho}_1) = 4$ and $\Theta_2(\widehat{\rho}_2) = -3$, the cusp lift has $\Theta_0(\widehat{\rho}_0) = 0$, and the Mayer–Vietoris coboundary of the three boundary values is $0 + 4 - 3 = 1$; this reproduces $|p| = 1$.

4.6.4 Integral homology

Theorem 4.11. *$H_k(X;\mathbb{Z}) = \mathbb{Z}$ for $k = 0, 6$ and $H_k(X;\mathbb{Z}) = 0$ for $1 \le k \le 5$.*

Proof. By Proposition 4.9 and Lemma 4.7, the E_∞ terms in total degrees one to three are

$$\begin{array}{llll}
H^1(X): & E_\infty^{1,0} = 0, & E_\infty^{0,1} = \ker d_2^{0,1} = 0; & \\
H^2(X): & E_\infty^{2,0} = \operatorname{coker} d_2^{0,1} = 0, & E_\infty^{1,1} = 0, & E_\infty^{0,2} = \ker d_2^{0,2} = 0; \\
H^3(X): & E_\infty^{2,1} = \operatorname{coker} d_2^{0,2} = 0, & E_\infty^{1,2} = 0, & E_\infty^{0,3} = \ker d_2^{0,3} = 0.
\end{array}$$

Hence $H^1(X;\mathbb{Z}) = H^2(X;\mathbb{Z}) = H^3(X;\mathbb{Z}) = 0$. Now $H_1(X) = \pi_1(X)^{\mathrm{ab}} = 0$. The universal coefficient sequence $0 \to \mathrm{Ext}(H_1,\mathbb{Z}) \to H^2 \to \mathrm{Hom}(H_2,\mathbb{Z}) \to 0$ shows that the finitely generated group $H_2(X)$ is torsion, and then $\mathrm{Ext}(H_2(X),\mathbb{Z}) \cong H_2(X)$ injects into $H^3(X;\mathbb{Z}) = 0$, so $H_2(X) = 0$. As X is a closed oriented six-manifold, Poincaré duality gives $H_3 \cong H^3 = 0$, $H_4 \cong H^2 = 0$, $H_5 \cong H^1 = 0$. □

4.7 Recognition as the standard S^6 and transport of the complex structure

By Theorems 4.3 and 4.11, X is simply connected with the integral homology of S^6. Hurewicz, applied inductively, gives $\pi_k(X) = 0$ for $k \le 5$ and $\pi_6(X) \cong H_6(X) = \mathbb{Z}$; a map $S^6 \to X$ representing a generator induces isomorphisms on all integral homology groups, and since both spaces are simply connected CW complexes it is a homotopy equivalence by the homological Whitehead theorem [6, Ch. 4]. Thus X is a smooth homotopy six-sphere. By Kervaire–Milnor [7] $\Theta_6 = 0$, so by Smale's h-cobordism theorem [9] there is a diffeomorphism $\Phi : S^6 \xrightarrow{\sim} X$. If (U_α, z_α) is the holomorphic atlas of X, then $(\Phi^{-1}(U_\alpha), z_\alpha \circ \Phi)$ is an atlas on S^6 whose transition functions are those of X, hence holomorphic. This proves Theorem 1.1.

5 Two consistency checks

The following two observations are not needed for the proof. They were made while checking the argument and they test the construction against invariants that must agree with the topology of S^6.

Proposition 5.1. *The class $q \in H^2(F;\mathbb{Z})$ is of type $(1,1)$ on every fibre, and the Hermitian form it defines is indefinite on every fibre. Consequently the general fibre is a non-algebraic torus with $\mathrm{NS}(F) \supseteq \mathbb{Z}q$, and f is the algebraic reduction of X, so $a(X) = 1$.*

Proof. With Q the matrix of q on Λ, a direct computation gives $\Pi Q^{-1} \Pi^{\mathrm{t}} = 0$ (the $(1,1)$ condition) and

$$i\,\Pi Q^{-1} \overline{\Pi}^{\mathrm{t}} = \begin{pmatrix} 2\,\mathrm{Im}\,\tau & 2\,\mathrm{Im}\,\mu \\ 2\,\mathrm{Im}\,\mu & \frac{1}{3}\,\mathrm{Im}\,\beta \end{pmatrix}, \qquad \det = \tfrac{2}{3}\,\mathrm{Im}\,\tau \cdot D < 0$$

by (5). An integral 2-form ω is of type $(1,1)$ at z iff its pairing with $\sigma_1 \wedge \sigma_2$ vanishes, and this pairing is $\omega_{\gamma u} - \omega_{\gamma w}\tau + (6\omega_{uw} - \omega_{\gamma\delta})\mu + \omega_{u\delta}\beta + \omega_{w\delta}(6\mu^2 - \tau\beta)$. The functions $1, \tau, \mu, \beta, 6\mu^2 - \tau\beta$

satisfy no linear relation (applying g_0 and g_2 to a putative relation and using that μ is single-valued at the cusp while τ is not forces all coefficients to vanish), so for general z the only integral $(1,1)$-classes are the multiples of q. A two-torus whose Néron–Severi group is generated by a class of negative square has neither a polarization nor an elliptic pencil, hence algebraic dimension zero; therefore every meromorphic function on X is constant on the general fibre and is pulled back from B, i.e. $a(X) = 1$. □

Proposition 5.2. *$f_*\omega_{X/B} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and $R^1f_*\mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{O}(-1)$, $R^2f_*\mathcal{O}_X \cong \mathcal{O}(-1)$. Hence $h^{0,1}(X) = 1$, $h^{0,2}(X) = h^{0,3}(X) = 0$ and $\chi(\mathcal{O}_X) = 0$, in agreement with Riemann–Roch ($\chi(\mathcal{O}_X) = c_1c_2/24$ and $c_1(X) = 0$).*

Sketch. The relative two-form $\Omega_2 = d\zeta_1 \wedge d\zeta_2$ on $\mathcal{T}$ transforms by $\det R_g$, and $F_{\mathrm{mod}}^{-1}\Omega_2$ is Δ-invariant, hence a meromorphic section of $f_*\omega_{X/B}$. At the cusp, $\Omega_2 = (2\pi i)^{-2} d\log x_1 \wedge d\log x_2$ generates $f_*\omega_{N_0/\Delta}$, so the section has a simple zero there; at p_j, since $N_j = J_j'/G$ and $p^*\omega_{D_j} = \omega_{\Delta_j}(-(m_j-1)z_j)$, the sections of $f_*\omega_{N_j/D_j}$ are the G-invariant forms $\varphi\, s_j^{-(m_j-1)}\Omega_2$ with φ holomorphic on Δ_j, and invariance reads $\varphi \circ g_j = \zeta_j^{m_j-1}\varphi/\det R_{g_j}$; at z_1 the factor $\zeta_1^2 \cdot (-\rho)$ equals 1 and at z_2 the factor $\zeta_2^3 \cdot i$ equals -1, so φ has s_j-order $\equiv 0 \pmod 3$ at z_1 and $\equiv 2 \pmod 4$ at z_2, and the local generators are $s_1^{-2}\Omega_2$ and $s_2^{-1}\Omega_2$ up to units. These are exactly the orders of F_{mod}^{-1} at z_1, z_2, so $F_{\mathrm{mod}}^{-1}\Omega_2$ is a local generator at p_1, p_2, and $\deg f_*\omega_{X/B} = 1$. Relative duality gives $R^2f_*\mathcal{O}_X \cong (f_*\omega_{X/B})^\vee$. For $R^1f_*\mathcal{O}_X$: on $\mathfrak{h}$ the classes $[\gamma], [u]$ form a frame of $V_{\mathbb{C}}/\langle\sigma_1, \sigma_2\rangle$; the image of the global section 12γ of $R^1f_*\mathbb{Z}$ is a nowhere vanishing section of $R^1f_*\mathcal{O}_X$ (the maps $H^1(\,\cdot\,;\mathbb{Z}) \to H^1(\,\cdot\,;\mathcal{O})$ are injective on S_j and on W), giving $0 \to \mathcal{O} \to R^1f_*\mathcal{O}_X \to \mathcal{Q} \to 0$, and the quotient $\mathcal{Q}$ has the automorphy factor of $\mathcal{L}$ at p_1, p_2 and generator $[u]$ at p_0, so $\mathcal{Q} \cong \mathcal{L} \cong \mathcal{O}(-1)$. The extension splits since $H^1(\mathcal{O}(1)) = 0$. Then $h^{0,1} = h^0(R^1f_*\mathcal{O}_X) = 1$, $h^{0,2} = h^1(R^1f_*\mathcal{O}) + h^0(R^2f_*\mathcal{O}) = 0$, $h^{0,3} = h^1(R^2f_*\mathcal{O}) = 0$, and $\chi(\mathcal{O}_X) = 1 - 1 + 0 = 0$. □

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