A COUNTEREXAMPLE TO POSITIVE LOWER HERMITIAN MONGE–AMPÈRE MASS

Abstract. We construct a compact complex threefold X, a Hermitian form ω on X, and smooth strictly ω-plurisubharmonic functions φ_j such that $\int_X (\omega + \mathrm{dd^c}\varphi_j)^3 \to 0$. Thus the lower Monge–Ampère mass $v_-(\omega)$ can vanish. The construction combines a Gauduchon-cone separation argument with accumulating deck translates of a local hypersurface, bounded-degree averages of rational curves, and the Hermitian Calabi–Yau theorem.

1. Introduction

We use

$$\mathrm{d^c} = \frac{\mathrm{i}}{2\pi}(\bar\partial - \partial), \qquad \mathrm{dd^c} = \frac{\mathrm{i}}{\pi}\partial\bar\partial.$$

For a Hermitian form ω on a compact complex n-fold X, set

$$v_-(\omega) := \inf_{\varphi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)} \int_X (\omega + \mathrm{dd^c}\varphi)^n.$$

The question addressed here is whether this number must be positive.

Theorem A. *There are a compact complex threefold X, a Hermitian form ω on X, and functions $\varphi_j \in C^\infty(X,\mathbb{R})$ such that*

$$\omega + \mathrm{dd^c}\varphi_j > 0, \qquad \sup_X \varphi_j = 0, \qquad \int_X (\omega + \mathrm{dd^c}\varphi_j)^3 \longrightarrow 0.$$

In particular, $v_-(\omega) = 0$.

The threefold is an elliptic quotient of the complement of two projective lines in $\mathbb{P}^3$. Its deck translates of one local hypersurface accumulate inside the covering space. Lamari-type separation and Siu's analyticity theorem then force arbitrarily large local traces in the normalized cone of positive $\mathrm{dd^c}$-closed $(2,2)$-forms. Holomorphic automorphisms spread such a trace over a fixed horizontal region. A second average, now of compact rational curves, supplies a transverse cofactor spike without increasing the Hermitian degree. A determinant inequality converts the overlap of the two concentrations into unbounded root volume. Finally, the Hermitian Calabi–Yau theorem turns these dual forms into the potentials in Theorem A.

The proof is arranged accordingly. The quotient and the dual Monge–Ampère estimate are established in Section 2. The trace concentration is proved in Section 3, and the bounded-degree root-volume construction occupies Section 4. The theorem is assembled in Section 5.

2. The quotient and the dual estimate

Write homogeneous coordinates on $\mathbb{P}^3$ as $[z_0 : z_1 : w_0 : w_1] = [z : w]$, and put

$$\Omega = \mathbb{P}^3 \setminus (\{z_0 = z_1 = 0\} \cup \{w_0 = w_1 = 0\}), \qquad q = \mathrm{e}^{2\pi}.$$

Let $g\colon \Omega \to \Omega$ be the biholomorphism

$$g[z : w] = [z : qw].$$

Proposition 2.1. *The quotient*

$$X = \Omega/\langle g\rangle$$

is a compact connected complex threefold. If

$$t = \log\frac{\|w\|^2}{\|z\|^2},$$

if α and β are the pullbacks from the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$ of the forms $\mathrm{dd^c}\log\|z\|^2$ and $\mathrm{dd^c}\log\|w\|^2$, respectively, and if

$$D = \mathrm{d}t \wedge \mathrm{d^c}t,$$

then

$$\omega = \alpha + \beta + D$$

is a Hermitian form on X.

Proof. Both blocks z and w are nonzero on Ω. If $g^k[z : w] = [z : w]$, projective equality gives one scalar that is simultaneously 1 on the z-block and q^k on the w-block. Hence $k = 0$, so the action is free. Moreover,

$$t \circ g = t + 4\pi.$$

This implies that the action is properly discontinuous: a compact subset of Ω has bounded t-image and therefore meets only finitely many translates of any other compact subset. The closed strip $0 \leq t \leq 4\pi$ is closed in $\mathbb{P}^3$ and stays a positive distance from the two omitted lines; it is compact and meets every orbit. Thus the quotient is compact. It is connected because Ω is connected.

The map $[z : w] \mapsto ([z], [w])$ descends to an elliptic fibration

$$\pi \colon X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

Indeed, after taking a logarithm of the relative scalar between the two blocks, a fiber is $\mathbb{C}/(2\pi\mathbb{Z} + 2\pi \mathrm{i}\mathbb{Z})$. The forms α and β are semipositive and are positive in the two base directions; with the present normalization, each has integral 2 on the corresponding $\mathbb{P}^1$. Although t itself changes by 4π, its differentials descend. On every fiber, t is twice the real part of the logarithmic fiber coordinate, up to an additive base term. Hence D is semipositive and positive on the vertical complex line. The kernels of $\alpha + \beta$ and D have zero intersection, so $\omega > 0$. □

We next record the pointwise convention used for positive $(2, 2)$-forms. In a holomorphic coframe on a threefold, the cofactor identification sends a real $(2, 2)$-form Θ to a Hermitian 3×3 matrix Q_Θ. If $\Theta = \eta^2$ and H is the coefficient matrix of η, then

$$Q_\Theta = 2\,\mathrm{cof}(H).$$

Lemma 2.2. *Every strictly positive real $(2, 2)$-form Θ on a complex threefold has a unique strictly positive real $(1, 1)$-form η satisfying $\eta^2 = \Theta$. If*

$$\nu = \left(\frac{\mathrm{i}}{\pi}\right)^3 \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}\bar{z}_3,$$

then

$$\Theta^{3/2} := \eta^3 = \frac{3}{\sqrt{2}}\sqrt{\det Q_\Theta}\,\nu. \tag{2.1}$$

Proof. For $H > 0$, the map $H \mapsto 2\,\mathrm{cof}(H)$ is bijective on the positive Hermitian cone: its inverse is obtained from $\det(2\,\mathrm{cof}\,H) = 8(\det H)^2$ and $H = 2(\det H)(2\,\mathrm{cof}\,H)^{-1}$. This proves existence and uniqueness. Since $\eta^3 = 6\det(H)\nu$, the determinant identity gives (2.1). □

The following proposition isolates the final Monge–Ampère argument.

Proposition 2.3. *Let Y be a compact connected complex threefold with Hermitian form h. Suppose that smooth strictly positive $\mathrm{dd^c}$-closed $(2, 2)$-forms Γ_j satisfy*

$$P_j := \int_Y h \wedge \Gamma_j \leq C, \qquad R_j := \int_Y \Gamma_j^{3/2} \longrightarrow \infty.$$

Then there are smooth functions u_j with $h + \mathrm{dd^c}u_j > 0$, $\sup_Y u_j = 0$, and

$$0 < \int_Y (h + \mathrm{dd^c}u_j)^3 \leq \frac{P_j^3}{R_j^2}. \tag{2.2}$$

In particular, $v_-(h) = 0$.

Proof. Let $\eta_j > 0$ be defined by $\eta_j^2 = \Gamma_j$. The Hermitian Calabi–Yau theorem [TW10, Corollary 1, equation (1.2)] gives unique $b_j \in \mathbb{R}$ and $u_j \in C^\infty(Y, \mathbb{R})$ such that

$$\gamma_j := h + \mathrm{dd^c} u_j > 0, \qquad \sup_Y u_j = 0, \qquad \gamma_j^3 = \mathrm{e}^{b_j} \eta_j^3.$$

Pointwise arithmetic–geometric mean, applied to the eigenvalues of γ_j relative to η_j, gives

$$\gamma_j \wedge \eta_j^2 \geq \mathrm{e}^{b_j/3} \eta_j^3.$$

Because $\mathrm{dd^c}\Gamma_j = 0$, integration by parts on Y yields

$$P_j = \int_Y \gamma_j \wedge \Gamma_j \geq \mathrm{e}^{b_j/3} R_j.$$

Therefore

$$\int_Y \gamma_j^3 = \mathrm{e}^{b_j} R_j \leq \frac{P_j^3}{R_j^2},$$

which is (2.2). Each u_j is bounded because it is smooth on the compact manifold Y. □

3. Trace concentration in the Gauduchon cone

We first prove the separation statement in the precise form needed below.

Lemma 3.1. *Let Y be a connected compact complex manifold of dimension $m \geq 2$, and let θ be a smooth real $(1,1)$-form. There is a real distribution u with*

$$\theta + \mathrm{dd^c} u \geq 0$$

if and only if

$$\int_Y \theta \wedge \Theta \geq 0$$

for every smooth strictly positive $\mathrm{dd^c}$-closed $(m-1, m-1)$-form Θ. The distribution u has a quasi-plurisubharmonic representative.

Proof. Let V be the real space of $(1,1)$-currents with its weak topology, $P \subset V$ the cone of positive currents, and

$$W = \{\mathrm{dd^c} u : u \text{ is a real distribution}\}.$$

The subspace W is closed. Indeed, after fixing a Hermitian metric h, the scalar operator $L = \mathrm{tr}_h \, \mathrm{dd^c}$ is elliptic and its kernel on functions consists of the constants by the maximum principle. Its inverse from its range to distributions normalized by one fixed mean is continuous. Thus convergence of $\mathrm{dd^c} u_j$ implies convergence, after normalization, of u_j as distributions.

Gauduchon's conformal theorem [Gau77, pp. A387–A390] supplies a smooth strictly positive $\mathrm{dd^c}$-closed $(m-1, m-1)$-form Θ_0. The cone $P + W$ is closed. Indeed, if $T_j + \mathrm{dd^c} u_j$ converges with $T_j \geq 0$, then $\int_Y T_j \wedge \Theta_0$ stays bounded. Strict positivity of Θ_0 bounds the masses of T_j, so a subnet converges weakly to a positive current. Closedness of W handles the remaining term. This argument applies to convergent nets and hence proves weak closedness.

If $\theta \notin P + W$, Hahn–Banach separation gives a smooth real $(m-1, m-1)$-form Θ that is nonnegative on P, vanishes on W, and satisfies $\int_Y \theta \wedge \Theta < 0$. The first two conditions say that Θ is semipositive and $\mathrm{dd^c}$-closed. Adding a sufficiently small positive multiple of Θ_0 makes it strictly positive while preserving the strict negative pairing, a contradiction. This proves the nontrivial implication; the converse is integration by parts. Finally, locally adding a sufficiently large smooth strictly plurisubharmonic function to u makes it plurisubharmonic, so u is quasi-psh. □

On the chart $z_0 w_0 \neq 0$ of X, use the coordinates

$$x = \frac{z_1}{z_0}, \qquad y = \frac{w_1}{w_0}, \qquad \zeta = \log \frac{w_0}{z_0} \in E := \mathbb{C}/(2\pi\mathbb{Z} + 2\pi\mathrm{i}\mathbb{Z}).$$

For every complex variable v, write

$$\sigma_v = \frac{\mathrm{i}}{\pi} \mathrm{d}v \wedge \mathrm{d}\bar{v}.$$

Let $a(s) = \mathrm{e}^{-1/s}$ for $s > 0$ and $a(s) = 0$ for $s \le 0$. Let p be the probability density, relative to σ_v, proportional to $a(1 - |v|^2)$. Define

$$\chi_0(v) = \frac{a(1 - |v|^2)}{a(1 - |v|^2) + a(|v|^2 - 1/4)}, \qquad \chi(x, y) = \chi_0(x)\chi_0(y).$$

Thus $\chi_0 = 1$ on $|v| \le 1/2$ and $\chi_0 = 0$ on $|v| \ge 1$. For $0 < \delta < 1/100$, set

$$B_\delta = \chi(x, y)\delta^{-2}p(\zeta/\delta)\sigma_\zeta, \tag{3.1}$$

extended by zero outside the coordinate tube. The extension is smooth because both cutoffs are flat at the boundaries of their supports.

Proposition 3.2. *For*

$$C_\delta := \sup\left\{\int_X B_\delta \wedge \Theta : \Theta > 0,\ \mathrm{dd^c}\Theta = 0,\ \int_X \omega \wedge \Theta = 1\right\},$$

one has $C_\delta \to \infty$ as $\delta \downarrow 0$.

Proof. For fixed δ, finiteness follows from $B_\delta \le M_\delta\omega$. Assume that $C_{\delta_\ell} \le C$ along a sequence $\delta_\ell \downarrow 0$. After normalizing any positive test form by its ω-degree, Lemma 3.1 applied to $C\omega - B_{\delta_\ell}$ gives a quasi-psh function u_ℓ satisfying

$$C\omega + \mathrm{dd^c}u_\ell \ge B_{\delta_\ell}.$$

Normalize $\sup_X u_\ell = 0$. The L^1_{loc} compactness theorem for psh functions, applied on a finite coordinate cover after adding fixed smooth strictly psh functions, gives an $L^1(X)$-convergent subsequence. Its limit u is a nontrivial $C\omega$-psh function. Since $B_{\delta_\ell} \to \chi[\zeta = 0]$ weakly as currents, passage to the distributional limit gives

$$C\omega + \mathrm{dd^c}u \ge \chi[\zeta = 0]. \tag{3.2}$$

On a smaller coordinate ball, add a smooth function ρ with $\mathrm{dd^c}\rho \ge C\omega$. Then $\mathrm{dd^c}(u + \rho) \ge [\zeta = 0]$ wherever $\chi = 1$. Hence the Lelong number of u is at least 1 on a nonempty open subset of this hypersurface. Siu's analyticity theorem [Siu74, Theorem I] shows that

$$Z = \{x \in X : \nu(u, x) \ge 1\}$$

is analytic. It is proper: positive Lelong number forces a quasi-psh function to be $-\infty$ at the point, whereas u is not identically $-\infty$.

Let $\widetilde{Z}$ be the inverse image of Z in Ω. It contains an open subset of

$$H_0 \cap \Omega, \qquad H_0 = \{w_0 = z_0\}.$$

The surface $H_0 \cap \Omega$ is $\mathbb{P}^2$ with two points removed and is connected and irreducible. The identity principle gives $H_0 \cap \Omega \subset \widetilde{Z}$. Deck invariance then gives

$$H_k \cap \Omega \subset \widetilde{Z}, \qquad H_k = \{w_0 = q^k z_0\}, \qquad k \in \mathbb{Z}.$$

Work in the chart $z_1 = 1$ near $[z_0 : z_1 : w_0 : w_1] = [0 : 1 : 1 : 0]$. For every large k, the hypersurface H_k has equation $z_0 = q^{-k}w_0$. If a local defining function for $\widetilde{Z}$ is restricted by fixing (w_0, w_1) near $(1, 0)$, it vanishes at the distinct points $q^{-k}w_0$ tending to zero. The one-variable identity theorem makes it vanish identically in z_0. Thus $\widetilde{Z}$ contains a polydisk, contradicting the properness of Z. Therefore $C_\delta \to \infty$. $\square$

We now fix the trace witnesses without leaving an unspecified choice. On X, define

$$R = \frac{zw^*}{\|z\|\|w\|}, \qquad s = \mathrm{e}^{\mathrm{i}t/2}.$$

The real and imaginary parts of the four entries of R and of s give functions $F_1, \dots, F_{10}$. The resulting map $F\colon X \to \mathbb{R}^{10}$ is a smooth embedding. Indeed, R recovers the two unit vectors $z/\|z\|$ and $w/\|w\|$ up to a common phase, while s recovers the ratio of their norms modulo $q^{\mathbb{Z}}$; these are precisely the equivalences defining X. The same reconstruction in local gauges proves immersion.

Enumerate, by increasing polynomial degree and rational coefficient height and then lexicographically, all pairs (A,ε) in which A is a 10×10 polynomial matrix with coefficients in $\mathbb{Q}+\mathrm{i}\mathbb{Q}$ and $\varepsilon\in\mathbb{Q}_{>0}$. For the m-th pair, put

$$h_m=\frac{\mathrm{i}}{\pi}\sum_{a,b=1}^{10}\left(A(F)^*A(F)+\varepsilon I\right)_{ab}\partial F_a\wedge\bar\partial F_b.$$

Since F is an immersion, h_m is a Hermitian metric. Gauduchon's theorem gives a unique positive function f_m, under the displayed normalization, such that

$$\mathrm{dd^c}(f_m h_m^2)=0,\qquad \int_X \omega\wedge f_m h_m^2=1.$$

Set $\Theta_m=f_m h_m^2$.

Lemma 3.3. *The sequence (Θ_m) is dense, for the C^∞ topology, in the set of smooth strictly positive $\mathrm{dd^c}$-closed $(2,2)$-forms of ω-degree one.*

Proof. The covectors ∂F_a generate the complex cotangent bundle. A Hermitian metric can therefore be expressed in them using a smooth positive Hermitian coefficient matrix: take a smooth right inverse of the generating map and add a positive matrix on its kernel. Extend the positive square root of this coefficient matrix to a neighborhood of the embedded compact manifold in $\mathbb{R}^{10}$. Multivariable polynomial approximation on a surrounding cube, with derivatives through any fixed order, followed by rational approximation of coefficients, shows that the h_m are C^∞-dense in the Hermitian metrics.

The normalized Gauduchon factor depends continuously in the C^∞ topology on the metric. To see this, its equation is a scalar linear elliptic equation with smoothly varying coefficients. The integral normalization, the Harnack inequality, and elliptic estimates give uniform bounds in every derivative; every subsequential limit is the uniquely normalized positive solution of the limiting equation. If $\Theta>0$ is $\mathrm{dd^c}$-closed and has ω-degree one, let $\eta>0$ be its square root. The normalized Gauduchon factor of η is 1. Approximating η by h_m proves the assertion. □

For $r\geq 1$, let $\delta_r=2^{-r}/100$ and let $B_r=B_{\delta_r}$. For each $j\geq 1$, let (r_j,m_j) be the first pair, ordered by increasing $r+m$ and then by increasing r, such that

$$r\geq j,\qquad J_j:=\int_X B_r\wedge\Theta_m>j^4. \tag{3.3}$$

The search terminates by Proposition 3.2 and Lemma 3.3.

4. Bounded degree and divergent root volume

Fix j, and abbreviate

$$\delta=\delta_{r_j},\qquad \Theta=\Theta_{m_j}.$$

For $a,b\in\mathbb{C}$ and for τ in the small coordinate disk of E, the map

$$T_{a,b,\tau}[z_0:z_1:w_0:w_1]=[z_0:z_1+az_0:\mathrm{e}^\tau w_0:\mathrm{e}^\tau(w_1+bw_0)]$$

is a holomorphic automorphism of X. It commutes with the deck transformation and acts on the product chart by

$$(x,y,\zeta)\longmapsto(x+a,y+b,\zeta+\tau).$$

Average $T_{a,b,\tau}^*\Theta$ against

$$3^{-4}p(a/3)p(b/3)\sigma_a\wedge\sigma_b,\qquad (3\delta)^{-2}p(\tau/(3\delta))\sigma_\tau,$$

and denote the resulting form by $\widehat\Theta_j$.

Lemma 4.1. *The forms $\widehat\Theta_j$ are smooth, strictly positive, and $\mathrm{dd^c}$-closed. There is a constant $C>0$, independent of j, such that*

$$\int_X\omega\wedge\widehat\Theta_j\leq C.$$

If $\widehat{F}_j$ is the coefficient of $\sigma_x \wedge \sigma_y$ in the restriction of $\widehat{\Theta}_j$ to a ζ-slice, then a constant $c > 0$, independent of j, satisfies

$$\widehat{F}_j \geq c J_j \tag{4.1}$$

on

$$V_j = \{|x|, |y| < 1/4,\ |\zeta| < \delta/4\}.$$

Proof. The first assertions follow by averaging pullbacks. All parameters lie in one compact set, so $(T_{a,b,\tau}^{-1})^*\omega \leq C\omega$ uniformly. Change of variables and $\int_X \omega \wedge \Theta = 1$ give the degree bound.

Let $F(x, y, \zeta) \geq 0$ be the corresponding slice coefficient of Θ. Pullback by the translations above makes $\widehat{F}_j$ its convolution with the displayed kernels. For a point of V_j, the base kernel has a fixed positive lower bound at every translation carrying the point into $\operatorname{supp}\chi$, because $|a|, |b| \leq 5/4$ there. The fiber kernel is at least a fixed multiple of δ^{-2} at every translation carrying ζ into $|\zeta'| < \delta$, because $|\tau| \leq 5\delta/4$. Since $\delta^{-2}p(\zeta'/\delta) \leq (\max p)\delta^{-2}$ and $0 \leq \chi \leq 1$, the integral in (3.3) gives (4.1). □

We next construct the transverse spike. Put

$$f(a, b) = p(4(a-1))p(b/2), \qquad w_j(\tau) = 4\delta^{-2}p(2\tau/\delta).$$

For $0 \leq \varepsilon \leq 1/16$, consider the rational curves

$$\iota_{\tau,u,a,b}([z_0 : z_1]) = [z_0 : z_1 : \mathrm{e}^{\tau}(z_0 + uz_1) : \mathrm{e}^{\tau}(az_1 + bz_0)].$$

Average their integration currents against

$$w_j(\tau)\sigma_\tau, \qquad f(a, b)\sigma_a \wedge \sigma_b, \qquad \varepsilon^{-2}p(u/\varepsilon)\sigma_u$$

when $\varepsilon > 0$. When $\varepsilon = 0$, set $u = 0$ and omit the last integration. Denote the result by $T_{j,\varepsilon}$.

Lemma 4.2. *For $\varepsilon > 0$, the current $T_{j,\varepsilon}$ is a globally smooth semipositive d-closed $(2, 2)$-form. There is a constant $C' > 0$, independent of j and ε, such that*

$$\int_X \omega \wedge T_{j,\varepsilon} \leq C'.$$

For fixed j, the forms $T_{j,\varepsilon}$ converge in C^∞ on compact subsets of the product chart as $\varepsilon \downarrow 0$. On that chart,

$$T_{j,0} = w_j(\zeta)\sigma_\zeta \wedge K_f, \tag{4.2}$$

where

$$K_f = \int_{\mathbb{C}} f(a, y - ax)\frac{\mathrm{i}}{\pi}(\mathrm{d}y - a\,\mathrm{d}x) \wedge (\mathrm{d}\bar{y} - \bar{a}\,\mathrm{d}\bar{x}) \wedge \sigma_a.$$

The form K_f is uniformly positive on $|x|, |y| \leq 1/4$.

Proof. On the parameter supports,

$$|a - bu| \geq \frac{3}{4} - \frac{2}{16} = \frac{5}{8}.$$

Thus the matrix $\left(\begin{smallmatrix} 1 & u \\ b & a \end{smallmatrix}\right)$ is uniformly nonsingular, so each map is an embedded compact rational curve in X. The evaluation map from $\mathbb{P}^1$ times the parameter space to X is a holomorphic submersion. At a point with $z_1 \neq 0$, the u- and a-variations span the two vector coordinates; at $z_1 = 0$, the τ- and b-variations do so. The $\mathbb{P}^1$ variable supplies the remaining direction. Fiber integration of the smooth compactly supported parameter density is therefore smooth. Positivity and closedness follow by averaging positive closed integration currents.

The matrices and all parameters range over one compact set, and the ω-area of the curves varies continuously there. Since the τ- and u-weights have mass one and the a, b-weight has fixed finite mass, the degree is uniformly bounded.

For $u = 0$, the curves have local equations

$$\zeta = \tau, \qquad y = ax + b.$$

Integrating first in b and τ gives (4.2). For $|x|, |y| \leq 1/4$, the function $f(a, y - ax)$ has a fixed positive lower bound as a ranges in a sufficiently small disk about 1. No nonzero horizontal tangent vector is annihilated by all covectors $\mathrm{d}y - a\,\mathrm{d}x$ in this disk. Compactness of the unit tangent sphere gives a uniform positive lower eigenvalue for K_f.

For general u, the local equations are

$$\zeta = \tau + \log(1+ux), \qquad y = \frac{ax+b}{1+ux}.$$

On a fixed compact subset and for small u, solve smoothly for

$$\tau = \zeta - \log(1+ux), \qquad b = (1+ux)y - ax.$$

The local coefficients and all their derivatives therefore depend smoothly on u. Averaging against the approximate identity $\varepsilon^{-2}p(u/\varepsilon)\sigma_u$ proves the asserted local C^∞ convergence. □

The following determinant estimate is the only pointwise inequality needed to combine the two averages.

Lemma 4.3. *If*

$$Q = \begin{pmatrix} A & p \\ p^* & F \end{pmatrix} > 0$$

is a Hermitian 3×3 matrix with a 2×2 block A, and if $B > 0$ is a Hermitian 2×2 matrix, then

$$\det\left(Q + \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\right) = F\det(B + A - pp^*/F) \geq F\det B. \tag{4.3}$$

Proof. The equality is the Schur-complement formula. Positivity of Q gives $A - pp^*/F > 0$, and determinant is monotone on the positive Hermitian cone. □

Proposition 4.4. *There are constants $C_2, c_1 > 0$ and a specified sequence of smooth strictly positive* $\mathrm{dd^c}$*-closed* $(2,2)$*-forms Γ_j such that*

$$P_j := \int_X \omega\wedge\Gamma_j \leq C_2, \qquad R_j := \int_X \Gamma_j^{3/2} \geq \frac{c_1}{2}j^2. \tag{4.4}$$

Proof. In the product coframe, write the cofactor matrix of $\widehat{\Theta}_j$ as

$$Q_j = \begin{pmatrix} A_j & p_j \\ p_j^* & \widehat{F}_j \end{pmatrix}.$$

By (4.2), the cofactor matrix of $T_{j,0}$ has the form $\operatorname{diag}(B_j, 0)$. The lower bound for K_f and the definition of w_j give

$$B_j \geq c_0\delta^{-2}I_2$$

on V_j. Combining Lemmas 4.1 and 4.3 with the root density formula (2.1), we find that $(\widehat{\Theta}_j + T_{j,0})^{3/2}$ is bounded below on V_j by

$$c\delta^{-2}\sqrt{J_j}\,\sigma_x\wedge\sigma_y\wedge\sigma_\zeta.$$

For

$$W_j = \{|x|, |y| < 1/8,\ |\zeta| < \delta/8\},$$

the base area is fixed and the fiber area is a fixed multiple of δ^2. Hence

$$I_j := \int_{W_j} (\widehat{\Theta}_j + T_{j,0})^{3/2} \geq c_1\sqrt{J_j} \geq c_1 j^2. \tag{4.5}$$

Let s_j be the least positive integer s such that

$$\int_{W_j} (\widehat{\Theta}_j + T_{j,2^{-s}/16})^{3/2} > \frac{I_j}{2}.$$

This integer exists by the local C^∞ convergence in Lemma 4.2 and the smooth dependence of the positive square root on a strictly positive $(2,2)$-form. Define

$$\Gamma_j = \widehat{\Theta}_j + T_{j,2^{-s_j}/16}.$$

The first summand is strictly positive and dd^c-closed; the second is smooth, semipositive, and d-closed. Thus Γ_j is smooth, strictly positive, and dd^c-closed. The two uniform degree bounds give the first inequality in (4.4), while (4.5) gives the second. □

5. The Monge–Ampère potentials

Proof of Theorem A. Apply Proposition 2.3 to the threefold and Hermitian form in Proposition 2.1, using the forms from Proposition 4.4. It gives smooth normalized functions φ_j with $\omega + dd^c\varphi_j > 0$ and

$$0 < \int_X (\omega + dd^c\varphi_j)^3 \leq \frac{P_j^3}{R_j^2} \leq \frac{4C_2^3}{c_1^2 j^4} \longrightarrow 0.$$

These smooth functions are bounded on X, so they belong to the class in the definition of $v_-(\omega)$. Since their Monge–Ampère masses are positive and converge to zero, $v_-(\omega) = 0$. □

References

[Gau77] P. Gauduchon. Le théorème de l'excentricité nulle. *C. R. Acad. Sci. Paris Sér. A-B* 285.5 (1977), A387–A390.

[Siu74] Y.-T. Siu. Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. *Invent. Math.* 27.1 (1974), pp. 53–156. URL: https://doi.org/10.1007/BF01389965.

[TW10] V. Tosatti and B. Weinkove. The complex Monge–Ampère equation on compact Hermitian manifolds. *J. Amer. Math. Soc.* 23.4 (2010), pp. 1187–1195. URL: https://doi.org/10.1090/S0894-0347-2010-00673-X.