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Constructions of complex analytic spaces

1. Introduction

2. Analytic spectra

Proposition 2.1. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Then the presheaf $F_{\mathcal{A}}$ on $\mathbb{C}\text{-An}/_S$ defined by

$$F_{\mathcal{A}}(T \xrightarrow{p} S) = \text{Hom}_{\mathcal{O}_T}(p^* \mathcal{A}, \mathcal{O}_T)$$

is representable.

PROOF. By the arguments of [Stacks, Tag 01JJ], the problem is local in S . So we may assume that \mathcal{A} has the following form

$$\mathcal{A} = \mathcal{O}_S[X_1, \dots, X_n]/\mathcal{I}$$

for some $n \in \mathbb{N}$ and $\mathcal{I} \subseteq \mathcal{O}_S(S)[X_1, \dots, X_n]$ an ideal sheaf of finite type.

Step 1. We first handle the case where $\mathcal{A} = \mathcal{O}_S[X_1, \dots, X_n]$.

In this case, we claim that $F_{\mathcal{A}}$ is represented by $S \times \mathbb{C}^n$. In fact, it suffices to observe that

$$\begin{aligned} F_{\mathcal{A}}(T \xrightarrow{p} S) &\xrightarrow{\sim} \text{Hom}_{\mathcal{O}_T}(\mathcal{O}_T[X_1, \dots, X_n], \mathcal{O}_T) \xrightarrow{\sim} \mathcal{O}_T(T)^n \\ &= \text{Hom}_{\mathbb{C}\text{-An}}(T, \mathbb{C}^n) = \text{Hom}_{\mathbb{C}\text{-An}/_S}(T, S \times \mathbb{C}^n). \end{aligned}$$

From this proof, it is easy to see that the universal morphism is

$$\eta : \mathcal{O}_{S \times \mathbb{C}^n}[X_1, \dots, X_n] \rightarrow \mathcal{O}_{S \times \mathbb{C}^n}$$

sending X_i to z_i , the i -th coordinate of \mathbb{C}^n .

Step 2. We handle the general case. We have a short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_S[X_1, \dots, X_n] \rightarrow \mathcal{A} \rightarrow 0.$$

For any $p : T \rightarrow S$ in $\mathbb{C}\text{-An}$, we have an exact sequence

$$p^* \mathcal{I} \rightarrow \mathcal{O}_T[X_1, \dots, X_n] \rightarrow p^* \mathcal{A} \rightarrow 0.$$

We then have

$$\begin{aligned} F_{\mathcal{A}}(T) &\xrightarrow{\sim} \{h \in \text{Hom}_{\mathcal{O}_T}(\mathcal{O}_T[X_1, \dots, X_n], \mathcal{O}_T) : h|_{p^* \mathcal{I}} = 0\} \\ &\xrightarrow{\sim} \{h \in F_{\mathcal{O}_S[X_1, \dots, X_n]}(T) : h|_{p^* \mathcal{I}} = 0\}. \end{aligned}$$

Let $\pi : S \times \mathbb{C}^n \rightarrow S$ be the projection. Then $F_{\mathcal{A}}(T)$ is represented by the closed subspace of $S \times \mathbb{C}^n$ defined by the ideal $\eta(\pi^* \mathcal{I})$, which is clearly of finite type. \square

Definition 2.2. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Then the complex analytic space representing the functor $F_{\mathcal{A}}$ in Proposition 2.1 is called the *analytic spectrum* of \mathcal{A} . We denote it by $\text{Spec}_S^{\text{an}} \mathcal{A}$. By construction, there is a canonical morphism $\text{Spec}_S^{\text{an}} \mathcal{A} \rightarrow S$.

By definition, we have a universal morphism $\xi \in F_{\mathcal{A}}(X) = \text{Hom}_{\mathcal{O}_X}(\mathcal{A}_X, \mathcal{O}_X)$ with $X = \text{Spec}_S^{\text{an}} \mathcal{A}$. It defines a morphism of ringed spaces $X \rightarrow (|S|_2, \mathcal{A})$. The pull-back of an \mathcal{A} -module \mathcal{M} is denoted by $\tilde{\mathcal{M}}$. The assignment $\mathcal{M} \mapsto \tilde{\mathcal{M}}$ is functorial in \mathcal{M} .

It is easy to see that $\text{Spec}_S^{\text{an}} \mathcal{A}$ is contravariant in \mathcal{A} .

Proposition 2.3. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Consider a morphism $g : S' \rightarrow S$ of complex analytic spaces. Then we have a Cartesian diagram

$$\begin{array}{ccc} \text{Spec}_{S'}^{\text{an}} g^* \mathcal{A} & \longrightarrow & \text{Spec}_S^{\text{an}} \mathcal{A} \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

PROOF. This is clear at the level of functor of points. \square

Corollary 2.4. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Take $s \in S$. Then $\text{Spec}_{\{s\}}^{\text{an}} \mathcal{A}_s \xrightarrow{\sim} (\text{Spec}_S^{\text{an}} \mathcal{A})_s$.

Moreover, the universal morphism $\mathcal{A}_{\text{Spec}_{\{s\}}^{\text{an}} \mathcal{A}_s} \rightarrow \mathcal{O}_{\text{Spec}_{\{s\}}^{\text{an}} \mathcal{A}_s}$ is the reduction of the universal morphism $\mathcal{A}_{\text{Spec}_S^{\text{an}} \mathcal{A}} \rightarrow \mathcal{O}_{\text{Spec}_S^{\text{an}} \mathcal{A}}$ modulo \mathfrak{m}_s .

PROOF. This follows from [Proposition 2.3](#). \square

Proposition 2.5. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Take $s \in S$. Write $X = \text{Spec}_S^{\text{an}} \mathcal{A}$ and $\mathcal{A}_s := \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S,s}$. Then the map from X_s to

$$\{\mathfrak{m} \in \text{Spm}_{\mathbb{C}} \mathcal{A}_s : \mathfrak{m} \supseteq \mathfrak{m}_s\}$$

sending $x \in X_s$ to the inverse image of \mathfrak{m}_x with respect to $\mathcal{A}_s \rightarrow \mathcal{O}_{X,x}$ is bijective.

If \mathfrak{m} corresponds to $x \in X_s$, then the natural homomorphism $\mathcal{A}_s \rightarrow \mathcal{O}_{X,x}$ factorizes through $\mathcal{A}_{s,\mathfrak{m}} \rightarrow \mathcal{O}_{X,x}$. The completion of the latter

$$\widehat{\mathcal{A}_{s,\mathfrak{m}}} \rightarrow \widehat{\mathcal{O}_{X,x}}$$

is an isomorphism.

PROOF. By [Corollary 2.4](#), we have natural bijections

$$X_s \xrightarrow{\sim} \text{Hom}_{\{s\}}(\{s\}, X_s) \xrightarrow{\sim} \text{Hom}_{\mathbb{C}\text{-Alg}}(\mathcal{A}_s/\mathfrak{m}_s \mathcal{A}_s, \mathbb{C}).$$

This gives the desired bijection.

Next we prove the latter part. The problem is local on S , we may assume that

$$\mathcal{A} = \mathcal{O}_S[X_1, \dots, X_n]/\mathcal{I}$$

for some $n \in \mathbb{N}$ and some ideal \mathcal{I} of finite type in $\mathcal{O}_S[X_1, \dots, X_n]$. Recall that the universal morphism

$$\eta : \mathcal{O}_{S \times \mathbb{C}^n}[X_1, \dots, X_n] \rightarrow \mathcal{O}_{S \times \mathbb{C}^n}$$

sends X_i to z_i , the i -th coordinate of \mathbb{C}^n .

By construction, we have

$$\mathcal{A}_s \xrightarrow{\sim} \mathcal{O}_{S,s}[X_1, \dots, X_n]/\mathcal{I}_s$$

and

$$\mathcal{O}_{X,x} = \mathcal{O}_{S \times \mathbb{C}^n, x}/\mathcal{J}_x,$$

where

$$\mathcal{J}_x = \eta_x(\mathcal{I}_s \mathcal{O}_{S \times \mathbb{C}^n, x}[X_1, \dots, X_n]).$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_s & \longrightarrow & \mathcal{O}_{S,s}[X_1, \dots, X_n] & \longrightarrow & \mathcal{A}_s \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{J}_x & \longrightarrow & \mathcal{O}_{S \times \mathbb{C}^n, x} & \longrightarrow & \mathcal{O}_{X,x} \longrightarrow 0 \end{array}.$$

The middle vertical map is induced by η_x . Let \mathfrak{p} be the inverse image of $\mathfrak{m}_{S \times \mathbb{C}^n, x}$ under the vertical map in the middle. Then \mathfrak{p} is generated by \mathfrak{m}_s and $X_1 - x_1, \dots, X_n - x_n$, where $x_i \in \mathbb{C}$ is the value of z_i at x for $i = 1, \dots, n$. By localization and completion, we find a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{(\mathcal{I}_s)}_{\mathfrak{p}} & \longrightarrow & (\widehat{\mathcal{O}_{S,s}[X_1, \dots, X_n]})_{\mathfrak{p}} & \longrightarrow & \widehat{(\mathcal{A}_s)}_{\mathfrak{m}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widehat{\mathcal{J}}_x & \longrightarrow & \widehat{\mathcal{O}_{S \times \mathbb{C}^n, x}} & \longrightarrow & \widehat{\mathcal{O}_{X,x}} \longrightarrow 0 \end{array}.$$

Observe that

$$(\widehat{\mathcal{O}_{S,s}[X_1, \dots, X_n]})_{\mathfrak{p}} \cong \widehat{\mathcal{O}_{S,s}}[[X_1 - x_1, \dots, X_n - x_n]]$$

and

$$\widehat{\mathcal{O}_{S \times \mathbb{C}^n, x}} \cong \widehat{\mathcal{O}_{S,s}} \hat{\otimes}_k \widehat{\mathcal{O}_{\mathbb{C}^n, (x_1, \dots, x_n)}} \cong \widehat{\mathcal{O}_{S,s}}[[X_1 - x_1, \dots, X_n - x_n]].$$

It is easy to see that the middle map is an isomorphism. As \mathcal{J}_x is generated by \mathcal{I}_s , the first vertical map is also an isomorphism. Our assertion follows. \square

Corollary 2.6. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra. Write $X = \text{Spec}_S^{\text{an}} \mathcal{A}$. Take $s \in S$. Then the fiber X_s is finite and is in bijection with $\text{Spm}_{\mathbb{C}} \mathcal{A}_s = \text{Spm} \mathcal{A}_s$. If \mathfrak{m} corresponds to $x \in X_s$, then we have a natural isomorphism

$$\mathcal{A}_{s, \mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{X,x}.$$

PROOF. We first observe that as \mathcal{A}_s is a finite $\mathcal{O}_{S,s}$ -algebra, its residue fields at maximal primes are finite extensions of the residue field \mathbb{C} of $\mathcal{O}_{S,s}$. So $\text{Spm}_{\mathbb{C}} \mathcal{A}_s = \text{Spm} \mathcal{A}_s$.

As $\mathcal{O}_{S,s} \rightarrow \mathcal{A}_s$ is finite, \mathcal{A}_s is semi-local. On the other hand, by [Proposition 2.5](#),

$$\mathcal{A}_{s, \mathfrak{m}} \rightarrow \mathcal{O}_{X,x}$$

is injective and $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{S,s}$. Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$ by [Theorem 5.4](#) in [Complex analytic local algebras](#). It follows from Nakayama's lemma that $\mathcal{A}_{s, \mathfrak{m}} \rightarrow \mathcal{O}_{X,x}$ is also surjective. \square

Corollary 2.7. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra. Then the image of $\text{Spec}_S^{\text{an}} \mathcal{A} \rightarrow S$ is $\text{Supp} \mathcal{A}$.

PROOF. This follows from [Corollary 2.6](#) and the fact that $\text{Spm}_{\mathbb{C}} \mathcal{A}_s = \text{Spm} \mathcal{A}_s$ for all $s \in S$. \square

Proposition 2.8. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra. Write $f : \text{Spec}_S^{\text{an}} \mathcal{A}$ for the structure map. Then we have the following assertions:

(1) for all \mathcal{A} -module \mathcal{M} , the natural morphism

$$\mathcal{M} \rightarrow f_*\tilde{\mathcal{M}}$$

is an isomorphism,

In particular, $\mathcal{A} \xrightarrow{\sim} f_*\mathcal{O}_X$.

(2) for all \mathcal{O}_X -module \mathcal{F} , the canonical morphism

$$\widehat{f_*\mathcal{F}} \rightarrow \mathcal{F}$$

is an isomorphism.

In particular, the category of \mathcal{A} -modules is equivalent to the category of \mathcal{O}_X -modules.

PROOF. By [Corollary 3.8](#), f is topologically finite. Take $s \in S$. Let x_1, \dots, x_n be the distinct points of $f^{-1}(s)$ and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ denote the maximal ideals of \mathcal{A}_s corresponding to x_1, \dots, x_n .

(1) By [Corollary 4.10](#) in [Topology and bornology](#) and [Corollary 2.6](#),

$$(f_*\tilde{\mathcal{M}})_s \cong \prod_{i=1}^n \widehat{M}_{x_i} \cong \prod_{i=1}^n \widehat{\mathcal{M}}_s \otimes_{\mathcal{A}_s} \mathcal{O}_{X,x_i} \cong \mathcal{M}_s \otimes_{\mathcal{A}_s} \prod_{i=1}^n \mathcal{A}_{s,\mathfrak{m}_i} \xrightarrow{\sim} \mathcal{M}_s.$$

(2) By [Corollary 4.10](#) in [Topology and bornology](#),

$$f_*\mathcal{F}_s \cong \prod_{i=1}^n \mathcal{F}_{x_i}.$$

It follows that

$$\widehat{f_*\mathcal{M}}_{x_i} \cong f_*\mathcal{F}_s \otimes_{\mathcal{A}_s} \mathcal{O}_{X,x_i} \cong \prod_{j=1}^n \mathcal{F}_{x_j} \otimes_{\mathcal{A}_s} \mathcal{A}_{s,\mathfrak{m}_i}$$

for $i = 1, \dots, n$. But the only non-zero term is when $j = i$, so

$$\widehat{f_*\mathcal{M}}_{x_i} \cong \mathcal{F}_{x_i}$$

for $i = 1, \dots, n$. □

Corollary 2.9. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra. Write $f : \text{Spec}_S^{\text{an}} \mathcal{A}$ for the structure map. Then for any coherent \mathcal{O}_X -module \mathcal{M} , $f_*\mathcal{F}$ is coherent.

Moreover, f_* is exact from $\text{Coh}(\mathcal{O}_X)$ to $\text{Coh}(\mathcal{O}_Y)$.

PROOF. The exactness of f_* follows from [Proposition 2.8](#).

We claim that up to shrinking S , we may assume that \mathcal{M} has a global presentation. Fix $s \in S$ and let x_1, \dots, x_n be the distinct points of $f^{-1}(s)$.

For each $j = 1, \dots, n$, we can find an open neighbourhood U_j of x_j in X , pairwise disjoint and an exact sequence

$$\mathcal{O}_{U_j}^{p_j} \rightarrow \mathcal{O}_{U_j}^{q_j} \rightarrow \mathcal{M}|_{U_j} \rightarrow 0$$

for some $p_j, q_j \in \mathbb{Z}_{>0}$. We may assume that $p_1 = \dots = p_n$ and $q_1 = \dots = q_n$. We denote the common values by p and q . Then $U = U_1 \cup \dots \cup U_n$ is a neighbourhood of $f^{-1}(s)$, and we have an exact sequence

$$\mathcal{O}_U^p \rightarrow \mathcal{O}_U^q \rightarrow \mathcal{M}|_U \rightarrow 0.$$

By Lemma 4.2 in [Topology and bornology](#), we may assume that $U = \pi^{-1}(V)$ for some open neighbourhood V of s in S . The induced map $f' : U \rightarrow V$ is finite and by Corollary 4.10 in [Topology and bornology](#).

Now let us take a presentation

$$\mathcal{O}^p \rightarrow \mathcal{O}^q \rightarrow \mathcal{M} \rightarrow 0.$$

By Proposition 2.8, we have an exact sequence

$$f_*\mathcal{O}^p \rightarrow f_*\mathcal{O}^q \rightarrow f_*\mathcal{M} \rightarrow 0.$$

By Proposition 2.8 again, this can be written as

$$\mathcal{A}^p \rightarrow \mathcal{A}^q \rightarrow f_*\mathcal{M} \rightarrow 0.$$

It follows that $f_*\mathcal{M}$ is coherent. \square

Proposition 2.10. Let S be a complex analytic space and \mathcal{A}, \mathcal{B} be \mathcal{O}_S -algebras of finite presentation. Assume that \mathcal{A} is finite. Then we have a natural bijection

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-An}/S}(\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}, \mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}).$$

PROOF. Let $f : X := \mathrm{Spec}_S^{\mathrm{an}} \mathcal{A} \rightarrow S$ be the natural map. We construct the bijection as

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, f_*\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{B}_X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-An}/S}(\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}, \mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}).$$

The first map is a bijection by Proposition 2.8 \square

Definition 2.11. Let S be a complex analytic space and \mathcal{E} be an \mathcal{O}_S -module of finite presentation. We define the *vector bundle* $\mathbf{V}(\mathcal{E})$ generated by \mathcal{E} as

$$\mathbf{V}(\mathcal{E}) = \mathrm{Spec}_S^{\mathrm{an}} \mathrm{Sym} \mathcal{E}.$$

We have a natural projection $\mathbf{V}(\mathcal{E}) \rightarrow S$.

We remind the readers that we are following Grothendieck's convention for $\mathbf{V}(\mathcal{E})$, which is different from Fulton's.

3. Analytic germs

Definition 3.1. A *pointed complex analytic space* is a pair (X, x) consisting of a complex analytic space X and a point $x \in X$. A morphism between pointed complex analytic spaces (X, x) and (Y, y) is a morphism $f : X \rightarrow Y$ of complex analytic spaces such that $f(x) = y$. The category of pointed complex analytic spaces is denoted by $\mathbb{C}\text{-An}_*$.

The category of *complex analytic germs* $\mathbb{C}\text{-Ger}$ is the right category of fractions of $\mathbb{C}\text{-An}$ with respect to the system of morphisms $f : (X, x) \rightarrow (Y, y)$ such that $f : X \rightarrow Y$ is an open immersion. An element in $\mathbb{C}\text{-Ger}$ is called a *complex analytic germ*. A complex analytic germ represented by (X, x) is denoted by X_x .

Given a complex analytic germ X_x , we write $\mathcal{O}_{X,x}$ for the local ring of X at x . Clearly, it does not depend on the choice of (X, x) . Given any morphism $f : X_x \rightarrow Y_y$ of complex analytic germs, we have an obvious local homomorphism $f^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$.

Definition 3.2. Given a complex analytic germ X_x , a *closed subgerm* of X_x is an isomorphism class in $\mathbb{C}\text{-Ger}/X_x$ of Y_x represented by a closed analytic subspace of X containing x for any representation (X, x) of X_x .

In particular, X_x is a closed subgerm of X_x . A closed subgerm Y_y of X_x is *proper* if Y_y is different from X_x as subgerms.

Given a closed subgerm Y_x of X_x , we have an induced surjective homomorphism $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$. The kernel is denoted by $I(Y, x)$ or $I_X(Y, x)$.

Theorem 3.3. The functor $\mathbb{C}\text{-Ger}^{\text{op}} \rightarrow \mathbb{C}\text{-LA}$ defined in [Definition 3.1](#) is an equivalence.

PROOF. Step 1. We show that the functor is faithfully.

In other words, let (X, x) and (Y, y) be two pointed complex analytic spaces and $f, g : (X, x) \rightarrow (Y, y)$ be two morphisms inducing the same map $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$, then f and g coincide on a neighbourhood of x in X .

The question is open on Y , so we may reduce to the case where Y is a complex model space. We then further reduce to the case where Y is a domain in \mathbb{C}^n for some $n \in \mathbb{N}$ and then to $Y = \mathbb{C}^n$.

By [Theorem 4.2](#) in [The notion of complex analytic spaces](#), f and g can be identified with systems $(f_1, \dots, f_n) \in \mathcal{O}_X(X)^n$ and $(g_1, \dots, g_n) \in \mathcal{O}_X(X)^n$. The assumption $f_x^\# = g_x^\#$ means $f_{i,x} = g_{i,x}$ for $i = 1, \dots, n$. So $f_i = g_i$ after shrinking X . We conclude by [Theorem 4.2](#) in [The notion of complex analytic spaces](#) again.

Step 2. We show that the functor is fully faithful.

In other words, let (X, x) and (Y, y) be two pointed complex analytic spaces and $\varphi : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ be a morphism in $\mathbb{C}\text{-LA}$. Then we can find an open neighbourhood U of x in X and a morphism $(U, x) \rightarrow (Y, y)$ inducing φ .

The problem is local on Y , so we may assume that Y is a complex model space, say Y is a closed subspace of a domain V in \mathbb{C}^n defined by a coherent ideal \mathcal{I} . We write $\psi : \mathcal{O}_{V,y} \rightarrow \mathcal{O}_{X,x}$ the homomorphism induced by φ , we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{V,y} & \xrightarrow{\psi} & \mathcal{O}_{Y,y} \\ \downarrow & \nearrow \varphi & \\ \mathcal{O}_{X,x} & & \end{array} .$$

Let z_1, \dots, z_n be the coordinates on V . Let $f_{i,x}$ be the image of $z_{i,x}$ under ψ for $i = 1, \dots, n$. Take an open neighbourhood U of x in X so that $f_{i,x}$ lifts to $f_i \in \mathcal{O}_X(U)$ for $i = 1, \dots, n$. By [Theorem 4.2](#) in [The notion of complex analytic spaces](#), f_1, \dots, f_n then defines a morphism $g : U \rightarrow \mathbb{C}^n$. Clearly $g(x) = y$. But $g_x^\#$ and ψ coincide on $z_{i,y}$ so $g_x^\# = \psi$ as $\mathcal{O}_{V,y} = \mathbb{C}\{z_{1,y} - a_1, \dots, z_{n,y} - a_n\}$ with $a_i = \epsilon(z_{i,y})$ for $i = 1, \dots, n$. Therefore, $g_x^\#(\mathcal{I}_y) = 0$. Up to shrinking U , we may guarantee that $g(U) \subseteq V$ and $g^*(\mathcal{I}) = 0$ on U . Namely, g factorizes through $f : U \rightarrow Y$ and $f_x^* = \varphi$.

Step 3. We show that the functor is essentially surjective.

In other words, let A be a complex analytic local algebra, then there is a pointed complex analytic space (X, x) with $\mathcal{O}_{X,x} \cong A$ in $\mathbb{C}\text{-LA}$.

We may assume that $A = \mathbb{C}\{z_1, \dots, z_n\}/I$ for some $n \in \mathbb{N}$ and ideal I in $\mathbb{C}\{z_1, \dots, z_n\}$. Then I is finitely generated as $\mathbb{C}\{z_1, \dots, z_n\}$ is noetherian. Take finitely many generators $f_1, \dots, f_m \in I$. We extend f_1, \dots, f_m to $g_1, \dots, g_m \in$

$\mathcal{O}_{\mathbb{C}^n}(U)$ for some open neighbourhood U of 0 in \mathbb{C}^n . Then the closed subspace X of U defined by f_1, \dots, f_m satisfies the required conditions. \square

Corollary 3.4. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

- (1) f is a local isomorphism;
- (2) $f_x^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is an isomorphism;
- (3) $\widehat{f_x^\#} : \widehat{\mathcal{O}}_{Y,f(x)} \rightarrow \widehat{\mathcal{O}}_{X,x}$ is an isomorphism.

Later on, we will see that Condition (3) means f is étale at x .

PROOF. (1) \Leftrightarrow (2): This follows from [Theorem 3.3](#).

(2) \implies (3): This is clear.

(3) \implies (2): As $f_x^\#$ is quasi-finite, the \mathfrak{m}_x -adic topology on $\mathcal{O}_{X,x}$ coincides with the $\mathfrak{m}_{f(x)}$ -adic topology on it regarded as an $\mathcal{O}_{Y,f(x)}$ -module. By [Theorem 5.4](#) in [Complex analytic local algebras](#), $f_x^\#$ is finite. So

$$\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \widehat{\mathcal{O}}_{Y,f(x)}.$$

So (2) follows from the fact that $\widehat{\mathcal{O}}_{Y,f(x)}$ is faithfully flat over $\mathcal{O}_{Y,f(x)}$, see [[Stacks, Tag 00MC](#)]. \square

Corollary 3.5. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

- (1) f is a local immersion at x ;
- (2) $f_x^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is surjective;
- (3) $\widehat{f_x^\#} : \widehat{\mathcal{O}}_{Y,f(x)} \rightarrow \widehat{\mathcal{O}}_{X,x}$ is surjective;
- (4) $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} \xrightarrow{\sim} \mathbb{C}$.

PROOF. (1) \implies (2): This is clear.

(2) \implies (1): Let I be the kernel of $f_x^\#$. Up to shrinking X , we may assume that I spreads to a coherent ideal sheaf \mathcal{I} on Y . Let Y' be the closed analytic subspace of Y defined by \mathcal{I} . Up to shrinking X , we may assume that f factorizes through $f' : X \rightarrow Y'$ by [Theorem 3.3](#). But $f_x'^\#$ is an isomorphism, so f' is a local isomorphism by [Corollary 3.4](#).

(2) \Leftrightarrow (3): This follows from the same arguments as in [Corollary 3.4](#).

(2) \Leftrightarrow (4): This follows from Nakayama's lemma. \square

Corollary 3.6. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces. Then the following are equivalent:

- (1) f is an immersion;
- (2) $|f|$ induces a homeomorphism of $|X|$ with a locally closed subset of $|Y|$ and for all $x \in X$, the homomorphism $f_x^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is surjective.

The condition in (2) is the usual definition of an immersion of ringed spaces. Our notion of immersion is usually called a locally closed immersion.

PROOF. (1) \implies (2): This is clear by definition.

(2) \implies (1): We may clearly assume that $f(X)$ is closed in Y . We need to show that the kernel of $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is of finite type. This follows from [Corollary 3.5](#). \square

Lemma 3.7. Let S be a complex analytic space and $s \in S$. For any finite $\mathcal{O}_{S,s}$ -algebra A , there is an open neighbourhood U of s in S and a finite \mathcal{O}_U -algebra such that $\mathcal{A}_s \cong A$.

PROOF. Let $s \in S$, as \mathcal{A}_s is a finite $\mathcal{O}_{S,s}$ -algebra, we can find finitely many generators $\sigma_1, \dots, \sigma_n$. As \mathcal{A}_s is integral over $\mathcal{O}_{S,s}$, we can find unitary polynomials $F_{i,s} \in \mathcal{O}_{S,s}[X_i]$ such that $F_{i,s}(\sigma_{i,s}) = 0$ for $i = 1, \dots, n$. Take a sufficient small neighbourhood U of s so that $\sigma_{i,s}$ lifts to $\sigma_i \in \mathcal{O}_S(U)$ and $F_{i,s}$ lifts to a unitary polynomial $F_i \in H^0(U, \mathcal{O}_S[X_i])$ for $i = 1, \dots, n$. Up to shrinking U , we may guarantee that $\sigma_1, \dots, \sigma_n$ generate $\mathcal{A}|_U$ at all points and $F_i(\sigma_i) = 0$ for $i = 1, \dots, n$. Then $\mathcal{B} := \mathcal{O}_U[X_1, \dots, X_n]/(F_1, \dots, F_n)$ is coherent and we have a surjective homomorphism $\mathcal{B} \rightarrow \mathcal{A}|_U$ sending X_i to σ_i for $i = 1, \dots, n$. As the kernel of this homomorphism is of finite type, up to shrinking U , we may take finitely many $G_1, \dots, G_m \in \mathcal{B}(U)$ that generate the kernel. Lift G_1, \dots, G_m to $H_1, \dots, H_m \in H^0(U, \mathcal{O}_S[X_1, \dots, X_m])$, then

$$\mathcal{A}|_U \cong \mathcal{O}_U[X_1, \dots, X_n]/(F_1, \dots, F_n, G_1, \dots, G_m).$$

This follows from the same arguments of the proof of [Theorem 3.3](#) Step 3. \square

Corollary 3.8. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra, then the map $\text{Spec}_S^{\text{an}} \mathcal{A} \rightarrow S$ is topologically finite.

PROOF. By [Corollary 2.6](#), the fibers of $\text{Spec}_S^{\text{an}} \mathcal{A} \rightarrow S$ is finite. The map $\text{Spec}_S^{\text{an}} \mathcal{A} \rightarrow S$ is separated by construction. It remains to show that the map is closed.

The problem is local on S . By the proof of [Lemma 3.7](#), we can find a closed immersion over S : $\text{Spec}_S^{\text{an}} \mathcal{A} \rightarrow \text{Spec}_S^{\text{an}} \mathcal{B}$, where $\mathcal{B} = \mathcal{O}_S[X_1, \dots, X_n]/(F_1, \dots, F_n)$ for some $n \in \mathbb{N}$, where F_i is a unitary polynomial in $\mathcal{O}_S(S)[X_i]$ for $i = 1, \dots, n$. It suffices to show that $\text{Spec}_S^{\text{an}} \mathcal{B} \rightarrow S$ is closed.

Observe that

$$\text{Spec}_S^{\text{an}} \mathcal{B} \cong \text{Spec}_S^{\text{an}} \prod_{j=1}^n \mathcal{O}_S[X_j]/(F_j)$$

in $\mathcal{A}n/S$ as can be seen from the functor of points. So the problem reduces to showing that

$$\text{Spec}_S^{\text{an}} \mathcal{O}_S[X]/(F) \rightarrow S$$

for a unitary polynomial is closed. This is the classical continuity of roots. \square

Next we describe the local structure of a complex analytic germ.

Theorem 3.9. Let X_x be a complex analytic germ, $n \in \mathbb{Z}_{>0}$ and $f_1, \dots, f_n \in \mathcal{O}_{X,x}$ be a system of parameters. We have a morphism $X_x \rightarrow \mathbb{C}_0^n$ induced by f_1, \dots, f_n . Then there is an open neighbourhood U of 0 in \mathbb{C}^n and a finite \mathcal{O}_U -algebra \mathcal{A} such that $\mathcal{A}_0 \cong \mathcal{O}_{X,x}$. The space $\text{Spec}_U^{\text{an}}(\mathcal{A})$ admits a unique point x' over 0 and X_x is isomorphic to $\text{Spec}_U^{\text{an}}(\mathcal{A})_{x'}$ in $\mathbb{C}\text{-Ger}/\mathbb{C}_0^n$.

PROOF. As f_1, \dots, f_n is a system of parameters, $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\mathbb{C}^n,0}$ is finite. By [Lemma 3.7](#), we can spread $\mathcal{O}_{X,x}$ to a finite \mathcal{O}_U -algebra on an open neighbourhood U of 0 in \mathbb{C}^n . Let $Y = \text{Spec}_U^{\text{an}}(\mathcal{A})$. It follows from [Corollary 2.6](#) that Y has a unique point x' over 0. By [Theorem 3.3](#), up to shrinking U , we may guarantee that X_x and $Y_{x'}$ are isomorphic over \mathbb{C}_0^n . \square

Proposition 3.10. Let X_x be a complex analytic germ. The map $Y_x \mapsto I_X(Y, x)$ defines a bijection between the set of closed subgerms of X_x and the set of ideals of $\mathcal{O}_{X,x}$.

In particular, we can view a germ Y_x as a closed subscheme $\text{Spec } \mathcal{O}_{X,x}/I_X(Y, x)$ of $\text{Spec } \mathcal{O}_{X,x}$.

PROOF. We construct a reverse map. Given an ideal I of $\mathcal{O}_{X,x}$, as $\mathcal{O}_{X,x}$ is noetherian, I is finitely generated. We can find an open neighbourhood U of x in X and an ideal sheaf of finite type \mathcal{I} of U with $\mathcal{I}_x = I$. Let Y be the closed analytic subspace of X defined by \mathcal{I} . We associated Y_x with I .

It is easy to verify that this map is the inverse of the given map. \square

Definition 3.11. Let X_x be a complex analytic germ and Y_x, Z_x be two closed subgerms of X_x . We say Y_x is contained in Z_x and write $Y_x \subseteq Z_x$ if $I(Y, x) \supseteq I(Z, x)$. This defines a partial order on the set of closed subgerms of X_x .

Definition 3.12. A complex analytic germ X_x is *integral* if $\mathcal{O}_{X,x}$ is integral.

We also say (X, x) is *integral*.

Theorem 3.13 (Nullstellensatz). Let X_x be an integral complex analytic germ and Y_y be a closed subgerm of X_x . Then the following are equivalent:

- (1) Y_x is a proper closed subgerm of X_x ;
- (2) $|Y|_x$ is a proper closed subgerm of $|X|_x$.

PROOF. (2) \implies (1): This is obvious.

(1) \implies (2): Consider a proper closed subgerm Y_x of X_x . By [Proposition 3.10](#), $I(Y, x) \neq 0$.

Step 1. We reduce to the case $I(Y, x) = (f)$ for some non-zero element $f \in \mathcal{O}_{X,x}$.

Take a non-zero element $f \in I(Y, x)$. Let Y'_x be the subgerm of X_x corresponding to the ideal (f) of $\mathcal{O}_{X,x}$. Then $Y_x \subseteq Y'_x$. It suffices to show that $|Y'_x| \neq |X|_x$. We may replace Y by Y' .

Step 2. We prove that $|Y|_x \neq |X|_x$.

Note that f is not a zero-divisor as $\mathcal{O}_{X,x}$ is integral. Write $n = \dim \mathcal{O}_{X,x}$. By Krull's Hauptidealsatz, $\dim \mathcal{O}_{X,x}/(f) = n - 1$. Let $\overline{f_1}, \dots, \overline{f_{n-1}}$ be a system of parameters ([\[Stacks, Tag 00KU\]](#)) of $\mathcal{O}_{X,x}/(f)$. Lift them to $f_1, \dots, f_{n-1} \in \mathcal{O}_{X,x}$. Then (f_1, \dots, f_{n-1}, f) is a system of parameters of $\mathcal{O}_{X,x}$. Let $\varphi : X_x \rightarrow \mathbb{C}_0^n$ and $\psi : Y_x \rightarrow \mathbb{C}_0^{n-1}$ be the morphisms defined by these systems of parameters. We then have a commutative diagram in \mathbb{C} -Ger:

$$\begin{array}{ccc} Y_x & \hookrightarrow & X_x \\ \downarrow \psi & & \downarrow \varphi \\ \mathbb{C}_0^{n-1} & \hookrightarrow & \mathbb{C}_0^n \end{array}$$

It induces a commutative diagram of topological germs:

$$\begin{array}{ccc} |Y|_x & \hookrightarrow & |X|_x \\ \downarrow |\psi| & & \downarrow |\varphi| \\ \mathbb{C}_0^{n-1} & \hookrightarrow & \mathbb{C}_0^n \end{array}$$

The morphism of topological germs of $\mathbb{C}_0^{n-1} \rightarrow \mathbb{C}_0^n$ is clearly not an isomorphism, so it suffices to show that $|\varphi| : |X|_x \rightarrow \mathbb{C}_0^n$ is surjective, in the sense that if we represent $|\varphi|$ by a morphism $(U, x) \rightarrow (\mathbb{C}^n, 0)$ from an open neighbourhood U of x in X to \mathbb{C}^n , then its image contains an open neighbourhood of 0 in \mathbb{C}^n .

By [Theorem 3.9](#), we may assume that $X = \text{Spec}_X^{\text{an}} \mathcal{A}$ for some finite \mathcal{O}_X -algebra \mathcal{A} and X has a unique point over 0 . Then by [Corollary 2.6](#), we have $\mathcal{A}_0 \xrightarrow{\sim} \mathcal{O}_{X,x}$. By [Corollary 5.5](#) in [Complex analytic local algebras](#), the natural homomorphism

$$\varphi^\# : \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{X_1, \dots, X_n\} \rightarrow \mathcal{A}_0$$

is injective.

By [Corollary 2.7](#), it remains to show that $\text{Supp } \mathcal{A}$ is a neighbourhood of s in S . But the kernel of $\mathcal{O}_S \rightarrow \mathcal{A}$ is 0 at s hence 0 in a neighbourhood of s since both \mathcal{O}_S and \mathcal{A} are coherent by [Corollary 7.4](#) in [The notion of complex analytic spaces](#). \square

Corollary 3.14. Let X_x be a complex analytic germ and I, J be two ideals in $\mathcal{O}_{X,x}$. We let $W(I), W(J)$ denote the topological germs of the closed analytic subgerms of X_x defined by I and J respectively. Then the following are equivalent:

- (1) $W(I) \subseteq W(J)$;
- (2) $J \subseteq \sqrt{I}$.

PROOF. If (2) is true, as $\mathcal{O}_{X,x}$ is noetherian, we can find $n \in \mathbb{Z}_{>0}$ such that $J^n \subseteq I$. Extend I, J to coherent ideals \mathcal{I}, \mathcal{J} on X up to shrinking X . Then $\text{Supp } \mathcal{O}_X/\mathcal{J} \subseteq \text{Supp } \mathcal{O}_X/\mathcal{I}$. Hence, (1) holds.

Suppose that (1) holds. In order to prove (2), we may assume that I is prime. Then the closed analytic subgerm Y_x of X_x defined by I is integral. Let Z_x denote the closed analytic subgerm of X_x defined by J . The intersection $Y_x \cap Z_x$ of the germs Y_x and Z_x is by definition the closed analytic subgerm of X_x defined by $I + J$. Then

$$|Y_x \cap Z_x| = |Y|_x \cap |Z|_x = W(I).$$

By [Theorem 3.13](#), $Y_x \subseteq Z_x$. Namely, (2) holds. \square

Corollary 3.15. Let X_x be a complex analytic germ and Y_x be a closed analytic subgerm. Then the following are equivalent:

- (1) $|X|_x = |Y|_x$;
- (2) $I_X(Y, x)$ is nilpotent.

In particular, if these conditions hold, $\dim \mathcal{O}_{Y,x} = \dim \mathcal{O}_{X,x}$.

PROOF. This follows immediately from [Corollary 3.14](#). \square

Corollary 3.16. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

- (1) x is isolated in X ;
- (2) $\mathcal{O}_{X,x}$ is artinian.

PROOF. (1) simply means that $X_x = \{x\}_x$. By [Corollary 3.15](#), this holds if and only if \mathfrak{m}_x is nilpotent. As $\mathcal{O}_{X,x}$ is noetherian, the latter is equivalent to that $\mathcal{O}_{X,x}$ is artinian. \square

Corollary 3.17. Let X be a complex analytic space and Y be a closed analytic subspace defined by a coherent ideal \mathcal{I} . Then the following are equivalent:

- (1) $|X| = |Y|$;

(2) \mathcal{I} is locally nilpotent.

PROOF. This follows immediately from [Corollary 3.15](#). \square

Corollary 3.18. Let X be a complex analytic space and $f \in \mathcal{O}_X(X)$. Then the following are equivalent:

- (1) $f(x) = 0$ for all $x \in X$;
- (2) f is locally nilpotent.

PROOF. This follows from [Corollary 3.17](#), where we take \mathcal{I} as the coherent ideal generated by f . \square

Corollary 3.19 (Rückert Nullstellensatz). Let X be a complex analytic space and \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Let $f \in \mathcal{O}_X(X)$ be a function that vanishes on $\text{Supp } \mathcal{F}$. Then for any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x and $m \in \mathbb{Z}_{>0}$ such that $f^m \mathcal{F}|_U = 0$.

PROOF. Let \mathcal{G} be the annihilator sheaf of \mathcal{F} :

$$\mathcal{G} := \ker(\mathcal{O}_X \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})),$$

where the map $\mathcal{O}_X \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ sends a local section f of \mathcal{O}_X to the endomorphism of multiplying by f of \mathcal{F} . Then \mathcal{G} is a coherent sheaf by Oka's coherence theorem [Theorem 7.3](#) in [The notion of complex analytic spaces](#). Let Y be the closed analytic subspace defined by \mathcal{G} . By our assumption, f is everywhere zero on Y , so f is locally nilpotent in $\mathcal{O}_X/\mathcal{G} \subseteq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. \square

Corollary 3.20. Let X be a complex analytic space and \mathcal{I} and \mathcal{J} be coherent ideal sheaves on X . Then the following are equivalent:

- (1) $\text{Supp } \mathcal{O}_X/\mathcal{I} \subseteq \text{Supp } \mathcal{O}_X/\mathcal{J}$;
- (2) For any $x \in X$, there is an open neighbourhood U of x in X and $n \in \mathbb{Z}_{>0}$ such that

$$\mathcal{J}^n|_U \subseteq \mathcal{I}|_U.$$

PROOF. This follows immediately from [Corollary 3.14](#). \square

4. Analytic subsets

Definition 4.1. Let X be a complex analytic space. A subset $A \subseteq X$ is *analytic at* $x \in X$ if there is an open neighbourhood U of x in X and finitely many $f_1, \dots, f_m \in \mathcal{O}_X(U)$ such that

$$A \cap U = \{x \in U : f_1(x) = \dots = f_m(x) = 0\}.$$

We will denote the set on the right-hand side as $N_U(f_1, \dots, f_m)$. A subset $A \subseteq X$ is *analytic* in X if it is analytic at all $x \in X$.

A subset $B \subseteq X$ is *co-analytic* in X if $X \setminus B$ is analytic in X .

We observe that given $A \subseteq X$, the set of points $x \in X$ such that A is analytic at x is open. Also observe that an analytic set is necessarily closed. Analytic sets are clearly closed under finite intersection and finite unions.

Example 4.2. Let X be a complex analytic space. The underlying set of a closed analytic subspace of X is an analytic set in X .

In particular, the support of a coherent sheaf of \mathcal{O}_X -modules is an analytic set in X .

Proposition 4.3. Let X be a complex analytic space and Y be a closed analytic subspace of X . Then each analytic set A in Y is also an analytic set in X .

Conversely, if A is an analytic subset of X , then $A \cap Y$ is an analytic set in Y .

PROOF. We prove the first part. Let A be an analytic set in Y . Then A is closed in Y . It follows that A is closed in X . Let $a \in A$, we can find an open neighbourhood V of a in Y and finitely many $g_1, \dots, g_k \in \mathcal{O}_Y(V)$ such that

$$A \cap V = N_V(g_1, \dots, g_k).$$

Up to shrinking V , we may find a neighbourhood U of a in X with $V = Y \cap U$ and $f_1, \dots, f_k \in \mathcal{O}_X(U)$ lifting g_1, \dots, g_k . Then

$$A \cap U = N_U(f_1, \dots, f_k) \cap Y.$$

So by [Example 4.2](#), $A \cap U$ is analytic at a as a subset of X .

The second part is obvious. \square

Definition 4.4. Let X be a complex analytic space and $A \subseteq X$ be an analytic set. We define the *sheaf of ideals* \mathcal{J}_A of A as the sheafification of the presheaf of ideals on X defined by

$$U \mapsto \{f \in \mathcal{O}_X(U) : N_U(f) \supseteq M \cap U\}$$

for any open subset $U \subseteq X$.

Observe that \mathcal{J}_A is reduced.

Lemma 4.5. Let X be a complex analytic space and $A, B \subseteq X$ be analytic sets. Take $x \in X$. Then the following are equivalent:

- (1) $\mathcal{J}_{A,x} \subseteq \mathcal{J}_{B,x}$;
- (2) $A \cap U \supseteq B \cap U$ for some neighbourhood U of x in X .

PROOF. (2) \implies (1): This is trivial.

(1) \implies (2): Choose a neighbourhood U of x and finitely many $f_1, \dots, f_k \in \mathcal{O}_X(U)$ such that $A \cap U = N_U(f_1, \dots, f_k)$. Then $f_{1,x}, \dots, f_{k,x} \in \mathcal{J}_{A,x} \subseteq \mathcal{J}_{B,x}$. Up to shrinking U , we may assume that $f_1, \dots, f_k \in \mathcal{J}_B(U)$. It follows that $A \cap U \supseteq B \cap U$. \square

Lemma 4.6. Let X be a complex analytic space and A be an analytic set in X . Take $a \in A$. Let \mathcal{I} be a coherent ideal sheaf on X with $\mathcal{I}_a = \mathcal{J}_{A,a}$. Then there is an open neighbourhood U of a in X such that

$$W(\mathcal{I}|_U) = A \cap U.$$

The lemma tells that an analytic set can always be locally written in the form $W(\mathcal{I})$ for some open set $U \subseteq X$ and a coherent ideal \mathcal{I} on U .

PROOF. Choose an open neighbourhood U of x in X and finitely many sections $f_1, \dots, f_k \in \mathcal{J}_A(U)$ such that

$$\mathcal{I}|_U = \mathcal{O}_U f_1 + \dots + \mathcal{O}_U f_k.$$

After shrinking U , we may assume that

$$A \cap U = N_U(g_1, \dots, g_l)$$

for finitely many $g_1, \dots, g_l \in \mathcal{J}_A(U)$. Then $g_{1,a}, \dots, g_{l,a} \in \mathcal{J}_{A,a} = \mathcal{I}_a$. So up to shrinking U , we can find equations for all $j = 1, \dots, l$:

$$g_j = \sum_{i=1}^k a_{ij} f_i$$

for some $a_{ij} \in \mathcal{O}_X(U)$ with $i = 1, \dots, k$, $j = 1, \dots, l$. This implies that $W(\mathcal{I}|_U) \subseteq A \cap U$. The reverse inclusion is clear. \square

5. Lasker–Noether decomposition

Definition 5.1. Let X be a complex analytic space. An analytic set A in X is *irreducible* at $a \in A$ if $\mathcal{J}_{A,a}$ is a prime ideal in $\mathcal{O}_{X,a}$.

Definition 5.2. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. A *local decomposition* of A at a consists of an open neighbourhood U of a in X and finitely many analytic sets A_1, \dots, A_s in U such that

(1)

$$A \cap U = A_1 \cup \dots \cup A_s;$$

(2) A_i is irreducible at a for $i = 1, \dots, s$;

(3) for any open neighbourhood V of a in U , $A_j \cap V \not\subseteq A_k \cap V$ for $j, k = 1, \dots, s$, $j \neq k$.

We also say $A_1 \cup \dots \cup A_s$ is a *local decomposition* of $A \cap U$.

Proposition 5.3. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. Let

$$\mathcal{J}_{A,a} = \bigcap_{j=1}^s \mathfrak{p}_j$$

be the Lasker–Noether decomposition. Then there is a local decompose of A at a :

$$A \cap U = A_1 \cup \dots \cup A_s$$

with $\mathcal{J}_{A_j,a} = \mathfrak{p}_j$ for $j = 1, \dots, s$.

Let $A \cap U' = A'_1 \cup \dots \cup A'_r$ be another local decomposition of A at a . Then $r = s$ and we can find an open neighbourhood $W \subseteq U \cap U'$ and a bijection $\sigma : \{1, \dots, s\} \rightarrow \{1, \dots, s\}$ such that

$$A'_j \cap W = A_{\sigma(j)} \cap W$$

for $j = 1, \dots, s$.

PROOF. We first prove the existence part. Take an open neighbourhood U of a in X and coherent ideal sheaves $\mathcal{I}_1, \dots, \mathcal{I}_s$ on U such that

$$\mathcal{I}_{j,a} = \mathfrak{p}_j$$

for $j = 1, \dots, s$. Define

$$\mathcal{I} = \bigcap_{j=1}^s \mathcal{I}_j.$$

Then $\mathcal{I}_a = \mathcal{J}_{A,a}$. By [Lemma 4.6](#), up to shrinking U , we may guarantee that

$$W(\mathcal{I}) = A \cap U.$$

We set $A_j = W(\mathcal{I}_j)$ for $j = 1, \dots, s$. Then A_j is an analytic set in U and

$$A \cap U = W(\mathcal{I}) = \bigcup_{j=1}^s W(\mathcal{I}_j) = A_1 \cup \dots \cup A_s.$$

Observe that $\mathfrak{p}_j = \mathcal{I}_{j,a} \subseteq \mathcal{J}_{A_j,a}$ for all $j = 1, \dots, s$. We need to prove the reverse inclusion. Assume that this is not true, say it fails for $j = 1$. Then there is $g_1 \in \mathcal{J}_{A_1,a} \setminus \mathfrak{p}_1$. As $\mathfrak{p}_j \not\subseteq \mathfrak{p}_1$ for $j = 2, \dots, s$, we can find $g_j \in \mathfrak{p}_j \setminus \mathfrak{p}_1$ for $j = 2, \dots, s$. Then

$$g_1 \cdots g_s \in \mathcal{J}_{A_1,a} \cap \dots \cap \mathcal{J}_{A_s,a} = \mathcal{J}_{A,a} \subseteq \mathfrak{p}_1.$$

This is a contradiction. So $\mathcal{J}_{A_j,a} = \mathfrak{p}_j$ for $j = 1, \dots, s$. We conclude that $A \cap U = A_1 \cup \dots \cup A_s$ is a local decomposition by [Lemma 4.5](#).

Next we prove the uniqueness statement. We take U' and A'_1, \dots, A'_r as in the statement of the theorem. Then

$$\mathcal{J}_{A,a} = \mathcal{J}_{A'_1,a} \cap \dots \cap \mathcal{J}_{A'_r,a}.$$

By [Lemma 4.5](#), we find that this is the Lasker–Noether decomposition of $\mathcal{J}_{A,a}$. The uniqueness follows from the uniqueness of Lasker–Noether decomposition and [Lemma 4.5](#). \square

Definition 5.4. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. Let

$$A \cap U = A_1 \cup \dots \cup A_s$$

be a local decomposition of A at a . We call $A_{1,a}, \dots, A_{s,a}$ the *prime components* of A at a .

By [Proposition 5.3](#), the prime components are uniquely determined by the germ of X at x .

Lemma 5.5. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. Let A_1, \dots, A_s be the prime components of A at a . Then A_1 is not contained in $A_2 \cup \dots \cup A_s$.

PROOF. If not, we have

$$\mathcal{J}_{A_1,a} \supseteq \bigcap_{j=2}^s \mathcal{J}_{A_j,a}.$$

So

$$\mathcal{J}_{A,a} = \bigcap_{j=2}^s \mathcal{J}_{A_j,a}.$$

This contradicts the uniqueness of the Lasker–Noether decomposition. \square

Proposition 5.6. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. The following are equivalent:

- (1) A is not irreducible at a ;
- (2) there is an open neighbourhood U of a in X and a decomposition

$$A \cap U = A' \cup A'',$$

where A' and A'' are analytic sets in U such that $A'_a \neq A_a$ and $A''_a \neq A_a$.

PROOF. (1) \implies (2): Let $A_{1,x}, \dots, A_{s,x}$ be the prime components of A at a . Then $s \geq 2$. Take an open neighbourhood U of a in X such that $A_{1,x}, \dots, A_{s,x}$ lifts to analytic subsets A_1, \dots, A_s of U . It suffices to let $A' = A_1$ and $A'' = A_2 \cup \dots \cup A_s$. By [Lemma 5.5](#), A' and A'' satisfies the conditions in (2).

(2) \implies (1): We have $\mathcal{J}_{A,a} \neq \mathcal{J}_{A',a}$ and $\mathcal{J}_{A,a} \neq \mathcal{J}_{A'',a}$. Take $f \in \mathcal{J}_{A',a} \setminus \mathcal{J}_{A,a}$ and $g \in \mathcal{J}_{A'',a} \setminus \mathcal{J}_{A,a}$. Then $fg \in \mathcal{J}_{A',a} \cap \mathcal{J}_{A'',a} = \mathcal{J}_{A,a}$. So $\mathcal{J}_{A,a}$ is not a prime ideal. \square

6. Diagonal morphism

Definition 6.1. Let $f : X \rightarrow Y$ be a morphism of complex analytic space. The *diagonal* of f is by definition the morphism:

$$\Delta_f = \Delta_{X/Y} : X \rightarrow X \times_Y X$$

induced by the identity maps $X \rightarrow X$ and $X \rightarrow X$.

When $Y = \mathbb{C}^0$, we write Δ_X instead of Δ_{X/\mathbb{C}^0} .

Example 6.2. Let $n \in \mathbb{N}$. The diagonal morphism $\mathbb{C}^n \rightarrow \mathbb{C}^n \times \mathbb{C}^n$ is a closed immersion corresponding to the ideal generated by $p_1^* z_1 - p_2^* z_1, \dots, p_1^* z_n - p_2^* z_n$, where $p_1, p_2 : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ are the two projections and z_1, \dots, z_n are the coordinates on \mathbb{C}^n .

This can be seen through the functor of points by [Theorem 4.2](#) in [The notion of complex analytic spaces](#).

Proposition 6.3. Let $f : X \rightarrow Y$ be a morphism of complex analytic space. Then $\Delta_{X/Y}$ is an immersion.

PROOF. **Step 1.** We first reduce to the case $Y = \mathbb{C}^0$.

By general abstract nonsense, we have a commutative diagram

$$\begin{array}{ccccc} & & \Delta_X & & \\ & \searrow & \curvearrowright & \searrow & \\ X & \xrightarrow{\Delta_{X/Y}} & X \times_Y X & \longrightarrow & X \times X \\ & & \downarrow & \square & \downarrow \\ & & Y & \xrightarrow{\Delta_Y} & Y \times Y \end{array}$$

So in order to show that $\Delta_{X/Y}$ is an immersion, it suffices to show that X and Y are.

Step 2. We reduce to the case $X = \mathbb{C}^n$ for some $n \in \mathbb{N}$.

We want to show that $\Delta_X : X \rightarrow X \times X$ is an immersion.

The problem is local on X , so we may assume that X is a complex model space, say X is a closed analytic subspace of an open set U in \mathbb{C}^n for some $n \in \mathbb{N}$. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times X \\ \downarrow & & \downarrow \\ U & \xrightarrow{\Delta_U} & U \times U \end{array} .$$

It suffices to show that Δ_U is an immersion. As the problem is local, it suffices to show that $\Delta_{\mathbb{C}^n}$ is an immersion.

Step 3. We show that $\Delta_{\mathbb{C}^n}$ is a closed immersion.

This is exactly [Example 6.2](#). \square

7. Conormal sheaf

Definition 7.1. Let $i : X \rightarrow Y$ be an immersion of complex analytic spaces. The *conormal sheaf* of f is a sheaf of \mathcal{O}_X -modules $\mathcal{C}_f = \mathcal{C}_{X/Y}$ with $i_*\mathcal{C}_{X/Y} \cong \mathcal{I}/\mathcal{I}^2$, where \mathcal{I} is the kernel of $i^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$.

The conormal sheaf is defined up to a unique isomorphism. A choice of a factorization of i into a closed immersion $i' : X \rightarrow Z$ followed by an open immersion $j : Z \rightarrow Y$ determines a realization of $\mathcal{C}_{X/Y}$. Namely, if \mathcal{J} is the ideal sheaf of i' , then $\mathcal{C}_{X/Y}$ is (isomorphic to) $i'^*\mathcal{J}$.

Proposition 7.2. Let $i : X \rightarrow Y$ be an immersion of complex analytic spaces. Then $\mathcal{C}_{X/Y}$ is coherent.

PROOF. We may assume that i is a closed immersion defined by a coherent ideal \mathcal{J} . Then $\mathcal{C}_{X/Y} \cong i^*\mathcal{J}$ is coherent by [Corollary 7.5](#) in [The notion of complex analytic spaces](#). \square

8. Kähler differentials

We will make free use of results and notations in [\[Stacks, Tag 08RL\]](#). In particular, for a morphism $f : X \rightarrow S$ of complex analytic spaces, $\Omega_{X/S}$ denotes the sheaf of Kähler differentials and $d_{X/S} : \mathcal{O}_X \rightarrow \Omega_{X/S}$ denotes the universal S -derivation.

[Include principal parts etc. here](#)

Proposition 8.1. Let $f : X \rightarrow S$ be a morphism of complex analytic spaces. Then there is a canonical isomorphism

$$\Omega_{X/S} \xrightarrow{\sim} \mathcal{C}_{\Delta_{X/S}}.$$

PROOF. We first define the map in question. Factorize $\Delta_{X/S}$ as $X \rightarrow W \rightarrow X \times_S X$, where $X \rightarrow W$ is a closed immersion defined by a coherent ideal \mathcal{I} and $W \rightarrow X \times_S X$ is an open immersion. We have a short exact sequence

$$0 \rightarrow \mathcal{C}_{X/X \times_S X} \rightarrow \Delta_{X/S}^{-1}(\mathcal{O}_W/\mathcal{I}^2) \rightarrow \mathcal{O}_X \rightarrow 0.$$

Let $p_1, p_2 : X \times_S X \rightarrow X$ be the two projection maps. Then the natural maps $p_i^\# : p_i^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{X \times_S X}$ induce $p_i^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_W/\mathcal{I}^2$ for $i = 1, 2$. Take Δ^{-1} , we find natural maps

$$s_i : \mathcal{O}_X \rightarrow \Delta^{-1}(\mathcal{O}_W/\mathcal{I}^2).$$

The difference $d = s_2 - s_1$ is clearly an S -derivation. By the universal property of $\Omega_{X/S}$, we get a unique \mathcal{O}_X -linear map $\Omega_{X/S} \rightarrow \mathcal{C}_{X/X \times_S X}$.

Now in order to verify

$$\Omega_{X/S} \xrightarrow{\sim} \mathcal{C}_{\Delta_{X/S}}$$

is an isomorphism, it suffices to work on each stalk. This reduces the problem to the corresponding problem of local rings, which is handled in [\[Stacks, Tag 08S2\]](#). \square

We will write $\mathcal{P}_{X/S}^{(1)}$ for $\Delta^{-1}(\mathcal{O}_W/\mathcal{I}^2)$ introduced in the proof.

Corollary 8.2. Let $f : X \rightarrow S$ be a morphism of complex analytic spaces. Then $\Omega_{X/S}$ is coherent.

PROOF. This follows from [Proposition 8.1](#) and [Proposition 7.2](#). \square

Proposition 8.3. Let $f : X \rightarrow Y$, $g : Y \rightarrow S$ be morphisms of complex analytic spaces. Then there is a canonical exact sequence

$$f^*\Omega_{Y/S} \rightarrow \Omega_{X/S} \rightarrow \Omega_{X/Y} \rightarrow 0.$$

PROOF. The existence of the morphisms is obvious. To prove that the sequence is exact, it suffices to localize along each $x \in X$. The result then follows from the algebraic case [Stacks, Tag 01UX]. \square

Proposition 8.4. Let $X \rightarrow S$ be a morphism of complex analytic spaces and $i : Z \rightarrow X$ be an immersion. Then we have a canonical exact sequence

$$\mathcal{C}_{Z/X} \rightarrow i^*\Omega_{X/S} \rightarrow \Omega_{Z/S} \rightarrow 0.$$

PROOF. The existence of the morphisms is obvious. To prove that the sequence is exact, it suffices to localize along each $x \in X$. The result then follows from the algebraic case [Stacks, Tag 01UZ]. \square

Proposition 8.5. Let $f : X \rightarrow S$, $g : S' \rightarrow S$ be morphisms of complex analytic spaces. Consider the Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & \square & \downarrow f \\ S' & \xrightarrow{f} & S \end{array}$$

Then we have a canonical isomorphism

$$g'^*\Omega_{X/S} \rightarrow \Omega_{X'/S'}.$$

PROOF. It suffices to show that the canonical morphism $g'^*\mathcal{P}_{X/S}^{(1)} \rightarrow \mathcal{P}_{X'/S'}^{(1)}$ is an isomorphism. For this purpose, it suffices to prove it after localizing around $x' \in X'$. But observe that the local rings of $\mathcal{P}_{X/S}^{(1)}$ are finite over the corresponding local rings of X , so the analytic tensor products reduce to usual tensor products. The result then follows from the corresponding algebraic results. \square

Corollary 8.6. Let $f : X \rightarrow S$, $g : X \rightarrow S$ be morphisms of complex analytic spaces. Consider the Cartesian diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p} & X \\ \downarrow q & \square & \downarrow f \\ Y & \xrightarrow{g} & S \end{array}$$

Then we have a canonical isomorphism

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \rightarrow \Omega_{X \times_S Y/S}.$$

PROOF. The existence of the morphism follows from [Stacks, Tag 08RU]. By Proposition 8.5, the composition

$$p^*\Omega_{X/S} \rightarrow \Omega_{X \times_S Y/S} \rightarrow \Omega_{X \times_S Y/Y}$$

is an isomorphism. In particular, $p^*\Omega_{X/S} \rightarrow \Omega_{X \times_S Y/Y}$ is injective. Similarly, we have a natural isomorphism

$$q^*\Omega_{Y/S} \xrightarrow{\sim} \Omega_{X \times_S Y/X}$$

By [Proposition 8.3](#), we have a short exact sequence

$$0 \rightarrow p^* \Omega_{X/S} \rightarrow \Omega_{X \times_S Y/S} \rightarrow q^* \Omega_{Y/S} \rightarrow 0,$$

which clearly splits. \square

Example 8.7. Let $n \in \mathbb{N}$. We claim that $\Omega_{\mathbb{C}^n}$ is the free $\mathcal{O}_{\mathbb{C}^n}$ -module generated by dz_1, \dots, dz_n , where $z_1, \dots, z_n \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ are the coordinates on \mathbb{C}^n .

By [Example 6.2](#), we know that $\Omega_{\mathbb{C}^n}$ is generated by dz_1, \dots, dz_n as an $\mathcal{O}_{\mathbb{C}^n}$ -module. Assume that there is $x \in \mathbb{C}^n$, $f_{1,x}, \dots, f_{n,x} \in \mathcal{O}_{X,x}$ such that

$$\sum_{i=1}^n f_{i,x} dz_i = 0.$$

We need to show that $f_{i,x} = 0$ for all $i = 1, \dots, n$. We may assume that $x = 0$. Observe that

$$\Omega_{\mathbb{C}^n,0}^1 \otimes_{\mathcal{O}_{\mathbb{C}^n,0}} \mathbb{C} \xrightarrow{\sim} \mathfrak{m}_0/\mathfrak{m}_0^2$$

by the algebraic results. Taking the residue of our linear relation at 0, we find

$$\sum_{i=1}^n f_{i,0} z_{i,0} \in \mathfrak{m}_0^2.$$

As $z_{1,0}, \dots, z_{n,0}$ form a basis of $\mathfrak{m}_0/\mathfrak{m}_0^2$, we have $f_{i,0} = 0$ for $i = 1, \dots, n$.

Bibliography

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