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Contents

Morphisms between complex analytic spaces

1. Introduction

2. Open morphisms

Definition 2.1. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. We say f is *open* at $x \in X$ if for any neighbourhood U of x in X, $f(U)$ is a neighbourhood of $f(x)$ in Y.

Proposition 2.2. Let $f : X \to Y$ be a morphism of complex analytic spaces. Assume that f is open at $x \in X$, then the kernel of $f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is nilpotent.

The converse fails.

PROOF. Let $g_{f(x)} \in \mathcal{O}_{Y,f(x)}$ be an element in the kernel of $f_x^{\#}$. Up to shrinking Y, we may spread $g_{f(x)}$ to $g \in \mathcal{O}_Y(Y)$. Then f^*g vanishes in a neighbourhood of x in X. As f is open at x, g vanishes in the neighbourhood $f(U)$ of $f(x)$. By Corollary 3.18 in Constructions of complex analytic spaces, $g_{f(x)}$ is nilpotent. $\square$

3. Quasi-finite morphisms

Definition 3.1. Let $f : X \to Y$ be a morphism of complex analytic spaces. We say f is *quasi-finite* at $x \in X$ if x is isolated in $f^{-1}(f(x))$. We say f is *quasi-finite* if f is quasi-finite at all $x \in X$.

This definition is purely topological. We will show that it is equivalent to an analytic definition.

Proposition 3.2. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

(1) f is quasi-finite at $x \in X$;
(2) $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$;
(3) $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,f(x)}$.

PROOF. (1) $\Leftrightarrow$ (2): By Corollary 3.16 in Constructions of complex analytic spaces, f is quasi-finite at $x \in X$ if and only if $\mathcal{O}_{X_{f(x)},x} = \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is artinian. In other words, $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is finite-dimensional over $\mathbb{C}$. The latter is equivalent to that $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$.

(2) $\Leftrightarrow$ (3): This follows from Theorem 5.4 in Complex analytic local algebras. $\square$

4. Finite morphisms

Definition 4.1. A morphism of complex analytic spaces $f : X \to Y$ is *finite* if its underlying map of topological spaces is topologically finite.

We say a morphism of complex analytic spaces $f : X \to Y$ is *finite at* $x \in X$ if there is an open neighbourhood U of x in X and V of $f(x)$ in Y such that $f(U) \subseteq V$ and the restriction $U \to V$ of f is finite.

Let S be a complex analytic space. A *finite analytic space over* S is a finite morphism $f : X \to S$ of complex analytic spaces. A morphism between finite analytic spaces over S is a morphism of complex analytic spaces over S.

Proposition 4.2. Let $n \in \mathbb{N}$ and D be an open neighbourhood of 0 in $\mathbb{C}^n$. Let X be a closed subspace of D which intersections $\{(0, \dots, 0)\} \times \mathbb{C}$ at and only at 0. Then there is a connected open product neighbourhood $B \times W \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ of 0 in D such that the projection $B \times W \to B$ induces a finite morphism $h : X' \to B$ with $X' = X \cap (B \times W)$.

Proof. We will denote the coordinates on $\mathbb{C}^{n-1} \times \mathbb{C}$ as (z, w).

Let $\mathcal{I}$ be the ideal of X in D. By our assumption, we can choose $f_0 \in \mathcal{I}_0$ such that $\deg_w f_0 < \infty$ and $f_0(0) = 0$. By Theorem 4.3 in Complex analytic local algebras, we can find a Weierstrass polynomial $\omega_0 = w^b + a_1 w^{b-1} + \cdots + a_b \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[w]$ such that $f_0 = e\omega_0$ for some unit e in $\mathbb{C}\{z_1, \dots, z_n\}$. We choose a product neighbourhood $B \times W \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ of 0 in D such that ω_0 can be represented by $\omega \in \mathcal{O}_{\mathbb{C}^{n-1}}(B)[w]$ with $\omega|_{B \times W} \in \mathcal{I}(B \times W)$. Let $\pi : A \to B$ be the Weierstrass map defined by ω. Then π is finite by Theorem 6.2 in The notion of complex analytic spaces. Up to shrinking B and W, we may assume that $A \cap (B \times W) \to B$ is finite as well. Set $X' := X \cap (B \times W)$. The restriction $h : X' \to B$ of π is then finite. □

Corollary 4.3. Let $n, k \in \mathbb{N}$ and D be an open neighbourhood of 0 in $\mathbb{C}^n$. Let X be a closed subspace of D which intersections $\{(0, \dots, 0)\} \times \mathbb{C}^k$ at and only at 0. Then there is a connected open product neighbourhood $B \times W \subseteq \mathbb{C}^{n-k} \times \mathbb{C}^k$ of 0 in D such that the projection $B \times W \to B$ induces a finite morphism $h : X' \to B$ with $X' = X \cap (B \times W)$.

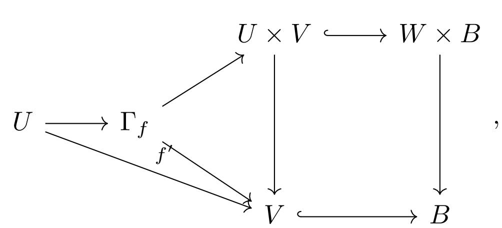
Proof. This follows from a repeted application of Proposition 4.2. □

Proposition 4.4. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

(1) f is quasi-finite at x;
(2) f is finite at x.

Proof. (2) $\implies$ (1): This follows from This follows from Proposition 4.5 in Topology and bornology.

(1) $\implies$ (2): Write $y = f(x)$. The assertion is local on both X and Y. So we may assume that U and V are complex model spaces in domains $W \subseteq \mathbb{C}^k$ and $B \subseteq \mathbb{C}^d$ respectively with $x = 0$ and $y = 0$. Moreover, we may assume that $\{x\} = f'^{-1}(y)$. We have the following commutative diagram:



where $\Gamma_{f'}$ denotes the graph of $f' : U \to V$. As $\{x\} = f'^{-1}(y)$, we have $\mathbb{C}^k \times \{0\}$ intersects Γ_f only at the origin. By Corollary 4.3, up to shrinking W and B, we may guarantee that the projection $W \times B \to B$ induces a finite morphism $\Gamma_f \to B$ and the pushforward under this map preserves coherence. Observe that $U \to \Gamma_f$ is a biholomorphism, we conclude that f' is finite. □

Corollary 4.5. Let $f : X \to Y$ be a morphism of complex analytic spaces. The following are equivalent:

(1) f is finite;
(2) f is quasi-finite and proper.

Proof. (1) $\Longrightarrow$ (2): This follows from Proposition 4.4.
(2) $\Longrightarrow$ (1): This follows from Proposition 4.5 in Topology and bornology. □

Corollary 4.6. Let $f : X \to Y$ be a morphism of complex analytic spaces. Then the set

$$\{x \in X : f \text{ is quasi-finite at } x\}$$

is open.

Proof. This follows from Proposition 4.4. □

Proposition 4.7. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Let $n = \dim_x f^{-1}(f(x))$. Then there is an open neighbourhood U of x in X and a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & Y \times \mathbb{C}^n \\ & \searrow^{f|_U} & \downarrow \\ & & Y \end{array},$$

where the right vertical morphism is the projection such that $U \to Y \times \mathbb{C}^n$ is finite at x.

Proof. Take an open neighbourhood U' of x in $f^{-1}(f(x))$ and a morphism $U' \to \mathbb{C}^n$ finite at x. For example, we can take the morphism induced by the spreading of a system of parameters $f_{1,x}, \ldots, f_{n,x} \in \mathcal{O}_{U',x}$. Up to shrinking U', we may assume that there is an open neighbourhood U of x in X with $U \cap f^{-1}(f(x)) = U'$ and a morphism $U \to \mathbb{C}^n$ extending $U' \to \mathbb{C}^n$. The induced morphism $U \to Y \times \mathbb{C}^n$ satisfies our requirements. □

Theorem 4.8. Let S be a complex analytic space. Then the functor $\mathrm{Spec}_S^{\mathrm{an}}$ defines an anti-equivalence from the category of finite $\mathcal{O}_S$-algebras to the category of finite analytic spaces over S.

Proof. We first observe that the functor is well-defined. This follows from Corollary 3.8 in Constructions of complex analytic spaces.

The functor is fully faithful by Proposition 2.10 in Constructions of complex analytic spaces. Suppose that $f : X \to S$ is a finite morphism of complex analytic spaces. We need to show that X is isomorphic to $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}$ for some finite $\mathcal{O}_S$-algebra $\mathcal{A}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}$.

By Proposition 2.8 in Constructions of complex analytic spaces, we necessarily have $\mathcal{A} \cong f_*\mathcal{O}_X$. So we need to show that the natural morphism $\mathrm{Spec}_S^{\mathrm{an}} f_*\mathcal{O}_X \to X$ over S is an isomorphism. The problem is local on S.

Fix $s \in S$. Write $x_1, \dots, x_n$ for the distinct points in $f^{-1}(s)$. Up to shrinking S, we may assume that X is the disjoint union of $V_1, \dots, V_n$, where V_i is an open neighbourhood of x_i in X. We need to show that X has the form $\mathrm{Spec}^{\mathrm{an}}_S \mathcal{B}$ for some $\mathcal{O}_S$-algebra $\mathcal{B}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}$.

It suffices to handle each V_i separately, so we may assume that $f^{-1}(s) = \{x\}$ consists of a single point. Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$ by Proposition 3.2. Up to shrinking S, we may assume that $\mathcal{O}_{X,x}$ spreads out to a finite $\mathcal{O}_S$-algebra $\mathcal{B}$. Let $X' = \mathrm{Spec}^{\mathrm{an}}_S \mathcal{B}$. There is a unique point x' of X' over s and $X'_{x'}$ is isomorphic to X_x over S_s. By Lemma 4.2 in Topology and bornology, up to shrinking S, we may assume that X is isomorphic to X' over S. We conclude. □

Corollary 4.9. Let $f : X \to Y$ be a finite morphism of complex analytic spaces and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules, then $f_*\mathcal{M}$ is coherent. Moreover, f_* is exact from $\mathcal{C}\mathrm{oh}(\mathcal{O}_X)$ to $\mathcal{C}\mathrm{oh}(\mathcal{O}_Y)$.

PROOF. This follows from Corollary 2.9 in Constructions of complex analytic spaces and Theorem 4.8. □

Corollary 4.10. Let X be a reduced complex analytic space. Then

(1) $\bar{X}$ is normal;
(2) $p : \bar{X} \to X$ is finite and surjective;
(3) There is a nowhere dense analytic set Y in X such that $p^{-1}(Y)$ is nowhere dense in $\bar{X}$ and the morphism $\bar{X} \setminus p^{-1}(Y) \to X \setminus Y$ induced by p is an isomorphism.

Conversely, these conditions determines $\bar{X}$ up to a unique isomorphism in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$.

PROOF. These properties are established in Proposition 7.8 in Local properties of complex analytic spaces. We need to prove the uniqueness.

Let $p : X' \to X$ be a morphism satisfying the three conditions. We need to show that X' is canonically isomorphic to $\bar{X}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$. By (2) and Theorem 4.8, it suffices to show that $p_*\mathcal{O}_{X'}$ is canonically isomorphic to $\bar{\mathcal{O}}_X$. By (1), and the universal property of normalization, there is a canonical morphism

$$p_*\mathcal{O}_{X'} \to \bar{\mathcal{O}}_X$$

of $\mathcal{O}_X$-algebras. We will show that this map is an isomorphism.

The problem is local. Let $x \in X$. By (3) and Corollary 3.14 in Constructions of complex analytic spaces, up to shrinking X, we can find $f \in \mathcal{O}_X(X)$ such that $f(y) = 0$ for all $y \in Y$ and f_x is a non-zero divisor in $(p_*\mathcal{O}_{X'})_x$. Up to shrinking X, we may assume that f_y is a non-zero divsior in $(p_*\mathcal{O}_{X'})_y$ for all $y \in X$. By (3), we have

$$\mathcal{O}_X|_{X\setminus Y} \to (p_*\mathcal{O}_{X'})|_{X\setminus Y}$$

is an isomorphism. It follows that

$$fp_*\mathcal{O}_{X'} \to \mathcal{O}_X$$

is injective. We then have an injective homomorphism:

$$p_*\mathcal{O}_{X'} \to \mathcal{O}_X \xrightarrow{\times f} \mathcal{O}_X \xrightarrow{\times f^{-1}} \mathcal{O}_X[f^{-1}].$$

For each $y \in X$, we deduce that $(p_*\mathcal{O}_{X'})_y$ is in the total ring of fraction of $\mathcal{O}_{X,y}[f_y^{-1}]$. But $(p_*\mathcal{O}_{X'})_y$ is finite and integral over $\mathcal{O}_{X,y}$, so is isomorphic to $\overline{\mathcal{O}_{X,y}}$ as $\mathcal{O}_{Y,y}$-algebras. □

Corollary 4.11. Let $f : X \to Y$ be a finite morphism of complex analytic spaces. Assume that $x \in X$ is a point such that $(f_*\mathcal{O}_X)_{f(x)}$ is torsion-free as an $\mathcal{O}_{Y,f(x)}$-module and Y is integral at $f(x)$. Then f is open at x.

PROOF. If not, we can choose open neighbourhoods U of x in X and V of $y := f(x)$ in Y such that $f(U) \subseteq V$ such that the induced morphism $g : U \to V$ is finite and $f(U)$ is not a neighbourhood of y in Y. Up to shrinking Y, we can find $h \in \mathcal{O}_Y(Y)$ such that $h_y \neq 0$ while h vanishes on $f(U)$. Observe that $f(U)$ is an analytic set in Y by Corollary 4.9. It follows from Corollary 3.18 in Constructions of complex analytic spaces that there is $t \in \mathbb{Z}_{>0}$ such that

$$h_y^t (g_*\mathcal{O}_U)_y = 0.$$

As $\mathcal{O}_{Y,y}$ is integral, this implies that $(g_*\mathcal{O}_U)_y$ is torsion as an $\mathcal{O}_{Y,f(x)}$-module. This is a contradiction, as $(f_*\mathcal{O}_X)_y$ as an $\mathcal{O}_{Y,f(x)}$-module is torsion-free by assumption. □

Lemma 4.12. Let X be an integral complex analytic space and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then

$$\{x \in X : \mathcal{M} \text{ is torsion-free at } x\}$$

is co-analytic in X.

PROOF. It suffices to show that $\operatorname{Supp} \mathcal{T}(\mathcal{M})$ is an analytic set in X. As X is integral, $\mathcal{T}(\mathcal{M})$ is just the kernel of the morphism $\mathcal{M} \to \mathcal{M}^{\vee\vee}$. □

Corollary 4.13. Let $f : X \to Y$ be a finite morphism of complex analytic spaces. Assume that Y is integral. Let $x \in X$ be a point such that X is integral at x and f is open at x, then there is an open neighbourhood U of x in X such that $f|_U : U \to Y$ is open.

PROOF. Let $y = f(x)$. The problem is local on Y. By Proposition 4.4, we may assume that $\{x\} = f^{-1}(y)$. By Corollary 4.9, $f_*\mathcal{O}_X$ is coherent. By Lemma 4.12, it suffices to show that it is torsion-free.

Observe that $(f_*\mathcal{O}_X)_y \xrightarrow{\sim} \mathcal{O}_{X,x}$. By Proposition 2.2, $f_x^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. As $\mathcal{O}_{X,x}$ is integral by our assumption, we conclude. □

Lemma 4.14. Let $f : X \to Y$ be a finite morphism of reduced complex analytic spaces and $x \in X$. Assume that $x \in X$, then there is a non-zero divisor $h \in \mathfrak{m}_{f(x)}$ such that $f_x^\#(h)$ is a non-zero divisor in $\mathcal{O}_{X,x}$.

PROOF. By Proposition 4.4, the problem is local on X. We may assume that X can be decomposed into irreducible components at x:

$$X = A_1 \cup \cdots \cup A_s.$$

By Corollary 4.9, $B_j := f(A_j)$ is an ananlytic set in Y for $j = 1, \ldots, s$. By our assumption, x is not an isolated point in A_j, so y is not an isolated point in B_j for $j = 1, \ldots, s$. Take a non-zero divisor $h \in \mathfrak{m}_{Y,y}$. Up to shrinking Y, we may assume that h spreads to $g \in \mathcal{O}_Y(Y)$. Observe that $W(f^*g) \cap A_j$ is not a neighbourhood of x in A_j for all $j = 1, \ldots, s$. So $f_x^\# h$ is not a zero divisor. □

Theorem 4.15. Let $f : X \to Y$ be a finite morphism of complex analytic spaces and $y \in Y$. Then

$$\dim_y f(X) = \max_{x \in f^{-1}(y)} \dim_x X.$$

The left-hand side makes sense because $f(X)$ is an analytic set in Y by Corollary 4.9.

PROOF. We may assume that X and Y are reduced and $f(X)=Y$.

Step 1. We reduce to the case where $f^{-1}(y)=\{x\}$ for some $x\in X$.

Let $x_1,\dots,x_t$ be the distinct points in $f^{-1}(y)$. The problem is local on Y. By Theorem 4.8 in Topology and bornology and Proposition 4.4, up to shrinking Y, we may assume that X is the disjoint union of open neighbourhoods $U_1,\dots,U_t$ of $x_1,\dots,x_t$ and $U_j\to V$ is finite for each $j=1,\dots,t$. It suffices to apply the special case to each $U_j\to V$ for $j=1,\dots,t$.

Step 2. We prove the theorem after the reduction in Step 1.

We make an induction on $d:=\dim_x X$. There is nothing to prove when $d=0$. Assume that $d\geq 1$. By Lemma 4.14, we can choose a non-zero divisor $g_y\in\mathfrak{m}_{Y,g_y}$ such that $f_x^{\#}(g_y)$ is a non-zero divisor in $\mathcal{O}_{X,x}$. Up to shrinking Y, we may assume that g spreads to $g\in\mathcal{O}_Y(Y)$. It suffices to apply our inductive hypothesis to $W(f_x^{\#}(g_y))\subseteq W(g_y)$. □

Corollary 4.16. Let $f:X\to Y$ be a finite open surjective morphism of complex analytic spaces. Assume that A is a thin subset of X of order $k\in\mathbb{Z}_{>0}$, then $f(A)$ is a thin subset of Y of order k.

PROOF. We may assume that X and Y are reduced. By Proposition 4.4 and the fact that f is open, the problem is local on X, we may assume that A is an analytic subset of X. Let $x\in A$. It suffices to handle the case where A is irreducible at x and x is the only point in $f^{-1}(f(x))$. By Corollary 4.9, $f(A)$ is an irreducible analytic subset of Y.

We may assume that Y is irreducible at $y:=f(x)$. Then

$$\operatorname{codim}_y(f(A),Y)=\dim_y Y-\dim_y f(A).$$

By Theorem 4.15, $\dim_y Y=\dim_x X$, $\dim_y f(A)=\dim_x A$. It follows that

$$\operatorname{codim}_y(f(A),Y)=\dim_x X-\dim_x A\geq\operatorname{codim}_x(A,X)\geq k.$$

□

Proposition 4.17. Let $f:X\to Y$ be a finite morphism of complex analytic spaces and $x\in X$. Assume that Y is unibranch at $f(x)$. Assume that $\dim_x X=\dim_{f(x)} Y$, then f is open at x.

PROOF. We may assume that X and Y are both reduced. Let $y=f(x)$. By Proposition 4.4, we may assume that $\{x\}=f^{-1}(y)$. By Corollary 4.9, $f(X)$ is an analytic set in Y. By Theorem 4.15,

$$\dim_y f(X)=\dim_x X.$$

As Y is irreducible at $f(x)$, we conclude that $f(X)_y=X_y$ and hence $f(X)$ is a neighbourhood of y. □

Corollary 4.18. Let $f:X\to Y$ be a quasi-finite morphism of equidimensional complex analytic spaces of dimension $d\in\mathbb{N}$. Assume that Y is unibranch. Then f is open.

The corollary fails if Y is not unibranch.

PROOF. By Proposition 4.4, f is finite at all $x \in X$. It suffices to apply Proposition 4.17. □

Corollary 4.19. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Then

$$\dim_x X - \dim_x f^{-1}(f(x)) \leq \dim_{f(x)} Y.$$

If equality holds and Y is unibranch at $f(x)$, then f is open at x.

PROOF. Let $y = f(x)$, $m = \dim_x X_y$ and $n = \dim_y Y$. The problem is local on X and Y, by Proposition 4.7, we may assume that there are morphisms

$$X \to Y \times \mathbb{C}^m, \quad Y \to \mathbb{C}^n$$

finite at x and y respective. In particular, the induced morphism

$$X \to \mathbb{C}^{m+n}$$

is finite at x and hence

$$\dim_x X \leq m + n.$$

This proves our inequality.

Assume that Y is unibranch at y and the equality holds. We want to show that f is open at x. As our problem is again local, we may assume that X and Y are connected.

Up to shrinking X and Y, we can choose a domain $U \subseteq \mathbb{C}^m$ such that $X \to Y \times U$ is finite. We conclude using Proposition 4.17. □

Corollary 4.20. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Assume that X is equidimensional at x, Y is unibranch at $f(x)$ and

$$\dim_x X - \dim_x f^{-1}(f(x)) = \dim_{f(x)} Y.$$

Then there is an open neighbourhood U of x in X such that $U \to Y$ induced by f is open.

PROOF. The problem is local on X. By Theorem 2.4 in Local properties of complex analytic spaces, up to shrinking X, we may assume that X is equidimensional of dimension $\dim_x X$. By Corollary 4.19,

$$\dim_x X - \dim_z f^{-1}(f(z)) \leq \dim_{f(z)} Y$$

for all $z \in X$. But as $\dim_z f^{-1}(f(z))$ is upper semi-continuous, the set where equality holds is open. Our assertion follows from Corollary 4.19. □

Lemma 4.21. Let $f : X \to Y$ be a finite open morphism of reduced complex analytic spaces. Assume that Y is a complex manifold. Then f is a branched covering.

PROOF. The statement is local on Y, so we may assume that Y is an open neighbourhood of 0 in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. By Proposition 4.4, we may assume that $\pi^{-1}\{0\}$ consists of a single point and X is a closed analytic subspace of a domain V in $\mathbb{C}^d$ for some $d \in \mathbb{N}$. Replacing X by the graph of f, we may assume that X is a closed analytic subspace of $V \times Y$ and f is the restriction of the projection map $V \times Y \to V$. In this case, the result follows from the local description lemma. Reproduce CAS p72! □

Corollary 4.22. Let X be an equidimensional complex analytic space of dimension d and $x \in X$. Then there is an open neighbourhood U of x in X and a connected domain $V \in \mathbb{C}^d$ such that there is a branched covering $U \to V$.

In fact, given any system of parameters $f_1, \ldots, f_d \in \mathcal{O}_{X,x}$, we can define sch a morphism sending x to 0 and the corresponding local ring homomorphism at x is

$$\mathcal{O}_{\mathbb{C}^d,0} \to \mathcal{O}_{X,x}$$

given by $f_1, \ldots, f_d$.

Proof. This follows from Theorem 3.9 in Constructions of complex analytic spaces, Lemma 4.21 and Corollary 4.18. □

Corollary 4.23. Let X be a complex analytic space and $x \in X$. Assume that X is unibranch at x. Let $f \in \mathcal{O}_{X,x}$. We assume that f is not constant and $\dim_x X \geq 1$, then for any open neighbourhood U of x in X such that f spreads to $g \in \mathcal{O}_X(U)$, there is $\epsilon > 0$ such that g takes all values $c \in \mathbb{C}$ with $|c - f(x)| < \epsilon$.

Proof. We may assume that X is reduced and $f(x) = 0$. Then f is a non-zero divisor in $\mathcal{O}_{X,x}$. We can find a system of parameters $f, g_1, \ldots, g_{n-1}$ with $n = \dim_x X$ such that $f, g_1, \ldots, g_{n-1}$ induce a branched covering $X \to V$ sending x to 0 after shrinking X, where V is an open neighbourhood of 0 in $\mathbb{C}^n$. This follows from Corollary 4.22. As the branched covering is open by Proposition 4.17, we conclude. □

Theorem 4.24. Let $f : X \to Y$ be a finite open surjective morphism of reduced complex analytic spaces, then f is a branched covering.

Proof. Let $x \in X$ and $y = f(x)$. As f is open, it suffices to find open neighbourhoods U of x in X and V of y in Y such that the morphism $U \to V$ induced by f is a branched covering. We first take U small enough so that U can be decomposed into prime components at x:

$$U = X_1 \cup \cdots \cup X_s.$$

We can assume that $X_i \cap X_j$ is thin in U for $i, j = 1, \ldots, s$, $i \neq j$. Up to shrinking U, we may assume that $U \to V$ is finite Proposition 4.4 for some open neighbourhood V of y in Y. As f is open, we may take $V = f(U)$. Observe that $f(X_i)$ is analytic in V for $i = 1, \ldots, s$ by Corollary 4.9. Moreover, $f(X_i)$ is irreducible at y for $i = 1, \ldots, s$. By Theorem 2.4 in Local properties of complex analytic spaces, we may assume that $f(X_i)$ is equidimensional of dimension $n_i \in \mathbb{N}$ for $i = 1, \ldots, s$.

By Corollary 4.22, up to shrinking V, we may assume that there is a branched covering $\eta_i : f(X_i) \to V_i$, where V_i is a connected domain in $\mathbb{C}^{n_i}$ for $i = 1, \ldots, s$. By Lemma 4.21, $\eta_i \circ f|_{X_i}$ is a branched covering for $i = 1, \ldots, s$. It follows that $X_i \to \pi(X_i)$ is a branched covering for $i = 1, \ldots, s$. This readily implies that f is a branched covering. □

Definition 4.25. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X. Take a critical locus T of f containing $f(A)$.

Consider $g \in \mathcal{O}_X(X \setminus A)$. We define a monic polynomial

$$\chi_g(w)(y) := \prod_{x \in f^{-1}(y)} (w - g(x)) \in \mathcal{O}_Y(Y \setminus T)[w].$$

By Theorem 3.7 in Local properties of complex analytic spaces, χ_g can be uniquely extended to $\chi_g \in \mathcal{O}_Y(Y \setminus f(A))[w]$. The monic polynomial χ_g is called the *characteristic polynomial* of g (with respect to f).

Proposition 4.26. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X and $g \in \mathcal{O}_X(X \setminus A)$. Let $\chi_g \in \mathcal{O}_Y(Y \setminus f(A))[w]$ be the characteristic polynomial of g. Then $\chi_g(g) = 0$.

If either of the following conditions hold:

(1) g is locally bounded near A;
(2) A is thin of order 2 in Y.

Then χ_g can be uniquely extended to $\chi_g \in \mathcal{O}_Y(Y)[w]$.

PROOF. Only the second part is non-trivial. By Corollary 4.16, f is open. By Corollary 4.16, $f(A)$ is thin in Y and under assumption (2), $f(A)$ is thin of order 2 in Y. It suffices to apply Theorem 3.7 in Local properties of complex analytic spaces. □

Proposition 4.27. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X and $e, g \in \mathcal{O}_X(X \setminus A)$. Take a critical locus T of f containing $f(A)$. Consider the $b \times b$-matrix

$$M(y) = \begin{bmatrix} 1 & e(x_1) & \dots & e(x_1)^{b-1} \\ 1 & e(x_2) & \dots & e(x_2)^{b-1} \\ & & \ddots & \\ 1 & e(x_b) & \dots & e(x_b)^{b-1} \end{bmatrix}$$

and $M_i(y)$ is $M(y)$ with the i-th colomn replace by

$$\begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_b) \end{bmatrix}$$

for $i = 0, \dots, b-1$, where $y \in Y \setminus T$ and $x_1, \dots, x_b$ are the distinct points in $f^{-1}(y)$. Then there are $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus f(A))$ such that for all $y \in Y \setminus T$,

$$\Delta_e(y) = (\det M(y))^2, \quad c_i(y) = \det M(y) \cdot \det M_i(y)$$

for $i = 0, \dots, b-1$. If either of the following conditions holds:

(1) e and g are locally bounded near A;
(2) A is thin of order 2 in X,

then we can take $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y)$

The function Δ_e is called the *discriminant* of e. We say e is *primitive* with respect to f if Δ is not identically 0.

PROOF. We first observe that $\det M(y)$ and $\det M_i(y)$ are independent of the ordering of $x_1, \dots, x_b$ by elementary lineary algebra, where $i = 1, \dots, b$. The entries of $M(y)$ and $M_i(y)$ can all be taken to be holomorphic outside T, so $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus T)$ are defined and the desired equation holds. By Theorem 3.7 in Local properties of complex analytic spaces, these functions can be extended uniquely into $\mathcal{O}_Y(Y \setminus f(A))$.

By Corollary 4.16, $f(A)$ is thin in Y and under assumption (2), $f(A)$ is thin of order 2 in Y. Applying Theorem 3.7 in Local properties of complex analytic spaces, we conclude the last assertion. $\square$

Corollary 4.28. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. A primitive element $e \in \mathcal{O}_X(X)$ exists if X is holomorphically separable.

Proof. Take a critical locus T of f. Let $y \in X \setminus T$. Let $x_1, \dots, x_b$ be distinct points of $f^{-1}(y)$. For each $i, j = 1, \dots, b$ with $i < j$, we can find a $g_{ij} \in \mathcal{O}_X(X)$ with $g(x_i) \neq g(x_j)$. A suitable linear combination of g_{ij}'s works. $\square$

Proposition 4.29. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X.

Let $e \in \mathcal{O}_X(X \setminus A)$ primitive element with respect to f. Then for each $g \in \mathcal{O}_X(X \setminus A)$, we have canonical polynomial $\Omega \in \mathcal{O}_Y(Y \setminus \pi(A))[X]$ such that

$$\Delta_e g = \Omega(e) \quad \text{on } X \setminus A.$$

If either of the following conditions holds:

(1) e and g are locally bounded near A;
(2) A is thin of order 2 in X,

then we can take $\Omega \in \mathcal{O}_Y(Y)[X]$.

In the traditional terminology, Δ_e is a *universal denominator* of the $\mathcal{O}_Y(Y)$-module $\mathcal{O}_X(X)$ if one of the two assumptons is satisfied.

Proof. Take a critical locus T of f containing $f(A)$. Consider $y \in Y \setminus T$ with fibers $x_1, \dots, x_b$. Consider the system of b-linear equations:

$$\Delta_e(y)g(x_i) = c_0(y) + c_1(y)e(x_i) + \cdots + c_{b-1}(y)e(x_i)^{b-1}$$

for $j = 1, \dots, b$. By Cramer's rule, if we use the notations of Proposition 4.27, if $\det M(y) \neq 0$, the unique solution is then

$$c_i(y) = (\det M(y))^{-1}\Delta(y) \det M_i(y) = \det M(y) \cdot \det M_i(y)$$

for $i = 0, \dots, b-1$. From Proposition 4.27, $c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus \pi(A))$. It suffices to take

$$\Omega = c_0 + c_1 X + \cdots + c_{b-1} w^{b-1}.$$

It is obvious that on $X \setminus (A \cup W(\Delta))$,

$$\Delta_e g = \Omega(e).$$

The same holds on $X \setminus A$ by continuity. The last asertion follows from Proposition 4.27. $\square$

Corollary 4.30 (Riemann extension theorem)**.** Let X be a reduced equidimensional complex analytic space of dimension $n \in \mathbb{N}$ and A be a thin set in X. Let $f \in \mathcal{O}_X(X \setminus A)$. Assume one of the following conditions holds:

(1) f is locally bounded near A;
(2) A is thin of order 2.

Then there is an element $g \in \overline{\mathcal{O}}_X(X)$ extending f.

PROOF. The uniquenss is obvious, we prove the existence. The problem is local on X, we may assume that X is holomorphically separable. By Corollary 4.22, we may take a connected complex manifold Y of dimension Y, $b \in \mathbb{Z}_{>0}$,a b-sheeted branched covering $f : X \to Y$. By Corollary 4.28, we can find a primitive element $e \in \mathcal{O}_X(X)$. By Proposition 4.29 and Proposition 4.26, it suffices to take $g = \Omega(e)/\Delta_e$, where Ω_e is the polynomial in Proposition 4.29. □

Corollary 4.31. Let X be a normal complex analytic space. Then the canonical map

$$\mathcal{O}_X(X) \to \mathcal{O}_X(X^{\mathrm{reg}})$$

is an isomorphism.

PROOF. By Proposition 6.9 in Local properties of complex analytic spaces, the map is injective. Take $f \in \mathcal{O}_X(X^{\mathrm{reg}})$, we need to extend it to $g \in \mathcal{O}_X(X)$. The problem is local on X. As X is normal, it is equidimensional at all points. By shrinking X, we may assume that X is equidimensional of some dimension $n \in \mathbb{N}$. Recall that X^{Sing} is thin of order 2 in X by Proposition 7.4 in Local properties of complex analytic spaces, so we can apply Corollary 4.30. □

Corollary 4.32. Let X be a connected normal complex analytic space then X^{reg} is connected.

PROOF. If not, we can find a continuous function $f : X^{\mathrm{reg}} \to \{0, 1\}$ which is not constant. By Corollary 4.31, f can be extended to $g \in \mathcal{O}_X(X)$. This contradicts the fact taht X is connected. □

Corollary 4.33. Let X be an irreducible complex analytic space and A be an analytic set in X. Suppose that there is $x \in A$ with $\dim_x A = \dim_x X$, then $A = X$.

PROOF. We may assume that X is irreducible. By Theorem 4.15, we may assume that X is normal.

Endow A with the reduced induced structure. As $\dim_x A = \dim_x X$, $\operatorname{Spec} \mathcal{O}_{X,x} = \operatorname{Spec} \mathcal{O}_{A,x}$ has a common irreducible component. By Nullstellensatz, $\operatorname{Int} A$ is non-empty. So $A' := A \setminus X^{\mathrm{Sing}}$ is non-empty and open in X^{reg}. We need to show that $A' = X^{\mathrm{reg}}$, taking closure we then conclude.

Suppose that $A' \neq X^{\mathrm{reg}}$. Then $\overline{A'} \cap X^{\mathrm{reg}}$ is a non-empty closed in X^{reg}, which is connected by Corollary 4.32. So

$$\overline{A'} \cap X^{\mathrm{reg}} \neq A',$$

as otherwise, $X^{\mathrm{reg}} = (\overline{A'} \cap X^{\mathrm{reg}}) \cup (X^{\mathrm{reg}} \setminus A')$. Take $a \in (\overline{A'} \cap X^{\mathrm{reg}}) \setminus A'$. Take a connected neighbourhood U of a in X^{reg} and finitely many holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}_X(U)$ so that $U \cap A = W(f_1, \ldots, f_k)$. As $U \cap A' \neq \emptyset$, $f_1, \ldots, f_k$ vanishes identically in U by Identitätssatz. In particular, $a \in A'$, which is a contradiction. □

Corollary 4.34. Let $f : X \to Y$ be a morphism of reduced complex analytic spaces. Let $Z \subseteq Y$ be the non-normal locus. Assume that $f^{-1}(Z)$ is nowhere dense in X (for example when X is irreducible and f is surjective), then there is a unique morphism $\bar{f} : \bar{X} \to \bar{Y}$ such that the following diagram commutes:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

Recall that Z is an analytic set in Y by Theorem 7.3 in Local properties of complex analytic spaces.

Proof. The uniqueness is clear. Let Z' be the inverse image of Z in $\bar{Y}$ and Z'' be the inverse image of Z in $\bar{X}$. By our assumption, Z'' is thin in $\bar{X}$. By construction, $\eta : \bar{Y} \setminus Z' \to Y \setminus Z$ is an isomorphism, so we get a morphism $g : \bar{X} \setminus Z'' \to \bar{Y} \setminus Z'$ completion the commutative diagram

$$\begin{array}{ccc} \bar{X} \setminus Z'' & \longrightarrow & \bar{Y} \setminus Z' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

Let $p \in Z''$. We need to extend g to a neighbourhood of p. Choose an open neighbourhood $V \subseteq \bar{Y}$ of the preimage of p in $\bar{X}$ which admits a closed immersion into a bounded domain $D \subseteq \mathbb{C}^n$ for some $n \in \mathbb{N}$. There is an open neighbourhood $U \subseteq \bar{X}$ of p such that g maps $U \setminus Z'' \to V$. The induced morphism $U \setminus Z'' \to D$ is given by bounded holomorphic functions in $\mathcal{O}_{U\setminus Z''}(U \setminus Z'')$. By Corollary 4.30, we get an extension $U \to D$. But this morphism factorizes through $U \to V$ as U is reduced, we conclude. □

Corollary 4.35. Let X be a complex analytic space. Then the following are equivalent:

(1) X is irreducible;
(2) If we write $X = Y_1 \cup Y_2$ with Y_1, Y_2 being analytic sets in X, then $X = Y_1$ or $X = Y_2$.

Proof. We may assume that X is reduced.

(1) $\implies$ (2): We may assume that X is normal. Suppose $X = Y_1 \cup Y_2$ with Y_1, Y_2 being analytic sets in X. Then $Y_1 \cap Y_2$ is not empty, as otherwise, X is not even connected. Let $x \in Y_1 \cap Y_2$. We then have $X_x = Y_{1,x} \cup Y_{2,x}$. This contradicts the fact that $\mathcal{O}_{X,x}$ is integral unless $Y_{1,x} \subseteq Y_{2,x}$ or $Y_{1,x} \subseteq Y_{2,x}$, which is impossible by Corollary 4.33.

(2) $\implies$ (1): Suppose that X is not irreducible. Then the normalization $\bar{X}$ is not connected, say $\bar{X} = Y_1' \cup Y_2'$, where Y_1, Y_2 are disjoint clopen sets in $\bar{X}$. Let $\pi : \bar{X} \to X$ be the normalization morphism. Then

$$X = \pi(Y_1') \cup \pi(Y_2').$$

By our assumption, either $X = \pi(Y_1')$ or $X = \pi(Y_2')$. We assume that the former holds. From Proposition 7.8 in Local properties of complex analytic spaces, we conclude that $Y_1' = \bar{X}$, which is a contradiction. □

Corollary 4.36. Let X be a connected complex analytic space. Then X is path-connected.

PROOF. We may assume that X is reduced.

If X is irreducible, after passing to the normalization, we may assume that X is normal. Then clearly X^{reg} is connected. So it suffices to apply Proposition 7.12 in Local properties of complex analytic spaces.

In general, take $x \in X$ and let X' be the set of all points of X that can be joined to x by a path. Then from the previous case, X' is the union of certain irreducible componenets of X. So is the complement $X \setminus X'$. As X is connected, we find that $X = X'$. □

Corollary 4.37. Let X be an irreducible complex analytic space. Then there is $n \in \mathbb{N}$ such that X is equidimensional of dimension n.

We remind the readers that X is not necessarily unibranch. For example, consider a nodal planar curve.

PROOF. We may assume that X is reduced. Taking nomralization, we can even assume that X is normal. Then X is connected. In particular, X^{reg} is connected by Corollary 4.32. But X^{reg} is then equidimensional of some dimension $n \in \mathbb{N}$. If $\dim_x X \neq n$ for some $x \in X^{\mathrm{Sing}}$, by Theorem 2.4 in Local properties of complex analytic spaces, $\dim_y X = \dim_x X$ whenever y is close to x. This is a contradiction. □

Corollary 4.38. Let X be a reduced irreducible complex analytic space, then X^{reg} is connected.

This corollary fails if X is not irreducible but only connected. For example, consider $\{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}$ endowed with the irreducible reduced structure.

PROOF. If not, we can find a continuous function $f : X^{\mathrm{reg}} \to \{0, 1\}$ which is not constant. By Corollary 4.30 and Corollary 4.37, f can be extended to $g \in \overline{O}_X(X)$. As X is irreducible and reduced, $\bar{X}$ is connected. It follows that g is constant and hence so is f, which is a contradiction. □

Corollary 4.39. Let $f : X \to Y$ be a finite surjective morphism between irreducible reduced complex analytic spaces. Then f is a branched covering.

PROOF. By Corollary 4.34, we have an obvious commutative diagram:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} .$$

If suffices to show that $\bar{f}$ is a branched covering, so we may assume that X and Y are normal.

By Proposition 4.17 and Corollary 4.37, f is open. So it suffices to apply Theorem 4.24. □

Corollary 4.40. Let $f : X \to Y$ be a finite surjective morphism between reduced complex analytic spaces. Then the following are equivalent:

(1) f is a branched covering;
(2) The image of each irreducible component of X has an interior point;
(3) The image of each irreducible component of X is an irreducible component of Y.

PROOF. (1) $\implies$ (2): Let $T \subseteq Y$ be a critical locus of f. Then $f^{-1}(T)$ is thin in X. Each irreducible component X' of X meets $X \setminus f^{-1}(T)$. It follows that $f(X' \setminus f^{-1(T)})$ is non-empty and open in Y.

(2) $\implies$ (3): Let X' be an irreducible component of X. Then $f(X)$ is an analytic set in Y. It is clearly irreducible. So $f(X)$ is contained in an irreducible component Y' of Y. But as $f(X')$ has an interior point, we find that $f(X') = Y'$ by Corollary 4.33.

(3) $\implies$ (1): The assertion is local, we may assume that the number of irreducible components of X is finite. Let $X_1, \ldots, X_s$ be the irreducible components of X. For each $i = 1, \ldots, s$, the induced map $X_i \to \pi(X_i)$ is finite and hence a branched covering by Corollary 4.39. It is enough to vefity that $\pi^{-1}(\pi(X_i \cap X_j))$ is thin in X for $i, j = 1, \ldots, s$ and $i \neq j$. If this fials, this set contains an interior point in X_k for some $k \in \{1, \ldots, s\}$. But then

$$X_k \subseteq \pi^{-1}(\pi(X_i \cap X_j)).$$

It follows that

$$\pi(X_i \cap X_j) \supseteq \pi(X_k).$$

This is impossible as $X_i \cap X_j \cap X_k$ is thin in X_k. □

Definition 4.41. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a normal complex analytic space. Take a critical locus $T \subseteq Y$ of f containing Y^{Sing}.

Consider $g \in \mathcal{O}_X(X)$. We define the *characteristic polynomial* $\chi_g \in \mathcal{O}_Y(Y)[w]$ of g (with respect to f) as follows: When Y is connected, by Corollary 4.32, Y^{reg} is a connected complex manifold. We define $\chi_g \in \mathcal{O}_Y(Y^{\mathrm{reg}})[w]$ as in Definition 4.25. We then extend χ_g to $\mathcal{O}_Y(Y^{\mathrm{reg}})[w]$ using Corollary 4.31. It is a monic polynomial of degree b. When Y is not connected, we just glue the characteristic polynomials defined using each connected components. Then we find a monic polynomial $\chi_g \in \mathcal{O}_Y(Y)[w]$ of degree b.

Proposition 4.42. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a normal complex analytic space. Let $g \in \mathcal{O}_X(X)$. Let $\chi_g \in \mathcal{O}_Y(Y)[w]$ be the characteristic polynomial of g. Then $\chi_g(g) = 0$.

PROOF. This follows immediately from Proposition 4.26. □

We give an alternative characterization of $\overline{\mathcal{O}}_X$.

Proposition 4.43. Let X be a reduced complex analytic space. Then for any open set $U \subseteq X$,

$$\overline{\mathcal{O}}_X(U) \xrightarrow{\sim} \{f : U \to \mathbb{C} : f \text{ is weakly holomorphic}\}.$$

PROOF. We temporarily denote the sheaf stated in theproposition by $\mathcal{O}'$. From the uniqueness in Proposition 7.5 in Local properties of complex analytic spaces, it suffices to show that $\mathcal{O}'_x$ is isomorphic to $\overline{\mathcal{O}_{X,x}}$ as $\mathcal{O}_{X,x}$-algebras for any $x \in X$.

We first observe that $\overline{\mathcal{O}}_X$ is a subsheaf of $\mathcal{O}'$. Let $U \subseteq X$ be an open subset and $f \in \overline{\mathcal{O}}_X(U)$. We need to show that f is locally bounded around $y \in U \cap X^{\mathrm{Sing}}$. Take an integral equation

$$f_y^n + a_{1,y} f_y^{n-1} + \cdots + a_{n,y} = 0$$

with $a_{1,y},\dots,a_{n,y}\in\mathcal{O}_{X,x}$. Take an open neighbourhood V of y in U such that $a_{1,y},\dots,a_{n,y}$ lift to $a_1,\dots,a_n\in\mathcal{O}_X(V)$ and

$$(f|_V)^n+a_1f|_V^{n-1}+\cdots+a_n=0.$$

Then for any $z\in V\setminus X^{\mathrm{Sing}}$,

$$|f(z)|\leq\max\{1,|a_1(z)|+\ldots+|a_n(z)|\}.$$

So $f\in\mathcal{O}'$.

Conversely, let $U\subseteq X$ be an open subset and $f\in\mathcal{O}'(U)$. By Proposition 7.8 in Local properties of complex analytic spaces, $p_*\mathcal{O}_{\overline{X}}=\mathcal{O}_X$, where $p:\overline{X}\to X$ is the normalization morphism. It follows from Proposition 7.8 in Local properties of complex analytic spaces and Corollary 4.30 that f can be uniquely extended to $g\in\mathcal{O}_{\overline{X}}(p^{-1}U)=\mathcal{O}_X(U)$. □

Proposition 4.44 (Rado, Cartan)**.** Let X be a normal complex analytic space and $f:X\to\mathbb{C}$ be a continuous map. Let $Z=f^{-1}(0)$. Assume that there is $g\in\mathcal{O}_X(X\setminus Z)$ such that $[g]=f|_{X\setminus Z}$, then $f=[g]$.

This result is proved in [Car52].

Proof. By Corollary 4.31, we may assume that X is a complex manifold. The problem is local on X, we may assume that X is the unit polydisk in $\mathbb{C}^n$ for some $n\in\mathbb{N}$. By Hartogs theorem, we may assume that $n=1$.

It remains to show that a continuous function $f:\{z\in\mathbb{C}:|z|<1\}$ which is holomorphic outside $Z:=\{f=0\}$ is holomorphic. This result is well-known. □

5. Flat morphisms

The notion of flat morphisms is defined for all ringed spaces. See [Stacks, Tag 02N2]. We will make use of these notions directly.

Proposition 5.1. Let $f:X\to Y$ be a morphism of complex analytic spaces and $x\in X$. Write $y=f(x)$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then the folloiwng are equivalent:

(1) $\mathcal{F}$ is f-flat at x;
(2) $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,y}$-module;
(3) For all $n\in\mathbb{N}$,
$$\hat{\mathcal{F}}_x\otimes_{\hat{\mathcal{O}}_{Y,y}}\hat{\mathcal{O}}_{Y,y}/\hat{\mathfrak{m}}_y^{n+1}$$
is a flat $\hat{\mathcal{O}}_{Y,y}/\mathfrak{m}_y^{n+1}$-module;
(4) We have
$$\mathrm{Tor}_1^{\mathcal{O}_{Y,y}}(\mathbb{C},\mathcal{F}_x)=0.$$

Proof. (1) ⇔ (2): This is the definition of flatness.
(2) ⇔ (3): This follows from [Stacks, Tag 0523].
(2) ⇔ (4): This follows from [Stacks, Tag 00MK]. □

Proposition 5.2. Let $f:X\to Y$ be a morphism of complex analytic spaces and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $g:Y'\to Y$ be a morphism of complex analytic spaces and consider the following Cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}.$$

Consider a point $x' \in X'$ defined by $x \in X$ and $y' \in Y'$ with common image $y \in Y$.

(1) If $\mathcal{F}$ is f-flat at x, then $g'^*\mathcal{F}$ is f'-flat at x'.

(2) If $g'^*\mathcal{F}$ is f'-flat at x' and $\hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_{Y',y'}$ is injective, then $\mathcal{F}$ is f-flat at x.

Proof. (1) Recall that

$$\hat{\mathcal{O}}_{X',x'} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x}\hat{\otimes}_{\hat{\mathcal{O}}_{Y,y}}\hat{\mathcal{O}}_{Y',y'}.$$

Let $n \in \mathbb{N}$, we then find

$$\hat{\mathcal{O}}_{X',x'}/\hat{\mathfrak{m}}_{y'}^{n+1}\hat{\mathcal{O}}_{X',x'} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x}\hat{\otimes}_{\hat{\mathcal{O}}_{Y,y}}\left(\hat{\mathcal{O}}_{Y',y'}/\hat{\mathfrak{m}}_{y'}^{n+1}\right) \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x}\otimes_{\hat{\mathcal{O}}_{Y,y}}\left(\hat{\mathcal{O}}_{Y',y'}/\hat{\mathfrak{m}}_{y'}^{n+1}\right).$$

By Proposition 5.1, $\hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_{Y,y}} \hat{\mathcal{O}}_{Y,y}/\hat{\mathfrak{m}}_y^{n+1}$ is a flat $\mathcal{O}_{Y',y'}$-module for each $n \in \mathbb{N}$. By Proposition 5.1 again, $\mathcal{F}$ is f'-flat at x'.

(2) For each $n \in \mathbb{N}$, let I_n be the inverse image of $\hat{\mathfrak{m}}_{y'}^{n+1}$ with respect to $\hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_{Y',y'}$. As the latter map is assumed to be injective, by Krull's intersection theorem, we find that

$$\bigcap_{n\in\mathbb{N}} I_n = 0.$$

It follows that the I_n's form a basis at 0 in $\hat{\mathcal{O}}_{Y,y}$. By Proposition 5.1, we are reduced to show that $\hat{\mathcal{F}}_x/I_n\hat{\mathcal{F}}_x$ is flat over $\hat{\mathcal{O}}_{Y,y}/I_n$. But by Proposition 5.1 again, we know that its base change along $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{Y',y'}/\hat{\mathfrak{m}}_{y'}^{n+1}$. So we are reduced to the well-known algebraic case. □

Proposition 5.3. Let $f : X \to Y$ be a flat morphism of complex analytic spaces and $x \in X$. Then

$$\dim_x X = \dim_{f(x)} Y + \dim_x X_{f(x)}.$$

Proof. Let $y = f(x)$. We may assume that X and Y are reduced. We make an induction on $n = \dim_y Y$. Note that the problem is local.

When $n = 0$, the result is obvious. Assume that $n > 0$. Take a non-zero divisor $g \in \mathcal{O}_{Y,y}$, then $h := f_x^{\#}(g) \in \mathcal{O}_{X,x}$ is a non-zero divisor as f is flat at x. Let X' and Y' be the closed analytic spaces of X and Y defined by h and g respectively. Up to shrinking X and Y, we may assume that there is a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}.$$

By inductive hypothesis,

$$\dim_x X' = \dim_x X'_y + \dim_y Y'.$$

We conclude using Krulls Hauptidealsatz. □

6. Separated morphisms

Recall that the notion of separated maps between topological spaces is defined in [Stacks, Tag 0CY1].

Definition 6.1. Let $f : X \to Y$ be a morphism of complex analytic spaces. We say f is *separated* if $|f| : |X| \to |Y|$ is separated.

If Y is the final object $\mathbb{C}^0$, we also say X is *separated*.

Proposition 6.2. Let $f : X \to Y$ be a morphism of complex analytic spaces. Then the following are equivalent:

(1) f is separated;
(2) $\Delta_{X/Y} : X \to X \times_Y X$ is closed;
(3) $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion.

PROOF. Recall that finite limits in $\mathbb{C}$-$\mathcal{A}$n commute with the forgetful functor to $\mathcal{T}$op by Corollary 8.6 in The notion of complex analytic spaces.

(1) ≡ (2): The undelrying morphism of topological spaces of $\Delta_{X/Y}$ is identified with

$$|X| \to |X| \times_{|Y|} |X|,$$

the topological diagonal. It suffices to apply [Stacks, Tag 0CY2].

(2) ≡ (3): We have shown that $\Delta_{X/Y}$ is always an immersion in Proposition 6.3 in Constructions of complex analytic spaces, so $\Delta_{X/Y}$ is a closed immersion if and only if it is closed. □

Corollary 6.3. Let X be a complex analytic space. The following are equivalent:

(1) X is separated;
(2) X is Hausdorff.

PROOF. This follows immediately from Proposition 6.2. □

Valuative criterion

7. Proper morphisms

Recall that the notion of separated maps between topological spaces is defined in [Stacks, Tag 01W1].

Definition 7.1. Let $f : X \to Y$ be a morphism of complex analytic spaces. We say f is *proper* if $|f| : |X| \to |Y|$ is proper.

If Y is the final object $\mathbb{C}^0$, we also say X is *proper*.

Valuative criterion

Definition 7.2. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. A *relative chart* at x is a closed immersion $U \to \Delta^n(r) \times V$, where U is an open neighbourhood of x in X, V is an open neighbourhood of $f(x)$ in Y, $n \in \mathbb{N}$, $r > 0$ such that $f(U) \subseteq V$ and the composition $U \to \Delta^n(r) \times V \to V$ is the restriction of f.

Recall that $\Delta^n(r) = (\Delta(r))^n$ is n-fold product of the disk of radius r.

Bibliography

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