Ymir

Contents

Topology and bornology

1. Introduction

In the whole project, a neighbourhood in a topology space is taken in Bourbaki's sense. In particular, a neighbourhood is not necessarily open.

We follow Bourbaki's convention about compact space. A comapct space is always Hausdorff.

On the other hand, we do not require locally compact spaces and paracompact spaces be Hausdorff.

A connected topological is always non-empty.

References to this chapter include [Ber93].

2. Nets

Let X be a set, $Y \subseteq X$ be a subset. Consider a collection τ of subsets of X, we write

$$\tau|_Y := \{V \in \tau : V \subseteq Y\}.$$

Definition 2.1. Let X be a topology space and τ be a collection of subsets of X. We say τ is

(1) *dense* if for any $V \in \tau$ and any $x \in V$, there is a fundamental system of neighbourhoods of x in V consisting of sets from $\tau|_V$;
(2) a *quasi-net* on X if for each $x \in X$, there exist $n \in \mathbb{Z}_{>0}$, $V_1, \dots, V_n \in \tau$ such that $x \in V_1 \cap \dots \cap V_n$ and that $V_1 \cup \dots \cup V_n$ is a neighbourhood of x in X;
(3) a *net* on X if it is a quasi-net and if for any $U, V \in \tau$, $\tau|_{U \cap V}$ is a quasi-net on $U \cap V$;
(4) *locally finite* if for any $x \in X$, there is a neighbourhood U of x in X such that $\{V \in \tau : V \cap U \neq \emptyset\}$ is finite.

We observe that if τ is a net, $\tau|_{U \cap V}$ is in fact a net.

Lemma 2.2. Let X be a topological space and τ be a quasi-net on X.

(1) A subset $U \subseteq X$ is open if and only if for each $V \in \tau$, $U \cap V$ is open in V.
(2) Suppose that τ consists of compact sets. Then X is Hausdorff if and only if for any $U, V \in \tau$, $U \cap V$ is compact.

We remind the readers that a compact space is Hausdorff by our convention.

Proof. (1) The direct implication is trivial. Suppose that $U \cap V$ is open in V for all $V \in \tau$. We want to show that U is open. Take $x \in U$, we can find $n \in \mathbb{Z}_{>0}$, $V_1, \dots, V_n \in \tau$ all containing x such that $V_1 \cup \dots \cup V_n$ is a neighbourhood of x in X. By our hypothesis, we can find open sets $W_1, \dots, W_n$ in W such that $W \cap V_i = U \cap V_i$

for $i = 1, \dots, n$. Then $W = W_1 \cap \cdots \cap W_n$ is an open neighbourhood of x in X. But then

$$U \cap (V_1 \cup \cdots \cup V_n) \supseteq W \cap (V_1 \cup \cdots \cup V_n),$$

the latter is a neighbourhood of x hence so is the former. It follows that U is open.

(2) The direct implication is trivial. Consider the quasi-net $\tau \times \tau := \{U \times V : U, V \in \tau\}$ on $X \times X$. By (1), it suffices to verify that the intersection of the diagonal with $U \times V$ is closed in $U \times V$ for any $U, V \in \tau$. But this intersection is homeomorphic to $U \cap V$, which is compact by our assumption and hence closed as U, V are both Hausdorff. □

Lemma 2.3. Let X be a Hausdorff space. Assume that X admits a quasi-net τ consisting of compact sets. Then X is locally compact.

Proof. Take $x \in X$. By assumption, we can find $n \in \mathbb{N}$ and $V_1, \dots, V_n \in \tau$ all containing x such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x. This neighbourhood is clearly compact. □

Lemma 2.4. Let X be a Hausdorff space and τ be a collection of compact subsets of X. Then the following are equivalent:

(1) τ is a quasi-net;
(2) For each $x \in X$, there are $n \in \mathbb{N}$ and $V_1, \dots, V_n \in \tau$ such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X.

Proof. (1) $\implies$ (2): This is trivial.

(2) $\implies$ (1): Given $x \in X$, take $V_1, \dots, V_n$ as in (2). We may assume that $x \in V_1, \dots, V_m$ and $x \notin V_{m+1}, \dots, V_n$ for some $1 \leq m \leq n$. Then $V_1 \cup \cdots \cup V_m$ is a neighbourhood of x in X: if U is an open neighbourhood of x in X contained in $V_1 \cup \cdots \cup V_n$, then $U \setminus (V_{m+1} \cup \cdots \cup V_n)$ is an open neighbourhood of x in X contained in $V_1 \cup \cdots \cup V_m$. □

Lemma 2.5. Let X be a topological space and τ be a net on X consisting of compact sets. Then

(1) for any pair $U, V \in \tau$, the intersection $U \cap V$ is locally closed in U and in V;
(2) If $n \in \mathbb{Z}_{>0}$, $V, V_1, \dots, V_n \in \tau$ are such that

$$V \subseteq V_1 \cup \cdots \cup V_n,$$

then there are $m \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_m \in \tau$ such that

$$V = U_1 \cup \cdots \cup U_m$$

and each U_j is contained in some V_i.

Proof. (1) It suffices to show that $U \cap V$ is locally compact in the induced topology. This follows from Lemma 2.3.

(2) For each $x \in V$ and each $i = 1, \dots, n$ such that $x \in V_i$, we take a neighbourhood of x in $V \cap V_i$ of the form $W_i V_{i1} \cup \cdots \cup V_{im_i}$ for some $m_i \in \mathbb{Z}_{>0}$ and $V_{ij} \in \tau$ for $j = 1, \dots, m_i$. Then the union of all W_i's is a neighbourhood of x of the form $U_1 \cup \cdots U_m$, where U_j belongs to τ and is contained in some V_i. Using the compactness of V, we conclude. □

3. Paracompact spaces

Definition 3.1. A topological space X is *paracompact* if any open covering of X admits a locally finite refinement.

A paracompact space is not necessarily Hausdorff according to our definition.

Proposition 3.2. Let X be a locally compact topological space.
(1) Assume that each connected component of X is σ-compact, then X is paracompact.
(2) If X is paracompact and Hausdorff, then each connected component of X is σ-compact.

If the conditions in (2) are satisfied, for any basis of neighbourhoods $\mathcal{B}$ of X, every open covering $\mathcal{U}$ of X can be refined into a locally finite covering $\mathcal{V}$ consisting of elements in $\mathcal{B}$.

We do not assume that the elements in $\mathcal{B}$ be open. The covering $\mathcal{V}$ is not necessarily open.

Theorem 3.3 (Michael)**.** Let $f : X \to Y$ be a closed continuous map of topological spaces. Assume that X is paracompact and Hausdorff, then $f(X)$ is also paracompact and Hausdorff.

This is a classical theorem of Ernest Michael. Reproduce the proof.

Proposition 3.4. Let X be a paracompact space and $Y \subseteq X$ be a closed subspace. Then Y is paracompact.

Proposition 3.5. Let X be a locally compact Hausdorff space and $Y \subseteq X$ be a subspace, then the following are equivalent:
(1) Y is locally compact and Hausdorff;
(2) Y is a locally closed subspace of X.

4. Closed maps and topologically finite maps

Definition 4.1 ([Stacks, Tag 004E],[Stacks, Tag 0CY1])**.** A map $f : X \to Y$ of topological spaces is *closed* if for each closed subset Z in X, $f(Z)$ is closed in Y.

A map $f : X \to Y$ of topological spaces is *separated* if it is continuous and the diagonal map $\Delta : X \to X \times_Y X$ is closed.

A closed map is not necessarily continuous.

Lemma 4.2. Let $f : X \to Y$ be a closed map of topological spaces, then for each $y \in Y$ and any open neighbourhood U of $f^{-1}(y)$ in X, there is an open neighbourhood V of y in Y such that $f^{-1}(V) \subseteq U$.

PROOF. It suffices to take $V = Y \setminus f(X \setminus U)$, □

Lemma 4.3. Let $f : X \to Y$ be a closed map of topological spaces. Then for any subspace V of Y, the map $f^{-1}(V) \to V$ induced by f is closed.

PROOF. Let A be a closed subset of $U := f^{-1}(V)$. We need to show that $f(A)$ is closed in V. Choose a closed subset B of X such that $A = B \cap U$, then $f(B)$ is closed in Y and $f(A) = f(B) \cap V$ is closed in V. □

Definition 4.4. A $f : X \to Y$ of topological spaces is *topologically finite* if

(1) f is separated and closed;
(2) for each $y \in Y$, the set $f^{-1}(y)$ is finite.

A map $f : X \to Y$ of topological spaces is *topologically finite at* $x \in X$ if there is an open neighbourhood U of x in X and an open neighbourhood V of $f(x)$ in Y such that $f(U) \subseteq V$ and the induced map $U \to V$ is topologically finite.

Proposition 4.5. Let $f : X \to Y$ be a map of topological spaces. Then the following are equivalent:

(1) f is topologically finite;
(2) f is proper and all fibers of f are discrete.

Here the properness is defined as in [Stacks, Tag 005O]. In particular, a proper map is always separated and hence continuous.

PROOF. Assume that f is topologically finite. As the fibers of f are finite and Hausdorff, they are discrete. We need to show that f is proper. This follows from [Stacks, Tag 005R].

Conversely, assume that f is proper with discrete fibers. By [Stacks, Tag 005R] again, the fibers of f are compact and hence finite. The map f is closed and separated as it is proper. So (1) follows. □

Lemma 4.6. Let $f : X \to Y$ be a continuous map between topologically spaces. Assume that Y is Hausdorff. Let W be an open relative quasi-compact subset of X, then the map

$$W \setminus f^{-1}(f(\partial W)) \to Y \setminus f(\partial W)$$

induced by f is proper.

PROOF. It is well-known that $f|_{\bar{W}} : \bar{W} \to Y$, as a continuous map from a quasi-compact space to a Hausdorff space is proper. The map in the lemma is a base change of the given map, hence is also proper. We apply [Stacks, Tag 005R]. □

Proposition 4.7. Let $f : X \to Y$ be a topologically finite map of topological spaces. Then for any subspace $V \subseteq Y$, the map $f^{-1}(V) \to V$ induced by f is topologically finite.

PROOF. This follows immediately from Lemma 4.3. □

Theorem 4.8. Let $f : X \to Y$ be a topologically finite map of topological spaces. Let $y \in f(X)$ and $x_1, \dots, x_n$ ($n \in \mathbb{Z}_{>0}$) denote the distinct points of $f^{-1}(y)$. Take pairwise disjoint open neighbourhoods $U'_1, \dots, U'_n$ of $x_1, \dots, x_n$ in X. Then any neighbourhood V' of y in Y contains an open neighbourhood V of y satisfying the following conditions:

(1) $U_1 := f^{-1}(V) \cap U'_1, \dots, U_n := f^{-1}(V) \cap U'_n$ are pairwise disjoint open neighbourhoods of $x_1, \dots, x_n$ in X;
(2) $f^{-1}V = \bigcup_{j=1}^n U_j$;
(3) The maps $U_j \to V$ for $j = 1, \dots, n$ induced from f are all topologically finite.

Let $\mathcal{F}$ be a sheaf of sets on X, then we have a functorial bijection

$$f_*\mathcal{F}(V) \xrightarrow{\sim} \prod_{j=1}^n \mathcal{F}(U_j).$$

The existence of $U'_1, \dots, U'_n$ is guaranteed by [Stacks, Tag 0CY2].

PROOF. As $\bigcup_{j=1}^n U'_j$ is an open neighbourhood of $f^{-1}(y)$ in X, by Lemma 4.2 and Lemma 4.3, we can find an open neighbourhood $V \subseteq V'$ of y in Y such that

$$f^{-1}V \subseteq \bigcup_{j=1}^n U'_j.$$

The conditions (1) and (2) are therefore satisfied.

In order to prove (3), it remains to show that the induced maps $U_j \to V$ are closed for $j = 1, \dots, n$. We may take $j = 1$. Let A be a closed subset of U_1. Then A is closed in $f^{-1}(V)$ by (1) and (2). It follows that $f(A)$ is closed in V by Lemma 4.3.

The last assertion follows from (1) and (2). □

Corollary 4.9. Let $f : X \to Y$ be a topologically finite map of topological spaces. Let $x \in X$ be U' be an open neighbourhood of x in X such that all other points in $f^{-1}(f(x))$ are in the interior of $X \setminus U'$. Then any neighbourhood V' of $f(x)$ in Y contains an open neighbourhood V of y such that for $U := f^{-1}(V) \cap U'$ the map $g : U \to V$ induced by f is topologically finite and $g^{-1}(g(x)) = \{x\}$.

PROOF. This follows immediately from Theorem 4.8. □

Corollary 4.10. Let $f : X \to Y$ be a topologically finite map of topological spaces. Let $\mathcal{F}$ be a sheaf of sets on X, $y \in f(X)$. Denote by $x_1, \dots, x_n$ ($n \in \mathbb{Z}_{>0}$) the distinct points of the fiber $f^{-1}(y)$. Then we have a canonical bijection

$$(f_*\mathcal{F})_y \xrightarrow{\sim} \prod_{j=1}^n \mathcal{F}_{x_j}.$$

In particular, $f_* : \mathcal{A}\mathrm{b}(X) \to \mathcal{A}\mathrm{b}(Y)$ is exact.

PROOF. This follows immediately from Theorem 4.8. □

5. Exhaustion

Definition 5.1. Let X be a space. A *quasi-compact exhaustion* of X is a sequence of quasi-compact sets $(K_i)_{i\in\mathbb{Z}_{>0}}$ in X such that

(1) For each $i \in \mathbb{Z}_{>0}$,

$$K_i \subseteq \operatorname{Int} K_{i+1};$$

(2)

$$X = \bigcup_{i=1}^\infty K_i.$$

When X is Hausdorff, we also say $(K_i)_{i\in\mathbb{Z}_{>0}}$ is a *compact exhaustion*.

Proposition 5.2. Let X be a topological space. Then the following are equivalent:

(1) there is a quasi-compact exhaustion of X;
(2) X is σ-compact and weakly locally compact;
(3) X is Lindelöf and weakly locally compact.

PROOF. To be included. □

Proposition 5.3. Let X be a locally compact Hausdorff topological space admitting a countable basis, then X admits a compact exhaustion.

Note that in the book of Grauret–Remmert, the condition of being Hausdorff is omitted.

Proof. Include a proof □

Lemma 5.4. Let X be a paracompact Hausdorff topological space and $\mathcal{F}$ be a sheaf of Abelian groups on X. Let $q \in \mathbb{Z}_{\geq 2}$ and $(K_i)_{i\in\mathbb{Z}_{>0}}$ be a compact exhaustion of X with the following property:

$$H^{q-1}(K_i, \mathcal{F}) = H^q(K_i, \mathcal{F}) = 0$$

for all $i \in \mathbb{Z}_{>0}$. Then $H^q(X, \mathcal{F}) = 0$.

Proof. Grauert–Remmert P103. □

6. Maps with discrete fibers

Lemma 6.1. Let X be a locally connected locally compact Hausdorff topological space and X_0 be a Hausdorff space with a basis β_0. Consider a continuous map $f : X \to X_0$ with discrete fiber. Then there is a basis of X made up of connected components of $f^{-1}U_0$ with $U_0 \in \beta_0$.

Proof. Let $x \in X$ and V be an open neighbourhood of x in X. We need to find $U_0 \in \beta_0$ and a component U of $f^{-1}(U_0)$ such that $U \subseteq V$.

For this purpose, we may assume that X is connected. Set $x_0 = f(x)$. Choose an open neighbourhood W of x in V with $\bar{W}$ compact and $B \cap f^{-1}(x_0) = \emptyset$, where $B = \bar{W} \setminus W$. Let $B_0 = f(B)$, then $x_0 \notin B_0$. As B_0 is compact, we can find $U_0 \in \beta_0$ containing x_0 such that $B_0 \cap U_0 = \emptyset$. Let U be the connected component of $f^{-1}(U_0)$ containing x. Then $B \cap U = \emptyset$ and hence $U \subseteq W \cup (X \setminus \bar{W})$. As X is connected and $W \cap U$ is non-empty, we find that $U \subseteq W$. □

Proposition 6.2. Let X be a connected, locally connected, first countable, locally compact Hausdorff space and X_0 be a topological space with countable basis. If there is a map $f : X \to X_0$ with discrete fibers, then X has countable topology as well.

This result is proved in [Jur59].

Proof. Let β_0 be a countable basis for the topology on X_0. Let β be the collection of open sets U in X such that

(1) There is $U_0 \in \beta_0$ such that U is a connected component of $f^{-1}(U_0)$;
(2) U is relatively compact in X.

By our assumption, any $U \in \beta$ has countable basis. By Lemma 6.1, β is a basis for the topology on X. It remains to show that β is countable.

Let $V \in \beta$. For each $n \in \mathbb{Z}_{>0}$, $\beta^{(n)}$ denotes the collection of $U \in \beta$ with the following property: there is a map $\{1, \ldots, n\} \to \beta$, say assigning $U_i \in \beta$ to i such that $U_1 = V$, $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$. As X is connected,

$$\beta = \bigcup_{n=1}^{\infty} \beta^{(n)}.$$

It remains to show that for each $n \in \mathbb{Z}_{>0}$, $\beta^{(n)}$ is countable. We make an induction. The case $n = 1$ is obvious. Assume that $n \geq 2$ and the assertion has been proved for $n-1$. Let $U_0 \in \beta_0$ and $U' \in \beta^{(n-1)}$. Let $\alpha^{(n)}(U_0, U')$ denote the collection of

$U \in \beta^{(n)}$ such that U is a connected component of $f^{-1}(U_0)$ and $U \cap U'$ is non-empty. Then

$$\beta^{(n)} = \bigcup_{U_0 \in \beta_0, U' \in \beta^{(n-1)}} \alpha^{(n)}(U_0, U').$$

But each $\alpha_{(n)}(U_0, U')$ is countable as U' has countable basis. It follows that $\beta^{(n)}$ is countable. □

7. Previlaged neighbourhoods

Definition 7.1. Let X be a topological space, $x \in X$ and $\mathcal{F}$ be a sheaf of sets on X. We say a neighbourhood U of x in X is *$\mathcal{F}$-previlaged at x* if the map

$$H^0(U, \mathcal{F}) \to \mathcal{F}_x$$

is injective. We also say U is an *$\mathcal{F}$-previlaged neighbourhood* of x in X.

Proposition 7.2. Let X be a topological space, $x \in X$ and

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H}$$

be an exact sequence of sheaves of Abelian groups on X. Let U be a neighbourhood of x in X.

(1) If U is $\mathcal{G}$-previlaged at x and $\mathcal{H}$-previlaged at x, then it is $\mathcal{F}$-previlaged at x;
(2) If U is $\mathcal{F}$-previlaged at x, then it is $\mathcal{G}$-previlaged at x.

PROOF. We have a commutative diagram of $\mathbb{C}$-linear spaces

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X,\mathcal{G}) & \longrightarrow & H^0(X,\mathcal{F}) & \longrightarrow & H^0(X,\mathcal{H}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{G}_x & \longrightarrow & \mathcal{F}_x & \longrightarrow & \mathcal{H}_x \end{array}.$$

Both assertions follow from simple diagram chasing. □

8. Stratification

Definition 8.1. Let M be a real analytic manifold and Ω be an open subset of M. A *stratification* $\mathcal{N}$ of Ω is a finite collection of connected locally closed analytic submanifold of Ω such that

(1) Ω is a disjoint union of the elements in $\mathcal{N}$;
(2) for each $\Gamma \in \mathcal{N}$, $(\bar{\Gamma} \setminus \Gamma) \cap \Omega$ is the union of elements in $\mathcal{N}$ of strictly smaller dimensions.

Elements in $\mathcal{N}$ are called the *strata* of the stratification.

For each open subset $U \subseteq \Omega$, we write $\mathcal{N}(U)$ for the collection of subsets of U consisting of all connected components of $U \cap \Gamma$ for all $\Gamma \in \mathcal{N}$.

Take $x \in \Omega$, an open neighbourhood U of x in Ω is *normal* with respect to $\mathcal{N}$ if

(1) $\mathcal{N}(U)$ is a stratification of U;
(2) for each $\Gamma \in \mathcal{N}(U)$, $x \in \bar{\Gamma}$.

Definition 8.2. Let X be a real analytic space. For each $x \in X$, we temporarily write $\mathcal{J}_a$ for the smallest family of germs of analytic subspaces of X stable under finite union, finite intersection, complement and contains all germs of the form $\{f < 0\}_a$ for some $f \in \mathcal{O}_{X,x}$.

A subset A of X is *real semi-analytic* if for all $x \in X$, $A_x \in \mathcal{J}_x$.

Theorem 8.3. Let X be a real analytic manifold, $x \in X$. Suppose that $\{A_0, \ldots, A_m\}$ is a finite collection of real semi-analytic subsets of X. Then there is a stratification $\mathcal{N}$ of an open neighbourhood U of x in X compatible with A_i in the sense that each stratum is either contained in A_i or is disjoint from A_i for $i = 1, \ldots, m$ such that each $y \in U$ admits a fundamental system of normal neighbourhoods with respect to $\mathcal{N}$.

PROOF. Include the ref. □

Proposition 8.4. Let X be a real analytic space, A be a semi-analytic set in X, $x \in A$. Let $Y_1, \ldots, Y_m$ be finitely many analytic sets in X passing x. Then there is a fundamental system $\mathcal{B}$ of neighbourhoods of a in A such that for each $V \in \mathcal{B}$, there is a fundamental system $\mathcal{B}_V$ of open neighbourhoods of V in X so that for any $W \in \mathcal{B}_V$, $W \cap Y_i$ is $\mathcal{O}_{Y_i}$-previlaged at x for $i = 1, \ldots, m$.

PROOF. The problem is local on X, we may assume that $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $x = 0$. Let U be an open neighbourhood of 0 in $\mathbb{R}^n$ compatible with the collection $\{A, Y_1, \ldots, Y_m\}$ and such that each $y \in U$ admits a fundamental system $\mathcal{U}(x)$ of normal neighbourhoods with respect to $\mathcal{N}$. The existence of U is guaranteed by Theorem 8.3. Let

$$\mathcal{B} = \{Q \cap A : Q \in \mathcal{U}(0)\}\,.$$

Suppose $V \in \mathcal{B}$, say $B = Q \cap A$ with $Q \in \mathcal{U}(0)$.

Let Ω be a neighbourhood of V in $\mathbb{R}^n$, we need to construct an open neighbourhood W of V in $\mathbb{R}^n$ contained in Ω such that $W \cap Y_i$ is $\mathcal{O}_{Y_i}$-previlaged at x for $i = 1, \ldots, m$.

For each $y \in V$, we let $Q(y)$ be an open neighbourhood of x in $\mathbb{R}^n$ normal with respect to $\mathcal{N}$ and contained in Ω. Let $W = \bigcup_{x \in V} Q(x)$.

Fix $i \in \{1, \ldots, m\}$. We will verify that $W \cap Y_i$ is $\mathcal{O}_{Y_i}$-previlaged at x.

Let $f \in \mathcal{O}_{Y_i}(W \cap Y_i)$ assume that f vanishes in a neighbourhood of 0, then we claim that $f = 0$ on $W \cap Y_i$. Now $Y_i \cap Q$ is a finite union of strata of $\mathcal{N}(Q)$ which are connected manifolds whose closures contain 0 and either contained in or disjoint from A. It is clear that f vanishes on all strata contained in $A \cap Y_i \cap Q$, hence on $A \cap Y \cap Q$. Also, f vanishes on $Y \cap Q(0)$, so it remains to prove that if $b \in V$, $b \neq 0$, then f vanishes on $Y \cap Q(b)$.

Let Γ be a stratum of $\mathcal{N}(Q(b))$ contained in Y. Then $b \in \bar{\Gamma}$. Take a stratum Γ' of $\mathcal{N}(Q)$ so that Γ is a connected component of $Q(b) \cap \Gamma'$. We may assume that b is on the boundary of A and $\Gamma' \cap A = \emptyset$. As $A \cap \Gamma \cap Q(b)$ is a union of strata of $\mathcal{N}(Q(b))$, one of them, say Γ_1 contains b. It is the intersection of a stratum Γ_1' of $\mathcal{N}(Q)$ with $Q(b)$. Let C be the connected component of $\Gamma' \cap W$ containing Γ. Consider the set

$$E = \left\{x \in \Gamma_1' \cap V : \Gamma'_{1,x} \subseteq \overline{C_x}\right\}.$$

Then $b \in E$ as $\Gamma_1' \cap Q(b) \subseteq \bar{\Gamma} \cap Q(b) \subseteq \bar{C} \cap Q(b)$. Moreover, E is an open subset of $\Gamma_1' \cap V$ by definition. We claim that E is also closed.

Let us postpone the proof of the claim. As Γ_1' is connected, we have $E = \Gamma_1' \cap V$. So

$$\Gamma_1' \cap V \subseteq \bar{C} \cap V$$

and

$$\Gamma_1' \cap W \subseteq \bar{C} \cap W.$$

But $a \in \overline{\Gamma_1}$, so $a \in \bar{C}$ and f vanishes on a clopen subset of C, namely $Q(a) \cap C$, so f vanishes on C. In particular on Γ.

It remains to verify the claim. In fact, we show that if $y \in \bar{E} \cap \Gamma'_1 \cap V$, then $\Gamma'_1 \cap Q(y) \subseteq \bar{C} \cap Q(y)$. To see this, observe that there is $z \in E \cap Q(y)$, so there is a non-empty open subset of $\Gamma'_1 \cap Q(y)$ contained in $\bar{C} \cap Q(y)$. But $\Lambda \cap Q(y)$ is a stratum of $\mathcal{N}(Q(y))$ as $y \in \Gamma'_1$, our claim follows since $\bar{C} \cap Q(y)$ is a union of strata of $\mathcal{N}(Q(y))$. □

9. Bornology

Definition 9.1. Let X be a set. A *bornology* on X is a collection $\mathcal{B}$ of subsets of X such that

(1) For any $x \in X$, there is $B \in \mathcal{B}$ such that $x \in \mathcal{B}$;
(2) For any $B \in \mathcal{B}$ and any subset $A \subseteq B$, $A \in \mathcal{B}$;
(3) $\mathcal{B}$ is stable under finite union.

The pair $(X, \mathcal{B})$ is called a *bornological set*. The elements of $\mathcal{B}$ are called the *bounded subsets* of $(X, \mathcal{B})$. When $\mathcal{B}$ is obvious from the context, we omit it from the notations.

A morphism between bornological sets $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ is a map of sets $f : X \to Y$ such that for any $A \in \mathcal{B}_X$, $f(A) \in \mathcal{B}_Y$. Such a map is called a *bounded map*.

Definition 9.2. Let $(X, \mathcal{B})$ be a bornological set. A *basis* for $\mathcal{B}$ is a subset $\mathcal{A} \subseteq \mathcal{B}$ such that for any $B \in \mathcal{B}$, there are $A_1, \ldots, A_n \in \mathcal{A}$ such that $B \subseteq A_1 \cup \cdots \cup A_n$.

Bibliography

[Ber93] V. G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 78.1 (1993), pp. 5–161.

[Jur59] M. Jurchescu. On a theorem of Stoilow. *Math. Ann.* 138 (1959), pp. 332–334. URL: https://doi.org/10.1007/BF01344153.

[Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.