Analytic Bertini theorem

Mingchen Xia

Chalmers Tekniska Högskola

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Notations

- X: complex projective manifold of dimension n.
- Λ: base-point free linear system on X.

Roughly speaking Λ is a family of linearly equivalent divisors on X. Being base-point free means that the intersection of all divisors is empty.

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Example

$X = \mathbb{P}^n$, Λ is the set of all hyperplanes in $\mathbb{P}^n$.

More generally, we require $\Lambda \cong \mathbb{P}^N$ to form a projective space as in the example.

Classical Bertini theorem

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Bertini type theorem: a theorem that relates properties (P) of objects on X to (P) of a generic member in Λ.
Here generic means the complement is small.

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Theorem (Fujino–Matsumura, X.)

There is a pluripolar set $\Sigma \subseteq \Lambda$ such that $\forall H \in \Lambda - \Sigma$, $\mathcal{J}(\phi|_H) = \mathcal{J}(\phi)|_H$.

Remark

$\mathcal{J}(\phi|_H) \subseteq \mathcal{J}(\phi)|_H$ is a consequence of Ohsawa–Takegoshi extension theorem.

A brief history

The analytic Bertini theorem was first considered by Fujino and Matsumura. They proved that the set of H satisfying $\mathcal{J}(\phi|_H) = \mathcal{J}(\phi)|_H$ is dense in the Euclidean topology of Λ.

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The current version was a conjecture of Boucksom.

Sketch of the proof

Goal: relate $\mathcal{I}(\phi|_H)$ to $\mathcal{I}(\phi)|_H$.

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Step 1. Group $\phi|_H$ for various H into a single object.
We will construct a universal family $\pi_1 : U \to \Lambda$ and a Hermitian psef line bundle $(\mathcal{L}, \Phi)$ on U, so that

- the fibre of $(U, \mathcal{L}, \Phi)$ over $H \in \Lambda$ is given by $(H, L|_H, \phi|_H)$.
- $\mathcal{J}(\Phi)$ is closely related to $\mathcal{J}(\phi)$.

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Step 2. We construct a psh metric h_H on $\pi_{1*}\left(\omega_{U/\Lambda} \otimes \mathcal{L} \otimes \mathcal{J}(\Phi)\right)$, so that h_H is singular at all H with $\mathcal{J}(\phi|_H) \neq \mathcal{J}(\phi)|_H$.
The main technique used in this step is (a generalization of) Berndtsson's theorem on the positivity of direct images.

Step 1

Kodaira map

The linear system induces a natural map $p : X \to \Lambda^*$, $x \in X$ is mapped to the set of divisors passing through x.

Let W $(\cong \Lambda)$ be the set of hyperplane on Λ^*.
The construction of $\pi_1 : U \to W$ is easy:

$$U = \{(H, x) \in W \times X : x \in H\}.$$

Take π_1 as the natural projection.
Take $(\mathcal{L}, \Phi)$ as the pull-back of (L, ϕ) along the projection $\pi_2 : U \to X$.

Step 1

Relation between $\mathcal{J}(\Phi)$ and $\mathcal{J}(\phi)$.

Lemma

Consider $H \in \Lambda$,

$$\mathcal{J}(\phi|_H) \subseteq \mathcal{J}(\phi)|_H \subseteq \mathcal{J}(\Phi)|_H .$$

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Lemma

Consider $H \in \Lambda$,

$$\mathcal{J}(\phi|_H) \subseteq \mathcal{J}(\phi)|_H \subseteq \mathcal{J}(\Phi)|_H .$$

Thus we have reduced the problem of understanding the set $\{H : \mathcal{J}(\phi|_H) \neq \mathcal{J}(\phi)|_H\}$ to that of $\{H : \mathcal{J}(\phi|_H) \neq \mathcal{J}(\Phi)|_H\}$.

Step 1

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What do we get after the reduction? We completely get rid of $\mathcal{J}(\phi)|_H$! The remaining problem is to relate $\mathcal{J}(\Phi)$ and the fibres of π_1.

Theorem (*)

Let $\pi : U \to W$ be a surjective morphism of smooth projective varieties. Let (L, ϕ) be a Hermitian psef line bundle on U, then for quasi-every $w \in W$, $\mathcal{J}(\phi|_{U_w}) = \mathcal{J}(\phi)|_{U_w}$.

Step 2

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Theorem (Kollár's torsion-free theorem à la Fujino–Matsumura)

Outside a proper Zariski closed subset of W, $\pi_*(\omega_{U/W} \otimes L \otimes \mathcal{J}(\phi))$ is locally free and the fibre at w is given by

$$H^0(\omega_{U_w} \otimes L|_{U_w} \otimes \mathcal{J}(\phi)|_{U_w}) .$$

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Our problem reduces to understanding the relation between the fibres of $\pi_*(\omega_{U/W} \otimes L \otimes \mathcal{J}(\phi))$ and $H^0(\omega_{U_w} \otimes L|_{U_w} \otimes \mathcal{J}(\phi|_{U_w}))$.

Hodge metric

We recall the definition of Hodge metric on $\pi_*(\omega_{U/W} \otimes L)$. Assuming that ϕ is smooth (and π is smooth for simplicity).

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Fact

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Given s in the fibre $\pi_*(\omega_{U/W} \otimes L)$ over $w \in W$, we can regard s as a section of $\omega_{U_w} \otimes L|_{U_w}$ near U_w. That is, s is a holomorphic $(n, 0)$-form with value in $L|_{U_w}$. Thus $s \wedge \bar{s}e^{-\phi}$ (after normalization) is a measure on U_w. Define

$$h_H(s, s) := \int_{U_w} s \wedge \bar{s}e^{-\phi} .$$

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Hodge metric

Theorem (Berndtsson–Păun)

h_H *is Griffiths positive.*

We do not need to recall the precise definition of positivity. For us it suffices to know that $\det h_H$ is a psh metric on the determinant line bundle $\det \pi_*(\omega_{U/W} \otimes L)$.

General case

For singular ϕ, the corresponding results are also known:

Fact (Kollár's torsion-free theorem)

$\pi_*(\omega_{U/W} \otimes L \otimes \mathcal{J}(\phi))$ is a vector bundle outside a proper Zariski closed set W_B.

On can mimic the above definition to define a metric h_H on $\pi_*(\omega_{U/W} \otimes L \otimes \mathcal{J}(\phi))$.

Properties of h_H

Theorem (Păun–Takayama)

The metric h_H is Griffiths positive (outside W_B). In particular, $\det h_H$ is a psh metric.

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Fact

The metric h_H (hence $\det h_H$) is singular at all general $w \in W$ satisfying

$$H^0(\omega_{U_w} \otimes L|_{U_w} \otimes \mathcal{I}(\phi|_{U_w})) \neq \pi_*(\omega_{U/W} \otimes L \otimes \mathcal{I}(\phi))_w\,.$$

Recall that by Kollár's torsion-free theorem, for a general $w \in W$,

$$\pi_*(\omega_{U/W} \otimes L \otimes \mathcal{I}(\phi))_w = H^0(\omega_{U_w} \otimes L|_{U_w} \otimes \mathcal{I}(\phi)|_{U_w})\,.$$

Properties of h_H

As a consequence, the set of $w \in W$ with

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A standard trick in algebraic geometry allows us to remove H^0 and conclude that the set of w so that

$$\mathcal{I}(\phi|_{U_w}) \neq \mathcal{I}(\phi)|_{U_w}$$

is pluripolar.

This finishes the proof of Theorem * and our analytic Bertini theorem follows.

Thank you for your attention!