Chern–Weil formulae of singular Hermitian vector bundles

Mingchen Xia

Chalmers Tekniska Högskola

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Classical Chern–Weil formula

- X: complex manifold of dimension n.
- (E,h): Hermitian vector bundle on X.
- Θ be the (normalized) curvature form of (E,h). Θ is a $\mathrm{End}(E)$-valued closed $(1,1)$-form.

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Write

$$\det(I + t\Theta) = \sum_i c_i(E, h)t^i \,.$$

Theorem (Chern–Weil formula)

$c_i(E, h)$ represents the i-th Chern class $c_i(E)$.

Classical Chern–Weil formula

Question

How to make sense of $c_i(E,h)$ when h is singular? What is the Chern–Weil formula in this case?

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Good=Generically smooth+mild growth at the boundary.

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Theorem (Mumford)

Chern–Weil formula still holds for good singularities.
More precisely, Chern forms defined on the smooth locus (considered as currents) represent the corresponding Chern classes.

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This notion is mostly useful in the case of Shimura varieties.
A Shimura variety is a quasi-projective variety parameterizing some meaningful arithmetic objects (like abelian varieties).
On a Shimura varieties, there are some automorphic vector bundles whose global sections correspond to certain modular forms.

Chern–Weil formula on Shimura varieties

Things to remember

1. A Shimura variety is a quasi-projective variety.
2. An automorphic vector bundle is a vector bundle on the Shimura variety.

Theorem (Mumford)

Automorphic vector bundles on Shimura varieties have a unique extension with good singularities to any smooth toroidal compactification.

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Corollary

There is a Chern–Weil formula on Shimura varieties.

In this case, Chern–Weil formula can be applied to count the dimension of automorphic forms.

More general cases?

Everything looks good so far, but

Caution!

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Roughly speaking, mixed Shimura varieties are natural generalization of Shimura varieties parameterizing more general objects (like one-motives). Autmorphism vector bundles on mixed Shimura varieties also have modular explanations.

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Roughly speaking, mixed Shimura varieties are natural generalization of Shimura varieties parameterizing more general objects (like one-motives). Autmorphism vector bundles on mixed Shimura varieties also have modular explanations.

We need a Chern–Weil formula dealing with more general singularities.

Difficulties

Question

a. How to make sense of $c_i(E, h)$ when h is singular?
b. What is the Chern–Weil formula in this case?

The most natural definition one may come up with is to generalize the curvature form Θ to some curvature current and mimic the smooth case. But

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No curvature currents!

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We will answer Question a in Part 2 and Question b in Part 3.

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Setup

Let X be a projective manifold of dimension n. Let (E, h) be a Hermitian vector bundle or rank $r+1$, h is smooth or singular.

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We want to make sense of $c_i(E, h)$ (and all other Chern polynomials) as a current.

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We will restrict our attention to (quasi-)positively curved cases, as required by our techniques.

Segre classes

Consider the projection $p : \mathbb{P}E^\vee \to X$ ($\mathbb{P}E$ does not follow Grothendieck's convention). Then from the surjective $p^*E \to \mathcal{O}(1)$, $\mathcal{O}(1)$ gets an induced Hermitian metric. We write $\hat{\mathcal{O}}(1)$ for $\mathcal{O}(1)$ equipped with this metric.

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Traditional definition

The i-th Segre class of E is by definition

$$s_i(E) := (-1)^i p_*(c_1(\mathcal{O}(1))^{r+i})\,.$$

Segre class contains the same information as Chern classes. They determine each other.

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This observation allows us to reduce to the line bundle case.

Non-pluripolar theory of line bundles

Consider a projective manifold X of dimension n, positively curved Hermitian line bundles (L_i, h_i) ($i = 1, \dots, m$, $m \leq n$) on X.

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Boucksom–Eyssidieux–Guedj–Zeriahi introduced the non-pluripolar product

$$c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_m, h_m)\,,$$

which is a closed positive (m, m)-current on X that puts no mass on pluripolar sets.

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Special cases

When h_i's are smooth, this product is the same as the usual wedge product of forms.
When h_i's are bounded, this product is the same as the Bedford–Taylor product.

Relative non-pluripolar products

We need a slight extension, introduced by Vu.
A closed dsh current is the difference of two closed positive currents. Vu defined the non-pluripolar product for any closed dsh current T:

$$c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_m, h_m) \cap T\,,$$

which reduces to the Bedford–Taylor theory when the h_i's are bounded and to the non-pluripolar product when $T = [X]$.

Segre currents

Suppose that we have a Griffiths positive singular Hermitian vector bundle (E,h) of rank $r+1$. Recall that $p:\mathbb{P}E^\vee \to X$ is the projection.

Segre currents

The i-th Segre class of (E,h) is by definition

$$s_i(E,h)\cap T := (-1)^i p_*(c_1(\hat{\mathcal{O}}(1))^{r+i}\cap p^*T)\,,$$

where T is any closed dsh current. The pull-back p^*T is an extension of the pull-back of differential forms.

Theorem

$s_i(E,h)\cap T$ is closed dsh.

Nice properties

Our Segre class behaves like the usual Segre class:

1. $s_i(E, h) \cap T = 0$ if $i < 0$.
2. $s_i(E, h) \cap s_j(E', h') \cap T = s_j(E', h') \cap s_i(E, h) \cap T$.
3. Projection formula and flat pull-back formula hold.

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3. Projection formula and flat pull-back formula hold.

These facts are not completely trivial, as one might have imagined. They fail in the theory of Chern currents of Lärkäng–Raufi–Sera–Wulcan.

Chern currents

As usual Chern forms are polynomials of Segre forms:

$$(1 + s_1(E)t + s_2(E)t^2 + \cdots)(1 + c_1(E)t + c_2(E)t^2 + \cdots) = 1\,.$$

Or $c_i(E) = P_i(s_1(E), \ldots)$ for some universal polynomials P_i.

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Chern polynomials

Set $T = [X]$ and by iteration, we can therefore make sense of

$$c_{i_1}(E_1, h_1) \cap \cdots \cap c_{i_m}(E_m, h_m)$$

or more generally, any Chern polynomial.

Chern currents

One can prove that $c_i(E,h)$ also behaves like the usual Chern classes, at least when h is locally bounded outside a closed pluripolar set.

Chern currents

One can prove that $c_i(E,h)$ also behaves like the usual Chern classes, at least when h is locally bounded outside a closed pluripolar set.
We have successfully answered Question a:

Question

a. How to make sense of $c_i(E,h)$ when h is singular?

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Chern–Weil problem

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Given positive Hermitian vector bundles (E_i, h_i), a Chern polynomial $P(c_j(E_i, h_i))$. How to interpret $P(c_j(E_i, h_i))$ in terms of intersection numbers?

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It is good to have a look at the final answer before we proceed.
Step 1. We introduce a notion of $\mathcal{I}$-good singularities extending Mumford's notion of goodness.
Step 2. We introduce certain algebraic objects (b-divisors) using the singularities of h_i.
Step 3. Solution to the problem:

Theorem

The Chern current $P(c_j(E_i, h_i))$ represents the algebraic intersection number of objects constructed in Step 2.

We begin with the line bundle case.

Question

Given a Hermitian pseudo-effective line bundle (L, h), what is the cohomology class of $c_1(L, h)^m$ (taken in the non-pluripolar sense)?

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The naive guess is $c_1(L)^m$, which fails for many obvious reasons: $c_1(L)^m$ does not necessarily support a closed positive current; the non-pluripolar products may lose mass.

General idea

$c_1(L,h)^m$ represents the product of the positive part of $c_1(L)$ relative to h.

b-divisors

The general idea can be made precise by introducing b-divisors.

b-divisor

A b-divisor on X is an assignment of numerical classes $\alpha_Y \in NS^1(Y)_{\mathbb{R}}$ for all projective birational resolution $\pi : Y \to X$, such that the $\alpha_\bullet$'s are compatible under pushforward between models.

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Intuitively, a b-divisor is the limit of divisors (or more precisely divisor classes) on the birational models of X.

b-divisors

Consider a Hermitian pseudo-effective line bundle (L, h) on X, one can construct a b-divisor $\mathbb{D}(L, h)$ as follows: given $\pi : Y \to X$, set

$$\mathbb{D}(L, h) := \pi^* L - \text{divisorial part of } \mathrm{dd}^{\mathrm{c}} \pi^* h \, .$$

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Recall that by Siu's decomposition,

$$\mathrm{dd}^{\mathrm{c}} \pi^* h = \sum_i a_i [E_i] + \text{ Residue part} \,,$$

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Good singularities

In this case of Mumford good singularities, $\mathbb{D}(L, h)_Y = \pi^* L$.

Dang–Favre's intersection theory

There is an ad hoc intersection theory of b-divisors, which although very unsatisfactory in many aspects, suffices for the moment.

Theorem (X.)

We have

$$(\mathbb{D}(L,h))^n = \mathrm{vol}(L,h) := \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{J}(kh)) \,.$$

Corollary

$c_1(L,h)^n$ *represents the algebraic intersection number* $(\mathbb{D}(L,h))^n$ *if*

$$c_1(L,h)^n = \mathrm{vol}(L,h) \,. \tag{1}$$

In general, (1) fails.

I-good singularities

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$\mathcal{I}$-good singularities

We say h is $\mathcal{I}$-good if one of the following equivalent conditions are satisfied:

1. h can be approximated by psh metrics with analytic singularities in a strong sense.
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The convergence in 1 implies the convergence of non-pluripolar masses.

I-good singularities

Theorem (Darvas–X.)

For any Hermitian pseudo-effective line bundle (L, h), we have

$$\operatorname{vol}(L,h) = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{I}(kh)) \geq \int_X c_1(L,h)^n \,.$$

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Corollary

If (L, h) is $\mathcal{I}$-good (by definition, has positive mass), then $c_1(L, h)^n$ represents the algebraic intersection number $(\mathbb{D}(L, h))^n$.

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Corollary

If (L,h) is $\mathcal{I}$-good (by definition, has positive mass), then $c_1(L,h)^n$ represents the algebraic intersection number $(\mathbb{D}(L,h))^n$.

This result was proved by me in 2020. A few months ago, a special case was rediscovered by Botero–Burgos Gil–Holmes–de Jong.

Chern–Weil formula for line bundles

Theorem

If (L_i, h_i) *are* $\mathcal{I}$*-good Hermitian pseudo-effective line bundles, then* $c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_n, h_n)$ *represents the algebraic intersection number* $(\mathbb{D}(L_1, h_1), \ldots, \mathbb{D}(L_n, h_n))$.

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If (L_i, h_i) *are* $\mathcal{J}$*-good Hermitian pseudo-effective line bundles, then* $c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_n, h_n)$ *represents the algebraic intersection number* $(\mathbb{D}(L_1, h_1), \ldots, \mathbb{D}(L_n, h_n))$.

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All of these discussions can be generalized to quasi-projective X, in particular, to mixed Shimura varieties.

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This is not a simple corollary of the previous corollary.
All of these discussions can be generalized to quasi-projective X, in particular, to mixed Shimura varieties.
It can been shown that the natural singularities on some automorphic line bundles (e.g. the line bundles of Siegal–Jacobi forms on the universal abelian variety) are $\mathcal{J}$-good. So our theorem gives a counting of automorphic forms.

More general singularities

We have only discussed positively curved singularities so far. In reality, the singularities on automorphic line bundles are usually not positively curved. In these cases, we can still talk about $\mathcal{I}$-goodness:

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Theorem

(Sum) The tensor products of $\mathcal{I}$-good line bundles are $\mathcal{I}$-good.
(Cancelation) If the tensor product of two Hermitian pseudo-effective line bundles are $\mathcal{I}$-good, then so is each factor (under the assumption of having positive mass).

Both parts are much harder than what they seem to be.

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Definition

Define a general $\mathcal{I}$-good Hermitian line bundle as the difference of two $\mathcal{I}$-good Hermitian pseudo-effective line bundles.

All constructions we mentioned can be extended to this setting.

Most general Chern–Weil for line bundles

Putting all efforts together, we conclude the Chern–Weil formula for line bundles:

Theorem

If (L_i, h_i) are $\mathcal{I}$-good Hermitian ~~pseudo-effective~~ line bundles, then $c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_n, h_n)$ represents the algebraic intersection number $(\mathbb{D}(L_1, h_1), \ldots, \mathbb{D}(L_n, h_n))$.

One can easily see that in the quasi-projective case, the b-divisors $\mathbb{D}(L_i, h_i)$ are independent of the choice of a compactication. So our theorem works on quasi-projective varieties as well.

Riemann–Zariski spaces

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Riemann–Zariski spaces

A more intrinsic way of describing this result is by introducing the Riemann–Zariski space $\mathcal{X}$. The projective limit of all birational $Y \to X$. A b-divisor is a divisor class on $\mathcal{X}$. An $\mathcal{I}$-good line bundle (L, h) defines a line bundle on $\mathcal{X}$. $\mathbb{D}$ is the first Chern class on $\mathcal{X}$. So out theorem becomes

Theorem

The intersection number of $\mathcal{I}$-good Hermitian line bundles is equal to the corresponding intersection number on the Riemann–Zariski space $\mathcal{X}$.

Chern–Weil for vector bundles

We say a Hermitian vector bundle (E, h) is $\mathcal{J}$-good if the induced metric on $\hat{\mathcal{O}}(1)$ is $\mathcal{J}$-good. As a non-trivial consequence of the case of line bundles, we conclude

Theorem

Assume that (E_j, h_j) are $\mathcal{J}$-good. Let $P(c_i(E_j, h_j))$ be a homogeneous Chern polynomial of degree n. Then $P(c_i(E_j, h_j))$ represents an algebraic intersection number on the Riemann–Zariski space $\mathfrak{X}$.

Thank you for your attention!

Some works in progress

There are three unsatisfactory features of our theory.
1. (mixed) Shimura varieties live on some canonically defined number fields instead of $\mathbb{C}$. So one need an intersection theory of b-divisors on CM field or totally real fields.
This is not worked out in Dang–Favre, but can be constructed easily using Galois descent.
2. The Dang–Favre intersection theory only works in codimension 1. We need a more general intersection theory on Riemann–Zariski spaces. Fortunately, there is a well-developed K-theory on Riemann–Zariski spaces. I am currently trying to make sense of the Bloch's formula:

$$\mathrm{CH}^p(\mathcal{X}) = H^p(\mathcal{X}, \mathcal{K}_p)\,.$$

3. It seems hard to make sense of Bott–Chern currents in our theory, as the latter forces us to leave the domain of quasi-positive vector bundles. Bott–Chern theory is necessary if we want to develop an Arakelov theory on mixed Shimura varieties.