A brief history of potential theory — From Poisson to Lelong

Mingchen Xia

Unemployed

September 4, 2024
ShanghaiTech University

Table of Contents

Elements of electrostatics

Electrostatics is the study of stationary electric charges.

Elements of electrostatics

Electrostatics is the study of stationary electric charges.

Coulomb's law

The (repulsive) force between two point charges is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}.$$

Elements of electrostatics

Electric field

Put a small test point charge q in the space, the electric field $\mathbf{E}$ at this point is

$$\mathbf{E} = \frac{\mathbf{F}}{q}.$$

Elements of electrostatics

Electric field

Put a small test point charge q in the space, the electric field $\mathbf{E}$ at this point is

$$\mathbf{E} = \frac{\mathbf{F}}{q}.$$

The Coulomb law can be restated in terms of the electric field $\mathbf{E}$:

Gauß's law

Given a electric charge distribution ρ, we have

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.$$

Elements of electrostatics

Next we integrate the electric field $\mathbf{E}$ into a scalar quantity ϕ (the electric potential):

$$-\nabla\phi = \mathbf{E}.$$

Elements of electrostatics

Next we integrate the electric field $\mathbf{E}$ into a scalar quantity ϕ (the electric potential):

$$-\nabla\phi = \mathbf{E}.$$

Gauß's law further becomes:

Gauß's law, 1813 by Poisson

$$\Delta\phi = -\frac{1}{\epsilon_0}\rho.$$

This equation is the celebrated Poisson equation.

Elements of electrostatics

Next we integrate the electric field $\mathbf{E}$ into a scalar quantity ϕ (the electric potential):

$$-\nabla\phi = \mathbf{E}.$$

Gauß's law further becomes:

Gauß's law, 1813 by Poisson

$$\Delta\phi = -\frac{1}{\epsilon_0}\rho.$$

This equation is the celebrated Poisson equation.
When $\rho = 0$ (that is, there are no charges), the equation

$$\Delta\phi = 0$$

is the Laplace equation.

Prescribing boundary potentials

When $\rho = 0$, a key problem investigated by Poisson:
Given the potential ϕ at the boundary of a domain, can we recover the interior values?

Prescribing boundary potentials

When $\rho = 0$, a key problem investigated by Poisson:
Given the potential ϕ at the boundary of a domain, can we recover the interior values?

Theorem (Poisson, 1820)

Yes for a 2D unit ball:

$$\phi\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\phi\left(e^{i\theta}\right)\,dt.$$

This is the so-called Poisson integration formula today. Poisson also proved the 3D version.

Prescribing boundary potentials

When $\rho = 0$, a key problem investigated by Poisson:
Given the potential ϕ at the boundary of a domain, can we recover the interior values?

Theorem (Poisson, 1820)

Yes for a 2D unit ball:

$$\phi\left(r\mathrm{e}^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta - t)}\phi\left(\mathrm{e}^{i\theta}\right)\,\mathrm{d}t.$$

This is the so-called Poisson integration formula today. Poisson also proved the 3D version.
More generally, if $\rho \neq 0$, using Green's function (1828), a similar solution with an extra term can be obtained.

Prescribing boundary potentials

What about domains other than balls?

Prescribing boundary potentials

What about domains other than balls?
It is in general extremely difficult to obtain explicit formulae for the Poisson kernels and the Green functions.

Prescribing boundary potentials

What about domains other than balls?
It is in general extremely difficult to obtain explicit formulae for the Poisson kernels and the Green functions.
These were one of the key topics of the school of Loo-Keng Hua.
Ref: Harmonic analysis of functions of several complex variables in the classical domains, L.-K. Hua.

Table of Contents

The foundational paper

A new era began with Gauß's 1840 paper
Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs-und-Abstossungs-Kräfte

The foundational paper

A new era began with Gauß's 1840 paper

Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs-und-Abstossungs-Kräfte

This paper had a strong historical impact.

The potential theory in the next few decades was based on this paper.

The foundational paper

A new era began with Gauß's 1840 paper

Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs-und-Abstossungs-Kräfte

This paper had a strong historical impact.

The potential theory in the next few decades was based on this paper.

The fundamental problems

Gauß proposed three main problems to study. Fix a domain $\Omega \subset \mathbb{R}^3$.

- The equilibrium problem Suppose that $\mathbb{R}^3 - \Omega$ is a conductor, how to find the equilibrium charge distribution (with a give mass) on $\partial\Omega$?
- The balayage problem Suppose that $\mathbb{R}^3 - \Omega$ is a conductor connected to the Earth, how does a given charge distribution in Ω induce charges on $\partial\Omega$?
- The Dirichlet problem

$$\begin{cases} \Delta\varphi = 0 \text{ on } \Omega, \\ \varphi = \varphi_0 \text{ on } \partial\Omega, \end{cases}$$

with φ_0 given.

The foundational paper

Gauß introduced the electric potential energy and reduce the equilibrium problem and the balayage problem to its study:

$$\int\int |x-y|^{-1}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

Subsequently, Riemann (1851) considered a variant for the Dirichlet problem:

$$\int |\nabla\varphi(x)|^2\,\mathrm{d}x,$$

which is known as the Dirichlet energy now.

Dirichlet principle

Dirichlet principle

A solution to the Dirichlet problem should minimize the Dirichlet energy.

The terminology Dirichlet principle was coined by Riemann (1857).

Dirichlet principle

Dirichlet principle

A solution to the Dirichlet problem should minimize the Dirichlet energy.

The terminology Dirichlet principle was coined by Riemann (1857).
The eponymy follows the celebrated Stigler's law:

Stigler's law

No scientific discovery is named after its original discoverer.

Dirichlet does not have any known contributions to the Dirichlet principle.
The principle was first derived by G. Green (1835).

Criticisms

Be cautious

Neither approach is rigorous as analysis was still premature at their time.

Weierstraß published the celebrated paper
Über das Sogenannte Dirichletsche Princip
raised doubts about the validity of Gauß and Riemann's works.

Criticisms

Be cautious

Neither approach is rigorous as analysis was still premature at their time.

Weierstraß published the celebrated paper
Über das Sogenannte Dirichletsche Princip
raised doubts about the validity of Gauß and Riemann's works.
In particular, Weierstraß pointed out that the existence of a minimizer requires a proof.

Impossible to solve in general

Counterexample

Lebesgue(1912) and Zaremba(1910) showed that Dirichlet problem is NOT always solvable!

Impossible to solve in general

Counterexample

Lebesgue(1912) and Zaremba(1910) showed that Dirichlet problem is NOT always solvable!

There are many attempts to justify Gauß and Riemann's works under certain restrictions in the subsequence decades.
Major contributions are due to Schwarz, Neumann, Poincaré, Hilbert, Lebesgue, Fredholm.
But we shall focus on a different approach.

Perron envelope

$$\begin{cases} \Delta\varphi = 0 \text{ on } \Omega, \\ \varphi = \varphi_0 \text{ on } \partial\Omega, \end{cases}$$

Perron envelope

$$\begin{cases} \Delta\varphi = 0 \text{ on } \Omega, \\ \varphi = \varphi_0 \text{ on } \partial\Omega, \end{cases}$$

Poincaré (1887) invented the so-called méthode du balayage.
Subsequent ideas of Perron (1923) and Wiener (1925) lead to a major revolution in the potential theory.

The ideas

Instead of studying strict solutions $\Delta\varphi = 0$, we study subsolutions instead:

$$\Delta\psi \geq 0.$$

Instead of studying $\varphi|_{\partial\Omega} = \varphi_0$, we require

$$\psi|_{\partial\Omega} \leq \varphi_0.$$

Perron envelope

The ideas

The maximal one among these subsolutions should be an actual solution.

The supremum of these subsolutions is the Perron envelope.

Perron envelope

The ideas

The maximal one among these subsolutions should be an actual solution.

The supremum of these subsolutions is the Perron envelope.
Subsequently, F. Riesz (1926) introduced the notion of subharmonic functions ψ:

- For a smooth function, this means $\Delta\psi \geq 0$;
- in general, a subharmonic function is allowed to take the value $-\infty$, and is defined using the sub-mean value property.

It is convenient to allow singular subharmonic functions in the definition of the Perron envelope.

Perron envelope

$$\begin{cases} \Delta\varphi = 0 \text{ on } \Omega, \\ \varphi = \varphi_0 \text{ on } \partial\Omega, \end{cases}$$

Wiener and later de la Vallée Poussin established:

Theorem

If φ_0 is continuous, the Perron envelope is a solution to the Dirichlet problem.

Table of Contents

Pathology

The study of Perron envelopes requires a supremum operation for subharmonic functions. But

Pathology

The increasing limit of a sequence/net of negative subharmonic functions φ_i is not always subharmonic.

So in particular, the Perron envelope is not subharmonic (and therefore not harmonic) in general!

The convergence theorem

Theorem (Szpilrajn–Radó, 1937)

The sequence φ_i converges to a subharmonic function φ outside a set E with 0 Lebesgue measure.

This is not fine enough for potential theory.

The convergence theorem

Theorem (Szpilrajn–Radó, 1937)

The sequence φ_i converges to a subharmonic function φ outside a set E with 0 Lebesgue measure.

This is not fine enough for potential theory.

Theorem (Brelot 1938, H. Cartan 1945)

The set E is polar. That is, there is a subharmonic function ψ with $\psi|_E \equiv -\infty$.

The convergence theorem

Theorem (Szpilrajn–Radó, 1937)

The sequence φ_i converges to a subharmonic function φ outside a set E with 0 Lebesgue measure.

This is not fine enough for potential theory.

Theorem (Brelot 1938, H. Cartan 1945)

The set E is polar. That is, there is a subharmonic function ψ with $\psi|_E \equiv -\infty$.

As a (non-trivial) consequence, up to modifying the Perron envelope by its values on a polar set, it becomes subharmonic.

Fine topology and thin sets

Since polar sets appear to be natural in the study of subharmonic functions, Brelot, Cartan among others began an in-depth study.

Observations

Subharmonic functions are not always continuous.
(Complete) polar sets are not always closed.

Fine topology and thin sets

Cartan introduced the notion of fine topology:

Fine topology

The fine topology is the coarsest topology rending all finite subharmonic functions continuous.

Both issues get solved naturally.

Fine topology and thin sets

Cartan introduced the notion of fine topology:

Fine topology

The fine topology is the coarsest topology rending all finite subharmonic functions continuous.

Both issues get solved naturally.
But the fine topology is rather abstract and difficult to use. Fortunately, we have

Theorem (Brelot, Cartan)

The fine convergence can be characterized using thin sets (introduced by Brelot).

Table of Contents

Several complex variables

With the advancements of several complex variables since the latter half of the 19th century, there is a natural need of applying potential theory in it.

Several complex variables

With the advancements of several complex variables since the latter half of the 19th century, there is a natural need of applying potential theory in it.

Example

Given a holomorphic function f (with one or more variables), the function $\log |f|^2$ is subharmonic.

This was also a motivation for Riesz to introduce subharmonic functions.

Several complex variables

Consider a holomorphic map F, $F^*f = f \circ F$ is holomorphic, so

$$\log |F^*f|^2 = F^* \left(\log |f|^2\right)$$

is also subharmonic. But this does not follow from the subharmonicity of $\log |f|^2$.

Several complex variables

Consider a holomorphic map F, $F^*f = f \circ F$ is holomorphic, so

$$\log |F^* f|^2 = F^* \left(\log |f|^2\right)$$

is also subharmonic. But this does not follow from the subharmonicity of $\log |f|^2$.
It is natural to reinforce the subharmonicity of $\log |f|$ to something functorial.

This is carried out by Pierre Lelong in 1945. Let $\Omega \subseteq \mathbb{C}^n$ be a domain.

Plurisubharmonic functions

A smooth function φ on Ω is plurisubharmonic if

$$\mathrm{i}\partial\bar{\partial}\varphi \geq 0.$$

Here $\mathrm{i}\partial\bar{\partial}\varphi$ is a $n \times n$-matrix. Positivity means the positivity as a matrix. The Laplacian Δ is just the trace (up to a universal constant).
Therefore,

Observation

A plurisubharmonic function is subharmonic.

Plurisubharmonic functions

Observation

Unlike the Laplacian Δ, both ∂ and $\bar{\partial}$ commute with holomorphic pullbacks.

This is nothing but the definition of a holomorphic map.

Plurisubharmonic functions

Observation

Unlike the Laplacian Δ, both ∂ and $\bar{\partial}$ commute with holomorphic pullbacks.

This is nothing but the definition of a holomorphic map.

Corollary

The holomorphic pull-back of a plurisubharmonic function is plurisubharmonic.

Plurisubharmonic functions

For singular functions, we have

Plurisubharmonic function

A function $\varphi\colon \Omega \to [-\infty, \infty)$ is plurisubharmonic if

1. φ is upper semi-continuous and not identically equal to $-\infty$;
2. for each complex line L in $\mathbb{C}^n$, the restriction of φ to each connected component of $L \cap \Omega$ is either subharmonic of identically $-\infty$.

All proceeding remarks work for these plurisubharmonic functions.

Plurisubharmonic functions

The functoriality of plurisubharmonic functions means that they can also be defined on complex manifolds. The study of these functions is known as the pluripotential theory.

A huge part of the potential theory has analogues in pluripotential theory.

Plurisubharmonic functions

The functoriality of plurisubharmonic functions means that they can also be defined on complex manifolds. The study of these functions is known as the pluripotential theory.

A huge part of the potential theory has analogues in pluripotential theory.

Moreover, there are quite a few unique features in pluripotential theory, like the L^2-estimates.

The pluripotential theory has become the cornerstone of the modern complex geometry.

References

- M. Brelot, Les étapes et les aspects multiples de la théorie du potentiel, 1972
- L. Gårding, The Dirichlet problem, 1979
- S. Deckelman, Electrostatic origins of the Dirichlet principle, arXiv:2408.12002

Thank you!