A mathematician's complaint about Hermitian operators

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Properties of Hermitian matrices.

- M is diagonalizable.
- All eigenvalues of M are real.

Physics explanation

In physical terms,

$$M = \sum_{i=1}^{n} \lambda_i \left| a_i \right\rangle \left\langle a_i \right| ,$$

where $|a_i\rangle$ is an complete orthonormal set of eigenstates, $\lambda_i \in \mathbb{R}$.
M corresponds to an observable.

Infinite dimensional Hilbert space

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But how?

Hermitian operator—Naive approach

The most common approach in physics textbooks is the following:

$$M = \sum_{i=1}^{\infty} \lambda_i \left| a_i \right\rangle \left\langle a_i \right| \,,$$

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Both are wrong!

An example

Consider the free Hamiltonian

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The operator H is not defined on the whole $\mathcal{H}$.

Hermitian operator—A less naive approach

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Symmetric operator

A densely defined operator A on $\mathscr{H}$ is symmetric if

$$(x, Ay) = (Ax, y) \quad \forall x, y \in D(A) .$$

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Consider $\mathscr{H} = L^2((0,\infty))$, $D(p) = C_c^\infty((0,\infty))$. Let $p = -\mathrm{i}\nabla$.
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More generally, we have

Theorem

The spectrum of a symmetric operator falls into one of the following four categories:

1. $\mathbb{C}$.
2. $\{z : \operatorname{Im} z \geq 0\}$.
3. $\{z : \operatorname{Im} z \leq 0\}$.
4. *A subset of* $\mathbb{R}$.

Self-adjoint operators

Consider a densely defined operator A on $\mathcal{H}$.

Adjoint

The domain $D(A^\dagger)$ is the set of $y \in \mathcal{H}$ such that

$$|(y, Ax)| \leq C\|x\|$$

for all $x \in D(A)$. By Riesz representation theorem, for each $y \in D(A^\dagger)$, there is a unique $A^\dagger y \in \mathcal{H}$ such that

$$(A^\dagger y, x) = (y, Ax) \quad \forall x \in D(A) .$$

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Slightly more generally,

Essentially self-adjoint

A symmetric operator A is essentially self-adjoint if A admits a (unique) self-adjoint extension.

Example

The Hamiltonian

$$H = -\frac{1}{2M}\nabla^2$$

is essentially self-adjoint if $D(H) = C_c^\infty(\mathbb{R}^3)$. The self-adjoint extension defined on the Sobolev space $H^2(\mathbb{R}^3)$.

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Similarly, the position operator x is self-adjoint on

$$D(x) = \left\{ f \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |x|^2 |f(x)|^2 \, \mathrm{d}x < \infty \right\}.$$

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The momentum operator $p = -\mathrm{i}\nabla$ is essentially self-adjoint on

$$D(p) = C_c^1(\mathbb{R}^3) \, .$$

Warning

The notion of self-adjointness is domain sensitive.
In general, it is fairly easy to determine if a given operator is symmetric, but there are no general methods to determine the self-adjointness of an operator.

Spectral theorem of self-adjoint operators

Theorem

A symmetric operator A is self-adjoint if and only if the spectrum of A is contained in $\mathbb{R}$.

This justifies λ_i's in

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Theorem

Assume that A is self-adjoint. The above equation makes sense if we replace the density operator by a projection-valued measure:

$$A = \int_{\mathbb{R}} \lambda \, \mathrm{d}\sigma(\lambda) \, .$$

Spectral calculus

Given the spectral theorem, we can rigorously define $f(A)$, where f is a measurable function on $\mathbb{R}$:

$$f(A) := \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\sigma(\lambda)$$

as long as the right-hand side is well-defined. In physical terms,

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In particular, $\exp(\mathrm{i}tA)$ is defined.

Stone's theorem

Theorem

$A \mapsto (\exp(\mathrm{i}tA))_{t\geq 0}$ *is a bijection between self-adjoint operators and strongly continuous one-parameter unitary groups.*

In particular, the evolution operator of Hamiltonian $\exp(itH)$ is a one-parameter unitary group, as expected.

Interactive theory

Only a few results about the self-adjointness in (non-relativistic) interactive theory are known. A good reference of Reed–Simon. Consider $H = -\frac{1}{2M}\nabla^2 + V(x)$ in 1D for simplicity.

Theorem

H is essentially self-adjoint if one of the following conditions are satisfied:

- $V = V_1 + V_2$, $V_1 \in L^2$, $V_2 \in L^\infty$.
- V *is locally* L^2, $V(x) \geq -V^*(|x|)$, *where* $V^*(r) = o(r^2)$ *as* $r \to \infty$.

Relativisitic free theory

When taking relativity into account, the one-particle Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ is replaced by the Bosonic Fock space: Consider the parabola $\Sigma = \{P \in \mathbb{R}^{1,3} : P^2 = M^2, P^0 > 0\}$. There is a natural measure $\mathrm{d}\lambda_M$ on Σ. The Bosonic Fock space is then

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In this case, by second quantization, one can enhance the free Hamiltonian H to a self-adjoint operator-valued Schwarz distribution on $\mathcal{H}$. Most of what physicists do still makes sense.

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In general, even the Hilbert space $\mathscr{H}$ is not expected to exist, due to the presence of Landau pole. Even in cases where $\mathscr{H}$ exists, it is not easy to verify that H is essentially self-adjoint.
Physicists' solution: QFT is developed only perturbatively in the interactive picture. Even though we do not have any information about $\mathscr{H}$, we can still calculate Green functions, S-matrices etc.
Mathematicians' dilemma: By Haag's theorem, interactive picture does not make sense. QFT only exists non-perturbatively. We do not have Hilbert spaces, the Hamiltonian is not expected to be self-adjoint, etc. We have no idea what the outcome of Feymann diagrams and renormalizations has to do with reality.

Thank you for your attention!