Partial Okounkov bodies and toric geometry

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K-stability and moment maps, Cambridge

References

1. Partial Okounkov bodies and Duistermaat–Heckman measures of non-Archimedean metrics (2021);
2. Singularities in global pluripotential theory — Lecture notes at Zhejiang university (2024).

Setup

- X: A smooth/normal projective variety of dimension n.
- L: A holomorphic line bundle on X.
- h: A (singular) positively-curved metric on L.

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Goal

Convex body

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Why do we study them?

Okounkov bodies translate the geometric properties of L to properties of convex bodies.

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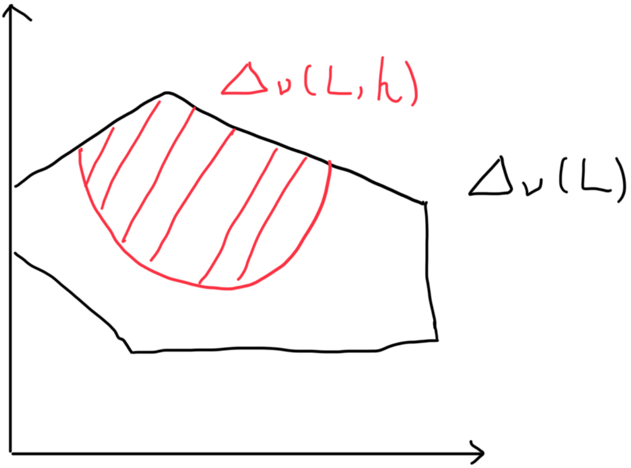
Goal

We want to construct convex bodies $\{\Delta_\nu(L,h)\}_\nu$ which transform the properties of (L,h) into the properties of convex bodies.

These convex bodies are the partial Okounkov bodies.

Goal



The parameter ν

The parameter ν

We begin with an (admissible) flag of subvarieties of X:

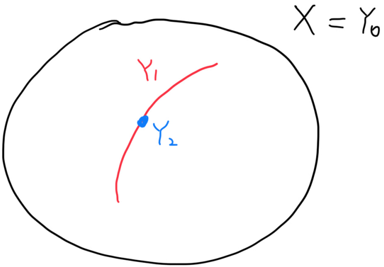
$$X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots Y_n = \{\text{pt}\}.$$

This flag induces a valuation $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}^n$: $\nu(f)$ is the successive order of vanishing of f along the flag.

$(\dim X = 1)$

$\nu(f) = \mathrm{ord}_{Y_1} f$

The parameter ν



dim $X = 2$

$$\nu_1(f) = \mathrm{ord}_{Y_1} f \;;\quad \nu_2(f) = \mathrm{ord}_{Y_2}\left(f \cdot \underbrace{t}_{Y_1=\{t=0\}}^{-\nu_1(f)}\right)\Big|_{Y_1}$$

The parameter ν

More generally, the parameter ν runs over all valuations $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ with similar properties.

To remember

The map ν transforms multiplications into additions.

Classic Okounkov bodies

Fix X, L, ν. Suppose that L is big (volume > 0). The construction of $\Delta_\nu(L) \subseteq \mathbb{R}^n$ consists of three steps:

- From the geometric data (X, L) to a ring:

$$(X, L) \mapsto R(X, L) = \bigoplus_{k=0}^{\infty} H^0(X, L^k);$$

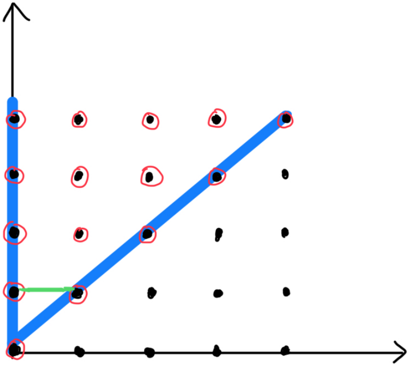
- From the ring to a semigroup:

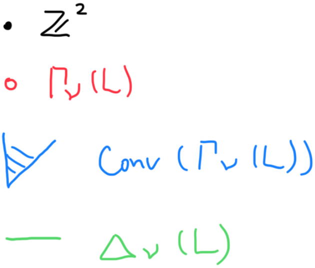
$$R(X, L) + \nu \mapsto \Gamma_\nu(L) = \left\{(\nu(s), k) : s \in H^0(X, L^k)^\times, k \in \mathbb{N}\right\} \subseteq \mathbb{Z}^{n+1};$$

- From semigroup to a convex body:

$$\Gamma \mapsto \Delta_\nu(L) = \Delta(\Gamma_\nu(L)) = \{x_{n+1} = 1\} \cap \mathrm{Conv}(\Gamma_\nu(L)).$$

Example





$X = \mathbb{P}^1$, $L = \mathcal{O}(1)$. Flag: $X \supseteq \{0\}$. $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}$ is the order of vanishing along 0.

$$\Gamma_\nu(L) = \left\{(a,b) \in \mathbb{Z}^2 : 0 \le a \le b\right\}, \quad \Delta_\nu(L) = [0,1].$$

Why are they useful?

Theorem (Lazarsfeld–Mustață)

The convex bodies $\Delta_\nu(L)$ depend only on the numerical class of L.

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Slogan

The Okounkov bodies are universal numerical invariants of the line bundles.

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Example

$\operatorname{vol} \Delta_\nu(L) = \frac{1}{n!} \operatorname{vol} L.$

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Partial Okounkov bodies

We next include h (a positively curved singular metric on L) into the picture.
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The analogue of the proceeding theorems is:

Theorem ([1])

$\Delta_\nu(L,h)$ depends only on the $\mathcal{J}$-equivalence class of h.
The family $\{\Delta_\nu(L,h)\}_\nu$ determines h up to $\mathcal{J}$-equivalence.

We say h and h' are $\mathcal{J}$-equivalent if all Lelong numbers of h and h' (on all birational models of X) are equal.

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Slogan

The partial Okounkov bodies are universal invariants of the singularities of h.

Construction

We could try to imitate the proceeding constructions: without h,

- $(X, L) \mapsto R(X, L)$ (ring);
- $R(X, L) + \nu \mapsto \Gamma_\nu(L)$ (semigroup);
- $\Gamma_\nu(L) \mapsto \Delta_\nu(L)$.

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- $(X, L) \mapsto R(X, L)$ (ring);
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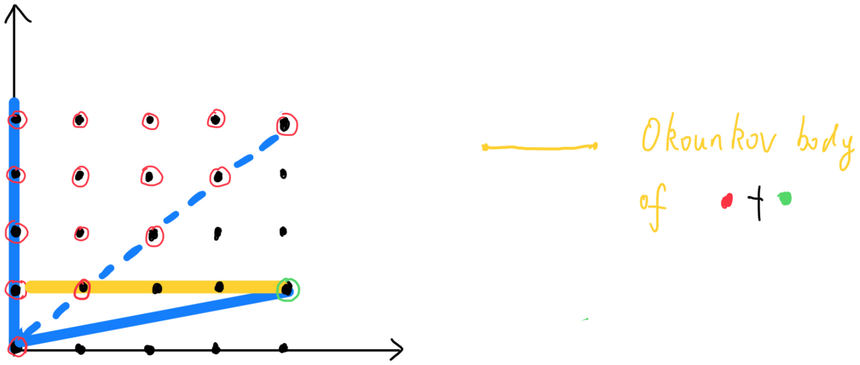
With h,

1. $(X, L, h) \mapsto R(X, L, h) = \bigoplus_{k=0}^{\infty} H^0(X, L^k \otimes \mathcal{I}(h^k))$ (no longer a ring);
2. $R(X, L, h) \mapsto \Gamma_\nu(L, h) = \{(\nu(s), k) : s \in H^0(X, L^k \otimes \mathcal{I}(h^k))^\times, k \in \mathbb{N}\}$ (no longer a semigroup);
3. ???.

Here $H^0(X, L^k \otimes \mathcal{I}(h^k))$ is the set of L^2-sections of L^k.

Construction



The Okounkov body construction fails to reflect the asymptotic behaviours of a non-semi-group!

The magic

The key observation is that $R(X, L, h)$ is not very far from a ring and $\Gamma_\nu(L, h)$ is not very far from a semigroup.

Theorem ((Essentially)Darvas–X., 2020+2021)

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Theorem ((Essentially)Darvas–X., 2020+2021)

$\Gamma_\nu(L, h)$ *is an almost semigroup.*

In concrete terms, $\Gamma_\nu(L, h)$ can be approximated by semigroups with respect to the following pseudometric:

$$d(S, S') = \varlimsup_{k\to\infty} k^{-n} \left(\#S_k + \#S'_k - 2\#(S_k \cap S'_k)\right).$$

Where $S_k = S \cap \{x_1 = k\}$.

The Magic

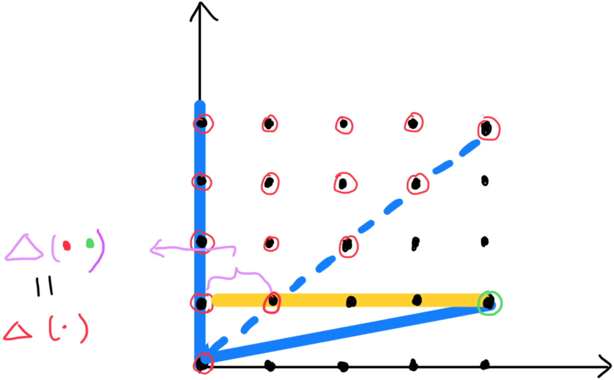
Theorem ([1])

The Okounkov body map extends continuously from semigroups to almost semigroups. In other words, we have a map

$$\Delta : \{\textit{Almost semigroups}\} \to \{\textit{Convex bodies}\}.$$

The topology on the set of convex bodies is induced by the Hausdorff metric.

Example



$\Delta = [0, 1].$

Construction of the partial Okounkov bodies

Recall our construction scheme:

1. $(X, L, h) \mapsto R(X, L, h) = \bigoplus_{k=0}^{\infty} H^0(X, L^k \otimes \mathcal{I}(h^k))$;
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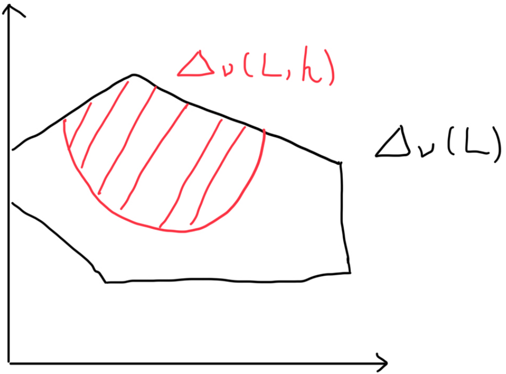
   $\{(\nu(s), k) : s \in H^0(X, L^k \otimes \mathcal{I}(h^k)), k \in \mathbb{N}\}$ (an almost semigroup);
3. $$\Delta_\nu(L, h) := \Delta(\Gamma_\nu(L, h)) \subseteq \Delta_\nu(L).$$

Construction of the partial Okounkov bodies



Example

$X = \mathbb{P}^1$, $L = \mathcal{O}(1)$. ν is the order of vanishing at 0. We have seen that $\Delta_\nu(L) = [0, 1]$.
When the singularities of h are like $a \log |z|^2$, we have

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Observation

The more singular h is, the smaller $\Delta_\nu(L, h)$ becomes.

Volumes

Recall that

$$\operatorname{vol}\Delta_\nu(L) = \lim_{k\to\infty}\frac{1}{k^n}h^0(X, L^k).$$

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Corollary

$\Delta_\nu(L,h) = \Delta_\nu(L)$ *if* h *has minimal singularities.*

Continuity

Theorem

The map $h \mapsto \Delta_\nu(L, h)$ *is continuous.*

Here the topology on the set of h is defined by Darvas–Di Nezza–Lu.

Other points of view

There are three other equivalent definitions of the partial Okounkov body.

Theorem ([1])

$\mathrm{Conv}(k^{-1}\Gamma_\nu(L,h)\cap\{x_{n+1}=1\}))\}$ *converge to* $\Delta_\nu(L,h)$ *as* $k\to\infty$.

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$\Delta_\nu(L,h)$ *is the Hausdorff limit of the net*

$$\Delta_\nu(\pi^*L - \textit{divisorial part of } \mathrm{dd^c}\pi^*h)+\nu(h),$$

where $\pi: Y\to X$ *runs over suitable birational models of* X.

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There is a valuative characterization as in Kewei's talk ([2]).

Toric case

There are a few well-studied cases of partial Okounkov bodies in the literature.
Suppose (X, L, h) are toric and the flag is toric-invariant.
In this case, h can be identified with a convex function $h\colon N_{\mathbb{R}} \to \mathbb{R}$.

Toric setting

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The partial Okounkov body $\Delta_\nu(L, h)$ can be canonically identified with

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The importance of this convex body is well-known to experts.

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Slicing of Okounkov bodies

The partial Okounkov bodies appear naturally when we study slices of Okounkov bodies.

Theorem ([2])

Suppose that L is big. Under mild assumptions, the intersection

$$\Delta(L) \cap \{x_1 = \cdots = x_k = 0\}$$

is given by the partial Okounkov body of the trace operator of $T_{\min}$ (the current with minimal singularities in $c_1(L)$).

Slicing of Okounkov bodies

As for the interior slices, we have

Theorem (Kewei Zhang)

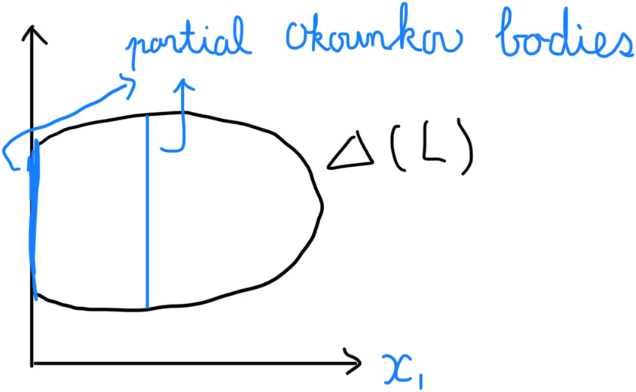
The slices

$$\Delta(L) \cap \{x_1 = t\}$$

are partial Okounkov bodies when t *does not take the two extreme values.*

This observation played a key role in the proof of the volume identity of transcendental Okounkov bodies (Darvas–Reboulet–Witt Nyström–X.–Zhang).

Slicing of Okounkov bodies



Computing the Lelong numbers

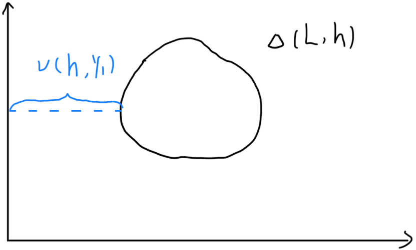
Theorem ([1])

Suppose that the valuation ν is induced by a flag $Y_1 \supseteq \cdots \supseteq Y_n$, we have

$$\min_{x \in \Delta(L,h)} x_1 = \nu(h, Y_1).$$

The right-hand side is the minimum of the Lelong number of h along Y_1.

This result seems to be new even in the toric setting.



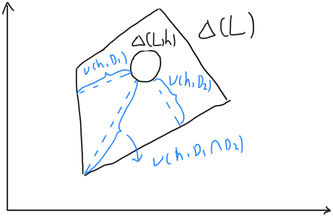
Computing the Lelong numbers

Corollary(Yi Yao)

In the toric situation, if D_1, D_2 are two different toric invariant prime divisors, then

$$\nu(h, D_1 \cap D_2) \geq \nu(h, D_1) + \nu(h, D_2).$$

This result can also be proved using the non-Archimedean point of view. But the Okounkov point of view gives more information.



Theorem ([2])

A non-Archimedean psh metric on the Berkovich analytification of L induces a canonical Radon measure on $\mathbb{R}$.

This construction extends the classical Duistermaat–Heckman measures of test configurations.

The Duistermaat–Heckman measures

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A non-Archimedean psh metric on the Berkovich analytification of L induces a canonical Radon measure on $\mathbb{R}$.

This construction extends the classical Duistermaat–Heckman measures of test configurations.
The interesting point is that the statement is completely independent of Okounkov bodies!
Eiji Inoue also made similar constructions.

The Duistermaat–Heckman measures

The proof consists of three steps:

1. A non-Archimedean metric can be identified with a concave curve $(\phi_\tau)_\tau$ of (complex) metrics (Darvas–X.–Zhang, 2023).
2. Choose a valuation and construct a corresponding concave curve of convex bodies $(\Delta(L, \phi_\tau))_\tau$.
3. Construct a Radon measure using an extension of Boucksom–Chen's method.

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In particular, we can show that the family of Okounkov bodies constructed from a filtered linear series are all partial Okounkov bodies.

Final comments

Almost everything explained in this talk can be extended to the transcendental setting. This is carried out in [2] based on the joint work with Darvas, Reboulet, Witt Nyström, Zhang.

Final comments

Conjecture

Suppose that $(L_1, h_1), \dots, (L_n, h_n)$ are Hermitian big line bundles equipped with $\mathcal{J}$-good metrics, then

$$\int_X c_1(L_1, h_1) \wedge \dots \wedge c_1(L_n, h_n) = \sup_{\nu} \mathrm{vol}(\Delta_\nu(L_1, h_1), \dots, \Delta_\nu(L_n, h_n)).$$

As a special case,

Conjecture

Assume that $L_1, \dots, L_n$ are big line bundles, then

$$\langle L_1, \dots, L_n \rangle = \sup_{\nu} \mathrm{vol}(\Delta_\nu(L_1), \dots, \Delta_\nu(L_n)).$$

$\langle L_1, \dots, L_n \rangle$ is the movable intersection number.

Thank you!