Pluripotential-theoretic approach to radial energy functionals

Mingchen, Xia

Chalmers Tekniska Högskola

Beijing University, 11/20/2020

Setup

Notations:

- X: compact Kähler manifold of dimension n.

Setup

Notations:

- X: compact Kähler manifold of dimension n.
- L: an ample line bundle on X.

Setup

Notations:

- X: compact Kähler manifold of dimension n.
- L: an ample line bundle on X.
- ω: a Kähler form in $c_1(L)$.

Setup

Notations:

- X: compact Kähler manifold of dimension n.
- L: an ample line bundle on X.
- ω: a Kähler form in $c_1(L)$.
- $V = \int_X \omega^n$.

Setup

Notations:

- X: compact Kähler manifold of dimension n.
- L: an ample line bundle on X.
- ω: a Kähler form in $c_1(L)$.
- $V = \int_X \omega^n$.
- h: a Hermitian metric on L, such that $\omega = c_1(L, h)$.

Setup

Notations:

- X: compact Kähler manifold of dimension n.
- L: an ample line bundle on X.
- ω: a Kähler form in $c_1(L)$.
- $V = \int_X \omega^n$.
- h: a Hermitian metric on L, such that $\omega = c_1(L, h)$.
- $\mathcal{E}^1(X, \omega)$: the set of ω-psh functions with finite energy:

$$\mathcal{E}^1(X,\omega) = \left\{ \varphi \in \mathrm{PSH}(X,\omega) : \int_X |\varphi|\, \omega_\varphi^n < \infty \,, \int_X \omega_\varphi^n = \int_X \omega^n \right\} .$$

Energy functionals

Various energy functionals have been shown to be important in the study of the geometry of (X, L, ω):

- Monge–Ampère energy $E(\varphi) = \frac{1}{V}\sum_{j=0}^{n}\int_X \varphi\, \omega_\varphi^j \wedge \omega^{n-j}$.

Energy functionals

Various energy functionals have been shown to be important in the study of the geometry of (X, L, ω):

- Monge–Ampère energy $E(\varphi) = \frac{1}{V} \sum_{j=0}^{n} \int_X \varphi \, \omega_\varphi^j \wedge \omega^{n-j}$.
- Ricci energy $E_R(\varphi) := -\frac{1}{V} \sum_{j=0}^{n-1} \int_X \varphi \, \mathrm{Ric}\, \omega \wedge \omega_\varphi^j \wedge \omega^{n-1-j}$.

Energy functionals

Various energy functionals have been shown to be important in the study of the geometry of (X, L, ω):

- Monge–Ampère energy $E(\varphi) = \frac{1}{V}\sum_{j=0}^{n}\int_X \varphi\, \omega_\varphi^j \wedge \omega^{n-j}$.
- Ricci energy $E_R(\varphi) := -\frac{1}{V}\sum_{j=0}^{n-1}\int_X \varphi\, \mathrm{Ric}\, \omega \wedge \omega_\varphi^j \wedge \omega^{n-1-j}$.
- Entropy $\mathrm{Ent}(\varphi) := \frac{1}{V}\int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n$ if $\omega_{\varphi^n} \ll \omega^n$, ∞ otherwise.

Energy functionals

Various energy functionals have been shown to be important in the study of the geometry of (X, L, ω):

- Monge–Ampère energy $E(\varphi) = \frac{1}{V}\sum_{j=0}^{n}\int_X \varphi\,\omega_\varphi^j \wedge \omega^{n-j}$.
- Ricci energy $E_R(\varphi) := -\frac{1}{V}\sum_{j=0}^{n-1}\int_X \varphi\,\mathrm{Ric}\,\omega \wedge \omega_\varphi^j \wedge \omega^{n-1-j}$.
- Entropy $\mathrm{Ent}(\varphi) := \frac{1}{V}\int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n$ if $\omega_{\varphi^n} \ll \omega^n$, ∞ otherwise.
- Mabuchi K-energy $M(\varphi) = \bar{S}E(\varphi) + E_R(\varphi) + \mathrm{Ent}(\varphi)$.

Energy functionals

Various energy functionals have been shown to be important in the study of the geometry of (X, L, ω):

- Monge–Ampère energy $E(\varphi) = \frac{1}{V}\sum_{j=0}^{n}\int_X \varphi\, \omega_\varphi^j \wedge \omega^{n-j}$.
- Ricci energy $E_R(\varphi) := -\frac{1}{V}\sum_{j=0}^{n-1}\int_X \varphi\, \mathrm{Ric}\,\omega \wedge \omega_\varphi^j \wedge \omega^{n-1-j}$.
- Entropy $\mathrm{Ent}(\varphi) := \frac{1}{V}\int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n$ if $\omega_{\varphi^n} \ll \omega^n$, ∞ otherwise.
- Mabuchi K-energy $M(\varphi) = \bar{S}E(\varphi) + E_R(\varphi) + \mathrm{Ent}(\varphi)$.

These functionals are defined on $\mathcal{E}^1$.

Weak geodesic in $\mathcal{H}$

Weak geodesics in $\mathcal{H}$. Let $\varphi_0, \varphi_1 \in \mathcal{H}$.

Definition

A weak geodesic from φ_0 to φ_1 is a curve $(\varphi_t)_{t\in[0,1]}$ of bounded psh functions such that the corresponding S^1-invariant potential Φ on $X \times \{z \in \mathbb{C} : e^{-1} \leq |z| \leq 1\}$ is qpsh and solves the homogeneous Monge–Ampère equation

$$\mathrm{MA}(\Phi) = 0\,.$$

Weak geodesic in $\mathcal{H}$

Weak geodesics in $\mathcal{H}$. Let $\varphi_0, \varphi_1 \in \mathcal{H}$.

Definition

A weak geodesic from φ_0 to φ_1 is a curve $(\varphi_t)_{t\in[0,1]}$ of bounded psh functions such that the corresponding S^1-invariant potential Φ on $X \times \{z \in \mathbb{C} : e^{-1} \leq |z| \leq 1\}$ is qpsh and solves the homogeneous Monge–Ampère equation

$$\mathrm{MA}(\Phi) = 0\,.$$

Theorem (Chen, Chu–Tosatti–Weinkove)

There is always a unique $C^{1,1}$ weak geodesic from φ_0 to φ_1.

Geodesic in $\mathcal{E}^1$

The notation of geodesics can be generalized to the case where $\varphi_0, \varphi_1 \in \mathcal{E}^1$.

Theorem (Darvas)

There is always a unique finite energy geodesic from φ_0 to φ_1.

Finite energy coincides with the weak geodesic if $\varphi_0, \varphi_1 \in \mathcal{H}$.

Geodesic rays

Definition

A geodesic ray is a map $\ell : [0, \infty) \to \mathcal{E}^1$ such that $\ell|_{[0,t]}$ is a finite energy geodesic for any $t \geq 0$.

The set of finite energy geodesic rays ℓ with $\ell_0 = 0$ is denoted by $\mathcal{R}^1$.

Theorem (Chen–Cheng, Darvas–Lu)

$\mathcal{R}^1$ has a complete metric:

$$d_1(\ell, \ell') := \lim_{t \to \infty} \frac{1}{t} d_1(\ell_t, \ell'_t) \,.$$

Radial functionals

For $F = E, E_R, \mathrm{Ent}, M$, we define

$$\mathbf{F}(\ell) = \lim_{t\to\infty} \frac{1}{t} F(\ell_t)\,, \quad \ell \in \mathcal{R}^1\,.$$

These radial functionals are important to the study of K-stability.

Radial functionals

For $F = E, E_R, \mathrm{Ent}, M$, we define

$$\mathbf{F}(\ell) = \lim_{t\to\infty} \frac{1}{t} F(\ell_t)\,, \quad \ell \in \mathcal{R}^1\,.$$

These radial functionals are important to the study of K-stability.

Theorem (Chen–Cheng)

Assume that $\mathrm{Aut}(X,L)/\mathbb{G}_m$ is discrete. Then there exists a cscK metric on L iff there exists $\delta > 0$, such that $\mathbf{M}(\ell) \geq \delta$ for any $\ell \in \mathcal{R}^1$, $d_1(0,\ell_1) = 1$.

Radial functionals

For $F = E, E_R, \mathrm{Ent}, M$, we define

$$\mathbf{F}(\ell) = \lim_{t\to\infty} \frac{1}{t} F(\ell_t)\,, \quad \ell \in \mathcal{R}^1\,.$$

These radial functionals are important to the study of K-stability.

Theorem (Chen–Cheng)

Assume that $\mathrm{Aut}(X, L)/\mathbb{G}_m$ is discrete. Then there exists a cscK metric on L iff there exists $\delta > 0$, such that $\mathbf{M}(\ell) \geq \delta$ for any $\ell \in \mathcal{R}^1$, $d_1(0, \ell_1) = 1$.

Theorem (Hisamoto, Xia)

Assume that (X, L) is (geodesically) unstable, there is a unique minimizer of

$$\inf_{\ell\in\mathcal{R}^2, d_2(0,\ell_1)=1} \mathbf{M}(\ell)\,.$$

Different approaches

So far, the following approaches to these radial functionals have been developed.

- Through test configurations: Odaka, Wang, Boucksom, Hisamoto, Jonsson, Li, Xu, ...

Different approaches

So far, the following approaches to these radial functionals have been developed.

- Through test configurations: Odaka, Wang, Boucksom, Hisamoto, Jonsson, Li, Xu, ...
- Through filtrations: Witt Nyström, Székelyhidi, ...

Different approaches

So far, the following approaches to these radial functionals have been developed.

- Through test configurations: Odaka, Wang, Boucksom, Hisamoto, Jonsson, Li, Xu, ….
- Through filtrations: Witt Nyström, Székelyhidi, ….
- Through non-Archimedean geometry: Boucksom, Favre, Jonsson, Berman, ….

Different approaches

So far, the following approaches to these radial functionals have been developed.

- Through test configurations: Odaka, Wang, Boucksom, Hisamoto, Jonsson, Li, Xu, ...
- Through filtrations: Witt Nyström, Székelyhidi, ...
- Through non-Archimedean geometry: Boucksom, Favre, Jonsson, Berman, ...
- Through test curves: Ross, Witt Nyström, Darvas, Di Nezza, Lu, ...

Different approaches

So far, the following approaches to these radial functionals have been developed.

- Through test configurations: Odaka, Wang, Boucksom, Hisamoto, Jonsson, Li, Xu, ...
- Through filtrations: Witt Nyström, Székelyhidi, ...
- Through non-Archimedean geometry: Boucksom, Favre, Jonsson, Berman, ...
- Through test curves: Ross, Witt Nyström, Darvas, Di Nezza, Lu, ...

The first two approaches can only handle some cases. They have the disadvantage that they do not map injectively to geodesic rays. Also they could not handle general geodesic rays.
We adopt a mixture of the third and the fourth approaches.

Test curves

A test curve is the Legendre transform of some geodesic ray:

$$\hat{\ell}_\tau := \inf_{t \geq 0} \ell_t - t\tau \,, \quad \tau \in \mathbb{R} \,.$$

How does the a test curve looks like?

- The curve $\hat{\ell}_\tau$ is concave in τ.

Test curves

A test curve is the Legendre transform of some geodesic ray:

$$\hat{\ell}_\tau := \inf_{t \geq 0} \ell_t - t\tau\,, \quad \tau \in \mathbb{R}\,.$$

How does the a test curve looks like?

- The curve $\hat{\ell}_\tau$ is concave in τ.
- The curve becomes $-\infty$ when $\tau > \tau^+$ for some τ^+, it approaches 0 when $\tau \to -\infty$.

Test curves

A test curve is the Legendre transform of some geodesic ray:

$$\hat{\ell}_\tau := \inf_{t\geq 0} \ell_t - t\tau\,, \quad \tau \in \mathbb{R}\,.$$

How does the a test curve looks like?

- The curve $\hat{\ell}_\tau$ is concave in τ.
- The curve becomes $-\infty$ when $\tau > \tau^+$ for some τ^+, it approaches 0 when $\tau \to -\infty$.
- The curve is usc as a function in (τ, x).

Test curves

A test curve is the Legendre transform of some geodesic ray:

$$\hat{\ell}_\tau := \inf_{t\geq 0} \ell_t - t\tau\,, \quad \tau \in \mathbb{R}\,.$$

How does the a test curve looks like?

- The curve $\hat{\ell}_\tau$ is concave in τ.
- The curve becomes $-\infty$ when $\tau > \tau^+$ for some τ^+, it approaches 0 when $\tau \to -\infty$.
- The curve is usc as a function in (τ, x).
- Each $\hat{\ell}_\tau$ is ω-psh and model.

Test curves

A test curve is the Legendre transform of some geodesic ray:

$$\hat{\ell}_\tau := \inf_{t\geq 0} \ell_t - t\tau\,, \quad \tau \in \mathbb{R}\,.$$

How does the a test curve looks like?

- The curve $\hat{\ell}_\tau$ is concave in τ.
- The curve becomes $-\infty$ when $\tau > \tau^+$ for some τ^+, it approaches 0 when $\tau \to -\infty$.
- The curve is usc as a function in (τ, x).
- Each $\hat{\ell}_\tau$ is ω-psh and model.
- $$\mathbf{E}(\psi_\bullet) := \frac{1}{V}\tau^+ + \frac{1}{V}\int_{-\infty}^{\tau^+} \left(\int_X \omega_{\psi_\tau}^n - \int_X \omega^n \right) \mathrm{d}\tau > -\infty\,.$$

Test curves

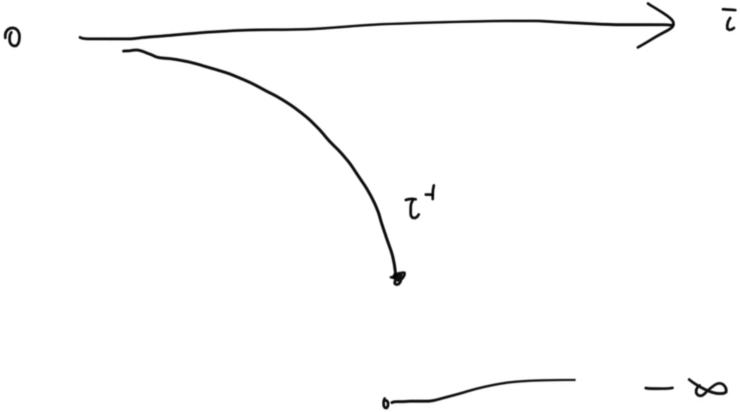
These conditions completely characterize the test curves.

Theorem (Ross–Witt Nyström, Darvas–Di Nezza–Lu, Darvas–Xia)

The Legendre is a bijection from $\mathcal{R}^1$ to the set of curves satisfying the five conditions above.

This theorem is the bridge between pluripotential theory of singular potentials and the theory of geodesic rays.

Image of a test curve



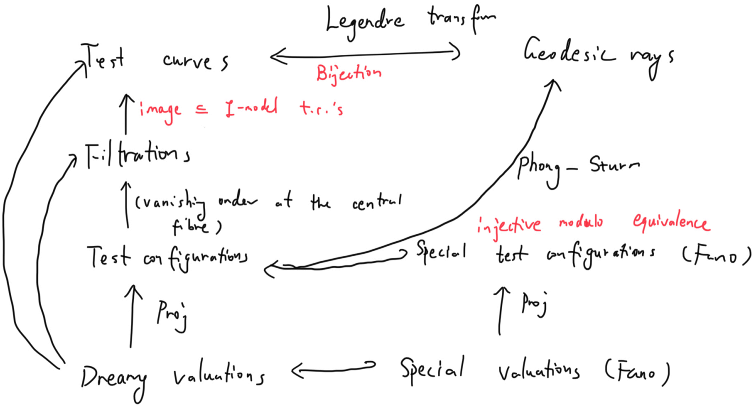
Deformation to the normal cone, an example

Let ψ be a potential with hyperplane singularities along some snc divisor D on X, such that $L - D$ is semi-ample.
Let $\mathcal{X} = \mathrm{Bl}_{D\times\{0\}} X \times \mathbb{C}$ with a natural map $\Pi : \mathcal{X} \to X \times \mathbb{C}$. Then $\mathcal{L} := \Pi^* p_1^* L - E$ is semi-ample and $(\mathcal{X}, \mathcal{L})$ is a test configuration.
The corresponding test curve is given by $P[(1+\tau)\psi]$ when $\psi \in [-1, 0]$, $-\infty$ if $\tau > 0$, 0 if $\tau < -1$.

Deformation to the normal cone, an example

Let ψ be a potential with hyperplane singularities along some snc divisor D on X, such that $L - D$ is semi-ample.
Let $\mathcal{X} = \mathrm{Bl}_{D\times\{0\}} X \times \mathbb{C}$ with a natural map $\Pi : \mathcal{X} \to X \times \mathbb{C}$. Then $\mathcal{L} := \Pi^* p_1^* L - E$ is semi-ample and $(\mathcal{X}, \mathcal{L})$ is a test configuration.
The corresponding test curve is given by $P[(1+\tau)\psi]$ when $\psi \in [-1, 0]$, $-\infty$ if $\tau > 0$, 0 if $\tau < -1$.
Lelong numbers are piecewise linear!

Relation to other approaches



Relation to other approaches

The map from filtrations to test curves is as follows: Let $\mathcal{F}^\bullet$ be a filtration of $R(X, L)$, then

$$\psi_\tau := \sup_{k \in \mathbb{Z}_{>0}}{}^* k^{-1} \sup{}^* \left\{ \log |s|^2_{h^k} : s \in \mathscr{F}^{k\tau} H^0(X, L^k), \sup_X |s|_{h^k} \leq 1 \right\} .$$

Goal

We want to express radial functionals in terms of functionals of the corresponding test curves.
Why?

Goal

We want to express radial functionals in terms of functionals of the corresponding test curves.

Why?

- The corresponding functional of test curves are much easier to understand.

Goal

We want to express radial functionals in terms of functionals of the corresponding test curves.

Why?

- The corresponding functional of test curves are much easier to understand.
- This will enable us to apply techniques developed in Kähler geometry to the study of singular potentials, and vice versa.

Goal

We want to express radial functionals in terms of functionals of the corresponding test curves.
Why?

- The corresponding functional of test curves are much easier to understand.
- This will enable us to apply techniques developed in Kähler geometry to the study of singular potentials, and vice versa.
- This is the correct setup to extend what have been studied for test configurations and filtrations.

Monge–Ampère energy — First example

Theorem (Ross–Witt Nyström, Darvas–Xia)

Let $\ell \in \mathcal{R}^1$. *Then*

$$\mathbf{E}(\ell) = \frac{1}{V}\tau^+ + \frac{1}{V}\int_{-\infty}^{\tau^+}\left(\int_X \omega_{\psi_\tau}^n - \int_X \omega^n\right) \mathrm{d}\tau\,.$$

This implies that we should define

$$E(\psi) := \frac{1}{V}\int_X \omega_\psi^n - 1\,.$$

Maximal geodesic rays

Given a test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L). One can always solve the homogeneous Monge–Ampère equation on $\mathcal{X}$ with boundary value 0 on $X \times S^1$. This geodesic ray is known as the Phong–Sturm geodesic ray. This is a $C^{1,1}$-geodesic ray (Chu–Tosatti–Weinkove).

Definition (Berman–Boucksom–Jonsson)

A geodesic ray $\mathcal{R}^1$ is *maximal* if is can be approximated by Phong–Sturm geodesic rays of test configurations.

Maximal geodesic rays

The most important geodesic rays in the study of K-stability are all maximal.

Theorem (Li)

A geodesic ray $\ell \in \mathcal{R}^1$ *with* $\mathbf{Ent}(\ell) < \infty$ *is maximal.*

A maximal geodesic ray is algebraic in a very strong sense.

Maximal geodesic rays under the Legendre transform

Theorem (Darvas–Xia)

Under the Legendre transform, maximal geodesic rays correspond to test curves $\psi_\bullet$, such that each ψ_τ ($\tau < \tau^+$) is $\mathcal{J}$-model.

Definition

A potential $0 \geq \psi \in \mathrm{PSH}(X, \omega)$ is *model* if there are no other potentials $\varphi \leq 0$ less singular than ψ while having the same mass.

Definition

A model potential $0 \geq \psi \in \mathrm{PSH}(X, \omega)$ is *$\mathcal{J}$-model* if there are no other potentials $\varphi \leq 0$ having the same multiplier ideal sheaves (in the sense that $\mathcal{J}(k\varphi) = \mathcal{J}(k\psi)$).

Roughly speaking, $\mathcal{J}$-model potentials are *algebraic* singularities.

$\mathcal{J}$-model potentials

Each of the following properties are characterizations of $\mathcal{J}$-model singularities among model singularities:

1. $$\int_X \omega_\psi^n = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{J}(k\psi)) \, .$$

$\mathcal{J}$-model potentials are extremely natural from the non-Archimedean point of view!

$\mathcal{J}$-model potentials

Each of the following properties are characterizations of $\mathcal{J}$-model singularities among model singularities:

1. $$\int_X \omega_\psi^n = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{J}(k\psi))\,.$$
2. There are no other potentials $\varphi \leq 0$ with the same generic Lelong numbers as ψ.

$\mathcal{J}$-model potentials are extremely natural from the non-Archimedean point of view!

$\mathcal{J}$-model potentials

Each of the following properties are characterizations of $\mathcal{J}$-model singularities among model singularities:

1. $$\int_X \omega_\psi^n = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{J}(k\psi))\,.$$
2. There are no other potentials $\varphi \leq 0$ with the same generic Lelong numbers as ψ.
3. Along the quasi-equisingular approximations of ψ, the mass converges.

$\mathcal{J}$-model potentials are extremely natural from the non-Archimedean point of view!

Ricci energy

Theorem (Xia)

Let $\ell \in \mathcal{R}^1$ be a maximal geodesic ray, sup-normalized, then

$$\mathbf{E}_R(\ell) = -\frac{n}{V}\int_{-\infty}^{0}\left(\int_X \operatorname{Ric}\omega \wedge \omega_{\psi_\tau}^{n-1} - \int_X \operatorname{Ric}\omega \wedge \omega^{n-1}\right) \mathrm{d}\tau\,.$$

Hence, one should define

$$E_R(\psi) := -\frac{n}{V}\left(\int_X \operatorname{Ric}\omega \wedge \omega_{\psi}^{n-1} - \int_X \operatorname{Ric}\omega \wedge \omega^{n-1}\right)\,.$$

Entropy

This is much more difficult!
By Li's theorem, $\mathbf{Ent}(\ell) < \infty$ implies that ℓ is maximal. So we only need to consider maximal geodesic rays and $\mathcal{J}$-model test curves.
Our approach depends essentially on the Berkovich space picture.

Berkovich analytification

We consider the Berkovich analytification X^{an} of X with respect to the trivial valuation on $\mathbb{C}$.

As a set, X^{an} is the disjoint union of valuations on the function fields of irreducible closed subschemes of X, extending the trivial valuation on $\mathbb{C}$.

As a topological space, X^{an} (with the Berkovich topology) is the inverse limit of a net of polytopes.

As a locally ringed site, X^{an} carries the Berkovich G-topology and a sheaf of rings.

Berkovich affine line over $\mathbb{C}$ with trivial valuation

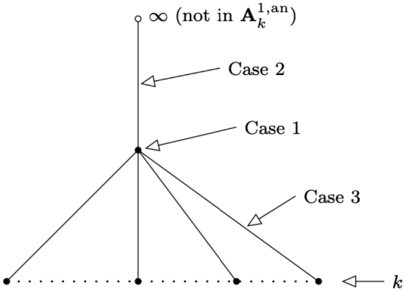


Figure 1.20: A picture of $\mathbf{A}_k^{1,\mathrm{an}}$.

This picture comes from the lecture notes of Mattias Jonsson.

How does the non-Archimedean picture show up?

A geodesic ray induces a non-Archimedean potential: when ℓ is sup-normalized,

$$\ell^{\mathrm{NA}}(v) = -G(v)(\Phi)\,,$$

where Φ is the S^1-invariant potential defined by ℓ. This establishes a bijection from the set of maximal geodesic rays to $\mathcal{E}^{1,\mathrm{NA}}$.

Theorem (Berman–Boucksom–Jonsson)

There is a bijection between maximal geodesic rays and $\mathcal{E}^1$ potentials on the Berkovich analytification of (X, L) (with respect to the trivial valuation on $\mathbb{C}$).

Hence every functional of maximal geodesic rays can be expressed in terms of the corresponding non-Archimedean potentials.

How does the non-Archimedean picture show up?

Another way to construct non-Archimedean potentials is as follows: let $\psi \in \mathrm{PSH}(X, \omega)$, then $\psi^{\mathrm{an}} : X^{\mathrm{an}} \to [-\infty, 0]$ is defined as

$$\psi^{\mathrm{an}}(v) = -\lim_{k \to \infty} \frac{1}{k} v\left(\mathcal{J}(k\psi)\right) .$$

How does the non-Archimedean picture show up?

Another way to construct non-Archimedean potentials is as follows: let $\psi \in \mathrm{PSH}(X, \omega)$, then $\psi^{\mathrm{an}} : X^{\mathrm{an}} \to [-\infty, 0]$ is defined as

$$\psi^{\mathrm{an}}(v) = -\lim_{k\to\infty} \frac{1}{k} v\left(\mathcal{J}(k\psi)\right) .$$

Theorem (Berman–Boucksom–Jonsson)

Let $\psi_\bullet$ be a test curve corresponding to a maximal geodesic ray ℓ, then

$$\ell^{\mathrm{NA}} = \sup_{\tau<\tau^+} \left(\psi_\tau^{\mathrm{an}} + \tau\right) .$$

Analysis on Berkovich spaces

Monge–Ampère operator on Berkovich spaces was first introduced by Chambert-Loir. The theory of differential forms on Berkovich spaces was studied in the celebrated *les Antoines* paper by Antoine Chambert-Loir and Antoine Ducros and in a series of papers by Gubler, Künnemann, etc.

Due to the lack of Demailly approximation in general, the theory is only well-behaved when the base field is either trivially valued or discretely valued, in which case, the theory was further studied by Boucksom, Favre and Jonsson.

Non-Archimedean Monge–Ampère energy

For $\phi \in \mathcal{E}^{1,\mathrm{NA}}$, define

$$E^{\mathrm{NA}}(\phi) := \frac{1}{V} \sum_{j=0}^{n} \int_X \phi \ \mathrm{MA}(\phi^{(j)}, \phi_{\mathrm{triv}}^{(n-j)}) \, .$$

Theorem

Let $\ell \in \mathcal{R}^1$ be a maximal geodesic ray. Then

$$\mathbf{E}(\ell) = E^{\mathrm{NA}}(\ell^{\mathrm{NA}}) \, .$$

Log discrepancy

Log discrepancy functional: $A_X : X^{\mathrm{an}} \to [0, \infty]$ (Jonsson–Mustaţă).

1. $v = c\,\mathrm{ord}_E$, E being a prime divisor over X. Then $A_X(v) := cA_X(\mathrm{ord}_E)$. Take a resolution of X, say $\pi : Y \to X$, such that E lies on Y, then $A_X(\mathrm{ord}_E) - 1$ is the coefficient of E in $K_{Y/X}$.

Log discrepancy

Log discrepancy functional: $A_X : X^{\mathrm{an}} \to [0, \infty]$ (Jonsson–Mustață).

1. $v = c\,\mathrm{ord}_E$, E being a prime divisor over X. Then $A_X(v) := cA_X(\mathrm{ord}_E)$. Take a resolution of X, say $\pi : Y \to X$, such that E lies on Y, then $A_X(\mathrm{ord}_E) - 1$ is the coefficient of E in $K_{Y/X}$.
2. $v = v$ is a quasi-monomial valuation defined on a log smooth model $(Y, D = \sum_i D_i)$ of X. Then

$$A_X(v) = \sum_i \alpha_i A_X(D_i)$$

Log discrepancy

Log discrepancy functional: $A_X : X^{\mathrm{an}} \to [0, \infty]$ (Jonsson–Mustață).

1. $v = c\,\mathrm{ord}_E$, E being a prime divisor over X. Then $A_X(v) := cA_X(\mathrm{ord}_E)$. Take a resolution of X, say $\pi : Y \to X$, such that E lies on Y, then $A_X(\mathrm{ord}_E) - 1$ is the coefficient of E in $K_{Y/X}$.
2. $v = v$ is a quasi-monomial valuation defined on a log smooth model $(Y, D = \sum_i D_i)$ of X. Then
$$A_X(v) = \sum_i \alpha_i A_X(D_i)$$
3. There is a retraction $r_{\mathcal{X}} : X^{\mathrm{an}} \to \Delta_{\mathcal{X}}$ from X^{an} to the dual complex of a model of X, X^{an} is the inverse limit, define
$$A_X := \sup_{\mathcal{X}} A_X \circ r_{\mathcal{X}} .$$

Log discrepancy

Important facts to remember:

- $A_X(v) = \infty$ if v is not a valuation (but just a semi-valuation).

Log discrepancy

Important facts to remember:

- $A_X(v) = \infty$ if v is not a valuation (but just a semi-valuation).
- $A_X(tv) = tA_X(v)$.

Log discrepancy

Important facts to remember:

- $A_X(v) = \infty$ if v is not a valuation (but just a semi-valuation).
- $A_X(tv) = tA_X(v)$.
- A_X is lsc.

Log discrepancy

Important facts to remember:

- $A_X(v) = \infty$ if v is not a valuation (but just a semi-valuation).
- $A_X(tv) = tA_X(v)$.
- A_X is lsc.
- $A_X \circ r_{\mathcal{X}}$ is continuous.

Log discrepancy

Important facts to remember:

- $A_X(v) = \infty$ if v is not a valuation (but just a semi-valuation).
- $A_X(tv) = tA_X(v)$.
- A_X is lsc.
- $A_X \circ r_{\mathcal{X}}$ is continuous.
- A_X can be viewed as a metric on the canonical sheaf on X^{an} (Temkin).

Non-Archimedean entropy

The non-Archimedean entropy $\mathrm{Ent}^{\mathrm{NA}} : \mathcal{E}^{1,\mathrm{NA}} \to [0, \infty]$ is defined as

$$\mathrm{Ent}^{\mathrm{NA}}(\phi) = \frac{1}{V} \int_{X^{\mathrm{an}}} A_X \, \mathrm{MA}(\phi) \, .$$

Non-Archimedean entropy

The non-Archimedean entropy $\mathrm{Ent}^{\mathrm{NA}} : \mathcal{E}^{1,\mathrm{NA}} \to [0,\infty]$ is defined as

$$\mathrm{Ent}^{\mathrm{NA}}(\phi) = \frac{1}{V}\int_{X^{\mathrm{an}}} A_X \, \mathrm{MA}(\phi)\,.$$

Conjecture

Let $\ell \in \mathcal{R}^1$ be a maximal geodesic ray. Then

$$\mathrm{Ent}^{\mathrm{NA}}(\ell^{\mathrm{NA}}) = \mathbf{Ent}(\ell)\,.$$

The proof of the direction $\leq$ was due to Li. The converse is highly non-trivial. It is known when ℓ is the Phong–Sturm ray.

Relation between NA energy and NA entropy

In the Archimedean case, at a smooth point $\varphi \in \mathcal{H}$,

$$\delta E|_{\varphi} = \frac{1}{V}\omega_{\varphi}^{n}.$$

Relation between NA energy and NA entropy

In the Archimedean case, at a smooth point $\varphi \in \mathcal{H}$,

$$\delta E|_\varphi = \frac{1}{V}\omega_\varphi^n .$$

At a non-smooth point,

Theorem (Berman–Boucksom)

Let $\varphi \in \mathcal{E}^1$, $f \in C^0(X)$. *Then*

$$\partial_t|_{t=0} E(P[\varphi + tf]) = \frac{1}{V}\int_X f\,\omega_\varphi^n .$$

Relation between NA energy and NA entropy

A formal computation when $\phi \in \mathcal{E}^{1,\mathrm{NA}}$, $\sup \phi = 0$,

$$
\begin{aligned}
\frac{1}{V}\int_X A_X \,\mathrm{MA}(\phi) =& \partial_t|_{t=0} E^{\mathrm{NA}}(P[\phi + tA_X]) \\
=& \frac{1}{V}\partial_t|_{t=0} \int_{-\infty}^{\tau^+} \left(\int_X \omega^n_{\psi^t_\tau} - \int_X \omega^n \right) \mathrm{d}\tau \\
=& \frac{1}{V}\int_{-\infty}^{\tau^+} \partial_t|_{t=0} \int_X \omega^n_{\psi^t_\tau} \,\mathrm{d}\tau \,.
\end{aligned}
$$

Algebraic reformulation of non-pluripolar mass

Let $\psi \in \mathrm{PSH}(X, \omega)$. Then ψ defines a Weil b-divisor (in the sense of Shokurov) on the Riemann–Zariski space associated to X:

$$(\mathrm{div}_{\mathfrak{X}}\, \psi)_Y = [\mathrm{div}_Y\, \psi]\,.$$

Here $[\bullet]$ denotes the numerical class, $\pi : Y \to X$ runs over birational models of X. Note that $\mathrm{div}_{\mathfrak{X}}\, \psi \in \lim_Y N^1_{\mathbb{R}}(Y)$.

Theorem (Xia)

Assume that ψ is $\mathcal{I}$-model, then

$$\int_X \omega_\psi^n = \mathrm{vol}(L - \mathrm{div}_{\mathfrak{X}}\, \psi)\,.$$

Formal computation, second part

$$\begin{aligned}\frac{1}{V}\int_X A_X \,\mathrm{MA}(\phi) &= \frac{1}{V}\int_{-\infty}^{\tau^+} \partial_t|_{t=0} \int_X \omega^n_{\psi^t_\tau}\,\mathrm{d}\tau \\ &= -\frac{1}{V}\int_{-\infty}^{\tau^+} \langle (L - \mathrm{div}_{\mathfrak{X}}\,\psi)^{n-1}\rangle \cdot \partial_t|_{t=0}\,\mathrm{div}_{\mathfrak{X}}\,\psi \\ &= \frac{1}{V}\int_{-\infty}^{\tau^+} \langle (L - \mathrm{div}_{\mathfrak{X}}\,\psi)^{n-1}\rangle \cdot (K_{\mathfrak{X}/X} + \mathrm{red}\,\mathrm{div}_{\mathfrak{X}}\,\psi)\,.\end{aligned}$$

Formal computation, second part

$$\frac{1}{V}\int_X A_X \operatorname{MA}(\phi) = \frac{1}{V}\int_{-\infty}^{\tau^+} \partial_t|_{t=0} \int_X \omega^n_{\psi^t_\tau}\, \mathrm{d}\tau$$

$$= -\frac{1}{V}\int_{-\infty}^{\tau^+} \langle (L - \operatorname{div}_{\mathfrak{X}} \psi)^{n-1}\rangle \cdot \partial_t|_{t=0} \operatorname{div}_{\mathfrak{X}} \psi$$

$$= \frac{1}{V}\int_{-\infty}^{\tau^+} \langle (L - \operatorname{div}_{\mathfrak{X}} \psi)^{n-1}\rangle \cdot (K_{\mathfrak{X}/X} + \operatorname{red}\operatorname{div}_{\mathfrak{X}} \psi)\,.$$

Definition

$$\operatorname{Ent}^{\mathrm{NA}}(\psi) := \frac{n}{V}\varinjlim_{Y} \left(\langle \pi^* L - \operatorname{div}_Y \psi \rangle^{n-1} \cdot (K_{Y/X} + \operatorname{red}\operatorname{div}_Y \psi) \right)\,.$$

Entropy theorem

Theorem (Xia)

Let $\ell \in \mathcal{R}^1$ be a maximal geodesic ray. Then

$$\mathrm{Ent}^{\mathrm{NA}}(\ell^{\mathrm{NA}}) \leq \int_{-\infty}^{\tau^+} \mathrm{Ent}^{\mathrm{NA}}(\psi_\tau)\,\mathrm{d}\tau\,.$$

Equality holds if ℓ is the Phong–Sturm ray of some test configuration.

What do we know?

Table 1. Comparison of functionals

Maximal geodesic rays	NA potentials	Test curves	Known facts
$\mathbf{E}$	E^{NA}	$\mathbf{E}$	All equal
$\mathbf{E}_R$	E_R^{NA}	$\mathbf{E}_R$	All equal
$\mathcal{C}_k^{\mathrm{NA}}$	?	$\mathcal{C}_k^{\mathrm{NA}}$	First=Third
$\mathbf{Ent}$	$\mathrm{Ent}^{\mathrm{NA}}$	$\mathrm{Ent}^{\mathrm{NA}}$	First$\leq$ Second First$\leq$ Third

Applications

- Trivially valued generalization of Vilsmeier's result.

Applications

- Trivially valued generalization of Vilsmeier's result.
- A new approach to the Yau–Tian–Donaldson conjecture.

Applications

- Trivially valued generalization of Vilsmeier's result.
- A new approach to the Yau–Tian–Donaldson conjecture.
- New approach to study non-Archimedean $\mathcal{L}$-functionals.

Applications

- Trivially valued generalization of Vilsmeier's result.
- A new approach to the Yau–Tian–Donaldson conjecture.
- New approach to study non-Archimedean $\mathcal{L}$-functionals.
- New stability thresholds.

References

1. T. Darvas and M. Xia, The closures of test configurations and algebraic singularity types, arXiv: 2003.04818
2. M. Xia, Pluripotential-theoretic stability thresholds, To appear.

Thank you for your attention!