Singularities in global pluripotential theory

夏 銘辰 (Xià Míng chén)

Institut de mathématiques de Jussieu – Paris Rive Gauche

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Table of Contents

Setup

- X: irreducible smooth (or normal) projective variety of dimension n.
- L: big line bundle on X (i.e. $\lim_{k\to\infty} k^{-n}h^0(X, L^{\otimes k}) > 0$).
- h_0: a smooth Hermitian metric on L with $\theta = c_1(L, h_0)$.

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Objects of interest

Positively-curved (singular) Hermitian metrics are in bijection with θ-plurisubharmonic functions:

$$\mathrm{PSH}(L) \leftrightarrow \mathrm{PSH}(X,\theta), \quad h_0 \exp(-\varphi) \mapsto \varphi.$$

Goal

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Understand the singularities of positively-curved singular Hermitian metrics on L (equivalently, of θ-plurisubharmonic functions).

Here singularities of $\varphi \in \mathrm{PSH}(X,\theta)$ mean the behaviour of φ near the set $\{\varphi = -\infty\}$.

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The singularities can be understood both locally and globally.

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Typical techniques

- In the local theory, one studies the invariants of the singularities: Lelong numbers, multiplier ideal sheaves, etc.;
- In the global theory, there are envelope operators classifying the singularities into rough classes; one could make use of the associated Berkovich(non-Archimedean) spaces.

The local theory

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Typical example

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We can compare a general psh singularity with this example:

Lelong number

Given a psh function φ and x in its domain of definition, the Lelong number is

$$\nu(\varphi, x) = \sup\{c \geq 0 : \varphi(y) \leq c \log|y-x| + \mathcal{O}(1) \text{ as } y \to x\}.$$

The local theory

A related invariant is the following:

Multiplier ideal sheaves

Given a psh function φ on X, the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the coherent ideal sheaf on X locally consisting of holomorphic functions f such that $|f|^2 \exp(-\varphi)$ is locally integrable.

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Theorem (Boucksom–Favre–Jonsson)

The multiplier ideal sheaves (of $k\varphi$ for all $k > 0$) and the Lelong numbers (of φ on all birational models of X) determine the same information.

These data do not completely determine the singularity of φ.

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There are interesting global invariants of singularities:

1. The analytic volume: $\int_X (\theta + \mathrm{dd^c}\varphi)^n$ (non-pluripolar sense);
2. The algebraic volume:

$$\mathrm{vol}(\theta,\varphi) = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^{\otimes k} \otimes \mathcal{I}(k\varphi)).$$

When φ gets more singular, both invariants get smaller.

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The global theory

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Observation

In the global setting, a function in $\mathrm{PSH}(X, \theta)$ is necessarily bounded from above, so after normalization, we may always consider negative functions. This allows us to take envelopes.

Envelopes

The two global invariants (analytic volume, algebraic volume) give rise to two envelopes:

$$P_\theta[\varphi] = \sup \left\{ \psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \int_X \theta_\varphi^n = \int_X \theta_\psi^n, \varphi \leq \psi + C \right\};$$

$$P_\theta[\varphi]_{\mathcal{I}} = \sup \left\{ \psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \mathcal{I}(k\varphi) = \mathcal{I}(k\psi) \text{ for all } k \in \mathbb{Z}_{>0} \right\}.$$

These envelopes reflect the essential information that can be detected from the two global invariants.

The two envelopes are due to Ross–Witt Nyström and Darvas–X. respectively.

In general, if $\varphi \leq 0$, we have

$$\varphi \leq P_\theta[\varphi] \leq P_\theta[\varphi]_{\mathcal{I}}.$$

Special singularities

We say φ is

- model if $\varphi = P_\theta[\varphi]$;
- $\mathcal{I}$-model if $\varphi = P_\theta[\varphi]_{\mathcal{I}}$;
- $\mathcal{I}$-good if $P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}$.

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Special singularities

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- $\mathcal{I}$-good if $P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}$.

A global singularity can be classified according to its model/$\mathcal{I}$-model type.

The general approach in global pluripotential theory

When studying the global singularities, we follow the general approach as follows:

General approach

1. Understand the model/$\mathcal{I}$-model singularities;
2. Understand the singularities with the same model/$\mathcal{I}$-model type.

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2. Understand the singularities with the same model/$\mathcal{I}$-model type.

Part 2 has been studied extensive by Darvas–Di Nezza–Lu in the last few years. In particular, the solvability of the Monge–Ampère equations with singularities is well-understood.
Here we focus on Part 1.

Table of Contents

$\mathcal{I}$-good singularities

The main result of Darvas and myself is the following highly non-trivial characterization of $\mathcal{I}$-good singularities:

Theorem (Darvas–X.)

Assume that $\int_X (\theta + \mathrm{dd^c}\varphi)^n > 0$, then the following are equivalent:

- *φ is $\mathcal{I}$-good;*
- *the analytic volume equals the algebraic volume:*

$$\int_X (\theta + \mathrm{dd^c}\varphi)^n = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^{\otimes k} \otimes \mathcal{I}(k\varphi));$$

- *φ can be approximated by analytic singularities (with respect to a pseudometric d_S).*

Analytic singularity means that locally $\varphi = c \log(|f_1|^2 + \cdots + |f_N|^2) + \mathcal{O}(1)$, where the f_i's are holomorphic functions and $c \in \mathbb{Q}_{>0}$.

The d_S-pseudometric

The pseudometric d_S introduced by Darvas–Di Nezza–Lu is a pseudometric on $\mathrm{PSH}(X,\theta)$, characterizing how far away two singularities are.
Instead of giving the precise definition, we only mention:

Example of d_S-convergence

If $\varphi_j \leq 0$ is an increasing sequence with a.e. limit φ, then $\varphi_j \xrightarrow{d_S} \varphi$.
If φ_j is a decreasing sequence with limit $\varphi \not\equiv -\infty$, then $\varphi_j \xrightarrow{d_S} \varphi$ iff $\int_X (\theta + \mathrm{dd^c}\varphi_j)^n \to \int_X (\theta + \mathrm{dd^c}\varphi)^n$.

0-distance

Two singularities have 0-distance iff they have the same P-envelope.

The d_S-pseudometric

The local and global invariants behave well under d_S-convergence.

Theorem (X.)

The Lelong numbers are continuous with respect to d_S; the non-pluripolar masses are continuous with respect to d_S.

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Theorem (X.)

The Lelong numbers are continuous with respect to d_S; the non-pluripolar masses are continuous with respect to d_S.

Together with the work of Boucksom–Favre–Jonsson, this theorem implies

Corollary (X. unpublished)

If $\varphi_j \xrightarrow{d_S} \varphi$, then the multiplier ideal sheaves of φ_j converges to those of φ.

Here converges means that a certain quasi-equisingular property holds.

Idea of the proof

Theorem

TFAE:

1. *φ is $\mathcal{I}$-good;*
2. *the analytic volume equals the algebraic volume;*
3. *φ can be approximated by analytic singularities.*

Given an $\mathcal{I}$-good singularity φ, we can take its Demailly approximation φ_j. The multiplier ideal sheaves of φ_j converge to those of φ.

One then shows that $\varphi_j \xrightarrow{d_S} \varphi$. This approach can be reversed yielding the equivalence of (1) and (3).

Idea of the proof

Theorem

TFAE:

1. φ *is* $\mathcal{I}$*-good;*
2. *the analytic volume equals the algebraic volume;*
3. φ *can be approximated by analytic singularities.*

In order to prove (2), we show that both volumes are continuous with respect to d_S, then by Demailly approximation again, we reduce to the case where φ has analytic singularities, a case essentially handled by Bonavero in his thesis.

Examples of $\mathcal{I}$-good singularities

- Analytic singularities;
- A potential with full Monge–Ampère mass;
- On toric varieties, all toric invariant metrics (unpublished result of Yi Yao);
- The singularities of the Mumford–Lear metric on the Siegel–Jacobi line bundle over (the toroidal compacitification of) the universal Abelian variety (Botero–Burgos Gil–Holmes–de Jong, the general form of their result implies Yao's result).

Examples of $\mathcal{I}$-good singularities

Non-examples

There are plenty of non-$\mathcal{I}$-good singularities, as constructed by Berman–Boucksom–Jonsson.

Scholie

The singularities that occur naturally in reality are $\mathcal{I}$-good. Non-$\mathcal{I}$-good singularities are pathological.

B-divisors

The singularities can also be understood using b-divisors:

B-divisor

A b-divisor on X is an assignment of a divisor/divisor class on each birational model of X satisfying certain natural compatibility conditions.

There is an intersection theory introduced by Dang–Favre of nef b-divisors.

Theorem (X.)

There is a natural way to construct a nef b-divisor $\mathbb{D}(\theta, \varphi)$ *associated with* $\varphi \in \mathrm{PSH}(X, \theta)$. *Moreover,*

$$(\mathbb{D}(\theta, \varphi)^n) = \lim_{k \to \infty} \frac{n!}{k^n} h^0(X, L^{\otimes k} \otimes \mathcal{I}(k\varphi)).$$

B-divisors

As a corollary,

Corollary

Assume that $\int_X (\theta + \mathrm{dd}^c\varphi)^n > 0$, then the following are equivalent:

- φ is $\mathcal{I}$-good;
- φ can be approximated by analytic singularities;
- $$\int_X (\theta + \mathrm{dd}^c\varphi)^n = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^{\otimes k} \otimes \mathcal{I}(k\varphi));$$
- $$\int_X (\theta + \mathrm{dd}^c\varphi)^n = (\mathbb{D}(\theta,\varphi)^n)\,.$$

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In the transcendental setting, the equivalence between the first three conditions still holds (with appropriate modifications).

Table of Contents

Ross–Witt Nyström correspondence

The theory of $\mathcal{I}$-good/$\mathcal{I}$-model potentials has been applied to the studied of K-stability/Ding stability by X., Darvas–Zhang, Dervan–Reboulet etc.

Key theorem

Concave curves of model potentials=geodesic rays in the space of Kähler potentials;
Concave curves of $\mathcal{I}$-model potentials=maximal geodesic rays.

The study of these correspondences was initiated by Ross–Witt Nyström and followed by Darvas–Di Nezza–Lu, Darvas–X..
The precise correspondence is given by the Legendre transform.

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Corollary (X.)

The δ-invariant (as explained in Odaka's talk) can be reformulated using the space of $\mathcal{I}$-model singularities.

The non-Archimedean point of view

An extension of the ideas in the previous slide allows one to describe Boucksom–Jonsson's non-Archimedean pluripotential theory using $\mathcal{I}$-model singularities.

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Theorem (Darvas–X.–Zhang)

There is a natural bijection between the set of non-Archimedean potentials and the set of concave curves of $\mathcal{I}$-model potentials.

We are in fact very sloppy when stating this theorem. In reality, we need to take certain projective limit!

The non-Archimedean point of view

Theorem (Darvas–X.–Zhang)

There is a natural bijection between the set of non-Archimedean potentials and the set of concave curves of $\mathcal{I}$-model potentials.

Corollary

Boucksom–Jonsson's envelope conjecture holds if X is smooth.

This result is also proved using algebraic methods by Boucksom–Jonsson.

Other applications

We give a brief summery of known applications in the literature.

- Darvas–X.: partial Bergman kernels converge to partial equilibrium measures;
- X.: There is a theory of partial Okounkov bodies;
- Botero–Burgos Gil–Holmes–de Jong: the ring of Siegel–Jacobi modular forms is not finitely generated.

ありがとうございます！