# A transcendental approach to non-Archimedean metrics

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# Setup

- $X$: irreducible normal projective variety of dimension $n$.
- $L$: big line bundle on $X$.

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- the complex analytification $X(\mathbb{C})$ (or $X$ for simplicity);
- the Berkovich analytification $X^{\mathrm{an}}$.

The line bundle $L$ induces analytic line bundles $L(\mathbb{C})$ (or $L$ for simplicity) and $L^{\mathrm{an}}$ on both $X(\mathbb{C})$ and $X^{\mathrm{an}}$.

# Goal

There are notions of plurisubharmonic metrics on both $L(\mathbb{C})$ and $L^{\mathrm{an}}$, giving rise to the Archimedean and non-Archimedean pluripotential theory respectively.

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## Goal

Understand the precise relation between these two theories.

# Intuitions of $X^{\mathrm{an}}$

We need some intuitions of $X^{\mathrm{an}}$ before proceeding.
As a set, $X^{\mathrm{an}}$ is a disjoint union:

$$X^{\mathrm{an}} = \coprod_Y Y^{\mathrm{val}},$$

where

- $Y$ runs over all irreducible reduced subvarieties of $X$;
- $Y^{\mathrm{val}}$ is the set of valuations of real valuations $\mathbb{C}(Y)^\times \to \mathbb{R}$ which are trivial on $\mathbb{C}$.

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**Information**

$X^{\mathrm{an}}$ is a set of valuations.

# Intuitions of $X^{\mathrm{an}}$

As a simple example:

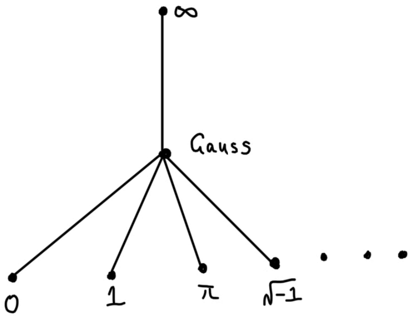


Figure: $\mathbb{P}^{1,\mathrm{an}}$

Here Gauss denotes the trivial valuation.

# Pluripotential theory

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1. The approach of Chambert-Loir–Ducros, Gubler, Jell, ..... This approach is based on local tropicalizations;
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## Drawbacks

The first approach fails for singular metrics. The second method does not yield a sheaf of psh functions.

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The first approach fails for singular metrics. The second method does not yield a sheaf of psh functions.

The two approaches agree for nice enough global psh functions.

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## Theorem

*Boucksom–Jonsson's non-Archimedean psh metrics can be realized as curves of singular Archimedean psh metrics.*

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# How to relate Archimedean to non-Archimedean

Assume that $L$ is ample for the time being.
On the Archimedean side, we have a notion of finite energy metrics:

## Finite energy

Fix a smooth strictly psh metric $h_0$ on $L = L(\mathbb{C})$, a psh metric $h$ has finite energy if

$$\int_X |h/h_0| (\mathrm{dd}^{\mathrm{c}} h)^n < \infty.$$

The space of such metrics is denoted by $\mathcal{E}^1(X, L)$.

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Intuitively, finite energy=almost regular.
The space $\mathcal{E}^1(X, L)$ has an intrinsic geometry. We have a notion of geodesics.
The space of geodesic rays $(\ell_t)_{t\geq 0}$ with $\ell_0 = 0$ is denoted by $\mathcal{R}^1(X, L)$.

# How to relate Archimedean to non-Archimedean

**Goal**

Get non-Archimedean information from $\ell \in \mathcal{R}^1(X, L)$.

After reparametrization $t = -\log|\tau|$, $\tau \in \Delta = \{z \in \mathbb{C} : |z| < 1\}$, we get a plurisubharmonic metric $\Phi$ on $p_1^* L$ on $X \times (\Delta - \{0\})$.
Assume that $\ell_t \leq 0$ for all $t \geq 0$, then $\Phi \leq 0$. By the Grauert–Remmert extension theorem, $\Phi$ admits a unique psh extension over the central fiber.

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## Goal, reformulated

Get non-Archimedean information from $\Phi \in \mathrm{PSH}(X \times \Delta, p_1^* L)$.

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**Goal**

Get non-Archimedean information from $\Phi \in \mathrm{PSH}(X \times \Delta, p_1^* L)$.

Given any divisorial valuation $c\,\mathrm{ord}_E \in X^{\mathrm{an}}$, there is a natural extension of $c\,\mathrm{ord}_E$ to a divisorial valuation $\sigma(c\,\mathrm{ord}_E)$ on $X \times \mathbb{C}$, known as the Gauss extension characterized by

- $\sigma(c\,\mathrm{ord}_E)(t) = 1$;
- $\sigma(c\,\mathrm{ord}_E)$ is $\mathbb{C}^*$-invariant;
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- The center of $\sigma(c \operatorname{ord}_E)$ is on the central fiber.

The non-Archimedean information associated with $\Phi$ is given by

$$\Phi^{\mathrm{an}}(c \operatorname{ord}_E) := -\sigma(c \operatorname{ord}_E)(\Phi).$$

# How to relate Archimedean to non-Archimedean

**Goal**

Get non-Archimedean information from $\Phi \in \mathrm{PSH}(X \times \Delta, p_1^* L)$.

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To summarize, we get

$$\ell \mapsto \Phi \mapsto \Phi^{\mathrm{an}}; \quad \mathcal{R}^1(X, L) \to \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}}).$$

This gives a map from Archimedean objects to non-Archimedean objects.

# How to relate Archimedean to non-Archimedean

**Theorem (Berman–Boucksom–Jonsson)**

*The map induces a bijection from a subclass of $\mathcal{R}^1(X,L)$ to finite energy metrics in $\mathrm{PSH}(X^{\mathrm{an}},L^{\mathrm{an}})$.*

These special geodesics are called maximal geodesic rays. Next we consider the Legendre transform of these maximal rays:

$$\psi_\tau := \inf_{t\geq 0}(\ell_t - t\tau).$$

Of course, here we regard $\ell_t$ as the associated potential after fixing a smooth reference metric.

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## Theorem (Ross–Witt Nyström, Darvas–Di Nezza–Lu, Darvas–Xia)

*The Legendre transform is a bijection from maximal geodesic rays to concave curves with finite energy of $\mathcal{I}$-model potentials .*

# How to relate Archimedean to non-Archimedean

Recall that $\varphi \in \mathrm{PSH}(X,\omega)$ is $\mathcal{I}$-model if $\varphi \leq 0$ and

$$\psi \in \mathrm{PSH}(X,\omega)_{\leq 0} + \mathcal{I}(t\varphi) = \mathcal{I}(t\psi)\ \forall t \implies \psi \leq \varphi.$$

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## Corollary

When $L$ is ample, there is a bijection between

- the non-Archimedean potentials in $\mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ with finite energy and
- concave curves with finite energy of $\mathcal{I}$-model potentials.

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- the non-Archimedean potentials in $\mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ with finite energy and
- concave curves with finite energy of $\mathcal{I}$-model potentials.

We reduce the non-Archimedean problems to Archimedean problems.

# Archimedean approach to non-Archimedean metrics

We have shown how to relate the Boucksom–Jonsson theory to the Archimedean theory. Conversely, we could give a new definition of Boucksom–Jonsson theory using the Archimedean theory.

# Archimedean approach to non-Archimedean metrics

We have shown how to relate the Boucksom–Jonsson theory to the Archimedean theory. Conversely, we could give a new definition of Boucksom–Jonsson theory using the Archimedean theory.
Why bother?

1. The Archimedean theory works for transcendental classes as well and makes sense even if $X$ is not projective;
2. The envelope conjecture (i.e. Hartogs' lemma) is trivial in the Archimedean theory.

# Archimedean approach to non-Archimedean metrics

How?

## Corollary

When $L$ is ample, there is a bijection between

- the non-Archimedean potentials in $\mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ with finite energy and
- concave curves with finite energy of $\mathcal{I}$-model potentials.

The naive idea is to remove the word finite energy.

# Archimedean approach to non-Archimedean metrics

Let $\xi$ be a pseudo-effective class. Then we define

$$\mathrm{PSH}^{\mathrm{an}}(\xi) := \varprojlim_{\omega} \text{Test curves}(X, \xi + \omega),$$

where $\omega$ runs over all Kähler forms on $X$ and Test curves$(X, \xi + \omega)$ is the set of concave curves of $\mathcal{I}$-model potentials not necessarily of finite energy (satisfying some mild assumptions).

Here the projective limit avoids the pathologies at $0$-mass.

**Theorem**

$\mathrm{PSH}^{\mathrm{an}}(\xi)$ *behaves exactly as the Archimedean potential theory. In particular, it satisfies Hartogs' lemma (=Envelope conjecture à la Boucksom–Jonsson).*

# Dictionary

Pluripotential theoretic constructions can be realized using the corresponding constructions in convex geometry. Here is a brief dictionary:

1. Maximum $\Leftrightarrow$ Concave envelope of pointwise maximum;
2. Sum $\Leftrightarrow$ infimal involution;
3. Inf along decreasing nets $\Leftrightarrow$ pointwise inf;
4. Regularized sup along increasing nets $\Leftrightarrow$ pointwise regularized sup;
5. Regularized sup $\Leftrightarrow$ Slightly more complicated to describe.
6. ...

# The envelope conjecture

**Theorem (Darvas–X.–Zhang)**

*When $X$ is smooth and $\xi = c_1(L)$ for a pseudo-effective $\mathbb{R}$-line bundle $L$, $\mathrm{PSH}^{\mathrm{an}}(\xi)$ is canonically isomorphic to $\mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ in the sense of Boucksom–Jonsson.*

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**Corollary**

When $X$ is smooth, Boucksom–Jonsson's envelope conjecture holds.

This result is proved independently by Boucksom–Jonsson using algebraic methods.

# The envelope conjecture

## Theorem (Unpublished)

*When* $X$ *is unibranch and* $\xi = c_1(L)$ *for a pseudo-effective* $\mathbb{R}$*-line bundle* $L$*, then the following are equivalent:*

1. *Boucksom–Jonsson's envelope conjecture holds;*
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2. $\mathrm{PSH}^{\mathrm{an}}(\xi)$ *is canonically isomorphic to* $\mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$.

In any case, the envelope conjecture always holds in the Archimedean theory, so it seems to be a better alternative than Boucksom–Jonsson's theory if the envelope conjecture remains open.

# Further problems

Although we have a well-developed basic theory of $\mathrm{PSH}^{\mathrm{an}}(\xi)$, a lot of important information is missing.

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## Conjecture

The energy functional $E$ is differentiable on the subspace of finite energy potentials in $\mathrm{PSH}^{\mathrm{an}}(\xi)$.

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**Conjecture/Problem**

The non-Archimedean Monge–Ampère equation can be formulated and solved.

**Conjecture/Ongoing work of Boucksom and Piccione**

There is a Berkovich like compactification of the space of divisorial valuations of a general unibranch Kähler space and $\mathrm{PSH}^{\mathrm{an}}(\xi)$ can be interpreted using this space.

Thank you for your attention!