Transcendental b-divisors

Mingchen Xia

Universitetet för arbetslösa

May 28, 2025
KASS, Chalmers Tekniska Högskola

References

1. Transcendental b-divisors (2025);
2. Singularities in global pluripotential theory — Lecture notes at Zhejiang university (2024).

Both are available on my homepage (but not on arXiv).

Table of Contents

Goal

Notation: X is a compact Kähler manifold.

Goal

Understand the relations between currents and cohomology classes on X.

In the whole talk, a current is understood as a closed positive $(1,1)$-current.

Examples

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

What do we mean?

Examples

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

What do we mean?

Trivial example 1

A Kähler class contains a Kähler form.

Examples

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

What do we mean?

Trivial example 1

A Kähler class contains a Kähler form.

Trivial example 2

The cohomology class of a current is pseudoeffective.

Examples

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

What do we mean?

Trivial example 1

A Kähler class contains a Kähler form.

Trivial example 2

The cohomology class of a current is pseudoeffective.

Trivial example 3

A current without Lelong numbers lives in a nef cohomology class.

Examples

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

A special case of Example 3

Suppose that T is a current with bounded local potentials, then the cohomology class of T is nef.

Examples

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

A special case of Example 3

Suppose that T is a current with bounded local potentials, then the cohomology class of T is nef.

Corollary

Suppose that T has log singularities along a divisor, then the non-divisorial part $\operatorname{Reg} T$ represents a nef cohomology class.

Examples

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

What if T has more general analytic singularities?

Observation

$\{\operatorname{Reg} T\}$ is NOT nef in general.

However,

Example

$\exists$ bimeromorphic modification $\pi\colon Y \to X$ so that $\pi^* T$ has log singularities along a divisor.
In particular, $\operatorname{Reg} \pi^* T$ is a nef cohomology class on Y.

We regard $\operatorname{Reg} \pi^* T$ as a resolution of the singularities of T.

Main theorem

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

For general T, no modification can resolve the whole singularities.
But, each modification π resolves parts of the singularities.

Main theorem

Goal

currents $\overset{?}{\Longleftrightarrow}$ cohomology classes.

For general T, no modification can resolve the whole singularities.
But, each modification π resolves parts of the singularities.

Expectation

The whole collection of the $\operatorname{Reg}\pi^*T$'s for various π forms a resolution of the singularities of T.
The collection of cohomology classes $\{\operatorname{Reg}\pi^*T\}$ should be nef in some sense.

The latter collection is a so-called nef b-divisor.

Main theorem

Theorem (Not very precise)

Given a cohomology class α. There is a canonical bijection between

- *Non-divisorial currents representing α modulo an equivalence relation, and*
- *Nef b-divisors rooted on α.*

Our notion of b-divisor is more commonly called a b-divisor class.

Main theorem

Theorem (Not very precise)

Given a cohomology class α. There is a canonical bijection between

- *Non-divisorial currents representing α modulo an equivalence relation, and*
- *Nef b-divisors rooted on α.*

Our notion of b-divisor is more commonly called a b-divisor class.

Remark

In the algebraic setting, this is a consequence of theorems of Boucksom–Jonsson (2018) and of Darvas–Xia–Zhang (2023). The novelty lies in the transcendental case.

Table of Contents

What is a b-divisor?

Definition

A b-divisor over X is a family $\mathbb{D}_Y$:

- Indexed by all modifications $\pi\colon Y \to X$ of X;
- $\mathbb{D}_Y \in \mathrm{H}^{1,1}(Y,\mathbb{R})$;
- If we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad p \quad} & Y' \\ & \searrow \quad \swarrow & \\ & X, & \end{array}$$

  then $p_*\mathbb{D}_Y = \mathbb{D}_{Y'}$.

The class $\mathbb{D}_X$ is called the root of $\mathbb{D}$.

What is a b-divisor?

Cartier b-divisors

Given

- a modification $\pi\colon Y \to X$, and
- a cohomology class $\alpha \in \mathrm{H}^{1,1}(Y, \mathbb{R})$,

we can construct a unique b-divisor $\mathbb{D}(\alpha)$ so that

- for each modification $Z \to X$ dominating Y, $\mathbb{D}_Z$ is the pull-back of α.

These b-divisors are called Cartier b-divisors.

What is a b-divisor?

Cartier b-divisors

Given

- a modification $\pi\colon Y \to X$, and
- a cohomology class $\alpha \in \mathrm{H}^{1,1}(Y, \mathbb{R})$,

we can construct a unique b-divisor $\mathbb{D}(\alpha)$ so that

- for each modification $Z \to X$ dominating Y, $\mathbb{D}_Z$ is the pull-back of α.

These b-divisors are called Cartier b-divisors.

Nef b-divisors

A Cartier b-divisor is nef if α can be taken as nef.
A b-divisor is nef if it can be approximated by nef Cartier b-divisors.

From current to b-divisors

Given a current T on X, we can construct a b-divisor $\mathbb{D}(T)$:

- Given a modification $\pi\colon Y \to X$, we set

$$\mathbb{D}(T)_Y := \{\operatorname{Reg} \pi^* T\}.$$

From current to b-divisors

Given a current T on X, we can construct a b-divisor $\mathbb{D}(T)$:

- Given a modification $\pi\colon Y \to X$, we set

$$\mathbb{D}(T)_Y := \{\operatorname{Reg} \pi^* T\}.$$

Special case

When T has analytic singularities and π is a log resolution, then $\mathbb{D}(T)$ is just the nef Cartier b-divisor generated by the class $\{\operatorname{Reg} \pi^* T\}$.

From current to b-divisors

Given a current T on X, we can construct a b-divisor $\mathbb{D}(T)$:

- Given a modification $\pi\colon Y \to X$, we set

$$\mathbb{D}(T)_Y := \{\operatorname{Reg} \pi^* T\}.$$

Special case

When T has analytic singularities and π is a log resolution, then $\mathbb{D}(T)$ is just the nef Cartier b-divisor generated by the class $\{\operatorname{Reg} \pi^* T\}$.

Theorem (Xia 2021)

The b-divisor $\mathbb{D}(T)$ is nef.

So we have constructed a map: currents $\to$ nef b-divisors.

From current to b-divisors

Given a current T on X, we can construct a b-divisor $\mathbb{D}(T)$:

- Given a modification $\pi\colon Y \to X$, we set

$$\mathbb{D}(T)_Y := \{\operatorname{Reg} \pi^* T\}.$$

Special case

When T has analytic singularities and π is a log resolution, then $\mathbb{D}(T)$ is just the nef Cartier b-divisor generated by the class $\{\operatorname{Reg} \pi^* T\}$.

Theorem (Xia 2021)

The b-divisor $\mathbb{D}(T)$ is nef.

So we have constructed a map: currents $\to$ nef b-divisors.

Private complaint

This theorem was reproved several times without citing my paper.

From current to b-divisors

The map

We have a map currents $\xrightarrow{\mathbb{D}}$ nef b-divisors.

The map is not injective for two reasons:

Reason 1

$\mathbb{D}(T) = \mathbb{D}(\operatorname{Reg} T)$.

So we restrict our attention to non-divisorial currents: $T = \operatorname{Reg} T$.

From current to b-divisors

The map

We have a map currents $\xrightarrow{\mathbb{D}}$ nef b-divisors.

The map is not injective for two reasons:

Reason 1

$\mathbb{D}(T) = \mathbb{D}(\operatorname{Reg} T)$.

So we restrict our attention to non-divisorial currents: $T = \operatorname{Reg} T$.

Reason 2

If two cohomologous non-divisorial currents T and T' have the same Lelong numbers (on all modifications), then $\mathbb{D}(T) = \mathbb{D}(T')$.

We say these currents are $\mathcal{J}$-equivalent.

From current to b-divisors

Modulo these issues, we get an injective map:

Theorem (Injectivity)

Given a cohomology class α, we have an injective map from

- *The $\mathcal{I}$-equivalence classes of non-divisorial currents in α, to*
- *the nef b-divisors rooted on α.*

This is not entirely obvious, though not very difficult.
Conversely,

Theorem (Surjectivity)

All nef b-divisors rooted on α with positive volume come from this construction.

Something to remember after forgetting most of the talk

We have achieved our goal!

Goal

currents $\iff$ cohomology classes.

Theorem

Non-divisorial currents$/\sim_{\mathcal{I}} \iff$ *nef b-divisors*.

Proof of the surjectivity

We begin with a nef b-divisor $\mathbb{D}$ with positive volume.
The proof is divided into steps.

1. Observation due to Boucksom's thesis. Although $\mathbb{D}_Y$ is not nef, it is nef in codimension 1. Very easy after giving the correct definition.
2. A current with minimal singularities T'_Y in $\mathbb{D}_Y$ is non-divisorial. Its pushforward T_Y to X is also non-divisorial. Very easy.
3. The singularities of T_Y are decreasing when we replace Y by further blowups. The proof relies on the theory of $\mathcal{I}$-good singularities of Darvas–Xia.
4. The decreasing T_Y's have a limit T. Completeness theorem of Darvas–Di Nezza–Lu.
5. Show that $\mathbb{D}(T) = \mathbb{D}$. This relies on a continuity theorem of mine.

Table of Contents

Applications

Dang–Favre (2020) developed the intersection theory of algebraic nef b-divisors, and asked if the same can be done in the transcendental setting. Via the correspondence currents $\iff$ nef b-divisors, we get

Theorem

The theory of mixed volumes of Darvas–Xia of currents gives the desired intersection theory of nef b-divisors.

Applications

The analytic techniques can be translated to give new insights for nef b-divisors.

For example, the theory of trace operators of Darvas–Xia can be translated into

Theorem

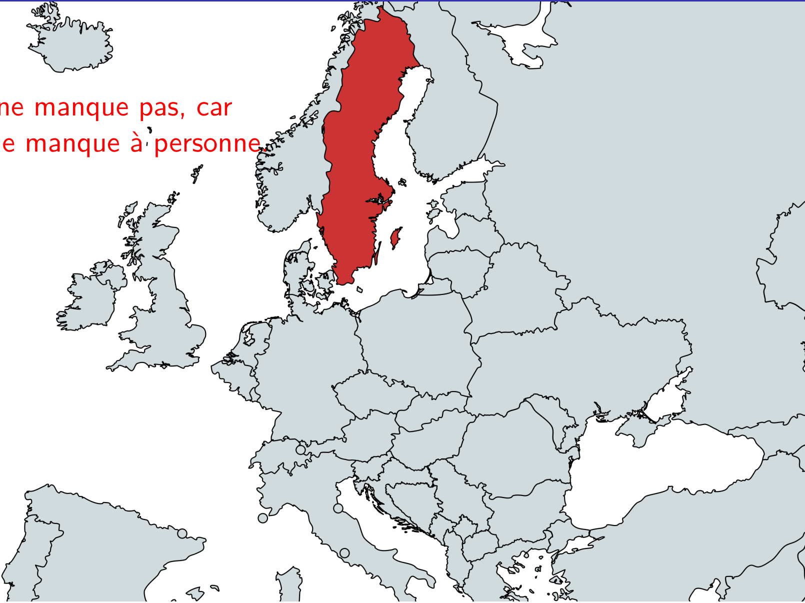
There is a way to restrict nef b-divisors over X to a submanifold of X.

This operation seems to be new.

Tack och adjö, Sverige!

My favorite pan

La France ne manque pas, car
la France ne manque à personne.



My favorite pan

La France ne manque pas, car
la France ne manque à personne.

Frankrike saknas inte, för
ingen saknar Frankrike.