Transcendental Okounkov bodies and the trace operator of currents

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References

We will be talking about results in the following two papers:

1. Transcendental Okounkov bodies (with Darvas, Witt Nyström, Reboulet, Zhang);
2. Restrictions of currents and transcendental partial Okounkov bodies (with Darvas).

Goal

We will consider

- a connected compact Kähler manifold X of dimension n;
- a big $(1,1)$-cohomology class α on X.

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Our goal is to understand the geometry of (X,α) using convex bodies in $\mathbb{R}^n$.

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A toy model

We begin with a toy model:

Toric varieties

A toric variety is a compactification X of $(\mathbb{C}^*)^n$ preserving the symmetry: $(\mathbb{C}^*)^n$-acts on X extending the action on $(\mathbb{C}^*)^n$.

We will also consider a $(\mathbb{C}^*)^n$-invariant line bundle L on X.

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We will also consider a $(\mathbb{C}^*)^n$-invariant line bundle L on X.
Thanks to the symmetry, we can usually reduce interesting problems related to (X, L) from complex dimension n to real dimension n.

Toric varieties

Fact

The geometric properties of L can be read from a polytope (Newton polytope) $\Delta(L) \subset \mathbb{R}^n$.

For example,

$$\operatorname{vol} L = n! \operatorname{vol} \Delta(L).$$

The very ampleness of L amounts to a particular shape of $\Delta(L)$, etc.

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The very ampleness of L amounts to a particular shape of $\Delta(L)$, etc.
In other words,

Complex to convex

Newton polytope construction translates complex geometry to convex geometry, which is *a priori* simpler.

Non-toric case

Now the question is:

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What about general smooth projective varieties?

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The answer is given by the Okounkov bodies, constructed by Lazarsfeld–Mustaţă and Kaveh–Khovanskii.

Theorem (Lazarsfeld–Mustaţă, Jow)

The numerical information of a big line bundle L on a smooth projective variety X of dimension n is reflected by a family of convex bodies in $\mathbb{R}^n$.

Okounkov bodies

Fix

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In order to construct a convex body in $\mathbb{R}^n$, we need an auxiliary object: a complete flag: $X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ of smooth subvarieties.
A flag relates complex geometry to convex geometry:

Flag valuations

A holomorphic section s of L has a valuation $\nu_{Y_\bullet}(s) \in \mathbb{Z}^n$: the successive orders of vanishing along the flag.

Okounkov bodies

Lemma-Definition

The closure $\Delta_\nu(L)$ or $\Delta_{Y_\bullet}(L)$ of the set

$$\{k^{-1}\nu(s) : k > 0, s \in H^0(X, L^k)\}$$

is a convex subset of $\mathbb{R}^n$, called the Okounkov body of L.

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Example

When $X = \mathbb{P}^1$, $L = \mathcal{O}(1)$ and $Y_0 = X$, $Y_1 = \{0\}$, this construction gives $[0, 1] \subseteq \mathbb{R}$.

More generally, the Okounkov body (with respect to a suitable flag) in the toric setting coincides with the Newton polytope.

Okounkov bodies

Why are Okounkov bodies important?

Theorem (Lazarsfeld–Mustață, Jow)

The Okounkov body $\Delta_{Y_\bullet}(L)$ depends only on the numerical class of L;
The family $\{\Delta_{Y_\bullet}(L)\}$ determines the numerical class of L.

In principle, we have reduced problems in complex geometry to convex geometry.

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Example

The volume of L: $\operatorname{vol} L = n! \operatorname{vol} \Delta_{Y_\bullet}(L)$ (Lazarsfeld–Mustață).
The asymptotic base locus of L can be read from $\{\Delta_{Y_\bullet}(L)\}$ (Choi–Hyun–Park–Won).

Question

What about transcendental $(1,1)$-classes?

Okounkov bodies

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What about transcendental $(1,1)$-classes?

The construction of $\Delta_{Y_\bullet}(L)$ depends on the holomorphic sections of L and hence is algebraic in nature. It fails in the transcendental world.

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Transcendental setting

The starting point is the following observation:

Observation

The valuation $\nu(s) \in \mathbb{Z}^n$ of a holomorphic section s of L can be recovered using the closed positive current $T = [s = 0]$ (with suitable multiplicity).

For example, $\nu_1(s)$ is the generic Lelong number of T along Y_1....

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For example, $\nu_1(s)$ is the generic Lelong number of T along Y_1....

$$\Delta_{Y_\bullet}(L) = \mathrm{Cl}\,\{\nu(T) : T = k^{-1}[s = 0] \text{ for some } s \in H^0(X, L^k)\}.$$

Transcendental setting

More generally, given a closed positive current $T \in c_1(L)$ with analytic singularities, we can define $\nu(T) \in \mathbb{R}^n$ as well.

Theorem (Deng)

$\Delta_{Y_\bullet}(L)$ is the closure of

- *all $\nu(T)$, where T has the form $k^{-1}[s = 0]$ for some $s \in H^0(X, L^k)$;*
- *all $\nu(T)$, where T is a closed positive current with analytic singularities in $c_1(L)$.*

Transcendental Okounkov bodies

Deng's construction works in the transcendental setting as well. Fix

- a connected compact Kähler manifold X;
- a big $(1,1)$-cohomology class α.

A smooth flag $Y_\bullet$ on X.

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Transcendental Okounkov bodies

The Okounkov body $\Delta_{Y_\bullet}(\alpha)$ is the closure of all $\nu(T)$, where T is a closed positive current with analytic singularities in α.

This construction was known for a long time. It was first written in the thesis of Deng, as suggested by Demailly.

Transcendental Okounkov bodies

Very few are known about transcendental Okounkov bodies. Our theorem is probably the first non-trivial result in this direction.

Theorem ([1])

The Okounkov body $\Delta_{Y_\bullet}(\alpha)$ has the expected volume:

$$\operatorname{vol} \Delta_{Y_\bullet}(\alpha) = \frac{1}{n!} \operatorname{vol} \alpha.$$

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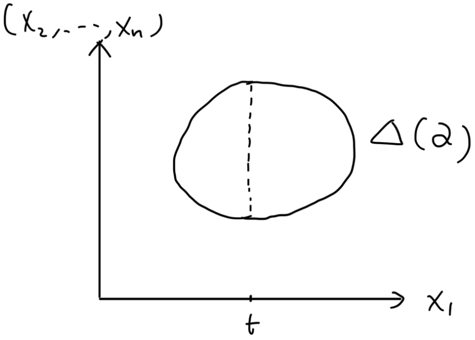
This answers the conjecture of Demailly, Deng and answers the question of Lazarsfeld–Mustață.

The proof

The proof is based on an induction on n. From the trivial base case of $n = 0$. Assume that the case $n - 1$ has been solved.



The idea is the compute the volume via

$$\mathrm{vol}\,\Delta(\alpha) = \int_t \mathrm{vol}(\Delta(\alpha) \cap \{x_1 = t\})\,\mathrm{d}t.$$

The proof

We have

$$\operatorname{vol}\Delta(\alpha) = \int_t \operatorname{vol}(\Delta(\alpha)\cap\{x_1 = t\})\,\mathrm{d}t.$$

Here we need a theorem of Witt Nyström (and Vu):

$$\operatorname{vol}\alpha = n\int_t \operatorname{vol}_{X|Y_1}(\alpha - t\{Y_1\})\,\mathrm{d}t.$$

This is a special case of the conjectured transcendental Morse inequality.

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This is a special case of the conjectured transcendental Morse inequality. Comparing them, we want

$$\operatorname{vol}_{X|Y_1}(\alpha - t\{Y_1\}) = (n-1)!\operatorname{vol}(\Delta(\alpha)\cap\{x_1 = t\}).$$

The proof

Difficulty

It is not possible to apply the inductive hypothesis directly, as $\Delta(\alpha) \cap \{x_1 = t\}$ is not an Okounkov body in general.

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It is not possible to apply the inductive hypothesis directly, as $\Delta(\alpha) \cap \{x_1 = t\}$ is not an Okounkov body in general.

But, we claim that it is a partial Okounkov body $\Delta(T_{\min,\alpha - t\{Y_1\}}|_{Y_1})$: an Okounkov body with respect to a singular psh metric.

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Theorem (X. 2021)

The volume of partial Okounkov bodies can be computed using the knowledge of Okounkov bodies.

One easily verifies that this gives the expeced identity.

The proof

The claim is technically the core of our proof. Its proof relies on two techniques:

- The extension theorem of Kähler currents with analytic singularities: it relates the slices of n-dimensional Okounkov bodies with $(n-1)$-dimensional partial Okounkov bodies;
- Bimeromorphic invariance of the Okounkov body: it allows us to avoid the complicated issues caused by the possibility that the flag lies in the null locus.

The proof is straightforward once these problems are solved.

The proof

In our paper, we developed both techniques and completed the proof.

- Extension problem: we proved an extension theorem of Kähler currents, extending the result of Collins–Tosatti;
- The bimeromorphic invariance: This leads to the trace operator construction.

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Bimeromorphic invariance of the Okounkov body

Problem

How to prove the bimeromorphic invariance of Okounkov bodies?

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How to prove the bimeromorphic invariance of Okounkov bodies?

Why is this difficult?
Recall that $\Delta_{Y_\bullet}(\alpha)$ is the closure of all $\nu(T)$, where T is a closed positive current with analytic singularities in α. But

Issue

The notion of analytic singularities is not bimeromorphically invariant!

Bimeromorphic invariance of the Okounkov body

There are two different solutions:

Observation by Boucksom

The notion of analytic singularities is almost bimeromorphically invariant.

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A more elegant approach

Prove that $\Delta_{Y_\bullet}(\alpha)$ is the closure of all $\nu(T)$, where T is a closed positive current in α with arbitrary singularities.

This is my original approach, which will be presented in [2].

The valuation of a current

Recall how $\nu(T)$ is defined when T has analytic singularities:

1. $\nu_1(T)$ is the generic Lelong number of T along Y_1. This works for general singularities as well;
2. up to replacing T by $T - \nu_1(T)[Y_1]$, we may assume that $\nu(T, Y_1) = 0$. In this case, the restriction $T|_{Y_1}$ is well-defined;
3. Induction.

The second step fails for general singularities! Consider the case of log-log singularities!.

The trace operator

Theorem ([2])

There is a well-defined trace operator: sending a closed positive $(1,1)$-current T on X with vanishing generic Lelong number on Y_1 to a closed positive current $\mathrm{Tr}\, T$ *on* Y_1.

The current $\mathrm{Tr}\, T$ is well-defined only up to d_S-equivalence. When T has analytic singularities, $\mathrm{Tr}\, T$ is represented by the naive restriction $T|_{Y_1}$.

Trace operator and Okounkov bodies

Using the trace operator instead of the naive restriction, we can therefore define the valuation $\nu(T)$ in general.

Theorem ([2])

$\Delta_{Y_\bullet}(\alpha)$ *is the closure of all* $\nu(T)$*, where* T *is a closed positive current in* α *with arbitrary singularities.*

This result is highly non-trivial as $T \mapsto \nu(T)$ is not continuous with respect to the natural topology.

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Corollary

The transcendental Okounkov bodies are bimeromorphically invariant.

An extra surprise

Finally, the trace operator has the following important property:

Theorem ([2])

We have

$$\mathcal{I}(\mathrm{Tr}\, T) \subseteq \mathcal{I}(T)|_{Y_1}.$$

When $T|_{Y_1}$ is defined, this theorem reduces to the classical Ohsawa–Takegoshi extension theorem.

Merci!