The volume of pseudo-effective line bundles and partial Okounkov bodies

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Singular Hermitian metrics on line bundles

- X: complex projective manifold of dimension n.
- L: a holomorphic line bundle on X.

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- L: a holomorphic line bundle on X.

Singular Hermitian metric on L

Locally on $U \subseteq X$, we can identify L with the trivial bundle $U \times \mathbb{C}$, a singular Hermitian is given by $e^{-\varphi}$, where φ is a locally integrable function on U: given any local section s of L over U, the square norm of s at $x \in U$ is given by

$$|s|^2_\phi(x) = |s(x)|^2 e^{-\varphi(x)} .$$

This definition can be globalized in the obvious manner, which gives the notion of a (singular) Hermitian metric ϕ on L.

Curvature form

Let $\mathrm{dd^c}\phi$ denote the curvature form/current of the metric ϕ: locally represent the metric ϕ by weights $e^{-\varphi}$, then

$$\mathrm{dd^c}\phi := \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\varphi$$

as currents.

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as currents.

Proposition

$\mathrm{dd^c}\phi$ is in the first Chern class $c_1(L)$ of L.

This explains our normalization of $\mathrm{dd^c}$.

Big line bundles

Definition

We say L is *big* if there exists $c > 0$ so that

$$h^0(X, L^k) > ck^n$$

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Example

All ample line bundles are big. In fact, by Riemann–Roch theorem

$$h^0(X, L^k) = \frac{(L^n)}{n!} k^n + \mathcal{O}(k^{n-1}) .$$

Pseudo-effective line bundles

Fix a Kähler form ω on X.

Theorem (Demailly)

The line bundle L is big if and only if L admits a singular upper semi-continuous Hermitian metric ϕ so that $\mathrm{dd}^{\mathrm{c}}\phi$ is a Kähler current, namely,

$$\mathrm{dd}^{\mathrm{c}}\phi \geq \epsilon\omega$$

for some $\epsilon > 0$.

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Definition

L is said to be pseudo-effective (psef) if there is an upper semi-continuous singular Hermitian metric ϕ on L with $\mathrm{dd^c}\phi \geq 0$ (a positive/psh metric). A pair (L, ϕ) consisting of a psef line bundle L and a positive metric ϕ on L is called a Hermitian pseudo-effective line bundle.

Goal of this talk

There are well-developed theories of psef line bundles: we have (among others)

- An intersection theory (movable intersection).
- A theory of volume.
- A theory of Okounkov bodies.

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- An intersection theory (movable intersection).
- A theory of volume.
- A theory of Okounkov bodies.

Each theory has a generalization in the Hermitian psef line bundle setting:

- The theory of non-pluripolar products (Boucksom–Eyssidieux–Guedj–Zeriahi, ...).
- A theory of volume (Darvas–X.).
- The theory of partial Okounkov bodies (X.).

We will explain the last two theories in this talk.

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The theory of volumes of pseudo-effective line bundles

Let L be a psef line bundle on X.

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The volume of L is defined as

$$\operatorname{vol}(L) := \lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k)\,.$$

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The existence of the limit is a non-trivial fact.

The volume of Hermitian pseudo-effective line bundles

Example

When L is ample/nef, $\operatorname{vol} L = \frac{1}{n!}(L^n)$. More generally, if L is only pseudo-effective, $\operatorname{vol} L = \frac{1}{n!}\langle L^n \rangle$ (movable intersection).

Proposition

1. The volume of a line bundle depends only on its numerical class: if $c_1(L) = c_1(L')$, then $\operatorname{vol} L = \operatorname{vol} L'$.
2. The volume is homogeneous: $\operatorname{vol} L^k = k^n \operatorname{vol} L$.
3. A psef line bundle L is big iff $\operatorname{vol} L > 0$.

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We want to generalize

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We can thus define

$$\operatorname{vol}(L, \phi) = \lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\phi))\,.$$

Existence of the limit is non-trivial!

Why and how do we study volumes?

Why?

From the perspective of algebraic geometry, we have the following vanishing theorem:

Theorem (Nadel–Cao+Darvas–X)

Suppose $\mathrm{vol}(L, \phi) > 0$, *then*

$$h^i(X, K_X \otimes L^k \otimes \mathcal{I}(k\phi)) = 0$$

for all $i > 0$, $k > 0$.

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How?

1. Treat the case of analytic singularities (Bonavero)
2. Treat the general case by Demailly approximation.

Bonavero's theorem

Assume that ϕ has analytic singularities: locally ϕ is of the form $c \log \sum_i |f_i|^2$ (up to a smooth term), where $f_1, \dots, f_M$ are holomorphic functions.

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Theorem (Bonavero)

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\phi)) = \frac{1}{n!} \int_X (\mathrm{dd^c}\phi)^n \,.$$

Here $(\mathrm{dd^c}\phi)^n$ is the non-pluripolar product.

Non-pluripolar products

Non-pluripolar products

When ϕ is smooth, $(\mathrm{dd}^{\mathrm{c}}\phi)^n$ is the usual wedge product of forms.
When ϕ is bounded, $(\mathrm{dd}^{\mathrm{c}}\phi)^n$ coincides with the Bedford–Taylor product.
For general ϕ, $(\mathrm{dd}^{\mathrm{c}}\phi)^n$ is a measure on X that puts no mass on pluripolar sets.

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The theory of non-pluripolar products is not the only extension of Bedford–Taylor theory (Dinh–Sibony products, residue currents, etc.). But non-pluripolar products have two advantages: $(\mathrm{dd^c}\phi)^n$ can be defined for all ϕ and

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Theorem (Witt Nyström)

If $\varphi \leq \psi$, then

$$\int_X (\mathrm{dd}^{\mathrm{c}}\varphi)^n \leq \int_X (\mathrm{dd}^{\mathrm{c}}\psi)^n .$$

Quasi-equisingular approximations

Assume that $\mathrm{dd^c}\phi$ is a Kähler current. We can find a decreasing sequence (ϕ^j) of psh metrics on L of analytic singularities converging to ϕ such that for any $\delta > 0$, $k > 0$, there is $j_0 > 0$ so that for $j \geq j_0$,

$$\mathcal{I}((1+\delta)k\phi^j) \subseteq \mathcal{I}(k\phi) \subseteq \mathcal{I}(k\phi^j)\,.$$

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To remember

Quasi-equisingular approximations are optimal analytic approximations of psh metrics in the sense of multiplier ideal sheaves.

From Bonavero to general potentials

By approximation,

$$\varlimsup_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{J}(k\phi)) \leq \lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{J}(k\phi^j)) \quad \text{as } \phi \leq \phi^j$$
$$= \frac{1}{n!} \int_X (\mathrm{dd^c} \phi^j)^n \quad \text{Bonavero}\,.$$

From Bonavero to general potentials

By approximation,

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By Witt Nyström, $\int_X(\mathrm{dd^c}\phi^j)^n$ is decreasing in j.

$$\overline{\lim_{k\to\infty}}\frac{1}{k^n}h^0(X,L^k\otimes\mathcal{I}(k\phi))\leq\frac{1}{n!}\lim_{j\to\infty}\int_X(\mathrm{dd^c}\phi^j)^n\,.$$

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In general,

$$\lim_{j\to\infty} \int_X (\mathrm{dd^c}\phi^j)^n \neq \int_X (\mathrm{dd^c}\phi)^n\,.$$

What is the limit? To answer this question, we need to introduce $\mathcal{I}$-model potentials.

I-model potentials

Given a smooth form $\theta \in c_1(L)$, then ϕ can be identified with $\varphi \in \mathrm{PSH}(X, \theta)$.

Definition

Define

$$P[\varphi]_{\mathcal{I}} := \sup \{\psi \le 0 : \mathcal{I}(k\varphi) = \mathcal{I}(k\psi)\} \ .$$

Equivalently,

$$P[\varphi]_{\mathcal{I}} := \sup \{\psi \le 0 : \nu(\pi^*\varphi, y) = \nu(\pi^*\psi, y)\} \ .$$

for any birational model $\pi : Y \to X$ and any $y \in Y$.

A potential φ with $\varphi = P[\varphi]_{\mathcal{I}}$ is called an $\mathcal{I}$-model potential.

I-model potential

Theorem (Darvas–X)

$$\lim_{j\to\infty}\int_X (\mathrm{dd}^{\mathrm{c}}\phi^j)^n = \int_X (\theta + \mathrm{dd}^{\mathrm{c}} P[\varphi]_{\mathcal{J}})^n \,.$$

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As a consequence,

$$\varlimsup_{k\to\infty}\frac{1}{k^n}h^0(X,L^k\otimes\mathcal{I}(k\phi))\leq\frac{1}{n!}\int_X(\theta+\mathrm{dd^c}P[\varphi]_{\mathcal{I}})^n\,.$$

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The reverse inequality is much harder. But the main idea is the same.

Theorem

When $\mathrm{dd^c}\phi$ *is a Kähler current, we have*

$$\mathrm{vol}(L,\phi)=\lim_{k\to\infty}\frac{1}{k^n}h^0(X,L^k\otimes\mathcal{I}(k\phi))=\frac{1}{n!}\int_X (\theta+\mathrm{dd^c}P[\varphi]_{\mathcal{I}})^n\,.$$

Main theorem

When $\mathrm{dd}^{\mathrm{c}}\phi$ is not a Kähler current, a trick allows us to reduce to the special case: there is always a potential $\psi \leq \phi$ such that $\mathrm{dd}^{\mathrm{c}}\psi$ is a Kähler current. We use $(1-\epsilon)\phi + \epsilon\psi$ to approximate ϕ.

Theorem

Let (L,ϕ) be a Hermitian psef line bundle on X, under the identification ϕ with $\varphi \in \mathrm{PSH}(X,\theta)$, $\mathrm{vol}(L,\phi)$ is well-defined and

$$\mathrm{vol}(L,\phi) = \frac{1}{n!}\int_X (\theta + \mathrm{dd}^{\mathrm{c}} P[\varphi]_{\mathcal{J}})^n \,.$$

Proof recap

The main steps in the proof can be sketched as follow:

- Treat the case of analytic singularities.
- Treat the case of Kähler currents by approximation.
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Proof recap

The main steps in the proof can be sketched as follow:

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This general reduction scheme will be applied in other contexts as well.

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Okounkov bodies

Fix a big line bundle L on X.
Okounkov body is a machinery that allows one to study numerical properties of L using convex geometry.

Definition

An admissible flag is a sequence

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

of subvarieties of X, Y_i of codimension i such that Y_i is smooth at the point Y_n.

Example

When X is a Riemann surface, an admissible flag is $X \supseteq \{\text{point}\}$.

Okounkov bodies

Fix an admissible flag $(Y_\bullet)$. Given any $s \in \mathbb{C}(X)^\times$, we can define the valuation of s along $(Y_\bullet)$:

$$\nu(s) = (\nu_1(s), \dots, \nu_n(s)) \in \mathbb{Z}^n \,.$$

Roughly speaking, $\nu_1(s)$ is the order of vanishing of s along Y_1. If $\nu_1(s) = 0$, $\nu_2(s)$ is the order of vanishing of $s|_{Y_1}$, etc..

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Set

$$\Gamma(L) := \left\{(\nu(s), k) \in \mathbb{Z}^{n+1} : s \in H^0(X, L^k)\right\} .$$

Then $\Gamma(L)$ is a semi-group in $\mathbb{Z}^{n+1}$ and generates a closed convex cone C in $\mathbb{R}^{n+1}$. We define the Okounkov body of L as

$$\Delta(L) := C \cap \{(x, 1) : x \in \mathbb{R}^n\} .$$

Example

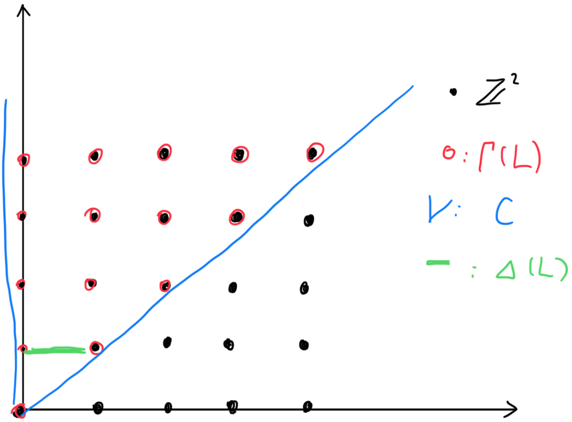
To understand the abstract definition, take $X = \mathbb{P}^1$. The flag is $X \supseteq \{0\}$. Given any rational function f, $\nu(f)$ is the order of vanishing of f at 0.

Example

To understand the abstract definition, take $X = \mathbb{P}^1$. The flag is $X \supseteq \{0\}$. Given any rational function f, $\nu(f)$ is the order of vanishing of f at 0. Set $L = \mathcal{O}(1)$, $H^0(X, L^k)$ is the set of homogeneous polynomials of degree k. Thus

$$\Gamma(L) = \left\{(a,b) \in \mathbb{Z}^2 : 0 \leq a \leq b\right\} .$$

Example



How to understand Okounkov bodies?

Example

When $n = 1$, flag is just a point and $\Delta(L) = [0, \deg L]$.

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Convex body

A convex body is a non-empty compact convex subset of $\mathbb{R}^n$.

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In general, $\Delta(L)$ is a convex body, but not a polytope!
Okounkov bodies is a way to apply techniques in toric geometry to non-toric varieties.

Why are Okounkov bodies important?

Theorem (Lazarsfeld–Mustață, Kaveh–Khovanskii)

$$\frac{1}{n!}\operatorname{vol}(L) = \operatorname{vol}\Delta(L)\,.$$

Why are Okounkov bodies important?

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Theorem (Lazarsfeld–Mustață, Jow)

$\Delta(L)$ depends only on the numerical class of L. Moreover, the information of $\Delta(L)$ for all admissible flags determines the numerical class of L uniquely.

In particular, we can (in principle) deduce all numerical information of L from $\Delta(L)$.

Partial Okounkov bodies

Based on the same ideas, we want to define Okounkov bodies $\Delta(L, \phi)$ of Hermitian psef line bundles (L, ϕ) with similar properties.

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Partial Okounkov bodies

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In contrast to concluding numerical information of L, we want to get information about the singularities of ϕ.

Theorem (X.)

There is a canonical convex body $\Delta(L,\phi) \subseteq \Delta(L)$ associated to each Hermitian psef line bundle with $\int_X (\mathrm{dd^c}\phi)^n > 0$. The assignment is continuous in ϕ.
Two psh metrics (of positive masses) ϕ and ϕ' have the same partial Okounkov bodies iff ϕ and ϕ' are $\mathcal{I}$-equivalent: $\mathcal{I}(k\phi) = \mathcal{I}(k\phi')$ for all k.

How to construct partial Okounkov bodies?

Recall our general construction scheme:

- Treat the case of analytic singularities.
- Treat the case of Kähler currents by approximation.
- Reduce the general case to the case of Kähler currents.

Partial Okounkov bodies—Analytic singularities

When ϕ has analytic singularities.
Assume for simplicity that ϕ has logarithmic singularities along some normal crossing divisor D: ϕ is locally $\log|s|^2$, where s is the defining equation of D.

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Assume for simplicity that ϕ has logarithmic singularities along some normal crossing divisor D: ϕ is locally $\log|s|^2$, where s is the defining equation of D. In this case,

$$H^0(X, L^k \otimes \mathcal{I}(k\phi)) = H^0(X, L^k \otimes \mathcal{O}(-kD))\,.$$

So

$$\bigoplus_k H^0(X, L^k \otimes \mathcal{I}(k\phi)) = \bigoplus_k H^0(X, L^k \otimes \mathcal{O}(-kD))\,.$$

Partial Okounkov bodies—Analytic singularities

Define

$$\Delta(L,\phi) := \Delta(L-D) + a\,,$$

where $a = \nu(D) \in \mathbb{R}^n$ is a fixed vector.

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$$\Delta(L,\phi) := \Delta(L-D) + a\,,$$

where $a = \nu(D) \in \mathbb{R}^n$ is a fixed vector.
The general case of analytic singularities can be handled in a similar fashion. This finishes Step 1 in our scheme.

Partial Okounkov bodies—Kähler currents

Step 2 in our scheme is the quasi-equisingular approximation (ϕ^j) of ϕ. We want to define

$$\Delta(L,\phi) := \lim_{j\to\infty} \Delta(L,\phi^j)\,.$$

But what is the topology on convex bodies $\Delta(\bullet)$?

Partial Okounkov bodies—Kähler currents

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We want to define

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But what is the topology on convex bodies $\Delta(\bullet)$?
The solution is given by the Hausdorff topology.

Definition

Given two convex bodies $K_1, K_2 \subseteq \mathbb{R}^n$, define their Hausdorff metric as

$$d_n(K_1,K_2) := \max\left\{ \sup_{x_1\in K_1} \inf_{x_2\in K_2} |x_1-x_2|,\ \sup_{x_2\in K_2} \inf_{x_1\in K_1} |x_1-x_2| \right\}.$$

Partial Okounkov bodies—General case

Step 3 in our scheme is carried our in a similar way as Step 2. We can reduce general potentials to those whose curvature currents are Kähler currents easily.

Partial Okounkov bodies—General case

Step 3 in our scheme is carried our in a similar way as Step 2. We can reduce general potentials to those whose curvature currents are Kähler currents easily.

The construction seems to depend on a lot of choices. But we can prove

Theorem

The convex body $\Delta(L,\phi)$ is independent of all choices we made! It depends on (L,ϕ) only through the current $\mathrm{dd^c}\phi$.

In other words, $\Delta(L,\phi)$ is a numerical invariant of (L,ϕ).

Why do we care about partial Okounkov bodies?

As we mentioned,

Theorem

Let ϕ, ϕ' be two psh metrics on L with positive masses. Then $\Delta(L,\phi)=\Delta(L,\phi')$ for all admissible flags (on all birational models of X) iff $\phi \sim_{\mathcal{I}} \phi'$.

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In other words,

Principle

Partial Okounkov bodies are universal invariants of the singularities of a psh metric.

Dictionary

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- Movable intersections   Non-pluripolar products
- Okounkov bodies   Partial Okounkov bodies
- Many others

How do we study/classify general psh singularities?

We discuss a general classification of the singularities of psh metrics of positive volumes.

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A few applications

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In pluripotential theory, our result yields a very general Bergman kernel convergence result: given a singularity type $[\varphi]$, a continuous metric ψ on L, it is possible to define partial Bergman measures $\beta^k_{[\varphi]}(\psi)$, generalizing the usual Bergman measure construction.

We proved that these partial Bergman measures converge weakly and the limit can be described explicitly. This result can be regarded as a local version of our volume theorem.

A few applications

When L is ample, our theory allows us to understand the non-Archimedean space $\mathcal{E}^1(L^{\mathrm{an}})$ defined by Boucksom–Jonsson in terms of the space of geodesic rays in the Archimedean (usual) setting.
In particular, we have a complete description of *maximal geodesic rays*.
One can also construct Duistermaat–Heckman measures of elements in $\mathcal{E}^1(L^{\mathrm{an}})$, which opens the gate to the study of non-Archimedean space $\mathcal{E}^p(L^{\mathrm{an}})$.

A few applications

Very recently, Botero–Burgos Gil–Holmes–De Jong made use of our results to study the canonical metric on the line bundle of Siegel–Jacobi forms: Let $A_{g,N}$ be the fine moduli space of PPAV of dimension g and level N. Consider the universal Abelian variety $U_{g,N}$ on $A_{g,N}$ and the line bundle $\mathcal{L}_{k,m}$ of Siegel–Jacobi forms on $U_{g,N}$ of weight k and index m. The latter is endowed with a natural metric h. They applied our results to study the behaviour of h on the compactification of $U_{g,N}$.

Thank you for your attention!