CHARACTERIZATIONS OF $\mathcal{I}$-GOOD SINGULARITIES

Contents

1. Introduction

These are the lecture notes for three lectures given in Chinese Academy of Science in the summer of 2023. The goal is to explain the proofs of [DX22; DX21] in detail.

2. Preliminaries

Fix a compact Kähler manifold X of pure dimension n.

2.1. **The space of finite energy potentials.** We fix a big $(1,1)$-class $\alpha \in H^{1,1}(X,\mathbb{R})$ on X and a closed smooth real $(1,1)$-form $\theta \in \alpha$.

Let

$$V_\theta = \sup\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0\}.$$

Observe that $V_\theta \in \mathrm{PSH}(X,\theta)$. In fact, the usc regularization V_θ^* of V_θ belongs to $\mathrm{PSH}(X,\theta)$ by Hartogs lemma, so $V_\theta^* \leq V_\theta$ and hence the equality holds. So $V_\theta \in \mathrm{PSH}(X,\theta)$.

We will be constantly using the non-pluripolar products. We refer to [BEGZ10] for the details. We write

$$\mathrm{PSH}(X,\theta)_{>0} = \left\{\varphi \in \mathrm{PSH}(X,\theta) : \int_X \theta_\varphi^n > 0\right\}.$$

The non-pluripolar theory is not the only extension of the Bedford–Taylor theory to unbounded qpsh functions, but two features indicate that it is probably the most natural theory: first of all, the non-pluripolar product is defined for all functions in $\mathrm{PSH}(X,\theta)$; secondly, there is a monotonicity theorem:

thm:mono **Theorem 2.1** ([WN19; DDNL18b]). *Suppose that $\varphi,\psi \in \mathrm{PSH}(X,\theta)$ and $\varphi \preceq \psi$ (see Definition 2.6), then*

$$\int_X \theta_\varphi^n \leq \int_X \theta_\psi^n.$$

More generally, if $\alpha_1,\dots,\alpha_n$ are pseudoeffective classes represented by $\theta_1,\dots,\theta_n$, $\varphi_j,\psi_j \in \mathrm{PSH}(X,\theta_j)$ $(j=1,\dots,n)$ and $\varphi_j \preceq \psi_j$ for $j=1,\dots,n$, then

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \leq \int_X \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{n,\psi_n}.$$

In particular, the non-pluripolar mass of any $\varphi \in \mathrm{PSH}(X,\theta)$ is always bounded from above by $V := \int_X \theta_{V_\theta}^n$. The number $V > 0$ is known as the *volume* of the class α.

The space of finite energy potentials is defined as

$$\mathcal{E}^1(X,\theta) := \left\{\varphi \in \mathrm{PSH}(X,\theta) : \int_X \theta_\varphi^n = V, \int_X |V_\theta - \varphi|\, \theta_\varphi^n < \infty\right\}.$$

We will need the Monge–Ampère energy functional $E : \mathcal{E}^1(X,\theta) \to \mathbb{R}$ defined as follows:

$$E(\varphi) \coloneqq \frac{1}{n+1}\sum_{j=0}^{n}\int_X (\varphi - V_\theta)\,\theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}.$$

The difference $\varphi - V_\theta$ is only defined outside the pluripolar set $\{V_\theta = -\infty\}$. The non-pluripolar product $\theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}$ does not put mass on pluripolar sets, so the integral is still defined.

It is useful to know that $E(\varphi)$ is increasing in φ and $E(V_\theta) = 0$.

We recall the definition of the metric d_1 on $\mathcal{E}^1(X,\theta)$. Take $\varphi, \psi \in \mathcal{E}^1(X,\theta)$. When $\varphi \leq \psi$, the metric is simply defined as

$$d_1(\varphi,\psi) = E(\psi) - E(\varphi). \tag{2.1}$$

By a simple argument using approximations and the integration by parts formula [Xia19; Lu21], one can show that

$$E(\psi) - E(\varphi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X (\psi - \varphi)\,\theta_\psi^j \wedge \theta_\varphi^{n-j}.$$

More generally, we need to use $\varphi \wedge \psi$, the maximal element in $\mathrm{PSH}(X,\theta)$ lying below both φ and ψ. It is shown in [DDNL18c] that $\varphi \wedge \psi \in \mathcal{E}^1(X,\theta)$. We define

$$d_1(\varphi,\psi) = d_1(\varphi\wedge\psi,\varphi) + d_1(\varphi\wedge\psi,\psi) = E(\varphi) + E(\psi) - 2E(\varphi\wedge\psi).$$

This is indeed a metric, as studied in [DDNL18a].

Next we recall the notion of geodesics in $\mathcal{E}^1(X,\theta)$. Let us fix $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\theta)$. A *subgeodesic* from φ_0 to φ_1 is a curve $(\varphi_t)_{t\in(0,1)}$ in $\mathcal{E}^1(X,\theta)$ such that

(1) if we define
$$\Phi : X \times \{z \in \mathbb{C} : \mathrm{e}^{-1} < |z| < 1\} \to [-\infty,\infty), \quad (x,z) \mapsto \varphi_{-\log|z|}(x),$$
then Φ is $p_1^*\theta$-psh, where $p_1 : X \times \{z \in \mathbb{C} : \mathrm{e}^{-1} < |z| < 1\} \to X$ is the natural projection;

(2) When $t \to 0+$ (resp. to $1-$), φ_t converges to φ_0 (resp. φ_1) with respect to L^1.

The maximal subgeodesic from φ_0 to φ_1 is called the *geodesic* (φ_t) from φ_0 to φ_1. The geodesic always exists and $\varphi_t \in \mathcal{E}^1(X,\theta)$ for all $t \in [0,1]$. We refer to [DDNL18c] for the details.

By abuse of language, we say that $(\varphi_t)_{t\in[0,1]}$ (with a closed interval instead of an open interval) is the geodesic from φ_0 to φ_1. More generally, given $t_0 \leq t_1$ in $\mathbb{R}$, we say a curve $(\varphi_t)_{t\in[t_0,t_1]}$ in $\mathcal{E}^1(X,\theta)$ is a geodesic from φ_{t_0} to φ_{t_1} if after a linear rescaling from $[t_0,t_1]$ to $[0,1]$, it becomes a geodesic. One can show that E is linear along a geodesic. In fact, by a simple perturbation argument, one can reduce this to [DDNL18c, Theorem 3.12].

2.2. The space of geodesic rays.

The notion of geodesics naturally gives us a notion of geodesic rays:

Definition 2.2. A *geodesic ray* is a curve $\ell = (\ell_t)_{t\in[0,\infty)}$ in $\mathcal{E}^1(X,\theta)$ such that for any $0 \leq t_1 < t_2$, the restriction $(\ell_t)_{t\in[t_1,t_2]}$ is a geodesic from ℓ_{t_1} to ℓ_{t_2}.

The space of geodesic rays ℓ with $\ell_0 = V_\theta$ is denoted by $\mathcal{R}^1(X,\theta)$.

The assumption $\ell_0 = V_\theta$ is not very restrictive. In fact, given any other $\varphi \in \mathcal{E}^1(X,\theta)$, we can always find a unique geodesic ray ℓ' with $\ell'_0 = \varphi$ such that $d_1(\ell_t,\ell'_t)$ is bounded. So if we are only interested in the asymptotic behaviour of a geodesic ray, we do not lose any information. We refer to [DL20] for the details.

Next we recall the metric d_1 on $\mathcal{R}^1(X,\theta)$. Given $\ell,\ell' \in \mathcal{R}^1(X,\theta)$, one can show as in [DL20] that $d_1(\ell_t,\ell'_t)$ is a convex function in $t \in [0,\infty)$. It follows that

$$d_1(\ell,\ell') \coloneqq \lim_{t\to\infty}\frac{1}{t}d_1(\ell_t,\ell'_t)$$

exists. It is not hard to show that d_1 is indeed a metric on $\mathcal{R}^1(X,\theta)$. In fact, it is a complete metric. We refer to [DL20; DDNL21] for the details.

Similarly, one can introduce $\mathbf{E} : \mathcal{R}^1(X,\theta) \to \mathbb{R}$ as

$$\mathbf{E}(\ell) = \lim_{t\to\infty} \frac{1}{t} E(\ell_t).$$

As we recalled above, the function $E(\ell_t)$ is linear in t, so the limit $\mathbf{E}(\ell)$ is nothing but the slope of this linear function. When $\ell, \ell' \in \mathcal{R}^1(X,\theta)$, $\ell \leq \ell'$, using (2.1), we have

$$d_1(\ell,\ell') = \mathbf{E}(\ell') - \mathbf{E}(\ell). \tag{2.2}$$

Example 2.3. *Given $\varphi \in \mathrm{PSH}(X,\theta)$, we construct a geodesic ray $\ell^\varphi \in \mathcal{R}^1(X,\theta)$. For each $C > 0$, let $(\ell_t^{\varphi,C})_{t\in[0,C]}$ be the geodesic from V_θ to $(V_\theta - C) \vee \varphi$. For each $t \geq 0$, it is not hard to see that $\ell_t^{\varphi,C}$ is increasing in $C \in [t,\infty)$. We let*

$$\ell_t^\varphi \coloneqq \sup_{C\geq t}{}^* \ell_t^{\varphi,C}.$$

One can show that $\ell^\varphi \in \mathcal{R}^1(X,\theta)$. A simple computation shows that

$$\mathbf{E}(\ell^\varphi) = \frac{1}{n+1}\left(\sum_{j=0}^n \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - V\right). \tag{2.3}$$

See [DDNL21, Theorem 3.1].

We need the following result concerning these geodesic rays: given $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, then $\ell^\varphi = \ell^\psi$ if and only if $\varphi \sim_P \psi$ (see Definition 2.6). This follows from [DDNL21, Proposition 3.2] and Remark 2.10.

Next we recall that $\vee$ operator at the level of geodesic rays. Given $\ell, \ell' \in \mathcal{R}^1(X,\theta)$. We define $\ell \vee \ell'$ as the minimal ray $\mathcal{R}^1(X,\theta)$ lying above both ℓ and ℓ'. In fact, it is easy to construct such a ray: for each $t > 0$, let $(\ell''^t_s)_{s\in[0,t]}$ be the geodesic from V_θ to $\ell_t \vee \ell'_t$. It is easy to see that for each fixed $s \geq 0$, ℓ''^t_s is increasing in $t \in [s,\infty)$. Let $(\ell\vee\ell')_s = \sup^*_{t\geq s} \ell''^t_s$. Then we get a geodesic ray $\ell \vee \ell'$. It is clear that this ray is minimal among all rays dominating ℓ and ℓ'. By construction, we have

$$E(\ell\vee\ell')_s = \lim_{t\to\infty} E(\ell''^t_s) = \lim_{t\to\infty} \frac{s}{t} E(\ell_t \vee \ell'_t).$$

In particular,

$$\mathbf{E}(\ell\vee\ell') = \lim_{t\to\infty} \frac{1}{t} E(\ell_t\vee\ell'_t). \tag{2.4}$$

Lemma 2.4. *For any $\ell, \ell' \in \mathcal{R}^1(X,\theta)$, we have*

$$d_1(\ell,\ell') \leq d_1(\ell,\ell\vee\ell') + d_1(\ell',\ell\vee\ell') \leq C_n d_1(\ell,\ell'), \tag{2.5}$$

where $C_n = 3(n+1)2^{n+2}$.

Proof. The first inequality is trivial. As for the second, we estimate

$$\begin{aligned} d_1(\ell,\ell\vee\ell') =& \mathbf{E}(\ell\vee\ell') - \mathbf{E}(\ell) \\ =& \lim_{t\to\infty} \frac{1}{t} E(\ell_t\vee\ell'_t) - \mathbf{E}(\ell) \\ =& \lim_{t\to\infty} \frac{1}{t} d_1(\ell_t\vee\ell'_t, \ell_t). \end{aligned}$$

where on the second line, we used (2.4), the third line follows from (2.2). In all, we find

$$d_1(\ell,\ell\vee\ell') + d_1(\ell',\ell\vee\ell') \leq \lim_{t\to\infty} \frac{1}{t}\left(d_1(\ell_t\vee\ell'_t,\ell_t) + d_1(\ell_t\vee\ell'_t,\ell'_t)\right).$$

By [DDNL18a, Theorem 3.7],

$$d_1(\ell_t\vee\ell'_t,\ell_t) + d_1(\ell_t\vee\ell'_t,\ell'_t) \leq 3(n+1)2^{n+2} d_1(\ell_t,\ell'_t).$$

Now (2.5) follows. $\square$

2.3. **The space of quasi-plurisubharmonic functions.** We write $\mathrm{QPSH}(X)$ for the direct limit in the category of sets

$$\mathrm{QPSH}(X) \coloneqq \varinjlim_{\theta} \mathrm{PSH}(X,\theta),$$

where θ runs over the set of all smooth real closed $(1,1)$-forms on X with $\theta \prec \theta'$ if $\theta' - \theta$ is a Kähler form. The transition maps are given by inclusions. In other words, $\mathrm{QPSH}(X)$ is the set of quasi-plurisubharmonic functions on X.

Remark 2.5. I am always curious about the possibility of enriching the set $\mathrm{QPSH}(X)$, but I have never been able to figure out the correct generality/category to work with. One should view the direct limit as in other categories instead of barely the category of sets.

A few failed options: pseudo-metric spaces, uniform spaces, topological spaces, condensed spaces. None of these options gives rise to the correct notion of convergence on $\mathrm{QPSH}(X)$ as we define later, which is closer to the strict direct limit as studied in functional analysis by Dieudonné–Schwarz.

Take a big class α on X with a representative θ, we will need the following envelope operators:

(1) Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, we set

$$\begin{aligned} P_\theta[\varphi] &= \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \leq \varphi + C \text{ for some } C \in \mathbb{R}\right\} \\ &= \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \int_X \theta_\varphi^n = \int_X \theta_\psi^n, \varphi \leq \psi + C \text{ for some } C \in \mathbb{R}\right\}; \end{aligned}$$

Observe that in the two conditions, the relation between φ and ψ are reversed.

(2) Let $\varphi \in \mathrm{PSH}(X,\theta)$, we set

$$P_\theta[\varphi]_{\mathcal{I}} = \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \mathcal{I}(k\varphi) = \mathcal{I}(k\psi) \text{ for all } k \in \mathbb{Z}_{>0}\right\}.$$

We refer to [DDNL18c] for a detailed study of the former envelope and to [DX21; DX22] for the latter.

The first envelop is pathological when $\int_X \theta_\varphi^n = 0$. There are multiple different ways to extend its definition. None of these seem to be natural to the author, so we will avoid them.

A potential $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ (resp. $\varphi \in \mathrm{PSH}(X,\theta)$) is *model* (resp. $\mathcal{I}$-model) if $P_\theta[\varphi] = \varphi$ (resp. $P_\theta[\varphi]_{\mathcal{I}} = \varphi$).

Both notions depends strongly on the choice of θ, which makes them not so natural. By contrast, the notion of $\mathcal{I}$-good potentials introduced in [Xia22] depends only on $\varphi \in \mathrm{QPSH}(X)$.

Definition 2.6. Let $\varphi, \psi \in \mathrm{QPSH}(X)$, we say

(1) φ is *more singular* than ψ and write $\varphi \preceq \psi$ if there is $C \in \mathbb{R}$ such that

$$\varphi \leq \psi + C;$$

(2) φ is *P-more singular* than ψ and write $\varphi \preceq_P \psi$ if for some Kähler form ω such that $\varphi, \psi \in \mathrm{PSH}(X,\omega)_{>0}$, we have

$$P_\omega[\varphi] \leq P_\omega[\psi];$$

(3) φ is *$\mathcal{I}$-more singular* than ψ and write $\varphi \preceq_{\mathcal{I}} \psi$ if for some Kähler form ω such that $\varphi, \psi \in \mathrm{PSH}(X,\omega)$, we have

$$P_\omega[\varphi]_{\mathcal{I}} \leq P_\omega[\psi]_{\mathcal{I}}.$$

All three relations define partial orders on $\mathrm{QPSH}(X)$. We denote the corresponding equivalence relation by $\sim$, $\sim_P$ and $\sim_{\mathcal{I}}$ respectively.

In (1), one cannot replace $\varphi \leq \psi + C$ by $\varphi - \psi \leq C$ without extra care. The problem is that $\varphi - \psi$ is only defined outside the pluripolar set $\{\varphi = \psi = -\infty\}$.

We observe that Condition (2) does not depend on the choice of ω by Lemma 2.7. On the other hand, Condition (3) is equivalent to $\mathcal{I}(k\varphi)\subseteq\mathcal{I}(k\psi)$ for all $k>0$ (either real or integral). So Condition (3) is also independent of the choice of ω.

Lemma 2.7. *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)_{>0}$. For any Kähler form ω on X, the following are equivalent:*

(1) $P_\theta[\varphi]\leq P_\theta[\psi]$;
(2) $P_{\theta+\omega}[\varphi]\leq P_{\theta+\omega}[\psi]$.

Proof. (1) implies (2): Observe that

$$P_\theta[\varphi]\leq P_{\theta+\omega}[\varphi],\quad \varphi\preceq P_\theta[\varphi].$$

It follows that

$$P_{\theta+\omega}[\varphi]=P_{\theta+\omega}[P_\theta[\varphi]]. \tag{2.6}$$

A similar formula holds for ψ. So we see that (2) holds.

(2) implies (1): By (2.6), we may assume that φ and ψ are both model potentials in $\mathrm{PSH}(X,\theta)$.

Observe that $\varphi\vee\psi\preceq P_{\theta+\omega}[\psi]$. It follows that $P_{\theta+\omega}[\varphi\vee\psi]\leq P_{\theta+\omega}[\psi]$. The reverse inequality is trivial, so

$$P_{\theta+\omega}[\varphi\vee\psi]=P_{\theta+\omega}[\psi].$$

From the direction we have proved, for any $C\geq 1$,

$$P_{\theta+C\omega}[\varphi\vee\psi]=P_{\theta+C\omega}[\psi].$$

So

$$\int_X\left(\theta+C\omega+\mathrm{dd^c}(\varphi\vee\psi)\right)^n=\int_X\left(\theta+C\omega+\mathrm{dd^c}\psi\right)^n.$$

In particular,

$$\int_X\theta^n_{\varphi\vee\psi}=\int_X\theta^n_\psi.$$

As ψ is model, it follows that $\varphi\vee\psi=\psi$. So (1) follows. □

Lemma 2.8. *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)_{>0}$. Then for any $t\in[0,1]$,*

$$t\varphi+(1-t)\psi\sim_P tP_\theta[\varphi]+(1-t)P_\theta[\psi].$$

Proof. By symmetry, it suffices to show that $t\varphi+(1-t)\psi\sim_P tP_\theta[\varphi]+(1-t)\psi$. As $t\varphi+(1-t)\psi\preceq tP_\theta[\varphi]+(1-t)\psi$ and both sides have positive masses, it suffices to show that

$$\int_X\theta^n_{t\varphi+(1-t)\psi}=\int_X\theta^n_{tP_\theta[\varphi]+(1-t)\psi}.$$

By binary expansion, it suffices to show that for any $j=0,\dots,n$,

$$\int_X\theta^j_\varphi\wedge\theta^{n-j}_\psi=\int_X\theta^j_{P_\theta[\varphi]}\wedge\theta^{n-j}_\psi,$$

which follows from [DDNL18b, Corollary 3.2]. □

Corollary 2.9. *Let $\varphi,\psi,\varphi',\psi'\in\mathrm{QPSH}(X)$. Assume that $\varphi\sim_P\varphi'$ and $\psi\sim_P\psi'$, then for any $a,b>0$, $a\varphi+b\psi\sim_P a\varphi'+b\psi'$.*

Proof. We may assume that $a+b=1$ by rescaling. Take a Kähler form ω on X so that $\varphi,\psi,\varphi',\psi'\in\mathrm{PSH}(X,\omega)_{>0}$. Then it suffices to apply Lemma 2.8. □

Remark 2.10. In [DDNL21], Darvas–Di Nezza–Lu introduced a different envelope operator C which is better behaved when the mass of a qpsh function is 0. We will show that for our purpose, it is not necessary to introduce it.

To be more precise, let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. We will show that the following are equivalent:

(1) $\varphi\sim_P\psi$;
(2) $C_\theta[\varphi]=C_\theta[\psi]$.

Assume (1). Then by Corollary 2.9, $t\varphi+(1-t)V_\theta \sim_P t\psi+(1-t)V_\theta$ for all $t\in[0,1)$. In particular,

$$P_\theta[t\varphi+(1-t)V_\theta]=P_\theta[t\psi+(1-t)V_\theta].$$

Let $t\to 1-$, we conclude (2).

Conversely assume (2). Let ω be a Kähler form on X. It suffices to show that

$$P_{\theta+\omega}[C_\theta[\varphi]]=P_{\theta+\omega}[\varphi].$$

In fact, the $\geq$ inequality is clear. As both sides are model potentials, it suffices to show that they have the same mass:

$$\int_X(\theta+\omega+\mathrm{dd^c}C_\theta[\varphi])^n=\int_X(\theta+\omega+\mathrm{dd^c}\varphi)^n.$$

After binary expansion, it suffices to show that for each $j=0,\dots,n$,

{eq:Cexpan}

$$\int_X\theta^j_{C_\theta[\varphi]}\wedge\omega^{n-j}=\int_X\theta^j_\varphi\wedge\omega^{n-j}. \tag{2.7}$$

As $C_\theta[\varphi]$ is the decreasing limit of $P_\theta[tV_\theta+(1-t)\varphi]$ as t decreases to 0, we have

$$\int_X\theta^j_{C_\theta[\varphi]}\wedge\omega^{n-j}\leq\lim_{t\to0+}\int_X\theta^j_{P_\theta[tV_\theta+(1-t)\varphi]}\wedge\omega^{n-j}=\lim_{t\to0+}\int_X\theta^j_{tV_\theta+(1-t)\varphi}\wedge\omega^{n-j}=\int_X\theta^j_\varphi\wedge\omega^{n-j}.$$

The reverse inequality follows from the monotonicity theorem Theorem 2.1. So (2.7) follows. We conclude the proof.

orm_preceqP

Lemma 2.11. *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi\preceq_P\psi$ (resp. $\varphi\preceq_{\mathcal{I}}\psi$);
(2) $\varphi\vee\psi\sim_P\psi$ (resp. $\varphi\vee\psi\sim_{\mathcal{I}}\psi$).

Proof. We may assume that $\int_X\theta_\varphi^n>0$, $\int_X\theta_\psi^n>0$. We only prove the P case, the $\mathcal{I}$ case is similar.

(2) implies (1): We may assume that φ,ψ are both model in $\mathrm{PSH}(X,\theta)$. By (2), $P_\theta[\varphi\vee\psi]=\psi$. But $\varphi\leq P_\theta[\varphi\vee\psi]$, so (1) follows.

(1) implies (2): We may still assume that φ,ψ are both model in $\mathrm{PSH}(X,\theta)$ as

$$P_\theta[\varphi\vee\psi]=P_\theta[P_\theta[\varphi]\vee P_\theta[\psi]].$$

Then $\varphi\leq\psi$ and (2) follows. □

3. The d_S-pseudometric

Let X be a compact Kähler manifold of pure dimension n.

3.1. The construction.

Let α be a big $(1,1)$-class on X represented by a smooth form θ.

Definition 3.1. For $\varphi,\psi\in\mathrm{PSH}(X,\theta)$, we define

$$d_S(\varphi,\psi):=d_1(\ell^\varphi,\ell^\psi).$$

When necessary, we also write $d_{S,\theta}$ instead. It turns out that this is never necessary once we finish the proof of Corollary 3.17.

By definition, d_S is a pseudo-metric on $\mathrm{PSH}(X,\theta)$. By Example 2.3, we have

Proposition 3.2. *For $\varphi,\psi\in\mathrm{PSH}(X,\theta)$, the following are equivalent:*

(1) $\varphi\sim_P\psi$;
(2) $d_S(\varphi,\psi)=0$.

The pseudo-metric d_S itself does not seem to be a natural choice, however, the convergence notion it defines is certainly natural, as we will see repeatedly in this note.

We derive a few elementary properties from the definition.

qpsi_metric

Lemma 3.3 ([DDNLmetric DDNL21, Lemma 3.4])**.** *Suppose that $\varphi,\psi\in\mathrm{PSH}(X,\theta)$ and $\varphi\preceq_P\psi$, then*

$$d_S(\varphi,\psi)=\frac{1}{n+1}\sum_{j=0}^n\left(\int_X\theta^j_\psi\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_\varphi\wedge\theta^{n-j}_{V_\theta}\right).$$

Proof. This follows trivially from (2.3). □

cdoubleineq **Lemma 3.4.** *For any $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we have*

:ds_biineq}

$$d_S(\varphi,\psi) \leq \sum_{j=0}^{n}\left(2\int_X \theta^j_{\varphi\vee\psi}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\psi}\wedge\theta^{n-j}_{V_\theta}\right) \leq C_n d_S(\varphi,\psi), \tag{3.1}$$

where $C_n = 3(n+1)2^{n+2}$.

Proof. It suffices to show that

eq:ellmax1}

$$\ell^{\varphi}\vee\ell^{\psi} = \ell^{\varphi\vee\psi}. \tag{3.2}$$

Assuming this, then (3.1) follows from Lemma 3.3 and Lemma 2.4.

Next we prove (3.2). Of course by definition, it is trivial that

$$\ell^{\varphi}\leq\ell^{\varphi\vee\psi},\quad \ell^{\psi}\leq\ell^{\varphi\vee\psi}.$$

So

$$\ell^{\varphi}\vee\ell^{\psi}\leq\ell^{\varphi\vee\psi}.$$

Conversely, if $\ell\in\mathcal{R}^1(X,\theta)$ and $\ell^{\varphi}\vee\ell^{\psi}\leq\ell$, then for any $C\geq 0$,

$$(V_\theta - C)\vee\varphi\leq\ell_C,\quad (V_\theta - C)\vee\psi\leq\ell_C.$$

It follows that

$$(V_\theta - C)\vee(\varphi\vee\psi)\leq\ell_C.$$

From this, we conclude that

$$\ell^{\varphi\vee\psi}\leq\ell.$$

□

From this lemma, we find that the d_S-convergence is characterized by numerical conditions of non-pluripolar masses. The criterion here is still way too complicated for applications, we will see a better criterion in Corollary 3.16. For now, let us record the following corollary.

:monodsconv **Corollary 3.5.** *Let $\varphi_j,\varphi\in\mathrm{PSH}(X,\theta)$ $(j\geq 1)$. Assume that one of the following conditions holds:*

(1) $\varphi_j\succeq\varphi$ for all j;
(2) $\varphi_j\preceq\varphi$ for all j.

Then the following are equivalent:

(1) $\varphi_j\xrightarrow{d_S}\varphi$;
(2) $\int_X\theta^k_{\varphi_j}\wedge\theta^{n-k}_{V_\theta}\to\int_X\theta^k_{\varphi}\wedge\theta^{n-k}_{V_\theta}$ for all $k=0,\dots,n$.

lma:dslor **Lemma 3.6.** *Let $\varphi,\psi,\eta\in\mathrm{PSH}(X,\theta)$, then*

{eq:dSmax}

$$d_S(\varphi\vee\eta,\psi\vee\eta)\leq C_n d_S(\varphi,\psi), \tag{3.3}$$

where $C_n = 3(n+1)2^{n+2}$.

Proof. According to Lemma 3.4, we may assume that $\varphi\leq\psi$.

We will show that for each $C\geq t\geq 0$,

:d1maxcomp}

$$d_1(\ell_t^{\varphi\vee\eta,C},\ell_t^{\psi\vee\eta,C})\leq d_1(\ell_t^{\varphi,C},\ell_t^{\psi,C}). \tag{3.4}$$

When $C\to\infty$, by [DDNL18a, Proposition 2.7], it follows that

$$d_1(\ell_t^{\varphi\vee\eta},\ell_t^{\psi\vee\eta})\leq d_1(\ell_t^{\varphi},\ell_t^{\psi}),$$

which implies (3.3).

It remains to argue (3.4). As $\varphi\leq\psi$, we know that

$$d_1(\ell_t^{\varphi},\ell_t^{\psi})=\frac{t}{C}d_1(\ell_C^{\varphi},\ell_C^{\psi}),\quad d_1(\ell_t^{\varphi\vee\eta},\ell_t^{\psi\vee\eta})=\frac{t}{C}d_1(\ell_C^{\varphi\vee\eta},\ell_C^{\psi\vee\eta}).$$

It suffices to handle the case $t=C$, namely,

$$d_1(\varphi\vee\eta\vee(V_\theta-C),\psi\vee\eta\vee(V_\theta-C))\leq d_1(\varphi\vee(V_\theta-C),\psi\vee(V_\theta-C)).$$

This is just [Xia21, Proposition 6.8]. ☐

3.2. Convergence theorems.

Lemma 3.7. *Let $(\varphi^k)_k$ be a sequence in* $\mathrm{PSH}(X,\theta)$ *and* $\varphi\in\mathrm{PSH}(X,\theta)$. *Assume that* $\varphi^k\xrightarrow{d_S}\varphi$ *as* $k\to\infty$. *Then for any* $t\in(0,1]$,

$$(1-t)\varphi^k+tV_\theta\xrightarrow{d_S}(1-t)\varphi+tV_\theta$$

as $k\to\infty$.

Proof. Fix $t\in(0,1]$, we write

$$\varphi_t^k=(1-t)\varphi^k+tV_\theta,\quad \varphi_t=(1-t)\varphi+tV_\theta.$$

By Lemma 3.4, it suffices to show that for each $j=0,\dots,n$,

$$2\int_X\theta^j_{\varphi^k_t\vee\varphi_t}\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_{\varphi^k_t}\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_{\varphi_t}\wedge\theta^{n-j}_{V_\theta}\to 0. \tag{3.5}$$

Observe that

$$\varphi^k_t\vee\varphi_t=(1-t)(\varphi\vee\varphi^k)+tV_\theta.$$

So after binary expansion, (3.5) follows from Lemma 3.4. ☐

We need the existence of an extraordinary envelope, which looks like a miracle to the author. This envelope plays a key role in reducing problems with general positive currents to problems with Kähler currents.

Lemma 3.8 ([DDNL21, Lemma 4.3]). *Let* $\varphi,\psi\in\mathrm{PSH}(X,\theta)$, $\varphi\preceq\psi$ *and* $\int_X\theta^n_\varphi>0$. *Then for any*

$$a\in\left(1,\left(\frac{\int_X\theta^n_\psi}{\int_X\theta^n_\psi-\int_X\theta^n_\varphi}\right)^{1/n}\right),$$

there is $\eta\in\mathrm{PSH}(X,\theta)$ *such that*

$$a^{-1}\eta+(1-a^{-1})\psi\leq\varphi.$$

The fraction is understood as ∞ if $\int_X\theta^n_\psi=\int_X\theta^n_\varphi$.

We write $P(a\varphi+(1-a)\psi)\in\mathrm{PSH}(X,\theta)$ for the regularized supremum of all such η's. In fact, observe that $\psi\geq\varphi-C$, so η is uniformly bounded from above. It follows that $P(a\varphi+(1-a)\psi)\in\mathrm{PSH}(X,\theta)$. On the other hand, by Hartogs lemma,

$$a^{-1}P(a\varphi+(1-a)\psi)+(1-a^{-1})\psi\leq\varphi$$

holds outside a pluripolar set, hence everywhere.

As a corollary is of crucial importance:

Proposition 3.9. *Let* $\varphi\in\mathrm{PSH}(X,\theta)$ *such that* $\int_X\theta^n_\varphi>0$. *Then there exists* $\psi\in\mathrm{PSH}(X,\theta)$ *such that* $\varphi\geq\psi$ *and* $\theta_\psi\geq\omega$ *for some Kähler form* ω.

Proof. We may assume that $\varphi\leq 0$. Since $\varphi\leq V_\theta$ and $\int_X\theta^n_{V_\theta}\geq\int_X\theta^n_\varphi>0$, by Lemma 3.8, there exists $b>0$ such that $h\coloneqq P((1+b)\varphi-bV_\theta)\in\mathrm{PSH}(X,\theta)$ and

$$\frac{b}{b+1}V_\theta+\frac{1}{b+1}h\leq u\,.$$

By [Bou02], there exists $w\in\mathrm{PSH}(X,\theta)$ such that $w\leq 0$ and $\theta_w\geq\delta\omega$ for some $\delta>0$. Since $w\leq V_\theta$, we obtain that

$$\psi\coloneqq\frac{b}{b+1}w+\frac{1}{b+1}h\leq\varphi$$

and $\theta_\psi\geq\frac{b\delta}{b+1}\omega$. ☐

Lemma 3.10. *Let* $\varphi_j,\varphi\in\mathrm{PSH}(X,\theta)$ $(j\geq 1)$. *Assume that* φ_j *is an increasing sequence converging almost everywhere to* φ. *Then* $d_S(\varphi_j,\varphi)\to 0$ *as* $j\to\infty$.

Proof. This follows from Lemma 3.4 and the lower semi-continuity of non-pluripolar products. ☐

lma:decdS **Lemma 3.11** ([DDNL21, Proposition 4.8])**.** *Let $\varphi_j,\varphi\in\mathrm{PSH}(X,\theta)$ $(j\geq 1)$. Assume that $\int_X\theta_{\varphi_j}^n$ is bounded from below by a positive constant, φ_j is model for each j and φ_j decreases pointwisely to φ, then $\varphi_j\xrightarrow{d_S}\varphi$.*

Proof. Let $b_j\in\mathbb{R}$ be a sequence converging to ∞ such that

$$b_j\in\left(1,\left(\frac{\int_X\theta_{\varphi_j}^n}{\int_X\theta_{\varphi_j}^n-\int_X\theta_{\varphi}^n}\right)^{1/n}\right).$$

The existence of this sequence of non-trivial. It requires the fact that $\int_X\theta_{\varphi_j}^n\to\int_X\theta_{\varphi}^n$. This is proved in [DDNL21, Proposition 4.6]. As the technique is quite unrelated to the techniques in this note, we do not reproduce the argument.

By Lemma 3.8, we can find $\eta_j\in\mathrm{PSH}(X,\theta)$ such that

$$b_j^{-1}\eta_j+(1-b_j^{-1})\varphi_j\leq\varphi.$$

It follows from Theorem 2.1 that for any $k=0,\dots,n$,

$$\int_X\theta_\varphi^k\wedge\theta_{V_\theta}^{n-k}\geq(1-b_j^{-1})^k\int_X\theta_{\varphi_j}^k\wedge\theta_{V_\theta}^{n-k}.$$

Together with Theorem 2.1, we conclude that

$$\lim_{j\to\infty}\int_X\theta_{\varphi_j}^k\wedge\theta_{V_\theta}^{n-k}=\int_X\theta_\varphi^k\wedge\theta_{V_\theta}^{n-k}.$$

Hence $\varphi_j\xrightarrow{d_S}\varphi$ by Lemma 3.3. □

The following proposition allows us to reduce a number of problems to monotone sequences.

p:incanddec **Proposition 3.12.** *Let $\varphi_j,\varphi\in\mathrm{PSH}(X,\theta)$ $(j\geq 1)$, $\varphi_j\xrightarrow{d_S}\varphi$. Assume that there is $\delta>0$ such that*

$$\int_X\theta_{\varphi_j}^n\geq\delta,\quad\int_X\theta_\varphi^n\geq\delta$$

for all j and $P_\theta[\varphi_j]=\varphi_j$, $P_\theta[\varphi]=\varphi$ for all j. Then up to replacing $(\varphi_j)_j$ by a subsequence, there is a decreasing sequence $\psi_j\in\mathrm{PSH}(X,\theta)$ and an increasing sequence $\eta_j\in\mathrm{PSH}(X,\theta)$ such that

(1)
$$d_S(\varphi,\psi_j)\to 0,\quad d_S(\varphi,\eta_j)\to 0$$
as $j\to\infty$;

(2) $\psi_j\geq\varphi_j\geq\eta_j$ for all j.

In fact, we will take

$$\eta_j=\varphi_j\wedge\varphi_{j+1}\wedge\cdots$$

and

$$\psi_j=\sup_{k\geq j}{}^*\varphi_k.$$

Proof. We are free to replace $(\varphi_j)_j$ by a subsequence. So we may assume that

$$d_S(\varphi_j,\varphi_{j+1})\leq C_n^{-2j},$$

where C_n is the constant in Lemma 3.4.

Step 1. We handle ψ_j's. For each $j\geq 1$ and $k\geq 1$, by Lemma 3.4 we have

$$\begin{aligned}d_S(\varphi_j,\varphi_j\vee\varphi_{j+1}\vee\cdots\vee\varphi_{j+k})\leq&C_nd_S(\varphi_j,\varphi_{j+1}\vee\cdots\vee\varphi_{j+k})\\\leq&C_nd_S(\varphi_j,\varphi_{j+1})+C_nd_S(\varphi_{j+1},\varphi_{j+1}\vee\cdots\vee\varphi_{j+k}).\end{aligned}$$

By iteration, we find

$$d_S(\varphi_j,\varphi_j\vee\varphi_{j+1}\vee\cdots\vee\varphi_{j+k})\leq\sum_{a=j}^{j+k-1}C_n^{a+1-j}d_S(\varphi_a,\varphi_{a+1})\leq\sum_{a=j}^{j+k-1}C_n^{a+1-j}C_n^{-2a}.$$

From this we see that

$$\lim_{j\to\infty}\lim_{k\to\infty} d_S(\varphi_j, \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) = 0.$$

By Lemma 3.10, we conclude that $d_S(\varphi, \psi_j) \to 0$.

Step 2. We consider the η_j's. This case is more tricky and the proof requires some different techniques, we omit the proof and refer to [DDNL21, Theorem 5.6] for the details. DDNLmetric $\square$

In fact the construction in Step 1 works more generally for any Cauchy sequence. This gives the following

hysubstract **Corollary 3.13.** *Let $\varphi_j \in \mathrm{PSH}(X, \theta)$ be a d_S-Cauchy sequence. Then up to replacing φ_j by a subsequence, there is a decreasing Cauchy sequence $\psi_j \in \mathrm{PSH}(X, \theta)$ such that $d_S(\varphi_j, \psi_j) \to 0$ and $\varphi_j \preceq \psi_j$.*

:dScomplete **Corollary 3.14.** *For any $\delta > 0$, the space*

$$\left\{\varphi \in \mathrm{PSH}(X, \theta) : \int_X \theta_\varphi^n \geq \delta\right\}$$

is complete with respect to d_S.

Proof. Take a Cauchy sequence $\varphi_j \in \mathrm{PSH}(X, \theta)$ $(j \geq 1)$ with $\int_X \theta_{\varphi_j}^n \geq \delta$. It suffices to show that each subsequence of φ_j admits a convergent subsequence. In turn, we are free to replace φ_j by a subsequence. By Corollary 3.13, we may therefore assume that we can find an equivalent decreasing Cauchy sequence $(\psi_j)_j$ with $\varphi_j \preceq \psi_j$. It suffices to show that ψ_j converges. But this follows from Lemma 3.11. $\square$

thm:convdS **Theorem 3.15.** *Let $\alpha_1, \ldots, \alpha_n$ be big $(1,1)$-classes on X represented by $\theta_1, \ldots, \theta_n$. Suppose that $(\varphi_j^k)_k$ are sequences in $\mathrm{PSH}(X, \theta_j)$ for $j = 1, \ldots, n$ and $\varphi_1, \ldots, \varphi_n \in \mathrm{PSH}(X, \theta)$. We assume that $\varphi_j^k \xrightarrow{d_S} \varphi_j$ as $k \to \infty$ for each $j = 1, \ldots, n$. Then*

ixedmassds}
$$\lim_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{3.6}$$

Proof. **Step 1**. We reduce to the case where φ_j^k, φ_j all have positive masses and there is a constant $\delta > 0$, such that for all j and k,

$$\int_X \theta_{j,\varphi_j^k}^n > \delta.$$

Take $t \in (0, 1)$. By Lemma 3.7, we have

$$(1-t)\varphi_j^k + tV_{\theta_j} \xrightarrow{d_S} (1-t)\varphi_j + tV_{\theta_j}$$

as $k \to \infty$. Assume that we have proved the special case of the theorem, we have

$$\lim_{k\to\infty} \int_X \theta_{1,(1-t)\varphi_1^k+tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi_n^k+tV_{\theta_n}} = \int_X \theta_{1,(1-t)\varphi_1+tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi_n+tV_{\theta_n}}.$$

From this, (3.6) follows easily.

Step 2. Now we may assume that φ_j^k and φ_j are all of positive mass and are model potentials.

It suffices to prove that any subsequence of $\int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k}$ has a converging subsequence with limit $\int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}$. Thus, by Proposition 3.12, we may assume that for each fixed i, φ_i^k is either increasing or decreasing. We may assume that for $i \leq i_0$, the sequence is decreasing and for $i > i_0$, the sequence is increasing.

Recall that in (3.6) the $\geq$ inequality always holds by Theorem 2.1, it suffices to prove

{eq:limsup}
$$\varlimsup_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{3.7}$$

By Theorem 2.1 in order to prove (3.7), we may assume that for $j > i_0$, the sequences φ_j^k are constant. Thus, we are reduced to the case where for all i, φ_i^k are decreasing.

In this case, for each i we may take an increasing sequence $b_i^k > 1$, tending to ∞, such that

$$(b_i^k)^n \int_X \theta^n_{i,\varphi_i} \geq \left((b_i^k)^n - 1\right) \int_X \theta^n_{i,\varphi_i^k}.$$

Let ψ_i^k be the maximal θ_i-psh function such that

$$(b_i^k)^{-1}\psi_i^k + \left(1-(b_i^k)^{-1}\right)\varphi_i^k \leq \varphi_i\,,$$

whose existence is guaranteed by Lemma 3.8.

Then by Theorem 2.1 again,

$$\prod_{i=1}^n \left(1-(b_i^k)^{-1}\right) \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

Let $k \to \infty$, we conclude (3.7).

□

:dsconvcrit **Corollary 3.16.** *Suppose that $\varphi, \varphi_i \in \mathrm{PSH}(X,\theta)$ $(i \geq 1)$. Then the following are equivalent:*

(1) $\varphi_i \xrightarrow{d_S} \varphi$;

(2) $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$ and

nv_varphii} (3.8)
$$\lim_{i\to\infty} \int_X \theta^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}$$

for each $j = 0, \dots, n$.

The corollary allows us to reduce a number of convergence problems related to d_S to the case $\varphi_i \geq \varphi$, which is much easier to handle by Lemma 3.3. This is the most handy way of establishing d_S-convergence in practice.

Proof. (1) implies (2): $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$ follows from Lemma 3.4. While (3.8) follows from Theorem 3.15.

(2) implies (1): By (3.1), we need to show that for each $j = 0, \dots, n$, we have

$$2\int_X \theta^j_{\varphi_i\vee\varphi} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} \to 0.$$

This follows from Theorem 3.15 and (3.8). □

changetheta **Corollary 3.17.** *Let $\varphi_k, \varphi \in \mathrm{PSH}(X,\theta)$ $(k \geq 1)$ and ω be a Kähler form on X. Then the following are equivalent:*

(1) $\varphi_k \xrightarrow{d_{S,\theta}} \varphi$;

(2) $\varphi_k \xrightarrow{d_{S,\theta+\omega}} \varphi$.

From now on, we mostly write d_S instead of $d_{S,\theta}$. This corollary shows that the d_S-convergence is the correct notion even at 0 mass.

Proof. (1) implies (2): It suffices to show that for each $j = 0, \dots, n$, we have

$$2\int_X (\theta+\omega)^j_{\varphi_k\vee\varphi} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X (\theta+\omega)^j_{\varphi_k} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X (\theta+\omega)^j_{\varphi} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} \to 0$$

as $k \to \infty$. Note that this quantity is a linear combination of terms of the following form:

$$2\int_X \theta^r_{\varphi_k\vee\varphi} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X \theta^r_{\varphi_k} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X \theta^r_{\varphi} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}}\,,$$

where $r = 0, \dots, j$. By Theorem 3.15, it suffices to show that $\varphi \vee \varphi_k \xrightarrow{d_S} \varphi$. But this follows from Corollary 3.16

(2) implies (1): From the direction we already proved, for each $C \geq 1$, we have that

$$\varphi_k \xrightarrow{d_{S,\theta+C\omega}} \varphi.$$

By Theorem 3.15, it follows that

$$\lim_{k\to\infty}\int_X (\theta+C\omega)^j_{\varphi_k}\wedge\theta^{n-j}_{V_\theta}=\int_X(\theta+C\omega)^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}$$

for all $j=0,\dots,n$. It follows that

$$\lim_{k\to\infty}\int_X \theta^j_{\varphi_k}\wedge\theta^{n-j}_{V_\theta}=\int_X\theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}. \tag{3.9}$$

By Corollary 3.16, it remains to show that $\varphi_k\vee\varphi\xrightarrow{d_{S,\theta}}\varphi$. By Corollary 3.16 again, we know that $\varphi_k\vee\varphi\xrightarrow{d_{S,\theta+\omega}}\varphi$. So it suffices to apply (3.9) to $\varphi_k\vee\varphi$ instead of φ_k and we conclude by Lemma 3.3. □

Theorem 3.18. *The map* $\mathrm{PSH}(X,\theta)_{>0}\to\mathrm{PSH}(X,\theta)_{>0}$ *given by* $\varphi\mapsto P[\varphi]_{\mathcal{I}}$ *is continuous with respect to* d_S.

Here $\mathrm{PSH}(X,\theta)_{>0}$ denotes the subset of $\mathrm{PSH}(X,\theta)$ consisting of φ with $\int_X\theta^n_\varphi>0$.

Proof. Let $\varphi_i,\varphi\in\mathrm{PSH}(X,\theta)_{>0}$, $\varphi_i\xrightarrow{d_S}\varphi$. We want to show that

$$P[\varphi_i]_{\mathcal{I}}\xrightarrow{d_S}P[\varphi]_{\mathcal{I}}. \tag{3.10}$$

We may assume that the φ_i's and φ are all model potentials. By Proposition 3.12, we may assume that $(\varphi_i)_i$ is either increasing or decreasing. Both cases follow from [DX22, Lemma 2.21] and Lemma 3.11. □

Lemma 3.19. *Let* $\varphi,\varphi_j,\psi_j,\eta_j\in\mathrm{PSH}(X,\theta)$ $(j\geq 1)$. *Assume that*

(1) $\psi_j\leq\varphi_j\leq\eta_j$;

(2) $\eta_j\xrightarrow{d_S}\varphi$, $\psi_j\xrightarrow{d_S}\varphi$.

Then $\varphi_j\xrightarrow{d_S}\varphi$.

Proof. Observe that for each $k=0,\dots,n$, we have

$$\int_X\theta^k_{\psi_j}\wedge\theta^{n-k}_{V_\theta}\leq\int_X\theta^k_{\varphi_j}\wedge\theta^{n-k}_{V_\theta}\leq\int_X\theta^k_{\eta_j}\wedge\theta^{n-k}_{V_\theta}$$

for all $j\geq 1$. By Theorem 3.15, the limit of the both ends are $\int_X\theta^k_\varphi\wedge\theta^{n-k}_{V_\theta}$ as $j\to\infty$. It follows that

$$\lim_{j\to\infty}\int_X\theta^k_{\varphi_j}\wedge\theta^{n-k}_{V_\theta}=\int_X\theta^k_\varphi\wedge\theta^{n-k}_{V_\theta}. \tag{3.11}$$

By Corollary 3.16, it remains to prove that $\varphi_j\vee\varphi\xrightarrow{d_S}\varphi$. By Corollary 3.16, up to replacing ψ_j (resp. φ_j, η_j) by $\psi_j\vee\varphi$ (resp. $\varphi_j\vee\varphi$, $\eta_j\vee\varphi$), we may assume from the beginning that $\psi_j,\varphi_j,\eta_j\geq\varphi$. Now $\varphi_j\xrightarrow{d_S}\varphi$ by (3.11) and Lemma 3.3. □

At this point, we can recall another fundamental property about d_S: the non-Archimedean data are continuous with respect d_S.

Theorem 3.20. *Let* $\varphi_j,\varphi\in\mathrm{PSH}(X,\theta)$ $(j\geq 1)$. *Assume that* $\varphi_j\xrightarrow{d_S}\varphi$, *then for any prime divisor* E *over* X, *we have*

$$\lim_{j\to\infty}\nu(\varphi_j,E)=\nu(\varphi,E). \tag{3.12}$$

Proof. By Corollary 3.17, we may assume that the masses of φ_j and of φ are bounded from below by a positive constant.

By Theorem 3.18, we may assume that φ_i and φ are both $\mathcal{I}$-model. When proving (3.12), we are free to pass to subsequences. By Proposition 3.12, up to passing to a subsequence, we may assume that $\varphi_i\to\varphi$ almost everywhere.

By Hartogs lemma, there is a null set $Z\subseteq X$ such that on $X\setminus Z$, we have

$$\sup_{j\geq i}{}^*\varphi_j=\sup_{j\geq i}\varphi_j$$

for all $i \geq 1$. It follows that

$$\varphi = \inf_{i\in\mathbb{N}} \sup_{j\geq i}{}^* \varphi_j$$

on $X \setminus Z$ hence everywhere on X. In fact, we can also assume that

$$\psi_i \coloneqq \sup_{j\geq i}{}^* \varphi_j \xrightarrow{d_S} \varphi$$

as $i \to \infty$ by Proposition 3.12.

It then follows that $P_\theta[\psi_i] \to \varphi$ everywhere. By Lemma 3.21, we then have

$$\lim_{i\to\infty} \nu(\psi_i, E) = \nu(\varphi, E).$$

By [DX22, Lemma 3.4], we have

$$\nu(\varphi, E) = \varliminf_{i\to\infty} \nu(\varphi_i, E).$$

Together with the upper semi-continuity of Lelong numbers, we find

$$\nu(\varphi, E) = \lim_{i\to\infty} \nu(\varphi_i, E).$$

□

eldeclelong **Lemma 3.21.** *Let $\varphi_j \in \mathrm{PSH}(X,\theta)$ ($j \geq 1$) be a decreasing sequence of model potentials. Let φ be the limit of φ_j. Assume that φ has positive mass. Then for any prime divisor E over X,*

$$\lim_{j\to\infty} \nu(\varphi_j, E) = \nu(\varphi, E).$$

Proof. Since $\varphi \coloneqq \lim_j \varphi_j$ and the φ_j's are model, we obtain that $\int_Y \theta_\varphi^n = \lim_j \int_Y \theta_{\varphi_j}^n > 0$ by Lemma 3.11. By Lemma 3.8, for any $\epsilon \in (0,1)$, for j big enough there exists $\psi_j \in \mathrm{PSH}(X,\theta)$ such that $(1-\epsilon)\varphi_j + \epsilon\psi_j \leq \varphi$. This implies that for j big enough we have

$$(1-\epsilon)\nu(\varphi_j, E) + \epsilon\nu(\psi_j, E) \geq \nu(\varphi, E) \geq \nu(\varphi_j, E).$$

However $\nu(\chi, E)$ is uniformly bounded (by some Seshadri type constant) for any $\chi \in \mathrm{PSH}(X,\theta)$ and E fixed. So letting $\epsilon \searrow 0$ we conclude. □

lor_dS_conv **Lemma 3.22.** *Let $\varphi_i, \varphi, \psi_j, \psi \in \mathrm{PSH}(X,\theta)$. Assume that $\varphi_i \xrightarrow{d_S} \varphi$, $\psi_i \xrightarrow{d_S} \psi$. Then*

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \varphi \vee \psi.$$

Proof. We compute

$$\begin{aligned} d_S(\varphi_i \vee \psi_i, \varphi \vee \psi) \leq& d_S(\varphi_i \vee \psi_i, \varphi_i \vee \psi) + d_S(\varphi_i \vee \psi, \varphi \vee \psi) \\ \leq& C_n \left(d_S(\psi_i, \psi) + d_S(\varphi_i, \varphi) \right), \end{aligned}$$

where the second inequality follows from Lemma 3.6. The right-hand side converges to 0 by our hypothesis. □

Sadditivity **Theorem 3.23.** *Let α_1, α_2 be big classes represented by θ_1, θ_2. Suppose that $\varphi, \varphi_i \in \mathrm{PSH}(X,\theta_1)$, $\psi, \psi_i \in \mathrm{PSH}(X,\theta_2)$. Consider the following three conditions:*

(1) $\varphi_i \xrightarrow{d_S} \varphi$;
(2) $\psi_i \xrightarrow{d_S} \psi$;
(3) $\varphi_i + \psi_i \xrightarrow{d_S} \varphi + \psi$.

Then any two of these conditions imply the third.

Proof. By Corollary 3.17, we may assume that θ_1, θ_2 are both Kähler forms. We denote them by ω_1, ω_2 instead.

(1) + (2) implies (3): Let $\omega = \theta_1 + \theta_2$. It suffices to show that for each $r = 0, \ldots, n$,

$$2\int_X \omega^r_{(\varphi_j+\psi_j)\vee(\varphi+\psi)} \wedge \omega^{n-r} - \int_X \omega^r_{\varphi_j+\psi_j} \wedge \omega^{n-r} - \int_X \omega^r_{\varphi+\psi} \wedge \omega^{n-r} \to 0.$$

Observe that

$$(\varphi_j + \psi_j) \vee (\varphi + \psi) \leq \varphi_j \vee \varphi + \psi_j \vee \psi.$$

Thus, it suffices to show that

$$2\int_X \omega^r_{\varphi_j\vee\varphi+\psi_j\vee\psi}\wedge\omega-\int_X \omega^r_{\varphi_j+\psi_j}\wedge\omega^{n-r}-\int_X \omega^r_{\varphi+\psi}\wedge\omega^{n-r}\to 0.$$

The left-hand side is a linear combination of

$$2\int_X \omega^a_{1,\varphi_j\vee\varphi}\wedge\omega^{r-a}_{2,\psi_j\vee\psi}\wedge\omega^{n-r}-\int_X \omega^a_{1,\varphi_j}\wedge\omega^{r-a}_{2,\psi_j}\wedge\omega^{n-r}-\int_X \omega^a_{1,\varphi}\wedge\omega^{r-a}_{2,\psi}\wedge\omega^{n-r}$$

with $a=0,\ldots,r$. Observe that $\varphi_j\vee\varphi\xrightarrow{d_S}\varphi$ and $\psi_j\vee\psi\xrightarrow{d_S}\psi$ by Lemma 3.4, each term tends to 0 by Theorem 3.15.

(1)+(3) implies (2): For each $C\geq 1$, from the direction we already proved,

$$C\varphi_i+\psi_i\xrightarrow{d_S}C\varphi+\psi.$$

By Theorem 3.15, for each $j=0,\ldots,n$,

$$\lim_{i\to\infty}\int_X (C\omega_1+\omega_2+\mathrm{dd}^c(C\varphi_i+\psi_i))^j\wedge\omega_2^{n-j}=\int_X (C\omega_1+\omega_2+\mathrm{dd}^c(C\varphi+\psi))^j\wedge\omega_2^{n-j}.$$

It follows that

quant_conv}
$$\lim_{i\to\infty}\int_X \omega^j_{2,\psi_i}\wedge\omega_2^{n-j}=\int_X \omega^j_{2,\psi}\wedge\omega_2^{n-j}. \tag{3.13}$$

Therefore, (2) follows if $\psi_i\geq\psi$ for each i by Lemma 3.3.

Next we prove the general case. By the direction that we already proved, we know that $\varphi_i+\psi\xrightarrow{d_S}\varphi+\psi$. By Lemma 3.22, we have that

$$\varphi_i+\psi_i\vee\psi\xrightarrow{d_S}\varphi+\psi.$$

It follows from the special case above that $\psi_i\vee\psi\xrightarrow{d_S}\psi$. It follows from (3.13) and Corollary 3.16 that (2) holds. □

Finally, let us show that the uniform structure defined by d_S is natural at mass 0.

Cauchy_pert
Lemma 3.24. *Let $\varphi_j,\varphi\in\mathrm{PSH}(X,\theta)$ ($j\geq 1$). Assume that the sequence $(\varphi_j)_j$ is Cauchy with respect to $d_{S,\theta}$, then it is also Cauchy with respect to $d_{S,\theta+\omega}$.*

Proof. Fix $t\in(0,1)$, we claim that $((1-t)\varphi_j+tV_\theta)_j$ is also a Cauchy sequence with respect to $d_{S,\theta}$. To see this, observe that for each $k=0,\ldots,n$,

$$\begin{aligned}&2\int_X \theta^k_{((1-t)\varphi_i+tV_\theta)\vee((1-t)\varphi_j+tV_\theta)}\wedge\theta^{n-k}_{V_\theta}-\int_X \theta^k_{(1-t)\varphi_i+tV_\theta}\wedge\theta^{n-k}_{V_\theta}-\int_X \theta^k_{(1-t)\varphi_j+tV_\theta}\wedge\theta^{n-k}_{V_\theta}\\ =&2\int_X \theta^k_{(1-t)\varphi_i\vee\varphi_j+tV_\theta}\wedge\theta^{n-k}_{V_\theta}-\int_X \theta^k_{(1-t)\varphi_i+tV_\theta}\wedge\theta^{n-k}_{V_\theta}-\int_X \theta^k_{(1-t)\varphi_j+tV_\theta}\wedge\theta^{n-k}_{V_\theta}\\ =&\sum_{a=0}^k\binom{k}{a}\left(2\theta^a_{\varphi_i\vee\varphi_j}\wedge\theta^{n-a}_{V_\theta}-\theta^a_{\varphi_i}\wedge\theta^{n-a}_{V_\theta}-\theta^a_{\varphi_j}\wedge\theta^{n-a}_{V_\theta}\right).\end{aligned}$$

By Corollary 3.14, we can find $\psi_t\in\mathrm{PSH}(X,\theta)$ so that

$$(1-t)\varphi_j+tV_\theta\xrightarrow{d_{S,\theta}}\psi_t.$$

It follows from Corollary 3.17 that

$$(1-t)\varphi_j+tV_\theta\xrightarrow{d_{S,\theta+\omega}}\psi_t.$$

In particular, $((1-t)\varphi_j + tV_\theta)_j$ is also a Cauchy sequence with respect to $d_{S,\theta+\omega}$. But observe that

$$\begin{aligned}
&\sum_{a=0}^{n}\left(2\int_X(\theta+\omega)^a_{((1-t)\varphi_i+tV_\theta)\vee((1-t)\varphi_j+tV_\theta)}\wedge\theta^{n-a}_{V_{\theta+\omega}}-\int_X(\theta+\omega)^a_{(1-t)\varphi_i+tV_\theta}\wedge\theta^{n-a}_{V_{\theta+\omega}}-\right.\\
&\left.\int_X(\theta+\omega)^a_{(1-t)\varphi_j+tV_\theta}\wedge\theta^{n-a}_{V_{\theta+\omega}}\right)\\
=&\sum_{a=0}^{n}\left(2\int_X(\theta+\omega)^a_{(1-t)\varphi_i\vee\varphi_j+tV_\theta}\wedge\theta^{n-a}_{V_{\theta+\omega}}-\int_X(\theta+\omega)^a_{(1-t)\varphi_i+tV_\theta}\wedge\theta^{n-a}_{V_{\theta+\omega}}-\int_X(\theta+\omega)^a_{(1-t)\varphi_j+tV_\theta}\wedge\theta^{n-a}_{V_{\theta+\omega}}\right)\\
\geq&\sum_{a=0}^{n}(1-t)^a\left(2\int_X(\theta+\omega)^{n-a}_{\varphi_i\vee\varphi_j}\wedge\theta^{n-a}_{V_{\theta+\omega}}-\int_X(\theta+\omega)^a_{\varphi_i}\wedge\theta^{n-a}_{V_{\theta+\omega}}-\int_X(\theta+\omega)^a_{\varphi_j}\wedge\theta^{n-a}_{V_{\theta+\omega}}\right).
\end{aligned}$$

It follows that $(\varphi_j)_j$ is also a Cauchy sequence with respect to $d_{S,\theta+\omega}$. □

3.3. Quasi-equisingular approximations.

Definition 3.25. Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ ($j \in \mathbb{Z}_{>0}$). We say φ_j is a *quasi-equisingular approximation* of φ if

(1) φ_j has analytic singularities for each j;
(2) φ_j is decreasing with limit φ;
(3) for each $\lambda' > \lambda > 0$, there is $j > 0$ such that

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi). \tag{3.14}$$

We prove that a general d_S-convergent sequence enjoys a quasi-equisingular property.

Theorem 3.26. *Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ ($j \in \mathbb{Z}_{>0}$). Assume that $\varphi_j \xrightarrow{d_S} \varphi$. Then for each $\lambda' > \lambda > 0$, there is $j_0 > 0$ so that for $j \geq j_0$, (3.14) holds.*

Proof. Fix $\lambda' > \lambda > 0$, we want to find $j_0 > 0$ so that for $j \geq j_0$, (3.14) holds.

Step 1. We first assume that φ has analytic singularities.

Let $\pi : Y \to X$ be a log resolution of φ and let $E_1, \dots, E_N$ be all prime divisors of the singular part of φ on Y. Recall that a local holomorphic function f lies in the right-hand side of (3.14) if and only if

$$\operatorname{ord}_{E_i}(f) > \lambda \operatorname{ord}_{E_i}(\varphi) - A_X(E_i) \tag{3.15}$$

whenever they make sense. Here A_X denotes the log discrepancy. Similarly, f lies in the left-hand side of (3.14) implies that there is $\epsilon > 0$ so that

$$\operatorname{ord}_{E_i}(f) \geq (1+\epsilon)\lambda' \operatorname{ord}_{E_i}(\varphi_j) - A_X(E_i).$$

As Lelong numbers are continuous with respect to d_S by Theorem 3.20, we can find $j_0 > 0$ so that when $j \geq j_0$, $\lambda' \operatorname{ord}_{E_i}(\varphi_j) \geq \lambda \operatorname{ord}_{E_i}(\varphi)$ for all i. In particular, (3.15) follows.

Step 2. We handle the general case.

By Corollary 3.17, we are free to increase θ and assume that θ_φ is a Kähler current.

Take a quasi-equisingular approximation ψ_k of φ. The existence is guaranteed by [DPS01]. Take $\lambda'' \in (\lambda, \lambda')$, then by definition, we can find $k > 0$ so that

$$\mathcal{I}(\lambda''\psi_k) \subseteq \mathcal{I}(\lambda\varphi).$$

Observe that $\varphi_j \vee \psi_k \xrightarrow{d_S} \psi_k$ as $j \to \infty$ by Lemma 3.22. By Step 1, we can find $j_0 > 0$ so that for $j \geq j_0$,

$$\mathcal{I}(\lambda'(\varphi_j \vee \psi_k)) \subseteq \mathcal{I}(\lambda''\psi_k).$$

It follows that for $j \geq j_0$,

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

□

Corollary 3.27. *Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$. Assume that φ_j have analytic singularities, φ_j decreases to φ and $\int_X \theta_\varphi^n > 0$. Then the following are equivalent:*

(1) $\varphi_j \xrightarrow{d_S} P[\varphi]_{\mathcal{I}}$;
(2) φ_j *is a quasi-equisingular approximation of* φ.

Proof. (2) implies (1): This is proved more generally in [DX21; DX22].
(1) implies (2): This follows from Theorem 3.26. □

This corollary shows in particular that being a quasi-equisingular approximation is invariant under blowing-ups with smooth centers, a fact which is not obvious by the very definition.

4. $\mathcal{I}$-GOOD SINGULARITIES

Let X be a connected compact Kähler manifold of dimension n.

4.1. **The closure of analytic singularity types.** Let θ be a smooth real closed $(1,1)$-form on X representing a big class.

pullback_PI **Lemma 4.1.** *Let* $\pi : X' \to X$ *be a bimeromorphic morphism from a connected compact Kähler manifold* Y *and* $\varphi \in \mathrm{PSH}(X,\theta)$. *Then we have*

$$\pi^* P_\theta[\varphi]_{\mathcal{I}} = P_{\pi^*\theta}[\pi^*\varphi]_{\mathcal{I}}.$$

The proof is left to the readers.

lgebraic_PI **Lemma 4.2.** *Suppose that* $\varphi \in \mathrm{PSH}(X,\theta)$ *has analytic singularities, then*

$$\varphi \sim P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}.$$

See [Kim15, Theorem 4.3].

p: PIlimit **Proposition 4.3.** *Let* $\varphi \in \mathrm{PSH}(X,\theta)$. *Assume that* θ_φ *is a Kähler current. Let* $\varphi_j \in \mathrm{PSH}(X,\theta)$ *be a quasi-equisingular approximation of* φ. *Then* $P_\theta[\varphi_j]_{\mathcal{I}} \searrow P_\theta[\varphi]_{\mathcal{I}}$ *as* $j \to \infty$. *In particular,* $\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$.

Proof. The last assertion follows from Lemma 4.2, Lemma 3.11 and the first assertion. It suffices to prove the first assertion.

We may assume that φ is $\mathcal{I}$-model. Let

$$\psi = \lim_{j\to\infty} P_\theta[\varphi_j]_{\mathcal{I}}.$$

Then ψ is $\mathcal{I}$-model and $\varphi \le \psi$. In order to conclude the equality, it suffices to show that for any $t > 0$,

iequalIpsi}
$$\mathcal{I}(t\varphi) = \mathcal{I}(t\psi). \tag{4.1}$$

We fix $t > 0$. By the quasi-equisingular property, for any $\delta > 0$, we can find $k_0 > 0$ so that

$$\mathcal{I}(t\delta\psi) \subseteq \mathcal{I}(t\delta\varphi_{k_0}) \subseteq \mathcal{I}(t\varphi).$$

Letting $\delta \searrow 1$ and using the strong openness, we conclude that

$$\mathcal{I}(t\psi) \subseteq \mathcal{I}(t\varphi).$$

The reverse inclusion is trivial and (4.1) is proved. □

S_A_closure **Theorem 4.4.** [1] *Let* $\varphi \in \mathrm{PSH}(X,\theta)$ *such that* $\int_X \theta_\varphi^n > 0$. *Then the following are equivalent:*

(1) φ *lies in the* d_S*-closure of analytic singularities;*
(2) φ *is* $\mathcal{I}$*-good.*

Proof. By Proposition 3.9, we can find $\psi \in \mathrm{PSH}(X,\theta)$ be such that $\psi \le \varphi$ and $\theta_\psi \ge \omega$ for some Kähler form ω on X. Let

$$\psi_t := (1-t)\psi + t\varphi$$

for $t \in [0,1]$. Then θ_{ψ_t} is a Kähler current for $t \in [0,1)$ and $\psi_t \nearrow \varphi$ a.e. as $t \nearrow 1$.

(2) $\implies$ (1): We may further assume that φ is $\mathcal{I}$-model. It is straightforward to verify that $P_\theta[\psi_t]_{\mathcal{I}} \nearrow P_\theta[\varphi]_{\mathcal{I}} = u$ a.e. as $t \to 1$. It follows that $P_\theta[\psi_t]_{\mathcal{I}} \xrightarrow{d_S} \varphi$. In particular, in order to

[1]There is an obvious typo in [DX21, Theorem 4.5]

prove (1), we may assume furthermore that θ_φ is a Kähler current. In this case, it suffices to apply Proposition 4.3.

(1) $\implies$ (2): Suppose there exists a sequence $\psi_j \in \mathrm{PSH}(X,\theta)$ with analytic singularities such that $\psi_j \xrightarrow{d_S} \varphi$. By Lemma 4.2, we can assume that ψ_j is $\mathcal{I}$-model for each j. In addition, we can assume that φ is model. Since $\int_X \theta_\varphi^n > 0$, after possibly restricting to a subsequence of ψ_j, we can use Proposition 3.12 to conclude existence of an increasing sequence of model potentials $\varphi_j \in \mathrm{PSH}(X,\theta)$ such that $\varphi_j \leq \psi_j$ and $\varphi_j \xrightarrow{d_S} \varphi$. Moreover, we can take

$$\varphi_j := \lim_{k\to\infty} \psi_j \wedge \psi_{j+1} \wedge \cdots \wedge \psi_{j+k}.$$

Since all the ψ_j's are $\mathcal{I}$-model, it is straightforward to verify that the φ_j's are $\mathcal{I}$-model as well. Lastly, since φ is the increasing limit of the φ_j a.e., we conclude that φ is $\mathcal{I}$-model as well. $\square$

The omitted parts should be easy to verify. You can also find the arguments in [DX22].

4.2. **The volumes of Hermitian pseudo-effective line bundles.** let T be an arbitrary holomorphic vector bundle on X, with rank r. Let L be a pseudoeffective line bundle on X. Let h be a smooth Hermitian metric on L such that $\theta := c_1(L,h)$. We fix a Kähler form ω on X such that $\omega - \theta$ is a Kähler form.

Proposition 4.5. *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$. Then*

$$\varlimsup_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \frac{r}{n!}\int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}. \tag{4.2}$$

Proof. We may assume that φ is $\mathcal{I}$-model.

Let $\varphi_j \in \mathrm{PSH}(X,\theta+\epsilon_j\omega)$ be a quasi-equisingular approximation of φ, where $\epsilon_j > 0$ is a decreasing sequence with limit 0. Let $\pi_k : Y_k \to X$ be a resolution of singularities of φ_j.

By [Dem12, Proposition 5.8] and [Bon98, Théorème 2.1][2] applied to $q = 0$ on Y_k, we obtain that

$$\begin{aligned}
\varlimsup_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) &\leq \varlimsup_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi_j)) \\
&= \varlimsup_{k\to\infty} \frac{1}{k^n} h^0(Y, \pi_k^*T \otimes (\pi_k^*L)^k \otimes K_{Y/X} \otimes \mathcal{I}(k\pi_k^*\varphi_j)) \\
&\leq \frac{r}{n!}\int_{Y_k(0)} \pi_k^*\theta^n_{\varphi_j} = \frac{r}{n!}\int_{\pi_k(Y_k(0))} \theta^n_{\varphi_j} \\
&\leq \frac{r}{n!}\int_{\pi_k(Y_k(0))} (\theta+\epsilon_j\omega)^n_{\varphi_j} \leq \frac{r}{n!}\int_X (\theta+\epsilon_j\omega)^n_{\varphi_j},
\end{aligned}$$

where $Y_k(0) \subseteq Y_k$ is the set contained in the smooth locus of the (1,1)-current $\pi_k^*\theta_{\varphi_j}$ where the eigenvalues of $\pi_k^*\theta_{\varphi_j}$ are all positive. Observe that $\lim_{j\to\infty}\int_X (\theta+\epsilon_j\omega)^n_{\varphi_j} = \int_X \theta_\varphi^n$ by the argument of Proposition 4.3. So (4.2) follows. $\square$

Lemma 4.6. *Let $\varphi \in \mathrm{PSH}(X,\theta)$ such that θ_φ is a Kähler current. Take a quasi-equisingular approximation $\varphi_j \in \mathrm{PSH}(X,\theta)$ of φ. Let $\beta \in (0,1)$. Then there exists $k_0 := k_0(u,\beta)$ such that for all $k \geq k_0$ there exists $v_{\beta,k} \in \mathrm{PSH}(X,\theta)$ satisfying the following:*

(1) $P_\theta[\varphi]_{\mathcal{I}} \geq (1-\beta)\varphi_k + \beta v_{\beta,k}$;
(2) $\int_X \theta^n_{v_{\beta,k}} > 0$.

Proof. Due to Proposition 4.3, we have that $\int_X \theta^n_{\varphi_k} \searrow \int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}$. In particular, there exists $k_0 > 0$ such that

$$\frac{1}{\beta^n} < \frac{\int_X \theta^n_{\varphi_k}}{\int_X \theta^n_{\varphi_k} - \int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}} \quad \text{for all } k \geq k_0.$$

[2]There is a subtle point. In Bonavero's thesis, he considered only analytic singularities with smooth remainders. We apply his results to analytic singularities with bounded remainders. This can be justified by first passing to a resolution of the singularity and then regularize the remainder term.

By Lemma 3.8 we obtain that

$$v_{k,\beta} := P\left(\frac{1}{\beta}P_\theta[\varphi]_{\mathcal{I}} - \frac{1-\beta}{\beta}\varphi_k\right) \in \mathrm{PSH}(X,\theta)$$

and

$$P_\theta[\varphi]_{\mathcal{I}} \geq (1-\beta)\varphi_k + \beta v_{\beta,k}.$$

Now we show that $v_{\beta,k}$ has positive mass. Pick $\beta' \in (0,\beta)$ such that

$$\frac{1}{\beta'^n} < \frac{\int_X \theta^n_{\varphi_k}}{\int_X \theta^n_{\varphi_k} - \int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}} \quad \text{for all } k \geq k_0.$$

Then

$$h := P\left(\frac{1}{\beta'}P_\theta[\varphi]_{\mathcal{I}} - \frac{1-\beta'}{\beta'}\varphi_k\right) \in \mathrm{PSH}(X,\theta)$$

is defined as well, and

$$v_{k,\beta} \geq \frac{\beta'}{\beta}h + \frac{\beta-\beta'}{\beta}\varphi_k \in \mathrm{PSH}(X,\theta),$$

implying that

$$\int_X \theta^n_{v_{k,\beta}} \geq \frac{(\beta-\beta')^n}{\beta^n}\int_X \theta^n_{\varphi_k} \geq \frac{(\beta-\beta')^n}{\beta^n}\int_X \theta^n_{\varphi} > 0.$$

□

ound_Kahler **Proposition 4.7.** *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$ with $\theta_\varphi \geq \delta\omega$ for some $\delta > 0$. Then*

$$\varliminf_{j\to\infty} \frac{1}{j^n} h^0(X, T\otimes L^j \otimes \mathcal{I}(j\varphi)) \geq \frac{r}{n!}\int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}.$$

Proof. To start, we fix a number $\beta = p/q \in (0,\min(\delta,1))\cap\mathbb{Q}$. It suffices to show that there is a constant $C>0$, only dependent on r, n and θ, such that

$$\varliminf_{j\to\infty} \frac{1}{j^n} h^0(X, T\otimes L^j \otimes \mathcal{I}(j\varphi)) \geq \frac{r}{n!}\int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}} - C\beta\,.$$

Writing $j = aq + b$ for some $b = 0,\ldots,q-1$, observe that

$$h^0(X, T\otimes L^j \otimes \mathcal{I}(j\varphi)) \geq h^0\left(X, T\otimes L^{b-q}\otimes L^{(a+1)q}\otimes \mathcal{I}((a+1)q\varphi)\right)\,.$$

Absorbing L^{b-q} into T, and noticing that $b-q$ can only take a finite number of values, we find that it suffices to prove the following

proofh0lim}
$$\varliminf_{j\to\infty} \frac{1}{j^n q^n} h^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jq\varphi)) \geq \frac{r}{n!}\int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}} - C\beta\,, \tag{4.3}$$

for arbitrary twisting bundle T.

By Lemma 4.6, there is $k_0 > 0$ depending on β and u, such that for $k \geq k_0$, there exists a potential $v_{\beta,k} \in \mathrm{PSH}(X,\theta)$ of positive mass such that

$$P_\theta[\varphi]_{\mathcal{I}} \geq w_{\beta,k} := (1-\beta)\varphi_k + \beta v_{\beta,k} \quad \text{for all } k \geq k_0.$$

For big enough k_0 we also have $\theta_{\varphi_k} \geq \beta\omega \geq \beta\theta$ for all $k \geq k_0$. In particular, $\varphi_k \in \mathrm{PSH}(X,(1-\beta)\theta)$. We have

$$H^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jq\varphi)) \supseteq H^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jqw_{\beta,k})),$$

hence

ineq_est1}
$$h^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jq\varphi)) \geq h^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jqw_{\beta,k})). \tag{4.4}$$

For each fixed $k > 0$, we can take a resolution of singularities $\pi : Y \to X$, such that $\pi^*\varphi_k$ has analytic singularities. By [Dem12, Proposition 5.8] and the projection formula,

ineq_est2}
$$h^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jqw_{\beta,k})) = h^0(Y, \pi^*T\otimes K_{Y/X}\otimes (\pi^*L)^{jq}\otimes \mathcal{I}(jq\pi^*w_{\beta,k})). \tag{4.5}$$

Since $\int_Y (\pi^*\theta + \mathrm{dd^c}\pi^* v_{\beta,k})^n = \int_X \theta^n_{v_{\beta,k}} > 0$, there exists a non-zero section

$$s_j \in H^0\left(Y, \pi^*L^{\beta jq}\otimes \mathcal{I}(\beta jq\pi^* v_{\beta,k})\right) = H^0\left(Y, \pi^*L^{jp}\otimes \mathcal{I}(jp\pi^* v_{\beta,k})\right)$$

for all j large enough, by Lemma 4.8. Hence applying Lemma 4.9 for

$$T \to \pi^*T \otimes K_{Y/X}, E_1 \to \pi^*L^{q-p}, E_2 \to \pi^*L^p, \chi_1 \to q\pi^*\varphi_k, \chi_2 \to p\pi^*v_{\beta,k}, s_j \to s_j, \epsilon \to \beta,$$

we find

ineq_est3}

$$\begin{aligned} & h^0(Y, \pi^*T \otimes K_{Y/X} \otimes \pi^*L^{jq} \otimes \mathcal{I}(jq\pi^* w_{k,\beta})) \\ (4.6) \qquad =& h^0(Y, \pi^*T \otimes K_{Y/X} \otimes \pi^*L^{(q-p)j} \otimes \pi^*L^{pj} \otimes \mathcal{I}((1-\beta)jq\pi^*\varphi_k + jp\pi^* v_{\beta,k}))) \\ \geq & h^0(Y, \pi^*T \otimes K_{Y/X} \otimes \pi^*L^{(q-p)j} \otimes \mathcal{I}(jq\pi^*\varphi_k)) \end{aligned}$$

for j large enough (depending on k).

Since $\theta_{\varphi_k} > \beta\omega \geq \beta\theta$, we notice that $q\varphi_k \in \mathrm{PSH}(X, \theta(q-p))$. Hence, by [Bon98, Théorème 2.1, Corollaire 2.2] (see also [DX22, Theorem 2.26]), we can write the following estimates.

ineq_est4}

$$\begin{aligned} & \varliminf_{j\to\infty} \frac{1}{j^n q^n} h^0(Y, \pi^*T \otimes K_{Y/X} \otimes \pi^*L^{(1-\beta)qj} \otimes \mathcal{I}(jq\pi^*\varphi_k)) \\ &= \varliminf_{j\to\infty} \frac{1}{j^n q^n} h^0\left(Y, \pi^*T \otimes K_{Y/X} \otimes \pi^*L^{(q-p)j} \otimes \mathcal{I}(jq\pi^*\varphi_k)\right) \\ (4.7) \qquad &= \frac{r}{q^n n!} \int_Y ((q-p)\pi^*\theta + q\mathrm{dd^c}\pi^*\varphi_k)^n \\ &= \frac{r}{n!} \int_X ((1-\beta)\theta + \mathrm{dd^c}\varphi_k)^n \\ &\geq \frac{r}{n!} \int_X \theta_{\varphi_k}^n - C\beta\,, \end{aligned}$$

where $C > 0$ depends only on r, n, θ. Putting together (4.4),(4.5), (4.6) and (4.7) we obtain

$$\varliminf_{j\to\infty} \frac{1}{j^n} h^0(X, T \otimes L^j \otimes \mathcal{I}(j\varphi)) \geq \frac{r}{n!} \int_X \theta_{\varphi_k}^n - C\beta\,.$$

Letting $k \to \infty$ and applying Proposition 4.3, we conclude (4.3). □

masssection

Lemma 4.8. *Suppose that $L \to X$ is a big line bundle, with smooth Hermitian metric h. Let $\theta = c_1(L, h)$. Let $v \in \mathrm{PSH}(X, \theta)$ with $\int_X \theta_v^n > 0$. Then for m big enough there exists $s \in H^0(X, L^m \otimes \mathcal{I}(mv))$ non-vanishing.*

Proof. By Proposition 3.9 there exists $w \in \mathrm{PSH}(X, \theta)$ such that $w \leq v$ and $\theta_w \geq \delta\omega$. By [Dem12, Theorem 13.21], for m big enough, there exists $s \in H^0(X, L^m \otimes \mathcal{I}(mw))$ non-zero. Since $w \leq v$, we get that $s \in H^0(X, L^m \otimes \mathcal{I}(mv))$. □

m:injective

Lemma 4.9. *Suppose that E_1, E_2, T are vector bundles over a connected complex manifold Y, with rank $E_2 = 1$, and χ_1, χ_2 are quasi-psh functions on Y, with χ_1 having normal crossing divisorial singularity type. Suppose that there exists a non-zero section $s_j \in H^0(Y, E_2^{\otimes j} \otimes \mathcal{I}(j\chi_2))$, for all j big enough. Then for any $\epsilon \in (0, 1)$ the map $w \mapsto w \otimes s_j$ between the vector spaces*

$$H^0(Y, T \otimes E_1^{\otimes j} \otimes \mathcal{I}(j\chi_1)) \to H^0\left(Y, T \otimes E_1^{\otimes j} \otimes E_2^{\otimes j} \otimes \mathcal{I}\left(j(1-\epsilon)\chi_1 + j\chi_2\right)\right)$$

is well-defined and injective, for all j big enough.

Proof. Suppose that the singularity type of χ_1 is given by the effective normal crossing $\mathbb{R}$-divisor $\sum_j \alpha_j D_j$ with $\alpha_j > 0$. By [Dem12, Remark 5.9] we have that

$$\mathcal{I}(j\chi_1) = \mathcal{O}_Y(-\sum_m \lfloor \alpha_m j \rfloor D_j)\,.$$

We obtain that $we^{-j(1-\epsilon)\chi_1}$ is bounded for any $w \in H^0(Y, T \otimes E_1^j \otimes \mathcal{I}(j\chi_1))$ and j big enough. Since $s_j \in H^0(Y, E_2^{\otimes j} \otimes \mathcal{I}(j\chi_2))$, we obtain that

$$w \otimes s_j \in H^0\left(Y, T \otimes E_1^{\otimes j} \otimes E_2^{\otimes j} \otimes \mathcal{I}(j(1-\epsilon)\chi_1 + j\chi_2)\right)\,.$$

Injectivity of $w \mapsto w \otimes s_j$ follows from the identity theorem of holomorphic functions. □

thm:Tsuji **Theorem 4.10.** *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$. Then*

q:Tsuji_eq}
$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) = \frac{r}{n!}\int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}. \tag{4.8}$$

Proof. We may assume that φ is $\mathcal{I}$-model. Proposition 4.5 implies (4.8) for $\int_X \theta_\varphi^n = 0$, so we can also assume that $\int_X \theta_\varphi^n > 0$. In particular, L is a big line bundle and X is projective.

By Proposition 3.9, there exists $\psi \le \varphi$ such that θ_ψ is a Kähler current. Let $\psi_t \coloneqq (1-t)\psi + t\varphi$. Then θ_{ψ_t} is a Kähler current for $t \in [0,1)$, so we can apply Proposition 4.7 to obtain that

$$\varliminf_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \ge \varliminf_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\psi_t)) \ge \frac{r}{n!}\int_X \theta^n_{P_\theta[\psi_t]_{\mathcal{I}}}.$$

DX22
Letting $t \to 0$ and using [DX22, Lemma 2.21(iii)], we obtain that

$$\varliminf_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \ge \frac{r}{n!}\int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}.$$

The reverse inequality, follows from Proposition 4.5. □

Corollary 4.11. *Let $\varphi \in \mathrm{PSH}(X,\theta)$ such that $\int_X \theta_\varphi^n > 0$. Then the following are equivalent:*

(1) φ lies in the d_S-closure of analytic singularities;
(2) φ is $\mathcal{I}$-good;
(3) The following equality holds:

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\varphi)) = \frac{1}{n!}\int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}.$$

Proof. This follows from Theorem 4.4 and Theorem 4.10. □

References

BEGZ10 [BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Monge-Ampère equations in big cohomology classes. *Acta Math.* 205.2 (2010), pp. 199–262. URL: https://doi.org/10.1007/s11511-010-0054-7.

Bon98 [Bon98] L. Bonavero. Inégalités de morse holomorphes singulières. *J. Geom. Anal.* 8.3 (1998), pp. 409–425. URL: https://doi.org/10.1007/BF02921793.

Bo02 [Bou02] S. Boucksom. On the volume of a line bundle. *Internat. J. Math.* 13.10 (2002), pp. 1043–1063. URL: https://doi.org/10.1142/S0129167X02001575.

DDNL18big [DDNL18a] T. Darvas, E. Di Nezza, and C. H. Lu. L^1 metric geometry of big cohomology classes. *Ann. Inst. Fourier (Grenoble)* 68.7 (2018), pp. 3053–3086. URL: http://aif.cedram.org/item?id=AIF_2018__68_7_3053_0.

DDNL18mono [DDNL18b] T. Darvas, E. Di Nezza, and C. H. Lu. Monotonicity of nonpluripolar products and complex Monge-Ampère equations with prescribed singularity. *Anal. PDE* 11.8 (2018), pp. 2049–2087. URL: https://doi.org/10.2140/apde.2018.11.2049.

L18fullmass [DDNL18c] T. Darvas, E. Di Nezza, and C. H. Lu. On the singularity type of full mass currents in big cohomology classes. *Compos. Math.* 154.2 (2018), pp. 380–409. URL: https://doi.org/10.1112/S0010437X1700759X.

DDNLmetric [DDNL21] T. Darvas, E. Di Nezza, and H.-C. Lu. The metric geometry of singularity types. *J. Reine Angew. Math.* 771 (2021), pp. 137–170. URL: https://doi.org/10.1515/crelle-2020-0019.

Dem12 [Dem12] J.-P. Demailly. Analytic methods in algebraic geometry. Vol. 1. Surveys of Modern Mathematics. International Press, Somerville, MA; Higher Education Press, Beijing, 2012, pp. viii+231.

DL20 [DL20] T. Darvas and C. H. Lu. Geodesic stability, the space of rays and uniform convexity in Mabuchi geometry. *Geom. Topol.* 24.4 (2020), pp. 1907–1967. URL: https://doi.org/10.2140/gt.2020.24.1907.

DPS01 [DPS01] J.-P. Demailly, T. Peternell, and M. Schneider. Pseudo-effective line bundles on compact Kähler manifolds. *Internat. J. Math.* 12.6 (2001), pp. 689–741. URL: https://doi.org/10.1142/S0129167X01000861.

DX21 [DX21] T. Darvas and M. Xia. The volume of pseudoeffective line bundles and partial equilibrium. *Geometry & Topology* (2021). arXiv: 2112.03827 [math.DG].

DX22 [DX22] T. Darvas and M. Xia. The closures of test configurations and algebraic singularity types. *Adv. Math.* 397 (2022), Paper No. 108198, 56. URL: https://doi.org/10.1016/j.aim.2022.108198.

Kim15 [Kim15] D. Kim. Equivalence of plurisubharmonic singularities and Siu-type metrics. *Monatsh. Math.* 178.1 (2015), pp. 85–95. URL: https://doi.org/10.1007/s00605-015-0742-7.

Lu21 [Lu21] C. H. Lu. Comparison of Monge-Ampère capacities. *Ann. Polon. Math.* 126.1 (2021), pp. 31–53. URL: https://doi.org/10.4064/ap200513-2-1.

WN19 [WN19] D. Witt Nyström. Monotonicity of non-pluripolar Monge-Ampère masses. *Indiana Univ. Math. J.* 68.2 (2019), pp. 579–591. URL: https://doi.org/10.1512/iumj.2019.68.7630.

Xia19b [Xia19] M. Xia. Integration by parts formula for non-pluripolar product. 2019. arXiv: 1907.06359 [math.DG].

Xia19 [Xia21] M. Xia. Mabuchi geometry of big cohomology classes with prescribed singularities. 2021. arXiv: 1907.07234 [math.DG].

Xia22 [Xia22] M. Xia. Non-pluripolar products on vector bundles and Chern–Weil formulae on mixed Shimura varieties. 2022. arXiv: 2210.15342 [math.AG].

Mingchen Xia, Department of Mathematics, Institut de Mathématiques de Jussieu-Paris Rive Gauche

Email address, mingchen@imj-prg.fr

Homepage, https://mingchenxia.github.io/home/.