LECTURES ON PLURIPOTENTIAL THEORY — LECTURE 2. THE NON-PLURIPOLAR THEORY

Contents

1. A quick recap

We defined the Monge–Ampère measure of a locally bounded psh function on a complex manifold X last time. Given locally bounded $\phi_1, \ldots, \phi_p \in \mathrm{PSH}(X)$, we defined

$$\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p = \mathrm{dd^c}\left(\phi_1\, \mathrm{dd^c}\phi_2 \cdots \wedge \mathrm{dd^c}\phi_p\right)$$

by induction on p. This approach fails if ϕ_1 is not locally bounded, as you cannot multiply the currents ϕ_1 and $\mathrm{dd^c}\phi_2 \cdots \wedge \mathrm{dd^c}\phi_p$.

Recall the following two facts mentioned last time.

Proposition 1.1. *Let $\phi_1, \ldots, \phi_p \in \mathrm{PSH}(X)$ be bounded. Then for any smooth $(n-p, n-p)$-form α and any pluripolar set $E \subseteq X$, we have*

$$\int_E (\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p) \wedge \alpha = 0.$$

In other words, the Bedford–Taylor product does not put mass on pluripolar sets.

Theorem 1.2. *Let $\phi_1, \ldots, \phi_p, \varphi_1, \ldots, \varphi_p \in \mathrm{PSH}(X)$ be bounded. Consider two psh functions $\psi_1, \psi_2 \in \mathrm{PSH}(X)$ (not necessarily bounded). Define $E = \{\psi_1 > \psi_2\}$. Assume that $\phi_i|_E = \varphi_i|_E$ for all $i = 1, \ldots, p$. Then*

$$\mathbb{1}_E \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p = \mathbb{1}_E \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p.$$

This result is usually expressed by saying that the Bedford–Taylor product is local in the pluri-fine topology.

The goal of this lecture is to extend the Monge–Ampère product to unbounded psh functions. There are several different ways to do so. We will focus only on the non-pluripolar product. We will explain later why it is the most natural one.

2. The non-pluripolar product

2.1. The local definition.

We fix a domain $\Omega \subseteq \mathbb{C}^N$, $1 \leq p \leq N$ and functions $\phi_1, \ldots, \phi_p \in \mathrm{PSH}(\Omega)$. We want to define a (p,p)-current $\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p$ such that

(1) the product coincides with the Bedford–Taylor product when the ϕ_i's are locally bounded;
(2) the product is non-pluripolar, namely it does not put mass on pluripolar sets;
(3) the product if local in the pluri-fine topology.

Lemma 2.1. *These conditions uniquely determine* $\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p$.

Sketch of the proof. The natural idea is to approach the ϕ_i's by bounded psh functions. For any $k \in \mathbb{R}$, consider the canonical approximations[1]:

$$\phi_i^k := \phi_i \vee k \in \mathrm{PSH}(\Omega).$$

Set

{eq:Ek}

$$E_k = \{x \in \Omega : \phi_1 > k, \dots, \phi_p > k\}. \tag{2.1}$$

This set is not open, but pluri-fine open. From Condition (3), we know that

$$\mathbb{1}_{E_k} \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p = \mathbb{1}_{E_k} \mathrm{dd^c}(\phi_1 \vee k) \wedge \cdots \wedge \mathrm{dd^c}(\phi_p \vee k).$$

From Condition (1), the right-hand side is uniquely determined. Hence so is the left-hand side. Letting $k \to -\infty$, we find that

$$\mathbb{1}_E \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p,$$

where

$$E = \{x \in \Omega : \phi_1 > -\infty, \dots, \phi_p > -\infty\}.$$

The complement of E is pluripolar, so the measure $\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p$ can only be the zero-extension of $\mathbb{1}_E \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p$ to Ω. □

The proof of the uniqueness also suggests how to define the product. We tend to define $\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p$ as the zero-extension of the weak limit of the currents

{eq:EkMA}

$$\mathbb{1}_{E_k} \mathrm{dd^c}(\phi_1 \vee k) \wedge \cdots \wedge \mathrm{dd^c}(\phi_p \vee k) \tag{2.2}$$

with E_k defined as in (2.1). But we have an immediate issue. The sequence (2.2) does not necessarily converge, nor is the zero-extension defined. We need a finite mass assumption to resolve the problem.

Definition 2.2. We say the product $\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p$ is *well-defined* if for any compact set $K \subseteq \Omega$, we have

$$\sup_{k \in \mathbb{R}} \int_{K \cap E_k} \mathbb{1}_{E_k} \mathrm{dd^c}(\phi_1 \vee k) \wedge \cdots \wedge \mathrm{dd^c}(\phi_p \vee k) \wedge \omega^{N-k} < \infty,$$

where ω is the standard Kähler form on $\mathbb{C}^N$.

In this case, we define the *non-pluripolar product* $\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_p$ as the weak limit of the measures in (2.2).

Exercise 2.3. Explain why the weak limit exists.

The fact that the non-pluripolar product is NOT defined for all tuples $(\phi_i)_i$ is annoying. But as we will see in a second, the problem goes away immediately when we work on compact Kähler manifolds.

Remark 2.4. The notion of non-pluripolar product is due to Bedford–Taylor [BT87]. It was subsequently studied and generalized by Guedj–Zeriahi [GZ07] and Boucksom–Eyssidieux–Guedj–Zeriahi [BEGZ10].

2.2. **The global setting.** Let X be a compact Kähler manifold of pure dimension n. We fix closed real smooth $(1,1)$-forms $\theta_1, \dots, \theta_p$ on X for some $1 \leq p \leq n$. Consider $\varphi_i \in \mathrm{PSH}(X, \theta_i)$ for $i = 1, \dots, p$.

Exercise 2.5. Explain how the definition in the local setting leads to a definition of $\theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{p,\varphi_p}$.

The most important fact is that

Proposition 2.6. *The product $\theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{p,\varphi_p}$ is always well-defined.*

[1] Recall that $\vee$ means the maximum

Proof. Fix a Kähler form ω on X. Write

$$\theta_{i,\varphi_i} = \theta_i + C\omega + \mathrm{dd^c}\varphi_i - C\omega$$

for some large enough $C > 0$. Using the multi-linearity of the non-pluripolar product (prove it!), we may assume that θ_i is in fact a Kähler form.

Now take an open set $U \subseteq X$ on which $\theta_i = \mathrm{dd^c} g_i$ for all i, where g_i are smooth negative psh functions on U.

For each $k \in \mathbb{R}$,

$$\{g_j + \varphi_j > k\} \subseteq \{\varphi_j > k\},$$

so for each compact subset $K \subseteq U$,

$$\begin{aligned}
&\int_K \mathbb{1}_{\cap_{j=1}^p \{g_j+\varphi_j>k\}} \bigwedge_{j=1}^p \mathrm{dd^c}\left((g_j+\varphi_j)\vee k\right)\wedge\omega^{n-p}\\
=&\int_K \mathbb{1}_{\cap_{j=1}^p \{g_j+\varphi_j>k\}} \bigwedge_{j=1}^p (\theta_j + \mathrm{dd^c}(\varphi_j\vee k))\wedge\omega^{n-p}\\
\leq&\int_X \bigwedge_{j=1}^p (\theta_j + \mathrm{dd^c}(\varphi_j\vee k))\wedge\omega^{n-p}\\
=&\int_X \bigwedge_{j=1}^p \theta_j\wedge\omega^{n-p}.
\end{aligned}$$

This concludes the proof. ☐

Exercise 2.7. Expand the last sentence of the proof.

From now on, for simplicity, we will work only in the global setting.

Exercise 2.8. Show that the product $\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{p,\varphi_p}$ is independent of the choice of θ_i and φ_i. It depends only on the current $\theta_i + \mathrm{dd^c}\varphi_i$.

The proceeding exercise therefore allows us to define $T_1\wedge\cdots\wedge T_p$ for closed positive $(1,1)$-currents $T_1,\dots,T_p$ on X.

prop:npp2 **Proposition 2.9.** *Let $T_1,\dots,T_p$ be closed positive $(1,1)$-currents on X. Then we have the following properties:*

(1) The product $T_1\wedge\cdots\wedge T_p$ is local in plurifine topology;
(2) The product $T_1\wedge\cdots\wedge T_p$ puts not mass on pluripolar sets.
(3) The current $T_1\wedge\cdots\wedge T_p$ is a closed positive (p,p)-current
(4) The product $T_1\wedge\cdots\wedge T_p$ is symmetric.
(5) The product is multi-linear: if T_1' is another closed positive $(1,1)$-current on X, then

$$(T_1+T_1')\wedge T_2\wedge\cdots\wedge T_p = T_1\wedge T_2\wedge\cdots\wedge T_p + T_1'\wedge T_2\wedge\cdots\wedge T_p.$$

The proofs are left to the readers. Part (3) is probably very challenging, you can find the proof in [BEGZ10, Theorem 1.8].

Remark 2.10. A remark for curious readers. In the definition of Bedford–Taylor product from the previous lecture, we allowed actually defined products of the form

$$\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{p,\varphi_p}\wedge T$$

for some closed positive (q,q)-current T. In the non-pluripolar theory, we have only defined the case without T. We do so only in order to avoid heavy technical burdens and to have better properties. The general theory with T is studied by Vu [Vu21]. Note that the general theory is not multi-linear.

3. Properties of the non-pluripolar product

Let X be a compact Kähler manifold of pure dimension n. We use $\theta, \theta_0, \theta_1, \ldots, \theta_n$ to denote various closed real $(1,1)$-forms on X.

We collect the most important properties of the non-pluripolar product, which are relevant to our next lecture. The results here are due to Boucksom–Eyssidieux–Guedj–Zeriahi, Darvas–Di Nezza–Lu, Witt Nyström and me. We refer to [DDNL23] for the proofs and the credits of each result.

Theorem 3.1 (Semi-continuity theorem)**.** *Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X, \theta_j)$ ($k \in \mathbb{Z}_{>0}$, $j = 1, \ldots, n$). Let $\chi \geq 0$ be a bounded continuous function on X. Assume that for any $j = 1, \ldots, n$, $i = 1, \ldots, m$, as $k \to \infty$, the sequence φ_j^k converges monotonically*[2] *to φ_j. Then*

$$\varliminf_{k\to\infty} \int_X \chi\, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_X \chi\, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{3.1}$$

If we compare this result with the corresponding result in the Bedford–Taylor theory, this result is much weaker: the Bedford–Taylor product is continuous along monotone sequences, while the non-pluripolar theory is *lower semi-continuous*.

Exercise 3.2. Find an example on $\mathbb{P}^1$ showing that the equality fails in general.

On the positive side, we have

Exercise 3.3. Assume in addition that

$$\varlimsup_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{3.2}$$

Then

$$\theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \rightharpoonup \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

Theorem 3.4 (Monotonicity theorem)**.** *Let $\varphi_j, \psi_j \in \mathrm{PSH}(X, \theta_j)$ for $j = 1, \ldots, n$. Assume that $\varphi_j \succeq \psi_j$ for every j, then*

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \theta_{n,\varphi_n} \geq \int_X \theta_{1,\psi_1} \wedge \cdots \theta_{n,\psi_n}.$$

Here $\varphi \succeq \psi$ means $\varphi \geq \psi - C$ for some $C \in \mathbb{R}$. In this case, we say φ is *less singular* than ψ or ψ is *more singular* than φ. When $\varphi \preceq \psi$ and $\psi \preceq \varphi$, we say φ and ψ have the same *singularity type*.

The non-pluripolar mass gets smaller when the potential gets more singular. In other words, the singularities account for the loss of non-pluripolar masses. The work of Darvas and myself that I will present in the next lecture can be seen as a rather precise version of this idea.

Assume that $\mathrm{PSH}(X, \theta)$ is not empty (we say the class $\{\theta\}$ is *pseudo-effective* in this case). If we set

$$V_\theta = \sup{}^*\{\varphi \in \mathrm{PSH}(X, \theta) : \varphi \leq 0\}, \tag{3.3}$$

we have

$$\int_X \theta_{V_\theta}^n \geq \int_X \theta_\varphi^n \tag{3.4}$$

for any $\varphi \in \mathrm{PSH}(X, \theta)$. The left-hand side is called the *volume* of the cohomology class of θ.

Exercise 3.5. Provide the details and explain why the usc envelope is not necessary in (3.3).

We say $\varphi \in \mathrm{PSH}(X, \theta)$ has *full mass* if equality holds in (3.4). The class of these functions is denoted by $\mathcal{E}(X, \theta)$. Potentials in this class can be considered as only mildly singular.

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[2] If $\varphi_j, \varphi \in \mathrm{PSH}(X, \theta)$ for $j = 1, 2, \ldots$, we say φ_j converges monotonically to φ if either

(1) φ_j decreases to φ everywhere or
(2) φ_j decreases to φ almost everywhere (then necessarily $\varphi = \sup^* \varphi_j$).

Exercise 3.6. Show that if a potential φ and V_θ have the same singularity type, then φ has full mass. In this case, we say φ has the *minimal singularity.*

thm:comp2 **Theorem 3.7** (Comparison principle)**.** *Assume that* $\varphi, \psi \in \mathcal{E}(X, \theta)$*, then*

$$\int_{\{\varphi<\psi\}} \theta_\psi^n \leq \int_{\{\varphi<\psi\}} \theta_\varphi^n.$$

:IBPgeneral **Theorem 3.8** (Integration by parts)**.** *Let* $\gamma_j \in \mathrm{PSH}(X, \theta_j)$ $(j = 2, \dots, n)$*. Let* $\varphi_1, \varphi_2 \in \mathrm{PSH}(X, \theta_0)$*,* $\psi_1, \psi_2 \in \mathrm{PSH}(X, \theta_1)$*. Let* $u = \varphi_1 - \varphi_2$*,* $v = \psi_1 - \psi_2$*. Assume that* φ_1 *and* φ_2 *have the same singularity type;* ψ_1 *and* ψ_2 *have the same singularity type. Then*

{eq:ibp4} (3.5)
$$\int_X u\, \mathrm{dd^c} v \wedge \theta_{2,\gamma_2} \wedge \dots \wedge \dots \wedge \theta_{n,\gamma_n} = \int_X v\, \mathrm{dd^c} u \wedge \theta_{2,\gamma_2} \wedge \dots \wedge \dots \wedge \theta_{n,\gamma_n}. \tag{3.5}$$

Here the notations in (3.5) are formal, but it should not be too difficult for the readers to figure out the precise meaning.

4. The envelope operators

Let X be a connected compact Kähler manifold of dimension n and θ be a closed real $(1,1)$-form on X. We assume that $\mathrm{PSH}(X, \theta)$ is non-empty and the volume of $\{\theta\}$ is positive. We say the class $\{\theta\}$ is *big* in this case.

This part concerns the notion of rooftop operators. As we have recalled last time, the minimum of two θ-psh functions is not θ-psh in general. In a number of situations, it is desirable to be able to take the minimum. This leads to the following definition:

Definition 4.1. Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$, their rooftop operator is defined as

$$\varphi \wedge \psi \coloneqq \sup{}^*\{\eta \in \mathrm{PSH}(X, \theta) : \eta \leq \varphi, \eta \leq \psi\}.$$

When the set on the right-hand side is empty, by convention, we set $\varphi \wedge \psi = -\infty$.

Exercise 4.2. Prove that $\wedge$ is associative, commutative and idempotent.

In order to imagine how the rooftop operator looks like, it is helpful to keep the following result in mind:

thm:ma **Theorem 4.3.** *Let* $\varphi, \psi \in \mathrm{PSH}(X, \theta)$*. Assume that* $\varphi \wedge \psi \neq -\infty$*, then*

eq:rooftop} (4.1)
$$\theta^n_{\varphi\wedge\psi} \leq \mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\theta^n_\varphi + \mathbb{1}_{\{\varphi\wedge\psi=\psi\}}\theta^n_\psi. \tag{4.1}$$

In particular, $\theta^n_{\varphi\wedge\psi}$ is supported on the set where $\varphi \wedge \psi$ is either equal to φ or to ψ.

It is a tricky question to determine when the rooftop operator of two potentials if not identically $-\infty$. We mention a one sufficient condition. We define the space of finite energy potentials as

{eq:E1} (4.2)
$$\mathcal{E}^1(X, \theta) \coloneqq \left\{\varphi \in \mathcal{E}(X, \theta) : \int_X |V_\theta - \varphi|\, \theta^n_\varphi < \infty\right\}. \tag{4.2}$$

Exercise 4.4. Explain why the integral in definition makes sense.

Theorem 4.5. *Suppose that* $\varphi, \psi \in \mathcal{E}^1(X, \theta)$*, then* $\varphi \wedge \psi \in \mathcal{E}^1(X, \theta)$*. In particular,* $\varphi \wedge \psi \not\equiv -\infty$*.*

Our interest in the rooftop operator comes from the fact that it can be used to define a projection operator:

def:Penv **Definition 4.6.** Let $\varphi \in \mathrm{PSH}(X, \theta)$. Assume that $\int_X \theta^n_\varphi > 0$. We define

$$\begin{aligned} P_\theta[\varphi] &= \sup_{C\in\mathbb{R}}{}^* V_\theta \wedge (\varphi + C) \\ &= \sup\left\{\psi \in \mathrm{PSH}(X, \theta) : \psi \leq 0, \int_X \theta^n_\varphi = \int_X \theta^n_\psi, \varphi \leq \psi + C \text{ for some } C \in \mathbb{R}\right\}; \end{aligned}$$

The equality is a very non-trivial result. This envelope is very pathological when φ has 0-mass. Instead of listing various possible definitions, we just leave it undefined.

Exercise 4.7. Show that $P_\theta[\bullet]$ is an idempotent operator. It depends only on the singularity type of the θ-psh function.

Roughly speaking, the P-operator gives an envelope of a θ-singularity. It is the finest information we can see from the comparison of singularity types and non-pluripolar masses.

Using the P-operator, the singular θ-psh functions can be grouped into several groups, each group having the same P-envelope (called the *prescribed singularity*). In particular, one of these groups is given by $\mathcal{E}(X,\theta)$.

The general idea of the series of works of Darvas–Di Nezza–Lu is that such a group is not very different from $\mathcal{E}(X,\theta)$.

Observe that the P-projection preserves the multiplier ideal sheaves:

op:misPproj **Proposition 4.8.** *Let $\varphi \in \mathrm{PSH}(X,\theta)$ and $\int_X \theta_\varphi^n > 0$. Then for any $k \in \mathbb{R}_{>0}$,*

$$\mathcal{I}(kP_\theta[\varphi]) = \mathcal{I}(k\varphi).$$

Proof. By the first equation in (4.6) and Guan–Zhou's strong openness theorem, it suffices to show that

$$\mathcal{I}(k(V_\theta \wedge (\varphi + C))) = \mathcal{I}(k\varphi).$$

But $V_\theta \wedge (\varphi + C)$ and φ have the same singularity type (why?), so we win. □

It is therefore natural to introduce a different envelope operator:

Definition 4.9. Let $\varphi \in \mathrm{PSH}(X,\theta)$, we set

$$P_\theta[\varphi]_{\mathcal{I}} = \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \mathcal{I}(k\varphi) \supseteq \mathcal{I}(k\psi) \text{ for all } k \in \mathbb{Z}_{>0}\right\}.$$

As we already recalled, the information of all multiplier ideal sheaves is equivalent to the information of all generic Lelong numbers, so this envelope can be equivalently characterized as follows

$$P_\theta[\varphi]_{\mathcal{I}} = \sup\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \nu(\varphi, E) \leq \nu(\psi, E) \text{ for all prime divisors } E \text{ over } X\}.$$

We can reformulate Proposition 4.8 by saying if φ has positive mass,

$$P_\theta[\varphi] \leq P_\theta[\varphi]_{\mathcal{I}}.$$

5. The space of finite energy potentials

Let X be a connected compact Kähler manifold of dimension n and θ be a closed real $(1,1)$-form θ on X representing a big cohomology class. We have introduced the space $\mathcal{E}^1(X,\theta)$ in (4.2). The goal of this section is to study the geometry of $\mathcal{E}^1(X,\theta)$.

We will need the Monge–Ampère energy functional $E : \mathcal{E}^1(X,\theta) \to \mathbb{R}$ defined as follows:

$$E(\varphi) \coloneqq \frac{1}{n+1}\sum_{j=0}^{n}\int_X (\varphi - V_\theta)\,\theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}.$$

The difference $\varphi - V_\theta$ is only defined outside the pluripolar set $\{V_\theta = -\infty\}$. The non-pluripolar product $\theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}$ does not put mass on pluripolar sets, so the integral is still defined.

Exercise 5.1. Show that $E(\varphi)$ is increasing in φ and $E(V_\theta) = 0$.

Exercise 5.2. Show that if $\psi \in \mathcal{E}^1(X,\theta)$,

$$E(\psi) - E(\varphi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X (\psi - \varphi)\,\theta_\psi^j \wedge \theta_\varphi^{n-j}.$$

Definition 5.3. We consider the following metric on $\mathcal{E}^1(X,\theta)$:

$$d_1(\varphi,\psi) \coloneqq d_1(\varphi \wedge \psi, \varphi) + d_1(\varphi \wedge \psi, \psi) = E(\varphi) + E(\psi) - 2E(\varphi \wedge \psi).$$

Theorem 5.4. *The space $(\mathcal{E}^1(X,\theta), d_1)$ is a complete metric space.*

You could find more details in [DDNL18a]. A vast extension can be found in [Xia23]. The latter can be very challenging to read if you are not very familiar with this field.

It turns out that the metric geodesics are not unique in $(\mathcal{E}^1(X,\theta),d_1)$. However, there is one special notion of geodesics that is closely related to what we will do in the next lecture.

Let us fix $\varphi_0,\varphi_1\in\mathcal{E}^1(X,\theta)$. A *subgeodesic* from φ_0 to φ_1 is a curve $(\varphi_t)_{t\in(0,1)}$ in $\mathcal{E}^1(X,\theta)$ such that

(1) if we define

$$\Phi: X\times\{z\in\mathbb{C}:\mathrm{e}^{-1}<|z|<1\}\to[-\infty,\infty),\quad (x,z)\mapsto\varphi_{-\log|z|}(x),$$

then Φ is $p_1^*\theta$-psh, where $p_1: X\times\{z\in\mathbb{C}:\mathrm{e}^{-1}<|z|<1\}\to X$ is the natural projection;

(2) When $t\to 0+$ (resp. to $1-$), φ_t converges to φ_0 (resp. φ_1) with respect to L^1.

The maximal subgeodesic from φ_0 to φ_1 is called the *geodesic* (φ_t) from φ_0 to φ_1. The geodesic always exists and $\varphi_t\in\mathcal{E}^1(X,\theta)$ for all $t\in[0,1]$. The construction/definition of (φ_t) is usually known as the *Perron–Bremermann envelope*. We refer to [DDNL18b] for the details.

By abuse of language, we say that $(\varphi_t)_{t\in[0,1]}$ (with a closed interval instead of an open interval) is the geodesic from φ_0 to φ_1. More generally, given $t_0\leq t_1$ in $\mathbb{R}$, we say a curve $(\varphi_t)_{t\in[t_0,t_1]}$ in $\mathcal{E}^1(X,\theta)$ is a geodesic from φ_{t_0} to φ_{t_1} if after a linear rescaling from $[t_0,t_1]$ to $[0,1]$, it becomes a geodesic. One can show that E is linear along a geodesic. In fact, by a simple perturbation argument, one can reduce this to [DDNL18b, Theorem 3.12].

Remark 5.5. When θ is a Kähler form and φ_0, φ_1 are smooth, the geodesic (φ_t) is the same as the usual geodesic as studied by X. Chen. We refer to [Blo13] for the details.

The notion of geodesics naturally gives us a notion of geodesic rays:

Definition 5.6. A *geodesic ray* is a curve $\ell=(\ell_t)_{t\in[0,\infty)}$ in $\mathcal{E}^1(X,\theta)$ such that for any $0\leq t_1<t_2$, the restriction $(\ell_t)_{t\in[t_1,t_2]}$ is a geodesic from ℓ_{t_1} to ℓ_{t_2}.

The space of geodesic rays ℓ with $\ell_0=V_\theta$ is denoted by $\mathcal{R}^1(X,\theta)$.

The assumption $\ell_0=V_\theta$ is not very restrictive. In fact, given any other $\varphi\in\mathcal{E}^1(X,\theta)$, we can always find a unique geodesic ray ℓ' with $\ell'_0=\varphi$ such that $d_1(\ell_t,\ell'_t)$ is bounded. So if we are only interested in the asymptotic behaviour of a geodesic ray, we do not lose any information. We refer to [DL20] for the details.

Next we recall the metric d_1 on $\mathcal{R}^1(X,\theta)$. Given $\ell,\ell'\in\mathcal{R}^1(X,\theta)$, one can show as in [DL20] that $d_1(\ell_t,\ell'_t)$ is a convex function in $t\in[0,\infty)$. It follows that

$$d_1(\ell,\ell')\coloneqq\lim_{t\to\infty}\frac{1}{t}d_1(\ell_t,\ell'_t)$$

exists. It is not hard to show that d_1 is indeed a metric on $\mathcal{R}^1(X,\theta)$. In fact, it is a complete metric. We refer to [DL20; DDNL21] for the details.

Similarly, one can introduce $\mathbf{E}:\mathcal{R}^1(X,\theta)\to\mathbb{R}$ as

$$\mathbf{E}(\ell)=\lim_{t\to\infty}\frac{1}{t}E(\ell_t).$$

As we recalled above, the function $E(\ell_t)$ is linear in t, so the limit $\mathbf{E}(\ell)$ is nothing but the slope of this linear function. When $\ell,\ell'\in\mathcal{R}^1(X,\theta)$, $\ell\leq\ell'$, using the definition of d_1, we have

$$d_1(\ell,\ell')=\mathbf{E}(\ell')-\mathbf{E}(\ell). \tag{5.1}$$

Example 5.7. *Given $\varphi\in\mathrm{PSH}(X,\theta)$, we construct a geodesic ray $\ell^\varphi\in\mathcal{R}^1(X,\theta)$. For each $C>0$, let $(\ell_t^{\varphi,C})_{t\in[0,C]}$ be the geodesic from V_θ to $(V_\theta-C)\vee\varphi$. For each $t\geq 0$, it is not hard to see that $\ell_t^{\varphi,C}$ is increasing in $C\in[t,\infty)$. We let*

$$\ell_t^\varphi\coloneqq\sup_{C\geq t}{}^*\,\ell_t^{\varphi,C}.$$

One can show that $\ell^\varphi \in \mathcal{R}^1(X,\theta)$. A simple computation shows that

{eq:Elphi}
$$\mathbf{E}(\ell^\varphi) = \frac{1}{n+1}\left(\sum_{j=0}^{n}\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - V\right). \tag{5.2}$$

See [DDNL21, Theorem 3.1].

Next we recall that $\vee$ operator at the level of geodesic rays. Given $\ell, \ell' \in \mathcal{R}^1(X,\theta)$. We define $\ell \vee \ell'$ as the minimal ray $\mathcal{R}^1(X,\theta)$ lying above both ℓ and ℓ'. In fact, it is easy to construct such a ray: for each $t>0$, let $(\ell''^t_s)_{s\in[0,t]}$ be the geodesic from V_θ to $\ell_t \vee \ell'_t$. It is easy to see that for each fixed $s \geq 0$, ℓ''^t_s is increasing in $t \in [s,\infty)$. Let $(\ell \vee \ell')_s = \sup^*_{t\geq s} \ell''^t_s$. Then we get a geodesic ray $\ell \vee \ell'$. It is clear that this ray is minimal among all rays dominating ℓ and ℓ'. By construction, we have

$$E(\ell\vee\ell')_s = \lim_{t\to\infty} E(\ell''^t_s) = \lim_{t\to\infty}\frac{s}{t}E(\ell_t\vee\ell'_t).$$

In particular,

{eq:Elor}
$$\mathbf{E}(\ell\vee\ell') = \lim_{t\to\infty}\frac{1}{t}E(\ell_t\vee\ell'_t). \tag{5.3}$$

a:d1rayineq **Lemma 5.8.** *For any $\ell, \ell' \in \mathcal{R}^1(X,\theta)$, we have*

:d1maxineq}
$$d_1(\ell,\ell') \leq d_1(\ell,\ell\vee\ell') + d_1(\ell',\ell\vee\ell') \leq C_n d_1(\ell,\ell'), \tag{5.4}$$

where $C_n = 3(n+1)2^{n+2}$.

Proof. The first inequality is trivial. As for the second, we estimate

$$\begin{aligned} d_1(\ell,\ell\vee\ell') =& \mathbf{E}(\ell\vee\ell') - \mathbf{E}(\ell) \\ =& \lim_{t\to\infty}\frac{1}{t}E(\ell_t\vee\ell'_t) - \mathbf{E}(\ell) \\ =& \lim_{t\to\infty}\frac{1}{t}d_1(\ell_t\vee\ell'_t,\ell_t). \end{aligned}$$

where on the second line, we used (5.3), the third line follows from (5.1). In all, we find

$$d_1(\ell,\ell\vee\ell') + d_1(\ell',\ell\vee\ell') \leq \lim_{t\to\infty}\frac{1}{t}\left(d_1(\ell_t\vee\ell'_t,\ell_t) + d_1(\ell_t\vee\ell'_t,\ell'_t)\right).$$

By [DDNL18a, Theorem 3.7],

$$d_1(\ell_t\vee\ell'_t,\ell_t) + d_1(\ell_t\vee\ell'_t,\ell'_t) \leq 3(n+1)2^{n+2}d_1(\ell_t,\ell'_t).$$

Now (5.4) follows. ∎

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