LECTURES ON PLURIPOTENTIAL THEORY — LECTURE 3. d_S-PSEUDOMETRIC AND THE VOLUME OF PSEUDO-EFFECTIVE LINE BUNDLES

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1. The d_S-pseudometric

Given a connected compact Kähler manifold X, a closed real $(1,1)$-form θ on X representing a big class, we defined the space $\mathcal{E}^1(X,\theta)$ of potentials with finite energy and the space $\mathcal{R}^1(X,\theta)$ of geodesic rays in $\mathcal{E}^1(X,\theta)$. We introduced a metric d_1 on $\mathcal{R}^1(X,\theta)$ as the slope at infinity of the metric d_1 on $\mathcal{E}^1(X,\theta)$.

We will need the following example a lot.

x:rayasspsh **Example 1.1.** *Given $\varphi \in \mathrm{PSH}(X,\theta)$, we construct a geodesic ray $\ell^\varphi \in \mathcal{R}^1(X,\theta)$. For each $C>0$, let $(\ell_t^{\varphi,C})_{t\in[0,C]}$ be the geodesic from V_θ to $(V_\theta - C)\vee\varphi$. For each $t\geq 0$, it is not hard to see that $\ell_t^{\varphi,C}$ is increasing in $C\in[t,\infty)$. We let*

$$\ell_t^\varphi := \sup_{C\geq t}{}^* \ell_t^{\varphi,C}.$$

One can show that $\ell^\varphi \in \mathcal{R}^1(X,\theta)$. A simple computation shows that

{eq:Elphi}
$$\mathbf{E}(\ell^\varphi) = \frac{1}{n+1}\left(\sum_{j=0}^{n}\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_{V_\theta}^n\right). \tag{1.1}$$

The point here is that we have an embedding from the space $\mathrm{PSH}(X,\theta)$ to $\mathcal{R}^1(X,\theta)$. The latter space admits a metric, so by transport of the structure, we get a pseudo-metric on $\mathrm{PSH}(X,\theta)$:

Definition 1.2. For $\varphi,\psi\in\mathrm{PSH}(X,\theta)$, we define

$$d_S(\varphi,\psi) := d_1(\ell^\varphi,\ell^\psi).$$

When necessary, we also write $d_{S,\theta}$ instead. We do not get a metric in general because the map $\mathrm{PSH}(X,\theta)\to\mathcal{R}^1(X,\theta)$ is not injective. For example, show that if φ and ψ have the same singularity type, then $\ell^\varphi=\ell^\psi$.

It turns out that this metric has a number of natural properties as we will explain below.

The first question is to understand how degenerate the pseudometric d_S is. Before doing so, we introduce a few equivalence relations, which will be of use later on as well.

ef:singcomp **Definition 1.3.** Let φ,ψ be qpsh functions on X, we say

(1) φ is *more singular* than ψ and write $\varphi\preceq\psi$ if there is $C\in\mathbb{R}$ such that

$$\varphi\leq\psi+C;$$

(2) φ is *P-more singular* than ψ and write $\varphi\preceq_P\psi$ if for some Kähler form ω such that $\varphi,\psi\in\mathrm{PSH}(X,\omega)_{>0}$, we have

$$P_\omega[\varphi]\leq P_\omega[\psi];$$

(3) φ is *$\mathcal{I}$-more singular* than ψ and write $\varphi \preceq_{\mathcal{I}} \psi$ if for some Kähler form ω such that $\varphi, \psi \in \mathrm{PSH}(X,\omega)$, we have

$$P_\omega[\varphi]_{\mathcal{I}} \leq P_\omega[\psi]_{\mathcal{I}}.$$

All three relations define partial orders on $\mathrm{QPSH}(X)$. We denote the corresponding equivalence relation by $\sim$, $\sim_P$ and $\sim_{\mathcal{I}}$ respectively.

Exercise 1.4. Show that these notions are all independent of the auxiliary choices of ω.

This exercise is not very easy, you need to know some tricks. You can find the detailed proofs in [Xia23]. Please contact me if you need a copy.

Proposition 1.5. *For $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, the following are equivalent:*

(1) $\varphi \sim_P \psi$;
(2) $d_S(\varphi,\psi) = 0$.

Originally this result was stated in [DDNL21] using the unnatural C-operator. This reformulation is due to [Xia23].

We derive a few elementary properties from the definition.

Lemma 1.6 ([DDNL21, Lemma 3.4])**.** *Suppose that $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ and $\varphi \preceq_P \psi$, then*

$$d_S(\varphi,\psi) = \frac{1}{n+1}\sum_{j=0}^{n}\left(\int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}\right).$$

Proof. This follows trivially from (1.1). ☐

In particular, for a monotone sequence, the d_S-convergence means the convergence of mixed non-pluripolar masses. This gives an intuitive idea about what d_S-convergence means.

Exercise 1.7. Suppose that φ_i is an increasing sequence in $\mathrm{PSH}(X,\theta)$ converging a.e. to $\varphi \in \mathrm{PSH}(X,\theta)$. Show that φ_i converges to φ with respect to d_S. You might need to have a look at the lecture notes from the last lecture.

Explain what goes wrong for a decreasing sequence.

Lemma 1.8. *For any $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we have*

$$d_S(\varphi,\psi) \leq \sum_{j=0}^{n}\left(2\int_X \theta_{\varphi\vee\psi}^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}\right) \leq C_n d_S(\varphi,\psi), \tag{1.2}$$

where $C_n = 3(n+1)2^{n+2}$.

From this lemma, we find that the d_S-convergence is characterized by numerical conditions of non-pluripolar masses. The criterion here is still way too complicated for applications, we will see a better criterion later on.

The pseudo-metric d_S has almost the best properties that one can dream of.

Theorem 1.9. *For any $\delta > 0$, the space*

$$\left\{\varphi \in \mathrm{PSH}(X,\theta) : \int_X \theta_\varphi^n \geq \delta\right\}$$

is d_S-complete.

Theorem 1.10. *Let $\alpha_1,\ldots,\alpha_n$ be big $(1,1)$-classes on X represented by $\theta_1,\ldots,\theta_n$. Suppose that $(\varphi_j^k)_k$ are sequences in $\mathrm{PSH}(X,\theta_j)$ for $j=1,\ldots,n$ and $\varphi_1,\ldots,\varphi_n \in \mathrm{PSH}(X,\theta)$. We assume that $\varphi_j^k \xrightarrow{d_S} \varphi_j$ as $k \to \infty$ for each $j=1,\ldots,n$. Then*

$$\lim_{k\to\infty}\int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{1.3}$$

Theorem 1.11. *Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ $(j \geq 1)$. Assume that $\varphi_j \xrightarrow{d_S} \varphi$, then for any prime divisor E over X, we have*

$$\lim_{j\to\infty}\nu(\varphi_j,E) = \nu(\varphi,E). \tag{1.4}$$

In general, the Lelong numbers are only usc with respect to the usual L^1-convergence.

Corollary 1.12. *Let $\varphi_j, \varphi \in \mathrm{PSH}(X, \theta)$ ($j \in \mathbb{Z}_{>0}$). Assume that $\varphi_j \xrightarrow{d_S} \varphi$. Then for each $\lambda' > \lambda > 0$, there is $j_0 > 0$ so that for $j \geq j_0$,*

$$\mathcal{I}(\lambda' \varphi_j) \subseteq \mathcal{I}(\lambda \varphi). \tag{1.5}$$

The first three theorems are well-documented in the literature. See [DDNL21; Xia21; Xia22]. The last corollary is proved in [Xia23]. You could also try do deduce it from Theorem 1.11. This can be challenging!

The proofs of these theorems are too complicated to be presented in this short course. We mention a key ingredient in each of these proofs.

Proposition 1.13. *Let $\varphi_j, \varphi \in \mathrm{PSH}(X, \theta)$ ($j \geq 1$), $\varphi_j \xrightarrow{d_S} \varphi$. Assume that there is $\delta > 0$ such that*

$$\int_X \theta^n_{\varphi_j} \geq \delta, \quad \int_X \theta^n_{\varphi} \geq \delta$$

for all j and $P_\theta[\varphi_j] = \varphi_j$, $P_\theta[\varphi] = \varphi$ for all j. Then up to replacing $(\varphi_j)_j$ by a subsequence, there is a decreasing sequence $\psi_j \in \mathrm{PSH}(X, \theta)$ and an increasing sequence $\eta_j \in \mathrm{PSH}(X, \theta)$ such that [1]

(1) As $j \to \infty$

$$d_S(\varphi, \psi_j) \to 0, \quad d_S(\varphi, \eta_j) \to 0;$$

(2) $\psi_j \geq \varphi_j \geq \eta_j$ for all j.

In other words, the behaviour of a general d_S-convergent sequence is dominated by the behaviours of monotone sequences! Usually in order to prove a general theorem about d_S-convergence, it suffices to prove it for monotone sequences. This is the common strategy for proving these results.

We sketch the proof of Theorem 1.10 as an example.

Sketch of the proof. **Step 1**. We reduce to the case where φ_j^k, φ_j all have positive masses and there is a constant $\delta > 0$, such that for all j and k,

$$\int_X \theta^n_{j, \varphi_j^k} > \delta.$$

Take $t \in (0, 1)$. Try to prove by yourself that

$$(1-t)\varphi_j^k + tV_{\theta_j} \xrightarrow{d_S} (1-t)\varphi_j + tV_{\theta_j}$$

as $k \to \infty$. Assume that we have proved the special case of the theorem, we have

$$\lim_{k\to\infty} \int_X \theta_{1,(1-t)\varphi_1^k + tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi_n^k + tV_{\theta_n}} = \int_X \theta_{1,(1-t)\varphi_1 + tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi_n + tV_{\theta_n}}.$$

From this, (1.3) follows easily.

Step 2. Now we may assume that φ_j^k and φ_j are all of positive mass and are model potentials.

It suffices to prove that any subsequence of $\int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k}$ has a converging subsequence with limit $\int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}$. Thus, by Proposition 1.13, we may assume that for each fixed i, φ_i^k is either increasing or decreasing. We may assume that for $i \leq i_0$, the sequence is decreasing and for $i > i_0$, the sequence is increasing.

Recall that in (1.3) the $\geq$ inequality always holds by the monotonicity theorem from the last time, it suffices to prove

$$\varlimsup_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{1.6}$$

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[1] In fact, we will take

$$\eta_j = \varphi_j \wedge \varphi_{j+1} \wedge \cdots$$

and

$$\psi_j = \sup_{k \geq j}{}^* \varphi_k.$$

By the monotonicity theorem again in order to prove (1.6), we may assume that for $j > i_0$, the sequences φ_j^k are constant. Thus, we are reduced to the case where for all i, φ_i^k are decreasing.

In this case, for each i we may take an increasing sequence $b_i^k > 1$, tending to ∞, such that

$$(b_i^k)^n \int_X \theta_{i,\varphi_i}^n \geq \left((b_i^k)^n - 1\right) \int_X \theta_{i,\varphi_i^k}^n.$$

Let ψ_i^k be the maximal θ_i-psh function such that

$$(b_i^k)^{-1}\psi_i^k + \left(1 - (b_i^k)^{-1}\right)\varphi_i^k \leq \varphi_i\,,$$

whose existence is guaranteed by Lemma 1.14.

Then by the monotonicity theorem again,

$$\prod_{i=1}^n \left(1 - (b_i^k)^{-1}\right) \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

Let $k \to \infty$, we conclude (1.6).

□

We have used the existence of an extraordinary envelope, which looks like a miracle to me. This envelope plays a key role in reducing problems with general positive currents to problems with Kähler currents.

thoenvelope **Lemma 1.14** ([DDNLmetric DDNL21, Lemma 4.3])**.** *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, $\varphi \preceq \psi$ and $\int_X \theta_\varphi^n > 0$. Then for any*

$$a \in \left(1, \left(\frac{\int_X \theta_\psi^n}{\int_X \theta_\psi^n - \int_X \theta_\varphi^n}\right)^{1/n}\right),$$

there is $\eta \in \mathrm{PSH}(X,\theta)$ such that

$$a^{-1}\eta + (1-a^{-1})\psi \leq \varphi.$$

The fraction is understood as ∞ if $\int_X \theta_\psi^n = \int_X \theta_\varphi^n$.

We write $P(a\varphi + (1-a)\psi) \in \mathrm{PSH}(X,\theta)$ for the regularized supremum of all such η's. In fact, observe that $\psi \geq \varphi - C$, so η is uniformly bounded from above. It follows that $P(a\varphi + (1-a)\psi) \in \mathrm{PSH}(X,\theta)$. On the other hand, by Hartogs lemma,

$$a^{-1}P(a\varphi + (1-a)\psi) + (1-a^{-1})\psi \leq \varphi$$

holds outside a pluripolar set, hence everywhere.

We remind the readers that in [DDNLmetric DDNL21, Lemma 4.3], the notation $P(a\varphi + (1-a)\psi)$ is used without rigorous justification. The above justification is necessarily as $a\varphi + (1-a)\psi$ is not everywhere defined.

Note that Theorem 1.10 shows that d_S-convergence is preserved by a great number of natural operations in pluripotential theory and it is not pathological at the mass 0. We record the following consequences:

:dsconvcrit **Corollary 1.15.** *Suppose that $\varphi, \varphi_i \in \mathrm{PSH}(X,\theta)$ $(i \geq 1)$. Then the following are equivalent:*

(1) $\varphi_i \xrightarrow{d_S} \varphi$;

(2) $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$ *and*

nv_varphii} (1.7)
$$\lim_{i\to\infty} \int_X \theta_{\varphi_i}^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}$$

for each $j = 0, \ldots, n$.

The corollary allows us to reduce a number of convergence problems related to d_S to the case $\varphi_i \geq \varphi$, which is much easier to handle by Lemma 1.6. This is the most handy way of establishing d_S-convergence in practice.

Proof. (1) implies (2): $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$ follows from Lemma 1.8. While (1.7) follows from Theorem 1.10.

(2) implies (1): By (1.2), we need to show that for each $j = 0, \dots, n$, we have

$$2\int_X \theta^j_{\varphi_i \vee \varphi} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} \to 0.$$

This follows from Theorem 1.10 and (1.7). ∎

changetheta **Corollary 1.16.** *Let $\varphi_k, \varphi \in \mathrm{PSH}(X,\theta)$ $(k \geq 1)$ and ω be a Kähler form on X. Then the following are equivalent:*

(1) $\varphi_k \xrightarrow{d_{S,\theta}} \varphi$;
(2) $\varphi_k \xrightarrow{d_{S,\theta+\omega}} \varphi$.

Proof. (1) implies (2): It suffices to show that for each $j = 0, \dots, n$, we have

$$2\int_X (\theta+\omega)^j_{\varphi_k \vee \varphi} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X (\theta+\omega)^j_{\varphi_k} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X (\theta+\omega)^j_{\varphi} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} \to 0$$

as $k \to \infty$. Note that this quantity is a linear combination of terms of the following form:

$$2\int_X \theta^r_{\varphi_k \vee \varphi} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X \theta^r_{\varphi_k} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X \theta^r_{\varphi} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}},$$

where $r = 0, \dots, j$. By Theorem 1.10, it suffices to show that $\varphi \vee \varphi_k \xrightarrow{d_S} \varphi$. But this follows from Corollary 1.15

(2) implies (1): From the direction we already proved, for each $C \geq 1$, we have that

$$\varphi_k \xrightarrow{d_{S,\theta+C\omega}} \varphi.$$

By Theorem 1.10, it follows that

$$\lim_{k\to\infty} \int_X (\theta + C\omega)^j_{\varphi_k} \wedge \theta^{n-j}_{V_\theta} = \int_X (\theta+C\omega)^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}$$

for all $j = 0, \dots, n$. It follows that

mass_limit} (1.8) $$\lim_{k\to\infty} \int_X \theta^j_{\varphi_k} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}.$$

By Corollary 1.15, it remains to show that $\varphi_k \vee \varphi \xrightarrow{d_{S,\theta}} \varphi$. By Corollary 1.15 again, we know that $\varphi_k \vee \varphi \xrightarrow{d_{S,\theta+\omega}} \varphi$. So it suffices to apply (1.8) to $\varphi_k \vee \varphi$ instead of φ_k and we conclude by Lemma 1.6. ∎

There are a few general results about d_S-convergence in [Xia22]. A number of other properties of d_S-convergence will appear in [Xia23].

2. $\mathcal{I}$-good singularities

Take a connected compact Kähler manifold X and a closed real $(1,1)$-form θ on X representing a big class.

We have seen that for $\varphi \in \mathrm{PSH}(X,\theta)$, we always have $d_S(\varphi, P_\theta[\varphi]) = 0$. We have a different envelope $P_\theta[\varphi]_{\mathcal{I}}$. What can we say about $d_S(\varphi, P_\theta[\varphi]_{\mathcal{I}})$? It turns out that this is not 0 in general. But it is not easy to write down even a single example! The following example is due to Berman–Boucksom–Jonsson [BBJ21].

ex:BBJ **Example 2.1.** *Take $X = \mathbb{P}^1$, $\theta = \omega$ is the Fubini–Study metric. It is possible to construct a polar Cantor set $K \subseteq \mathbb{P}^1$. The set K carries an atom-free probability measure μ. Write $\mu = \omega + \mathrm{dd^c}\phi$. Then ϕ has no non-zero Lelong numbers. It follows that $P_\omega[\phi]_{\mathcal{I}} = 0$.*

But $\phi \notin \mathcal{E}(X,\omega)$ as it puts mass on the polar set K, so $P_\omega[\phi] \neq 0$. It follows that

$$d_S(\phi, P_\omega[\phi]_{\mathcal{I}}) \neq 0.$$

This motivates the following definition:

Definition 2.2. We say a potential $\varphi \in \mathrm{PSH}(X,\theta)$ is *$\mathcal{I}$-good* if $\int_X \theta_\varphi^n > 0$ and

$$d_S(\varphi, P_\theta[\varphi]_{\mathcal{I}}) = 0. \tag{2.1}$$

Note that (2.1) can also be written as

$$P_\theta[\varphi]_{\mathcal{I}} = P_\theta[\varphi].$$

As a first example, please do the following exercise. The relevant notions will be recalled in a second.

Exercise 2.3. Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$ has analytic singularities and has positive mass, then φ is $\mathcal{I}$-good.

Definition 2.4. We say $\varphi \in \mathrm{PSH}(X,\theta)$ has *analytic singularities* if for each $x \in X$, we can find an open neighbourhood U of x such that $\varphi|_U$ has the form:

$$c \log\left(|f_1|^2 + \cdots + |f_N|^2\right) + R,$$

where $f_1, \ldots, f_N$ are holomorphic functions on U, $c \in \mathbb{Q}_{>0}$ and R is a bounded function on U. We also say the current θ_φ has analytic singularities.

Remark 2.5. We cannot take $c \in \mathbb{R}_{>0}$ if we want to guarantee that the maximum of two potentials with analytic singularities has analytic singularities. We do not take R to be smooth in general, as this condition depends heavily on the choice of the generators $f_1, \ldots, f_N$.

One of the first main results proved in my joint papers with T. Darvas gives a complete characterization of $\mathcal{I}$-good singularities.

Theorem 2.6. *Let $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with $\int_X \theta_\varphi^n > 0$. Then the following are equivalent:*

(1) φ is $\mathcal{I}$-good;
(2) φ can be d_S-approximated by a sequence of analytic singularities $\varphi_j \in \mathrm{PSH}(X,\theta)$.

Sketch of the proof. (2) $\implies$ (1): This is the easier direction. By definition, we know that

$$d_S(\varphi_j, P_\theta[\varphi_j]_{\mathcal{I}}) = 0.$$

We want to take $j \to \infty$ to conclude (2.1). The first term does not cause any trouble as we have assumed that φ_j converges to φ with respect to d_S. In order to handle the second term, we need to prove the continuity of $P_\theta[\bullet]_{\mathcal{I}}$ with respect to d_S.

As we mentioned earlier, following the general strategy, we can reduce the problem to proving the continuity of $P_\theta[\bullet]_{\mathcal{I}}$ along monotone sequences. The increasing case does not cause any trouble. We leave it as an exercise. The decreasing case is more tricky. See [DDNL21, Proposition 4.8].

(1) $\implies$ (2): This is the difficult direction. We first assume that θ_φ is a Kähler current. In this case, one can construct a so-called *quasi-equisingular approximation* φ_j of φ:

(1) φ_j has analytic singularities for each j;
(2) φ_j is decreasing with limit φ;
(3) for each $\lambda' > \lambda > 0$, there is $j > 0$ such that (1.5) holds.

See [DPS01]. We deduce that $P_\theta[\varphi_j]$ decreases pointwisely to $P_\theta[\varphi]$. A non-trivial result proved in [DDNL21] then implies the d_S-convergence of φ_j to φ.

When θ_φ is not a Kähler current. We can apply a trick discovered by Darvas. There is always $\psi \in \mathrm{PSH}(X,\theta)$ such that $\psi \leq \varphi$ and θ_ψ is a Kähler current. We approximate a general φ by $(1-\epsilon)\varphi + \epsilon\psi$. □

3. The volume of Hermitian pseudo-effective line bundles

As a first application of the theory of $\mathcal{I}$-good singularities, we study the volume of a line bundle endowed with e singular psh metric.

Let X be a connected compact Kähler manifold and L be a big line bundle on X. Assume that L is endowed with a singular psh metric h.

In order to reduce to the language of qpsh functions, we fix an arbitrary smooth metric h_0 on L. Let $\theta = c_1(L, h_0)$, then h can be identified with $h_0 \exp(-\varphi)$ for some $\varphi \in \mathrm{PSH}(X, \theta)$. The volume we have in mind is

{eq:voldef}

$$\mathrm{vol}(L, h) := \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^{\otimes k} \otimes \mathcal{I}(h^{\otimes k})). \tag{3.1}$$

The motivation is of course the Riemann–Roch formula. When L is ample and h is smooth, $\mathrm{vol}(L, h)$ is nothing but (L^n), the usual volume of L. When the multiplier ideal sheaf is not presented in (3.1), the volume is known as the volume of the line bundle L. It is well studied in the literature.

There are plenty of motivations to study the quantity $\mathrm{vol}(L, h)$. The very first motivation comes from arithmetic theory. When X is a suitable moduli space (say the moduli space of PPAV) and L is a suitable automorphic line bundle, the holomorphic sections of L can usually be identified with certain (weak) modular forms. Now imposing a singular metric h means imposing a boundary condition at infinity, hence giving rise to certain cusp forms. The volume (3.1) characterizes the asymptotic behaviour of cusp forms.

Another motivation comes from the so-called Witt Nyström correspondence that I will explain in the next lecture, the volume can be regarded as the differentiated version of the Monge–Ampère energy.

Anyway, let us begin the study of (3.1). One first difficulty is the existence of the limit (3.1). We know that this limit exists *a posteriori*, but for now, let us try to avoid the notation $\mathrm{vol}(L, h)$ before knowing its existence.

Let us first look into the literature. There is a positive result due to Bonavero [Bon98]:

thm:Bon **Theorem 3.1** (Bonavero). *Assume that h has analytic singularities, then the limit in* (3.1) *exists and*

{eq:volLh}

$$\mathrm{vol}(L, h) = \int_X \theta_\varphi^n. \tag{3.2}$$

This result can proved by a suitable resolution of singularity. In general, a potential with analytic singularities can be resolved to a potential with log singularities along a divisor. In the latter case, (3.2) is essentially the algebraic Riemann–Roch formula.

One might wonder if (3.2) holds in general. Unfortunately, this is not true.

Exercise 3.2. Explain why (3.2) fails in the example of Example 2.1.

The reason is easy to understand: the volume (3.1) is defined using only the data related to $\mathcal{I}(h^{\otimes k})$. We know that these data are rougher than the data given by non-pluripolar masses, as reflected in the fact that $P[\bullet] \neq P[\bullet]_{\mathcal{I}}$ in general. Therefore, a more reason guess is the following:

thm:volLh **Theorem 3.3.** *For a general h, the limit in* (3.1) *exists and*

{eq:volLh2}

$$\mathrm{vol}(L, h) = \int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}. \tag{3.3}$$

Sketch of the proof. The case where $\int_X \theta_\varphi^n = 0$ is not so difficult, we assume that $\int_X \theta_\varphi^n > 0$. The same reduction as in Theorem 2.6 allows us to assume that θ_φ is a Kähler current.

By Theorem 2.6 and its proof, φ can be d_S-approximated by a decreasing sequence $\varphi_j \in \mathrm{PSH}(X, \theta)$ with analytic singularities. Then by Theorem 3.1, we have

$$\int_X \theta^n_{\varphi_j} = \mathrm{vol}(L, h_j),$$

where $h_j = h \exp(-\varphi_j)$. We want to let $j \to \infty$ to conclude. The left-hand side is easy: recall that we have a general convergence theorem of the non-pluripolar masses with respect to d_S. It remains to show that $\mathrm{vol}(L, h_j) \to \mathrm{vol}(L, h)$. The proof is very difficult. It is the technical core of [DX21]. This also explains why in our previous paper [DX22] we needed a different approach and only the case where L is ample is proved. □

Now putting everything together, we have

Corollary 3.4. *Assume that $\int_X c_1(L,h)^n > 0$, then the following are equivalent:*

1. $\mathrm{vol}(L,h) = \int_X c_1(L,h)^n$;
2. *h can be approximated by psh metrics with analytic singularities with respect to d_S;*
3. *h is $\mathcal{I}$-good.*

There are plenty of $\mathcal{I}$-good singularities. We mention the most important instance, as proved by Y. Yao:

Example 3.5. *In the toric setting, all toric psh metrics are $\mathcal{I}$-good.*

If you are familiar with toric geometry, this is not a super difficult exercise. A slightly more general result is proved by Botero–Burgos Gil–Holmes–de Jong.

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