LECTURES ON PLURIPOTENTIAL THEORY — LECTURE 4. THE THEORY OF B-DIVISORS AND APPLICATIONS

Contents

1. The b-divisor point of view

Our next goal is to make precise the idea that the loss of non-pluripolar masses is caused by the singularities.

Let X be a connected compact Kähler manifold and L be a big line bundle on X. Assume that L is endowed with a singular psh metric h. The notation h_0, θ, φ are as before.

Example 1.1. *Consider an effective $\mathbb{Q}$-divisor D on X. Assume that locally the singularities of φ look like $\log|s_D|^2$, where s_D is a local holomorphic function whose zero locus is D (counting multiplicity), we say φ has log singularities along D in this case. The non-pluripolar mass of (L,h) is $((L-D)^n)$, as follows from the results of the last time.*

In particular,

$$\operatorname{vol}(L,h) = ((L-D)^n).$$

This formula strongly resembles the Riemann–Roch formula $\operatorname{vol} L = (L^n)$. It suggests that the loss of mass is caused by the singularity D.

In general, we want something similar. Let us consider the case of analytic singularities.

Example 1.2. *Assume that φ has analytic singularities. As we recalled earlier, there is a sequence of blowing-ups with smooth centers $\pi : Y \to X$ such that $\pi^*\varphi$ has log-singularities along a $\mathbb{Q}$-divisor D on Y. Then from the previous example, we know that*

$$\operatorname{vol}(\pi^*L, \pi^*h) = ((\pi^*L - D)^n).$$

It is not hard to show that $\operatorname{vol}(\pi^*L, \pi^*h) = \operatorname{vol}(L,h)$. (This requires a twisted version of the results from Lecture 3, which you can find in [DX21; DX22].) So

$$\operatorname{vol}(L,h) = ((\pi^*L - D)^n).$$

The key point is that one could not rely only on the divisors on X, but we need divisors on birational models of X as well.

Remark 1.3. Careful readers should have already noticed that we are talking about birational models instead of bimeromorphic models. This is because our X is necessarily projective: it is a Kähler manifold and it admits a big line bundle, we could apply Moishezon's criterion. By GAGA, X comes from an algebraic variety, so it makes sense to talk about birational models.

In general, when the singularity is no longer analytic, it is therefore natural to consider all birational models at the same time. This leads to the notion of b-divisors. We first recall that the Néron–Severi group $\mathrm{NS}^1(X)_{\mathbb{Q}}$ of X is defined as the quotient of the space of $\mathbb{Q}$-divisors on X by the numerical equivalence relation. The space $\mathrm{NS}^1(X)_{\mathbb{R}}$ is defined as $\mathrm{NS}^1(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. This space is always of finite dimension.

Definition 1.4. A *b-divisor* on X is an assignment $Y \mapsto \mathbb{D}_Y$ that assigns to each smooth birational model Y of X a class in $\mathrm{NS}^1(Y)_{\mathbb{R}}$ such that if Y and Z are two models such that there is a morphism $\pi : Y \to Z$ over X (necessarily unique), we have

$$\pi_* \mathbb{D}_Y = \mathbb{D}_Z.$$

In other words, the space of b-divisors is the projective limit

$$\varprojlim_{Y \to X} \mathrm{NS}^1(Y)_{\mathbb{R}}.$$

This gives the space of b-divisors a projective limit topology.

From now on, a birational model is always assumed to be smooth.

A simple example is as follows: if D is an $\mathbb{R}$-divisor on X, then define $\mathbb{D}(D)$ as follows: for each birational model $\pi : Y \to X$, set

$$\mathbb{D}(D)_Y = \pi^* D.$$

Similarly, an $\mathbb{R}$-divisor D on a birational model of X also defines a b-divisor on X, such b-divisors are called *Cartier b-divisors*.

Exercise 1.5. Explain the details.

Now let us see how we can get a b-divisor $\mathbb{D}(L, h)$ from h such that

$$\mathrm{vol}(L, h) = (\mathbb{D}(L, h)^n),$$

whatever the right-hand side means.

From our examples, we see that $\mathbb{D}(L, h)_X$ should take the form $L - D$, where D is a divisor associated with the singularities of h. It turns out that D can be constructed through Siu's decomposition:

$$\theta_\varphi = \sum_i c_i D_i + R,$$

where D_i is a countable family of prime divisors and $c_i > 0$, where R does not have Lelong numbers along divisors. We define

$$\mathbb{D}(L, h)_X \coloneqq L - \sum_i c_i D_i.$$

It can be shown that the right-hand side converges.

Similarly, if $\pi : Y \to X$ denotes a birational model, then

$$\mathbb{D}(L, h)_Y \coloneqq \pi^* L - \text{divisorial part of } (\pi^*\theta + \mathrm{dd^c}\pi^*\varphi).$$

Exercise 1.6. Prove that $\mathbb{D}(L, h)$ is a b-divisor on X.

Exercise 1.7. Compute $\mathbb{D}(L, h)$ when h has analytic singularities.

The b-divisor $\mathbb{D}(L, h)$ has a special property: it can be approximated by nef Cartier b-divisors with respect to the projective limit topology. By a nef Cartier b-divisor, we me a Cartier b-divisor determined by a nef divisor on a birational model. This is a reformulation of the existence of quasi-equisingular approximations.

In the case of nef b-divisors, Dang–Favre [DF20] introduced an intersection theory. We do not recall the precise definition here.

Theorem 1.8. *The b-divisor $\mathbb{D}(L, h)$ is nef. Assume that $\int_X c_1(L, h)^n > 0$, then*

$$\mathrm{vol}(L, h) = (\mathbb{D}(L, h)^n).$$

This result is essentially established in [Xia22b]. The general statement can be found in [Xia22a].

2. Ross–Witt Nyström correspondence

We fix a connected compact Kähler manifold X and a closed real $(1,1)$-form θ on X representing a big class. In the previous lecture, we defined $\mathcal{R}^1(X,\theta)$. We will make full use of the techniques developed previously to understand this space.

We will see that there is a bijection from $\mathcal{R}^1(X,\theta)$ to a space of concave curves in $\mathrm{PSH}(X,\theta)$ (known as *test curves*):

{eq:bij}
$$\mathcal{R}^1(X,\theta) \xrightarrow{\sim} \{\text{test curves}\}. \tag{2.1}$$

This duality is rather surprising: an element in $\mathcal{R}^1(X,\theta)$ is a curve with mild singularities, while a test curve is a curve of (very) singular potentials.

By contrast, the concavity/convexity duality in this statement is less surprising. A curve $\ell = (\ell_t)_t$ in $\mathcal{R}^1(X,\theta)$ is convex in t, while a test curve $\psi = (\psi_\tau)_\tau$ is concave in τ. This phenomenon is classically known as the *Legendre duality*. This also suggests how to construction the bijection (2.1).

Definition 2.1. Let $\ell \in \mathcal{R}^1(X,\theta)$. The *Legendre transform* of ℓ is defined as

$$\hat{\ell}_\tau := \inf_{t\geq 0} (\ell_t - t\tau)\,, \quad \tau \in \mathbb{R}.$$

It is a non-trivial result, known as Kiselman's minimum principle that $\hat{\ell}_\tau \in \mathrm{PSH}(X,\theta)\cup\{-\infty\}$. We get a concave curve $\psi = \hat{\ell}$ which satisfies the following:

(1) $\psi_\bullet$ is concave in $\bullet$.
(2) ψ is usc as a function $\mathbb{R}\times X \to [-\infty,\infty)$.
(3) $\lim_{\tau\to-\infty}\psi_\tau = V_\theta$ in L^1.
(4) $\psi_\tau = -\infty$ for τ large enough.

The surprising fact is the following:

(5) $P_\theta[\psi_\tau] = \psi_\tau$ (we say ψ_τ is *model* in this case) for all $\tau < \tau^+$, where
$$\tau^+ := \inf\{\tau \in \mathbb{R} : \psi_\tau \equiv -\infty\}.$$

You should find no difficulty when verifying (1) to (4), as for (5), see [Dar17, Proposition 5.1].

Definition 2.2. A *test curve* in $\mathrm{PSH}(X,\theta)$ is a curve $\mathbb{R} \to \mathrm{PSH}(X,\theta)\cup\{-\infty\}$ satisfying (1) (5) as above.

The *energy* of a test curve $\psi_\bullet$ is defined as

{eq:defE}
$$\mathbf{E}(\psi_\bullet) := \tau^+ + \frac{1}{V}\int_{-\infty}^{\tau^+}\left(\int_X \theta_{\psi_\tau}^n - \int_X \theta_{V_\theta}^n\right)\mathrm{d}\tau. \tag{2.2}$$

A test curve ψ is said to be of *finite energy* if $\mathbf{E}(\psi) > -\infty$. We denote the set of finite energy test curves by $\mathcal{TC}^1(X,\theta)$.

Now we can state the Ross–Witt Nyström correspondence.

thm:RWN **Theorem 2.3** ([DX22, Theorem 3.7])**.** *The Legendre transform establishes a bijection from $\mathcal{R}^1(X,\theta)$ to $\mathcal{TC}^1(X,\theta)$. For $\ell \in \mathcal{R}^1(X,\theta)$, We have $\sup_X \ell_1 = \tau^+$ and $\mathbf{E}(\ell) = \mathbf{E}(\hat{\ell})$.*

This result is due to a lot of people. The preliminary form is due to Ross–Witt Nyström. Successive generalizations are due to Darvas–Di Nezza–Lu, Darvas and myself. The most general version is written in the joint paper by Darvas, K. Zhang and myself [DXZ23].

Here we find a rather surprising fact: the rigid condition of being a geodesic (corresponding to the homogeneous Monge–Ampère equation) corresponds to the soft condition of a test curve. There are no PDEs involved at all!

This result is also amusing because it tells us that the study of geodesic rays is not very different from the study of model metrics. This philosophy has led to a number of interesting results of mine in the last few years. Already we see from (2.2) that Monge–Ampère energy is roughly the same as the integral of non-pluripolar masses. In [Xia22b], you will find many similar results.

For readers knowing the concept of maximal geodesic rays in the sense of Berman–Boucksom–Jonsson, we recall the following result:

Theorem 2.4. *Under the bijection Theorem 2.3, maximal geodesic rays correspond to test curves satisfying the following*

*(5') $P_\theta[\psi_\tau]_{\mathcal{I}} = \psi_\tau$ (we say ψ_τ is $\mathcal{I}$-*model *in this case) for all $\tau < \tau^+$.*

Maximal geodesic rays are rather mysterious objects for complex geometriers, this theorem shows that they boil down to very concrete objects: the $\mathcal{I}$-model potentials, which are relatively well understood. This point of view turns out to be very fruitful. Let us mention in particular that in [DX22; DXZ23], we managed to prove Boucksom–Jonsson's envelope conjecture on smooth projective varieties following this idea. For applications in Ding stability/K-stability, we refer to the recent work of Darvas–Zhang, Dervan–Reboulet.

3. The partial Okounkov bodies

Let X be an irreducible smooth projective variety of dimension n. Let L be a big holomorphic line bundle on X.

Let us consider an admissible flag $X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ on X: each Y_i is a connected normal projective subvariety of X of codimension i satisfying that Y_i is smooth at the point Y_n.

One can associate a natural convex body $\Delta(L)$ of dimension n to L, generalizing the classical Newton polytope construction in toric geometry. This construction was first considered by Lazarsfeld–Mustață [LM09] and Kaveh–Khovanskii [KK12] and $\Delta(L)$ is known as the *Okounkov body* or *Newton–Okounkov body* associated with L (with respect to the given flag).

We briefly recall its definition: given any non-zero $s \in H^0(X, L^k)$, let $\nu_1(s)$ be the vanishing order of s along Y_1. Then s can be regarded as a section of $H^0(X, L^k \otimes \mathcal{O}_X(-\nu_1(s)Y_1))$ after possible shrinking X around the point Y_n. It follows that $s_1 := s|_{Y_1}$ is a non-zero section of $L|_{Y_1}^k \otimes \mathcal{O}_X(-\nu_1(s)Y_1)|_{Y_1}$. We can then repeat the same procedure with s_1, Y_2 in place of s, Y_1. Repeating this construction, we end up with $\nu(s) = (\nu_1(s), \ldots, \nu_n(s)) \in \mathbb{N}^n$. In fact, ν extends naturally to a rank n valuation on $\mathbb{C}(X)$ of rational rank n. Consider the semigroup

$$\Gamma(L) := \left\{ (a, k) \in \mathbb{Z}^{n+1} : k \in \mathbb{N}, a = \nu(s) \text{ for some } s \in H^0(X, L^k)^\times \right\}.$$

Then $\Delta(L)$ is the intersection of the closed convex cone in $\mathbb{R}^{n+1}$ generated by $\Gamma(L)$ and $\{(x, 1) : x \in \mathbb{R}^n\}$. A key property of $\Delta(L)$ is that the Lebesgue volume of $\Delta(L)$ is proportional to the volume of the line bundle L:

$$\operatorname{vol} \Delta(L) = \frac{1}{n!} \operatorname{vol} L. \tag{3.1}$$

The theory we developed in the last lecture allows us to extend this theory to a pair (L, h), where h is a singular psh metric.

Theorem 3.1. *Let (L, h) be as above. Assume that $\int_X c_1(L, h)^n > 0$. Then there is a canonical convex body $\Delta(L, h) \subseteq \Delta(L)$ associated with (L, h) satisfying*

$$\operatorname{vol} \Delta(L, h) = \operatorname{vol}(L, h). \tag{3.2}$$

Moreover, $\Delta(L, h)$ is continuous in h. Here the set of h is endowed with the d_S-pseudometric and the set of convex bodies is endowed with the Hausdorff metric.

Define

$$\Gamma_k := \left\{ k^{-1}\nu(s) \in \mathbb{R}^n : s \in H^0(X, L^k \otimes \mathcal{I}(kh))^\times \right\}$$

and let Δ_k denote the convex hull of Γ_k. Then Δ_k converges to $\Delta(L, h)$ with respect to the Hausdorff metric.

This convex body $\Delta(L, h)$ is known as the *partial Okounkov body*. It played an important role in our proof of Lazarsfeld–Mustață's conjecture [DRWN+23].

There are various different but equivalent ways of constructing $\Delta(L,h)$. The most elementary way goes as follows: when h has analytic singularities, we can resolve h and assume that h has log singularities along a $\mathbb{Q}$-divisor D. Then we simply set

$$\Delta(L,h) \coloneqq \Delta(D)+\nu(D),$$

where $\nu(D)$ is a vector in $\mathbb{R}^n$.

Assume that $c_1(L,h)$ is a Kähler current. Take a quasi-equisingular approximation h_j of h. Then we set

$$\Delta(L,h) \coloneqq \bigcap_j \Delta(L,h_j).$$

The general case follows by approximation. This approach has an obvious drawback: one has to verify that the eventual definition is independent of the choices we made. In [Xia21], I adopted a slightly different approach.

The partial Okounkov bodies are important due to the following result:

Theorem 3.2. *Let L be a big line bundle on X. Let ϕ, ϕ' be two psh metrics on L. Then the following are equivalent:*

(1) $\phi \sim_{\mathcal{I}} \phi'$.
(2) $\Delta(L,\phi)=\Delta(L,\phi')$ for all valuations on $\mathbb{C}(X)$ of rank n and rational rank n.

4. Partial Bergman kernels

Let X be a smooth projective variety and (L,h) be pseudo-effective line bundle together with a psh metric. Fix a smooth reference metric h_0 on L as before and write $\theta = c_1(L,h_0)$. The metric h is then identified with $\varphi \in \mathrm{PSH}(X,\theta)$. Let $v \in C^0(X)$. Let ν be a smooth positive volume form of volume 1 on X.

We introduce the associated *partial Bergman kernels*: for any $k\in\mathbb{N}$, $x\in K$,

$$B^k_{v,h,\nu}(x) := \sup\left\{ |s|^2_{h_0^{\otimes k}} \mathrm{e}^{-kv}(x) : \int_K |s|^2_{h_0^{\otimes k}} \mathrm{e}^{-kv}\,\mathrm{d}\nu \leq 1, s\in H^0(X, L^{\otimes k}\otimes \mathcal{I}(h^{\otimes k}))\right\}.$$

The associated partial Bergman measures on X is

$$\beta^k_{v,h,\nu} := \frac{n!}{k^n} B^k_{v,h,\nu}\,\mathrm{d}\nu\,. \tag{4.1}$$

Theorem 4.1. *We have $\beta^k_{u,v,\nu}$ converges to the partial equilibrium measure weakly as $k\to\infty$.*

Here the partial equilibrium measure is defined as

$$P_\theta[\varphi]_{\mathcal{I}}(v) := \sup{}^*\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq v, \psi \preceq_{\mathcal{I}} \varphi\right\}. \tag{4.2}$$

One can show that $\theta^n_{P[u]_{\mathcal{I}}(v)}$ is supported on $\{P[u]_{\mathcal{I}}(v)=v\}$.

When h is not presented (i.e. when h has minimal singularity), this result is proved by Berman–Boucksom [BB10]. The general case is proved in [DX21].

It is not very hard to understand the relation between this result and our previous result:

$$\int_K \beta^k_{v,h,\nu} = \frac{n!}{k^n} h^0(X, L^{\otimes k}\otimes \mathcal{I}(kh)). \tag{4.3}$$

So we have the convergence of the total mass from our previous result. In particular, in the proof of Theorem 4.1 we only have to establish one inequality.

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