LECTURE II. BERKOVICH SPACES AND PLURISUBHARMONIC FUNCTIONS

Contents

1. Introduction

In the last lecture, we explained the basic example of Berkovich spaces. The goal of today is to extend the previous example to higher dimensions.

We will always fix the base field $\mathbb{C}$ with the trivial valuation $|\bullet|$. We will associate a Berkovich space X^{an} to each algebraic scheme X over $\mathbb{C}$. The construction is similar to the classical GAGA. As in the classical GAGA, X^{an} comes with the structure of a $\mathbb{C}$-ringed space and a morphism $X^{\mathrm{an}} \to X$ of ringed spaces. Moreover, it is characterized by the fact that any morphism from a Berkovich space to X factorizes through $X^{\mathrm{an}} \to X$. This is the content of Berkovich GAGA [Ber12, Chapter 3].

This general approach requires some deep understanding of the Berkovich geometry. In this lecture, we will follow the *ad hoc* approach of Boucksom–Jonsson. As a consequence, for us X^{an} will only be a topological space instead of a $\mathbb{C}$-ringed space.

2. Berkovich spaces

Let X be an algebraic scheme over $\mathbb{C}$. We first construct X^{an} as a set. We will need the following definition:

Definition 2.1. Let Y be an irreducible algebraic scheme over $\mathbb{C}$. A (non-Archimedean) *valuation* on Y is a valuation $v\colon \mathbb{C}(Y) \to \mathbb{R} \cup \{\infty\}$ which extends the trivial valuation $|\bullet|$ on $\mathbb{C}$. Recall that this means the following: for any $a, b \in \mathbb{C}(Y)$

(1) $v(a-b) \geq \min\{v(a), v(b)\}$ (non-Archimedean seminorm);
(2) $v(a) = \infty$ iff $a = 0$ (norm);
(3) $v(ab) = v(a) + v(b)$ (multiplicativity);
(4) $v(c) = 0$ for any $c \in \mathbb{C}^{\times}$.

The set of valuations on Y is denoted by Y^{val}.

When Y is a single point, Y^{val} gives a single point.

Definition 2.2. The underlying set of X^{an} is defined as follows:

$$X^{\mathrm{an}} := \coprod_{Y \subseteq X, Y \text{ integral}} Y^{\mathrm{val}}.$$

The *support* of $v \in X^{\mathrm{an}}$ is defined as Y if $v \in Y^{\mathrm{val}}$.

For each Zariski open subset $U \subseteq X$, U^{an} denotes the subset of X^{an} whose support meets U.

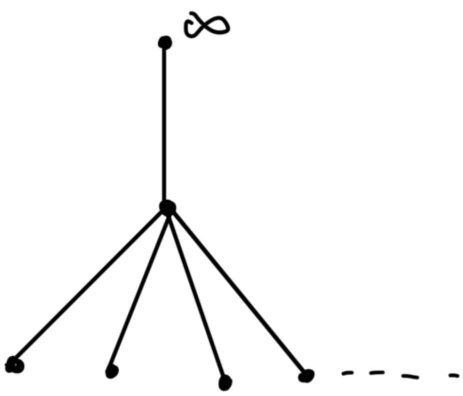
When X is proper, each valuation $v \in X^{\mathrm{an}}$ admits a center: a scheme-theoretic point x in the support Y of x such that v is non-negative on $\mathcal{O}_{Y,x}$ and positive on its maximal ideal.

In particular, if X is irreducible, we see that X^{an} can be decomposed as

$$X^{\mathrm{an}} = X^{\mathrm{val}} \amalg \coprod_{Y \subsetneq X} Y^{\mathrm{val}}.$$

We find that X consists of two parts: a top-dimensional part X^{val} and some lower-dimensional boundaries. Inside X^{val}, there is a subset consisting of explicit valuations, which will play an important role. A valuation $v \in X^{\mathrm{val}}$ is a divisorial valuation if there exists $c \in \mathbb{Q}_{>0}$ and a prime divisor E over X such that $v = c\,\mathrm{ord}_E$. The set of divisorial points will be denoted by X^{div}.

Figure 1. Berkovich $\mathbb{P}^1$ over a trivially valued field



In our favorite example Fig. 1, X^{val} is exactly the who pro-tree minus all ∞-ends of the legs.

When $U \subseteq X$ is an affine Zariski open subset, say $U = \operatorname{Spec} A$, we could interpret U^{an} as the set of semi-valuations on A (extending the trivial valuation on $\mathbb{C}$ by our non-standard definition of semi-valuations).

Next we describe the Berkovich topology on X^{an}.

Definition 2.3. The Zariski topology on X^{an} is the topology where the open subsets are given by U^{an} for all Zariski open subsets $U \subseteq X$.

Definition 2.4. The *Berkovich topology* on X^{an} is the weakest topology which refines the Zariski topology such that for each regular function f on a Zariski open subset $U \subseteq X$, the function $|f|\colon U^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$ sending v to $\exp(-v(f))$ is continuous.

Remark 2.5. To avoid during the first reading. The Berkovich topology is not the most useful topology in the general theory of Berkovich spaces. The correct topology is a Grothendieck topology, called Berkovich G-topology.

On the other hand, the Berkovich spaces constructed from analytification satisfy a very nice property: they are *good* Berkovich spaces. In this case, the Berkovich topology and the Berkovich G-topology have the same topos, so one could avoid the much deeper machinery developed by Berkovich.

Proposition 2.6. *Assume that X is projective, then the Berkovich space X^{an} is compact and Hausdorff.*

Proof. We may assume that X is integral (explain why).

We first argue that X^{an} is Hausdorff. Take $v, v' \in X^{\mathrm{an}}$. Assume that $v \neq v'$. We want to find disjoint open sets containing each of them. Take an affine open subset $U = \operatorname{Spec} A \subsetneq X$ that intersects the supports of both v and v', so that $v, v' \in U^{\mathrm{an}}$. We interpret U^{an} as the set of semi-valuations on U, then we see immediately that there exists $f \in A$ such that $|f(v)| \neq |f(v')|$, say $|f(v)| < |f(v')|$. Take $c \in \mathbb{R}$ such that $|f(v)| < c < |f(v')|$. Then the two open sets can be constructed as

$$\{w \in U : |f(w)| < c\}, \quad \{w \in U : |f(w)| > c\}.$$

The compactness is harder. Here we only give a sketch.

Step 1. Consider a closed immersion $X \hookrightarrow \mathbb{P}^N$. Show that the analytification $X^{\mathrm{an}} \hookrightarrow \mathbb{P}^{N,\mathrm{an}}$ identifies the topology on X^{an} with the subspace topology induced from $\mathbb{P}^{N,\mathrm{an}}$. So we have reduced to the case $X = \mathbb{P}^N$. This step is not difficult, do it by yourself.

Step 2. Show that $\mathbb{P}^{N,\mathrm{an}}$ can be seen as the gluing of two Berkovich polydisks.

Step 3. Show that the polydisk is compact. This is essentially a consequence of the Tychonoff theorem. □

In fact, Berkovich GAGA guarantees that if X is proper, then X^{an} is automatically compact and Hausdorff.

Exercise 2.7. *Identify our general definition of the topology on $\mathbb{P}^{1,\mathrm{an}}$ with the pro-tree topology.*

There are several natural structures on the Berkovich spaces. In the sequel, the triviality of the valuation on $\mathbb{C}$ is essential.

We shall assume that X is integral and projective from now on.

We have a scaling action of $\mathbb{R}_{>0}$ on X^{an} given by simple application.

Exercise 2.8. *Show that the scaling action is continuous.*

Remark 2.9. If the base field is non-trivially valued, we do not have the scaling action. In general, the semi-valuations occurring in the definition of the Berkovich spaces have to extend the given valuation on the base field. But no other valuations are not invariant under scaling.

There is always a special point $v_{\mathrm{triv}} \in X$, sending all non-zero elements in $\mathbb{C}(X)$ to 0. This valuation is known as the *trivial valuation*.

Exercise 2.10. *What are the center and the support of v_{triv}?*

There is a partial order as well: given $v, v' \in X^{\mathrm{an}}$, we say $v \geq v'$ if $v(\mathcal{I}) \geq v'(\mathcal{I})$ for all coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$. Here

$$v(\mathcal{I}) \coloneqq \min\left\{v(x) : x \in \mathcal{I}_{c(v)}\right\} \in [0,\infty],$$

where $c(v) \in X$ denotes the center of v.

Exercise 2.11. *Show that v_{triv} is the minimal element in X^{an}.*

We shall write $\log|\mathcal{I}| \colon X^{\mathrm{an}} \to [-\infty, 0]$ sending v to $-v(\mathcal{I})$.

3. Piecewise linear functions

In this section, X will always be an integral projective variety of dimension n over $\mathbb{C}$.

The general structure of X^{an} is quite similar to the one-dimensional picture in Fig. 1. Basically, if X is smooth, then X can be realized as the projective limit of a family of finite simplicial complexes of dimension n. It therefore makes sense to talk about the piecewise linear functions. We shall again adopt the approach as in Boucksom–Jonsson's paper.

Recall that a *flag ideal* is a fractional ideal on $X \times \mathbb{A}^1$, given by

$$\mathcal{I} = \sum_{j\in\mathbb{Z}} \mathcal{I}_j \pi^{-j},$$

where $\mathcal{I}_j$ is a decreasing sequence of coherent ideals on X and π is the coordinate on $\mathbb{A}^1$. Moreover, $\mathcal{I}_j$ is $\mathcal{O}_X$ for small enough j and 0 for large enough j.

A flag ideal $\mathcal{I}$ induces a continuous function on X^{an} in the following way:

$$\varphi_{\mathcal{I}}(v) = \max_j \left(\log|\mathcal{I}_j|(v) + j\right).$$

Exercise 3.1. *Prove that this is a continuous function.*

The set of piecewise linear functions on X^{an} is then defined as the $\mathbb{Q}$-linear span (in $C^0(X^{\mathrm{an}})$) of the $\varphi_{\mathcal{I}}$'s for various flag ideals $\mathcal{I}$. The set of piecewise linear functions is denoted by $\mathrm{PL}(X^{\mathrm{an}})$. If we replace $\mathbb{Q}$-coefficients by $\mathbb{R}$-coefficients, we get a set $\mathrm{PL}(X^{\mathrm{an}})_{\mathbb{R}}$.

Similarly, $\mathrm{PL}^+(X^{\mathrm{an}})$ will denote the subset of $\mathrm{PL}(X^{\mathrm{an}})$ consisting of all functions like $m^{-1}\varphi_{\mathcal{I}}$ for some flag ideal $\mathcal{I}$.

Exercise 3.2. *Show that on $\mathbb{P}^1$, these notions coincide with the ones defined in the last lecture.*

Both $\mathrm{PL}(X^{\mathrm{an}})$ and $\mathrm{PL}^+(X^{\mathrm{an}})$ can be understood more abstractly using test configurations, making connection with the general theory of model metrics (over a non-trivially valued field).

We want to relate piecewise linear functions with test configurations. This will allow us to handle the energy pairing in the next lecture.

Definition 3.3. A *test configuration* of X is a flat projective morphism of schemes $\pi\colon \mathcal{X} \to \mathbb{A}^1$ together with a $\mathbb{G}_m$-action on $\mathcal{X}$ lifting the standard action on $\mathbb{A}^1$ and an isomorphism $\mathcal{X}_1 \cong X$.

Test configurations play the role of models in the non-trivially valued case. There is a canonical embedding $\sigma\colon X^{\mathrm{an}} \hookrightarrow \mathcal{X}^{\mathrm{an}}$ known as the *Gauss extension*: given $v \in Y^{\mathrm{val}}$ for some integral closed subscheme Y of X, we let $\mathcal{Y}$ be the reduced scheme with underlying set given by the Zariski closure of the $\mathbb{G}_m$-orbit of $Y \subseteq X_1$ in $\mathcal{X}$, then we can identify $\mathbb{C}(\mathcal{Y})$ with $\mathbb{C}(Y)(t)$, where t is the coordinate on $\mathbb{A}^1$. We then define

$$\sigma(v)\left(\sum_{i\in\mathbb{N}} f_i t^i\right) = \min_{i\in\mathbb{N}} \left(v(f_i) + i\right)$$

for any $f_i \in \mathbb{C}(Y)$. This extends to a valuation $\sigma(v)$ on $\mathbb{C}(\mathcal{Y})$.

The semi-valuations ω in the image of the map σ are characterized by the conditions ω is $\mathbb{C}^\times$-invariant, $\omega(\mathcal{X}_0) = 1$ and $\omega(t) = 1$

This construction is a special case of a more general construction in non-Archimedean geometry associated with base extension.

A *vertical $\mathbb{Q}$-Cartier divisor* on $\mathcal{X}$ is a $\mathbb{G}_m$-invariant $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ supported on $\mathcal{X}_0$. Such a divisor D induces a function $\varphi_D\colon X^{\mathrm{an}} \to \mathbb{R}$ as follows

$$\varphi_D(v) \coloneqq \sigma(v)(D).$$

Exercise 3.4. *Show that $\varphi_D \in \mathrm{PL}(X^{\mathrm{an}})$.*

In fact, functions like φ_D exhaust the whole $\mathrm{PL}(X^{\mathrm{an}})$. Similar results hold with $\mathbb{R}$-coefficients as well.

The readers can find the details in [BJ21, Section 2.2]. But we shall recall the following consequence of this connection:

Theorem 3.5. *The subset X^{div} is dense in X^{an}.*

4. Fubini–Study functions

Now we can finally begin to talk about some pluripotential theory. We shall fix a holomorphic line bundle L on X. The Berkovich GAGA allows us to analytify L as well, obtaining an invertible sheaf L^{an} on X^{an}. A Hermitian metric can be defined similar to the complex setting. But there are more subtleties to consider. When we move on to general base field in the last lecture, I will come back to this point.

The most important observation is that in the trivially valued case, there is always a trivial metric on L^{an}: given each nowhere vanishing local regular section $s \in H^0(U, L)$, we just define $|s| = 1$ everywhere on U^{an}. This allows us to trivialize each section of L^{an} and instead of considering metrics on L^{an}, we just have to consider actual functions on X^{an}.

The idea of defining metrics on L^{an} comes from Shouwu Zhang. Basically, the metrics are defined using models of (X, L). Models do not quite make sense in the trivially valued case, so let us digress a bit to the general valued base field k. A model means a pair $(\mathcal{X}, \mathcal{L})$ consisting of a flat scheme over k° and a line bundle $\mathcal{L}$ on $\mathcal{X}$ together with an identification of the general fiber of this pair with (X, L). Each model induces a metric on L^{an}. Plurisubharmonicity corresponds to the positivity of $\mathcal{L}$.

Let us come back to the trivially valued $\mathbb{C}$. Then in this case, the definition of models still makes sense, but there is essentially only one model: (X, L) itself. So we cannot define model metrics as we desired. The idea of introducing the Fubini–Study functions is exactly to remedy this. These metrics become the model metrics in the non-trivially valued case, as we will see in the last lecture.

Having made all these preparations, we can finally come to our main definition:

Definition 4.1. The set $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ consists of functions $\varphi\colon X^{\mathrm{an}}\to[-\infty,\infty)$ with the following property:

1. φ is not identically $-\infty$;
2. there is $m\in\mathbb{Z}_{>0}$, $s_1,\dots,s_N\in H^0(X,L^m)$, $\lambda_i\in\mathbb{Q}$ such that
$$\varphi=m^{-1}\max_{j=1,\dots,N}\left(-v(s_j)+\lambda_j\right).$$

A function in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ is called a *$\mathbb{Q}$-rational generically finite Fubini–Study function* on L.

A *$\mathbb{Q}$-rational Fubini–Study function* on L is a $\mathbb{Q}$-rational generically finite Fubini–Study function on L taking finite value. The set of $\mathbb{Q}$-rational Fubini–Study functions on L is denoted by $\mathcal{H}(L)$.

Exercise 4.2. *Show that $\varphi(x)\neq-\infty$ for any $\varphi\in\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ and $x\in X^{\mathrm{val}}$.*

Exercise 4.3. *Show that any $\mathcal{H}(L)\subseteq\mathrm{PL}^+(X^{\mathrm{an}})$.*

More detailed discussions can be found in [BJ21, Section 2.5].

All what we said in this section works for $\mathbb{Q}$-line bundles L as well, as one can easily check.

5. Plurisubharmonic functions

On a convex polyhedron in the Euclidean space, a bounded from above convex function can be easily realized as the decreasing limit of a sequence of piecewise linear functions. Similarly, plurisubharmonic functions will be realized as decreasing limits of some piecewise linear functions.

The space $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ can be regarded as model psh functions with controllable singularities and $\mathcal{H}(L)$ are the "regular" psh functions. As in the complex case, if L is ample, it suffices to define general psh metrics using decreasing sequences of metrics in $\mathcal{H}(L)$. But the general case requires a more complicated construction.

As in the complex case, pluripotential theory is only well-behaved if X does not contain singularities like nodes. So we shall assume that X is unibranch in the sequel. Here unibranch means that locally (either analytically or in the Zariski topology) X has only one branch. So nodes are not unibranch while cusps are.

Definition 5.1. Consider a class $\theta\in N^1(X)_{\mathbb{R}}$ (the Néron–Severi group with real coefficients). A θ-psh function is a function $\varphi\colon X^{\mathrm{an}}\to[-\infty,\infty)$ satisfying the following conditions:

1. φ is not identically $-\infty$;
2. φ is upper semi-continuous;
3. φ can be written as the limit of a decreasing net $\varphi_i\in\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L_i)$, where L_i is a $\mathbb{Q}$-line bundle on X such that $\lim_i c_1(L_i)=\theta$.

We write the set of θ-psh metrics as $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$.

The theory of [DXZ23] allows us to take more generally $\theta\in H^{1,1}(X,\mathbb{R})$ as well. But I will not talk about this theory in these lectures.

Now the job is to show that the θ-psh functions defined in this way behave as their complex counterparts.

Proposition 5.2. *The set $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ is convex, stable under finite maxima, uniform limits, directed decreasing limits, addition by $\mathbb{R}$ and the scaling action of $\mathbb{R}_{>0}$.*

We have $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')\supseteq\mathrm{PSH}^{\mathrm{NA}}(X,\theta)+\mathrm{PSH}^{\mathrm{NA}}(X,\theta')$ and $\mathrm{PSH}^{\mathrm{NA}}(X,t\theta)=t\mathrm{PSH}^{\mathrm{NA}}(\theta)$ for any $t>0$.

For any birational map from another integral unibranch projective variety $\pi\colon Y\to X$, $\pi^*\mathrm{PSH}^{\mathrm{NA}}(X,\theta)\subseteq\mathrm{PSH}^{\mathrm{NA}}(Y,\pi^*\theta)$[1].

[1]One expects equality, but this is unfortunately an open problem.

The real constants are in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *iff* θ *is nef.*

If $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)\neq\varnothing$*, then* θ *is pseudo-effective. Conversely, if* θ *is big, then* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)\neq\varnothing$*.*

The t-scaling of a function φ on X^{an} is given by $t\varphi(t^{-1}\bullet)$.

These results can be proved by reducing to the Fubini–Study case.

Next let us talk about some more sophisticated facts.

Theorem 5.3. *If* θ *is ample, then any* $\varphi\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *is the decreasing limit of a sequence in* $\mathcal{H}(L)$ *for any* $\mathbb{Q}$*-line bundle* L *with* $c_1(L)=\theta$*.*

This is [BJ21, Theorem 4.15, Corollary 12.18].

Theorem 5.4. *A non-Archimedean* θ*-psh function is uniquely determined by its restriction to* X^{div}*.*

More precisely, if $\varphi\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$*, then* φ *is the smallest usc extension of* $\varphi|_{X^{\mathrm{div}}}$*. Moreover,* $\varphi(x)\neq-\infty$ *for any* $x\in X^{\mathrm{div}}$*.*

This is [BJ21, Theorem 4.22]. Note that φ can actually take the value $-\infty$ on X^{val}.

Unfortunately, we do not have time to explain the proofs of these results.

There is a topology on $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$: a net φ_i converges to φ if $\varphi_i(x)\to\varphi(x)$ for any $x\in X^{\mathrm{div}}$. This is a Hausdorff topology.

The following result is deep:

Theorem 5.5. *Suppose that* X *is smooth. Then any increasing net* φ_i *of non-positive* θ*-psh functions converges.*

This result was first proved in [BJ22]. A more analytic proof was given shortly after in [DXZ23]. The corresponding statement for unibranch X is widely open and is known as the *envelope conjecture*.

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