LECTURE III. ENERGY PAIRING, MONGE–AMPÈRE OPERATOR AND CALABI–YAU THEOREM

Contents

1. Introduction

In the last lecture, we defined the Berkovich spaces and the psh functions. As in the complex pluripotential theory, one wishes to define the Monge–Ampère operator and study the Monge–Ampère equations. Unfortunately, we did not develop the theory of differential forms and currents yet, so it is impossible to do so by the usual formulae (for the time being). Today I will talk about an indirect method of defining the Monge–Ampère operator.

In complex geometry, we know that a primitive of the Monge–Ampère operator is given by the Monge–Ampère energy (or Aubin–Yau energy if you prefer). We shall define the energy first and then deduce the Monge–Ampère operator as an application.

2. Energy pairing in the piecewise linear case

As before we fix X an irreducible projective variety of dimension n over $\mathbb{C}$, θ a pseudo-effective class in $N^1(X)_\mathbb{R}$.

We begin with the case of piecewise linear functions $\varphi \in \mathrm{PL}(X^{\mathrm{an}})_\mathbb{R}$. Recall that any such function has the form φ_D, where D is a vertical $\mathbb{R}$-Cartier divisor on a test configuration $\mathcal{X}$ of X. We may assume that $\mathcal{X}$ dominates the trivial test configuration $X \times \mathbb{A}^1$. In this case, θ can be pulled-back to a cohomology class $\theta_\mathcal{X}$ on $\mathcal{X}$. We say φ is θ-psh if $\theta_\mathcal{X} + D$ is nef over $\mathbb{A}^1$.

We can also compactify the test configuration by gluing the trivial product $X \times (\mathbb{P}^1 \setminus \{0\}) \to \mathbb{P}^1 \setminus \{0\}$. We therefore obtain a morphism $\overline{\mathcal{X}} \to \mathbb{P}^1$. The class $\theta_\mathcal{X} + D$ extends naturally to a $\mathbb{P}^1$-nef class $\theta_{\overline{\mathcal{X}}} + D$.

Exercise 2.1. *Verify that this condition does not depend on the choices we made.*

Exercise 2.2. *Verify that this definition is compatible with the general definition given in the last lecture.*

We shall define a polarized version of the energy pairing. Suppose that $\theta_0, \dots, \theta_n$ are pseudo-effective classes in $N^1(X)_\mathbb{R}$ and $\varphi_0, \dots, \varphi_n \in \mathrm{PL}(X^{\mathrm{an}})_\mathbb{R}$. As above, represent each of them as vertical divisors $D_0, \dots, D_n$ on the same test configuration $\mathcal{X}$. We define their energy pairing as the intersection number

$$(\theta_0, \varphi_0) \cdot \dots \cdot (\theta_n, \varphi_n) \coloneqq (\theta_{\overline{\mathcal{X}}} + D_0, \dots, \theta_{\overline{\mathcal{X}}} + D_n) \in \mathbb{R}.$$

Next we proceed to define the energy pairing of general θ-psh functions. The approach of Boucksom–Jonsson only works when the classes are ample of nef. We will content ourselves to this case.

Theorem 2.3. *Assume that $\omega_0,\dots,\omega_n$ are Kähler classes in $N^1(X)_{\mathbb{R}}$. Then there is a unique map*

$$\prod_{i=0}^{n}\mathrm{PSH}^{\mathrm{NA}}(X,\omega_i)\to[-\infty,\infty)$$

satisfying the following properties:

(1) *this pairing extends the pairing recalled above;*
(2) *the pairing is usc;*
(3) *the pairing is increasing in each variable.*

We refer to [BJ21, Theorem 7.1] for the details.

Definition 2.4. Assume that ω is a Kähler class in $N^1(X)_{\mathbb{R}}$. Then the *Monge–Ampère energy* $E\colon \mathrm{PSH}^{\mathrm{NA}}(X,\omega)\to[-\infty,\infty)$ is defined as

$$E(\varphi)\coloneqq\frac{1}{n+1}(\omega,\varphi)\cdot\dots\cdot(\omega,\varphi).$$

The set $\mathcal{E}^{1,\mathrm{NA}}(X,\omega)$ is the subset of $\mathrm{PSH}^{\mathrm{NA}}(X,\omega)$ consisting of φ such that $E(\varphi)>-\infty$.

Our definition of E differs from Boucksom–Jonsson's convention by a constant. At this stage, one could easily verify that E satisfies the familiar properties of the Monge–Ampère energy as in the complex setting.

The energy is extended in Boucksom–Jonsson's paper to general pseudoeffective classes θ_i with a somewhat subtle condition on φ_i. We wish to avoid the extra technical burden and restrict to the simple case of ample classes in the these lectures.

After defining the Monge–Ampère energy, the definition of the Monge–Ampère measure is immediate. We still work out a slightly more general version.

Theorem 2.5. *Assume that $\omega_1,\dots,\omega_n$ are Kähler classes in $N^1(X)_{\mathbb{R}}$ and $\varphi_i\in\mathrm{PSH}^{\mathrm{NA}}(X,\omega_i)$. We define $(\omega_1+\mathrm{dd^c}\varphi_1)\wedge\dots\wedge(\omega_n+\mathrm{dd^c}\varphi_n)$ as the unique Radon measure on X^{an} satisfying the following: given any $\varphi\in\mathrm{PL}(X^{\mathrm{an}})_{\mathbb{R}}$, we have*

$$\int_{X^{\mathrm{an}}}\varphi\,(\omega_1+\mathrm{dd^c}\varphi_1)\wedge\dots\wedge(\omega_n+\mathrm{dd^c}\varphi_n)=(0,\varphi)\cdot(\omega_1,\varphi_1)\cdot\dots\cdot(\omega_n,\varphi_n).$$

This theorem is a simple consequence of the Riesz–Markov–Kakutani representation theorem. We leave its proof as an exercise. Of course, you need to develop a few basic properties of the energy pairing by yourself.

Let us consider a very particular case:

Example 2.6. *Assume that $\omega=c_1(L)$ for an ample line bundle L and $\varphi\in\mathrm{PSH}^{\mathrm{NA}}(X,\omega)$ is piecewise linear and represented by a vertical $\mathbb{Q}$-divisor D on some test configuration $\mathcal{X}$. In this case, the Monge–Ampère measure is given by*

$$(\omega+\mathrm{dd^c}\varphi)^n=\sum_E c_E\,\delta_{v_E},$$

where E runs over all irreducible components of $\mathcal{X}_0$ and

$$c_E=\mathrm{ord}_E(\mathcal{X}_0)\cdot(\omega_{\mathcal{X}}+D)|_E^n.$$

Here v_E denotes the unique valuation of $\mathbb{C}(X)$ whose Gauss extension is given by $(\mathrm{ord}_E\,\mathcal{X}_0)^{-1}\,\mathrm{ord}_E$.

This resembles the original definition of the Chambert-Loir measure.

Exercise 2.7. *Prove the above assertions.*

One could develop the general theory in parallel with its complex counterpart, as done in Boucksom–Jonsson's paper.

3. The envelope

Let us make a pose to study some different object: the envelope operator. This is required when solving the Monge–Ampère equation.

We will fix an ample class $\omega \in N^1(X)_{\mathbb{R}}$. Given a function $f\colon X^{\mathrm{an}} \to \mathbb{R}$ its envelope is defined to be the function $P_\omega(f)\colon X^{\mathrm{an}} \to [-\infty, \infty]$ given by

$$P_\omega(f) \coloneqq \sup\left\{u \in \mathrm{PSH}^{\mathrm{an}}(X,\omega) : u \leq f\right\}.$$

The following *envelope conjecture* lies at the heart of the non-Archimedean pluripotential theory.

Conjecture 3.1. *Given any $f \in C^0(X^{\mathrm{an}})$[1], $P_\omega(f)$ is also continuous.*

This conjecture concerns not only with the trivially valued case, as we shall see in the last lecture.

Why is this conjecture important? We give a list of properties equivalent to this condition.

(1) For any bounded above (non-empty) family of ω-psh functions φ_i, the usc regularized sup is still ω-psh;
(2) The space $\mathcal{E}^{1,\mathrm{NA}}(X,\omega)$ is complete with respect to the d_1-metric (we did not introduce it yet);
(3) The space $\mathrm{PSH}^{\mathrm{NA}}(X,\omega)$ is invariant under blowing-up X;
(4) There is a canonical isomorphism between $\mathrm{PSH}^{\mathrm{NA}}(X,\omega)$ and the corresponding space defined in [DXZ23].

Part (1) of this conjecture makes sense for more general pseudo-effective class as well. This is the *envelope conjecture* for general pseudo-effective class.

The conjecture is known when X is smooth. When X is only unibranch, it is widely open. This also explains why one needs to assume the smoothness in [BBJ21].

4. Differentiablity of the volume

The idea of solving the non-Archimedean Calabi–Yau theorem is similar to the complex case [BBGZ13]. The variational approach is preferable because it does not depend on local estimates. Recall that in the non-Archimedean world, psh functions are not defined locally!

The key to the variational approach is the differentiability of the energy, which unfortunately depends on the envelope conjecture.

Theorem 4.1. *Assume that the envelope conjecture holds for an ample class $\omega \in N^1(X)_{\mathbb{R}}$, then for any $\varphi \in \mathcal{E}^{1,\mathrm{NA}}(X,\omega)$ and any $f \in C^0(X^{\mathrm{an}})$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(P(\varphi + tf)) = \int_{X^{\mathrm{an}}} f\,(\omega + \mathrm{dd}^{\mathrm{c}}\varphi)^n.$$

The envelope P is defined in a way similar to the continuous case. We will talk about the more general result of Boucksom–Gubler–Martin in the last lecture.

The rest of the story is completely standard, as in the Archimedean case. We want to solve the Monge–Ampère equation $(\omega + \mathrm{dd}^{\mathrm{c}}\varphi)^n = \mu$ for a Radon measure μ with volume $\int_X \omega^n$ on X^{an}. We view this problem as a variational problem: finding the maximizer of the functional

$$\varphi \mapsto E(\varphi) - \int_{X^{\mathrm{an}}} \varphi\,\mu$$

on $\mathcal{E}^{1,\mathrm{NA}}(X,\omega)$.

Theorem 4.2. *As long as this functional is bounded from above (in other words, μ has finite energy), there is always a unique (up to constant) solution $\varphi \in \mathcal{E}^{1,\mathrm{NA}}(X,\omega)$ to the Monge–Ampère equation*

$$(\omega + \mathrm{dd}^{\mathrm{c}}\varphi)^n = \mu.$$

This approach was due to [BFJ15].

[1] There is an obvious typo in [BJ21, Lemma 5.17]

References

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