LECTURE IV. BERKOVICH GEOMETRY OVER NON-TRIVIALLY VALUED BASES

Contents

1. Introduction

In the previous lectures, we have only developed the theory of Berkovich spaces over the trivially valued field. But as we recalled in the very first lecture, it makes sense to consider general base fields as well.

For people working on complex geometry, there is at least one case of interest: the base field $\mathbb{C}((t))$ with the t-adic valuation. Why? Let us simply consider a smooth projective morphism $\mathcal{X} \to \Delta^*$ onto the punctured disk, meromorphic at 0. It is always of interest to understand the degeneration at $t = 0$. We can view $\mathcal{X}$ as a subspace of $\Delta^* \times \mathbb{P}^N$ defined by finitely many equation of definitions with meromorphic coefficients. So $\mathcal{X}$ can be viewed as a scheme over $\mathbb{C}((t))$. The degeneration behaviour of many problems can be characterized using the Berkovich analytification in these cases. My favorite examples include [Shi24] and [BJ17].

The goal of today's talk is to give a general introduction to the non-Archimedean pluripotential theory in this case with an emphasis on the difference with the trivially valued case.

2. The Berkovich analytification

We will fix a complete non-Archimedean valued field (k, v_k). We assume that the valuation is non-trivial. Let X be an integral projective k-variety. As in the first lecture, we will still realize X^{an} as a topological space instead of a ringed space.

Definition 2.1. As a set, X^{an} is defined as before

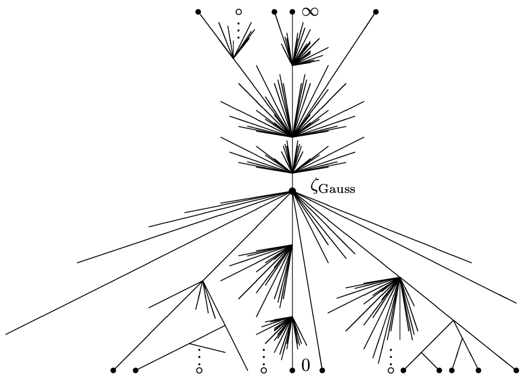
$$X^{\mathrm{an}} = \coprod_{Y \subseteq X} Y^{\mathrm{val}},$$

where Y runs over all integral closed subschemes of X. The set Y^{val} consists of valuations $v \colon k(Y) \to (-\infty, \infty]$ extending the given valuation on k.

The Berkovich topology is defined in exactly the same way as before. One can show that X^{an} is compact and Hausdorff. We also have the GAGA type results. In particular, there is a canonical morphism of k-ringed spaces $X^{\mathrm{an}} \to X$.

To give the readers a feeling of what $\mathbb{P}^1$ looks like, I steal Fig. 1 from Baker's lecture notes. This strange looking picture still slightly resembles the trivially valued case: there is a center point ζ_{Gauss}, the Gauss point; there are still lines connecting the Gauss point to every point in k, the latter appears as the outer ends of the corresponding legs. These points are known as type 1 points. But there are more branches along these legs. The branched points are the type 2 points. There are also some other points on the legs, known at type 3 points. Some branches extend all the way down, the end point would be the type 4 points. The picture itself requires a full lecture to explain. I will avoid doing this in this last lecture.

Figure 1. Berkovich $\mathbb{P}^1$ over a non-trivially valued field



Let me also mention that in Tate's theory, only type 1 points are preserved. In Huber's theory, there is one more type of points to add. So Tate's rigid spaces are not connected and Huber's adic spaces are not Hausdorff (in general).

One important remark is that we no longer have the scaling operator as in the trivially valued case. This is because points in X^{an} are supposed to be compatible with the given valuation on k, which itself is not scaling invariant!

Now given each $v \in X^{\mathrm{an}}$, say $v \in Y^{\mathrm{val}}$, by definition v is a valuation of $k(Y)$. Let $\mathcal{H}(v)$ denote the completion of $k(Y)$ with respect to v. It is a complete valued field associated with v. It should be regarded as the analogue of the residue field at a point in classical algebraic geometry.

3. Models

The next subject is the piecewise linear metrics. In the trivially valued case, we know that they are essentially the same as metrics induced by test configurations. In the non-trivially valued case, we have the well-developed machinery of *models*, taking the roles of test configurations.

We shall write k° for the subring of k consisting of elements with non-negative valuation. Write $\tilde{k}$ for the quotient of k° by its maximal ideal.

Definition 3.1. A *model* of X is a separated, flat morphism $\pi\colon \mathcal{X} \to \operatorname{Spec} k^\circ$ of finite type between schemes together with an identification $\mathcal{X}_K \cong X$.

The special fiber $\mathcal{X}_s$ of the model is the fiber of $\mathcal{X}$ over $\operatorname{Spec} \tilde{k}$. Given a model $\mathcal{X}$, there is an *anti-continuous* reduction map $\mathrm{red}_{\mathcal{X}}\colon X^{\mathrm{an}} \to \mathcal{X}_s$.

Similarly, if L is a line bundle on X, a model of (X, L) consists of $\pi\colon \mathcal{X} \to \operatorname{Spec} k^\circ$ as above and a line bundle $\mathcal{L}$ on $\mathcal{X}$, together with an identification $(\mathcal{X}_K, \mathcal{L}_K) \cong (X, L)$. One can similarly talk about $\mathbb{Q}$-models by allowing $\mathcal{L}$ to be a $\mathbb{Q}$-line bundle.

We now need to understand what a metric means. In the trivially valued case, L^{an} admits a canonical reference metric, but now things are different. By definition, L^{an} is the pull-back with respect to the morphism $X^{\mathrm{an}} \to X$ of k-ringed spaces. At any $x \in X^{\mathrm{an}}$, the fiber of L^{an} can be identified with the $\mathcal{H}(x)$-vector space $L \otimes_X \mathcal{H}(x)$. A metric on L is therefore a family of norms (in the usual sense) parameterized by $x \in X^{\mathrm{an}}$.

A special type of metrics are given by the model metrics: suppose that $(\mathcal{X}, \mathcal{L})$ is a model of (X, L). We shall define a metric $\phi_{(\mathcal{X},\mathcal{L})}$ on L^{an} as follows. For any $x \in X^{\mathrm{an}}$, take an affine open subset $U \subseteq \mathcal{X}$ such that $x \in \mathrm{red}_{\mathcal{X}}^{-1}(U_s)$ and that $\mathcal{L}$ admits a trivializing section τ over U. Then we require that $|\tau|_{\phi_{(\mathcal{X},\mathcal{L})}}(x) = 1$. This construction already dates back to S.Zhang.

In fact, it can be shown that model metrics determined by semi-ample $\mathbb{Q}$-models ($\mathcal{L}$ is a semi-ample $\mathbb{Q}$-line bundle) are exactly the same as the Fubini–Study metrics (defined as in the trivially valued case). A proof can be found in [BE21, Theorem 5.14]. From this class of metrics, one can similarly define the piecewise linear metrics.

This class of metrics serves as the smooth metrics in the general theory. Now plurisubharmonic metrics can be defined in exactly the same way as in the trivially valued case.

The Monge–Ampère energy is more difficult, as we cannot simply realize the piecewise linear case as an intersection product. One needs to rely on the general theory of [CLD12]. The details can be found in [BE21].

The precise relation with the trivially valued case is as follows: a continuous metric in the trivially valued case is psh if and only if after base change to a non-trivially valued field, the metric becomes psh.

4. The differentiability of energy

Here we consider the problem of solving the Monge–Ampère equation. Similar to the trivially valued case, it is important to understand the differentiability of the energy functional.

The key is again to understand the continuity of envelopes. The latter conjecture can be formulated exactly as in the trivially valued case. So far, it is known in the following cases:

(1) X is a curve;
(2) k is discretely valued and $\tilde{k}$ has characteristic 0;
(3) k is discretely valued of characteristic p, (X, L) is defined over a function field of transcendence degree d and resolution of singularities hold in dimension $\dim X + d$;
(4) X is smooth and the characteristic of k is 0;
(5) X is a smooth surface;
(6) X smooth, k has mixed characteristic under extra assumptions.

It is expected to hold as long as X is unibranch.

Theorem 4.1. *Assume the continuity of envelope, then we have the differentiability of the energy as in the trivially valued case.*

This general result is proved in [BGM22].

Based on this result, one could try to solve the Monge–Ampère equation using the variational method. As far as I know, in general, only the following result is known:

Theorem 4.2. *Under some suitable conditions.*

Assume that μ is a Radon measure on X^{an} with appropriate mass. Assume in addition that μ is supported on the dual complex of some snc model of X. Then the Monge–Ampère equation

$$(\omega + \mathrm{dd}^{\mathrm{c}}\varphi)^n = \mu$$

has a unique continuous solution (up to constant).

Here the the suitable condition could mean that X is smooth and the characteristic of $\tilde{k}$ is 0. See [BFJ15] and [BGJKM20].

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