NOTE ON DUISTERMAAT–HECKMAN MEASURES OF NON-ARCHIMEDEAN METRICS

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This is an informal note. Please contact me at mingchen@imj-prg.fr for comments.

Contents

1. Introduction

In this note, we define the Duistermaat–Heckman measure of a non-Archimedean metric using the theory of partial Okounkov bodies developped in [Xia21; DX24]. The main result Theorem 4.3 states that the Duistermaat–Heckman measure is canonical (independent of the choice of the flag).

sec:pre

2. Preliminaries

In this section, we recall the theory of Hausdorff metrics on the set of convex bodies following [Sch14, Section 1.8]. Fix $n \in \mathbb{N}$. Recall that a convex body in $\mathbb{R}^n$ is a non-empty compact convex subset of $\mathbb{R}^n$, which may have empty interior. Let $\mathcal{K}_n$ denote the set of convex bodies in $\mathbb{R}^n$. We will fix the Lebesgue measure $\mathrm{d}\lambda$ on $\mathbb{R}^n$, normalized so that the unit cube has volume 1.

Recall the definition of the Hausdorff metric between $K_1, K_2 \in \mathcal{K}_n$:

$$d_n(K_1, K_2) \coloneqq \max\left\{ \sup_{x_1\in K_1} \inf_{x_2 \in K_2} |x_1 - x_2|, \sup_{x_2\in K_2} \inf_{x_1 \in K_1} |x_1 - x_2| \right\}.$$

We extend d_n to an extended metric on $\mathcal{K}_n \cup \{\emptyset\}$ by setting

$$d_n(K, \emptyset) = \infty$$

for all $K \in \mathcal{K}_n$.

Theorem 2.1. *The metric space $(\mathcal{K}_n, d_n)$ is complete.*

thm:Bst **Theorem 2.2** (Blaschke selection theorem)**.** *Every bounded sequence in $\mathcal{K}_n$ has a convergent subsequence.*

thm:contvol **Theorem 2.3.** *The Lebesgue volume* $\mathrm{vol} : \mathcal{K}_n \to \mathbb{R}_{\geq 0}$ *is continuous.*

ausconvcond **Theorem 2.4.** *Let $K_i, K \in \mathcal{K}_n$ ($i \in \mathbb{N}$). Then $K_i \xrightarrow{d_n} K$ if and only if the following conditions hold*

(1) Each point $x \in K$ is the limit of a sequence $x_i \in K_i$.
(2) The limit of any convergent sequence $(x_{i_j})_{j\in\mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ lies in K, where i_j is a subsequence of $1, 2, \ldots$.

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The proofs of all these results can be found in [Sch14, Section 1.8].

:volcbimpeq **Lemma 2.5.** *Let $K_0, K_1 \in \mathcal{K}_n$. Assume that $K_0 \subseteq K_1$ and*

$$\operatorname{vol} K_0 = \operatorname{vol} K_1 > 0.$$

Then $K_0 = K_1$.

Proof. In fact, if $K_1 \neq K_0$, then $K_1 \setminus K_0$ is a non-empty open subset of K_1. As $\operatorname{vol} K_1 > 0$, $(K_1 \setminus K_0) \cap \operatorname{Int} K_1 \neq \emptyset$. Thus, $\operatorname{vol} K_1 > \operatorname{vol} K_0$, which is a contradiction. □

3. Okounkov test curves

Let $\Delta \in \mathcal{K}^n$. Assume that $V = n! \operatorname{vol} \Delta > 0$.

def:Otc **Definition 3.1.** An *Okounkov test curve* relative to Δ is an assignment $(\Delta_\tau)_{\tau<\tau^+}$ ($\tau^+ \in \mathbb{R}$) such that

1. Δ_τ is a decreasing assignment of convex bodies in $\mathbb{R}^n$ for $\tau < \tau^+$;
2. Δ_τ converges to Δ as $\tau \to -\infty$ with respect to the Hausdorff metric;
3. Δ_τ is concave in the τ variable.

rop:Otccont **Proposition 3.2.** *Any Okounkov test curve $(\Delta_\tau)_{\tau\leq\tau^+}$ relative to Δ is continuous for $\tau < \tau^+$.*

This is proved in [Xia21] for finite energy curves, but the proof works in general as well.

def:tf **Definition 3.3.** A *test function* on Δ is a function $F : \Delta \to [-\infty, \infty)$ such that

1. F is concave;
2. F is finite on $\operatorname{Int} \Delta$;
3. F is usc.

Let $\Delta_\bullet$ be an Okounkov test curve relative to Δ. We define the *Legendre transform* of $\Delta_\bullet$ as

$$G[\Delta_\bullet]\colon \Delta \to [-\infty, \infty), \quad a \mapsto \sup\left\{\tau < \tau^+ : a \in \Delta_\tau\right\}.$$

Conversely, a test function F on Δ, set $\tau^+ = \sup_\Delta F$. We define the *inverse Legendre transform* of F as

$$\Delta[F]\colon (-\infty, \tau^+] \to \mathcal{K}_n, \quad \Delta[F]_\tau = \{F \geq \tau\}.$$

kotestcurve **Theorem 3.4.** *The Legendre transform and inverse Legendre transform are inverse to each other, defining a bijection between the set of Okounkov test curves relative to Δ and test functions on Δ.*

The proof is essentially contained in [Xia21].

Definition 3.5. Let $\Delta_\bullet$ be an Okounkov test curve relative to Δ. We define the *Duistermaat–Heckman measure* $\mathrm{DH}(\Delta_\bullet)$ as

$$\mathrm{DH}(\Delta_\bullet) \coloneqq G[\Delta_\bullet]_*(\mathrm{d}\lambda).$$

It is a Radon measure on $\mathbb{R}$.

In other words, $\mathrm{DH}(\Delta_\bullet)$ is the probability distribution of the random variable $G[\Delta_\bullet]$ on the measure space $(\Delta, \mathrm{d}\lambda)$.

lma:DHmconv **Lemma 3.6.** *Suppose that $\Delta^k_\bullet$ is a decreasing sequence of Okounkov test curves relative to Δ with the same τ^+. Assume that the pointwise Hausdorff limit $\Delta_\bullet$ is still a Okounkov test curve relative to Δ. Then $\mathrm{DH}(\Delta^k_\bullet) \rightharpoonup \mathrm{DH}(\Delta_\bullet)$ as $k \to \infty$.*

Proof. Observe that

$$G[\Delta^k_\bullet] \to G[\Delta_\bullet]$$

pointwisely as $k \to \infty$. It follows from the dominated convergence theorem that $\mathrm{DH}(\Delta^k_\bullet) \rightharpoonup \mathrm{DH}(\Delta_\bullet)$ as $k \to \infty$. □

Observe that

{eq:massDH}
$$\int_{\mathbb{R}} \mathrm{DH}(\Delta_\bullet) = \operatorname{vol}\Delta. \tag{3.1}$$

More generally, we compute the characteristic function of $G[\Delta_\bullet]$ as follows: for any $t \in \mathbb{C}$,

{eq:char}
$$\int_\Delta \mathrm{e}^{\mathrm{i}tG[\Delta_\bullet]}\,\mathrm{d}\lambda = \mathrm{e}^{\mathrm{i}t\tau^+}\operatorname{vol}\Delta - \mathrm{i}t\int_{-\infty}^{\tau^+} (\operatorname{vol}\Delta - \operatorname{vol}\Delta_\tau)\,\mathrm{e}^{\mathrm{i}t\tau}\,\mathrm{d}\tau. \tag{3.2}$$

In particular, the moments are given by

$$\int_{\mathbb{R}} x^m \mathrm{DH}(\Delta_\bullet)(x) = \int_\Delta G[\Delta_\bullet]^m\,\mathrm{d}\lambda = (\tau^+)^m \operatorname{vol}\Delta - \int_{-\infty}^{\tau^+} m\tau^{m-1}(\operatorname{vol}\Delta - \operatorname{vol}\Delta_\tau)\,\mathrm{d}\tau.$$

4. The Duistermaat–Heckman measure of a non-Archimedean metric

Let X be an connected compact Kähler manifold of dimension n and θ be a closed real smooth $(1,1)$-form on X such that $\mathrm{PSH}(X,\theta) \neq \varnothing$. We will define the Duistermaat–Heckman measure of elements in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as studied in [DXZ23; Xia23]. We will follow the notations in [Xia23].

4.1. **Non-Archimedean metrics.** Consider an element $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, recall that by definition, Γ is an inverse system $(\Gamma^{\theta+\omega})_\omega$ indexed by the directed set of Kähler forms on X ordered by reverse of the usual comparison. For each ω,

$$\Gamma^{\theta+\omega}\colon (-\infty, \Gamma_{\max}) \to \mathrm{PSH}(X, \theta+\omega)$$

is a decreasing concave curve of $\mathcal{I}$-model potentials. The number $\Gamma_{\max} \in \mathbb{R}$ is independent of the choice of ω. The transition map from the index ω to $\omega+\omega'$ sends $\Gamma^{\theta+\omega}$ to the following map

$$(-\infty, \Gamma_{\max}) \to \mathrm{PSH}(X, \theta+\omega+\omega'), \quad \tau \mapsto P_{\theta+\omega+\omega'}\left[\Gamma^{\theta+\omega}_\tau\right]_{\mathcal{I}}.$$

The volume of Γ is defined as the limit

$$\lim_\omega \left(\theta+\omega+\mathrm{dd^c}\Gamma^{\theta+\omega}_{-\infty}\right)^n.$$

Here $\Gamma^{\theta+\omega}_{-\infty} = \sup^*_{\tau<\Gamma_{\max}} \Gamma^{\theta+\omega}_\tau$.

The subset $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ of $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ consisting of elements with positive volume can be identified with the set of concave curves of $\mathcal{I}$-model potentials $(\Gamma_\tau)_{\tau<\Gamma_{\max}}$ in $\mathrm{PSH}(X,\theta)$ for some $\Gamma_{\max}\in\mathbb{R}$ such that the volume $\int_X(\theta+\mathrm{dd^c}\Gamma_{-\infty})^n > 0$.

4.2. **The Duistermaat–Heckman measure.** We fix a smooth flag $Y_\bullet$ on X.

Now suppose that $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. We define the Okounkov test curve $(\Delta_{Y_\bullet}(\Gamma)_\tau)_{\tau<\Gamma_{\max}}$ associated with Γ as follows: given $\tau < \Gamma_{\max}$, we set

$$\Delta_{Y_\bullet}(\Gamma)_\tau \coloneqq \Delta_{Y_\bullet}(\theta+\mathrm{dd^c}\Gamma_\tau).$$

The right-hand side is the partial Okounkov body studied in [DX24].

Proposition 4.1. *Given $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, the curve $(\Delta_{Y_\bullet}(\Gamma)_\tau)_{\tau<\Gamma_{\max}}$ is an Okounkov test curve relative to $\Delta_{Y_\bullet}(\theta+\mathrm{dd^c}\Gamma_{-\infty})$.*

Proof. This is a simple consequence of the properties proved in [DX24]. □

Definition 4.2. The *Duistermaat–Heckman measure* $\mathrm{DH}(\Gamma)$ of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ is defined as the Duistermaat–Heckman measure of the Okounkov test curve $\Delta_{Y_\bullet}(\Gamma)$.

thm:DHindep
Theorem 4.3. *The Duistermaat–Heckman measure $\mathrm{DH}(\Gamma)$ of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ is independent of the choice of the flag $Y_\bullet$.*

Proof. Assume further more that Γ is bounded ($\Gamma_\tau = \Gamma_{-\infty}$ for small enough τ), we observe that the characteristic function of the random variable $G[\Delta_{Y_\bullet}(\Gamma)]$ as computed in (3.2) is independent of the choice of the flag and is entire. It is a classical result that in this case, the corresponding probability distribution is determined by the moments.

In general, Γ is the decreasing limit of the sequence $\Gamma \vee \Gamma^k$ as $k \to \infty$, where $\Gamma^k \colon (-\infty, -k) \to \mathrm{PSH}(X, \theta)$ takes the constant value $\Gamma_{-\infty}$. It follows from the general continuity result proved in [DX24] that $\Delta_{Y_\bullet}(\Gamma)_\tau$ is the decreasing limit of $\Delta_{Y_\bullet}(\Gamma \vee \Gamma^k)_\tau$ for any $\tau < \Gamma_{\max}$. So $\mathrm{DH}(\Gamma \vee \Gamma^k) \rightharpoonup \mathrm{DH}(\Gamma)$ by Lemma 3.6. It follows that $\mathrm{DH}(\Gamma)$ is independent of the choice of the flag. □

More generally, when X does not admit a smooth flag, we could make a sequence of blowingups with smooth centers $\pi \colon Y \to X$ so that Y admits a flag. We define

$$\mathrm{DH}(\Gamma) = \mathrm{DH}(\pi^* \Gamma).$$

It follows from the bimeromorphic invariance of the partial Okounkov bodies that this measure is independent of the choice of π.

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