NOTE ON DUCROS' BOOK — CHAPTER 3

MINGCHEN XIA

Contents

1. Introduction

These are a series of notes on the book [DucCurve Duc24].

2. Notes

Let k be a non-Archimedean analytic field, namely a complete non-Archimedean real-valued field. We allow the valuation of k to be trivial.

3.1.1.3.

Definition 2.1. A morphism $\varphi\colon Y\to X$ between k-analytic spaces is *constant* if one of the following conditions holds:

(1) $Y=\varnothing$;
(2) $\varphi(Y)$ is a singleton consisting of a *rigid* point.

An analytic function $f\in \mathrm{H}^0(X,\mathcal{O}_X)$ on a k-analytic space X is *constant* if the induced morphism

{eq:YtoA1}
$$\tilde{f}\colon X\to \mathbb{A}^{1,\mathrm{an}}_k. \tag{2.1}$$

is constant.

Recall that the morphism (2.1) is constructed on the underlying sets as follows: Suppose that $P\in k[T]$, then

$$|P|_{\tilde{f}(x)}\coloneqq |P(f)|_x.$$

st_fun_char
Proposition 2.2. *Let X be a connected k-analytic space and $f\in \mathrm{H}^0(X,\mathcal{O}_X)$. Then the following are equivalent:*

(1) f is constant;
(2) the image of f in $\mathrm{H}^0(X_{\mathrm{red}},\mathcal{O}_{X_{\mathrm{red}}})$ is algebraic over k.

When X is quasi-compact, Condition (2) is satisfied if and only if f is algebraic over k.

The non-trivial direction is the direct one. Suppose that P is a monic polynomial in $k[T]$ with $P(f)=0$ in $\mathrm{H}^0(X_{\mathrm{red}},\mathcal{O}_{X_{\mathrm{red}}})$. Then $P(f)^n=0$ for some positive integer n. Hence f is algebraic.

Proof. Note that f is constant if and only if its image in $\mathrm{H}^0(X_{\mathrm{red}},\mathcal{O}_{X_{\mathrm{red}}})$ is constant. So we may assume that X is reduced.

(1) $\implies$ (2). Let $t\in\mathbb{A}^{1,\mathrm{an}}_k$ be the image of X. Then t is a rigid point by assumption. So we can find a maximal ideal $\mathfrak{m}$ in $k[T]$ so that t is the rigid point corresponding to $\mathfrak{m}$. Take a monic generator $P\in k[T]$ of $\mathfrak{m}$. We claim that $P(f)=0$.

Since X is reduced, it suffices to verify that $|P(f)|_x=0$ for each $x\in X$. But this holds trivially by our assumption.

(2) $\implies$ (1). Let $Q \in k[T]$ be the minimal polynomial of f. Then we claim that $\tilde{f}$ is constant with image being the rigid point defined by (Q).

Let $x \in X$. Then $f(x)$ clearly descends to a valuation of $k[T]/(Q)$, which extends the valuation on k. But such valuations are unique since $k[T]/(Q)$ is a finite extension of k. Our assertion follows. □

Let $\mathrm{Cst}(X) \subseteq \mathrm{H}^0(X, \mathcal{O}_X)$ denote the k-algebra of constant analytic functions on X.

ro_cst_char *Remark* 2.3. We observe that a constant function $f \in \mathrm{Cst}(X)$ is corresponds to $a \in k$ if and only if the image of $\tilde{f}$ is the rigid point corresponding to the maximal ideal $(T-a)$ in $k[T]$.

This can be reduced immediately to the case where X is connected, then it is a direct consequence of the proof of Proposition 2.2.

3.1.1.4. Let k be a non-Archimedean analytic field. Let k^a denote an algebraic closure of k and k^s be the separable closure of k in k^a. Let X be a k-analytic space.

Definition 2.4. Define the k-algebra $\mathfrak{s}(X) \subseteq \mathrm{H}^0(X, \mathcal{O}_X)$ as follows: An analytic function $f \in \mathrm{H}^0(X, \mathcal{O}_X)$ lies in $\mathfrak{s}(X)$ if G-locally f is separable and algebraic over k.

Thanks to Proposition 2.2, the functions in $\mathfrak{s}(X)$ are G-locally constant.

rop:sXinCst **Proposition 2.5.** *Assume that X is connected. Then $\mathfrak{s}(X)$ is a finite separable field extension of k.*

Moreover,

{eq:sXcst}
$$\mathfrak{s}(X) \subseteq \mathrm{Cst}(X). \tag{2.2}$$

If $f \in \mathfrak{s}(X)$, then the image of f in $\mathrm{H}^0(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$ is separable and algebraic over k.

Proof. There are two non-trivial facts to prove: $\mathfrak{s}(X)$ is a field. Let $f \in \mathfrak{s}(X)$ be a non-zero element. Then thanks to Remark 2.3, f is G-locally non-zero as well. So our assertion follows.

Next, let $f \in \mathfrak{s}(X)$. Thanks to Proposition 2.2, the image f' of f in $\mathrm{H}^0(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$ is algebraic over k. Let $P \in k[T]$ be its minimal polynomial. We want to show that P is separable. By assumption, we can find an affinoid domain $U \subseteq X$ so that $f|_U$ is separable and algebraic over k. Let $Q \in k[T]$ be its minimal polynomial. Note that Q is separable. Observe that

$$Q\left(f'|_{U_{\mathrm{red}}}\right) = 0, \quad P\left(f'|_{U_{\mathrm{red}}}\right) = 0.$$

Since both P and Q are monic and irreducible, it follows that $P = Q$. our assertion follows. □

Lemma 2.6. *Assume that X is connected and geometrically reduced (namely, if $X_{\widehat{k^a}}$ is reduced), then (2.2) is an equality.*

Proof. Note that X is reduced by [Duc18, Proposition 2.6.7].

Let $f \in \mathrm{Cst}(X)$. We want to show that f is G-locally separable over k. The problem is clearly G-local on X, so we may assume that X is k-affinoid, say $X = \mathrm{Sp}\, A$ for some k-affinoid algebra A. By assumption, $A \hat{\otimes}_k \widehat{k^a}$ is reduced. In particular, due to Gruson's theorem [Gru66, Théorème 1], A is geometrically reduced.

By Proposition 2.2, f is algebraic over k. Let $P \in k[T]$ be the minimal polynomial of f. Then P has to be separable, since otherwise, $k[f] \otimes_k k^a$ is not reduced. □

Lemma 2.7. *The following are equivalent:*

(1) X is geometrically connected (namely, if $X_{\widehat{k^a}}$ is connected);
(2) $\mathfrak{s}(X) = k$.

Proof. (1) $\implies$ (2). Let $f \in \mathfrak{s}(X)$. Since X is clearly connected, f is constant. Let $P \in k[T]$ be the minimal polynomial of f' over k, where f' is the image of f in $\mathrm{H}^0(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$. Then P is separable due to Proposition 2.5. Note that the image of f in $\mathrm{H}^0(X_{\widehat{k^a}}, \mathcal{O}_{X_{\widehat{k^a}}})$ is constant, thanks to Proposition 2.2.

Note that P cannot have a root different from f' in k^a. Since otherwise, by Chinese remainder theorem, $\mathrm{H}^0(X_{\widehat{k^a}}, \mathcal{O}_{X_{\widehat{k^a}}})$ would have to contain a non-trivial idempotent, which contradicts the connectedness of $X_{\widehat{k^a}}$. As a result, P has degree 1 and hence $f' \in k$. It follows from Remark 2.3 that $f \in k$ as well.

(2) $\implies$ (1). Assume (2). Then clearly X is connected. Suppose that (1) fails. Then one can construct a non-trivial fixed point of the $\mathrm{Gal}(k^s/k)$-action on $(k^a)^{\pi_0(X_{\widehat{k^a}})}$, which contradicts (2). □

3.1.1.5. Let k be a non-Archimedean analytic field, X be a k-analytic space and $x \in X$.

This part contains an annoying typo, which confused me for a week: On line 5, Si $[\mathfrak{s}(x) = k]$ est finie should be Si $[\mathfrak{s}(x) : k]$ est fini.

Definition 2.8. We write $\mathfrak{s}(x)$ for the separable closure of k in $\mathcal{H}(x)$.

The evaluation map

$$\chi_x \colon \mathcal{O}_{X,x} \to \mathcal{H}(x)$$

induces a homomorphism of k-algebras

$$\mathfrak{s}(X) \to \mathcal{H}(x).$$

Note that contrary to Ducros' claim, this map is not injective in general.

This map can be factorized through

$$\mathfrak{s}(X) \to \mathfrak{s}(x).$$

Next we assume that X is good.

If U is a *connected* k-analytic neighborhood of X containing x. Then we get an injective homomorphism of k-algebras

{eq:sUsx}
$$\mathfrak{s}(U) \hookrightarrow \mathfrak{s}(x). \tag{2.3}$$

Recall that $\mathcal{O}_{X,x}$ is Henselian ([Berk93 Ber93, Theorem 2.1.5]). It follows that each element in $\mathfrak{s}(x)$ can be lifted to $\mathfrak{s}(U)$ for small enough U. We conclude that

Proposition 2.9. *Assume that X is good and $[\mathfrak{s}(x) : k]$ is finite. Then there is a basis of connected k-affinoid neighborhoods U of x such that the natural homomorphism (2.3) is an isomorphism.*

References

Berk93 [Ber93] V. G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.* 78 (1993), 5–161 (1994). URL: http://www.numdam.org/item?id=PMIHES_1993__78__5_0.

DucFam [Duc18] A. Ducros. Families of Berkovich spaces. *Astérisque* 400 (2018), pp. vii+262.

DucCurve [Duc24] A. Ducros. La structure des courbes analytiques. 2024. arXiv: 2405.10619 [math.AG].

Gru66 [Gru66] L. Gruson. Théorie de Fredholm p-adique. *Bull. Soc. Math. France* 94 (1966), pp. 67–95. URL: http://www.numdam.org/item?id=BSMF_1966__94__67_0.

Mingchen Xia, Chalmers Tekniska Högskola and Institute of Geometry and Physics, USTC

Email address, xiamingchen2008@gmail.com

Homepage, https://mingchenxia.github.io/home/.