HAUSDORFF CONVERGENCE PROPERTY OF PARTIAL OKOUNKOV BODIES

MINGCHEN XIA

Contents

1. Introduction

This note is a refinement of [Xia21, Theorem A]. We prove the Hausdorff convergence property in full generality.

This note is motivated by a discussion with Sébastien Boucksom.

2. Hausdorff convergence property

Let X be a connected smooth projective variety of dimension n. Let (L,h) be a Hermitian pseudo-effective line bundle on X with $\int_X (\mathrm{dd^c}h)^n > 0$. Fix $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ a valuation of rank n and rational rank n. Take a smooth Hermitian metric h_0 on L and set $\theta = c_1(L,h_0)$. We can then identify h with $\varphi \in \mathrm{PSH}(X,\theta)$.

For each $k \in \mathbb{Z}_{>0}$, we introduce

$$\Delta^k_\nu(\theta,\varphi) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

Here Conv denotes the convex hull. The convex hull is necessarily closed by [LM09, Lemma 1.4].

For later use, we introduce a twisted version as well. If T is a holomorphic line bundle on X, we introduce

$$\Delta^{k,T}_\nu(\theta,\varphi) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, T\otimes L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

We also write

$$\Delta^{k,T}_\nu(L) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, T\otimes L^k) \right\} \subseteq \mathbb{R}^n$$

and

$$\Delta^{k}_\nu(L) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, L^k) \right\} \subseteq \mathbb{R}^n$$

We write $\mathcal{I}_\infty(\varphi) = \mathcal{I}_\infty(h)$ for the ideal sheaf on X locally consisting of holomorphic functions f such that $|f|_h$ is locally bounded.

We endow the space $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ with the Hausdorff metric. We remind the readers that the Minkowski sum is continuous with respect to the Hausdorff metric. We will always use this property without explicitly mentioning. For further backgrounds on convex bodies, we refer to [Sch14].

For notations related to Okounkov bodies and partial Okounkov bodies, we refer to [Xia21].

We first extend [Xia21, Theorem 3.13] to the twisted case.

Proposition 2.1. *For any holomorphic line bundle T on X,*

$$\Delta_\nu^{k,T}(L) \to \Delta_\nu(L)$$

as $k \to \infty$.

Proof. As L is big, we can take $k_0 \in \mathbb{Z}_{>0}$ so that

(1) $T^{-1} \otimes L^{k_0}$ admits a non-zero global holomorphic section s_0;
(2) $T \otimes L^{k_0}$ admits a non-zero global holomorphic section s_1.

Then for $k \in \mathbb{Z}_{>k_0}$, we have injective linear maps

$$H^0(X, L^{k-k_0}) \xrightarrow{\times s_1} H^0(X, T \otimes L^k) \xrightarrow{\times s_0} H^0(X, L^{k+k_0}).$$

It follows that

$$(k-k_0)\Delta_\nu^{k-k_0}(L) + \nu(s_1) \subseteq k\Delta_\nu^{k,T}(L) \subseteq (k+k_0)\Delta_\nu^{k+k_0}(L) - \nu(s_0).$$

By [Xia21, Theorem 3.13], we conclude. □

Lemma 2.2. *Let T be a holomorphic line bundle on X. Assume that φ has analytic singularities and φ has positive mass, then*

$$\Delta_\nu^{k,T}(\theta, \varphi) \to \Delta_\nu(\theta, \varphi)$$

as $k \to \infty$.

Proof. Up to replacing X by a birational model and twisting T accordingly, we may assume that φ has log singularities along a nc $\mathbb{Q}$-divisor D. Take $\epsilon \in (0,1) \cap \mathbb{Q}$. In this case, by Ohsawa–Takegoshi theorem, for any $k \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} H^0(X, T \otimes L^k \otimes \mathcal{I}_\infty(k\varphi)) &\subseteq H^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \\ &\subseteq H^0(X, T \otimes L^k \otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi)). \end{aligned}$$

Take an integer $N \in \mathbb{Z}_{>0}$ so that ND is a divisor and $N\epsilon$ is an integer.

Let Δ' be the limit of a subsequence of $(\Delta_\nu^{k,T}(\theta, \varphi))_k$, say the sequence defined by the indices $k_1, k_2, \ldots$. We want to show that $\Delta' = \Delta_\nu(\theta, \varphi)$.

There exists $t \in \{0, 1, \ldots, N-1\}$ such that $k_i \equiv t$ modulo N for infinitely many i, up to replacing k_i by a subsequence, we may assume that $k_i \equiv t$ modulo N for all i. Write $k_i = Ng_i + t$.

Now we have

$$\begin{aligned} \Delta_\nu^{g_i, T\otimes L^t}(NL - ND) + N\nu(D) &\subseteq N\Delta_\nu^{k,T}(\theta, \varphi) \\ \subseteq \Delta_\nu^{g_i, T\otimes L^t}(NL - N(1-\epsilon)D) &+ N(1-\epsilon)\nu(D). \end{aligned}$$

By Proposition 2.1,

$$\Delta_\nu(L - D) + \nu(D) \subseteq \Delta' \subseteq \Delta_\nu(L - (1-\epsilon)D) + (1-\epsilon)\nu(D).$$

Let $\epsilon \to 0+$, we find that

$$\Delta_\nu(L - D) + \nu(D) = \Delta'.$$

It follows from Blanschke selection theorem that

$$\Delta_\nu^{k,T}(\theta, \varphi) \to \Delta_\nu(L - D) + \nu(D) = \Delta_\nu(\theta, \varphi)$$

as $k \to \infty$. □

Lemma 2.3. *Assume that θ_φ is a Kähler current, then as $\mathbb{Q} \ni \beta \to 0+$, we have*

$$\Delta_\nu((1-\beta)\theta, \varphi) \to \Delta_\nu(\theta, \varphi).$$

Proof. By [Xia21, Proposition 5.15], we have

$$\Delta_\nu((1-\beta)\theta, \varphi) + \beta\Delta_\nu(L) \subseteq \Delta_\nu(\theta, \varphi).$$

In particular, if Δ' is a limit of a subsequence of $(\Delta_\nu((1-\beta)\theta, \varphi))_\beta$, then

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi).$$

But

$$\operatorname{vol}\Delta' = \lim_{\beta\to 0+} \Delta_\nu((1-\beta)\theta, \varphi) = \lim_{\beta\to 0+} \int_X ((1-\beta)\theta + \mathrm{dd^c} P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n.$$

We claim that

$$\lim_{\beta\to 0+} \int_X ((1-\beta)\theta + \mathrm{dd^c} P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n = \int_X (\theta + \mathrm{dd^c} P^{\theta}[\varphi]_{\mathcal{I}})^n.$$

Note that this finishes the proof as $\operatorname{vol}\Delta_\nu(\theta, \varphi)$ is exactly equal to the right-hand side.

Next we prove our claim. We make use of the b-divisors introduced in [Xia22b; Xia22a]. By [Xia22a, Theorem 0.6], the claim is equivalent to

$$\lim_{\beta\to 0+} \operatorname{vol}\mathbb{D}((1-\beta)\theta, \varphi) = \operatorname{vol}\mathbb{D}(\theta, \varphi),$$

which is obvious. □

Theorem 2.4. *Let T be a holomorphic line bundle on X. As $k \to \infty$, $\Delta_\nu^{k,T}(\theta, \varphi) \to \Delta_\nu(\theta, \varphi)$.*

Proof. Fix a Kähler form $\omega \geq \theta$ on X.

Step 1. We first handle the case where $\mathrm{dd^c} h$ is a Kähler current, say $\mathrm{dd^c} h \geq \beta_0\omega$ for some $\beta_0 \in (0, 1)$.

Take a decreasing quasi-equisingular approximation φ_j of φ. Up to replacing β_0 by $\beta_0/2$, we may assume that $\theta_{\varphi_j} \geq \beta_0\omega$ for all $j \geq 1$.

Let Δ' be a limit of a subsequence of $(\Delta_\nu^{k,T}(\theta, \varphi))_k$. Let us say the indices of the subsequence are $k_1 < k_2 < \cdots$. By Blaschke selection theorem, it suffices to show that $\Delta' = \Delta_\nu(\theta, \varphi)$.

As $\varphi \leq \varphi_j$ for each $j \geq 1$, we have

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi_j)$$

by Lemma 2.2. Let $j \to \infty$, we find

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi).$$

In particular, it suffices to prove that

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta_\nu(\theta, \varphi).$$

Take $\beta \in (0, \beta_0) \cap \mathbb{Q}$. Write $\beta = p/q$ with $p, q \in \mathbb{Z}_{>0}$. Observe that for any $j \geq 1$,

$$\theta_{\varphi_j} \geq \beta\omega \geq \beta\theta.$$

Namely, $\varphi_j \in \mathrm{PSH}(X,(1-\beta)\theta)$. Similarly, $\varphi \in \mathrm{PSH}(X,(1-\beta)\theta)$. By Lemma 2.3, it suffices to argue that

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta_\nu((1-\beta)\theta,\varphi). \tag{2.1}$$

For this purpose, we are free to replace k_i's by a subsequence, so we may assume that $k_i \equiv a$ modulo q for all $i \geq 1$, where $a \in \{0,1,\ldots,q-1\}$. We write $k_i = g_i q + a$. Observe that for each $i \geq 1$,

$$H^0(X, T\otimes L^{k_i}\otimes \mathcal{I}(k_i\varphi)) \supseteq H^0(X, T\otimes L^{-q+a}\otimes L^{g_iq+q}\otimes \mathcal{I}((g_iq+q)\varphi)).$$

Up to replacing T by $T\otimes L^{-q+a}$, we may therefore assume that $a=0$.

By [DX21, Lemma 4.2], we can find $k' \in \mathbb{Z}_{>0}$ such that for all $k \geq k'$, there is $v_{\beta,k} \in \mathrm{PSH}(X,\theta)$ satisfying

(1)
$$P[\varphi]_{\mathcal{I}} \geq (1-\beta)\varphi_k + \beta v_{\beta,k};$$
(2) $v_{\beta,k}$ has positive mass.

Fix $k \geq k'$. It suffices to show that

$$\Delta_\nu((1-\beta)\theta,\varphi_k) + v' \subseteq \Delta' \tag{2.2}$$

for some $v' \in \mathbb{R}^n_{\geq 0}$. In fact, if this is true, we have

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta_\nu((1-\beta)\theta,\varphi_k).$$

Let $k\to\infty$, by [Xia21, Theorem A], we conclude (2.1).

It remains to prove (2.2). Let $\pi : Y \to X$ be a log resolution of the singularities of φ_k. By the proof of [DX21, Proposition 4.3], there is $j_0 = j_0(\beta,k) \in \mathbb{Z}_{>0}$ such that for any $j \geq j_0$, we can find a non-zero section $s_j \in H^0(Y,\pi^*L^{pj}\otimes\mathcal{I}(jp\pi^*v_{\beta,k}))$ such that we get an injective linear map

$$H^0(Y,\pi^*T\otimes K_{Y/X}\otimes\pi^*L^{(q-p)j}\otimes\mathcal{I}(jq\pi^*\varphi_k)) \xrightarrow{\times s_j} H^0(X,T\otimes L^{jq}\otimes\mathcal{I}(jq\varphi)).$$

In particular, when $j=k_i$ for some i large enough, we then find

$$\Delta_\nu^{k_i,\pi^*T\otimes K_{Y/X}}((1-\beta)q\pi^*\theta, q\pi^*\varphi_k) + (k_i)^{-1}\nu(s_{k_i}) \subseteq q\Delta_\nu^{k_i,T}(\theta,\varphi).$$

We observe that $(k_i)^{-1}\nu(s_{k_i})$ is bounded as the right-hand side is bounded when i varies. Then by Lemma 2.2, there is a vector $v' \in \mathbb{R}^n_{\geq 0}$ such that

$$\Delta_\nu((1-\beta)\pi^*\theta,\pi^*\varphi_k) + v' \subseteq \Delta'.$$

By the birational invariance of the partial Okounkov bodies, we find (2.2).

Step 2. Next we handle the general case.

Let Δ' be the limit of a subsequence of $(\Delta_\nu^{k,T}(\theta,\varphi))_k$, say the subsequence with indices $k_1 < k_2 < \cdots$. By Blaschke selection theorem, it suffices to prove that $\Delta' = \Delta_\nu(\theta,\varphi)$.

Take $\psi \in \mathrm{PSH}(X,\theta)$ such that

(1) θ_ψ is a Kähler current;
(2) $\psi \leq \varphi$.

The existence of ψ is proved in [DX21, Proposition 3.6].

Then for any $\epsilon \in \mathbb{Q}\cap(0,1)$,

$$\Delta_\nu^{k,T}(\theta,\varphi) \supseteq \Delta_\nu^{k,T}(\theta,(1-\epsilon)\varphi+\epsilon\psi)$$

for all k. It follows from Step 1 that

$$\Delta' \supseteq \Delta_\nu(\theta,(1-\epsilon)\varphi+\epsilon\psi).$$

Letting $\epsilon \to 0$ and applying [Xia21, Theorem A], we have

$$\Delta' \supseteq \Delta_\nu(\theta, \varphi).$$

It remains to establish that

$$\operatorname{vol} \Delta' \leq \operatorname{vol} \Delta_\nu(\theta, \varphi). \tag{2.3}$$

For this purpose, we are free to replace $k_1 < k_2 < \cdots$ by a subsequence. Fix $q > 0$, we may then assume that $k_i \equiv a$ modulo q for all $i \geq 1$ for some $a \in \{0, 1, \ldots, q-1\}$. We write $k_i = g_i q + a$. Observe that

$$H^0(X, T \otimes L^{k_i} \otimes \mathcal{I}(k_i\varphi)) \subseteq H^0(X, T \otimes L^a \otimes L^{g_i q} \otimes \mathcal{I}(g_i q\varphi)).$$

Up to replacing T by $T \otimes L^a$, we may assume that $a = 0$.

Take a very ample line bundle H on X and fix a Kähler form $\omega \in c_1(H)$, take a non-zero section $s \in H^0(X, H)$.

We have an injective linear map

$$H^0(X, T \otimes L^{jq} \otimes \mathcal{I}(jq\varphi)) \xrightarrow{\times s^j} H^0(X, T \otimes H^j \otimes L^{jq} \otimes \mathcal{I}(jq\varphi))$$

for each $j \geq 1$. In particular, for each $i \geq 1$,

$$k_i \Delta_\nu^{k_i, T}(q\theta, q\varphi) + k_i \nu(s) \subseteq k_i \Delta_\nu^{k_i, T}(\omega + q\theta, q\varphi).$$

Let $i \to \infty$, by Step 1,

$$q\Delta' + \nu(s) \subseteq \Delta_\nu(\omega + q\theta, q\varphi).$$

So

$$\operatorname{vol} \Delta' \leq \operatorname{vol} \Delta_\nu(q^{-1}\omega + \theta, \varphi) = \int_X (q^{-1}\omega + \theta + \mathrm{dd^c} P^{q^{-1}\omega + \theta}[\varphi]_{\mathcal{I}})^n.$$

By [Xia21, Corollary 4.4],

$$\operatorname{vol} \Delta' \leq \int_X (q^{-1}\omega + \theta + \mathrm{dd^c} P^{\theta}[\varphi]_{\mathcal{I}})^n.^*$$

Let $q \to \infty$, we conclude (2.3). □

*This can be argued more directly using b-divisors, as in the proof of Lemma 2.3.

References

[DX21] T. Darvas and M. Xia. The volume of pseudoeffective line bundles and partial equilibrium. *Geometry & Topology* (2021). arXiv: 2112.03827 [math.DG].

[LM09] R. Lazarsfeld and M. Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)* 42.5 (2009), pp. 783–835. URL: https://doi.org/10.24033/asens.2109.

[Sch14] R. Schneider. Convex bodies: the Brunn–Minkowski theory. 151. Cambridge university press, 2014.

[Xia21] M. Xia. Partial Okounkov bodies and Duistermaat–Heckman measures of non-Archimedean metrics. 2021. arXiv: 2112.04290 [math.AG].

[Xia22a] M. Xia. Non-pluripolar products on vector bundles and Chern–Weil formulae on mixed Shimura varieties. 2022. arXiv: 2210.15342 [math.AG].

[Xia22b] M. Xia. Pluripotential-Theoretic Stability Thresholds. *International Mathematics Research Notices* (2022). rnac186. URL: https://doi.org/10.1093/imrn/rnac186.

Mingchen Xia, Department of Mathematics, Institut de Mathématiques de Jussieu-Paris Rive Gauche

Email address, mingchen@imj-prg.fr

Homepage, https://mingchenxia.github.io/home/.