NOTES ON HODGE THEORY — CARLSON'S CORRESPONDENCE

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Contents

1. Introduction

The result in this note is due to [Car80]. Unfortunately, I have never been able to get a copy of [Car80]. A lot of papers and lecture notes on this subject indicate the construction of this bijection. I spend some time to write down the full details.

2. Carlson's correspondence

Let MHS be the category of $\mathbb{Z}$-mixed Hodge structures. An object of MHS then consists of $(V, \mathcal{F}^\bullet, \mathcal{W}_\bullet)$, where V is a free $\mathbb{Z}$-module of finite rank, $\mathcal{F}^\bullet$ is a filtration on $V_\mathbb{C}$ and $\mathcal{W}_\bullet$ is a filtration on $V_\mathbb{Q}$. We require the usual axioms. We can also regard $\mathcal{W}_\bullet$ as a saturated filtration on V. By abuse of language, we say $V \in \mathrm{MHS}$. When we refer to the filtered $\mathbb{Z}$-module underlying V, we mean $(V, \mathcal{W}_\bullet)$.

We define the *Jacobian* of V as

$$\mathrm{J}V = \mathcal{W}_0 V_\mathbb{C} / \left(\mathcal{W}_0 V + \mathcal{F}^0 V_\mathbb{C} \cap \mathcal{W}_0 V_\mathbb{C} \right).$$

Theorem 2.1 (Carlson). *Let $V, W \in \mathrm{MHS}$. There is a group isomorphism from $\mathrm{Ext}^1_{\mathrm{MHS}}(W, V)$ to $\mathrm{J}\,\mathrm{Hom}_\mathbb{Z}(W, V)$.*

Proof. **Step 1**. We construct the map

{eq:map1} (2.1)
$$\mathrm{Ext}^1_{\mathrm{MHS}}(W, V) \to \mathrm{J}\,\mathrm{Hom}_\mathbb{Z}(W, V).$$

Let

{eq:extension} (2.2)
$$0 \to V \to E \xrightarrow{\pi} W \to 0$$

be a short exact sequence in MHS. As W is a projective object in the category of filtered $\mathbb{Z}$-modules, we can find a splitting

$$r : E \to V$$

of (2.2) in the category of filtered $\mathbb{Z}$-modules. Let

$$s : W_\mathbb{C} \to E_\mathbb{C}$$

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be a section of π, which is a morphism of $\mathbb{C}$-mixed Hodge structures. The existence of s follows from the functoriality of the Deligne decomposition. We let $e \in \operatorname{Hom}_{\mathbb{C}}(W_{\mathbb{C}}, V_{\mathbb{C}})$ be the composition $r \circ s$. By our choices of r and s, we have $e \in \mathcal{W}_0 \operatorname{Hom}_{\mathbb{C}}(W, V)$. We define the image of E under (2.1) as the coset defined by e.

We need to show that this coset is well-defined.

We first handle the freedom in choosing r. If $r' : E \to V$ is another splitting of (2.2) in the category of filtered $\mathbb{Z}$-modules, then $r - r' : E \to V$ is a morphism of filtered $\mathbb{Z}$-modules that vanishes on V. We can therefore view $r - r'$ as a linear map $a : \operatorname{Hom}_{\mathbb{Z}}(W, V)$. As π is strict (This is a theorem of Deligne!), we see that $\mathcal{W}_k W$ is exactly $\pi(\mathcal{W}_k E)$ for each $k \in \mathbb{Z}$, so it follows that $a \in \mathcal{W}_0 \operatorname{Hom}_{\mathbb{Z}}(W, V)$. If we replace r by r', we will replace e by $e + a \circ \pi \circ s = e + a$. So we see that the cosets in $\mathrm{J} \operatorname{Hom}_{\mathbb{Z}}(W, V)$ remain the same.

Next we handle the freedom in choosing s. If $s' : W_{\mathbb{C}} \to E_{\mathbb{C}}$ is another section of π, which is a morphism of $\mathbb{C}$-mixed Hodge structures, then $s - s' : W_{\mathbb{C}} \to E_{\mathbb{C}}$ has image lying in V. In other words, we identify $s - s'$ with $b \in \operatorname{Hom}_{\mathbb{C}}(W, V)$. Again, using the strictness of $V \to E$, we find that $b \in \mathcal{F}^0 \operatorname{Hom}_{\mathbb{C}}(W, V) \cap \mathcal{W}_0 \operatorname{Hom}_{\mathbb{C}}(W, V)$. If we replace s by s', then e becomes

$$e + r \circ b = e + b,$$

where in the first equation, we omit the inclusion map $V \to E$. Again, we end up with the same coset in $\mathrm{J} \operatorname{Hom}_{\mathbb{Z}}(W, V)$.

We conclude that (2.1) is well-defined.

For later use, we observe that we have an isomorphism of filtered $\mathbb{Z}$-modules $E \to V \oplus W$ given by (r, π). Under this isomorphism

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$$\mathcal{F}^p E_{\mathbb{C}} \mapsto \{(v, w) \in V_{\mathbb{C}} \oplus W_{\mathbb{C}} : e(w) - v \in \mathcal{F}^p V_{\mathbb{C}}, w \in \mathcal{F}^p W_{\mathbb{C}}\}. \tag{2.3}$$

Step 2. We construct the map

{eq:map2}
$$\mathrm{J} \operatorname{Hom}_{\mathbb{Z}}(W, V) \to \operatorname{Ext}^1_{\mathrm{MHS}}(W, V). \tag{2.4}$$

Let $\varphi \in \mathcal{W}_0 \operatorname{Hom}_{\mathbb{Z}}(W, V)$. We define an extension $E \in \operatorname{Ext}^1_{\mathrm{MHS}}(W, V)$ as follows: the underlying filtered $\mathbb{Z}$-module of E is the direct sum of the underlying filtered $\mathbb{Z}$-modules of W and V. The Hodge filtration is defined as follows:

{eq:FpEC}
$$\mathcal{F}^p E_{\mathbb{C}} = \{(v, w) \in V_{\mathbb{C}} \oplus W_{\mathbb{C}} : \varphi(w) - v \in \mathcal{F}^p V_{\mathbb{C}}, w \in \mathcal{F}^p W_{\mathbb{C}}\}. \tag{2.5}$$

We first verify that $(E, \mathcal{F}^\bullet, \mathcal{W}_\bullet)$ is indeed a mixed Hodge structure. Fix $k \in \mathbb{Z}$, then

$$\mathcal{F}^p \operatorname{Gr}^{\mathcal{W}}_k E_{\mathbb{C}} = \left(\mathcal{F}^p E_{\mathbb{C}} \cap \mathcal{W}_k E_{\mathbb{C}} + \mathcal{W}_{k-1} E_{\mathbb{C}}\right) / \mathcal{W}_{k-1} E_{\mathbb{C}}.$$

for any $p \in \mathbb{Z}$. We rewrite the right-hand side as

$$\left\{(v, w) \in \operatorname{Gr}^{\mathcal{W}}_k V_{\mathbb{C}} \times \operatorname{Gr}^{\mathcal{W}}_k W_{\mathbb{C}} : \varphi(w) - v \in \mathcal{F}^p \operatorname{Gr}^{\mathcal{W}}_k V_{\mathbb{C}}, w \in \mathcal{F}^p \operatorname{Gr}^{\mathcal{W}}_k W_{\mathbb{C}}\right\}.$$

Now let $p, q \in \mathbb{Z}$, $p + q = k + 1$. Take $(v, w) \in \operatorname{Gr}^{\mathcal{W}}_k V_{\mathbb{C}} \times \operatorname{Gr}^{\mathcal{W}}_k W_{\mathbb{C}}$, then we can uniquely decompose

$$w = w_1 + \overline{w_2}, \quad w_1 \in \mathcal{F}^p \operatorname{Gr}^{\mathcal{W}}_k W_{\mathbb{C}}, w_2 \in \mathcal{F}^q \operatorname{Gr}^{\mathcal{W}}_k W_{\mathbb{C}}.$$

Then

$$\varphi(w) = \varphi(w_1) + \overline{\varphi(w_2)}.$$

Similarly, we uniquely decompose

$$v - \varphi(w) = v_1 + \overline{v_2}, \quad v_1 \in \mathcal{F}^p \operatorname{Gr}_k^{\mathcal{W}} V_{\mathbb{C}}, v_2 \in \mathcal{F}^q \operatorname{Gr}_k^{\mathcal{W}} V_{\mathbb{C}}.$$

Then we find that

$$(v, w) = (v_1 + \varphi(w_1), w_1) + \overline{(v_2 + \varphi(w_2), w_2)}.$$

Clearly, this decomposition is unique. That is, $\mathcal{F}^\bullet \operatorname{Gr}_k^{\mathcal{W}} E_{\mathbb{C}}$ is a pure Hodge structure of weight k. It follows that E is a $\mathbb{Z}$-mixed Hodge structure. We can view $E \in \operatorname{Ext}^1_{\mathrm{MHS}}(W, V)$ in the obvious way:

:extensionstructure}

$$0 \to V \xrightarrow{v \mapsto (v,0)} E \xrightarrow{(v,w) \mapsto w} W \to 0. \tag{2.6}$$

Next we verify that $\mathcal{F}^\bullet$ does not depend on the choice of the representative of φ. There are two types of freedoms in the definition of φ.

If we modify φ by an element in $\mathcal{F}^0 V_{\mathbb{C}} \cap \mathcal{W}_0 V_{\mathbb{C}}$, it it clear from (2.5) that we end up with the same Hodge filtration. On the other hand, if we take $a \in \mathcal{W}_0 \operatorname{Hom}_{\mathbb{Z}}(W, V)$ and replace φ by $\varphi + a$, let us denote the resulting mixed Hodge structure by E'. We have an isomorphism of $\mathbb{Z}$-mixed Hodge structures:

$$E \to E', \quad (v, w) \mapsto (v - a(w), w).$$

Of course, this isomorphism preserves the extension structure in (2.6). Now we see that (2.4) is well-defined.

Step 3. We verify that the two maps (2.1) and (2.4) are inverse to each other.

We begin with an extension E of V by W as in (2.2). We construct e as in Step 1. By (2.3), we see that the image of e under (2.4) is exactly E.

Conversely, if we begin with $\varphi \in \mathcal{W}_0 \operatorname{Hom}_{\mathbb{Z}}(W, V)$ as in Step 2, we define E as in Step 2, then we can define $r : E \to V$ in Step 1 as the usual projection and $s : W_{\mathbb{C}} \to E_{\mathbb{C}}$ as $w \mapsto (\varphi(w), w)$. Then we see that e in Step 1 is exactly φ.

Step 4. We show that the map constructed in Step 1 is a group homomorphism. We let E_1, E_2 be two extensions of V by W in MHS. Recall that the Baer sum $E_1 + E_2$ is constructed as follows

$$\begin{tikzcd}
0 \arrow[r] & V \oplus V \arrow[r] \arrow[d, "\Sigma"] & E_1 \oplus E_2 \arrow[r] \arrow[d] & W \oplus W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & W \oplus W \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] & E_1 + E_2 \arrow[r] & W \arrow[r] \arrow[u, "\Delta"] & 0
\end{tikzcd},$$

where the upper left square is a pushout square and the lower right square is a pull-back square. The map $\Sigma : V \oplus V \to V$ sends (v, v') to $v + v'$ and $\Delta : W \to W \oplus W$ send w to (w, w). They are both morphisms in MHS. More explicitly, the underlying filtered $\mathbb{Z}$-module of E' is just the pushforward of the underlying filtered $\mathbb{Z}$-modules of the other objects in the upper left square. Similarly, the underlying filtered $\mathbb{Z}$-module of $E_1 + E_2$ is just the pull-back of the underlying filtered $\mathbb{Z}$-modules of the other objects in the lower right square.

We construct $r_1, r_2, s_1, s_2, e_1, e_2$ as in Step 1. Then $r_1 \oplus r_2 : E_1 \oplus E_2 \to V \oplus V$ induces a morphism $E' \to V$ of filtered $\mathbb{Z}$-modules and then a morphism $E_1 + E_2 \to V$ of filtered $\mathbb{Z}$-modules. Similarly, $s_1 \oplus s_2 : W_{\mathbb{C}} \oplus W_{\mathbb{C}} \to E_{1,\mathbb{C}} \oplus E_{2,\mathbb{C}}$ induces a morphism $W_{\mathbb{C}} \oplus W_{\mathbb{C}} \to E'_{\mathbb{C}}$ of $\mathbb{C}$-mixed Hodge structures and then $W_{\mathbb{C}} \to E_{1,\mathbb{C}} \oplus E_{2,\mathbb{C}}$ of $\mathbb{C}$-mixed Hodge structures. We want to understand the composition

$$W_{\mathbb{C}} \to E_{1,\mathbb{C}} \oplus E_{2,\mathbb{C}} \to V_{\mathbb{C}}.$$

We will see explicitly what the curved maps are in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & V_{\mathbb{C}} \oplus V_{\mathbb{C}} \arrow[r] \arrow[d, "\Sigma"] & E_{1,\mathbb{C}} \oplus E_{2,\mathbb{C}} \arrow[r] & W_{\mathbb{C}} \oplus W_{\mathbb{C}} \arrow[r] \arrow[d] \arrow[ll, bend right, "(r_1 \oplus r_2) \circ (s_1 \oplus s_2)"'] & 0 \\
0 \arrow[r] & V_{\mathbb{C}} \arrow[r] \arrow[d] & E'_{\mathbb{C}} \arrow[r] & W_{\mathbb{C}} \oplus W_{\mathbb{C}} \arrow[r] \arrow[ll, bend right] & 0 \\
0 \arrow[r] & V_{\mathbb{C}} \arrow[r] & (E_1 + E_2)_{\mathbb{C}} \arrow[r] & W_{\mathbb{C}} \arrow[r] \arrow[u, "\Delta"] \arrow[ll, bend right] & 0
\end{tikzcd} \cdot$$

The composition

$$W_{\mathbb{C}} \oplus W_{\mathbb{C}} \to E'_{\mathbb{C}} \to V_{\mathbb{C}}$$

is clearly given by $(w_1, w_2) \mapsto e_1(w_1) + e_2(w_2)$. Similarly, the composition

$$W_{\mathbb{C}} \to (E_1 \oplus E_2)_{\mathbb{C}} \to V_{\mathbb{C}}$$

is given by $w \mapsto e_1(w) + e_2(w)$. So we see that the map in Step 1 is indeed a group homomorphism. $\square$

References

Car80 [Car80] J. A. Carlson. Extensions of mixed Hodge structures. *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 107–127.

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