NON-PLURIPOLAR CURRENTS ARE NOT NECESSARILY $\mathcal{I}$-GOOD

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Contents

1. Introduction

In this note, we give an example of a non-$\mathcal{I}$-good non-pluripolar quasi-plurisubharmonic function, which has small unbounded locus.

2. Witt Nyström construction

Let X be a connected compact Kähler manifold of dimension n and ω be a Kähler form on X.

For each $N \geq 1$, let ω_N denote the Fubini–Study metric on $\mathbb{P}^N$. By abuse of notation, we also denote the corresponding Hermitian metric on $\mathcal{O}_{\mathbb{P}^N}(1)$ as ω_N.

Let $X_N = X \times \mathbb{P}^N$. Let

$$\pi_1^N : X_N \to X, \quad \pi_2^N : X_N \to \mathbb{P}^N$$

be the projections. Let

$$\Omega_N := \pi_1^{N*}\omega + \pi_2^{N*}\omega_N.$$

Note that Ω_N is a Kähler form on X_N.

Consider $\varphi \in \mathrm{PSH}(X, \omega)$. We define

$$\Phi_N[\varphi] := \sup_{\substack{\alpha\in\mathbb{R}^N_{>0},\\ |\alpha|<1}}{}^{*} \left((1-|\alpha|)\pi_2^{N*}\log|Z_0|^2_{\omega_N} + \sum_{i=1}^N \alpha_i \left(\pi_1^{N*}\varphi + \pi_2^{N*}\log|Z_i|^2_{\omega_N} \right) \right),$$

where we adopted the multi-index notation, $\alpha = (\alpha_1, \dots, \alpha_N)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. The elements $Z_0, \dots, Z_N$ are a basis of $H^0(\mathbb{P}^N, \mathcal{O}(1))$.

Note that $\Phi_N[\varphi] \in \mathrm{PSH}(X_N, \Omega_N)$ and $\Phi_N[\varphi] \geq \pi_2^{N*}\log|Z_0|^2_{\omega_N}$. It follows that $\Phi_N[\varphi]$ has small unbounded locus.

thm:WN **Theorem 2.1** (Witt Nyström)**.** *As $N \to \infty$, we have the strong convergence of measures*

$$\frac{n!}{N^n}\pi^N_{1*}\left(\Omega_N + \mathrm{dd^c}\Phi_N[\varphi]\right)^{N+n} \to \omega_\varphi^n.$$

See [WN19].

3. A counterexample

Let us take $\varphi \in \mathrm{PSH}(X, \omega)$ which is not $\mathcal{I}$-good and $\int_X \omega_\varphi^n > 0$. Let $\psi = P_\omega[\varphi]_{\mathcal{I}}$.

First observe that $\pi_1^{N*}\varphi \sim_{\mathcal{I}} \pi_1^{N*}\psi$, as a consequence of [Xia, Proposition 1.4.5]. Therefore,

$$\Phi_N[\varphi] \sim_{\mathcal{I}} \Phi_N[\psi]$$

due to [Xia, Proposition 6.1.5, Proposition 6.1.6]. Also observe that

$$\Phi_N[\varphi] \leq \Phi_N[\psi]$$

We claim that there are infinitely many $\Phi_N[\varphi]$ which are not $\mathcal{I}$-good.

Otherwise, we must have

$$\int_{X_N} (\Omega_N + \mathrm{dd^c}\Phi_N[\varphi])^{N+n} = \int_{X_N} (\Omega_N + \mathrm{dd^c}\Phi_N[\psi])^{N+n}$$

for all sufficiently large N. It follows from Theorem 2.1 that

$$\int_X \omega_\varphi^n = \int_X \omega_\psi^n,$$

which is a contradiction.

Note that $\Omega_N + \mathrm{dd^c}\Phi_N$ is non-pluripolar since

$$\mathbb{1}_{\{Z_0=0\}}(\Omega_N + \mathrm{dd^c}\Phi_N) = \nu(\Phi_N, \{Z_0 = 0\})[\{Z_0 = 0\}] = 0.$$

Here we have identified Z_0 with $\pi_2^{N*}Z_0$.

References

WN19 [WN19] D. Witt Nyström. Monotonicity of non-pluripolar Monge-Ampère masses. *Indiana Univ. Math. J.* 68.2 (2019), pp. 579–591. URL: https://doi.org/10.1512/iumj.2019.68.7630.

Xiabook [Xia] M. Xia. Singularities in global pluripotential theory. URL: https://mingchenxia.github.io/home/Lectures/SGPT.pdf.

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