NOTE ON LAGERBERG FORMS

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Contents

1. Introduction

There are two distinct groups of mathematicians working on the theory of Lagerberg forms (also known as superforms): A group of complex/convex geometriers, and a group of non-Archimedean geometriers.

In convex geometry, they provide natural languages to study valuations of convex bodies, as in [Ber25]; while in non-Archimedean geometry, (pre-)forms on Berkovich spaces are the pulled back of Lagerberg forms through (smooth or harmonic) tropicalizations, see [CLD12] and the more recent [GJR21].

Very frequently, the basic notions in these languages differ from each other in subtle ways, making the comparisons difficult. Moreover, many results have been proved in both languages repeatedly. In this note, I will summarize and compare the results from the two groups.

2. Linear algebra

Let V be a real vector space of finite dimension n. The dual vector space of V will be denoted by $V^\vee$. We write

$$\wedge^\bullet V^\vee := \bigoplus_{p=0}^{n} \wedge^p V^\vee .$$

Note that $\wedge^\bullet V^\vee$ has the natural structure of a graded $\mathbb{R}$-algebra.

Definition 2.1. A *linear Lagerberg (p,q)-form* on V is an element in $\wedge^p V^\vee \otimes \wedge^q V^\vee$.

Using the Koszul sign convention, we can make $\wedge^\bullet V^\vee \otimes \wedge^\bullet V^\vee$ a bi-graded $\mathbb{R}$-algebra. In other words, given $p,q,p',q' \in \mathbb{N}$, the multiplication $\wedge$ is the composition of

$$\left(\wedge^p V^\vee \otimes \wedge^q V^\vee\right) \otimes \left(\wedge^{p'} V^\vee \otimes \wedge^{q'} V^\vee\right) \xrightarrow{(-1)^{qp'}} \left(\wedge^p V^\vee \otimes \wedge^{p'} V^\vee\right) \otimes \left(\wedge^q V^\vee \otimes \wedge^{q'} V^\vee\right) \to \wedge^{p+p'} V^\vee \otimes \wedge^{q+q'} V^\vee ,$$

where the latter arrow is induced by the usual wedge product.

def:Lag_inv **Definition 2.2.** Let $\mathrm{J}\colon \wedge^p V^\vee \otimes \wedge^p V^\vee \to \wedge^p V^\vee \otimes \wedge^p V^\vee$ be the unique homomorphism of $\mathbb{R}$-linear spaces such that

{eq:Jdef} (2.1) $$\mathrm{J}(\alpha \otimes \beta) = (-1)^{p(q+1)} \beta \otimes \alpha$$

for all $\alpha \in \wedge^p V^\vee$ and $\beta \in \wedge^q V^\vee$. The morphism J is called the *Lagerberg involution*.

We have to be extra careful. The morphism J is not an involution: We have

$$\mathrm{J}^2 = (-1)^{p+q} \quad \text{on } \wedge^p V^\vee \otimes \wedge^q V^\vee.$$

Moreover, J is not a homomorphism of algebras. In fact, given linear Lagerberg (p,q)-form α and linear Lagerberg (p',q')-form β, we have

$$\mathrm{J}(\alpha \wedge \beta) = (-1)^{pq'+p'q} \mathrm{J}\alpha \wedge \mathrm{J}\beta. \tag{2.2}$$

Remark 2.3. Here we followed the convention of [Ber25]. This convention has the advantage that symmetric forms look symmetric, as in Definition 2.4.

There is a different convention in the literature, as in [Lag12; GJR21], where one takes

$$\mathrm{J}(\alpha \otimes \beta) = \beta \otimes \alpha$$

instead of (2.1). This definition has the advantage that J is indeed an involution. However, J is *not* a morphism of algebras either. In fact, (2.2) continues to hold, contrary to the assertion in [GJR21, Section 2.7].

Definition 2.4. Let $p \in \mathbb{N}$. A linear Lagerberg (p,p)-form α on V is *symmetric* if

$$\mathrm{J}\alpha = \alpha.$$

Definition 2.5. Let $p \in \mathbb{N}$. An element $\alpha \in \wedge^p V^\vee \otimes \wedge^p V^\vee$ is

(1) *elementary* if there exist $\beta_1, \dots, \beta_p \in V^\vee$ such that

$$\alpha = (-1)^{p(p-1)/2} (\beta_1 \wedge \cdots \wedge \beta_p) \otimes (\beta_1 \wedge \cdots \wedge \beta_p);$$

(2) *strongly positive* if there exists a sequence $(\alpha_i)_i$ of elementary linear Lagerberg (p,p)-forms on V and a sequence $(c_i)_i$ of non-negative constants such that

$$\alpha = \sum_i c_i \alpha_i; \tag{2.3}$$

(3) *weakly positive* if for any elementary linear Lagerberg $(n-p,n-p)$-form β and any elementary linear Lagerberg (n,n)-form γ on V, we have

$$\alpha \wedge \beta = c\gamma$$

for some $c > 0$;

(4) *positive* if there are finitely many elements $\beta_1, \dots, \beta_N \in \wedge^p V^\vee$ and $c_1, \dots, c_N \geq 0$ such that

$$\alpha = \sum_{i=1}^N c_i (-1)^{p(p-1)/2} \beta_i \wedge \beta_i.$$

Note that by definition, positive forms are closed under products.

Remark 2.6. In complex geometry, weakly positive forms are usually called positive.

Remark 2.7. Our notion of strongly positive forms seems to be the most common one, as in the complex setting of [Dem12, Section 3.1]. It is also the definition used in Lagerberg's original paper [Lag12]. Note that Demailly claimed that the cone of strongly positive forms is closed, which does not seem to hold in general.

As in the complex setting, there are two other incompatible definitions of strongly positive forms:

(2') In (2.3), we require in addition that the sum is finite, or

(2") defining the set of strongly positive forms as the closure of the cone of forms in (2).

In the complex setting, (2') is introduced in Lelong's original exposé in [SL62]. Its Lagerberg form version is used in [BGGJK21].

The notion (2") is considered in [Ber25].

By contrary, all definitions of positive forms in the literature agree.

The following proposition is proved exactly as in the complex setting.

Proposition 2.8. *A strongly positive linear Lagerberg form is positive, and a positive linear Lagerberg form is weakly positive. Conversely, a weakly positive linear Lagerberg (p,p)-form on V is strongly positive if $p = 0, 1, n-1, n$.*

The wedge product of a strongly positive linear Lagerberg form and a weakly positive linear Lagerberg form is weakly positive.

A weakly positive Lagerberg form is symmetric.

See [Lag12, Section 2].

Definition 2.9. Let $p \in \mathbb{N}$. A symmetric linear Lagerberg (p,p)-form α on U is *strong* if there exist strongly positive Lagerberg (p,p)-forms β_1, β_2 on U such that $\alpha = \beta_1 - \beta_2$.

Unlike the complex setting, a symmetric Lagerberg form is not necessarily strong (see Example 3.15). So the statement of [CLD12, Lemme 5.2.3] is wrong.

A first counterexample is constructed [BGGJK21, Example 2.3.6]. We shall explain the more elegant way to understand this failure as studied in [Ber25] in the next section.

Recall that an *orientation* of V is an equivalence class of non-zero elements in $\det V = \wedge^n V$, where two elements are considered equivalent if they differ by a positive multiple. The set of orientations of V is denoted by $\mathrm{Or}(V)$. It has the natural structure of a $\mathbb{Z}/(2)$ homogeneous space if $n > 0$.

3. Lagerberg forms on domains

Let V be a real vector space of finite dimension n and $\mathcal{A}^\bullet$ denote the sheaf of real smooth differential forms on V with the usual grading.

3.1. The definitions.

Definition 3.1. Let $p, q \in \mathbb{Z}$. The sheaf of *Lagerberg (p,q)-forms* on V is the sheaf

$$\mathcal{A}^{p,q} := \mathcal{A}^p \otimes_{C^\infty} \mathcal{A}^q \tag{3.1}$$

of C^∞-modules.

Correspondingly, a local section of the sheaf $\mathcal{A}^{p,q}$ is called a Lagerberg (p,q)-form.

Given a Lagerberg (p,q)-form α on an open set $U \subseteq V$ and $x \in U$, the fiber α_x defines a linear Lagerberg (p,q)-form on V.

Remark 3.2. Similarly, a measurable Lagerberg (p,q)-form can be defined with the sheaves $\mathcal{A}^p$ and $\mathcal{A}^q$ in (3.1) replaced by the sheaves of forms with measurable coefficients.

The constructions below not involving differentiations also work for these forms.

Definition 3.3. Let $U \subseteq V$ be an open subset and α be a Lagerberg form on U. The *support* $\mathrm{Supp}\,\alpha$ of α is defined as the smallest closed subset of U outside of which α restricts to 0.

Example 3.4. *A Lagerberg $(0,0)$-form is just a smooth (real) function.*

More generally, for any $p \in \mathbb{N}$, we can identify usual p-forms with Lagerberg $(p,0)$-forms.

Example 3.5. *Choose two bases $x^1, \dots, x^n$ and $\xi^1, \dots, \xi^n$ of the dual space $V^\vee$.*

Let $U \subseteq V$ be an open set. Then a Lagerberg (p,q)-form on U can be written as

$$\sum_{|I|=p, |J|=q} \alpha_{I,J} \mathrm{d}x^I \otimes \mathrm{d}\xi^J. \tag{3.2}$$

Here I, J run over subsets of $\{1, \dots, n\}$ with the assigned cardinalities, $\alpha_{I,J} \in C^\infty(U)$. Here we used the usual multi-index notation.

In the literature, it is very common to take $\xi^i = x^i$ for all $i = 1, \dots, n$.

The change of variable formula for the coefficients $\alpha_{I,J}$ is obvious.

Note that we have two natural differential operators:

$$\mathrm{d}\colon \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}, \quad \mathrm{d}^{\mathrm{c}}\colon \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}. \tag{3.3}$$

Both are easy to define in terms of local coordinates with $\xi^i = x^i$ as in (3.5):

$$\begin{aligned}\mathrm{d}\left(\sum_{|I|=p,|J|=q}\alpha_{I,J}\mathrm{d}x^I\otimes\mathrm{d}\xi^J\right) &:= \sum_{|I|=p,|J|=q}\sum_{k=1}^{n}\frac{\partial\alpha_{I,J}}{\partial x^k}\mathrm{d}x^k\wedge\mathrm{d}x^I\otimes\mathrm{d}\xi^J,\\ \mathrm{d}^{\mathrm{c}}\left(\sum_{|I|=p,|J|=q}\alpha_{I,J}\mathrm{d}x^I\otimes\mathrm{d}\xi^J\right) &:= \sum_{|I|=p,|J|=q}\sum_{k=1}^{n}(-1)^p\frac{\partial\alpha_{I,J}}{\partial x^k}\mathrm{d}x^I\otimes\mathrm{d}\xi^k\wedge\mathrm{d}\xi^J.\end{aligned} \tag{3.4}$$

Remark 3.6. It is more common to denote the two operators in (3.3) as d' and d''. But due to the specific identification in Example 3.4, we prefer the current notations.

Definition 3.7. Given $p,q,p',q'\in\mathbb{N}$, we can define a morphism of C^∞-modules:

$$\mathcal{A}^{p,q}\otimes_{C^\infty}\mathcal{A}^{p',q'}\to\mathcal{A}^{p+p',q+q'}$$

as the following composition:

$$\begin{aligned}\mathcal{A}^{p,q}\otimes_{C^\infty}\mathcal{A}^{p',q'} = (\mathcal{A}^p\otimes_{C^\infty}\mathcal{A}^q)\otimes_{C^\infty}\left(\mathcal{A}^{p'}\otimes_{C^\infty}\mathcal{A}^{q'}\right)\xrightarrow{(-1)^{p'q}}\\ \left(\mathcal{A}^p\otimes_{C^\infty}\mathcal{A}^{p'}\right)\otimes_{C^\infty}\left(\mathcal{A}^q\otimes_{C^\infty}\mathcal{A}^{q'}\right)\to\mathcal{A}^{p+p'}\otimes_{C^\infty}\mathcal{A}^{q+q'} = \mathcal{A}^{p+p',q+q'}.\end{aligned}$$

Note that here we employed the Koszul sign convention.

Koszul_sign *Remark* 3.8. In terms of local coordinates (3.2), this means that we are treating $\otimes$ formally as $\wedge$. Therefore, we will sometimes write (3.2) as

q:Lag_loc2}
$$\sum_{|I|=p,|J|=q}\alpha_{I,J}\mathrm{d}x^I\wedge\mathrm{d}\xi^J. \tag{3.5}$$

More generally, an arbitrary wedge product of the $\mathrm{d}x^i$'s and the $\mathrm{d}\xi^j$'s in any order makes sense: We re-order the product so that the $\mathrm{d}x^i$'s proceed the $\mathrm{d}\xi^j$'s and then add an extra Koszul sign.

As a straightforward consequence of the Koszul sign convention,

$$\mathcal{A}^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{N}}\mathcal{A}^{p,q}$$

is a sheaf of doubly differential graded C^∞-algebras (with respect to d and d^{c}).

Definition 3.9. The *Lagerberg involution* $\mathrm{J}\colon\mathcal{A}^{\bullet,\bullet}\to\mathcal{A}^{\bullet,\bullet}$ is the unique morphism of the sheaf of C^∞-modules such that fiberwise J is given by the operation in Definition 2.2.

In local coordinates as in Example 3.5, if we take $\xi^i = x^i$ for $i=1,\dots,n$, then

:alpha_exp}
$$\mathrm{J}\left(\sum_{|I|=p,|J|=q}\alpha_{I,J}\mathrm{d}x^I\wedge\mathrm{d}\xi^J\right) = \sum_{|I|=p,|J|=q}\alpha_{I,J}(-1)^{p(q+1)}\mathrm{d}x^J\wedge\mathrm{d}\xi^I. \tag{3.6}$$

In particular,

$$\mathrm{J}\mathrm{d}x^i = -\mathrm{d}\xi^i,\quad \mathrm{J}\mathrm{d}\xi^i = \mathrm{d}x^i$$

for $i=1,\dots,n$.

Observe that J restricts to an isomorphism of C^∞-modules $\mathcal{A}^{p,q}\to\mathcal{A}^{q,p}$ for all $p,q\in\mathbb{N}$. In fact, $\mathrm{J}^2 = (-1)^{p+q}$.

ef:sym_form **Definition 3.10.** Let $p\in\mathbb{N}$ and $U\subseteq V$ be an open subset. A Lagerberg (p,p)-form α on U is *symmetric* if

symm_alpha}
$$\alpha = \mathrm{J}\alpha. \tag{3.7}$$

If we expand α using a coordinate with $\xi^i = x^i$ as in (3.6), then (3.7) means $\alpha_{I,J} = \alpha_{J,I}$ for all I,J.

Definition 3.11. Let $U \subseteq V$ be an open subset. Let $\mathcal{A}_c^{p,q}(U)$ denote the subset of $\mathcal{A}^{p,q}(U)$ consisting of Lagerberg (p,q)-forms with compact supports in U. We endow $\mathcal{A}_c^{p,q}(U)$ with the canonical LF topology.

A *Lagerberg* $(n-p, n-q)$*-current* on U is a continuous linear functional on $\mathcal{A}_c^{p,q}(U)$.

Definition 3.12. Let $p \in \mathbb{N}$, $U \subseteq V$ be an open subset and $\alpha \in \mathcal{A}^{p,p}(U)$. We say α is *strongly positive* (resp. *positive*, resp. *positive*, resp. *strong*) if each fiber of α is so in the sense of Definition 2.5.

As in Remark 2.7, the definitions of strongly positive Lagerberg forms in the literature do not agree.

Definition 3.13. Let $p, q \in \mathbb{N}$ and $q \geq 1$. The *Berndtsson operator* $T_B \colon \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q-1}$ is a C^∞-module homomorphism such that if locally we choose a coordinate as in Example 3.5 with $\xi^i = x^i$, then

{eq:TB}
$$T_B\left(\sum_{|I|=p,|J|=q} \alpha_{I,J} \mathrm{d}x^I \wedge \mathrm{d}\xi^J\right) = \sum_{i=1}^n \sum_{|I|=p,|J|=q} \alpha_{I,J} \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \left(\frac{\partial}{\partial \xi^i} \lrcorner \, \mathrm{d}\xi^J\right). \tag{3.8}$$

It is clear that (3.8) is independent of the choice of the coordinates.

The following observation is due to Berndtsson.

Proposition 3.14. *The operator T_B annihilates all strong forms.*

For linear forms, the vanishing of T_B characterizes all strong forms, as proved in [Ber25, Theorem 3.1].

Proof. Due to the obvious C^∞-linearity of T_B, it suffices to show that T_B eliminates all elementary linear Lagerberg (p,p)-forms α. But T_B has the obvious derivative like property, which allows us to assume that $p = 1$. Take $\beta \in V^\vee$ so that

$$\alpha = \beta \otimes \beta.$$

We want to show that $T_B(\alpha) = 0$. We use the local coordinates as in Example 3.5 with $\xi^i = x^i$. In particular, we can expand

$$\beta = \sum_{i=1}^n c_i \mathrm{d}x^i.$$

We compute

$$T_B(\alpha) = \sum_{i,j=1}^n c_i c_j T_B(\mathrm{d}x^i \wedge \mathrm{d}\xi^j) = \sum_{i,j=1}^n c_i c_j \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge 1 = 0.$$

□

ex:Ber **Example 3.15.** *Now consider the following symmetric linear Lagerberg form on $\mathbb{R}^4$:*

$$\alpha = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}\xi^3 \wedge \mathrm{d}\xi^4 + \mathrm{d}x^3 \wedge \mathrm{d}x^4 \wedge \mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2.$$

It is easy to see that $T_B(\alpha) \neq 0$, so α is not strong.

Note that we have the following replacement.

Proposition 3.16. *Let $U \subseteq V$ be an open subset and $p \in \mathbb{N}$. Assume that α is a symmetric Lagerberg (p,p)-form on U. Then there exist positive Lagerberg (p,p)-forms α_1, α_2 on U such that $\alpha = \alpha_1 - \alpha_2$.*

This result is stated in [GJR21, Section 2.7], while the proposed proof only works for linear forms.

Proof. We use the coordinates as in Example 3.5 with $\xi^i = x^i$. Then α can be expanded as

$$\alpha = \sum_{|I|=p,|J|=p} \alpha_{I,J} \mathrm{d}x^I \wedge \mathrm{d}\xi^J.$$

The fact that α is symmetric means that the matrix $(\alpha_{I,J})_{I,J}$ is symmetric. Now take a smooth function $f \colon U \to \mathbb{R}_{\geq 0}$ satisfying the following condition:

(1) $f+\alpha_{I,I}\geq 0$ for each $|I|=p$;
(2)
$$f+\alpha_{I,I}\geq \sum_{|J|=p,J\neq I}\alpha_{J,I}.$$

Then we take $\alpha_2=f\sum_{|I|=p}\mathrm{d}x^I\wedge \mathrm{d}\xi^I$ and $\alpha_1=\alpha+\alpha_2$. It follows from elementary linear algebra and [Lag12, Proposition 2.1] that both α_2 and α_1 are positive. □

3.2. **Berezin integrals.** Let $W\subseteq V$ be an affine subspace of dimension $m>0$. An orientation of W is understood as an orientation of the vector subspace of V parallel to W. Let $U\subseteq V$ be an open subset.

Definition 3.17. Given $\lambda\in\det V$, $p\in\mathbb{N}$ and a Lagerberg (p,n)-form α on U, we define the *contraction* $\lambda\lrcorner\alpha$ as a p-form with measurable coefficients on U as follows: Take a coordinate as in Example 3.5 with $\xi^i=x^i$, expand
$$\lambda=\lambda_0\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^n},\quad \alpha=\sum_{|I|=p}\alpha_I\mathrm{d}x^I\wedge\mathrm{d}\xi^1\wedge\cdots\wedge\mathrm{d}\xi^n.$$
Then
$$\lambda\lrcorner\alpha\coloneqq\lambda_0(-1)^{n(n-1)/2}\sum_{|I|=p}\alpha_I\mathrm{d}x^I.$$

The extra coefficient $(-1)^{n(n-1)/2}$ is due to our Koszul sign convention.

Definition 3.18. Let α be a Lagerberg (m,n)-form with measurable coefficients on U and $\mu\in\mathrm{Or}(W)\times^{\mathbb{Z}/(2)}(\det V)$. We say α is *integrable* with respect to μ if the following holds: Choose a representative $(o,\lambda)\in\mathrm{Or}(W)\times\det V$ of μ, the restriction $(\lambda\lrcorner\alpha)|_{U\cap W}$ is integrable. In this case, we define the *integral* as
$$\int_{U\cap W}(\alpha,\mu)\coloneqq\int_{U\cap W}(\lambda\lrcorner\alpha)|_{U\cap W},$$
where the orientation of $U\cap W$ is induced by o.

Remark 3.19. More generally, suppose that W is an oriented submanifold of V of pure dimension m and α is a Lagerberg (m,n)-form with measurable coefficients on U, if we choose an element λ in $\det V$, then we can also define $\int_{U\cap W}(\alpha,\mu)$.

ex:Ber_coor **Example 3.20.** *We take the coordinate system as in Example 3.5 with $\xi^i=x^i$. We expand a Lagerberg (m,n)-form α with measurable coefficients on U as*
$$\alpha=\sum_{|I|=m}\alpha_I\mathrm{d}x^I\wedge\mathrm{d}\xi^1\wedge\cdots\wedge\mathrm{d}\xi^n.$$
Fix an orientation o on W. Then we write μ as the equivalence class of
$$\left(o,\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^n}\right).$$
Then
$$\int_{U\cap W}(\alpha,\mu)=(-1)^{n(n-1)/2}\sum_{|I|=m}\int_{U\cap W}\left(\alpha_I\mathrm{d}x^I\right)|_{U\cap W},$$
where the orientation on $U\cap W$ is induced by o.

Example 3.21. *As a particular case of Example 3.20, if $W=V$ and the orientation o is the orientation given by $\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^n}$, then*

{eq:int_nn}
$$\int_U(\alpha,\mu)=(-1)^{n(n-1)/2}\sum_{|I|=n}\int_U\alpha_I\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n.\tag{3.9}$$

In particular, our sign convention agrees with [Ber25]. The integral in (3.9) is denoted by $\int_U\alpha$ in [Ber25], we choose to follow the notations of [CLD12] instead, which make the functoriality transparent.

Example 3.22. *Next we consider the case where $m = n - 1 > 0$. Given $\mu \in \mathrm{Or}(V) \times^{\mathbb{Z}/(2)} \det V$, it induces a canonical element $\partial\mu \in \mathrm{Or}(W) \times^{\mathbb{Z}/(2)} \det V$. To see, take $\ell \in V^\vee$ and $c \in \mathbb{R}$ so that $W = \{\ell = c\}$. Fix an orientation o of W and an oriented basis $w_2, \dots, w_n$ of $\{\ell = 0\}$. Choose a vector $v \in V$ so that $\ell(v) < 0$. Then $v, w_2, \dots, w_n$ form a basis of V and hence determines an orientation o' of V. Then μ is represented by (o', λ) for some $\lambda \in \det V$. Then $\partial\mu$ is defined as the equivalence class of (o, λ).*

There is a Green formula in this case, see [CLD12, Lemme 1.3.8].

ex:latt_mu **Example 3.23.** *Let N be a lattice in V. Then for any integral affine polyhedron σ (the defining conditions of σ have the form $m + c$, where $m \in N^\vee$ and $c \in \mathbb{R}$) of dimension $d > 0$, there is a well-defined element $\mu_\sigma \in \mathrm{Or}(\mathrm{Aff}\,\sigma) \times^{\mathbb{Z}/(2)} \det V(\sigma)$, where $\mathrm{Aff}\,\sigma$ is the affine space spanned by σ and $V(\sigma)$ is the corresponding vector space.*

In fact, let $v_1, \dots, v_d$ be a basis of $N \cap V(\sigma)$, we write o for the corresponding orientation of $\mathrm{Aff}\,\sigma$. then μ_σ is represented by $(o, v_1 \wedge \cdots \wedge v_d)$.

Given a Lagerberg (d, d)-form α on an open subset $U \subseteq V$ with measurable coefficients, we write

$$\int_{U \cap \mathrm{Aff}\,\sigma} \alpha := \int_{U \cap \mathrm{Aff}\,\sigma} (\alpha|_{\mathrm{Aff}\,\sigma}, \mu_\sigma)$$

if the right-hand side is integrable. Note that $\mathrm{Aff}\,\sigma$ is just an affine space, the right-hand side makes sense only after an arbitrary identification $\mathrm{Aff}\,\sigma \cong V(\sigma)$.

4. Lagerberg forms on polyhedral complex

Let V be a real vector space of finite dimension n.

4.1. **Generalities.** Let S be a polyhedral complex in V. In other words, S is a finite collection of polyhedrons in V such that

(1) if $\sigma \in S$, then each face of σ is also in V, and
(2) if $\sigma, \sigma' \in S$, then so is $\sigma \cap \sigma'$.

The set of faces of dimension $a \in \mathbb{N}$ in S is denoted by $S^{(a)}$.

We say S has dimension d if all maximal polyhedra (with respect to inclusion) in S are of dimension d. We shall assume that S has dimension d in the sequel.

We endow S with a weight, namely, a map $m\colon S^{(d)} \to \mathbb{Z}$. Given any $\sigma \in S^{(d)}$, the corresponding weight is denoted by m_σ.

Recall that two weighted polyhedral complexes (S, m) and (S', m') are equivalent if after a subdivision, m and m' agree on the polyhedra with non-zero weights.

Definition 4.1. Let $p, q \in \mathbb{N}$. We define the sheaf $\mathcal{A}_S^{p,q}$ on $|S|$ as follows: Let $\Omega \subseteq V$ be an open subset, then

$$\Gamma(\Omega \cap |S|, \mathcal{A}_S^{p,q}) := \Gamma(\Omega, \mathcal{A}^{p,q}) / \sim,$$

where for $\alpha, \alpha \in \Gamma(\Omega, \mathcal{A}^{p,q})$, we say $\alpha \sim \alpha'$ if

$$\alpha|_{\mathrm{relint}\,\sigma \cap \Omega} = \alpha'|_{\mathrm{relint}\,\sigma \cap \Omega}$$

for all $\sigma \in S$.

Given $U \subseteq |S|$, a section $\alpha \in \Gamma(U, \mathcal{A}_S^{p,q})$ is called a *Lagerberg (p, q)-form* on U.

The standard partition of unity argument shows that this definition is independent of the choice of Ω and $\mathcal{A}^{p,q}$ is indeed a sheaf of Abelian groups on $|S|$.

Definition 4.2. Let $U \subseteq |S|$ be an open subset, $p, q \in \mathbb{N}$ and α be a Lagerberg (p, q)-form on U. The *support* $\mathrm{Supp}\,\alpha$ of α is the smallest closed subset of U outside of which α restricts to 0.

The Abelian group $\mathcal{A}_c^{p,q}(U)$ is the subgroup of $\mathcal{A}^{p,q}(U)$ consisting of $\alpha \in \mathcal{A}^{p,q}(U)$ with compact support.

Definition 4.3. The Lagerberg involution J, the differentials d, d^{c} all descend to $\mathcal{A}_S^{p,q}$, which we denote by the same notations.

We write $C^\infty_S = \mathcal{A}^{0,0}_S$ as usual. Then

$$\mathcal{A}^{\bullet,\bullet}_S = \bigoplus_{p,q\in\mathbb{N}} \mathcal{A}^{p,q}_S$$

is a sheaf of doubly differential graded C^∞_S-algebras (with respect to d and d^{c}).

Definition 4.4. Let $p \in \mathbb{N}$, $U \subseteq |S|$ be an open subset and α be a Lagerberg (p,p)-form on U. We say α is *strongly positive* (resp. *positive*, resp. *weakly positive*, resp. *strong*) if for each $x \in U$, there is an open neighborhood Ω of x in V and a representation $\tilde{\alpha} \in \Gamma(\Omega, \mathcal{A}^{p,p})$ of $\alpha|_{U\cap\Omega}$ such that $\tilde{\alpha}$ satisfies the same property.

Remark 4.5. Using a standard partition of unity argument, our definitions of positivity agree with the ones in [GJR21, Section 2.7].

The following notion is introduced in [GK17, Definition 3.2].

Definition 4.6. Let $U \subseteq |S|$ be an open subset, $p, q \in \mathbb{N}$. A *Lagerberg* $(d-p, d-q)$*-current* on U is a linear map $T\colon \mathcal{A}^{p,q}_c(U) \to \mathbb{R}$ such that there is an open subset Ω of V with $V \cap |S| = U$ and a Lagerberg $(n-p, n-q)$-current $\tilde{T}$ on Ω such that for each $\beta \in \mathcal{A}^{p,q}_c(V)$, we have

$$T(\beta|_V) = \tilde{T}(\beta).$$

The set of Lagerberg $(d-p, d-q)$-currents on U is denoted by $\mathcal{D}'^{d-p,d-q}(U)$ or $\mathcal{D}'_{p,q}(U)$.

Note that $\mathcal{D}'_{p,q}(U)$ is a subset of $\mathcal{D}'_{p,q}(\Omega)$ for each open subset $\Omega \subseteq V$ with $\Omega \cap |S| = U$.

4.2. **With chosen lattice.** Fix a lattice N in V. Assume that (S, m) is a weighted $\mathbb{Z}$-affine polyhedral complex in V of dimension $d > 0$. Then we have a canonical current $\delta_S \in \mathcal{D}'^{0,0}(|S|)$ as follows:

$$\delta_S = \sum_\sigma m_\sigma \delta_\sigma,$$

where δ_σ is the current of integration along σ as in Example 3.23.

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