NOTE ON L^2-METHODS IN GLOBAL PLURIPOTENTIAL THEORY

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Abstract. In this note, we collect the proofs of various fundamental results related to Nadel's multiplier ideal sheaves in global pluripotential theory proved using L^2-methods.

Contents

1. Introduction

In this note, we collect the proofs of various fundamental results related to Nadel's multiplier ideal sheaves in global pluripotential theory proved using L^2-methods. Most proofs are just reproduction of the known proofs in the literature, apart from fixing typos and miscalculations.

Some results in this note are more general than one find in the literature. To be more precise, in Theorem 7.1, we prove the positivity of direct images for proper morphisms instead of projective ones. In Theorem 8.1, we prove the relative version of Bertini theorem without requiring the base be projective. Corollary 8.6 seems to be new.

Some of the proofs are not self-contained. I intend to include more details in the future and make all arguments self-contained.

Some comments on the terminologies: All complex analytic spaces are assumed to be reduced.

Given a general complex analytic space X, when we want to talk about a small part in the Zariski topology, we avoid saying that a subset is a *proper closed analytic subset*, as people usually do in the literature. Instead, we say a subset is a *nowhere dense closed analytic subset*. The reason is that when X has more than one connected components, the former does not exclude sets like a whole connected component!

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The notation Δ denotes the open unit disk in $\mathbb{C}$.

2. Preliminaries

2.1. **Complex analytic spaces.** Recall that all complex analytic spaces are assumed to be reduced.

Recall the following generic flatness theorem.

thm:genflat **Theorem 2.1** ([BS76, Theorem V.4.10])**.** *Let $f : X \to Y$ be a morphism of complex analytic spaces. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $A \subseteq X$ be the non-flat locus of $\mathcal{F}$. Then A is an analytic subset of X. Moreover, if*

(1) X is σ-compact, then $f(A)$ is non-where dense.

(2) f is proper, then $f(A)$ is a nowhere dense proper closed analytic subset in Y.

We also have the cohomology and base change theorem.

thm:cbc **Theorem 2.2** ([BS76, Theorem III.3.4, Corollary III.3.7])**.** *Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Let $\mathcal{F}$ be an f-flat coherent $\mathcal{O}_X$-module. Let $q \geqslant 0$ be an integer and $y \in Y$. Assume that the canonical map*

{eq:phiq} $$\phi_q(y) : R^q f_*(\mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y) \to H^q(X_y, \mathcal{F}|_{X_y}) \tag{2.1}$$

is surjective. Then ϕ_q is an isomorphism in a neighbourhood of y. Moreover, the following are equivalent:

(1) $\phi_{q-1}(y)$ is surjective.

(2) $R^q f_*(\mathcal{F})_y$ is a free $\mathcal{O}_{Y,y}$-module.

cor:cbc **Corollary 2.3.** *Let $f : X \to Y$ be a proper morphism of complex analytic spaces and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is a nowhere dense proper analytic subset Z of Y such that*

(1) $\mathcal{F}|_{f^{-1}(Y \setminus Z)}$ is f-flat.

(2) $R^q f_*(\mathcal{F})|_{Y\setminus Z}$ is locally free for all $q \geqslant 0$.

(3) For any $y \in Y \setminus Z$, the canonical morphism $\phi_q(y)$ is an isomorphism for all $q \geqslant 0$.

We say $\mathcal{F}$ has *the base change property* with respect to f on $Y \setminus Z$ if (1), (2) and (3) are all satisfied.

Proof. The problem is local on Y, so we may assume that the dimension of the fibers of f are bounded by a constant N. By Theorem 2.1, we may further assume that $\mathcal{F}$ is f-flat. Recall that the $R^i f_*(\mathcal{F})$'s for $i = 0, \dots, N$ are all coherent, so up to subtracting a closed analytic subset from Y, we may further assume that all of these sheaves are locally free. Observe that for any $y \in Y$, $\phi_{N+1}(y)$ is surjective, so we can apply Theorem 2.2 to conclude that $\phi_i(y)$ is surjective for all $i = N, N-1, \dots, 0$. Applying Theorem 2.2 again, we conclude that (3) is also both satisfied. □

Recall the theorem of generic flatness:

thm:gensm **Theorem 2.4** ([GPR94, Theorem II.1.22], [BF93, Corollary 2.1])**.** *Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Then the set N of $y \in Y$ such that X_y is not a manifold is a closed negligible subset of Y.*

Assume furthermore that X is smooth and Y is irreducible. Then N is a proper closed analytic subset of Y.

2.2. **Kähler morphisms.**

Definition 2.5. Let $f : X \to Y$ be a morphism of complex analytic spaces. A smooth function φ defined on X is *strictly f-plurisubharmonic* if for each $x \in X$, we can find a neighbourhood $U \subseteq X$ of x, a neighbourhood V of $f(x)$ satisfying $f(U) \subseteq V$ and a smooth strictly plurisubharmonic function ψ on V such that $\varphi + f^*\psi$ is smooth and strictly plurisubharmonic on U.

Definition 2.6. A morphism $f : X \to Y$ of complex analytic spaces is *Kähler* if there is an open covering $\{U_\alpha\}_\alpha$ of X, smooth strictly f-psh functions φ_α defined on U_α such that for each α, β, $\varphi_\alpha - \varphi_\beta$ is pluriharmonic on $U_\alpha - U_\beta$.

Definition 2.7. A complex analytic space X is *weakly 1-complete* if there is a smooth psh exhaustion ψ on X.

A holomorphically convex space is always weakly 1-complete. A weakly 1-complete Kähler manifold carries a complete Kähler metric.

2.3. **Singular Hermitian line bundles.** Let X be a complex manifold. Recall that a singular Hermitian metric on a one-dimensional vector space V is either the quadratic form of a Hermitian inner product on V or the map that maps $V^\times$ to ∞ and 0 to 0. A singular Hermitian metric on a line bundle is a collection of singular Hermitian metrics on each fiber.

Definition 2.8. A *singular Hermitian line bundle* on X is a pair (L, h) consisting of a holomorphic line bundle L on X and a singular Hermitian metric h on L, such that if locally take a smooth Hermitian metric h_0 on L and identify h with $h_0 \exp(-\varphi)$, then φ takes value in $[-\infty, \infty)$, is locally integrable and usc.

A *(smooth) Hermitian line bundle* on X is a singular Hermitian line bundle (L, h) in which h is smooth.

A singular Hermitian line bundle (L, h) is called a *Hermitian psef line bundle* (resp. *Hermitian quasi-psef line bundle*) if $\mathrm{dd^c} h \geqslant \gamma$ for some smooth real closed $(1,1)$-form γ on X in the sense of currents.

Given a local section f of L over $U \subseteq X$, we write $|f|_h^2$ for the map $U \to [0, \infty]$: $x \mapsto h_x(f_x, f_x)$. When $h_x = \infty$, $f_x = 0$, the right-hand side is understood as 0. According to our normalization

$$|f|_h^2 = |f|_{h_0}^2 \mathrm{e}^{-\varphi} .$$

Be careful, we do not put 2 in front of φ.

Next we recall the definition of several basic invariants of a singular Hermitian line bundle.

Definition 2.9. Let (L, h) be a Hermitian quasi-psef line bundle on X. The *multiplier ideal sheaf* of h in the sense of Nadel is the sheaf of ideals $\mathcal{I}(h)$ on X, locally generated by sections f of h satisfying $|f|_h^2$ is locally integrable.

By a theorem of Nadel, $\mathcal{I}(h)$ is a coherent ideal sheaf.

Definition 2.10. Given two Hermitian quasi-psef line bundles (L,h) and (L,h') with the same underlying line bundle, we say $h\sim_{\mathcal{I}} h'$ if $\mathcal{I}(kh)=\mathcal{I}(kh')$ for all real $k>0$.

Definition 2.11. Assume that X is compact and Kähler. Let ω be a Kähler form on X. Let (L,h) be a Hermitian psef line bundle on X. Take a quasi-equisingular approximation h^i of h as in Theorem 2.14. For each $a=0,\dots,n$, define *the mixed mass in the sense of Cao* as

$$\langle \mathrm{dd^c} h^a\wedge\omega^{n-a}\rangle := \lim_{i\to\infty}\int_X (\mathrm{dd^c} h_i)^a\wedge\omega^{n-a}\,,$$

where on the right-hand side, the product is taken in the non-pluripolar sense. It is easy to see that $\langle \mathrm{dd^c} h^a\wedge\omega^{n-a}\rangle$ is independent of the choice of the approximation h^i.

We define the *numerical dimension* $\mathrm{nd}(L,h)$ of (L,h) as the maximum of a such that $\langle \mathrm{dd^c} h^a\wedge\omega^{n-a}\rangle>0$.

Definition 2.12. Assume that Assume that X is compact and Kähler. The *volume* of a Hermitian psef line bundle (L,h) is defined as

$$\mathrm{vol}(L,h)=\lim_{k\to\infty}\frac{n!}{k^n}h^0(X,L^k\otimes\mathcal{I}(kh))\,.$$

The existence of the limit is a non-trivial result, proved in [DX21; DX22].

2.4. Equisingular approximations.

thm:equising

Theorem 2.13. *[[DPS01]] Let X be a complex manifold. Let ω be a smooth closed positive real $(1,1)$-form on X and γ be a smooth real $(1,1)$-form on X. Let (L,h) be a singular Hermitian line bundle on X. Assume that $T:=\mathrm{dd^c}h$ satisfies $T\geqslant\gamma$. Then for any relative compact open subset $U\Subset X$, there are currents T_k $(k\in\mathbb{Z}_{>0})$ defined on U satisfying T_k-T is exact on a neighbourhood of U and a decreasing sequence ϵ_k of positive real numbers converging to 0 satisfying*

1. *Each T_k has a smooth potential outside a proper subvariety Z_k of U.*
2. *$T_{k''}$ is more singular than $T_{k'}$ on K when $k''>k'$. Any T_k is less singular than T on U.*
3. *$\mathcal{I}(T)|_K=\mathcal{I}(T_k)|_U$ for all k.*
4. *$T_k\geqslant\gamma-\epsilon_k\omega$ on U.*

Moreover, if ω_U is a complete positive real $(1,1)$-form on U (instead of on X), we may assume that (d) holds for ω_K.

thm:quasiequising

Theorem 2.14. *[[DPS01]] Let X be a compact Kähler manifold and θ be a closed smooth real $(1,1)$-form on X. Consider $\varphi\in\mathrm{PSH}(X,\theta)$. For any open set $U\Subset X$, there is a decreasing sequence of quasi-psh functions φ^j on U satisfying*

1. *φ^j has analytic singularities.*
2. *φ^j is decreasing in j and converges to φ everywhere.*
3. *There is a sequence $\tau_j\to 0$ so that*
$$\theta_{\varphi^j}\geqslant-\tau_j\omega\,.$$
4. *$\mathcal{I}((1+2/j)\varphi^j)=\mathcal{I}(\varphi)$.*

2.5. **Bochner–Kodaira–Nakano identities.**

prop:Bochner **Proposition 2.15.** *Let X be a complex manifold and ω be a complete Kähler form on X. Let (L,h) be a singular Hermitian line bundle on X such that $\mathrm{dd^c}h \geqslant -C_1\omega$ for some constant C_1. Let Φ be a smooth function on X such that*

$$\sup_X |\mathrm{d}\Phi|_\omega < \infty, \quad \mathrm{dd^c}\Phi > -C_2\omega$$

for some constant C_2. Then for any

$$u \in \operatorname{Dom}\bar{\partial}^*_{h,\omega} \cap \operatorname{Dom}\bar{\partial} \subseteq L^{n,q}_{(2)}(X,L)_{h,\omega}\,,$$

we have

{eq:twisBochner}

$$\|\sqrt{\eta}(\bar{\partial}+\bar{\partial}\Phi)u\|^2_{h,\omega}+\|\sqrt{\eta}\bar{\partial}^*_{h,\omega}u\|^2_{h,\omega} = \|\sqrt{\eta}(D'^*_{h,\omega}-(\partial\Phi)^*)u\|^2_{h,\omega}+2\pi\langle\eta(\mathrm{dd^c}h+\mathrm{dd^c}\Phi)\Lambda_\omega u,u\rangle_{h,\omega}\,, \tag{2.2}$$

where $\eta = \exp(\Phi)$,

Here we clarify some definitions: for any L-valued forms u,v of the same bi-degree

$$\langle u,v\rangle_{h,\omega} := \frac{1}{n!}\int_X (u,v)_{h,\omega}\omega^n\,.$$

The notation D'_h denotes the $(1,0)$-part of the Chern connect of (L,h) and $D'^*_{h,\omega}$ is its formal adjoint. We define Λ_ω as the adjoint of $\omega\wedge$.

We observe that when $\Phi=0$, (2.2) reduces to the usual Bochner's formula

{eq:Bochuntw}

$$\|\bar{\partial}u\|^2_{h,\omega}+\|\bar{\partial}^*_{h,\omega}u\|^2_{h,\omega} = \|D'^*_{h,\omega}u\|^2_{h,\omega}+2\pi\langle\mathrm{dd^c}h\wedge\Lambda_\omega u,u\rangle_{h,\omega}\,. \tag{2.3}$$

In order to render this formula useful, we need the following simple computation:

Lemma 2.16. *Let X be a complex manifold of pure dimension n and ω be a complete Kähler form on X. Let (L,h) be a smooth Hermitian line bundle on X. We denote the eigenvalues of $\mathrm{dd^c}h$ with respect to ω by $\lambda_1 \leqslant \cdots \leqslant \lambda_n$. Then for any smooth (p,q)-form u with valued in L on X, we have*

eq:thetalambdalower}

$$([\mathrm{dd^c}h,\Lambda_\omega]u,u)_{h,\omega} \geqslant (\lambda_1+\cdots+\lambda_q-\lambda_{n-p+1}-\cdots-\lambda_n)|u|^2_{h,\omega}\,. \tag{2.4}$$

In particular, when $p=n$,

hetalambdalowerpeqn}

$$([\mathrm{dd^c}h,\Lambda_\omega]u,u)_{h,\omega} \geqslant (\lambda_1+\cdots+\lambda_q)|u|^2_{h,\omega}\,. \tag{2.5}$$

Proof. The problem is local on X, so we may replace X be a small coordinate chart so that

$$\omega = \mathrm{i}\sum_{j=1}^n \mathrm{d}z_j\wedge \mathrm{d}\bar{z}_j, \quad \mathrm{dd^c}h = \mathrm{i}\sum_{j=1}^n \lambda_j \mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j\,.$$

Also, we may assume that L admits a nowhere vanishing holomorphic section e. Expand the form u as

$$u = \sum_{|\alpha|=p,|\beta|=q} u_{\alpha\beta}\mathrm{d}z_\alpha\wedge\mathrm{d}\bar{z}_\beta\otimes e\,,$$

where $u_{\alpha\beta}$ are smooth functions on X. Then

$$([\mathrm{dd^c}h,\Lambda_\omega]u,u)_{h,\omega} = \sum_{|\alpha|=p,|\beta|=q}\left(\sum_{j\in\alpha}\lambda_j+\sum_{j\in\beta}\lambda_j-\sum_{j=1}^n\lambda_j\right)|u_{\alpha\beta}|^2\,.$$

For this formula, (2.4) follows. □

2.6. **Čech Cocycles.** Let X be a complex manifold of pure dimension n, ω be a Kähler form on X. Let (L,h) be a singular Hermitian line bundle on X such that $\mathrm{dd^c}h+\omega \geqslant 0$. Let $\mathcal{U}=\{B_i\}_{i\in I}$ be a locally finite Stein cover of X such that $B_i \Subset X$ for each $i\in I$.

For each compact subset $K\Subset B_{i_0\cdots i_q}$, we define a seminorm on $\check{C}^q(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$ by

$$\|\beta\|_{K,i_0\cdots i_q}=\left(\frac{1}{n!}\int_K |\beta_{i_0\cdots i_q}|^2_{h,\omega}\,\omega^n\right)^{1/2}.$$

Observe that this seminorm is independent of the choice of ω.

lma:Fre **Lemma 2.17.** *The set $\check{C}^q(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$ and $Z^q(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$ are both Fréchet spaces with respect to the family of semi-norms $\|\bullet\|_{K,i_0\cdots i_q}$. The Čech coboundary*

$$\partial^q:\check{C}^q(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))\rightarrow\check{C}^{q+1}(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$$

is bounded.

If X is holomorphically convex, then so is $B^q(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$.

Proof. The first part is just some well-known complex analysis. For the latter statement, see [Mat18a, Lemma 2.14]. In a later version, I plan to include the proof. □

ormestitotrivialcech **Lemma 2.18.** *Assume that X is holomorphically convex. Let $\beta\in\check{H}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$. Fix a smooth metric h' on L. Assume that there exists $\beta^j\in\check{C}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$ in the cohomology class of β satisfying*

$$\lim_{j\to\infty}\int_K|\beta^j_{i_0\cdots i_p}|_{h'}=0 \tag{2.6}$$

for any compact subset $K\subseteq U_{i_0\cdots i_p}$. Then $\beta=0$ in $\check{H}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$.

Proof. Observe that the coboundary map

$$\partial^p:\check{C}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(\varphi))\rightarrow\check{Z}^{p+1}(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(\varphi))$$

is continuous and has closed images by Lemma 2.17. It follows that the natural quotient map

$$\check{Z}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(\varphi))\rightarrow\check{H}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(\varphi))$$

is continuous. Our assumption guarantees that $\beta_p\to 0$ in $\check{Z}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(\varphi))$. It follows that the corresponding classes in $\check{H}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(\varphi))$ also converge to 0. But by our assumption, these classes are all equal to β, so $\beta=0$. □

Next we assume that Z is a nowhere dense closed analytic subset of X and $Y=X\setminus Z$. Assume that there is a Kähler form $\tilde{\omega}$ on Y such that

(1) $\tilde{\omega}\geqslant\omega$ on Y.
(2) $\tilde{\omega}$ has locally bounded potentials on X (not Y).

thm:CechtoDeRham **Theorem 2.19.** *There are continuous maps*

$$f : \ker \bar\partial \text{ in } L^{n,q}_{2,\mathrm{loc}}(F)_{h,\tilde\omega} \to \ker \partial^q \text{ in } \check{C}^q(\mathcal{U}, \omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h))$$

and

$$g : \ker \partial^q \text{ in } \check{C}^q(\mathcal{U}, \omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h)) \to \ker \bar\partial \text{ in } L^{n,q}_{(2),\mathrm{loc}}(F)_{h,\tilde\omega}$$

inducing isomorphisms $\bar f$, $\bar g$ on the level of cohomology. Moreover, $\bar f$ and $\bar g$ are inverse to each other.

Here $\bar\partial$ is the closed operator defined in Definition 3.4. We omit the proof and refer to [Mat16 Mat18a, Proposition 2.16].

cor:imgbpclosed **Corollary 2.20.** *Assume furthermore that X is holomorphically convex, then $\operatorname{Im}\bar\partial$ is a closed subspace of $L^{n,q}_{(2),\mathrm{loc}}(F)_{h,\tilde\omega}$.*

prop:cpt **Proposition 2.21.** *Assume furthermore that X is holomorphically convex, then the natural map*

$$\ker \bar\partial \text{ in } L^{n,q}_{(2)}(Y, L)_{h,\tilde\omega} \to \ker \bar\partial / \operatorname{Im} \bar\partial \text{ in } L^{n,q}_{(2),\mathrm{loc}}(Y, L)_{h,\tilde\omega}$$

is compact.

See [Mat16 Mat18a, Proposition 2.19].

2.7. **Uniform integrability.** The following is usually known as the comparision of integral technique.

lma:compint **Lemma 2.22.** *Let X be a compact Kähler manifold of pure dimension n, ω be a Kähler form on X. Let $(\gamma_i)_{i\in I}$ and $(\varphi_i)_{i\in I}$ be families of quasi-psh functions on X satisfying*

(1) *There is a Kähler form ω' on X so that $\omega' + \mathrm{dd^c}\gamma_i \geqslant 0$.*
(2) $\sup_{i\in I}\sup_X \gamma_i < \infty$.

Let (L, h) be a smooth Hermitian line bundle on X and f be a smooth (n, q)-form with value in L on an open subset $U \subseteq X$, then for any $s_1 > 0$, there exists $s > 0$ such that there is a constant $C = C(s_1, s, \|f\|_{L^\infty(h,\omega)}, (\gamma_i)_i)$ such that

{eq:compint}
$$\int_U |f|^2_{h,\omega} \mathrm{e}^{-s\gamma_i - \varphi_i}\, \omega^n \leqslant C \left(\int_U |f|^2_{h,\omega} \mathrm{e}^{-(1+s_1)\varphi_i}\, \omega^n \right)^{1/(1+s_1)}. \tag{2.7}$$

Proof. By uniform Skoda estimate, we can find $a > 0$ so that

$$\sup_{i\in I} \int_X \mathrm{e}^{-a\gamma_i} \leqslant C_0$$

for some $C_0 > 0$. For any given $s_1 > 0$, take $s > 0$ small enough so that $s(1 + s_1)/s_1 \leqslant a$. Then by Hölder's inequality

$$\begin{aligned}\int_U |f|^2_{h,\omega} \mathrm{e}^{-s\gamma_i-\varphi_i}\,\omega^n \leqslant & \left(\int_U |f|^2_{h,\omega}\mathrm{e}^{-(1+s_1)\varphi_i}\,\omega^n\right)^{1/(1+s_1)} \left(\int_U |f|^2_{h,\omega}\mathrm{e}^{-s(1+s_1)/s_1}\right)^{s_1/(s_1+1)}\\ \leqslant & C\left(\int_U |f|^2_{h,\omega}\mathrm{e}^{-(1+s_1)\varphi_i}\,\omega^n\right)^{1/(1+s_1)}.\end{aligned}$$

□

3. L^2-METHODS

We fix a complex manifold X of pure dimension n.

3.1. L^2-spaces of differential forms.

Let (L,h) be a singular Hermitian line bundle on X and ω be a smooth positive real $(1,1)$-form on X.

Definition 3.1. For any smooth L-valued (p,q)-forms u and v at $x\in X$, we introduce the inner product $(u,v)_{h,\omega}$ as follows: take a holomorphic coordinate $z_1,\dots,z_n$ on X and a nowhere vanishing holomorphic section e of L near x, write

$$u=\sum_{|\alpha|=p,|\beta|=q} u_{\alpha,\beta}\mathrm{d}z_\alpha\wedge \mathrm{d}\bar{z}_\beta\otimes e\,,$$
$$v=\sum_{|\alpha|=p,|\beta|=q} v_{\alpha,\beta}\mathrm{d}z_\alpha\wedge \mathrm{d}\bar{z}_\beta\otimes e\,.$$

Then we define

$$(u,v)_{h,\omega}:=\left(\sum_{|\alpha|=p,|\beta|=q} u_{\alpha,\beta}\mathrm{d}z_\alpha,\sum_{|\alpha|=p,|\beta|=q} v_{\alpha,\beta}\mathrm{d}z_\alpha\right)_\omega h(e,e)\,.$$

Here the bracket $(\bullet,\bullet)_\omega$ is the usual inner product induced by the Hermitian metric associated with ω.

We write

$$|u|_{h,\omega}=(u,u)^{1/2}_{h,\omega}\,.$$

When (L,h) is trivial, we usually omit h from the notations. When we want to emphasize X, we will replace the subindex h,ω by h,ω,X. The same convention applies to all later definitions.

Definition 3.2. Define the space $L^{p,q}_{(2)}(X,L)_{h,\omega}$ as the space of L-valued (p,q)-forms u with measurable coefficients such that

$$\int_X (u,u)_{h,\omega}\,\omega^n<\infty\,.$$

Similarly, define $L^{p,q}_{(2),\mathrm{loc}}(X,L)_{h,\omega}$ as the space of F-valued (p,q)-forms u with measurable coefficients such that

$$\int_K (u,u)_{h,\omega}\,\omega^n<\infty$$

for any compact subset $K\subseteq X$.

Define $C^{p,q}_\infty(X,L)$ as the space of smooth L-valued (p,q)-forms.

Definition 3.3. Given L-valued (p,q)-forms u and v on X, we define the inner product

$$<u,v>_{h,\omega}:=\int_X (u,v)_{h,\omega}\,\omega^n\,.$$

Of course, $(u,v)_{h,\omega}$ is only defined almost everywhere.

Next we introduce the $\bar\partial$-operator.

def:bp **Definition 3.4.** The operator $\bar{\partial} : L^{p,q}_{(2)}(X,L)_{h,\omega} \to L^{p,q+1}_{(2)}(X,L)_{h,\omega}$ is a densely defined operator with

$$\operatorname{Dom}\bar{\partial} := \left\{ u \in L^{p,q}_{(2)}(X,L)_{h,\omega} : \bar{\partial}u \in L^{p,q+1}_{(2)}(X,L)_{h,\omega} \right\} ,$$

where $\bar{\partial}$ on the right-hand side means $\bar{\partial}$ in the sense of distribution. We then define the unbounded operator $\bar{\partial}$ on $\operatorname{Dom}\bar{\partial}$ as the $\bar{\partial}$ in the sense of distribution.

It is obvious that $\bar{\partial}$ is closed*. Similarly, when h is smooth, we introduce the densely defined operator

$$D'_{h,\omega} : L^{p,q}_{(2)}(X,L)_{h,\omega} \to L^{p+1,q}_{(2)}(X,L)_{h,\omega} ,$$

which is the $(1,0)$-part of the Chern connection of (L,h).

Let $* : C^{p,q}_{\infty}(X,L) \to C^{n-p,n-q}_{\infty}(X,L)$ be the Hodge star normalized by

$$\frac{1}{n!}(u,v)_{h,\omega}\,\omega^n = (u \wedge \overline{*v})_h ,$$

where on the right-hand side $(\bullet)_h$ means that we contract the indices in L with h.

Recall the following density lemma of Andreotti–Vesentini. We refer to Hor90 [Hör90, Lemma 5.2.1] for a proof.

lma:den **Lemma 3.5.** *Assume that ω is complete and h is smooth. The set of compactly supported smooth (p,q)-forms with value in L is dense in $\operatorname{Dom}\bar{\partial}^*_{h,\omega}$, $\operatorname{Dom}\bar{\partial}$, $\operatorname{Dom}\bar{\partial}^*_{h,\omega} \cap \operatorname{Dom}\bar{\partial}$ respectively with respect to the graph norm of $\bar{\partial}$, the graph norm of $\bar{\partial}^*_{h,\omega}$ and the norm $u \mapsto \|u\|_{h,\omega} + \|\bar{\partial}^*_{h,\omega}u\|_{h,\omega} + \|\bar{\partial}u\|_{h,\omega}$.*

Here $\bar{\partial}^*_{h,\omega}$ denotes the Hilbert space adjoint of $\bar{\partial}$.

cor:adjcoin **Corollary 3.6.** *Assume that ω is complete and h is smooth. On the space of smooth forms with compact supports, $\bar{\partial}^*_{h,\omega}$ coincides with the formal adjoint*

$$\bar{\partial}^*_{h,\omega} = - * \bar{\partial} * .$$

When h is smooth, we let $D'^*_{h,\omega}$ denote the formal adjoints of $D'_{h,\omega}$:

$$D'^*_{h,\omega} = - * D'_{h,\omega} *$$

defined on the space of smooth forms.

For any smooth (s,t)-form θ, θ acts on $C^{p,q}_{\infty}(X,L)$ by wedge product on the left, its pointwise adjoint is given by

$$\theta^* = (-1)^{p+q}(s+t+1) * \bar{\theta} * .$$

We introduce the Lefschetz operators:

$$\Lambda_\omega : C^{p,q}_{\infty}(X,L) \to C^{p-1,q-1}_{\infty}(X,L)$$

is the pointwise adjoint of $\omega \wedge \bullet$.

lma:speceq **Lemma 3.7.** *If ω is a Kähler form (i.e. if ω is closed), then*

(1) $\theta^* = \mathrm{i}[\bar{\theta}, \Lambda_\omega]$ for any smooth $(1,0)$-form θ.
(2) $\theta^* = -\mathrm{i}[\bar{\theta}, \Lambda_\omega]$ for any smooth $(0,1)$-form θ.

*Matsumura talks about maximal extensions, but an unbounded operator not defining on the whole space never has a maximal extension.

(3) If h is smooth, for any smooth function Φ on X,

$$[\bar{\partial}, (\bar{\partial}\Phi)^*] + [D'*_{h,\omega}, \partial\Phi] = 2\pi[\mathrm{dd^c}, \Lambda_\omega]\,.$$

All equalities are in the sense of operators on smooth forms with value in L.

lma:diffomegacomp **Lemma 3.8.** *Let ω' and ω be smooth positive real $(1,1)$-forms such that $\omega' \geqslant \omega$. Then there is a constant $C > 0$ so that $|\theta^* u|_\omega \leqslant C|\theta|_\omega |u|_\omega$ for all smooth forms θ and u. Moreover,*

(1) $|u|_{\omega'} \leqslant |u|_\omega$ for smooth forms u.
(2) $|u|_{\omega'}\omega'^n \leqslant |u|_\omega \omega^n$ for smooth (n,q)-forms u.
(3) $|u|_{\omega'}\omega'^n = |u|_\omega \omega^n$ for smooth $(n,0)$-forms u.

Similarly, when h is smooth, the same holds for forms with value in L.

Both results follow from simple computations, which we omit.

3.2. **Adjoint operators on domains with boundaries.** Let (L,h) be a smooth Hermitian line bundle on X, ω be a Hermitian form on X and Φ be a smooth function on X. For each $c \in \mathbb{R}$, we write $X_c := \{x \in X : \Phi(x) < c\}$. When $X_c \Subset X$ and $\mathrm{d}\Phi$ does not vanish on ∂X_c, we define an inner product

$$<u, v>_{h,\omega,\partial X_c} := \int_{\partial X_c} (u,v)_{h,\omega}\,\mathrm{d}S_\omega$$

for smooth L-valued (p,q)-forms u, v defined in a neighbourhood of ∂X_c. Here $\mathrm{d}S_\omega$ is the volume form on ∂X_c defined by $\mathrm{d}S_\omega = \frac{1}{|\mathrm{d}\Phi|^2_\omega} * \mathrm{d}\Phi$. Then

$$\frac{1}{n!}\omega^n = \mathrm{d}\Phi \wedge \mathrm{d}S_\omega\,.$$

We reformulate Stokes formula as follows:

prop:Stokes **Proposition 3.9.** *Let u (resp. v) be a smooth L-valued $(p,q-1)$-form (resp. (p,q)-form) on X. Given c as above, we have*

$$<\bar{\partial} u, v>_{h,\omega,X_c} = <u, \bar{\partial}^*_{h,\omega} v>_{h,\omega,X_c} + <u, (\bar{\partial}\Phi)^* v>_{h,\omega,\partial X_c}\,.$$

More generally,

prop:Stokesgen **Proposition 3.10.** *Let Y be the complement of a nowhere dense closed analytic subset of X. Assume that there is a complete positive $(1,1)$-form ω' on Y. Let u (resp. v) be a smooth L-valued $(p,q-1)$-form (resp. (p,q)-form) on Y satisfying*

$$\|u\|_{h,\omega'}, \|v\|_{h,\omega'}, \|\bar{\partial} u\|_{h,\omega'}, \|\bar{\partial}^*_{h,\omega'} v\|_{h,\omega'} < \infty\,.$$

Take c as above. Then there is a sufficiently small positive number $\epsilon > 0$ such that

(1) $\mathrm{d}\Phi$ does not vanish on ∂X_d for every $d \in (c-\epsilon, c+\epsilon)$.
(2) For almost every $d \in (c-\epsilon, c+\epsilon)$,

$$<\bar{\partial} u, v>_{h,\omega',X_d} = <u, \bar{\partial}^*_{h,\omega} v>_{h,\omega,X_d} + <u, (\bar{\partial}\Phi)^* v>_{h,\omega,\partial X_d}\,. \tag{3.1}$$

The proof involves a simple cutoff argument, we refer to [Mat16 Mat18a, Proposition 2.5] for the details.

3.3. L^2-estimates.

prop:L2esti **Proposition 3.11.** *Let X be a connected compact Kähler manifold of dimension n, ω be a Kähler form on X. Let (L,h) be a singular Hermitian line bundle on X satisfying the following conditions:*

(1) h is smooth outside a proper closed analytic subset Z of X.

(2) $\mathrm{dd^c} h \geqslant -\epsilon\omega$ for some $\epsilon>0$ on $X\setminus Z$.

Let f be a holomorphic (n,q)-form with value in L satisfying

$$\int_X |f|^2_{h,\omega}\,\omega^n<\infty\,.$$

Let $\lambda_1\leqslant\cdots\leqslant\lambda_n$ be the eigenvalues of $\mathrm{dd^c}h$ and set $\hat\lambda_i=\lambda_i+2\epsilon$. Then there exist a $(n,q-1)$-form u with L^2-coefficients with value in L and a (n,q)-form v with L^2-coefficients with value in L such that

{eq:ftodupv}
$$f=\bar\partial u+v \tag{3.2}$$

and

{eq:uvestimate}
$$\int_X |u|^2_{h,\omega}\,\omega^n+\frac{1}{4\pi p\epsilon}\int_X|v|^2_{h,\omega}\,\omega^n\leqslant\frac{1}{2\pi}\int_X\frac{1}{\hat\lambda_1+\cdots+\hat\lambda_p}|f|^2_{h,\omega}\,\omega^n\,. \tag{3.3}$$

Proof. As $Y:=X\setminus Z$ is weakly 1-complete, we can fix a complete Kähler metric ω' on Y. For any $\delta>0$, set $\omega_\delta=\omega+\delta\omega'$. Our assumption on f implies that $f\in L^{n,q}_{(2)}(Y,L)_{h,\omega_\delta}$. We also observe that $L^{n,q}_{(2)}(Y,L)_{h,\omega_\delta}$ gets smaller as δ decreases to 0.

For any $s\in\operatorname{Dom}\bar\partial^*_{h,\omega_\delta}$, we decompose it as s_1+s_2, where $s_1\in\ker\bar\partial$ and $s_2\in(\ker\bar\partial)^\perp\subseteq\ker\bar\partial^*_{h,\omega_\delta}$.

eq:Bochunteq:thetalambdalowerpeqn

By Bochner's formula (2.3) and (2.5),

$$\|\bar\partial s_1\|^2_{h,\omega_\delta}+\|\bar\partial^*_{h,\omega}s_1\|^2_{h,\omega_\delta}\geqslant 2\pi\int_Y\left(\hat\lambda_1+\cdots+\hat\lambda_p-2p\epsilon\right)|s_1|^2_{h,\omega_\delta}\,\omega^n_\delta\,.$$

By assumption, $f\in\ker\bar\partial$, so

$$\begin{aligned}
|{<}f,s{>}_{h,\omega_\delta}|^2=&|{<}f,s_1{>}_{h,\omega_\delta}|^2\\
\leqslant&\left(\int_Y\frac{1}{\hat\lambda_1+\cdots+\hat\lambda_p}|f|^2_{h,\omega_\delta}\,\omega^n_\delta\right)\left(\int_Y(\hat\lambda_1+\cdots+\hat\lambda_p)|s_1|^2_{h,\omega_\delta}\,\omega^n_\delta\right)\\
\leqslant&\left(\int_Y\frac{1}{\hat\lambda_1+\cdots+\hat\lambda_p}|f|^2_{h,\omega_\delta}\,\omega^n_\delta\right)\left(2p\epsilon\|s_1\|^2_{h,\omega_\delta}+(2\pi)^{-1}\|\bar\partial^*_{h,\omega}s_1\|^2_{h,\omega_\delta}\right)\\
\leqslant&\frac{1}{2\pi}\left(\int_Y\frac{1}{\hat\lambda_1+\cdots+\hat\lambda_p}|f|^2_{h,\omega_\delta}\,\omega^n_\delta\right)\left(4\pi p\epsilon\|s\|^2_{h,\omega_\delta}+\|\bar\partial^*_{h,\omega}s\|^2_{h,\omega_\delta}\right)\,.
\end{aligned}$$

By Hahn–Banach theorem applied to

$$L^{n,q}_{(2)}(Y,L)_{h,\omega_\delta}\times L^{n,q}_{(2)}(Y,L)_{h,\omega_\delta}$$

and the subspace $\operatorname{Dom}\bar\partial^*_{h,\omega_\delta}$ (embedded into the former space by $s\mapsto((4\pi p\epsilon)^{1/2}s,\bar\partial^*_{h,\omega}s))$, we can find L^2-forms u_δ,v_δ on Y with value in L of appropriate degrees so that

{eq:fsdecomp}
$${<}f,s{>}_{h,\omega_\delta}={<}u_\delta,\bar\partial^*_{h,\omega_\delta}s{>}_{h,\omega_\delta}+{<}v_\delta,s{>}_{h,\omega_\delta} \tag{3.4}$$

and

$$\|u_\delta\|^2_{h,\omega_\delta} + \frac{1}{4\pi p\epsilon}\|v_\delta\|^2_{h,\omega_\delta} \leqslant \frac{1}{2\pi}\left(\int_Y \frac{1}{\hat{\lambda}_1+\cdots+\hat{\lambda}_p}|f|^2_{h,\omega_\delta}\,\omega_\delta^n\right).$$

Fix $\delta' > 0$. Take a sequence $\delta_i \to 0$ of positive numbers so that u_{δ_i} (resp. v_{δ_i}) converges weakly to some u (resp. v) in $L^{n,q}_{(2)}(Y,L)_{h,\omega_{\delta'}}$. It follows that

$$\begin{aligned}\|u\|^2_{h,\omega_{\delta'}} + \frac{1}{4\pi p\epsilon}\|v\|^2_{h,\omega_{\delta'}} \leqslant & \frac{1}{2\pi}\left(\int_Y \frac{1}{\hat{\lambda}_1+\cdots+\hat{\lambda}_p}|f|^2_{h,\omega_{\delta'}}\,\omega_{\delta'}^n\right) \\ \leqslant & \frac{1}{2\pi}\left(\int_Y \frac{1}{\hat{\lambda}_1+\cdots+\hat{\lambda}_p}|f|^2_{h,\omega}\,\omega^n\right).\end{aligned}$$

Let $\delta' \to 0+$, we find

$$\|u\|^2_{h,\omega} + \frac{1}{4\pi p\epsilon}\|v\|^2_{h} \leqslant \frac{1}{2\pi}\left(\int_Y \frac{1}{\hat{\lambda}_1+\cdots+\hat{\lambda}_p}|f|^2_{h,\omega}\,\omega^n\right),$$

which is exactly (3.3). It remains to prove (3.2). By (3.4) and the fact that the maximal extension of $\bar{\partial}^*_{h,\omega_\delta}$ coincides with the Hilbert space adjoint of $\bar{\partial}$ (which follows from the density lemma, applicable as ω_δ is complete), we have

$$f = \bar{\partial}u_\delta + v_\delta$$

in the sense of currents. Let $\delta \to 0$ along δ_i, we have

$$f = \bar{\partial}u + v$$

as currents on Y. Using the estimate (3.3), we may extend u and v to the whole X as forms with L^2-coefficients and (3.2) follows. □

In the proof of Proposition 3.11, we take an arbitrary complete Kähler metric on Y. We can make this more explicit:

Proposition 3.12. *Let X be a Kähler manifold and ω be a complete Kähler form on X. Let Z be a nowhere dense closed analytic subset of X. Write $Y = X \setminus Z$. Assume there is a larger Kähler manifold X' such that $X \Subset X'$, a nowhere dense closed analytic subset Z' of X' such that $Z' \cap X = Z$. Then there is a complete Kähler metric ω' on Y satisfying*

(1) $\omega' \geqslant \omega$ on Y.
(2) The local potentials of ω' on X (not on Y) are locally bounded.

Proof. Fix a quasi-psh function ψ on X which has log poles along Z and smooth outside Z. From out assumption about X' and Z', we may assume that ψ is bounded from above on X, say $\psi < -\mathrm{e}$ on X. Set

$$\tilde{\psi} = \frac{1}{\log(-\psi)},$$

which is a quasi-psh function on X with $\tilde{\psi} < 1$. Take a positive constant α such that

$$\alpha\omega + \mathrm{dd^c}\tilde{\psi} > 0$$

on Y. Then we claim that

$$\omega' := \omega + (\alpha\omega + \mathrm{dd^c}\tilde{\psi})$$

is the Kähler form we need. All we need to show is that this Kähler form is complete on Y. This follows from the inequality

{eq:omegaplower} (3.5)
$$\omega' \geqslant \frac{\mathrm{i}}{2\pi}\partial(\log\log(-\psi)) \wedge \bar{\partial}(\log\log(-\psi)) \tag{3.5}$$

on Y_k as long as α is large enough and the fact that $\log\log(-\psi)$ tends to ∞ on Z. The equation (3.5) itself follows from a direct computation, which we leave to the readers. □

4. The Ohsawa–Takegoshi extension theorem

We need the following theorem due to Cao [Cao17] and Guan–Zhou [GZ15].

thm:OTadj **Theorem 4.1.** *Let $f : X \to B^m$ be a proper Kähler morphism from a connected complex manifold X of pure dimension n to B^m, the unit ball in $\mathbb{C}^m$. Let (L, h) be a Hermitian psef line bundle on X such that X_0 is smooth of pure codimension m and the restriction of h to X_0 is not identically ∞ on any connected component of X_0.*

We also assume that there is a proper Kähler morphism $f' : X' \to B^m(r)$ ($r > 1$) extending f such that (L, h) extends to a Hermitian psef line bundle on X'. † *Then for any $\alpha \in H^0(\omega_{X_0} \otimes \mathcal{L} \otimes \mathcal{I}(h|_{X_0}))$, there is a section $s \in H^0(X, \omega_X \otimes \mathcal{L})$ such that $\alpha = s|_{X_0}$ and*

$$\frac{1}{n!}\int_X |s \wedge \bar{s}|_h \leqslant \frac{\mu(B^m)}{(n-m)!}\int_{X_0} |\alpha \wedge \bar{\alpha}|_h\,. \tag{4.1}$$

Here μ is the Lebesgue measure, so $\mu(B^m) = \pi^m/m!$.

We will prove the following stronger result.

thm:OT2 **Theorem 4.2.** *Let X be a connected weakly 1-complete Kähler manifold of dimension n and ω be a complete Kähler metric on X. Assume that X admits a finite covering by domains biholomorphic to pseudoconvex domains in $\mathbb{C}^n$* ‡. *Let (E, h_E) be a smooth Hermitian holomorphic vector bundle or rank r on X. Fix a non-zero holomorphic section v of E. We assume that the zero locus Z of v is smooth of pure codimension r and $|v|^{2r}_{h_E} \leqslant 1$. Set $\Psi := \log|v|^{2r}_{h_E}$. Assume that $\mathrm{dd^c}\Psi \geqslant 0$.*

Let (L, h) be a Hermitian psef line bundle on X. We assume that there is a sequence of increasing analytic approximations h^k of h satisfying

$$\mathrm{dd^c}h^k \geqslant -\epsilon_k\omega$$

with $\epsilon_k \to 0+$.

Then for any $f \in H^0(Z, \omega_X|_Z \otimes \mathcal{L}|_Z \otimes \mathcal{I}(h|_Z))$ and any $\delta > 0$, there is $F \in H^0(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h))$ extending f and

{eq:est3} (4.2)
$$\frac{1}{n!}\int_X |F \wedge \bar{F}|_h \leqslant \frac{1+\delta}{(n-r)!}\int_Z |f|^2_{h,\omega}|\Lambda^r \mathrm{d}v|^{-2}\omega|^{n-r}_Z\,, \tag{4.2}$$

†This condition is omitted in [Cao17]. It seems necessary to include it in order to apply Theorem 4.2. Otherwise, s can still be defined, but only over a smaller polydisk.

‡This assumption is omitted in [Cao17]. We include it because we need a uniform constant σ in (4.3)

where $\Lambda^r(\mathrm{d}v)$ *is define by*

$$\frac{1}{(n-r)!}\int_Z \frac{G}{|\Lambda^r(\mathrm{d}v)|^2}\,\omega|_Z^{n-r} = \frac{1}{n!}\lim_{m\to\infty}\int_{\{-m-1\leqslant \log|v|^{2r}_{h_E}\leqslant -m\}} \frac{G}{|v|^{2r}_{h_E}}\,\omega^n$$

for any smooth function G *on* X.

This theorem clear implies Theorem 4.1.

We introduce a few notations that will be used in the proof. For each $m \geqslant 1$, we define

$$b_m(t) = \begin{cases} 1, & t\geqslant -m;\\ t+m+1, & t\in[-m-1,-m);\\ 0, & t\in(-\infty,-m-1). \end{cases}$$

In particular, $0 \leqslant b_m \leqslant 1$.

Proof of Theorem 4.2. Fix a smooth metric h_0 on L and identify h with $\varphi \in \mathrm{PSH}(X,\theta)$, where $\theta = \mathrm{dd^c} h_0$.

Step 1. We claim that there is a smooth section $\tilde f$ of $K_X \otimes L$ such that

(1) $\tilde f|_Z = f$.
(2) $\bar\partial \tilde f|_Z = 0$.
(3) There is a constant $\sigma > 0$ such that

$$\int_X \frac{|\bar\partial \tilde f|^2_{h_0,\omega}}{|v|^{2r}_{h_E}(\log|v|_{h_E})^2}\mathrm{e}^{-(1+\sigma)\varphi}\,\omega^n \leqslant C\int_Z \frac{|f|^2_{h,\omega}}{|\Lambda^r(\mathrm{d}v)|^2}\,\omega|_Z^{n-r}\,.$$

Taking a finite Stein cover $\{U_i\}$ such that each U_i is biholomorphic to a pseudo-convex domain in $\mathbb{C}^n$ of X and a partition of unity χ_i subordinate to $\{U_i\}$. Locally on each U_i, by strong openness, we can find $\sigma > 0$ so that

{eq:strongop1} (4.3) $$\int_{U_i\cap Z} |f|^2_{h_0,\omega}\mathrm{e}^{-(1+\sigma)\varphi}\,\omega^{n-r} \leqslant 2\int_Z |f\wedge\bar f|_h\,\omega^{n-r}\,.$$

By the usual version of the Ohsawa–Takegoshi theorem [Dem12, Theorem 12.6], we can find holomorphic sections f_i of $K_X \otimes L$ on U_i extending such that

$$\int_{U_i} \frac{|f_i|^2_{h_0,\omega}}{|v|^{2r}_{h_E}(\log|v|_{h_E})^2}\mathrm{e}^{-(1+\sigma)\varphi}\,\omega^n \leqslant C\int_{U_i\cap Z} \frac{|f|^2_{h,\omega}}{|\Lambda^r(\mathrm{d}v)|^2}\,\omega|_Z^{n-r}\,.$$

It suffices to take $\tilde f = \sum_i \chi_i f_i$.

Step 2. Set $g_m = \bar\partial((1-b_m\circ\Psi)\tilde f)$. We claim that there is a sequence of positive integers $a_m \to \infty$ satisfying $m/a_m \to 0$ and L-valued locally L^2-forms γ_m, β_m of appropriate degrees such that

{eq:bpgm} (4.4) $$\bar\partial\gamma_m + (m/a_m)^{1/2}\beta_m = g_m$$

and

{eq:est2} (4.5)

$$\varlimsup_{m\to\infty}\left(\frac{1}{n!}\int_X |\gamma_m|^2_{h_{a_m},\omega}\,\omega^n + C\int_X |\beta_m|^2_{h_{a_m},\omega}\mathrm{e}^{-\Phi}\,\omega^n\right) \leqslant \frac{1}{(n-r)!}\int_Z \frac{|f|^2_{h,\omega}}{|\Lambda^r(\mathrm{d}v)|^2}\,\omega|_Z^{n-r}\,.$$

Moreover,

{eq:gammamz0} (4.6) $$\gamma_m|_Z = 0\,.$$

Up to passing to a subsequence, we assume that $\gamma_m - (1 - b_m \circ \Psi)\tilde{f}$ converges weakly to some F in $L^{n,0}_{(2)}(L)_{h_0,\omega}$.

The proof of the claim is a long and tedious calculation, we refer to [Cao17, Lemma 2.1] for the details.

Step 3. We verify that F satisfies the desired properties. By (4.5) and (4.4), we conclude that

$$\bar{\partial}(\gamma_m - (1 - b_m \circ \Psi)\tilde{f}) = -(m/a_m)^{1/2}\beta_m$$

converges weakly to 0 in $L^{n,1}_{(2)}(L)_{h_{a_m}\exp(-\Phi),\omega}$. As

$$\bar{\partial} : L^{n,0}_{(2)}(X, L)_{h_0,\omega} \to L^{n,1}_{(2)}(X, L)_{h_{a_m}\exp(-\Phi),\omega}$$

is a closed operator, it follows that F is holomorphic.

Next we show that F extends f. Fix the Stein covering $\{U_i\}$ as before. We need to show that $F|_{U_i \cap Z} = f$. We solve the $\bar{\partial}$-equation: $\bar{\partial} w_m = \beta_m$ on U_i such that

$$\int_{U_i} |w_m|^2_{h_{a_m},\omega} \mathrm{e}^{-\Psi}\omega^n \leqslant C \int_{U_i} |\beta_m|^2_{h_{a_m},\omega} \mathrm{e}^{-\Psi}\omega^n \leqslant C\,.$$

Here the second inequality follows from (4.5).

It follows that

$$F_m := (1 - b_m \circ \Psi)\tilde{f} - \gamma_m - (m/a_m)^{1/2} w_m$$

is a holomorphic function on U_i. Moreover, F_m converges to F weakly in $L^{n,0}_{(2)}(U_i, L)_{h_{a_m}\exp(-\Phi),\omega}$. By (4.6), $F_m|_{U_i \cap Z} = f|_{U_i \cap Z}$, so it follows that $F|_{U_i \cap Z} = f|_{U_i \cap Z}$ as well.

It remains to establish the estimate (4.2). By (4.5) and the monotonicity of h_k, we have

$$\varlimsup_{m\to\infty} \frac{1}{n!} \int_X |\gamma_m|^2_{h_k,\omega}\,\omega^n \leqslant \frac{1}{(n-r)!} \int_Z \frac{|f|^2_{h,\omega}}{|\Lambda^r(\mathrm{d}v)|^2}\,\omega|_Z^{n-r}$$

for any fixed $k > 0$. By Fatou's lemma, we find

$$\frac{1}{n!} \int_X |F|^2_{h_k,\omega}\,\omega^n \leqslant \frac{1}{(n-r)!} \int_Z \frac{|f|^2_{h,\omega}}{|\Lambda^r(\mathrm{d}v)|^2}\,\omega|_Z^{n-r}\,.$$

Let $k \to \infty$, we conclude (4.2). ☐

5. Nadel–Cao vanishing theorem

In this section, we fix a compact Kähler manifold X of pure dimension n. We will prove Nadel–Cao vanishing theorem.

Theorem 5.1 ([Cao14]). *Let (L, h) be a Hermitian psef line bundle on X. Then*

$$H^q(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h)) = 0 \quad \textit{for } p > n - \mathrm{nd}(L, h)\,.$$

Here and in the whole paper, we use the caligraphic font $\mathcal{L}$ to denote $\mathcal{O}_X(L)$. The same convention applies to other line bundles as well.

The proof of Theorem 5.1 relies on Lemma 2.18. We want to represent a general element β in $H^q(X, \omega_X \otimes \mathcal{O}_X(L) \otimes \mathcal{I}(h))$ by suitable Čech cocycles v^j so that the local norms of v^j tend to 0. Under the Čech to de Rham isomorphism, this amounts to choosing holomorphic forms representing β with small norms, which can be carried out by L^2-estimates.

The problem is that in order to apply L^2-estimates as in Proposition 3.11, we need some regularity of the metric. So we need to carry out a suitable approximation preserving $\mathcal{I}(h)$ at first, such approximations are called *equisingular approximations*:

lma:esapp **Lemma 5.2.** *Let (L,h) be a Hermitian psef line bundle on X and $p > n - \mathrm{nd}(L,h)$. Fix a Kähler metric ω on X. Then there is a sequence of metrics h^j with analytic singularities on L with the following properties:*

(1) $\mathcal{I}(h^j) = \mathcal{I}(h)$. Moreover, take a smooth metric h_0 on L and write h with $h_0\exp(-\varphi)$, then for any small enough $s_1 > 0$, there exists $s > 0$ such that for any smooth bounded (n,q)-form on an open subset U of X, we have

{eq:locintfhj}
$$\int_U |f|^2_{h^j,\omega}\,\omega^n \leqslant C(\|f\|_{L^\infty,h_0,\omega})\left(\int_U |f|^2_{h_0,\omega}\mathrm{e}^{-(1+s_1)\varphi}\,\omega^n\right)^{1/(1+s_1)}. \tag{5.1}$$

(2) We write Z_j for the singular locus of h^j Let $\lambda_1^j \leqslant \lambda_2^j \leqslant \cdots \leqslant \lambda_n^j$ be the eigenvalues of $\mathrm{dd^c}h^j$ with respect to ω, defined on $X \setminus Z_j$. Then there is a sequence of positive numbers $\epsilon_j \to 0$ such that

{eq:lambdaesti}
$$\lambda_1^j + \epsilon_j \geqslant \frac{1}{2}\epsilon_j \quad \text{on } X \setminus Z_j\,. \tag{5.2}$$

(3) We can choose $\beta > 0$ and $\alpha \in (0,1)$ such that for all $j \geqslant 1$, there is an open subset $U_j \subseteq X \setminus Z_j$ satisfying

$$\lim_{j\to\infty}\int_{U_j}\omega^n = 0$$

and

$$\lambda_p^j + 2\epsilon_j \geqslant \epsilon_j^\alpha \quad \text{on } X \setminus (U_j \cup Z_j)\,.$$

(4) There is a smooth metric H on L such that $H \leqslant h^j$ for all j.

Let us deduce Theorem 5.1 from this lemma.

Proof of Theorem 5.1. Let h_0 be a smooth metric on L and let $\theta = c_1(L,h_0)$. We will identify h with $\varphi \in \mathrm{PSH}(X,\theta)$ through $h = h_0\exp(-\varphi)$.

Let h_j be the approximations constructed in Lemma 5.2. Let f be a holomorphic (n,p)-form representing a class $\alpha \in H^p(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{I}(\varphi))$. Take $s_1, s > 0$ so that (5.1) holds. We assume that s_1 is small enough so that $\mathcal{I}((1+s_1)\varphi) = \mathcal{I}(\varphi)$.

By L^2-estimates Proposition 3.11, we can write $f = \bar\partial u_j + v_j$ such that

{eq:temp1}
$$\int_X |u_j|^2_{h^j,\omega}\,\omega^n + \frac{1}{4\pi p\epsilon_j}\int_X |v_j|^2_{h^j,\omega}\,\omega^n \leqslant \frac{1}{2\pi}\int_X \frac{1}{\hat\lambda_1^j + \cdots + \hat\lambda_p^j}|f|^2_{h^j,\omega}\,\omega^n\,. \tag{5.3}$$

Here $\hat\lambda_p^j = \lambda_p^j + 2\epsilon_j$.

We estimate the right-hand side using Lemma 5.2. By Lemma 5.2(2), $\hat{\lambda}_1^j \geqslant c_1\epsilon_j$ for some $c_1 > 0$ independent of j, so

$$\begin{aligned}\int_X \frac{1}{\hat{\lambda}_1^j+\cdots+\hat{\lambda}_p^j}|f|^2_{h^j,\omega}\,\omega^n =& \int_{U_j} \frac{1}{\hat{\lambda}_1^j+\cdots+\hat{\lambda}_p^j}|f|^2_{h^j,\omega}\,\omega^n + \int_{X\setminus U_j} \frac{1}{\hat{\lambda}_1^j+\cdots+\hat{\lambda}_p^j}|f|^2_{h^j,\omega}\,\omega^n\\ \leqslant & C\int_{U_j}\frac{1}{\epsilon_j}|f|^2_{h^j,\omega}\,\omega^n + C\int_{X\setminus U_j}\frac{1}{\epsilon_j^\alpha}|f|^2_{h^j,\omega}\,\omega^n\,.\end{aligned}$$

It follows that

$$\int_X |v_j|^2_{h^j,\omega}\,\omega^n \leqslant C\int_{U_j}|f|^2_{h^j,\omega}\,\omega^n + C\epsilon_j^{1-\alpha}\int_X |f|^2_{h^j,\omega}\,\omega^n\,.$$

As the volume of U_j tends to 0, the first term tends to 0 as $j\to\infty$. The second term tends to 0 as by (5.1), $\int_X |f|^2_{h^j,\omega}\,\omega^n$ is uniformly bounded. It follows that

{eq:liml20}
$$\lim_{j\to\infty}\int_X |v_j|^2_{h^j,\omega}\,\omega^n = 0\,. \tag{5.4}$$

Now take a Stein covering $\mathcal{U}=\{U_i\}_{i\in I}$ of X. Represent v_j by a Čech cocycle

$$v_j\in \check{C}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h^j)) = \check{C}^p(\mathcal{U},\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(\varphi))\,.$$

The components of this cocycle satisfy

$$\int_{U_{i_0\cdots i_p}} |v_{j,i_0\cdots i_p}|^2_{h^j,\omega}\,\omega^n \leqslant C\int_X |v_j|^2_{h^j,\omega}\,\omega^n\,.$$

It follows from this inequality and (5.4) that

$$\lim_{j\to\infty}\int_{U_{i_0\cdots i_p}} |v_{j,i_0\cdots i_p}|^2_{h^j,\omega}\,\omega^n = 0\,.$$

In particular,

$$\lim_{j\to\infty}\int_{U_{i_0\cdots i_p}} |v_{j,i_0\cdots i_p}|^2_{H,\omega}\,\omega^n = 0\,.$$

By Lemma 2.18 we conclude that the cohomology class of f is trivial. □

Proof of Lemma 5.2. Fix a smooth metric h_0 on L. Let $\theta = c_1(L,h_0)$. Fix a Kähler form ω on X. We identify h with $\varphi\in \mathrm{PSH}(X,\theta)$. Let φ^j be a quasi-equisingular approximation of φ as in Theorem 2.14. To be more precise, we require the following properties:

(1) φ^j has analytic singularities.
(2) φ^j is decreasing in j and converges to φ everywhere.
(3) There is a sequence $\tau_j\to 0$ so that
$$\theta_{\varphi^j}\geqslant -\tau_j\omega\,.$$
(4) $\mathcal{I}((1+2/j)\varphi^j)=\mathcal{I}(\varphi)$.

Take $C_1>0$ so that $\theta\leqslant C_1\omega$.

Step 1. Construction of the metric h^j.

For each j, choose a log resolution $\pi_j : X_j\to X$ of φ^j. Write

$$\mathrm{dd^c}\pi_j^*\varphi^j = [E_j]+f_j\,,$$

where E_j is a nc $\mathbb{Q}$-divisor on X_j and f_j is smooth. Fix a smooth metric h_j on $\mathcal{O}_{X_j}(-E_j)$ so that $\pi^*\omega + \delta \mathrm{dd^c} h_j$ is a Kähler form on X_j for all $\delta > 0$ small enough. We write s_j for the canonical section of $\mathcal{O}(E_j)$.

Take two sequences $\delta_j \to 0$, $\epsilon_j \to 0$ of positive numbers so that

(1) $\pi^*\omega + \delta_j \mathrm{dd^c} h_j$ is a Kähler form on X_j.
(2) $(\epsilon_j - \tau_j)\pi^*\omega + \delta_j \mathrm{dd^c} h_j$ is a Kähler form on X_j.
(3)

{eq:temp2}
$$\frac{1}{2}\epsilon_j - 3\tau_j - \frac{2C_1}{j} \geqslant 0\,. \tag{5.5}$$

We consider the following Monge–Ampère type equation on X_j with respect to ψ_j:

{eq:MAXj}
$$\begin{cases} (\pi^*\theta_{\varphi^j} + \epsilon_j \pi^*\omega + \delta_j \mathrm{dd^c} h_j + \mathrm{dd^c}\psi_j)^n = C_j \epsilon_j^{n-\mathrm{nd}(L,h)} (\omega + \delta_j \mathrm{dd^c} h_j)^n, \\ \sup_{z\in X_j} (\pi^*\varphi^j + \psi_j + \delta_j \log |s_j|_{h_j^{-1}})(z) = 0\,. \end{cases} \tag{5.6}$$

Here C is a constant making the two sides having same masses. By Yau's theorem, this equation has a unique smooth solution ψ_j. We introduce $\gamma_j := \pi^*\varphi^j + \psi_j + \delta_j \log |s_j|_{h_j^{-1}}$.

Observe that by definition of $\mathrm{nd}(L,h)$, C_j is bounded away from 0. We get immediately from the definition that

$$\pi^*\theta_{\varphi^j} + \delta_j \mathrm{dd^c} h_j + \mathrm{dd^c}\psi_j \geqslant -\epsilon_j \pi^*\omega\,.$$

By Lelong–Poincaré formula,

$$\mathrm{dd^c} \log |s_j|_{h_j^{-1}} = [E_j] + \mathrm{dd^c} h_j\,.$$

So

{eq:pistarthetalower}
$$\pi^*\theta + \mathrm{dd^c}\gamma_j \geqslant -\epsilon_j \pi^*\omega\,. \tag{5.7}$$

In particular, γ_j descends to a $\theta + \epsilon_j\omega$-psh function on X, which we still denote by γ_j.

Now we can define

{eq:etajdef}
$$\eta^j := (1 + 2j^{-1} - s)\varphi^j + s\gamma_j \tag{5.8}$$

for some small enough $s > 0$. The exact condition of s will be clear from the next step. We will regard η^j as a metric on $\pi_j^* L$, namely, $\pi_j^* h_0 \exp(-\eta^j)$ is a metric on $\pi_j^* L$.

We can easily compute the curvature current of this metric:

$$\begin{aligned} \pi^*\theta_{\eta^j} =& \pi^*\theta + (1 + 2j^{-1} - s)\pi^*\mathrm{dd^c}\varphi^j + s\mathrm{dd^c}\gamma_j \\ \geqslant& (1-s)\pi^*\theta_{\varphi^j} + \frac{2}{j}\pi^*\mathrm{dd^c}\varphi^j - s\epsilon_j\pi^*\omega \\ \geqslant& \left(-s\epsilon_j - (1+2j^{-1})\tau_j - 2C_1 j^{-1}\right)\pi^*\omega \\ \geqslant& \left(-\epsilon_j - 3\tau_j - 2C_1 j^{-1}\right)\pi^*\omega \end{aligned}$$

for some constant $C > 0$ independent of j. Here we used (5.7) eq:pistarthetalower. It follows that η^j descends to a quasi-psh function on X, still denoted by η^j. We then have

$$\theta_{\eta^j} \geqslant \left(-\epsilon_j - 3\tau_j - 2C_1 j^{-1}\right)\omega\,. \tag{5.9}$$

We can also regard η^j as a metric h^j on L, namely by considering $h_0 \exp(-\eta^j)$.

Step 2. Verification of the properties.

(1) Fix $s_1 > 0$ so that

$$\mathcal{I}(\varphi) = \mathcal{I}((1+s_1)\varphi)\,. \tag{5.10}$$

We first observe that by Lemma 2.22, for any smooth (n,p)-form with value in L on an open subset U of X, when $s > 0$ is small enough,

$$\int_U |f|^2_{h_j,\omega} \mathrm{e}^{-\eta_j}\,\omega^n \leqslant C(\|f\|_{L^\infty,h_0}) \left(\int_U |f|^2_{h_0,\omega} \mathrm{e}^{-(1+s_1)\varphi^j}\,\omega^n\right)^{1/(1+s_1)}$$

for all large j. Here C is independent of j. Using the fact $\varphi^j \geqslant \varphi$, we obtain (5.1).

It is by now clear that $\mathcal{I}(\varphi) \subseteq \mathcal{I}(\eta_j)$. By construction, η_j is more singular than $(1+2j^{-1})\varphi^j$, so $\mathcal{I}(\eta_j) \subseteq \mathcal{I}(\varphi)$. We complete the proof of the property (1).

(2) (5.2) follows from (5.9) and (5.5).

(3) Let $\hat{\lambda}^j_i := \lambda^j_i + 2\epsilon_j$. Observe that by (5.6),

$$\prod_{i=1}^n \hat{\lambda}^j_i \geqslant c\epsilon_j^{n-\mathrm{nd}(L,h)} \quad \text{on } X \setminus Z_j,$$

where $c > 0$ is a constant independent of j. Choose $\alpha \in (0,1)$ so that $n - \mathrm{nd}(L,h) < \alpha p$. Let $U_j = \{x \in X \setminus Z_j : \hat{\lambda}^j_p(x) < \epsilon_j^\alpha\}$.

Observe that

$$\int_{X\setminus Z_j} \sum_{i=1}^n (\hat{\lambda}^j_i)\,\omega^n = \int_{X\setminus Z_j} (\Lambda_\omega \mathrm{dd}^\mathrm{c} h^j)\,\omega^n + C = \int_{X\setminus Z_j} (\mathrm{dd}^\mathrm{c} h^j, \omega)_\omega\,\omega^n + C \leqslant C\,,$$

where $C > 0$ is independent of j. It follows that

$$\int_{U_j} \sum_{i=1}^n (\hat{\lambda}^j_i)\,\omega^n \leqslant C\,.$$

But on U_j,

$$\prod_{i=p+1}^n \hat{\lambda}^j_i \geqslant c\frac{\epsilon_j^{n-\mathrm{nd}(L,h)}}{\epsilon_j^{\alpha p}}\,.$$

Therefore,

$$\sum_{i=p+1}^n \hat{\lambda}^j_i \geqslant c\left(\frac{\epsilon_j^{n-\mathrm{nd}(L,h)}}{\epsilon_j^{\alpha p}}\right)^{1/(n-p)}\,.$$

We find that

$$\int_{U_j} \left(\frac{\epsilon_j^{n-\mathrm{nd}(L,h)}}{\epsilon_j^{\alpha p}}\right)^{1/(n-p)} \omega^n \leqslant M$$

for a different M, still independent of j. As $n - d < \alpha p$, we find that

$$\int_{U_j} \omega^n \leqslant M\epsilon_j^\beta$$

for some $\beta > 0$. We complete the proof of (3).

(4) This follows directly from our definition (5.8) and our normalization of γ_j in (5.6). ☐

6. Kollár's injectivity theorem

thm:Kolinj1 **Theorem 6.1** ([Mat18a]). *Let $f : X \to Y$ be a surjective proper Kähler morphism from a complex manifold X of pure dimension n to a complex analytic space Y. Let (L, h) be a Hermitian psef line bundle on X. Then for any section $s \in H^0(X, L^m)$ ($m \in \mathbb{Z}_{\geqslant 0}$) satisfying*

(1) s is not identically 0 on each connected component of X.

(2) $\sup_K |s|_{h^m} < \infty$ for each compact subset $K \subseteq X$.

Then the multiplication by s map induces an injection

{eq:Rqinj}
$$R^q f_*(\omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h)) \to R^q f_*(\omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h^{m+1})) \tag{6.1}$$

for all $q \geqslant 0$.

Observe that our problem is local on Y, so we may assume that Y is a Stein space and *a fortiori* X is holomorphically convex. In this case, (6.1) reduces to the map

{eq:Rqinj2}
$$H^q(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h)) \to H^q(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h^{m+1})) \tag{6.2}$$

induced by tensoring with s. So Theorem 6.1 is equivalent to the following theorem:

thm:Kolinj2 **Theorem 6.2.** *Let X be a holomorphically convex Kähler manifold. Let (L, h) be a Hermitian psef line bundle on X. Then for any section $s \in H^0(X, L^m)$ ($m \in \mathbb{Z}_{\geqslant 0}$) satisfying*

(1) s is not identically 0 on each connected component of X.

(2) $\sup_K |s|_{h^m} < \infty$ for each compact subset $K \subseteq X$.

Then for all $q \geqslant 0$, the map (6.2) induced by multiplication by s is injective.

Let us observe that the above reduction procedure gives more: by considering a smaller relatively compact Stein space Y' in Y and replacing X by $f^{-1}Y'$. We will repeatedly use this kind of simplifications in the following proof.

Proof. We may assume that $X \Subset \tilde{X}$, where $\tilde{X}$ satisfies the same conditions as $\tilde{X}$. Similarly, we may assume that L, h, s are all defined on $\tilde{X}$ and the assumptions in the theorem are met on $\tilde{X}$.

It follows that

$$\sup_X |s|_{h^m} < \infty\,.$$

Fix a complete Kähler metric ω on X. Fix a smooth psh exhaustion function Φ on X satisfying

$$\sup_X \Phi < \infty\,.$$

Let α be a cohomology class in the kernel of (6.2). We need to show that $\alpha = 0$.

Step 1. We construct suitable Kähler metrics in this step.

Take an equisingular approximation of h. The existence of such approximations is guaranteed by Theorem 2.13. More precisely, we take singular metrics h_k ($k \in \mathbb{Z}$) on L and a decreasing sequence of positive numbers ϵ_k converging to 0 with the following properties:

(1) h_k is smooth outside some nowhere dense closed analytic subset Z_k of X.
(2) $h \geqslant h_{k''} \geqslant h_{k'}$ for $k' < k''$.
(3) $\mathcal{I}(h) = \mathcal{I}(h_k)$.
(4) $\mathrm{dd^c} h_k \geqslant -\epsilon_k \omega$.

Here we are using the tricks at the beginning the proof again to embed X into a bigger space to achieve these properties.

We let $Y_k = X \setminus Z_k$.

Step 1.1 By Proposition 3.12, we can construct complete Kähler metrics ω_k on Y_k satisfying

(1) $\omega_k \geqslant \omega$ on Y_k.
(2) The local potentials of ω_k on X (not on Y_k) are locally bounded.

We will consider the following Kähler forms

$$\omega_{k,\delta} := \omega + \delta\omega_k \tag{6.3}$$

on Y_k for all $0 < \delta < \delta_{k,0}$, where $\delta_{k,0}$ is a suitable positive real number such that $\delta_{k,0} \ll \epsilon_k$. Then we have

(1) $\omega_{k,\delta}$ is a complete Kähler form on Y_k.
(2) $\omega_{k,\delta} \geqslant \omega$ on Y_k.
(3) For any $x \in X$, there is an open neighbourhood U of x, bounded functions $\Psi_{k,\delta}$ on U such that $\mathrm{dd^c}\Psi_{k,\delta} = \omega_{k,\delta}$ and $\lim_{\delta\to 0+} \Psi_{k,\delta}$ exists and is a local potential of ω.

We may assume that our psh exhaustion function Φ on X satisfies

$$\sup_X |\mathrm{d}\Phi|_{\omega_{k,\delta}} \leqslant C \tag{6.4}$$

for some constant C independent of k and $\delta < \delta_{k,0}$.

In fact, we may assume that

$$\sup_X |\mathrm{d}\Phi|_{\omega'} \leqslant C$$

for some Kähler form ω' on $\tilde{X}$. As ω is complete on X, we may assume that $\omega \geqslant \omega'$ up to a rescaling of ω'. Then as $\omega_{\epsilon,\delta} \geqslant \omega \geqslant \omega'$, we have

$$\sup_X |\mathrm{d}\Phi|_{\omega_{k,\delta}} \leqslant \sup_X |\mathrm{d}\Phi|_{\omega'} \leqslant C$$

by Lemma 3.8. In particular, the Bochner formula Proposition 2.15 applies to $\omega_{k,\delta}$ on Y_k.

Step 2. We represent α by suitable harmonic forms.

We first represent α by a closed (n,q) form u with locally L^2-coefficients.

Step 2.1

Fix an increasing convex function $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\|u\|_{h\exp(-\chi\circ\Phi),\omega} < \infty\,.$$

We let

$$H := h\exp(-\chi\circ\Phi), \quad H_k := h_k\exp(-\chi\circ\Phi)\,.$$

Moreover, let $\|\bullet\|_{k,\delta} := \|\bullet\|_{H_k,\omega_{k,\delta}}$. Then by Lemma 3.8,

{eq:uunifbdd}

$$\|u\|_{k,\delta,Y_k} \leqslant \|u\|_{H,\omega_{k,\delta}} \leqslant \|u\|_{H,\omega} < \infty\,. \tag{6.5}$$

Step 2.2 Consider the space

$$L^{n,q}_{(2)}(L)_{k,\delta} := L^{n,q}_{(2)}(Y_k, L)_{H_k,\omega_{k,\delta}}\,.$$

Then we find that $u \in L^{n,q}_{(2)}(L)_{k,\delta}$ as a consequence of (6.5). Also observe that

$$L^{n,q}_{(2)}(L)_{k,\delta} \supseteq L^{n,q}_{(2)}(L)_{H,\omega_{k,\delta}} \supseteq L^{n,q}_{(2)}(L)_{H,\omega}$$

and

$$L^{n,q}_{(2)}(L)_{k,\delta} \supseteq L^{n,q}_{(2)}(L)_{k,\delta'} \supseteq L^{n,q}_{(2)}(L)_{H_k,\omega}$$

for any $0 < \delta' < \delta < \delta_{k,0}$. Here we omit the canonical embeddings.

Recall the general orthogonal decomposition

$$L^{n,q}_{(2)}(L)_{k,\delta} = \overline{\operatorname{Im}\bar\partial} \oplus \mathcal{H}^{n,q}_{k,\delta}(L) \oplus \overline{\operatorname{Im}\bar\partial^*_{k,\delta}}\,,$$

where

$$\mathcal{H}^{n,q}_{k,\delta}(L) = \left\{ v \in L^{n,q}_{(2)}(L)_{k,\delta} : \bar\partial v = \bar\partial^*_{k,\delta} v = 0 \right\}$$

and $\bar\partial^*_{k,\delta}$ is the formal adjoint of $\bar\partial$. As u lies in the kernel of $\bar\partial$, its orthogonal projection to $\overline{\operatorname{Im}\bar\partial^*_{k,\delta}}$ vanishes (this follows from Corollary 3.6). So we find a decomposition

$$u = w_{k,\delta} + u_{k,\delta} \quad \text{for some } w_{k,\delta} \in \overline{\operatorname{Im}\bar\partial} \text{ and } u_{k,\delta} \in \mathcal{H}^{n,q}_{k,\delta}(L)\,. \tag{6.6}$$

Step 2.3

We claim that there exists a decreasing sequence $\delta^v > 0$ converging to 0 and $\alpha^k \in L^{n,q}_{(2)}(L)_{H_k,\omega}$ with the following properties:

(1) For any $k \geqslant 1$, $\delta' \in (0, \delta_{0,k})$, as $v \to \infty$, u_{k,δ^v} converges to α^k weakly in $L^{n,q}_{(2)}(L)_{k,\delta'}$.

(2) For any $k \geqslant 1$, we have

$$\|\alpha^k\|_{H_k,\omega} \leqslant \varliminf_{\delta'\to 0+} \|\alpha^k\|_{k,\delta'} \leqslant \varliminf_{v\to\infty} \|u_{k,\delta^v}\|_{k,\delta^v} \leqslant \|u\|_{H,\omega}\,. \tag{6.7}$$

We first observe that for any $k \geqslant 1$, $\delta' \in (0, \delta_{0,k})$, any $\delta \in (0, \delta')$, we have

$$\|u_{k,\delta}\|_{k,\delta'} \leqslant \|u_{k,\delta}\|_{k,\delta} \leqslant \|u\|_{k,\delta} \leqslant \|u\|_{H,\omega}\,.$$

Therefore, there is a decreasing sequence $\delta^{v,\delta'} \to 0$ such that $u_{k,\delta^{v,\delta'}}$ converges weakly to some $\alpha^k_{\delta'}$ in $L^{n,q}_{(2)}(L)_{k,\delta'}$. We take $M_k \in \mathbb{Z}_{>0}$ large enough so that $M_k^{-1} < \delta_{0,k}$. By repeatedly choosing subsequence of $\delta^{v,\delta'}$ and using a simple diagonal argument, we may guarantee that for $\delta' = 1/M$ ($M \in \mathbb{Z}_{>0}$ large enough), we have a decreasing sequence of positive numbers $\delta^v \to 0$ satisfying

$$u_{k,\delta^v} \overset{v}{\rightharpoonup} \alpha^k_{\delta'}$$

in $L^{n,q}_{(2)}(L)_{k,\delta'}$. Observe that $\alpha^k_{\delta'}$ is independent of δ' as $L^{n,q}_{(2)}(L)_{k,\delta'} \to L^{n,q}_{(2)}(L)_{k,\delta''}$ is bounded when $\delta' < \delta''$. We will write α^k for this common value. Then

$$u_{k,\delta^v} \overset{v}{\rightharpoonup} \alpha^k$$

in $L^{n,q}_{(2)}(L)_{k,\delta'}$. Part (1) of the claim follows.

As for the estimate, for any $k \geqslant 1$, $\delta' = 1/M \in (0, \delta_{0,k})$ for some integer M,

$$\|\alpha^k\|_{k,\delta'} \leqslant \varliminf_{v\to\infty} \|u_{k,\delta^v}\|_{k,\delta'} \leqslant \varliminf_{v\to\infty} \|u_{k,\delta^v}\|_{k,\delta^v} \leqslant \|u\|_{H,\omega}\,.$$

By Fatou's lemma,

$$\|\alpha^k\|^2_{H_k,\omega} = \frac{1}{n!}\int_{Y_k} |\alpha^k|^2_{H_k,\omega}\,\omega^n \leqslant \varliminf_{M\to\infty} \frac{1}{n!}\int_{Y_k} |\alpha^k|^2_{H_k,\omega_{k,1/M}}\,\omega^n_{k,1/M} = \varliminf_{M\to\infty} \|\alpha_k\|^2_{k,1/M}\,.$$

This proves (6.7) when δ' in the second term has the form $1/M$. But the general case follows as $\|\alpha^k\|_{k,\delta'}$ is decreasing in δ'.

Step 2.4. Fix $k_0>0$, for sufficiently large k, we have

$$\|\alpha^k\|_{H_{k_0},\omega} \leqslant \|\alpha^k\|_{H_k,\omega} \leqslant \|u\|_{H,\omega}\,.$$

There is therefore a sequence $k_v\to\infty$ such that α^{k_v} converges weakly to some $a \in L^{n,q}_{(2)}(L)_{H_{k_0},\omega}$.

Assume that $a=0$, then we claim that $\alpha=0$.

Note that this is just a slightly more complicated version of Lemma 2.18.

To prove the claim, take $\delta'>0$ of the form $1/M$ with M being a sufficiently large integer, consider the de Rham to Čech isomorphism

$$\ker\bar\partial/\operatorname{Im}\bar\partial \text{ of } L^{n,q}_{2,\mathrm{loc}}(L)_{k,\delta'} \to \check H^q(X,\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h_k)) = \check H^q(X,\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$$

constructed in Theorem 2.19. By Corollary 2.20, $\operatorname{Im}\bar\partial$ is closed in $L^{n,q}_{2,\mathrm{loc}}(L)_{k,\delta'}$.

Now for $\delta\in(0,\delta')$, we have

$$u-u_{k,\delta} \in \overline{\operatorname{Im}\bar\partial} \text{ in } L^{n,q}_2(L)_{k,\delta} \subseteq \overline{\operatorname{Im}\bar\partial} \text{ in } L^{n,q}_2(L)_{k,\delta'} \subseteq \overline{\operatorname{Im}\bar\partial} \text{ in } L^{n,q}_{2,\mathrm{loc}}(L)_{k,\delta'} = \operatorname{Im}\bar\partial \text{ in } L^{n,q}_{2,\mathrm{loc}}(L)_{k,\delta'}$$

Take limit in δ along δ^v, we find

$$u-\alpha^k \in \overline{\operatorname{Im}\bar\partial} \text{ in } L^{n,q}_2(L)_{k,\delta'} \subseteq \overline{\operatorname{Im}\bar\partial} \text{ in } L^{n,q}_{2,\mathrm{loc}}(L)_{k,\delta'} = \operatorname{Im}\bar\partial \text{ in } L^{n,q}_{2,\mathrm{loc}}(L)_{k,\delta'}\,.$$

Write $q_1 : \ker\bar\partial$ in $L^{n,q}_{(2)}(L)_{k,\delta'} \to \ker\bar\partial/\operatorname{Im}\bar\partial$ in $L^{n,q}_{(2)}(L)_{k,\delta'}$. Then $q_1(u-\alpha^k)=0$.

We will need the following basic fact: each element U of $\ker\bar\partial$ in $L^{n,q}_{(2)}(L)_{H_k,\omega}$ admits a canonical extension to an element of $\ker\bar\partial$ in $L^{n,q}_{(2)}(L)_{H_{k_0},\omega}$. See [Dem82, Lemme 6.9]. On the other hand, by Proposition 2.21, $q_2 : \ker\bar\partial$ in $L^{n,q}_{(2)}(L)_{H_{k_0},\omega} \to \ker\bar\partial/\operatorname{Im}\bar\partial$ in $L^{n,q}_{(2),\mathrm{loc}}(L)_{H_{k_0},\omega}$ is compact. In particular,

$$\lim_{v\to\infty} q_2(u-\alpha^{k_v}) = q_2(u-a) = q_2(u)\,.$$

Under the canonical identifications

$$\ker\bar\partial/\operatorname{Im}\bar\partial \text{ in } L^{n,q}_{(2),\mathrm{loc}}(L)_{k,\delta'} = \check H^q(X,\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h)) = \ker\bar\partial/\operatorname{Im}\bar\partial \text{ in } L^{n,q}_{(2),\mathrm{loc}}(L)_{H_{k_0},\omega}\,,$$

we have $q_1(u-\alpha^k)$ corresponds to $q_2(u-\alpha^k)$. It follows that $q_2(u)=0$. In other words, $u\in\operatorname{Im}\bar\partial$ in $L^{n,q}_{(2)}(L)_{H_{k_0},\omega}$. Using the canonical identifications

$$\ker\bar\partial/\operatorname{Im}\bar\partial \text{ in } L^{n,q}_{(2),\mathrm{loc}}(L)_{H_{k_0},\omega} = \check H^q(X,\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h)) = \ker\bar\partial/\operatorname{Im}\bar\partial \text{ in } L^{n,q}_{(2),\mathrm{loc}}(L)_{H_{k_0},\omega}$$

and the fact that $u\in\ker\bar\partial$ in $L^{n,q}_{(2)}(L)_{H,\omega}$, we find that $u\in\operatorname{Im}\bar\partial$ in $L^{n,q}_{(2),\mathrm{loc}}(L)_{H_{k_0},\omega}$. The claim is then proved.

Therefore, it remains to show that $a=0$.

Step 3. We make a further reduction in this step.

Fix $k_0>0$. Define $Y^j_{k_0} := \{y\in Y_{k_0} : |s|_{h^m_{k_0}}(y) > 1/j\}$. Write X_c for the set $\{\Phi<c\}$. Observe that $Y^j_{k_0}$ is an open subset of Y_{k_0}.

We will show that if

{eq:doubleliminf}

$$\varliminf_{k\to\infty}\varliminf_{\delta\to 0+}\|su_{k,\delta}\|_{k,\delta,X_c}=0 \tag{6.8}$$

for all $c<\sup_X\Phi$, then $a=0$ hence completing the proof.

Recall that α^{k_v} converges weakly to a in $L^{n,q}_{(2)}(L)_{H_{k_0},\omega}$, therefore, $\alpha^{k_v}|_{X_c\cap Y^j_{k_0}}$ converges weakly to $a|_{X_c\cap Y^j_{k_0}}$ in $L^{n,q}_{(2)}(X_c\cap Y^j_{k_0},L)_{H_{k_0},\omega}$. It follows that

$$\|a\|_{H_{k_0},\omega,X_c\cap Y^j_{k_0}}\leqslant \varliminf_{v\to\infty}\|\alpha^{k_v}\|_{H_{k_0},\omega,X_c\cap Y^j_{k_0}}\leqslant \varliminf_{v\to\infty}\|\alpha^{k_v}\|_{H_k,\omega,X_c\cap Y^j_{k_0}}\,.$$

Similarly,

$$\|\alpha^k\|_{k,\delta',X_c\cap Y^j_{k_0}}\leqslant \varliminf_{v\to\infty}\|u_{k,\delta^v}\|_{k,\delta',X_c\cap Y^j_{k_0}}\leqslant \varliminf_{v\to\infty}\|u_{k,\delta^v}\|_{k,\delta_v,X_c\cap Y^j_{k_0}}\,.$$

By Fatou's lemma,

$$\|\alpha^k\|_{H_k,\omega,X_c\cap Y^j_{k_0}}\leqslant \varliminf_{\delta'\to 0+}\|\alpha^k\|_{k,\delta',X_c\cap Y^j_{k_0}}\leqslant \varliminf_{v\to\infty}\|u_{k,\delta^v}\|_{k,\delta_v,X_c\cap Y^j_{k_0}}\,.$$

Putting these estimates together, we find

$$\|a\|_{H_{k_0},\omega,X_c\cap Y^j_{k_0}}\leqslant \varliminf_{v'\to\infty}\varliminf_{v\to\infty}\|u_{k^v,\delta^v}\|_{k,\delta_v,X_c\cap Y^j_{k_0}}\,.$$

On the other hand, $1/j<|s|_{h^m_{k_0}}\leqslant |s|_{h^m_k}$ on $Y^j_{k_0}$, so

$$\|u_{k^v,\delta^v}\|_{k,\delta_v,X_c\cap Y^j_{k_0}}\leqslant j\|su_{k^v,\delta^v}\|_{k,\delta_v,X_c\cap Y^j_{k_0}}\leqslant j\|su_{k^v,\delta^v}\|_{k,\delta_v,X_c}\,,$$

We therefore conclude that $u=0$ on $X_c\cap Y^j_{k_0}$. As c,j are arbitrary, we conclude that $a=0$.

Now it remains to establish (6.8).

Step 4. Next we carry out and $\bar\partial$-estimate.

We will prove the following claim: there is a solution to the $\bar\partial$-equation

$$\bar\partial w_{k,v}=u-u_{k,\delta^v}$$

with uniformly bounded local L^2-norm:

$$\varliminf_{v\to\infty}\|w_{k,v}\|_{k,\delta^v,X_c}\leqslant C_c \tag{6.9}$$

for any $c<\sup_X\Phi$, where C_c is independent to k.

We omit the complicated proof and just refer to [Mat18a, Proposition 3.9].

We will need the following consequence: for any $c<\sup_X\Phi$, there is $V_{k,v}\in L^{n,q-1}_{(2)}(L^{m+1})_{k,\delta^v}$ such that

(1) $\bar\partial V_{k,v}=su_{k,\delta^v}$.
(2) $\varlimsup_{v\to\infty}\|V_{k,v}\|_{k,\delta^v,X_c}<C_c$, where C_c is independent to k.

Recall that we have assumed that $s\alpha$ is exact, so there exists w with $\bar\partial w=su$ and $\|w\|_{Hh^m,\omega,X_c}<\infty$. It suffices to take $V_{k,v}=w-sw_{k,v}$.

Step 5. We will establish (6.8). Fix $c<c'<\sup_X\Phi$ so that $\mathrm{d}\Phi$ does not vanish on $\partial X_{c'}$. Such c' exists by Sard's theorem. We consider $V_{k,v}$ as in Step 4, with c' in place of c. We take regularizations of $V_{k,v}$, say $V_{k,v,j}$ so that as $j\to\infty$, $V_{k,v,j}\to V_{k,v}$ and $\bar\partial V_{k,v,j}\to\bar\partial V_{k,v}$, both in $L^{n,\bullet}_{(2)}(L^{m+1})_{k,\delta^v}$, as follows from Lemma 3.5.

For a generic $d' > d$, we have

{eq:limlimsplittwo}

$$
\begin{aligned}
&\varlimsup_{k\to\infty}\varlimsup_{\delta\to 0+}\|su_{k,\delta}\|_{k,\delta,X_c}\\
\leqslant\ &\varlimsup_{k\to\infty}\varlimsup_{\delta\to 0+}\|su_{k,\delta}\|_{k,\delta,X_d}\\
=\ &\varlimsup_{k\to\infty}\varlimsup_{v\to\infty}\lim_{j\to\infty}<su_{k,\delta},\bar{\partial}V_{k,v,j}>_{k,\delta,X_d}\\
=\ &\varlimsup_{k\to\infty}\varlimsup_{v\to\infty}\left(\lim_{j\to\infty}<\bar{\partial}^*_{k,\delta^v}su_{k,\delta^v},V_{k,v,j}>_{k,\delta^v,X_d}+\lim_{j\to\infty}<(\bar{\partial}\Phi)^*su_{k,\delta^v},V_{k,v,j}>_{k,\delta^v,\partial X_d}\right).
\end{aligned}
\tag{6.10}
$$

Here we applied the general Stokes' formula Proposition 3.10.

Next we will show that both terms vanish.

Step 5.1 Define

$$g_{k,\delta}:=2\pi<\mathrm{dd^c}H_k\wedge\Lambda_{k,\delta}u_{k,\delta},u_{k,\delta}>_{k,\delta}\,.$$

We claim that

{eq:glowerbound}

$$g_{k,\delta}\geqslant-\frac{2\pi q}{k}|u_{k,\delta}|^2_{k,\delta}\,. \tag{6.11}$$

In fact, for any $x\in X$, we can pick up a local coordinates in a neighbourhood of x, say $z_1,\dots,z_n$ so that

$$2\pi\mathrm{dd^c}H_k=\frac{\mathrm{i}}{2}\sum_{i=1}^n\lambda_i\mathrm{d}z_i\wedge\mathrm{d}\bar{z}_i$$

and

$$\omega_{k,\delta}=\frac{\mathrm{i}}{2}\sum_{i=1}^n\mathrm{d}z_i\wedge\mathrm{d}\bar{z}_i\,.$$

Here $\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n$ are the eigenvalues of $2\pi\mathrm{dd^c}H_k$ with respect to $\omega_{k,\delta}$. Locally write

$$u_{k,\delta}=\sum_{|\gamma|=q}u^\gamma_{k,\delta}\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n\wedge\mathrm{d}\bar{z}_\gamma\,.$$

Then

$$g_{k,\delta}=\sum_{|\gamma|=q}(\sum_{j\in\gamma}\lambda_j)|u^\gamma_{k,\delta}|^2_{H_k}\,.$$

On the other hand,

$$\mathrm{dd^c}H_k=\mathrm{dd^c}h_k+\mathrm{dd^c}\chi\circ\Phi\geqslant-\frac{1}{k}\omega\geqslant-\frac{1}{k}\omega_{k,\delta}\,.$$

So $\lambda_1\geqslant-\frac{2\pi}{k}$ and (6.11) follows.

As a consequence of (6.11),

{eq:glowerbound2}

$$
\begin{aligned}
0\geqslant\frac{1}{n!}\int_{\{y\in Y_k:g_{k,\delta}(y)\leqslant 0\}}g_{k,\delta}\,\omega^n_{k,\delta}&\geqslant-\frac{2\pi q}{n!k}\int_{\{y\in Y_k:g_{k,\delta}(y)\leqslant 0\}}|u_{k,\delta}|^2_{k,\delta}\,\omega^n_{k,\delta}\\
&\geqslant-\frac{2\pi q}{k}\|u_{k,\delta}\|^2_{k,\delta}\geqslant-\frac{2\pi q}{k}\|u\|^2_{H,\omega}\,.
\end{aligned}
\tag{6.12}
$$

Step 5.2 We prove the following preliminary result:

{eq:Dpstarkdeltato0}

$$\lim_{k\to\infty}\varlimsup_{\delta\to 0+}\|D'^*_{k,\delta}u_{k,\delta}\|_{k,\delta}=0\,. \tag{6.13}$$

Moreover,

$$\lim_{k\to\infty}\overline{\lim_{\delta\to 0+}}\,\|D'^{*}_{k,\delta}su_{k,\delta}\|_{k,\delta}=0 \tag{6.14}$$

and

$$\lim_{k\to\infty}\overline{\lim_{\delta\to 0+}}\,\|\bar{\partial}^{*}_{k,\delta}u_{k,\delta}\|_{k,\delta}=0\,. \tag{6.15}$$

By Bochner's formula (2.3) applied to $u_{k,\delta}$, we have

$$0=\|D'^{*}_{k,\delta}u_{k,\delta}\|^2_{k,\delta}+\frac{1}{n!}\int_{Y_k}g_{k,\delta}\,\omega^n_{k,\delta}\,.$$

So

$$\|D'^{*}_{k,\delta}u_{k,\delta}\|^2_{k,\delta}+\frac{1}{n!}\int_{\{y\in Y_k:g_{k,\delta}(y)\geqslant 0\}}g_{k,\delta}\,\omega^n_{k,\delta}=-\frac{1}{n!}\int_{\{y\in Y_k:g_{k,\delta}(y)\leqslant 0\}}g_{k,\delta}\,\omega^n_{k,\delta}\leqslant\frac{2\pi q}{k}\|u\|^2_{H,\omega}\,.$$

Therefore, (6.13) follows.

We obtain moreover that

$$\lim_{k\to\infty}\overline{\lim_{\delta\to 0+}}\int_{\{y\in Y_k:g_{k,\delta}(y)\geqslant 0\}}g_{k,\delta}\,\omega^n_{k,\delta}=0\,.$$

Next, we apply Bochner formula (2.3) to $su_{k,\delta}$ to obtain

$$\|\bar{\partial}^{*}_{k,\delta}su_{k,\delta}\|^2_{k,\delta}=\|D'^{*}_{k,\delta}su_{k,\delta}\|^2_{k,\delta}+\frac{1}{n!}\int_{Y_k}|s|^2_{h^m_k}g_{k,\delta}\,\omega^n_{k,\delta}\,.$$

Observe that

$$\int_{Y_k}|s|^2_{h^m_k}g_{k,\delta}\,\omega^n_{k,\delta}\leqslant\int_{\{y\in Y_k:g_{k,\delta}\geqslant 0\}}|s|^2_{h^m_k}g_{k,\delta}\,\omega^n_{k,\delta}\leqslant\sup_X|s|^2_{h^m}\int_{\{y\in Y_k:g_{k,\delta}\geqslant 0\}}g_{k,\delta}\,\omega^n_{k,\delta}\,.$$

On the other hand,

$$\|D'^{*}_{k,\delta}su_{k,\delta}\|_{k,\delta}=\|sD'^{*}_{k,\delta}u_{k,\delta}\|_{k,\delta}\leqslant\sup_X|s|_{h^m}\|D'^{*}_{k,\delta}u_{k,\delta}\|_{k,\delta}\,.$$

From these estimates (6.14) and (6.14) follow.

As a consequence, we have

$$\lim_{k\to\infty}\overline{\lim_{\delta\to 0+}}\lim_{j\to\infty}<\bar{\partial}^{*}_{k,\delta}su_{k,\delta},V_{k,\delta,j}>_{k,\delta,X_d}=0\,. \tag{6.16}$$

Recall that $d\in(c',\sup_X\Phi)$ is a general element.

By Cauchy–Schwarz inequality, it suffices to estimate two norms. From the construction of $v_{k,\delta}$, we know that

$$\lim_{k\to\infty}\overline{\lim_{\delta\to 0+}}\lim_{j\to\infty}\|V_{k,\delta,j}\|_{k,\delta,X_d}<\infty\,.$$

On the other hand,

$$\lim_{k\to\infty}\overline{\lim_{\delta\to 0+}}\,\|\bar{\partial}^{*}_{k,\delta}su_{k,\delta}\|_{k,\delta,X_d}=0$$

by (6.14). So (6.16) follows. We have completed the estimate of the first term in (6.10).

Step 5.3 We estimate the second term in (6.10). Namely

$$\underline{\lim}_{k\to\infty}\,\underline{\lim}_{v\to\infty}\lim_{j\to\infty}<(\bar{\partial}\Phi)^{*}su_{k,\delta^v},V_{k,v,j}>_{k,\delta^v,\partial X_d}\,.$$

Applying Cauchy–Schwarz inequality, we find that it suffices to prove the following two statements:

{eq:split1} (6.17)
$$\varliminf_{k\to\infty}\varliminf_{v\to\infty}\lim_{j\to\infty}\|V_{k,v,j}\|_{k,\delta^v,\partial X_d}<\infty$$

and

{eq:split2} (6.18)
$$\varliminf_{k\to\infty}\varliminf_{v\to\infty}\|(\bar\partial\Phi)^* s u_{k,\delta^v}\|_{k,\delta^v,\partial X_d}=0\,.$$

Recall that we use the measure $\mathrm{d}S:=*\mathrm{d}\Phi/|\mathrm{d}\Phi|_{\omega_{k,\delta^v}}$ on the boundary ∂X_d, so that $\omega_{k,\delta^v}^n/n!=\mathrm{d}\Phi\wedge\mathrm{d}S$.

We first prove (6.17). By Fubini's theorem,

$$\int_{c'-a}^{c'+a}\int_{\partial X_d}(V_{k,v,j},V_{k,v,j})_{k,\delta^v,\partial X_d}\,\mathrm{d}\Phi\,\mathrm{d}d=\frac{1}{n!}\int_{\{c'-a<\Phi<c'+a\}}|V_{k,v,j}|_{k,\delta^v}\,\omega_{k,\delta^v}^n$$

By Fatou's lemma, we have

$$\int_{c'-a}^{c'+a}\int_{\partial X_d}\varliminf_{k\to\infty}\varliminf_{v\to\infty}\lim_{j\to\infty}(V_{k,v,j},V_{k,v,j})_{k,\delta^v,\partial X_d}\,\mathrm{d}\Phi\,\mathrm{d}d\leqslant\varliminf_{k\to\infty}\varliminf_{v\to\infty}\|v_{k,\delta^v}\|^2_{k,\delta,X_{c'+a}}\,.$$

The right-hand side is finite by assumption and hence for a general d, the integrand is also finite. This proves (6.17).

It remains to prove (6.18). As $(\bar\partial\Phi)^* s u_{k,\delta^v}=s(\bar\partial\Phi)^* u_{k,\delta^v}$, it suffices to prove

{eq:split3} (6.19)
$$\varliminf_{k\to\infty}\varliminf_{v\to\infty}\|(\bar\partial\Phi)^* u_{k,\delta^v}\|_{k,\delta^v,\partial X_d}=0\,.$$

Applying Stokes formula Proposition 3.9, we have

$$\begin{aligned}<\bar\partial((\bar\partial\Phi)^*u_{k,\delta}),u_{k,\delta}>_{k,\delta,X_d}&=<(\bar\partial\Phi)^*u_{k,\delta},\bar\partial^*_{k,\delta}u_{k,\delta}>_{k,\delta,X_d}+<(\bar\partial\Phi)^*u_{k,\delta},(\bar\partial\Phi)^*u_{k,\delta}>_{k,\delta,\partial X_d}\\&=<(\bar\partial\Phi)^*u_{k,\delta},(\bar\partial\Phi)^*u_{k,\delta}>_{k,\delta,\partial X_d}\,.\end{aligned}$$

So we are reduced to prove

{eq:split4} (6.20)
$$\varliminf_{k\to\infty}\varliminf_{v\to\infty}<\bar\partial((\bar\partial\Phi)^*u_{k,\delta^v}),u_{k,\delta^v}>_{k,\delta,X_d}=0\,.$$

We observe that

$$\bar\partial u_{k,\delta}=0,\quad \partial\Phi\wedge u_{k,\delta}=0,\quad \mathrm{dd^c}\Phi\wedge u_{k,\delta}=0\,.$$

It follows from the twisted Kähler identity Lemma 3.7 that

$$<\bar\partial((\bar\partial\Phi)^*u_{k,\delta}),u_{k,\delta}>_{k,\delta,X_d}=-<\partial\Phi\wedge D'^*_{k,\delta}u_{k,\delta},u_{k,\delta}>_{k,\delta,X_d}+2\pi<\mathrm{dd^c}\Phi\wedge\Lambda_{k,\delta}u_{k,\delta},u_{k,\delta}>_{k,\delta,X_d}\,.$$

By (6.14) and Cauchy–Schwarz inequality, the first term tends to 0 if $\delta\to 0$ along δ^v.

So it remains to establish

{eq:split5} (6.21)
$$\varliminf_{k\to\infty}\varliminf_{v\to\infty}<\mathrm{dd^c}\Phi\wedge\Lambda_{k,\delta}u_{k,\delta},u_{k,\delta}>_{k,\delta,X_d}=0\,.$$

Here we need the twisted version of Bochner's formula Proposition 2.15:

$$\|\sqrt{\eta}(\bar\partial\Phi)u_{k,\delta}\|^2_{k,\delta}=\|\sqrt{\eta}(D'^*_{k,\delta}-(\bar\partial\Phi)^*)u_{k,\delta}\|^2_{k,\delta}+2\pi<\eta(\mathrm{dd^c}H_k+\mathrm{dd^c}\Phi)\Lambda_{k,\delta}u_{k,\delta},u_{k,\delta}>_{k,\delta,X_d}\,.$$

Using (6.11), we find

{eq:etapartPhi} (6.22)
$$\|\sqrt{\eta}(\bar\partial\Phi)u_{k,\delta}\|^2_{k,\delta}\geqslant\|\sqrt{\eta}(D'^*_{k,\delta}-(\bar\partial\Phi)^*)u_{k,\delta}\|^2_{k,\delta}-k^{-1}C\|u_{k,\delta}\|^2_{k,\delta}+2\pi<\eta\,\mathrm{dd^c}\Phi\wedge\Lambda_{k,\delta}u_{k,\delta},u_{k,\delta}>_{k,\delta,X_d}\,,$$

By Cauchy–Schwarz inequality,

$$\|\sqrt{\eta}(D'^{*}_{k,\delta}-(\bar{\partial}\Phi)^{*})u_{k,\delta}\|^2_{k,\delta} \geqslant -2\|\sqrt{\eta}D'^{*}_{k,\delta}u_{k,\delta}\|_{k,\delta}\|\sqrt{\eta}(\partial\Phi)^{*}u_{k,\delta}\|_{k,\delta}+\|\sqrt{\eta}(\partial\Phi)^{*}u_{k,\delta}\|^2_{k,\delta}\,,$$

where $C>0$ is independent of k and δ. From the twisted Kähler identity, we have

$$\|\sqrt{\eta}(\partial\Phi)^{*}u_{k,\delta}\|^2_{k,\delta} \geqslant \|\sqrt{\eta}(\bar{\partial}\Phi)u_{k,\delta}\|^2_{k,\delta}\,.$$

Therefore,

$$\|\sqrt{\eta}(D'^{*}_{k,\delta}-(\bar{\partial}\Phi)^{*})u_{k,\delta}\|^2_{k,\delta} \geqslant -2\|\sqrt{\eta}D'^{*}_{k,\delta}u_{k,\delta}\|_{k,\delta}\|\sqrt{\eta}(\partial\Phi)^{*}u_{k,\delta}\|_{k,\delta}+\|\sqrt{\eta}(\bar{\partial}\Phi)u_{k,\delta}\|^2_{k,\delta}\,.$$

Substituting back to (6.22), we find

$$2\|\sqrt{\eta}D'^{*}_{k,\delta}u_{k,\delta}\|_{k,\delta}\|\sqrt{\eta}(\partial\Phi)^{*}u_{k,\delta}\|_{k,\delta}+k^{-1}C\|u\|^2_{H,\omega} \geqslant 2\pi\langle\eta\mathrm{dd^c}\Phi\wedge\Lambda_{k,\delta}u_{k,\delta},u_{k,\delta}\rangle_{k,\delta,X_d}\,.$$

The term $\|\sqrt{\eta}D'^{*}_{k,\delta}u_{k,\delta}\|_{k,\delta}$ converges to 0 by (6.13) and the term $\|\sqrt{\eta}(\partial\Phi)^{*}u_{k,\delta}\|_{k,\delta}$ is bounded from above by the elementary estimate $|(\partial\Phi)^{*}u_{k,\delta}|_{k,\delta} \leqslant C|\partial\Phi|_{k,\delta}|u_{k,\delta}|_{k,\delta}$, which is uniformly bounded. On the other hand, by [Dem12, Discussion after (4.8)], we have $(\mathrm{dd^c}\Phi\wedge\Lambda_{k,\delta}u_{k,\delta},u_{k,\delta})_{k,\delta} \geqslant 0$. Together with the fact that η is bounded away from 0 on X_d, we conclude (6.21). □

As a consequence, we have the torsion-free theorem.

cor:torsionfree

Corollary 6.3. *Let $f: X\to Y$ be a surjective proper Kähler morphism from a complex manifold X of pure dimension n to a complex analytic space Y. Let (L,h) be a Hermitian psef line bundle on X. Then for any $q\geqslant 0$, the sheaf $R^qf_*(\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))$ is torsion-free.*

Proof. It suffices to apply the $m=0$ case of Theorem 6.1 to holomorphic functions on X of the form f^*g, where g is a holomorphic function on an open subset V of Y, not identically 0 on each connected component of V. □

Corollary 6.4. *Let $f: X\to Y$ be a surjective proper Kähler morphism from a complex manifold X of pure dimension n to a complex analytic space Y. Let (L,h) be a Hermitian psef line bundle on X. Assume that a general fiber of f has dimension at most N. Then*

$$R^qf_*(\omega_X\otimes\mathcal{L}\otimes\mathcal{I}(h))=0,\quad q>N\,.$$

Proof. By Corollary 2.3, $R^qf_*(\omega_X\otimes L\otimes\mathcal{I}(h))=0$ is supported on a nowhere dense proper closed analytic subspace of Y. This contradicts the fact that this sheaf is torsion-free Corollary 6.3. □

7. Positivity of direct images

thm:posdirima

Theorem 7.1 ([HPS18]). *Let $f: X\to Y$ be a proper surjective Kähler morphism from between complex manifolds X and Y. Let (L,h) be a Hermitian pseudo-effective line bundle on X. Then there is a canonical Griffiths positive metric on the torsion-free sheaf $f_*(\omega_{X/Y}\otimes\mathcal{L}\otimes\mathcal{I}(h))$.*

Observe that $f_*(\omega_{X/Y}\otimes\mathcal{L}\otimes\mathcal{I}(h))$ is always torsion-free as the push-forward of a torsion-free sheaf. We will call the metric constructed in this theorem the *Hodge metric*.

The general idea is to construct the metric on a Zariski open subset of Y, prove the positivity there and extend. Conditions guaranteeing the existence of extensions of psh metrics on line bundles is well-known, see [GR56]. The case of Griffiths positive metrics on vector bundles follows from the bijective correspondence between Griffiths positive metrics and Finsler metrics. The case of torsion-free sheaves follows trivially from the case of vector bundles.

Proof. By considering each connected component of Y separately, we may assume that Y is a connected manifold of dimension m. We can then assume that X is connected and of dimension n. Write

$$\mathcal{F} := f_*(\omega_{X/Y} \otimes \mathcal{L} \otimes \mathcal{I}(h)) .$$

For any $y \in Y$, we write X_y for the fiber of f over y. Similarly, we write $L_y = L|_{X_y}$, $\mathcal{L}_y = \mathcal{L}|_{X_y}$ and $h_y = h|_{L_y}$.

Step 1. We construct the metric H on $\mathcal{F}$ outside a proper closed analytic subset $Z \subsetneq Y$.

Choose a proper closed analytic subset $Z \subsetneq Y$ such that

(1) f is smooth outside Z. This is possible by Theorem 2.4.
(2) Both $\mathcal{F}$ and $f_*(\omega_{X/Y} \otimes \mathcal{L})/\mathcal{F}$ are locally free on $Y \setminus Z$. Here we use the properness of f.
(3) $\omega_{X/Y} \otimes \mathcal{L}$ has the base change property with respect to f on $Y \setminus Z$. Here we use Corollary 2.3.

Let F be the vector bundle on $Y \setminus Z$ so that $\mathcal{O}_{Y\setminus Z}(F) = \mathcal{F}|_{Y\setminus Z}$. Then we find

{eq:Eysubh0}

$$E_y \subseteq H^0(X_y, \omega_{X_y} \otimes L_y) . \tag{7.1}$$

By the Ohsawa–Takegoshi extension theorem,

$$H^0(X_y, \omega_{X_y} \otimes \mathcal{L}_y \otimes \mathcal{I}(h_y)) \subseteq E_y .$$

Next we define a singular Hermitian inner product H_y on E_y for $y \in Y \setminus Z$: given $\alpha \in E_y$, we can regard α as an element in $H^0(X_y, \omega_{X_y} \otimes L_y)$ by (7.1). eq:Eysubh0 We then define

$$|\alpha|^2_{H_y} := \int_{X_y} |\alpha \wedge \bar{\alpha}|^2_{h_y} \in [0, \infty] .$$

We observe that $\{\alpha \in E_y : |\alpha|_{H_y} = 0\} = 0$, so $H_y^\vee$ is everywhere finite.

Step 2. We want to prove that H is Griffiths positive.

Take an open set $U \subseteq Y$ and a section $g \in H^0(U, \mathcal{F}^\vee)$. We want to show that

$$\psi := \log |g|_{H^\vee} : U \setminus Z \to [-\infty, \infty)$$

is psh and has a psh extension to U. This amounts to three different claims, as proved in each of the following substeps.

Step 2.1. We prove that ψ is locally bounded from above near Z.

Choose open sets $V_1 \Subset V_2 \Subset U$ so that for any $x \in V_1$, there is an embedding $\iota : B^m \hookrightarrow V_2$ with $\iota(0) = x$.

Fix $y \in V_1 \setminus Z$, we want to find an upper bound of $\psi(y)$. Of course, we may assume that $\psi(y) > -\infty$. Choose $\alpha \in E_y$ with $|\alpha|_{H_y} = 1$ and $|g|_{H_y^\vee} = |g(\alpha)|$. So that

$$\psi(y) = \log |g(\alpha)| .$$

Choose an embedding $\iota : B^m \to V_2$ with $\iota(0) = y$. We will omit ι from our notations and regard B^m as an open subset of V_2. By the Ohsawa–Takegoshi extension theorem Theorem 4.1, we can find $s \in H^0(B^m, \mathcal{F})$ with $s(0) = \alpha$ and

$$\int_{B^m \setminus Z} |s|_H^2 \, \mathrm{d}\mu \leqslant \mu(B^m)\,,$$

where $\mathrm{d}\mu$ is the Lebesgue measure on B^m. It follows that $g(s)$ on B^m is bounded from above by a constant depending only on C_0.

Step 2.2 We show that ψ is usc on $Y \setminus Z$. This problem is local, so we may assume that $Y = B^m$ and $Z = \emptyset$. We show that ψ is usc at $y = 0$:

$$\varlimsup_{k\to\infty} \psi(y_k) \leqslant \psi(0)$$

for any sequence $y_k \to 0$ in B^m. We may assume that $\psi(y_k) \neq -\infty$ for all k and the limsup is an actual limit. Take $\alpha_k \in E_{y_k}$ such that $\psi(y_k) = \log|g(\alpha_k)|$ and $|\alpha_k|_{H_{y_k}} = 1$. Extend α_k to a holomorphic section $s_k \in H^0(B^m, \mathcal{F})$ so that $\int_{B^m} |s_k \wedge \bar{s}_k|_H \, \mathrm{d}\mu \leqslant \mu(B^m)$ by the Ohsawa–Takegoshi theorem Theorem 4.1. By compactness, there is sequence $k_i \to \infty$ such that s_{k_i} converges to some s with respect to the compact-open topology. It follows that $g(s_{k_i})$ converges to $g(s)$ with respect to the compact-open topology. By definition of the dual metric, $\psi \geqslant \log|g(s)| - \log|s|_H$, so what we need to show is that $|s(0)|_{H_0} \leqslant 1$. As $f : X \to B^m$ is smooth, by Ehresmann's fibration theorem, X is diffeomorphic to $X_0 \otimes B^m$. Choose a Kähler metric ω_0 on X_0, then we can find a lsc and locally integrable function $F : X_0 \times B^m \to [0, \infty]$ such that

$$|s_k \wedge \bar{s}_k \wedge \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_m|_h^2 = F_k \frac{\omega_0^{m-n}}{(m-n)!}\,. \tag{7.2}$$

In particular,

$$|s_k|^2_{H_{y_k}} = \int_{X_0} F_k(\bullet, y) \frac{\omega_0^{m-n}}{(m-n)!}\,.$$

Similarly define $F : X_0 \times B^m \to [0, \infty]$ using s instead of s_k. As the local weights of h is usc and s_k converges to s uniformly on compact sets, we have

$$F(\bullet, 0) \leqslant \varliminf_{i\to\infty} F_{k_i}(\bullet, y_{k_i})\,.$$

The desired inequality then follows from Fatou's lemma.

Step 2.3. We show that ψ is plurisubharmonic on $Y \setminus Z$. By Fornaess–Narasimhan theorem, we may assume replace Y by a disk Δ and assume that $Z = \emptyset$.

We will verify the mean-value inequality:

$$\psi(0) \leqslant \frac{1}{\pi} \int_\Delta \gamma^\psi \, \mathrm{d}\mu\,. \tag{7.3}$$

Of course, we may assume that $\psi(0)$ is not $-\infty$. Choose $\alpha \in E_0$ with $|\alpha|_{H_0} = 1$ and $\psi(0) = \log|g(\alpha)|$. By the Ohsawa–Takegoshi extension theorem Theorem 4.1, we may extend α to a holomorphic section $s \in H^0(\Delta, E)$ such that $s(0) = \alpha$ and

$$\int_\Delta |s|_H^2 \, \mathrm{d}\mu \leqslant \pi\,.$$

By definition of the dual metric,

$$\psi \geqslant \log|g(s)| - \log|s|_H$$

for any holomorphic function g on Δ with $g(0) = g(\alpha)$. It follows that

(7.4)
$$\frac{2}{\pi}\int_\Delta \psi \,\mathrm{d}\mu \geqslant \frac{1}{\pi}\int_\Delta \log|g(s)|^2\,\mathrm{d}\mu - \frac{1}{\pi}\int_\Delta \log|s|_H^2\,\mathrm{d}\mu \geqslant 2\psi(0) - \log\left(\frac{1}{\pi}\int_\Delta |s|_H^2\,\mathrm{d}\mu\right) \geqslant 2\psi(0)\,.$$

This proves the desired result. □

As an immediate consequence of our construction, we have the following explicit description of the Hodge metric.

Corollary 7.2. *Under the assumptions of Theorem 7.1, there is a nowhere dense closed analytic subset $Z \subseteq Y$ such that the following are satisfied*

(1) f is smooth outside Z.
(2) $f_*(\omega_{X/Y} \otimes \mathcal{L} \otimes \mathcal{I}(h))$ is locally free on $Y \setminus Z$. We write F for the vector bundle on $Y \setminus Z$ associated with this sheaf.
(3) For any $y \in Y \setminus Z$, any $\alpha \in F_y$, we have

(7.5)
$$|\alpha|^2_{H_y} = \int_{X_y} |\alpha \wedge \bar{\alpha}|_h\,,$$

where we identify α with an element in $H^0(X_y, \omega_{X_y} \otimes L_y)$.
(4)
$$H^0(X_y, \omega_{X_y} \otimes \mathcal{L}_y \otimes \mathcal{I}(h_y)) \subseteq F_y \subseteq H^0(X_y, \omega_{X_y} \otimes \mathcal{L}_y)\,.$$

8. Bertini type theorems

Theorem 8.1 ([Xia21]). *Let $f : X \to Y$ be a projective surjective morphism between complex manifolds X and Y. Let (L, h) be a Hermitian psef line bundle on X. Then for quasi-every $y \in Y$, the fiber X_y is smooth and*

(8.1)
$$\mathcal{I}(kh_y) = \mathcal{I}(kh)|_{X_y}$$

for all real $k > 0$.

Remark 8.2. Due to the lack of Chow's lemma in the complex analytic setting (which fails unless the proper morphism is bimeromorphic), it is not clear if Theorem 8.1 holds for a proper morphism f.

On the other hand, for a general proper surjective morphism $f : X \to Y$ from a complex manifold X to a complex analytic space Y, it is obvious that (8.1) holds almost everywhere. Here properness guarantees that outside a null subset of Y, the fibers of f are smooth.

Proof. We take $Z \subseteq Y$ as in Corollary 7.2. We use the notation F as in Corollary 7.2. For any $y \in Y \setminus Z$, we have

(8.2)
$$H^0(X_y, \omega_{X_y} \otimes \mathcal{L}_y \otimes \mathcal{I}(h_y)) \subseteq F_y \subseteq H^0(X_y, \omega_{X_y} \otimes \mathcal{L}_y)\,.$$

Observe that an element $\alpha \in F_y$ lies in $H^0(X_y, \omega_{X_y} \otimes \mathcal{L}_y \otimes \mathcal{I}(h_y))$ if and only if $\|\alpha\|_{H_y} < \infty$. It follows that if the first inclusion of (8.2) is strict, then H_y is singular and *a fortiori* $\det H$ is singular at y. But we already know that the Hodge metric H is Griffiths positive Theorem 7.1, so $\det H$ is positively curved. It follows that the first inclusion in (8.2) is an equality almost everywhere. On the other hand, by Corollary 2.3, outside a nowhere dense closed analytic subset of $Y \setminus Z$, $F_y = H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{I}(h)|_{X_y})$. It follows that

$$H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{I}(h)|_{X_y}) = H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{I}(h|_{X_y}))$$

for every $y \in Y \setminus \Sigma(L, h, f)$, where $\Sigma(L, h, f)$ is a pluripolar subset of Y.

Now we need to use the projectivity of f (instead of proper Kähler) for the first time. As our problem is local in Y, we may assume that Y is Stein. Take an f-ample line bundle S on X with associated invertible sheaf $\mathcal{S}$. Take a smooth positively curved metric h_S on S.

Assume that the cokernel $\mathcal{J}$ of the inclusion $\mathcal{I}(h|_{X_y}) \to \mathcal{I}(h)|_{X_y}$ is non-zero for some $y \in Y \setminus \bigcup_{C \in \mathbb{Z}_{\geqslant 0}} \Sigma(L \otimes S^C, h \otimes h_S^C, f)$. Then there is a large integer C such that

$$H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{S}_y^{\otimes C} \otimes \mathcal{J}) \neq 0$$

and

$$H^1(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{S}_y^{\otimes C} \otimes \mathcal{I}(h|_{X_y})) = 0\,.$$

It then follows from the exact sequence

$$\begin{aligned} 0 \to H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{S}_y^{\otimes C} \otimes \mathcal{I}(h|_{X_y})) &\to H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{S}_y^{\otimes C} \otimes \mathcal{I}(h)|_{X_y}) \\ &\to H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{S}_y^{\otimes C} \otimes \otimes \mathcal{J}) \to 0 \end{aligned}$$

that

$$H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{S}_y^{\otimes C} \otimes \mathcal{I}(h|_{X_y})) \neq H^0(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{S}_y^{\otimes C} \otimes \mathcal{I}(h)|_{X_y})\,,$$

which contradicts our choice of y. It follows that $\mathcal{I}(h)|_{X_y} = \mathcal{I}(h|_{X_y})$ outside the pluripolar set $\bigcup_{C \in \mathbb{Z}_{\geqslant 0}} \Sigma(L \otimes S^C, h \otimes h_S^C, f)$.

Next we prove (8.1), by strong openness theorem, we only need to consider countably many $k \in \mathbb{Q}_{>0}$. As countable unions of pluripolar sets are still pluripolar, it suffices to prove (8.1) for a single $k \in \mathbb{Q}_{>0}$. It suffices to regard kh_y as a positively curved metric on $L \otimes S^C$ for a large enough C and apply what we have proved. □

cor:ndgencons **Corollary 8.3** ([Mat18b]). *Let $f : X \to Y$ be a proper Kähler morphism from a connected complex manifold X to a connected complex analytic space Y. Let (L, h) be a Hermitian psef line bundle on X. Then for almost all $y \in Y$, X_y is smooth and* $\mathrm{nd}(L_y, h_y)$ *is independent of the choice of y.*

If moreover f is projective and Y is smooth, then for quasi-every $y \in Y$, X_y is smooth and $\mathrm{nd}(L_y, h_y)$ *is independent of the choice of y.*

Proof. The problem is local on Y, so we may assume that Y is Stein. In fact, by further localization, we may assume that $Y \Subset Y'$ for some Y' and X' satisfying the same assumptions. In particular, we may assume that there is a quasi-equisingular approximation h^i of h on X. Fix a Kähler form ω on X. Up to removing a nowhere dense closed analytic subset from Y, we may assume that f is smooth of pure relative dimension r.

We only prove the latter statement, as the first is similar using Remark 8.2 instead of Theorem 8.1.

By Theorem 8.1, h^i restricts to a quasi-equisingular approximation of h_y for quasi-every y. Take a log resolution $\pi_i : X_i \to X$ of h^i and write $\mathrm{dd^c}\pi_i^* h^i = [E_i] + \alpha_i$, where α_i is smooth and E_i is a nc divisor on X_i. Up to removing a nowhere dense closed analytic subset from Y, we may assume that the restriction of π_i to all fibers X_y are still log resolutions of $h^i|_{X_y}$ and $\pi_i^{-1}(X_y)$ is not contained in E_i. Observe that

$$\int_{X_y} \left(\mathrm{dd^c} h^i|_{X_y}\right)^a_{\mathrm{ac}} \wedge \omega|_{X_y}^{r-a} = \int_{\pi_i^{-1}(X_y)} (P_i^a \wedge f_i^* \omega^{r-a})_{\pi_i^{-1}(X_y)} .$$

The right-hand side, as a closed fiber integration, is constant outside a nowhere dense closed analytic subset. It follows that the left-hand side is also constant outside a nowhere dense closed analytic subset. But $h^i|_{X_y}$ is a quasi-equisingular approximation of $h|_{X_y}$ for quasi-every $y \in Y$, so we conclude that Cao's mixed mass $\langle (\mathrm{dd^c} h|_{X_y})^a \wedge \omega^{r-a}|_{X_y}\rangle$ is constant quasi-everywhere. In particular, $\mathrm{nd}(L_y, h_y)$ is constant quasi-everywhere. □

Definition 8.4. Let $f : X \to Y$ be a proper Kähler morphism from a connected complex manifold X to a connected complex analytic space Y. Let (L, h) be a Hermitian psef line bundle on X. Take a null set $\Sigma \subseteq Y$ so that for $y \in Y \setminus \Sigma$, X_y is smooth $\mathrm{nd}(L_y, h_y)$ is constant. We define the numerical dimension $\mathrm{nd}_f(L, h)$ of f as this constant value.

We can now state the relative version of Theorem 5.1.

Corollary 8.5. *Let $f : X \to Y$ be a proper Kähler morphism from a connected complex manifold X to a connected complex analytic space Y. Let (L, h) be a Hermitian psef line bundle on X.*

$$R^q f_*(\omega_X \otimes \mathcal{L} \otimes \mathcal{I}(h)) = 0 \quad \textit{for } p > \dim X - \dim Y - \mathrm{nd}_f(L, h) .$$

Proof. This is a simple consequence of the torsion-free theorem Corollary 6.3 and Corollary 8.3. □

cor:ieqfib **Corollary 8.6.** *Let X be a complex manifold, $f : X \to \Delta^*$ be a projective surjective morphism. Let (L, h), (L, h') be Hermitian pseudo-effective line bundles on X with the same underlying line bundle. Assume that there is a biholomorphic S^1-action on (X, L) making f equivariant and such that h and h' are invariant under this action. Assume that for quasi-every $z \in \Delta^*$, X_z is smooth and $h|_{X_z} \sim_{\mathcal{I}} h|'_{X_z}$, then $h \sim_{\mathcal{I}} h'$.*

Proof. We need to show that $\mathcal{I}(kh) = \mathcal{I}(kh')$ for all positive integer k. Clearly, it suffices to prove the case $k = 1$. We will therefore prove $\mathcal{I}(h) = \mathcal{I}(h')$. First observe that it suffices to prove that

{eq:fstarcoin}

$$f_*(K_X \otimes L \otimes \mathcal{I}(h)) = f_*(K_X \otimes L \otimes \mathcal{I}(h')) \tag{8.3}$$

as subsheaves of $f_*(K_X \otimes L)$. In fact, suppose that (8.3) holds. Let H be a f-ample invertible sheaf on X, then (8.3) also holds with $L \otimes H^m$ in place of L. It follows from Grauert–Remmert's version of Serre vanishing theorem [BS76, Theorem 2.1(A)] that $\mathcal{I}(h) = \mathcal{I}(h')$.

It remains to prove (8.3). Observe that both sides of (8.3) are locally free as they are clearly torsion-free, we claim that it suffices to show that

{eq:fstarcoin2}

$$f_*(K_X \otimes L \otimes \mathcal{I}(h))_z = f_*(K_X \otimes L \otimes \mathcal{I}(h'))_z \tag{8.4}$$

for one $z \in \Delta^*$. In fact, this implies that the same holds outside a countable subset of Δ^*. But the set where (8.4) fails has to be S^1-invariant, it has to be empty.

In fact, we will prove (8.4) for quasi-every $z \in \Delta^*$. By cohomology and base change together with Theorem 8.1, for quasi-every $z \in \Delta^*$, we have

$$
\begin{aligned}
f_*(K_X \otimes L \otimes \mathcal{I}(h))_z &= H^0(X_z, K_X|_{X_z} \otimes L|_{X_z} \otimes \mathcal{I}(h|_{X_z})),\\
f_*(K_X \otimes L \otimes \mathcal{I}(h'))_z &= H^0(X_z, K_X|_{X_z} \otimes L|_{X_z} \otimes \mathcal{I}(h'|_{X_z}))\,.
\end{aligned}
$$

But we assumed that for quasi-every z, $h|_{X_z} \sim_{\mathcal{I}} h|'_{X_z}$, it follows that for quasi-every $z \in \Delta^*$, (8.4) holds. The proof is complete. □

It is of interest to understand more general types of analytic Bertini theorems. In particular, we ask the following question: given a morphism of complex manifolds $f : X \to Y$ with smooth fibers and two quasi-psh functions φ, ψ on X. Assume that $\varphi|_{X_y} \sim_{\mathcal{I}} \psi|_{X_y}$ for all $y \in Y$, then is it true that $\varphi \sim_{\mathcal{I}} \psi$.

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